

# Essays on Macroeconomics

Keshav Dogra

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## ABSTRACT

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The three chapters of my dissertation study the effect of access to credit on economic volatility and welfare, and the implications for policy. Chapter 1 presents a unified framework to analyze debt relief and macroprudential policies in a liquidity trap when households have private information. I develop a model with a deleveraging-driven recession and a liquidity trap in which households differ in their impatience, which is unobservable. Ex post debt relief stimulates the economy, but anticipated debt relief encourages overborrowing ex ante, making savers worse off. Macroprudential taxes and debt limits prevent the recession, but can harm impatient households, since the planner cannot directly identify and compensate them. I solve for optimal policy, subject to the incentive constraints imposed by private information. Optimal allocations can be implemented either by providing debt relief to moderate borrowers up to a maximum level, combined with a marginal tax on debt above the cap, or with ex ante macroprudential policy - a targeted loan support program, combined with a tax on excessive borrowing. These policies are ex ante Pareto improving in a liquidity trap; in normal times, however, they are purely redistributive. These results extend to economies with aggregate uncertainty, alternative sources of heterogeneity, and endogenous labor supply.

The second chapter of my dissertation presents a theoretical framework to understand sovereign debt crises in a monetary union and the optimal policy response to these crises. The risk of default encourages indebted countries to pay down their short term debt, depressing consumption demand throughout the union. This fall in demand can cause the monetary union to hit the zero lower bound on nominal interest rates, leading to a union-wide recession. I evaluate three policies to prevent such a recession: debt relief, which writes off a portion of short term debt; lending policy, which allows indebted countries to issue new debt at above-market prices; and debt postponement, which converts short into long term debt. I show that if countries can be prevented from retrading in secondary markets after debt restructuring, all three policies are equivalent, and are welfare improving. If retrading is possible, lending policy and debt postponement are superior to debt relief.

The final chapter of my dissertation evaluates the impact of increased income uncertainty and financial liberalization in the US on consumption volatility and welfare at the household level. In this joint work with Olga Gorbachev, we estimate Euler equations using consumption data from the Panel Study of Income Dynamics, and measure the volatility of unpredictable changes in consumption as the squared residuals. We directly control for liquidity constraints using data on access to credit from the Survey of Consumer Finances, and document that despite the increase in household debt between 1983 and 2007, there was no decline in the proportion of liquidity constrained households. Consumption volatility increased significantly over this period, especially for liquidity constrained households, indicating substantial welfare losses.

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*To my parents*

# *Liquidity traps, debt relief, and macroprudential policy: a mechanism design approach*

## **1.1 Introduction**

The Great Recession saw an extraordinary contraction in output, employment and consumption, driven in large part by household deleveraging. There are two obvious remedies for a debt-driven recession: prevent borrowing ex ante, or write off debt ex post. A recent theoretical literature makes the case for both policies. Debt relief is an ex post optimal policy in a liquidity trap: transfers to indebted households, who have a high propensity to consume, stimulate demand, raise aggregate income, and benefit everyone, even the households who are taxed to pay for these transfers. Alternatively, macroprudential taxes or limits on borrowing can prevent the overborrowing that leads to a recession in the first place.

But both debt relief and macroprudential regulation face criticisms, which are fundamentally linked to the existence of private information. The literature mentioned above asks whether debt relief is optimal in a liquidity trap, taking as given the distribution of household debt. However, a common concern is that bailouts encourage households to borrow even more ex ante, making the recession deeper. A social planner would like to write off debt for households who would have borrowed anyway, without inducing anyone to borrow more than they would have done in the absence of policy, but this is not possible, since households' propensities to borrow are private information. Thus transfers targeted to debtors inevitably encourage even patient households to take on more debt, making the recession worse. Equally, one criticism of macroprudential limits on borrowing is that they harm households who want to borrow. This is not a concern under full information, since the planner can directly identify and compensate these households, leaving them no worse off. However, if the government cannot observe a household's type, compensating transfers are not possible, and there is an efficiency-equity tradeoff: macroprudential

policy prevents a recession, but harms borrowers. Given the constraints imposed by private information, can *any* transfer policies avert a liquidity trap and make everyone better off?

To answer this question, I take a mechanism design approach to study debt relief and macroprudential policy. I build a model with three key ingredients. First, there is a distribution of households who differ in their propensity to borrow, which can be interpreted as impatience, and which is private information. Second, interest rates are constrained by a zero lower bound (ZLB); when the ZLB binds, output is demand determined. Finally, there is an exogenous contraction in the borrowing constraint - which is perfectly anticipated in the baseline model - which can make the ZLB bind. The borrowing constraint also generates heterogeneity in households' marginal propensity to consume (MPC): highly indebted households will be liquidity constrained, and have a higher MPC than savers, who are not constrained. Thus in this economy, unanticipated transfers from savers to borrowers increase aggregate demand, and this in turn increases aggregate income when the ZLB binds, making all households better off *ex post*.

However, I show that the concerns raised above are valid: *anticipated* debt relief may not be *ex ante* Pareto improving, because it encourages overborrowing on both an intensive and an extensive margin. On the intensive margin, if the government does not commit *ex ante* to limit the scale of debt relief, borrowers take on more debt, since they are now richer in the future. This means the government must tax savers more heavily to write off borrowers' debt, making savers worse off, relative to a world without debt relief. On the extensive margin, even if the government commits to a cap on debt relief, patient savers may overborrow in order to mimic impatient borrowers and qualify for the transfer. Macroprudential policy also faces constraints. *Ex ante* debt limits or taxes on borrowing prevent borrowers from taking on too much debt, increase aggregate demand, and mitigate the liquidity trap. However, limits on borrowing may not be *ex ante* Pareto improving, because they harm households who want to borrow.

To study optimal policy, I consider the problem of a social planner who chooses allocations subject to the ZLB, the borrowing constraint, and private information, which imposes incentive compatibility constraints stating that no household's allocation can be so generous that another household wants to mimic them. By varying the weight the planner puts on each agent's utility, I trace out the constrained Pareto frontier. I prove an equivalence result: any solution to the social planner's problem can be implemented as an equilibrium with transfers that depend on

a household's debt level, either at date 1 (ex post redistribution) or at date 0 (ex ante macroprudential policy). Furthermore, efficient allocations can be implemented with particular simple policies. First, they can be implemented with ex post debt relief with a cap. Under such a policy, the government writes off debt up to some maximum level. Above that amount, additional borrowing is taxed, discouraging overborrowing on the intensive margin. Equivalently, efficient allocations can be implemented with ex ante targeted loan support programs, which provide a transfer to households who borrow above some minimum level, coupled with a macroprudential tax on borrowing above that level.

In fact, debt relief with a cap (equivalently, targeted loan support) can be ex ante Pareto improving relative to the competitive equilibrium. In an economy with two types, there always exists a Pareto improving debt relief policy when the ZLB binds in equilibrium. Incentive constraints eventually restrict transfers from savers to borrowers, once these transfers become too large. But in competitive equilibrium, there are no transfers, and incentive constraints are slack: each individual strictly prefers her own allocation. Starting from equilibrium, there is always *some* room to redistribute to borrowers without violating incentive constraints. However, when borrowers are not too impatient relative to savers, the ZLB does not bind, and the competitive equilibrium is constrained efficient. In this case, debt relief (equivalently, targeted loan support) has a purely redistributive role: it implements allocations which are better for borrowers, but worse for savers, relative to the competitive equilibrium.

One concern is that in a two agent economy, it may be too easy to design debt relief programs which induce no extensive margin overborrowing. To address this concern, I also study optimal policy with a continuous distribution of types, using Lagrangian methods similar to those developed by [Amador et al. \[2006\]](#). In this economy, transfers targeted to highly indebted households always induce some less indebted households to borrow more; optimal policy trades off these distortions against the benefits from debt relief. Nonetheless, I show that an equilibrium in which the ZLB binds is always ex ante Pareto inefficient, and debt relief with a cap (equivalently, targeted loan support) remains ex ante Pareto efficient. However, simple linear policies may not be Pareto improving. In particular, they may make the most indebted borrowers worse off, since they impose a marginal tax on excessive levels of debt.

I then address three further concerns regarding these results. First, while in the baseline

model the contraction in borrowing constraints is perfectly anticipated, a more realistic assumption is that this shock only occurs with some probability. In this case, there is an even stronger case for debt relief. The less agents anticipate the crisis, the less incentive concerns restrict debt relief: if the crisis is completely unanticipated, concerns about ex ante incentives vanish completely. Moreover, with aggregate risk and incomplete markets, there is an additional role for debt relief, namely to complete markets and insure agents against a contraction in borrowing constraints. A second concern might be that debt relief or macroprudential taxes might not be desirable if households have different motives for borrowing - perhaps if borrowers borrow because they expect high future income, rather than because they are impatient, writing off their debt will not stimulate demand. I show that even when I extend the model to include alternative motives for borrowing, all the results go through. Thirdly, while my baseline model is an endowment economy, I show that all the results go through in a more standard economy with endogenous labor supply.

The rest of the chapter is structured as follows. Section 1.2 presents the model. Section 1.3 shows that inefficient overborrowing can occur in equilibrium, and ex post debt relief can be Pareto improving; however, such a policy may not be incentive compatible. Section 1.4 characterizes constrained efficient allocations in an economy with two types, discusses how they can be implemented, and demonstrates conditions under which debt relief can be ex ante Pareto improving. Section 1.5 describes how these results generalize to a continuous distribution of types, and presents a numerical example. Section 1.6 discusses the benefits and costs of macroprudential policy, and shows how macroprudential policies can implement optimal allocations. Section 1.7 considers three extensions: a probability of crisis less than 1, alternative sources of heterogeneity, and endogenous labor supply. Section 1.8 concludes.

### 1.1.1 Related literature

Many recent contributions consider models in which deleveraging leads to a liquidity trap and debt relief is ex post optimal. Eggertsson and Krugman [2012] and Guerrieri and Lorenzoni [2011] were among the first to present models in which an exogenous shock to borrowing constraints causes a recession due to the zero lower bound. My model features the same shock, but asks a different question: what is the optimal policy in response to this shock?

There is a well-established literature on the use of monetary policy at the zero lower bound (the classic papers are [Krugman \[1998\]](#) and [Eggertsson and Woodford \[2003\]](#); for a recent contribution, see [Werning \[2012\]](#)). There is also a more recent literature on government spending. [Eggertsson and Krugman \[2012\]](#) themselves advocated government purchases, noting that while transfers from savers to borrowers might be stimulative, such transfers are hard to target in practice. [Bilbiie et al. \[2013b\]](#) show that government spending is never Pareto improving in a [Eggertsson and Krugman \[2012\]](#)-type model, since it hurts savers by lowering interest rates. I study transfer policy, rather than monetary policy or government spending.

My paper is also related to a growing literature on the positive effects of targeted transfers. [Oh and Reis \[2012\]](#) emphasize that government transfers increased much more than government spending during the Great Recession, and provide a model to understand the effects of targeted transfers. [McKay and Reis \[2013\]](#) assess the extent to which automatic stabilizers reduce aggregate volatility. A related empirical literature documents that the marginal propensity to consume varies across households and is correlated with debt ([Misra and Surico \[2014\]](#), [Jappelli and Pistaferri \[2014\]](#), [Cloyne and Surico \[2013\]](#)); [Kaplan and Violante \[2014\]](#) present a model which can match these facts. Relative to these authors, I focus on the effect of transfers in a liquidity trap.

A number of recent contributions discuss the role of targeted transfers in a liquidity trap. [Giambattista and Pennings \[2013\]](#) and [Mehrotra \[2013\]](#) compare the multiplier effects of targeted transfers and government spending in a liquidity trap. [Bilbiie et al. \[2013a\]](#) show that both balanced-budget redistribution and uniform, debt financed tax cuts are expansionary. [Bilbiie et al. \[2013b\]](#) show that debt-financed tax cuts are Pareto improving, as they relax borrowers' credit constraint. Rather than studying transfers in general, I focus on debt relief.

A few recent contributions discuss debt relief. [Fornaro \[2013a\]](#) shows that debt relief is expansionary, and may be Pareto improving, at the zero lower bound. [Guerrieri and Iacoviello \[2013\]](#) also show numerically that debt relief can be Pareto improving in a rather different model of housing and collateral constraints. These papers study ex post debt relief, and do not consider whether the anticipation of debt relief can distort incentives ex ante. My contribution, relative to this whole literature, is to consider how ex post redistribution itself distorts incentives ex ante, and to characterize optimal policy taking these distortions into account.

In this sense, my results are most similar to those of [Bianchi \[2012\]](#) who considers optimal

bailouts of firms in a small open economy model. Ex post, bailouts relax collateral constraints and increase output, but ex ante, bailouts induce overborrowing: optimal policy combines ex ante macroprudential policy and ex post bailouts. In [Bianchi \[2012\]](#)'s model, the planner can mitigate moral hazard effects by making bailouts conditional on a systemic crisis, rather than individual borrowing. Since firms are identical, there is no need to target particular firms. In my model, the central friction is that debt relief must be targeted at particular households based on observable debt, and savers can mimic borrowers if the bailout is too large. Another difference is that I consider debt relief targeted to households, rather than firms.<sup>1</sup>

Most closely related to my paper are [Korinek and Simsek \[2014\]](#) and [Farhi and Werning \[2013\]](#). These authors describe in detail the macroeconomic externality that arises in models such as that of [Eggertsson and Krugman \[2012\]](#): households overborrow and then deleverage, without internalizing that their deleveraging reduces aggregate income. They show that unanticipated ex post redistribution (debt relief) can be Pareto improving. In contrast, I study the design of ex post policies, taking into account how these policies affect borrowing ex ante. The main focus of [Korinek and Simsek \[2014\]](#) and [Farhi and Werning \[2013\]](#) is to consider how ex ante macroprudential policies can prevent overborrowing under full information.<sup>2</sup> I study macroprudential policy under the (realistic) assumption that preferences are private information, so it is not possible to directly target taxes and transfers to households based on their unobservable type.

A vast literature in mechanism design and optimal taxation ([Mirrlees \[1971\]](#)) considers the problem of a social planner who must redistribute among agents whose preferences or skills are private information.<sup>3</sup> The key insight from this literature is that private information reduces the

---

<sup>1</sup>This also differentiates my results from a few recent papers which take a mechanism design approach to study transfer policies targeted at banks. [Philippon and Schnabl \[2013\]](#) study efficient recapitalization in an economy with debt overhang, in which government does not observe banks' asset quality. [Tirole \[2012\]](#) takes a mechanism design approach to analyze how targeted purchases can rejuvenate asset markets. [Farhi and Tirole \[2012\]](#) analyze optimal bailouts when the government cannot perfectly observe a bank's need for liquidity. I also use a mechanism design approach to study targeted transfers, but consider different transfer policies (debt relief, rather than debt-equity swaps or asset purchases), different recipients (households, rather than banks), and a different rationale for intervention (aggregate demand externalities, rather than debt overhang or adverse selection).

<sup>2</sup>There is also a much larger literature on macroprudential policy which focuses on pecuniary externalities, rather than aggregate demand externalities. The mechanism design approach I follow in this chapter could also be applied to consider the equity-efficiency tradeoffs associated with these policies.

<sup>3</sup>Formally, my model is closest to the literature on Pareto-efficient income taxation ([Werning \[2007\]](#)); in particular, results for the two-type economy are similar to [Stiglitz \[1982\]](#), who considers a model with two agents.

ability of the planner to redistribute. However, it is still possible to achieve some redistribution, by distorting allocations away from the first-best. I apply a mechanism design approach to study optimal redistribution and macroprudential policy in an economy with macroeconomic externalities.<sup>4</sup> While macroeconomic externalities provide a new motive for redistribution, private information still constrains the planner's ability to redistribute.

## 1.2 Model

This section presents the baseline model and defines the equilibrium in the absence of policy.

### 1.2.1 Agents

Time is discrete and indexed by  $t = 0, 1, \dots$ . There exists a distribution of households with total measure 1. Households have preferences over consumption

$$U(c_0^i, \theta_i) + \sum_{t=1}^{\infty} \beta^t u(c_t^i) \quad (1.1)$$

where  $u' > 0, u'' < 0, \beta \in (0, 1), U_c > 0, U_{cc} < 0$ .  $\theta_i$  measures household  $i$ 's demand for date 0 consumption, with  $U_{c\theta} > 0$ . Agents with a higher  $\theta_i$  are more impatient, have a more urgent need for consumption at date 0, and will be borrowers in equilibrium.<sup>5</sup> In all subsequent periods, agents have the same preferences (this ensures that a well-defined steady state exists). In the benchmark model, this is the only source of heterogeneity between agents. For now, I allow  $\theta_i$  to have a general distribution function  $F(\theta)$ . Later, I will focus on two special cases, which I define here.

**Definition 1.2.1.** *In the **two type economy**,  $F(\theta)$  is a discrete distribution with probability mass  $f(\theta_S) = f(\theta_B) = 1/2, \theta_B > \theta_S = 1$ .*

*In the **continuous type economy**,  $\theta$  has a continuous density  $f(\theta)$  with support  $[\underline{\theta}, \bar{\theta}]$ .*

Agents face a standard budget constraint

$$c_t^i = y_t^i - d_t^i + \frac{d_{t+1}^i}{1 + r_t} \quad (1.2)$$

---

<sup>4</sup>In this sense, my results are also related to the literature on mechanism design with externalities (Baliga and Maskin [2003]).

<sup>5</sup>For now, I interpret  $\theta$  as a preference or discount factor shock; in Section 1.7, I show that it can be reinterpreted in terms of income, so high- $\theta$  households borrow because they have temporarily low income at date 0.

where  $d_{t+1}^i$  is the face value of debt agent  $i$  takes out in period  $t$  and promises to repay in period  $t + 1$ ,  $r_t$  is the real interest rate on a loan between periods  $t$  and  $t + 1$ , and  $y_t^i$  is  $i$ 's income. Each agent  $i$  can costlessly produce up to  $y^*$  of their own differentiated variety of the output good. Each agent's consumption  $c_t^i$  is an aggregate of all these varieties  $y_t^j, j \in [0, 1]$ , providing a motive for trade.  $y_t^i$  is not a choice variable of the household: instead, each household takes the demand for its good as given, and produces whatever is necessary to meet demand. Agents have no initial debt:

$$d_0^i = 0, \forall i \quad (1.3)$$

Agents also face an ad hoc borrowing constraint  $\phi_t \geq 0$  in the spirit of [Aiyagari \[1994\]](#):

$$d_{t+1}^i \leq \phi_t, t = 1, \dots \quad (1.4)$$

Implicitly,  $\phi_t$  reflects the collateralized value of durable goods such as housing, as in [Kiyotaki and Moore \[1997\]](#) (although this is not explicitly modelled here). As in [Korinek and Simsek \[2014\]](#), [Eggertsson and Krugman \[2012\]](#), I model a financial crisis as an exogenous tightening of the constraint. Specifically, households are unconstrained at date 0 ( $\phi_0 = \infty$ ) but the constraint permanently falls to  $\phi > 0$  at date 1:  $\phi_t = \phi \geq 0, t \geq 1$ . In the baseline model, this tightening is perfectly anticipated; in Section 6, I relax this assumption.

## 1.2.2 Equilibrium

First, I consider a Walrasian equilibrium, without any frictions (besides the borrowing constraint). I then add the zero lower bound constraint on interest rates. This forces me to modify the standard Walrasian equilibrium concept, as I describe later.

**Definition 1.2.2.** *A Walrasian equilibrium is  $\{c_t^i, d_t^i, y_t, r_t\}$  such that*

1. *each household  $i$  chooses  $\{c_t^i, d_t^i\}$  to maximize (1.1) s.t. (1.2), (1.3), (1.4)*
2.  $\int c_t^i di = y^* = y_t^i, \forall i, t = 0, 1, \dots$

I now characterize equilibrium in the two type economy starting in date 1, taking debt at the start of date 1 as given.<sup>6</sup>

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<sup>6</sup>The proof of this Proposition, and all subsequent Propositions, is in the Appendix.

**Proposition 1.2.3.** *In a Walrasian equilibrium, in the two type economy:*

1. *If  $d_1^B \leq \phi$ , consumption, debt and interest rates are constant in periods  $t \geq 1$ :  $r_t = r^* := \beta^{-1} - 1$ ,  $c_t^i = y^* - (1 - \beta)d_1^i$ ,  $d_t^i = d_1^i$ .*
2. *If  $d_1^B > \phi$ , B is borrowing constrained in period 1:  $d_2^B = \phi$ . Consumption, debt and interest rates are constant in periods  $t \geq 2$ :  $r_t = r^*$ ,  $c_t^i = y^* - (1 - \beta)\phi$ ,  $d_t^i = \phi$ .  $r_1 = r(d_1^B)$ , implicitly defined by*

$$1 + r_1 = \frac{u' \left( y^* + d_1^B - \frac{\phi}{1 + r_1} \right)}{\beta u' (y^* + (1 - \beta)\phi)}$$

*$r(d_1^B)$  is decreasing in  $d_1^B$ , with  $r(\phi) = r^*$ .*

Equilibrium interest rates are decreasing in  $d_1$ . If debt is sufficiently low, or the tightening of borrowing constraints is not too severe, the economy immediately converges to a steady state at date 1. If debt is too high, borrowers are no longer able to roll over their debt, and are forced to pay back some debt, temporarily reducing their consumption. In order for markets to clear, savers must consume more at date 1 than they do at date 2. Interest rates must fall to induce them to do so, thus  $r_1$  is a decreasing function of  $d_1$ .

Figure 1.1 illustrates. Aggregate date 1 consumption is decreasing in  $r_1$ . For a given interest rate, aggregate consumption is also decreasing in borrowers' debt  $d_1^B$ , when debt is high enough that the borrowing constraint binds. When borrowers are liquidity constrained, their marginal propensity to consume is 1, and an increase in debt reduces their consumption one for one. The corresponding increase in savers' net worth increases savers' consumption, but less than one for one, because savers' MPC is much less than 1. Consequently, an increase in debt tends to reduce aggregate consumption. Interest rates must fall to keep aggregate consumption equal to  $y^*$ .

I now introduce a constraint on interest rates,  $r_t \geq \bar{r}$ . For simplicity, in what follows I assume  $\bar{r} = 0$ . The interest rate  $r_t$  required to clear markets may be negative, violating this zero lower bound (ZLB) constraint. In this case, the above equilibrium is no longer possible, and a new equilibrium concept is required. I assume that when the ZLB binds, households cannot sell their whole endowment, and output (i.e., the amount they do sell) is the variable that adjusts to clear markets. Aggregate consumption is still equal to aggregate output. However, aggregate output  $y_t$  can fall below potential output  $y^*$  when the zero lower bound binds. Formally:

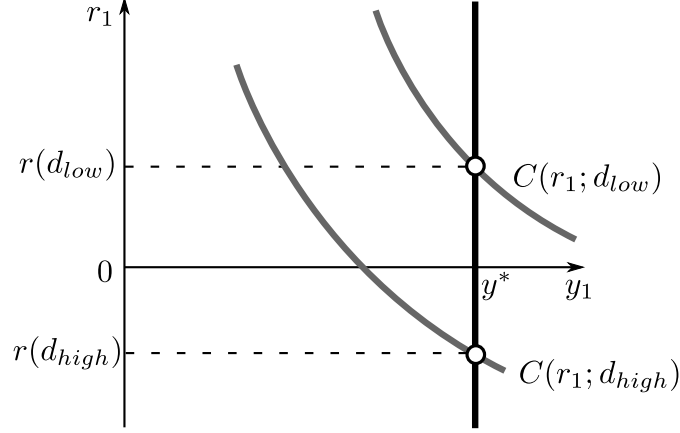


Figure 1.1: Walrasian equilibrium

**Definition 1.2.4.** A *ZLB-constrained equilibrium* is  $\{c_t^i, d_t^i, y_t, r_t\}$  such that

1. each household  $i$  chooses  $\{c_t^i, d_t^i\}$  to maximize (1.1) s.t. (1.2), (1.3), (1.4)
2.  $\int c_t^i di = 2y_t$
3.  $r_t \geq 0, y_t^i = y_t \leq y^*, r_t(y^* - y_t) = 0$

When interest rates can adjust to clear markets, they do, and agents sell all of their endowment. When the ZLB prevents interest rates from falling enough to clear markets, agents sell less than their total endowment, and income  $y_t$  is the variable that adjusts to clear markets.

Since this is a real model, some justification for the constraint  $r_t \geq 0$  is in order. The constraint is a tractable way to model the effect of a zero lower bound on nominal interest rates, combined with a limit on the expected rate of inflation that the central bank can or will target. In Appendix A I show that this equilibrium is isomorphic to the limit of a standard New Keynesian model as prices become infinitely sticky, as in [Korinek and Simsek \[2014\]](#). In Appendix B I present two alternative economies providing a microfoundation for this equilibrium concept. The first draws on the extensive literature on rationing or non-Walrasian equilibria (see e.g. [Benassy \[1993\]](#) for a survey, and [Kocherlakota \[2013\]](#), [Caballero and Farhi \[2014\]](#) for two recent papers employing a similar concept). The second is an economy with downward nominal wage rigidity drawing on [Schmitt-Grohé and Uribe \[2011\]](#).

The following Proposition characterizes ZLB-constrained equilibria in the two-type economy.

**Proposition 1.2.5.** *In the two-type economy:*

1. If  $r(d_1) \geq 0$ , the Walrasian equilibrium is also the ZLB-constrained equilibrium.
2. If  $r(d_1) < 0$ , then in a ZLB-constrained equilibrium:

$$r_1 = 0$$

$$u'(c_1^S) = \beta u'(y^* + (1 - \beta)\phi)$$

$$y_1 = c_1^S - d_1 + \phi < y^*$$

$$c_1^B = c_1^S - 2d_1 + 2\phi$$

*The economy enters a steady state in period 2, as in the Walrasian equilibrium.*

*Proof.* The first part of the proposition is obvious. To prove the second part, first note that if  $r(d_1) < 0$ , we cannot have a Walrasian equilibrium satisfying the ZLB. We must have  $r_1 = 0$  and  $y_1 < y^*$ . Since borrowers are constrained at  $t = 1$ ,  $c_1^B = y_1 - d_1 + \phi$ . Substituting this into the market clearing condition  $c_1^S + c_1^B = 2y_1$ , we get the above result.  $\square$

When debt is not too high, interest rates are positive, the ZLB does not bind, and markets clear. When debt is too high, borrowers' consumption falls sharply in period 1, and the ZLB prevents interest rates from falling enough to induce savers to consume the remaining output. The fall in income forces borrowers to reduce spending further. Figure 1.2 illustrates.

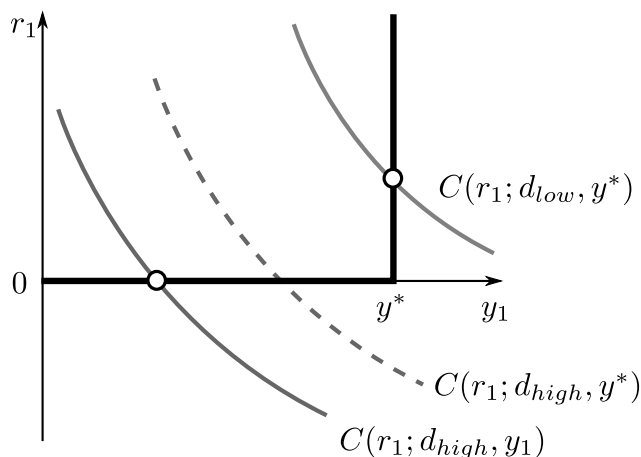


Figure 1.2: ZLB-constrained equilibrium

### 1.3 Liquidity traps and overborrowing

In this section, I first restate two results from the recent literature on liquidity traps and debt relief (in particular, [Korinek and Simsek \[2014\]](#)). First, overborrowing can occur in equilibrium: if the motive for borrowing is sufficiently strong, borrowers may take on so much debt that they trigger the ZLB, even though they know this will happen. Second, any equilibrium in which the ZLB binds is Pareto inefficient. Ex post, taxing savers and writing off borrowers' debt restores output to potential, makes borrowers better off, and leaves savers no worse off.

I then show that such a policy, while ex post optimal, may have adverse incentive effects ex ante. First, lump sum redistribution induces overborrowing, as borrowers take out more debt, anticipating that they will be richer in the future. Second, even if the government commits to a cap on debt relief, a transfer large enough to restore full employment may not be incentive compatible, since savers may take on debt in order to qualify for the transfer.<sup>7</sup>

In this section, and in [Section 1.4](#), I focus on the two type economy to build intuition. In [Section 1.5](#), I show how these results generalize to a continuum of types.

#### 1.3.1 Overborrowing and the potential gains from transfers

A natural question, posed by [Korinek and Simsek \[2014\]](#), is whether borrowers may take on so much debt that the ZLB binds at date 1 even though they anticipate this happening. The next proposition shows that this is indeed possible, under the following assumption:

**Assumption 1.3.1.**

$$u'(2y^*) < \beta u'(y^* + (1 - \beta)\phi)$$

The market clearing interest rate is decreasing in  $d_1$ . As borrowers become more impatient, they borrow more: in the limit as they become infinitely impatient, they promise to repay all their income at date 1, so savers must consume the whole aggregate endowment. [Assumption \(1.3.1\)](#) ensures that they would only do so if the date 1 interest rate was negative. This guarantees that if borrowers are sufficiently impatient, they will take on so much debt that  $r(d) < 0$ .

**Proposition 1.3.2.** *There exists  $\theta^{ZLB} \in (1, \infty)$  such that  $r(d_1) < 0$  if  $\theta_B > \theta^{ZLB}$ .*

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<sup>7</sup>In [section 1.6](#), I show that private information introduces similar problems for macroprudential policy.

If the ZLB binds at date 1, the recession makes households poorer, and ceteris paribus they want to borrow more at date 0. But if borrowers are sufficiently impatient, their impatience outweighs this wealth effect, and they take on so much debt that the ZLB binds.

An equilibrium in which the ZLB binds is ex post Pareto inefficient, since unanticipated redistribution from savers to borrowers can be Pareto improving. Suppose savers' income is unexpectedly reduced to  $y_t - T$ , and borrowers' income is unexpectedly increased to  $y_t + T$ . This is identical to an unanticipated reduction in the borrowers' debt. Redistribution directly reduces savers' income by  $T$ . However, since output is decreasing one for one in borrowers' debt, writing off  $T$  debt increases savers' income by  $T$ , leaving them no worse off. Borrowers, meanwhile, benefit twice from the redistribution: their debt falls by  $T$ , and their income rises by  $T$  due to the multiplier effect of their own spending. So we have a Pareto improvement. Given a large enough redistribution  $T$ , it is possible to restore full employment, as the following Proposition states:

**Proposition 1.3.3.** *If the borrowers receive an unanticipated increase in income  $T^{FE}(d_1) = d_1 - (c_1^B + \phi - y^*)$  and the savers face an unanticipated fall in income  $T^{FE}(d_1)$ :*

1. *There is full employment:  $y_1 = y^*$*
2. *Borrowers' consumption increases to  $2y^* - c_1^S > c_1^S - 2d_1 + 2\phi$*
3. *Savers' consumption is unchanged.*

*If the redistribution is equal to  $T < T^{FE}(d_1)$ ,  $y_1 = c_1^S + T + \phi - d$ .*

*Proof.* The unanticipated redistribution is equivalent to a change in  $d_1^B$ . The result follows by replacing  $d_1$  with  $d_1 - T$  in the equilibrium described in Proposition 1.2.5. □

### 1.3.2 Equilibrium with transfers

The government may attempt to implement a lump sum redistribution by writing off borrowers' debt and taxing savers. However, if redistribution is implemented through policy, it will be anticipated, and may distort decisions ex ante. To consider this possibility, it is necessary to define an equilibrium with policy.

I now replace the budget constraint (1.2) with the following:

$$c_1^i = y_1^i - d_1^i + \frac{d_2^i}{1+r_1} + T(d_1^i, \theta_i) - \bar{T} \quad (1.5)$$

$$c_t^i = y_t^i - d_t^i + \frac{d_{t+1}^i}{1+r_t} \text{ for } t \neq 1$$

where for any debt level  $d_1^i$ , the transfer to agent  $i$  in period 1 is  $T(d_1^i, \theta_i) - \bar{T}$ .<sup>8</sup> For now, I allow the government to observe households' type, and target transfers directly. The bulk of this chapter will consider the case where  $\theta$  is private information, and transfers can only depend on  $d_1^i$ .

The planner cannot make any taxes or transfers to agents starting in date 2, and must run a balanced budget:

$$\int T(d_1^i, \theta_i) di = \bar{T} \quad (1.6)$$

This assumption is crucial. If the government could impose taxes and transfers forever, it would be possible to completely undo the effect of the liquidity constraint, for example through a deficit-financed transfer to all households (Woodford [1990], Yared [2013], Bilbiie et al. [2013b]). I rule out such policies in order to isolate the effects of debt relief. Government credit policies may be a powerful tool in responding to recessions caused by a contraction in private credit. My goal is to evaluate whether non-credit policies, such as debt relief, can also be effective (not to seriously compare debt relief to other fiscal or monetary policies, a task which I leave to future work).<sup>9</sup>

I now define a ZLB-constrained equilibrium with debt-contingent date 1 transfers.<sup>10</sup>

**Definition 1.3.4.** *A ZLB-constrained equilibrium with transfers is  $\{c_t^i, d_t^i, y_t, r_t, \bar{T}\}$  such that, given a transfer function  $T(d, \theta)$ :*

1. each household  $i$  chooses  $\{c_t^i, d_t^i\}$  to maximize (1.1) s.t. (1.5), (1.3), (1.4)

---

<sup>8</sup>It is always possible to normalize  $\bar{T} = 0$ . I write the transfer in this general form to ensure that balanced-budget equilibrium is defined for any transfer function  $T(d, \theta)$ .

<sup>9</sup>One motivation for employing non-credit policy in addition to deficit-financed transfers might be a concern with the distortionary effects of non-lump sum transfers in the long run, which is not modelled here. Non-credit policies such as debt relief are also feasible even when the government faces borrowing constraints, in addition to the private sector.

<sup>10</sup>Equilibria with debt-contingent date 0 transfers are defined analogously, and are considered in Section 1.6.

2.  $\int c_t^i di = y_t$
3.  $r_t \geq 0, y_t \leq y^*, r_t(y^* - y_t) = 0$
4. *the government budget constraint (1.6) is satisfied.*

### 1.3.3 Debt relief induces overborrowing

Having defined ZLB-constrained equilibrium with transfers, I describe how the prospect of debt relief distorts decisions ex ante. In this stylized model, there are two ways in which this can happen: the intensive and the extensive margin. Along the intensive margin, the anticipation of debt relief causes borrowers to borrow more, because they will be richer in period 1, and want to borrow against that wealth at date 0. I show that the ex post optimal policy is never Pareto improving ex ante: some commitment is necessary if we are to obtain a Pareto improvement.

Suppose first that the government does not commit ex ante to a particular level of transfers. Instead, after observing the equilibrium level of debt  $d_1^*$ , it makes whatever transfer  $T^{FE}(d_1^*)$  to borrowers restores full employment, and finances this with a lump sum tax on savers.<sup>11</sup> Note that since an individual borrower is measure zero, the transfer she receives does not depend on her own debt, but only on aggregate debt  $d_1^*$ , which she is too small to affect.

**Proposition 1.3.5.** *Consider a constant transfer function  $T(d, \theta_B) = T^*$ ,  $T(d, \theta_S) = -T^*$ ,  $\forall d$ . Suppose  $T^* = T^{FE}(d^*)$ , where  $d^*$  is the level of  $d_1^B$  in the equilibrium with transfers, given  $T^*$ . This equilibrium is not Pareto improving relative to the equilibrium without transfers. Savers are strictly worse off.*

*Proof.* Combining the savers' and borrowers' Euler equations and using market clearing

$$\frac{U_c(c_0^S, \theta_S)}{U_c(2y^* - c_0^S, \theta_B)} = \frac{u'(c_1^S)}{u'(2y^* - c_1^S)} > \frac{u'(c_1^S)}{u'(\hat{c}_1^B)}$$

where  $\hat{c}_1^B < 2y^* - c_1^S$  denotes the equilibrium without policy. It follows that  $c_0^S < \hat{c}_0^S$ , and since  $c_0^S = \hat{c}_0^S$ , savers are worse off.  $\square$

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<sup>11</sup>Throughout this chapter, I consider debt relief policies in which the government assumes responsibility for borrowers' debt, and makes payments to savers on the borrowers' behalf, financing these payments with lump sum taxes on the savers. Crucially, since an individual saver is measure zero, her lending decision does not affect the lump sum tax required to pay for the debt relief. One could consider an alternative debt relief policy, in which the government decrees that borrowers no longer have to make some promised payments to savers (effectively, legislating a mass default). Under this alternative policy, debt relief would create default risk, which will be priced into the interest rates charged by savers at date 0. In contrast, under the policy considered in this chapter, interest rates are not directly affected by debt relief, since there is no default.

If the government makes a transfer to borrowers, they will be richer at date 1. Anticipating this, they borrow against this future income to consume more at date 0, and in equilibrium, savers consume less. In this sense, debt relief encourages overborrowing.

Note that this result holds even if borrowers do not perceive that their individual debt will be written off: if they did, there would be even more overborrowing ex ante. Suppose borrowers expect the government to write off a fraction  $\tau$  of their debt: that is,  $T_B(d) = \tau d$ . Then they face an effective gross interest rate of  $(1 + r_0)(1 - \tau)$ , while savers face an interest rate  $1 + r_0$ . This wedge between the borrowers' and savers' Euler equations makes the borrowers' consumption even higher at date 0 and makes savers even worse off.

### 1.3.4 Is debt relief incentive compatible?

Above, I showed that debt relief without commitment induces overborrowing, and is not ex ante Pareto improving. In this section, I show that even if the government commits to limit the amount of debt relief, the full employment transfer may not be incentive compatible. If debt relief is too generous, savers switch to become borrowers, and the equilibrium breaks down. This problem arises if borrowers are sufficiently impatient, so they take on so much debt that the required amount of debt relief would tempt the savers to borrow.

One way to limit debt relief is as follows. Take the level of debt in the equilibrium without policy,  $\hat{d}$ . Let the government give a transfer  $T^{FE}(\hat{d})$  to borrowers with exactly  $\hat{d}$  debt.<sup>12</sup> Since borrowers only receive a transfer if they borrow exactly  $\hat{d}$ , this policy obviously cannot induce overborrowing. Clearly, this transfer function is unrealistic. My goal is to show that even if I allow the government to completely avoid overborrowing in this way, another problem remains.

Assume that household type,  $\theta_i$ , is private information: the government cannot directly distinguish type  $S$  and type  $B$  agents. In this case, transfers must be anonymous:  $T(d, \theta_S) = T(d, \theta_B) = T(d)$ . Formally:

**Definition 1.3.6.** *A balanced budget equilibrium with anonymous transfers (henceforth, an equilibrium with transfers) is a balanced budget equilibrium with transfers in which  $T(d, \theta) = T(d), \forall \theta$ .*

---

<sup>12</sup>Formally, this corresponds to the transfer functions  $T_B(d) = T^{FE}(\hat{d})\mathbf{1}(d = \hat{d})$ ,  $T_S(d) = -T^{FE}$ .

With anonymous transfers, the only way to transfer funds to borrowers is to reward agents who take on more debt, and tax savers, i.e. to make  $T(d)$  positive for some  $d > 0$ , and negative for some  $d < 0$ . If the transfer  $T(\hat{d})$  is sufficiently large, savers will receive strictly higher utility by mimicking borrowers, taking on debt  $\hat{d}$  instead of saving  $\hat{d}$ . Then the ex post optimal debt relief policy is not incentive compatible, and cannot be implemented. If the government offered such generous debt relief, at least some savers would mimic borrowers, and borrow  $\hat{d}$ . Then the government will be forced to raise taxes at date 1, and the net transfer to borrowers will not be enough to restore full employment

Figure 1.3 illustrates a case in which the full employment transfer is not incentive compatible. Date 1 output is below potential,  $y_1 < y^*$ . Savers' date 1 consumption is constrained by the ZLB. Through taxes and transfers, the government could essentially transfer all the surplus output,  $y^* - y$ , to the borrowers, increasing their date 1 consumption and leaving everything else unchanged. However, then borrowers' allocation would give strictly higher utility to savers than their own allocation, and savers would rather mimic borrowers than choose their own allocation. At least some savers will take on the same debt as borrowers in order to qualify for the transfer at date 1, and the government will be forced to reduce the net transfer to borrowers.

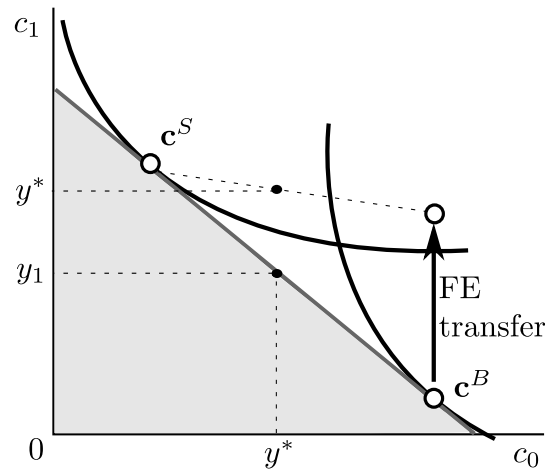


Figure 1.3: Full employment transfer is not incentive compatible

When will the FE transfer violate incentive compatibility? That is, when will savers prefer borrowers' allocation (including the date 1 transfer) to their own allocation? Borrowers consume less than savers at date 1, despite the transfer, but consume more at date 0. If borrowers are

sufficiently impatient ( $\theta_B$  is large enough), they consume so much more at date 0 that savers would want to mimic them and do the same, if the government writes off their debt at date 1. As  $\beta \rightarrow 1$ , steady state interest rates go to zero, so savers' steady state consumption converges to their income  $y^*$ ; at the same time, as savers become more patient, they become unwilling to consume more at date 1 than they do in the steady state, so their date 1 consumption also converges to  $y^*$ . The cost of mimicking borrowers - lower consumption at date 1 and in steady state - converges to zero, so savers become increasingly willing to mimic borrowers.

**Proposition 1.3.7.** *Consider the transfer function  $T(d) = T^{FE}(\hat{d})$  if  $d = \hat{d}$ ,  $T(d) = -T^{FE}(\hat{d})$  if  $d \neq \hat{d}$ , where  $\hat{d}$  is the debt level in the equilibrium without policy. There exists a continuous function  $\theta^{FE}(\beta, \phi)$ , which may equal  $\infty$ , such that:*

1. *If  $\theta_B \leq \theta^{FE}$ , the transfer is incentive compatible. There exists an equilibrium with transfers with full employment which is a Pareto improvement over the equilibrium without policy.*
2. *If  $\theta_B > \theta^{FE}$ , the transfer is not incentive compatible.*
3.  *$\theta^{FE}(\beta, \phi) \geq \theta^{ZLB}$ . If the FE transfer is not incentive compatible, the ZLB must bind in equilibrium.*
4.  *$\theta^{FE}(\beta, \phi)$  is increasing in  $\beta$  and decreasing in  $\phi$ , with  $\theta^{FE}(1, 0) = 1$ . That is, if  $\beta = 1, \phi = 0$ , the transfer is not incentive compatible for any  $\theta_B > 1$ . For any  $\theta_B \in [1, \infty)$ , there exist  $\bar{\beta}, \bar{\phi}$  sufficiently close to 1, 0 such that the FE transfer is not incentive compatible if  $\beta > \bar{\beta}, \phi < \bar{\phi}$ .*

## 1.4 Optimal policy in the two type economy

In the previous section, I showed that poorly designed debt relief policies, while optimal ex post, can have harmful incentive effects ex ante. The question remains: what is optimal policy in this economy? Can sophisticated debt relief programs avoid these adverse incentives?

To answer these questions, I now analyze optimal policy. First I solve the Pareto problem, subject to the constraints imposed by incentive compatibility and the zero lower bound. I let the government choose *any* system of taxes and transfers which depend only on an agent's observable debt, not on her unobservable type, and show that solutions to the Pareto problem can be implemented with such debt-contingent transfers. I show that debt relief always implements

some constrained efficient allocations (in particular, those which are relatively favorable for borrowers). When the ZLB binds, some debt relief policy is always Pareto improving. When the ZLB does not bind, debt relief is purely redistributive, taking from savers and giving to borrowers.

### 1.4.1 Pareto problem

To characterize constrained efficient allocations, I consider the problem of a social planner who puts weight  $\alpha$  on savers and  $1 - \alpha$  on borrowers, and faces four sets of constraints. First, resource feasibility. Second, the liquidity constraint at date 1, which (combined with the assumption of no transfers at date 2) puts a lower bound on the date 2 consumption of borrowers. Third, the zero lower bound, which imposes that the savers' Euler equations must be satisfied with a nonnegative interest rate. And fourth, incentive compatibility, which states that neither agent can strictly prefer the other agent's allocation. As discussed above, since my focus is on non-credit policies, I assume the planner cannot make any taxes or transfers to agents starting in date 2, and must run a balanced budget. Consequently, the economy always enters a steady state at date 2.

$$\max \alpha \left\{ U(c_0^S, \theta_S) + \beta u(c_1^S) + \frac{\beta^2}{1-\beta} u(c_2^S) \right\} + (1-\alpha) \left\{ U(c_0^B, \theta_B) + \beta u(c_1^B) + \frac{\beta^2}{1-\beta} u(c_2^B) \right\} \quad (1.7)$$

$$\text{s.t. } c_0^S + c_0^B \leq 2y^* \quad (\text{RC0})$$

$$c_1^S + c_1^B \leq 2y^* \quad (\text{RC1})$$

$$c_2^S + c_2^B = 2y^* \quad (\text{RC2})$$

$$c_2^B \geq y^* - (1-\beta)\phi \quad (\text{BC})$$

$$u'(c_1^S) \geq \beta u'(c_2^S) \quad (\text{ZLB})$$

$$U(c_0^S, \theta_S) + \beta u(c_1^S) + \frac{\beta^2}{1-\beta} u(c_2^S) \geq U(c_0^B, \theta_S) + \beta u(c_1^B) + \frac{\beta^2}{1-\beta} u(c_2^B) \quad (\text{ICS})$$

$$U(c_0^B, \theta_B) + \beta u(c_1^B) + \frac{\beta^2}{1-\beta} u(c_2^B) \geq U(c_0^S, \theta_B) + \beta u(c_1^S) + \frac{\beta^2}{1-\beta} u(c_2^S) \quad (\text{ICB})$$

Constrained efficient allocations solve (1.7) for some  $\alpha \in (0, 1)$ . Varying  $\alpha$  traces out the constrained Pareto frontier. The following proposition characterizes constrained efficient allocations.

**Proposition 1.4.1.** *There are ten classes of constrained efficient allocation.*

1. (RC0) and (RC2) always bind.

2. Either **(ICS)** binds, **(ICB)** binds, or no incentive constraints bind. There exist  $\alpha_B, \alpha_S$  with  $1 > \alpha_B > \alpha_S > 0$  such that **(ICB)** binds iff  $\alpha > \alpha_B$  and **(ICS)** binds iff  $\alpha < \alpha_S$ .
3. Either neither **(BC)** nor **(ZLB)** bind, only **(BC)** binds, or **(BC)** and **(ZLB)** both bind.
4. **(RC1)** binds unless **(ICS)**, **(BC)** and **(ZLB)** all bind. In this case, **(RC1)** may be slack.

Figure 1.4 illustrates point 2.<sup>13</sup> It shows which incentive constraints bind, as a function of  $\alpha$  and  $\theta_B$ . The dark grey region on the left shows the set of  $(\alpha, \theta_B)$  for which **(ICS)** binds; the unshaded middle region shows the set where neither constraint binds; and the light grey region on the right shows the set where **(ICB)** binds. Setting  $\alpha = 0$  selects an allocation which maximizes  $B$ 's utility, subject to the remaining constraints, putting no weight on  $S$ . Absent incentive compatibility constraints, this allocation would have  $B$  consuming everything and  $S$  nothing. Incentive compatibility rules this out, since  $S$  would want to mimic  $B$ . So when  $\alpha = 0$ , **(ICS)** always binds. Increasing  $\alpha$ , we go from left to right, moving along the Pareto frontier towards allocations that are better for  $S$  and worse for  $B$ . Eventually,  $S$ 's utility increases so much that **(ICS)** no longer binds, and each individual strictly prefers his own allocation.<sup>14</sup> Increasing  $\alpha$  further, eventually  $B$ 's utility falls so much that he prefers  $S$ 's allocation, and **(ICB)** binds.

Figure 1.5 illustrates point 3. It shows whether the borrowing constraint is slack, the borrowing constraint binds, or both the borrowing constraint and the ZLB binds, as a function of  $\alpha$  and  $\theta_B$ . When  $\theta_B$  is close to  $\theta_S = 1$ , borrowers are almost as patient as savers, and have similar consumption profiles. The borrowing constraint does not bind at date 1, and the economy enters steady state immediately. As we raise  $\theta_B$ , borrowers become more impatient, consuming more at date 0 and less at dates 1 and 2. Eventually, the borrowing constraint binds: the planner would like to reduce borrowers' steady state consumption, but this would increase steady state debt above  $\phi$ . Increasing  $\theta_B$  further, borrowers become yet more impatient, and the planner gives less to borrowers and more to savers at date 1. Date 2 allocations, however, remain fixed. Savers must tolerate an increasingly steep decline in consumption between dates 1 and 2; interest rates fall to induce them to do so. Eventually, **(ZLB)** binds, and we enter the light grey region at the

<sup>13</sup>This figure is not to scale. In particular, note that the two shaded regions meet at the point  $\alpha = 0.5, \theta = 1$ , which should lie in the middle of the horizontal axis.

<sup>14</sup>This region only exists when agents have different preferences, i.e.  $\theta_B > \theta_S = 1$ .

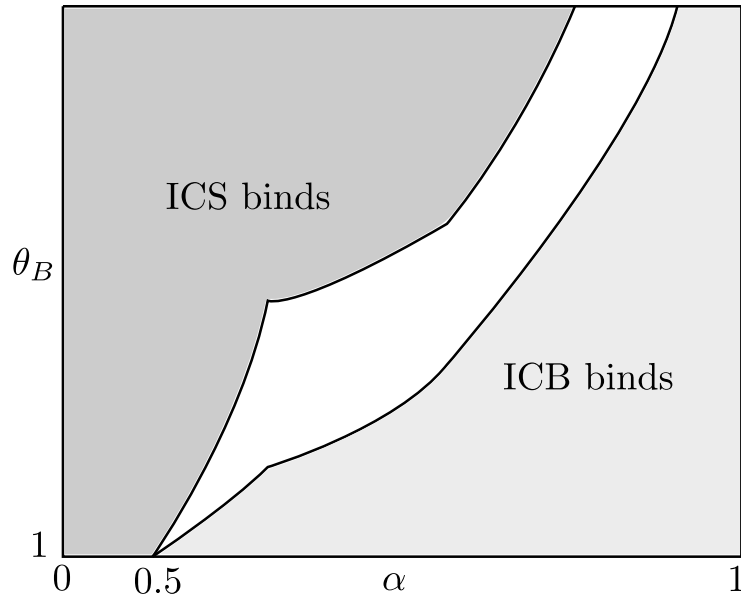


Figure 1.4: Optimal allocations in  $(\alpha, \theta_B)$ -space

top of Figure 1.5. Savers consume  $\bar{c}_1^S$ , the maximum they can be induced to consume with zero interest rates. Unlike in the competitive equilibrium, this does not generally mean that aggregate consumption falls below output: the planner recognizes the ZLB constraint, gives the remaining consumption to the borrowers, and distributes date 0 consumption between B and S.

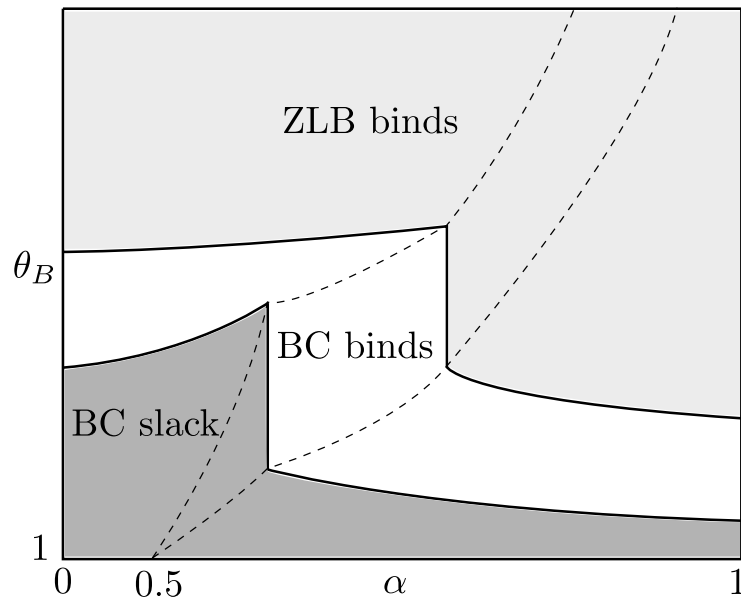


Figure 1.5: Optimal allocations in  $(\alpha, \theta_B)$ -space

Finally, Figure 1.6 illustrates point 3. It shows the region of parameter space (shaded black)

in which the date 1 resource constraint is slack, so aggregate consumption is less than potential output. The figure shows that this may be constrained efficient, but only if (ZLB) and (ICS) both bind. This will be the case if  $\theta_B$  is high enough, in allocations which are relatively favorable for borrowers (corresponding to a low  $\alpha$ ). Suppose borrowers are very impatient, so  $\theta_B$  is very high, and suppose (ICS) and (ZLB) bind. Borrowers would like more date 0 consumption, but that would tempt savers to choose the borrowers' allocation, violating incentive compatibility. To get more date 0 consumption, borrowers can throw away some of their date 1 consumption. (They cannot give it to savers because (ZLB) binds.) This makes their allocation less attractive to savers, who put a high weight on date 1 consumption, which in turn means that borrowers can get away with higher date 0 consumption, which they value more. So if  $\theta_B$  is sufficiently high, there are some constrained efficient allocations in which (ZLB) and (ICS) bind but (RC1) is slack.

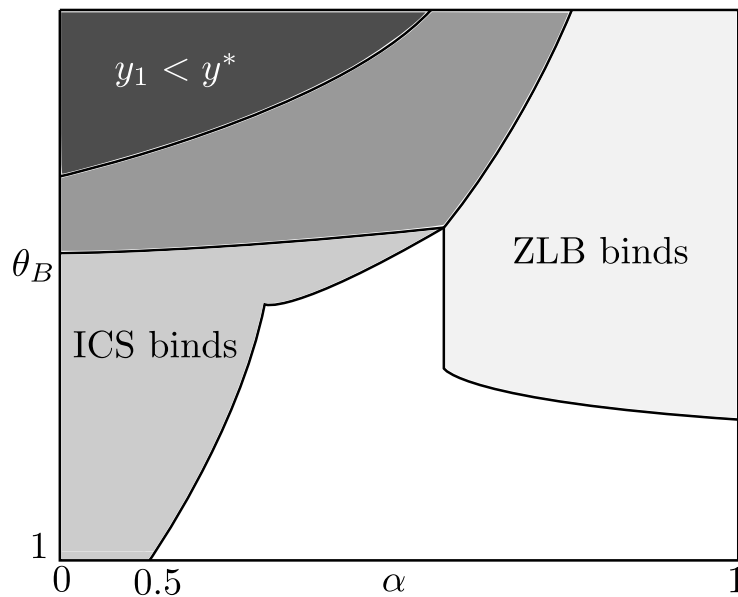


Figure 1.6: Optimal allocations in  $(\alpha, \theta_B)$ -space

While I will discuss implementation below, note for now that the difference between potential output and consumption,  $2y^* - c_1^S - c_1^B$ , can be interpreted as unproductive government spending. Point 5 then states that unproductive spending may be optimal, provided not only that the economy is in a demand-driven slump (as in the standard Keynesian argument), but also that incentive constraints prevent the government from achieving full employment with targeted transfers alone. More generally, if I introduced government spending explicitly and allowed it to

have some value for households, it would be optimal to increase government spending above the normal efficient level when incentive constraints bind. Intuitively, one advantage of spending on pure public goods is that they benefit all agents equally, and do not create incentive problems.

To summarize, the optimal allocation has three important properties. First, when moving along the Pareto frontier, in allocations which are relatively favorable for borrowers, (ICS) binds; in intermediate allocations, neither incentive constraint binds; and in allocations which are better for savers, (ICB) binds. Second, when differences in preferences between the two households are large, creating a motive for borrowing, it is optimal for (ZLB) to bind. Unlike in the equilibrium without policy, however, even when (ZLB) binds, the planner generally uses targeted transfers to prevent unemployment. Finally, it may be constrained optimal to allow some unemployment when (ZLB) binds, but only if (ICS) also binds, so the planner cannot give the surplus output to  $B$  without making  $S$  prefer  $B$ 's allocation.

## 1.4.2 Implementation

Next, I consider how constrained efficient allocations can be implemented. First I show that any solution to the Pareto problem can be implemented as an equilibrium with debt-contingent transfers. Then I ask when it is optimal for these transfers to take the form of debt relief.

The next proposition states that any constrained efficient allocation can be implemented as an equilibrium with debt-contingent transfers at date 1.<sup>15</sup>

**Proposition 1.4.2.** *Any solution to (1.7) can be implemented as an equilibrium with transfers.*

Intuitively, every transfer function  $T(d)$  maps out a nonlinear budget constraint in consumption space. In any constrained efficient allocation, each agent prefers her own consumption allocation to the other agent's allocation. The implementability problem is to construct a nonlinear budget set such that each agent prefers her own allocation to all other allocations in the budget set.<sup>16</sup> Figure 1.7 provides an illustration.  $c^B, c^S$  are arbitrary consumption allocations in  $(c_0, c_1)$ -space satisfying incentive compatibility, so  $S$ 's allocation lies weakly below  $B$ 's indifference curve, and vice versa. The gray line shows one particular nonlinear budget constraint which

<sup>15</sup>In Section (1.6), I show that efficient allocations can also be implemented with date 0 transfers.

<sup>16</sup>There are multiple ways to do this. A trivial solution is to offer only two points in the budget set, corresponding to the desired debt levels of  $S$  and  $B$ , and set  $T(d) = -\infty$  for all off-equilibrium levels of debt.

implements this allocation. Graphically, it is clear that for any allocation, we can find some non-linear budget set which lies below the lower envelope of both agents' indifference curves, and intersects the indifference curves at their intended allocations.

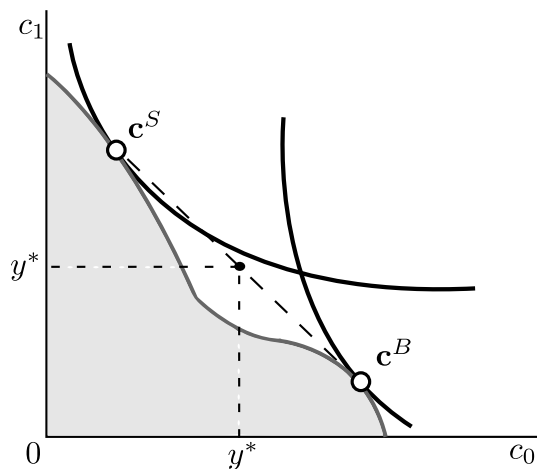


Figure 1.7: Implementation with debt-contingent transfers

### 1.4.3 Debt relief implements constrained efficient allocations

Next, I ask whether constrained efficient allocation can be implemented with particular simple transfer functions.

It turns out that any constrained efficient allocation can be implemented with a piecewise linear transfer function with at most three segments. The transfer function is one of two kinds. I call the first a debt relief transfer function:

**Definition 1.4.3.**  $T(d)$  is a *debt relief transfer function* if it has the form

$$\begin{aligned} T^{DR}(d; \bar{T}, \underline{d}, \bar{d}) &= -\bar{T} \text{ if } d < \underline{d} \\ &= -\bar{T} + (d - \underline{d}) \text{ if } d \in [\underline{d}, \bar{d}] \\ &= -\bar{T} + (\bar{d} - \underline{d}) - \tau(d - \bar{d}) \text{ if } d > \bar{d} \end{aligned}$$

where  $\bar{T} > 0$ ,  $\underline{d}, \bar{d} > \underline{d}$ ,  $\tau$  are parameters.

There is a lump sum tax on all households with debt below a certain level  $\underline{d}$ . The government writes off all debt above  $\underline{d}$  up to a maximum level  $\bar{d}$ . Above that point, further borrowing is

penalized: the transfer falls by  $\tau$  dollars for each dollar of debt above the maximum level  $\bar{d}$ .<sup>17</sup> Debt relief is offered only to borrowers with a moderate level of debt: excessive borrowing is discouraged. Figure 1.8 shows the budget constraint induced by a debt relief transfer function.

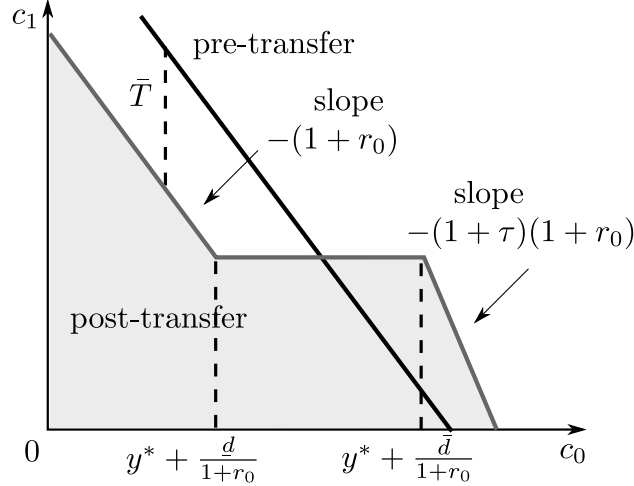


Figure 1.8: A debt relief transfer function

The next proposition states conditions under which debt relief is Pareto optimal.

**Proposition 1.4.4.** *There exists  $\alpha(\theta_B)$  such that*

1. *debt relief implements the optimal allocation iff  $\alpha < \alpha(\theta_B)$ .*
2. *If (ICS) binds,  $\alpha < \alpha(\theta_B)$  and debt relief implements the optimal allocation.*

First, consider a constrained efficient allocation in which  $S$ 's incentive compatibility constraint binds. Figure 1.9 shows that debt relief implements such an allocation. In equilibrium, borrowers all choose exactly the maximum level of debt. While there are other transfer schemes that might implement allocations in which (ICS) binds, in any such scheme,  $T(d)$  must be nondifferentiable at  $d_1^B$ ,  $B$ 's equilibrium debt level. In this case,  $S$  and  $B$ 's indifference curve intersect at  $\mathbf{c}^B$ . Because

<sup>17</sup>One could imagine a situation in which borrowers can verify that they have a debt, but can hide part of their debt - if their true debt level is  $d > 0$ , they can claim to have any debt level  $\bar{d} \in [0, d]$ . In this case, it would not be possible to offer a transfer function where  $T(d)$  is decreasing over some range. Instead, it would be necessary to combine debt relief with a 'macroprudential' debt limit  $d \leq \bar{d}$  at date 0.

Crucially, I assume that all asset trades are observable. If agents could engage in secret asset trades, savers could reduce their tax burden. For example, consider two savers who save  $a > 0$  in equilibrium, and both pay the lump sum tax  $\bar{T}$ . Consider the following deviation: one saver saves  $3a$  and pays  $\bar{T}$ , the other saver borrows  $a$  and receives a transfer equal to  $\bar{T}$ , and they pool their resources at each date. This deviation reduces their tax burden to zero. I rule out such deviations by prohibiting secret asset trades.

the two households have different preferences, the indifference curves have different slopes at this point. It follows that the budget set must have a kink at  $c^B$ .

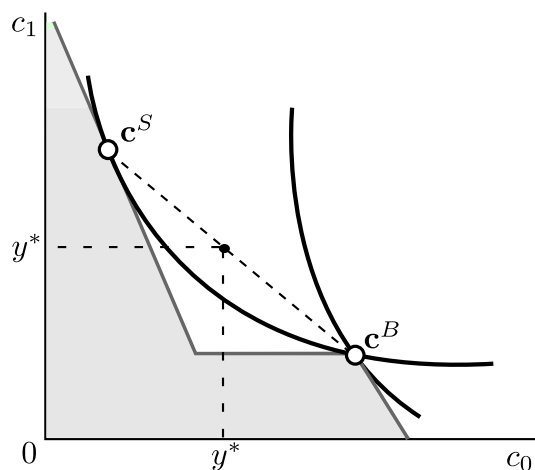


Figure 1.9: Debt relief implements allocations in which (ICS) binds

Debt relief can also be used to implement allocations in which (ICS) does not bind, but which are still relatively favorable to borrowers. Figure 1.10 provides an example. In this case, it is not strictly necessary for the transfer function to be nondifferentiable, since both agents will locate at interior points. If the ZLB does not bind, borrowers and savers face the same interest rates in the first period, and we can set the tax on excessive debt,  $\tau$ , equal to zero. If the ZLB binds, it will in general be necessary to tax either borrowers or savers.

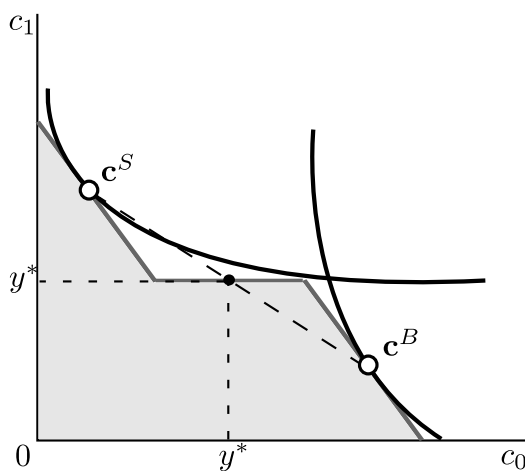


Figure 1.10: Debt relief implements some allocations in which no ICs bind

#### 1.4.4 Savings subsidies

Intuitively, it seems unlikely that debt relief implements every constrained efficient allocation. Consider the extreme case in which  $\alpha = 1$ , so we look for the constrained efficient allocation which is best for the saver, ignoring the borrower's welfare altogether. Clearly, savers have little to gain by offering debt relief to the borrowers. If anything, they would prefer to tax borrowers, and transfer resources to themselves - subject to the *borrowers'* incentive compatibility constraints. In this section, I show that efficient allocations which are relatively favorable for savers can be implemented with a subsidy to savers, as this intuition suggests.

Define savings subsidies as follows:

**Definition 1.4.5.**  $T(d)$  is a *savings subsidy* if it has the form

$$\begin{aligned} T^{SS}(d; T_s, T_b, d^*, \tau) &= T_b \text{ if } d > d^* \\ &= T_s - \tau(d^* - d) \text{ if } d \leq d^* \end{aligned}$$

where  $T_s, T_b, d^*, \tau$  are parameters.

There is a lump sum tax on households with debt above a certain level  $d^*$ . There is a lump sum subsidy to households with debt equal to  $d^*$ . Excessive saving is penalized by a tax  $\tau$  on saving above this level.

**Proposition 1.4.6.** 1. A savings subsidy implements the optimal allocation iff  $\alpha > \alpha(\theta_B)$ .

2. If (ICB) binds,  $\alpha > \alpha(\theta_B)$  and a savings subsidy implements the optimal allocation.

Consider a constrained efficient allocation in which  $B$ 's incentive compatibility constraint binds. Figure 1.11 shows that savings subsidies implement such an allocation. In equilibrium, savers all choose exactly the maximum level of savings. In any transfer scheme implementing an optimal allocation of this kind,  $T(d)$  must be nondifferentiable at  $d_1^S$ ,  $S$ 's equilibrium debt level. As in the case when (ICS) binds,  $S$  and  $B$ 's indifference curve intersect, this time at  $c^S$ . Again, because the two households have different preferences, the indifference curves have different slopes at this point, and the budget set must be kinked at  $c^S$ .

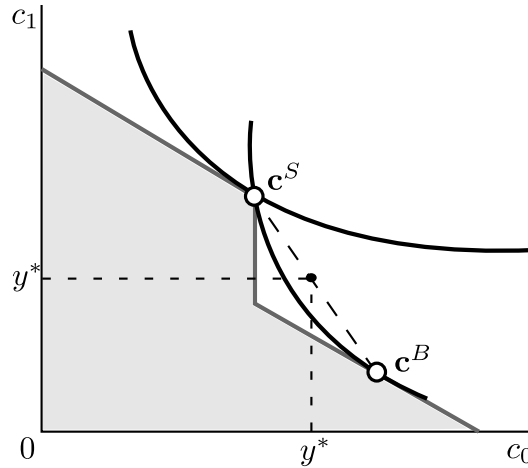


Figure 1.11: Savings subsidies implement allocations in which (ICB) binds

### 1.4.5 Pareto improving debt relief

While debt relief always implements some Pareto optimal allocations, we have just seen that savings subsidies - the opposite of debt relief - always implement efficient allocations. In what sense is debt relief a desirable policy?

In this section, I ask whether debt relief is Pareto improving relative to the competitive equilibrium without policy. The following proposition states that this is always the case, provided the zero lower bound binds in equilibrium. Even if  $\alpha = 1$ , so the planner only values  $S$ 's welfare, there exists a debt relief policy which is welfare improving at the zero lower bound. However, if the ZLB does not bind, debt relief is purely redistributive: it increases utility for borrowers but reduces utility for savers.

**Proposition 1.4.7.** 1. If  $\theta_B \leq \theta^{ZLB}$ , the competitive equilibrium is Pareto optimal (it is the solution to the planner's problem with  $\alpha = \alpha(\theta_B)$ ). Debt relief is not Pareto improving.

2. If  $\theta_B > \theta^{ZLB}$ , the competitive equilibrium is Pareto inefficient. Debt relief is always Pareto improving.

Intuitively, suppose the zero lower bound binds in equilibrium, and date 1 consumption is below potential output. In any competitive equilibrium, each agent strictly prefers their own allocation to the other agent's allocation, and both incentive constraints are slack. Suppose we attempt to increase date 1 consumption. The savers cannot consume more, since the zero lower

bound binds. However, we can increase borrowers' consumption by some amount before we make savers' incentive compatibility constraint bind. This is a Pareto improvement.

If the zero lower bound does not bind in equilibrium, the competitive equilibrium is Pareto optimal, for the usual reasons. We already know that debt relief remains constrained efficient in this case: relative to the competitive equilibrium, it provides higher utility to borrowers and lower utility to savers. But it does not offer a Pareto improvement over the competitive equilibrium.

### 1.4.6 Characterizing optimal debt relief

In Section 1.4.5, I showed that there always exists some Pareto improving debt relief policy at the zero lower bound. In this section, I focus on one particular Pareto improving policy, namely the one which is most favorable to borrowers. I explain what determines the amount of debt relief which is optimal. In Section 1.3 I showed that poorly designed debt relief causes overborrowing; in this section, I explain how optimal policy avoids overborrowing.

As we have seen above, there are a continuum of constrained efficient allocations, indexed by  $\alpha$ , the Pareto weight on savers. I focus on one particular allocation, namely, the allocation which maximizes borrowers' utility, subject to the constraint that savers are no worse off than in the equilibrium without policy. That is, the allocation solves

$$\max_{c_0^S, c_1^S, c_2^S, c_0^B, c_1^B, c_2^B} U(c_0^B, \theta_B) + \beta u(c_1^B) + \frac{\beta^2}{1-\beta} u(c_2^B) \quad (1.8)$$

$$\text{s.t. } U(c_0^S, \theta_S) + \beta u(c_1^S) + \frac{\beta^2}{1-\beta} u(c_2^S) \geq \bar{U}(\theta_S, \theta_B, \phi) \quad (\text{US})$$

$$(\text{RC0}), (\text{RC1}), (\text{RC2}), (\text{BC}), (\text{ZLB}), (\text{ICS})$$

where  $\bar{U}(\theta_S, \theta_B, \phi)$  is the savers' utility in the equilibrium without policy.<sup>18</sup> I call the solution to this program the **borrower-optimal** allocation. To be clear, this always solves our original Pareto problem (1.7) for some  $\alpha$ . All the results in Sections 1.4 and 1.4.2 therefore apply.

To guarantee that borrower-optimal allocations are continuous in  $\theta_B$ , we make the following assumption.

**Assumption 1.4.8.**  $U_{cc\theta} < 0$ .

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<sup>18</sup>Note that we can omit (ICB), since we are trying to maximize the borrowers' utility.

This is satisfied by  $U(c, \theta) = \theta u(c)$  with  $u$  concave, and by  $U(c, \theta) = u(c - \theta)$  if  $u''' > 0$ .

The following proposition characterizes borrower-optimal allocations.

**Proposition 1.4.9.** *The solution to the borrower-optimal problem is in one of five classes.*

1. *If the ZLB does not bind in equilibrium, the optimal allocation is the equilibrium without policy.*
2. *If the ZLB binds in equilibrium, and the full employment transfer is incentive compatible, i.e.*

$$U(c_0^S, \theta_S) + \beta u(\bar{c}_1^S) + \frac{\beta^2}{1 - \beta} u(\bar{c}_2^S) \geq U(c_0^B, \theta_S) + \beta u(2y^* - \bar{c}_1^S) + \frac{\beta^2}{1 - \beta} u(\bar{c}_2^B)$$

*then the full employment transfer is optimal. That is, the optimal allocation is identical to the equilibrium without policy, except that  $c_1^B$  is increased to  $2y^* - c_1^B$ . (US) and (RC1) bind.*

*If the full employment transfer is not incentive compatible, there are three possibilities. Let  $\hat{c}(\theta_B)$  solve*

$$U_c(\hat{c}, \theta_B) = U_c(\hat{c}, \theta_S) + U_c(2y^* - \hat{c}, \theta_S) \quad (1.9)$$

*And let  $\underline{c}(\phi)$  be the smallest level of  $c_0^B$  which satisfies (ICS) and (RC1) with equality:*

$$U(2y^* - \underline{c}, \theta_S) + \beta u(\bar{c}_1^S) + \frac{\beta^2}{1 - \beta} u(\bar{c}_2^S) = U(\underline{c}, \theta_S) + \beta u(2y^* - \bar{c}_1^S) + \frac{\beta^2}{1 - \beta} u(\bar{c}_2^B)$$

*Then:*

3. *If  $\hat{c}(\theta_B) < \underline{c}(\phi) < c_0^B$ , the optimal allocation is  $\underline{c}(\phi)$ , and (ICS) and (RC1) bind.*
4. *If  $\underline{c}(\phi) < \hat{c}(\theta_B) < c_0^B$ , the optimal allocation is  $\hat{c}(\theta_B)$ , and (ICS) binds.*
5. *If  $\underline{c}(\phi) < c_0^B < \hat{c}(\theta_B)$ , the optimal allocation is  $c_0^B$ , and (ICS) and (US) bind.*

When the ZLB is slack, this allocation is identical to the equilibrium, since the equilibrium is already efficient, and there is no policy. When the ZLB binds, Proposition 1.4.7 implies that the borrower-optimal allocation involves debt relief.

Even when the full employment transfer is not incentive compatible, it is still possible to increase borrowers' date 1 consumption until (ICS) binds. In fact, it is sometimes possible to do even better, as Figure 1.12 illustrates. When (ICS) binds, we can increase  $c_1^B$  further by reducing  $c_0^B$  and increasing  $c_0^S$ , to keep savers indifferent between their own allocation and the borrowers'

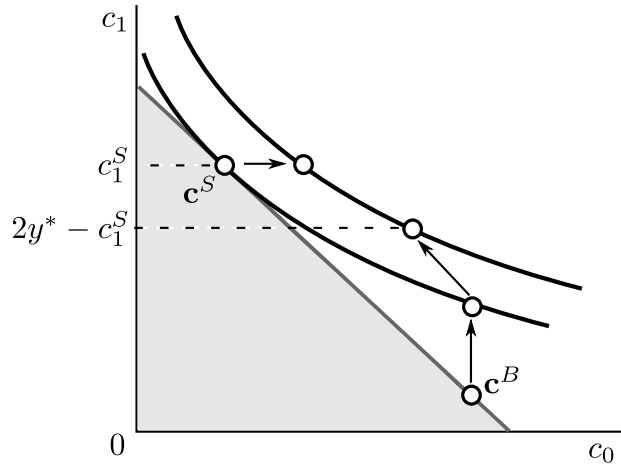


Figure 1.12: Borrower-optimal policy reduces  $c_0^B$

allocation. This can be implemented by reducing the borrowers' debt, i.e. setting the cap on debt relief,  $\bar{d}$ , below the level of debt in the equilibrium without policy.

When is it optimal to reduce debt in this way? When borrowers are not too impatient ( $\theta_B$  is not too high), the gains from higher date 1 consumption outweigh the cost of lower date 0 consumption. When  $\theta_B$  is slightly higher, it is optimal to reduce debt to some extent, but not all the way to full employment. Finally, if borrowers are sufficiently impatient, it is never optimal to reduce debt in return for higher date 1 consumption.

In section 1.3, I showed that debt relief can induce overborrowing, on both the intensive and extensive margins. On the intensive margin, debt relief increases borrowers' lifetime wealth. Absent any change in interest rates, this increase in wealth would induce borrowers to consume more at date 0, which would mean that savers consume less in equilibrium, making savers worse off. On the extensive margin, debt relief might induce some households who would otherwise save, to borrow instead, in order to qualify for this transfer.

The optimal debt relief policy avoids both these pitfalls. To prevent overborrowing on the intensive margin, the optimal policy requires a wedge  $\tau$  between the shadow interest rates faced by borrowers and savers, as the following proposition states.

**Proposition 1.4.10.** *Define the wedge*

$$\tau := \frac{U_c(c_0^B, \theta_B) u'(c_1^S)}{U_c(c_0^S, \theta_S) u'(c_1^B)} - 1.$$

*If  $c^S, c^B$  is Pareto improving relative to the equilibrium without policy,  $\tau > 0$ .*

A positive wedge  $\tau > 0$  increases the effective marginal interest rate faced by borrowers at date 0, making them borrow less. As I showed in Section 1.4.2, this wedge can be implemented by making debt relief conditional on having a relatively moderate level of debt, below some maximum  $\bar{d}$ . Above  $\bar{d}$ , an additional dollar of debt reduces the transfer that households receive by  $\tau$  dollars.<sup>19</sup> Making debt relief conditional prevents borrowers from borrowing more than they would have done in the equilibrium without policy.

Optimal policy avoids overborrowing on the extensive margin in two ways: by keeping  $T$  at a moderate level, and by lowering debt. Conditional debt relief can reduce equilibrium debt by setting  $\bar{d}$  lower than the equilibrium level of debt, and charging a high tax on debt above  $\bar{d}$ . However, this is not optimal for borrowers if they are too impatient. In this case, the only way to prevent overborrowing is to keep  $T$  below the level required for full employment.

## 1.5 Optimal policy with a continuous distribution of types

A concern with the model presented above is that incentive compatibility conditions are not too demanding when there are only two types. The planner can always design allocations in which no agent strictly prefers to mimic another agent's allocation. In this sense, it is possible to provide debt relief without encouraging any agents to overborrow. With a distribution of types, any debt relief policy will always induce some agent to borrow more. Debt relief may still be optimal, but the planner must now trade off the benefits of debt relief against the cost of distorting incentives towards overborrowing. Does the striking result presented above - that some debt relief is always Pareto improving when the zero lower bound binds - still hold?

To answer this question, I modify the model to include a continuous distribution of types. Agents have date 0 preferences  $U(c, \theta) = \theta u(c)$ , with  $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ .  $\theta$  has a continuous density  $f(\theta)$  with support  $\Theta = [\underline{\theta}, \bar{\theta}]$ . Equilibrium is defined as before.

As in the discrete type economy, the ZLB binds in equilibrium if agents are sufficiently impatient. Index households by  $i \in [0, 1]$ , and let  $\theta(i) = F^{-1}(i)$  be the type of household  $i$ , so the most patient agent is 0, and he has type  $\underline{\theta}$ . The following proposition states that if we make each of the remaining agents  $i > 0$  sufficiently impatient, then the ZLB binds in equilibrium: the impatient

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<sup>19</sup>Any tax greater than or equal to  $\tau$  would suffice.

households borrow so much that the patient households accumulate large savings, and it would take a negative real interest rate at date 1 to make them consume all their wealth.

**Proposition 1.5.1.** *Take any sequence of functions  $\theta_N : [0, 1] \rightarrow [1, \infty)$  such that for all  $N = 0, 1, \dots$   $\theta_N(0) = \underline{\theta}$ ,  $\theta_N$  is increasing, and for all  $i \in (0, 1]$ ,  $\theta_N(i) \rightarrow \infty$  as  $N \rightarrow \infty$ . There exists  $N^*$  such that  $r_1 < 0$  if  $N > N^*$ . Informally, the ZLB binds if agents' types are high enough.*

When the ZLB binds, equilibrium consumption falls below potential output, and it would be ex post Pareto improving to transfer wealth from the most patient household, whose consumption is limited by the ZLB, to an impatient household, who is liquidity constrained. As in the discrete type economy, such a transfer may not be incentive compatible. In fact, incentive compatibility is a much stronger constraint in the continuous type economy. Any transfer targeted at high  $\theta$  individuals will induce some households with slightly lower  $\theta$  to borrow more. To put this another way, any debt relief policy induces some overborrowing on the extensive margin, as well as overborrowing on the intensive margin.

### 1.5.1 Pareto problem

I now proceed to set up the Pareto problem in the continuous type economy. It is useful to write this problem in terms of households' compensated demand functions, which I now define.

Define the date 1 expenditure function to be

$$\begin{aligned} E(v_1, r_1) &= \min_{c_1, c_2} c_1 + \frac{c_2}{(1+r_1)(1-\beta)} & (\text{EMP}) \\ \text{s.t. } u(c_1) + \frac{\beta}{1-\beta} u(c_2) &\geq v_1 \\ c_2 &\geq \underline{c}_2 \phi \end{aligned}$$

where  $\underline{c}_2 = y^* - (1-\beta)\phi$ . Let  $C_1(v_1, r_1), C_2(v_1, r_1)$  be the solutions to this cost minimization problem.

These compensated demand functions will enter the Pareto problem: the planner will choose the value  $v_1(\theta)$  to provide to each type, and the resource cost of providing this value at dates 1 and 2 will be  $C_1(v_1, r_1), C_2(v_1, r_1)$ . It will be more intuitive, however, to relate the first order conditions of the planner's problem to the Marshallian uncompensated demand functions, which

I now define. Let the date 1 value function to be

$$\begin{aligned}
V(a_1, r_1) &= \max_{c_1, c_2} u(c_1) + \frac{\beta}{1-\beta} u(c_2) & (\text{UMP}) \\
\text{s.t. } c_1 + \frac{c_2}{(1+r_1)(1-\beta)} &\leq a_1 \\
c_2 &\geq \underline{c}_2
\end{aligned}$$

Let  $X_1(a_1, r_1), X_2(a_1, r_1)$  be the solutions to this utility maximization problem. The following results are, for the most part, standard.

**Lemma 1.5.2.** 1. *Convexity.*  $E$  is convex in  $v_1$ .  $V$  is concave in  $a_1$ .

2. *Duality.*  $E(V(a_1, r_1), r_1) = a_1$ .  $V(E(v_1, r_1), r_1) = v_1$ .  $C_t(v_1, r_1) = X_t(E(v_1, r_1), r_1), t = 1, 2$ .

3. *Envelope theorem.*  $V_a(a_1, r_1) = u'(c_1)$ .

4. *Marginal cost of utility and MPC.*  $\frac{\partial C_t(v_1, r_1)}{\partial v_1} = \frac{\partial X_t(E(v_1, r_1), r_1)/\partial a_1}{u'(c_1)}, t = 1, 2$ .

5. *Borrowing constraint eventually binds.* There exists  $\bar{v}_1(r_1)$  such that

$$C_2(v_1, r_1) = \underline{c}_2 \text{ if } v_1 \leq \bar{v}_1(r_1).$$

There exists  $\bar{a}_1(r_1) = E(\bar{v}_1(r_1), r_1)$  such that

$$X_2(a_1, r_1) = \underline{c}_2, \frac{\partial X_1}{\partial a_1} = 1, \frac{\partial X_2}{\partial a_1} = 0 \text{ if } a_1 < \bar{a}_1(r_1)$$

6. *Unconstrained CRRA households have constant MPC.* If  $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ , then if  $a_1 > \bar{a}_1(r_1)$ ,

$$\frac{\partial X_1}{\partial a_1} = \frac{1-\beta}{1-\beta + \beta^{1/\sigma}(1+r_1)^{\frac{1}{\sigma}-1}}$$

7. *Convex savings function.* If  $X_1(a_1, r_1)$  is concave in  $a_1$ ,  $C_2(v_1, r_1)$  is convex in  $v_1$ . A sufficient condition for this is that  $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ .

With these definitions in hand, I consider the social planner problem. The social planner puts weight  $a(\theta)$  on type  $\theta$  households. As before, the planner faces resource and incentive compatibility constraints. Now, however, the borrowing constraint and the constraints imposed by liquidity unconstrained households' Euler equations are embodied in the Hicksian demand

functions in constraints (RC1) and (RC2). I write the zero lower bound constraint (ZLB) explicitly as a constraint on the real interest rate.

$$\max_{c_0, v_1, r_1} \int a(\theta) [\theta u(c_0(\theta)) + \beta v_1(\theta)] d\theta \quad (\text{PP})$$

$$\text{s.t. } \int c_0(\theta) f(\theta) d\theta \leq y^* \quad (\text{RC0})$$

$$\int C_1(v_1(\theta), r_1) f(\theta) d\theta \leq y^* \quad (\text{RC1})$$

$$\int C_2(v_1(\theta), r_1) f(\theta) d\theta \leq y^* \quad (\text{RC2})$$

$$\theta u(c_0(\theta)) + \beta v_1(\theta) \geq \theta u(c_0(\hat{\theta})) + \beta v_1(\hat{\theta}), \forall \theta, \hat{\theta} \quad (\text{IC})$$

$$r_1 \geq 0 \quad (\text{ZLB})$$

First I transform this problem into an equivalent, concave programming problem; I then characterize solutions to the problem using Lagrangian theorems.<sup>20</sup> It is convenient to work in terms of utilities, rather than consumption allocations. Define the convex, increasing cost of utility function  $C_0(u_0) = u^{-1}(u_0)$ , and the date 0 value function  $v(\theta) = \theta u_0(\theta) + \beta v_1(\theta)$ . It is possible to express the incentive compatibility constraint (IC) as an integral condition, using the result of [Milgrom and Segal \[2002\]](#).  $u_0, v_1$  satisfies (IC) if and only if

$$v(\theta) = v(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} u_0(z) dz \quad (1.10)$$

and  $u_0$  is nondecreasing.

Instead of choosing functions  $u_0$  and  $v_1$ , we can equivalently choose a function  $u_0$  and a scalar  $\underline{v} := v(\underline{\theta})$ , subject to the constraint that  $u_0 \in \Omega$ , the space of nondecreasing functions.  $v, v_1$  are then implicitly defined by

$$v(\theta) = \underline{v} + \int_{\underline{\theta}}^{\theta} u_0(z) dz$$

$$v_1(\theta) = \beta^{-1} [\underline{v} + \int_{\underline{\theta}}^{\theta} u_0(z) dz - \theta u_0(\theta)]$$

The objective function can be rewritten as

$$\int a(\theta) v(\theta) d\theta = \underline{v} + \int (1 - A(\theta)) u_0(\theta) d\theta$$

where  $A(\theta) := \int_{\underline{\theta}}^{\theta} a(z) dz$ , and I normalize  $\int_{\underline{\theta}}^{\bar{\theta}} a(\theta) d\theta = 1$ .

<sup>20</sup>The Lagrangian optimization approach used here follows that of [Amador et al. \[2006\]](#).

Putting all this together, we can rewrite the social planner's problem as

$$\mathcal{W}^* = \max_{u_0 \in \Omega, \underline{v}, r_1} \underline{v} + \int (1 - A(\theta)) u_0(\theta) d\theta \quad (\text{PP}')$$

$$\text{s.t. } \int C_0(u_0(\theta)) f(\theta) d\theta \leq y^* \quad (\text{RC0}')$$

$$\int C_1 \left( \beta^{-1} \left[ \underline{v} + \int_{\underline{\theta}}^{\theta} u_0(z) dz - \theta u_0(\theta) \right], r_1 \right) f(\theta) d\theta \leq y^* \quad (\text{RC1}')$$

$$\int C_2 \left( \beta^{-1} \left[ \underline{v} + \int_{\underline{\theta}}^{\theta} u_0(z) dz - \theta u_0(\theta) \right], r_1 \right) f(\theta) d\theta \leq y^* \quad (\text{RC2}')$$

$$r_1 \geq 0 \quad (\text{ZLB})$$

## 1.5.2 Necessary and sufficient conditions

It is now almost possible to apply Lagrangian theorems to (PP'). There are two remaining problems. First, the date 1 consumption function  $C_1(v_1, r_1)$  is not convex in  $v_1$ : it has a kink at  $\bar{v}_1(r_1)$ . Second, the consumption functions need not be convex in  $r_1$ . In the following proposition, I show that it is nonetheless possible to use Lagrangian methods.

**Proposition 1.5.3.**  $u_0, \underline{v}, r_1$  solves (PP') if and only if there exist Lagrange multipliers  $\lambda_0, \lambda_1, \lambda_2$  such that  $u_0, \underline{v}$  solve

$$\begin{aligned} \mathcal{W}^* = \max_{u_0 \in \Omega, \underline{v}} \underline{v} + \int (1 - A(\theta)) u_0(\theta) d\theta - \lambda_0 \int C_0(u_0(\theta)) f(\theta) d\theta \\ - \int M \left( \beta^{-1} \left[ \underline{v} + \int_{\underline{\theta}}^{\theta} u_0(z) dz - \theta u_0(\theta) \right] | \lambda_1, \lambda_2, r_1 \right) f(\theta) d\theta \end{aligned} \quad (1.11)$$

where

$$M(v_1 | \lambda_1, \lambda_2, r_1) := \lambda_1 C_1(v_1, r_1) + \lambda_2 C_2(v_1, r_1)$$

and the Lagrange multipliers satisfy the following conditions. If the ZLB is slack,  $\frac{\lambda_1}{\lambda_2} = (1 + r_1)(1 - \beta)$ .

If the ZLB binds,  $\frac{\lambda_1}{\lambda_2} < 1 - \beta$ .

Before providing a sketch of the proof, I explain how to interpret the function  $M$ .  $M(v_1 | \lambda_1, \lambda_2, r_1)$  represents the total social cost of providing date 1 utility  $v_1$  to a household, given that the household will choose its spending at dates 1 and 2 according to the interest rate  $r_1$ , and given that the social planner's shadow price of date 1 and date 2 output are  $\lambda_1$  and  $\lambda_2$  respectively. When  $\frac{\lambda_1}{\lambda_2} = (1 + r_1)(1 - \beta)$ , the relative shadow price of date 1 and 2 output, from the planner's perspective, is the same as the relative price of output faced by agents. In this case,  $M$  is simply a

rescaled version of the expenditure function  $M = \lambda_1 E(v_1, r_1)$ . Proposition 1.5.3 states that the planner sets interest rates to equalize the private and social relative price of output, whenever this is not prevented by the ZLB.

However, when  $\frac{\lambda_1}{\lambda_2} < (1 + r_1)(1 - \beta)$ , the planner perceives that date 1 output is socially cheaper than date 2 output - the economy is in recession at date 1 - but private agents do not internalize this, because the relative price of date 1 output is still too high, because of the ZLB. This provides a motive for the planner to redistribute utility towards households with a higher propensity to spend at date 1 consumption, when consumption is socially cheap. In this economy, households with relatively low date 1 utility and wealth (i.e. with  $v_1 \leq \bar{v}_1$ ) have a higher propensity to spend at date 1. In fact, their consumption functions are kinked at  $\bar{v}_1$ , which means that  $M$  is kinked at  $\bar{v}_1$ . Given that some households have date 1 utility  $v_1(\theta)$  below  $\bar{v}_1$ , the planner would like  $v_1(\theta)$  to be relatively high, since it is relatively cheap to supply this utility. As a result, it may be optimal to redistribute towards households with a higher propensity to consume date 1 consumption (which is socially cheap), or to give those households incentives to save at date zero so they have more wealth to spend at date 1.<sup>21</sup>

The proof of Proposition 1.5.3 has six steps. First we show that solutions to (PP') also solve a modified problem in which we replace the date 1 resource constraint with the aggregate expenditure function. Second, the modified problem can be solved in two stages: first maximize social welfare given  $r_1$ , yielding welfare  $\mathcal{W}(r)$ , and then choose  $r$  to maximize  $\mathcal{W}(r)$  subject to the ZLB. Third, the first stage of this problem is concave, and Lagrangian theorems (Luenberger [1969]) apply. Fourth, we can also express the expenditure functions as maximized sub-Lagrangians. Substituting these sub-Lagrangians into the main Lagrangian, we see that  $W(r)$  is also the maximum of an expanded Lagrangian. Fifth, returning to our two stage problem, we can switch the order of maximization, first choosing  $r$  to minimize a certain function, subject to the ZLB, and then choosing utilities to maximize social welfare. Sixth, and finally, I show that when the ZLB is slack, one constraint in the planner's problem becomes slack, and the expanded Lagrangian is equivalent to (1.11), with  $\frac{\lambda_1}{\lambda_2} = (1 + r_1)(1 - \beta)$ . When the ZLB binds, we have  $\frac{\lambda_1}{\lambda_2} < 1 - \beta$ .

It is possible to express necessary and sufficient conditions for an optimum in terms of

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<sup>21</sup>This is exactly the result of Farhi and Werning [2013]. Relative to their framework, however, here the social planner faces additional incentive compatibility constraints resulting from private information, which make it harder to redistribute.

Gateaux differentials of the Lagrangian.<sup>22</sup> Before doing so, it is first necessary to show that these differentials can be computed.

**Lemma 1.5.4.** *The Gateaux differential of the Lagrangian (1.11) is*

$$\begin{aligned} \delta\mathcal{L}(u_0, \underline{v}; \Delta_0, \underline{\Delta}) &= \underline{\Delta} + \int (1 - A(\theta))\Delta_0(\theta) \, d\theta - \lambda_0 \int C'_0(u_0(\theta))\Delta_0(\theta)f(\theta) \, d\theta \\ &\quad - \int_{\Theta_+} M'_+(v_1(\theta)|\lambda_1, \lambda_2, r_1)\beta^{-1} \left[ \underline{\Delta} + \int_{\theta}^{\theta} \Delta_0(z) \, dz - \theta\Delta_0(\theta) \right] f(\theta) \, d\theta \\ &\quad - \int_{\Theta_-} M'_-(v_1(\theta)|\lambda_1, \lambda_2, r_1)\beta^{-1} \left[ \underline{\Delta} + \int_{\theta}^{\theta} \Delta_0(z) \, dz - \theta\Delta_0(\theta) \right] f(\theta) \, d\theta \end{aligned}$$

where  $v_1(\theta) = \beta^{-1} \left[ \underline{v} + \int_{\theta}^{\theta} u_0(z) \, dz - \theta u_0(\theta) \right]$ , and

$$\begin{aligned} \Theta_+ &= \left\{ \theta \in \Theta : \underline{\Delta} + \int_{\theta}^{\theta} \Delta_0(z) \, dz - \theta\Delta_0(\theta) > 0 \right\} \\ \Theta_- &= \left\{ \theta \in \Theta : \underline{\Delta} + \int_{\theta}^{\theta} \Delta_0(z) \, dz - \theta\Delta_0(\theta) < 0 \right\} \end{aligned}$$

Putting all these results together, we can now characterize constrained efficient allocations in terms of first order conditions.

**Lemma 1.5.5.**  $u_0, \underline{v}, r_1$  solves (PP') if and only if there exist Lagrange multipliers  $\lambda_0, \lambda_1, \lambda_2$  such that, for all  $\underline{\Delta}, \Delta_0$  such that  $u_0 + \Delta_0 \in \Omega$ ,

$$\delta\mathcal{L}(u_0, \underline{v}; \Delta_0, \underline{\Delta}) \leq 0$$

where  $\delta\mathcal{L}(u_0, \underline{v}; \Delta_0, \underline{\Delta})$  is defined as in Lemma 1.5.4, and the Lagrange multipliers satisfy the following conditions. If the ZLB is slack,  $\frac{\lambda_1}{\lambda_2} = (1 + r_1)(1 - \beta)$ . If the ZLB binds,  $\frac{\lambda_1}{\lambda_2} < 1 - \beta$ .

### 1.5.3 Constrained efficient debt relief

I now show that the main results from the two type economy considered above generalize to a continuous distribution of types. When the ZLB binds in the equilibrium without policy, the equilibrium is constrained inefficient. Somewhat surprisingly, the simple, piecewise linear debt

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<sup>22</sup>Given a real valued functional  $f$  defined on a vector space  $X$ , if the limit

$$\lim_{\alpha \downarrow 0} \frac{1}{\alpha} [f(x + \alpha h) - f(x)]$$

exists, then it is called the Gateaux differential of  $f$  at  $x$  with increment  $h$  and is denoted by  $\delta f(x; h)$ .

relief transfer functions which implemented optimal allocations in the two type economy also implement some (but by no means all) optimal allocations with a continuous distribution of types. In particular, if a debt relief transfer function has a positive marginal tax above the cap  $\tau > 0$ , and implements full employment, then it is constrained efficient.

**Proposition 1.5.6.** *If the ZLB binds in the equilibrium without policy:*

1. *The equilibrium is constrained inefficient.*
2. *Debt relief transfer functions with  $\tau > 0$  implementing full employment allocations are constrained efficient.*

*If the ZLB is slack, the equilibrium is constrained efficient.*

A sketch of the proof of part 1 is as follows. Suppose by contradiction that a competitive equilibrium in which the ZLB binds, and the date 1 resource constraint is slack, is constrained efficient. Consider the following deviation: make a small transfer to households whose date 0 consumption is such that they are 'just' borrowing constrained. Households with slightly higher date 0 consumption will consume less at date 0, and more at date 1, to qualify for the transfer; households with slightly lower date 0 consumption will consume more at date 0, and less at dates 1 and 2. To first order, the effect on date 0 consumption cancels out, but aggregate date 1 consumption increases, and date 2 consumption falls. This is clearly feasible, since the date 1 resource constraint is slack, and it increases utility for the households who change their behavior. Thus the original allocation cannot have been constrained efficient.

To show that debt relief transfer functions with  $\tau > 0$  implement constrained efficient allocations, two steps are necessary. First, it is necessary to show that such allocations exist. The proof proceeds by showing that aggregate consumption demand at dates 0,1 and 2 is a continuous function of the parameters of the debt relief transfer function,  $\bar{T}$ ,  $\underline{d}$ ,  $\bar{d}$  and  $\tau$ . A fixed point argument then shows that there exists a transfer function implementing a full employment allocation. Second, it is necessary to show that such allocations, if they exist, are constrained efficient. The proof proceeds by showing that the allocations satisfy the condition in Lemma 1.5.5.

### 1.5.4 How the optimal policy prevents overborrowing

Debt relief encourages overborrowing on the intensive margin, inducing borrowers who had enough debt to qualify for the transfer, even in the equilibrium without policy, to borrow and consume more through a wealth effect. In addition, any debt relief policy induces some overborrowing on the extensive margin. That is, if households with higher debt receive a higher transfer, some households will take on more debt in order to benefit from this transfer. This was not true in the two type economy: in that economy, there was always some room to give borrowers a transfer without encouraging savers to overborrow.

Figure 1.13 illustrates. Although there are a continuum of types, to simplify the figure I only show the indifference curves of two types,  $\theta_M$  and  $\theta_B > \theta_M$ . Suppose households' debt is written off one-for-one in some range, shifting the budget set to the right as shown in the figure. This induces some households to increase their date zero consumption and borrowing. For high types like  $\theta_B$ , debt relief acts through a wealth effect, and these types increase their date 0 and date 1 consumption. Intermediate types like  $\theta_M$ , however, will borrow up to the kink in the budget constraint. Thus there is overborrowing on both an intensive and an extensive margin. Finally, there are some more patient types who are not affected by the policy.

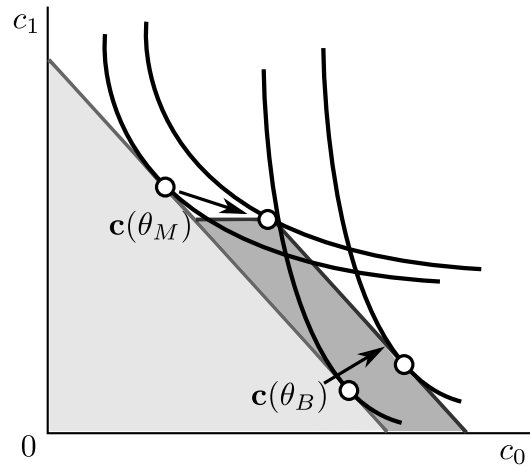


Figure 1.13: Debt relief induces intensive and extensive margin overborrowing

The budget set shown in 1.13 does not implement a feasible allocation, since aggregate date 0 consumption has increased and markets do not clear. To prevent overborrowing, and clear markets, the government can tax borrowing above the cap at rate  $\tau > 0$ . This reduces the date

0 consumption and borrowing of the most impatient households, such as  $\theta_B$ . That is, this debt relief policy reduces the debt of extreme borrowers, through a marginal tax on debt, to balance out the overborrowing of moderate borrowers such as  $\theta_M$ . Figure 1.14 illustrates.

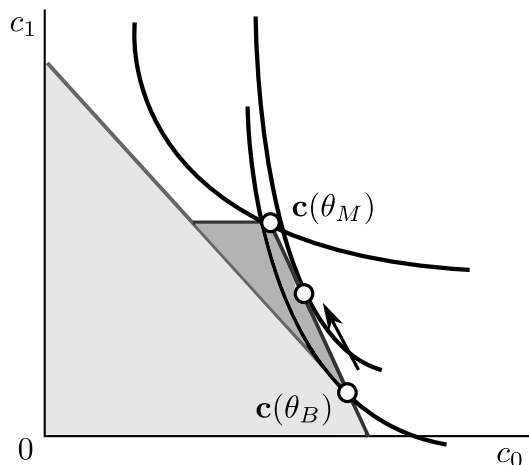


Figure 1.14:  $\tau$  prevents overborrowing

By increasing the transfer to moderate borrowers, and increasing the tax on extreme borrowers to ensure markets clear at date 0, it is always possible to return the economy to full employment at date 1. However, the further the economy is from full employment, the larger the transfer required to return it to full employment, and the higher the marginal tax on debt required to balance out the overborrowing induced by this transfer. It is possible that the tax on debt required to prevent overborrowing is so high that some impatient types (such as  $\theta_B$ ) are made worse off by this policy. Debt relief transfer functions which implement full employment are always constrained efficient, but they may not be Pareto improving. Figure 1.15 illustrates such a case.

### 1.5.5 Numerical exercise

How large are the optimal transfer and the marginal debt tax  $\tau$  likely to be? Is linear debt relief policy likely to be Pareto improving? In this section I present a numerical exercise to give a rough answer to these questions.

I interpret 1 period as 5 years.  $\theta$  is lognormal, with  $\ln \theta \sim \mathcal{N}(\mu_\theta, \sigma_\theta^2)$ . I set  $\beta = 0.975$ , corresponding to a steady state real interest rate of 0.5%. I set  $\phi = 0.2$ . I then vary  $\sigma$ , the inverse

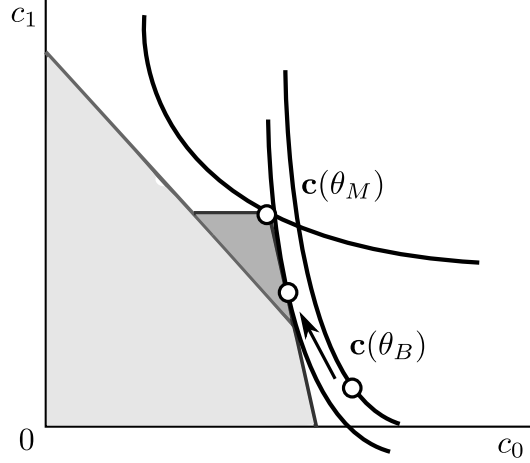


Figure 1.15: Debt relief transfer function may not be Pareto improving

of the intertemporal elasticity of substitution, and choose  $\mu_\theta, \sigma_\theta^2$  to roughly match the 2008-2012 fall in output and deleveraging, and the 2007 distribution of debt, in the United States.

Potential output  $y^*$  is normalized to 1. The first variable to compare to the data is  $y_1$ , the shortfall in output relative to potential output. I measure potential output by fitting either a linear trend or an exponential trend to 1984-2007 real GDP. I then measure  $y_1$  as the average value of annual output divided by potential output over 2008-2012. Using a linear trend, this yields  $y_1 = 0.95$ ; using an exponential trend yields  $y_1 = 0.90$ . In the table below, I report a simple average of these,  $y_1 = 0.92$ .

Data on aggregate household debt comes from the Financial Accounts of the United States. I divide total household debt by 5 times trend annual GDP. Since I interpret the crisis period  $t = 1$  as 2008-2012, I interpret aggregate debt at date 1,  $D_1 = \int d_1^i \mathbb{1}\{d_1^i \geq 0\} di$  as household debt in 2008, and aggregate debt at date 2,  $D_2 = \int d_2^i \mathbb{1}\{d_2^i \geq 0\} di$  as household debt in 2012. This yields  $D_1 = 0.19, D_2 = 0.14$ .

Data on the distribution of debt comes from the 2007 Survey of Consumer Finances. I restrict the sample to heads of household aged between 25 and 65 who are not students and who are not retired. I interpret  $d_t^i$  in the model as total household debt minus financial assets, divided by 5 times the average family income for households in the sample.<sup>23</sup> As explained above, whether

<sup>23</sup>The SCF collects data from two samples: a standard multistage area-probability sample selected from the 48 contiguous US states, and a list sample designed to disproportionately sample wealthy families. The SCF provides probability weights which account for the sample design, and also for differential patterns of non-response among families with different characteristics. All SCF data presented here is weighted using these probability weights.

simple linear debt relief policies are Pareto improving depends crucially on the right tail of the distribution of debt. I therefore attempt to roughly match the 90th percentile of the distribution of the debt.

To solve the model, I approximate the distribution of types as a discrete type economy with 500 types, using an unequally spaced grid for  $\theta$ . I truncate the distribution of  $\theta$ , setting  $\underline{\theta} = 0.05$ ,  $\bar{\theta} = 20$ . For each set of parameter values, I first verify that the ZLB binds in equilibrium, then search for the debt relief transfer policy that ensures full employment at date 1, while keeping the date 0 rate of interest,  $1 + r_0$ , the same as in the equilibrium without policy. In what follows, I call this the ‘optimal policy’, but it is important to bear in mind that there are a large set of constrained efficient policies, and this is only one of them.

Table 1.1: Optimal policy

Parameters			Moments				Policy			
$\sigma$	$\mu_\theta$	$\sigma_\theta$	$y_1$	$D_1$	$D_2$	$p_{90}$	$\bar{T}$	$\underline{d}$	$\bar{d}$	$\tau$
0.5	0.1	0.3	0.94	0.16	0.11	0.46	0.00	0.15	0.17	0.16
1	0.5	0.5	0.92	0.16	0.11	0.44	0.02	0.16	0.20	0.53
2	1	1	0.93	0.14	0.10	0.40	0.03	0.16	0.23	1.66
			<b>Data</b>							
			0.92	0.19	0.14	0.42				

Table 1.1 presents the results. The maximum transfer is 2 – 4% of 5 years’ income, or \$8,000 – 16,000. The marginal tax above the cap is a 3 – 33% annual spread: it is higher if the IES,  $\frac{1}{\sigma}$ , is low.

Next, I evaluate the welfare effect of these policies. Following Lucas [1987], define the welfare benefit of debt relief for type  $\theta$ ,  $\lambda(\theta)$ , as the percentage increase in consumption, in every period  $t$ , such that household  $\theta$  would be indifferent between the equilibrium without policy (plus the percentage increase in consumption) and the equilibrium with policy. Let  $c_t^*$  denote allocations in the equilibrium without policy, and  $c_t^{**}$  allocations in the equilibrium with policy. For each type  $\theta$ ,  $\lambda(\theta)$  solves

$$\theta u(c_0^{**}(\theta)) + \sum_{t=1}^{\infty} \beta^t u(c_t^{**}(\theta)) = \theta u((1 + \lambda(\theta))c_0^*(\theta)) + \sum_{t=1}^{\infty} \beta^t u((1 + \lambda(\theta))c_t^*(\theta))$$

Figure 1.16 plots  $\lambda(\theta)$ , for  $\sigma = 0.5, 1, 2$ . Since the distribution of types is different under the three experiments, I plot  $\lambda(\theta)$  against  $F(\theta)$ , the rank of type  $\theta$  in the whole distribution. In each case,

the optimal debt relief policy increases welfare for most households, particularly for ‘moderate’ types with intermediate values of  $\theta$ . However, the optimal policy is never Pareto improving: extremely high types are made worse off by debt relief with a cap. These households have an extremely high demand for date 0 consumption; anything that reduces date 0 consumption, such as a marginal tax on debt, makes them worse off.

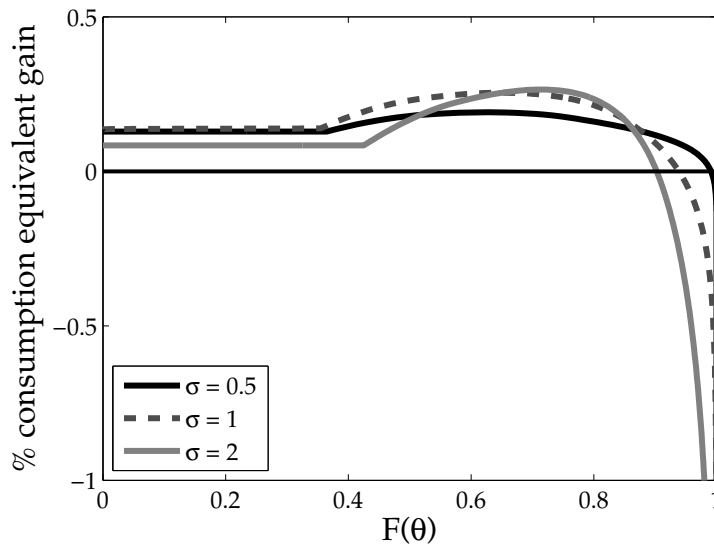


Figure 1.16: Consumption equivalent gains under optimal debt relief

This result should be interpreted with caution, for two reasons. First, the exercise here is to interpret the empirical distribution of debt in 2007 through the lens of a model in which all households started with zero debt in 2002, and in some cases took out large amounts of debt, knowing with certainty that the financial crisis would happen in 2008, and they would be forced to deleverage. Clearly, such a model requires substantial dispersion in preferences  $\theta$  to generate the significant wealth inequality observed in the data. In reality, households did not expect the crisis to happen with probability one; had they done so, they would surely have taken out less debt. Second, under the more realistic assumption that households do not expect the crisis to happen for sure (which I consider in Section 1.7.1 below), the incentive problems associated with debt relief are less severe, and it is easier to find an ex ante Pareto improving policy. In the extreme case, if the crisis is a zero probability event, there are no incentive concerns associated with debt relief. For both these reasons, one should not necessarily conclude that it is hard to design Pareto improving debt relief policies.

However, Figure 1.16 does highlight that any policy to reduce overborrowing runs the risk of harming households who need to borrow.<sup>24</sup> This risk is even greater for alternative macroprudential policies, which attempt solely to prevent overborrowing, as I discuss in section 1.6 below. Debt relief with a cap is less of a culprit in this regard, since it combines a transfer to indebted households with a marginal tax to reduce their borrowing.

## 1.6 Macroprudential policy

In this section I compare debt relief to macroprudential policies. Korinek and Simsek [2014] and Farhi and Werning [2013] have discussed the role of taxes on overborrowing, debt limits, and insurance requirements in preventing liquidity traps. Such macroprudential policies prevent overborrowing ex ante, while debt relief corrects overborrowing ex post. Korinek and Simsek [2014] and Farhi and Werning [2013] both consider optimal policy in the case when the planner can observe households' type. In contrast, I assume that type is private information.

### 1.6.1 Macroprudential policy with full information

Is macroprudential policy superior or inferior to debt relief, or are the two policies equivalent? To answer this question, I define equilibrium with two macroprudential policies considered by Korinek and Simsek [2014] and Farhi and Werning [2013] which are an alternative to debt relief: debt limits with compensating transfers at date 0, and linear debt taxes and transfers at date 1. I allow transfers and taxes to depend on an agent's type.

**Definition 1.6.1.** *An equilibrium with macroprudential taxes is a ZLB-constrained equilibrium with transfers in which transfer functions have the form  $T(d, \theta) = T_1(\theta) - \tau(\theta)d$ .*

*An equilibrium with debt limits is  $\{c_t^i, d_t^i, y_t, r_t\}$  such that, given a date 0 debt limit  $\phi_0$  and date 0 transfers  $T_0(\theta)$ ,*

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<sup>24</sup>As Zinman [2014] emphasizes, we still understand little about why some households borrow as much as they do.

1. each household  $i$  chooses  $\{c_t^i, d_t^i\}$  to maximize (1.1) s.t. (1.2), (1.4), and

$$d_0^i = -T_0(\theta_i)$$

$$d_1^i \leq \phi_0$$

2.  $c_t^B + c_t^S = 2y_t$

3.  $r_t \geq 0, y_t \leq y^*, r_t(y^* - y_t) = 0$

Farhi and Werning [2013] and Korinek and Simsek [2014] show that these macroprudential policies implement first-best allocations when household type is observable to the planner. The following proposition merely restates their result.

**Proposition 1.6.2.** *Consider the relaxed Pareto problem without constraints (ICS) and (ICB).*

1. *Every solution to the relaxed Pareto problem can be implemented as an equilibrium with macroprudential taxes. If the ZLB does not bind,  $\tau(\theta_S) = \tau(\theta_B) = 0$ . If the ZLB binds,*

$$\frac{1 + \tau(\theta_B)}{1 + \tau(\theta_S)} = \frac{u'(c_1^S) U_c(c_0^B, \theta_B)}{u'(c_1^B) U_c(c_0^S, \theta_S)} > 1 \quad (1.12)$$

*In particular, the optimal allocation can be implemented with a tax on debt targeted only at borrowers ( $\tau(\theta_B) > 0, \tau(\theta_S) = 0$ ).*

2. *Every solution to the relaxed Pareto problem can be implemented as an equilibrium with debt limits. If the ZLB binds, then the debt limit binds, and it is equal to*

$$\phi_0 = \bar{d}_1 := \bar{c}_1^S - y^* + \phi \quad (1.13)$$

Since these transfers depend on a household's type  $\theta$ , they cannot be implemented when  $\theta$  is private information. Next, I study macroprudential policies under private information, and compare them to the ex post policies considered throughout this chapter. Once we restrict the simple linear taxes and transfers considered in Proposition 1.6.2 to be anonymous, macroprudential taxes are sub-optimal, and debt limits only implement particular optimal allocations. However, modified macroprudential policies with nonlinear taxes and transfers are equivalent to ex post policies, and implement optimal allocations. Thus the results in Sections 1.4 and 1.4.2 can also be interpreted as describing optimal macroprudential policy under private information.

## 1.6.2 Macprudential taxes

First, I study macroprudential taxes under private information. Under full information, linear debt taxes implement optimal allocations, and satisfy equation (1.12) (only borrowers face a tax on debt). Under private information, linear debt taxes are not optimal. Nonlinear debt taxes implement optimal allocations, since they are equivalent to the debt-contingent transfers considered throughout this chapter; however optimal marginal tax rates may not satisfy equation (1.12).

Anonymous *linear* taxes on debt which do not depend on a household's type are redundant: they change the equilibrium interest rate, but implement the same allocations as an equilibrium without policy.<sup>25</sup> To see this, note that taxes only enter households' Euler equations:

$$(1 + r_0)(1 + \tau) = \frac{U_c(c_0^i, \theta_i)}{\beta u'(c_1^i)}$$

If  $\tau$  is increased,  $r_0$  falls to clear markets, and the equilibrium is unchanged. When this equilibrium is inefficient, anonymous linear taxes fail to implement optimal allocations.

If anonymous macroprudential taxes are to implement optimal allocations, they must be nonlinear. Nonlinear taxes on debt are equivalent to the debt-contingent transfers considered throughout this chapter, and implement the same allocations. All the results above about optimal debt-contingent transfers can be interpreted in terms of nonlinear macroprudential taxes. Equation (1.12), which describes the optimal marginal debt tax under full information, must now be modified to take into account incentive constraints. We can interpret  $-T'(d)$  as the analogue of  $\tau$ , the marginal tax on debt. (The negative sign is present because  $T(d) > 0$  denotes a positive transfer, whereas  $\tau > 0$  denotes a positive tax on debt.)

**Proposition 1.6.3.** 1. If (ZLB), (ICS) and (ICB) do not bind,  $T'(d_1^S) = T'(d_1^B) = 0$ .<sup>26</sup>

2. If (ZLB) binds and neither incentive constraint binds,  $T'(d_1^S) > T'(d_1^B)$ .

3. If (ZLB) does not bind and either (ICS) or (ICB) binds,  $T'(d_1^S) < T'(d_1^B)$ .

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<sup>25</sup>This is noted by Farhi and Werning [2013] who emphasize that only the relative financial taxes faced by different agents affect the allocation: the average level of taxes is indeterminate.

<sup>26</sup> $T'(d)$  denotes either the left-hand derivative or the right-hand derivative of  $T(d)$ ; as noted above, allocations in which an incentive constraint binds cannot be implemented with a differentiable value function.

4. If (ZLB) binds and either (ICS) or (ICB) binds, we may have  $T'(d_1^S) < T'(d_1^B)$ ,  $T'(d_1^S) > T'(d_1^B)$ , or  $T'(d_1^S) = T'(d_1^B) = 0$ .

If neither incentive constraint binds, and (ZLB) binds, we want to induce borrowers to take on less debt, since excessive debt imposes a macroeconomic externality. To do this, we impose a marginal tax on debt, as in the case with linear macroprudential taxes and perfect information. However, when incentive constraints bind, marginal taxes play a different role: they make the allocations of savers and borrowers different, so that savers do not want to mimic borrowers (or vice versa). If for example (ICS) binds, we distort allocations so that  $\frac{\beta u'(c_1^S)}{U_c(c_0^S, \theta_S)} < \frac{\beta u'(c_1^B)}{U_c(c_0^B, \theta_B)}$ . Absent incentive constraints, it would be Pareto improving to have  $B$  consume more at date 1, and have  $S$  consume more at date 0. But this would make  $B$ 's allocation more attractive to  $S$ , who values date 1 consumption more. To deter  $S$ , who prefers later consumption, from mimicking  $B$ , optimal policy front-loads  $B$ 's consumption by offering him a *lower* marginal interest rate than  $S$ , or a *negative* marginal tax on debt. When both (ZLB) and one incentive constraint bind, both motives are in play, and the sign of the marginal tax on debt is ambiguous.

### 1.6.3 Debt limits and date 0 transfers

Next, I consider debt limits under private information. Under full information, debt limits together with compensating transfers implement optimal allocations, and satisfy equation (1.13) (debt is low enough to ensure full employment). Under private information, debt limits implement one particular optimal allocation, which may not be Pareto improving. A modified version of the debt limit policy, with debt-contingent date 0 transfers, implements any optimal allocation. However, the optimal debt limit may not satisfy equation (1.13).

If transfers do not depend on a household's type, they must be equal to zero, under our maintained assumption that the government runs a balanced budget. A debt limit without any compensating transfer implements one particular constrained efficient allocation. This policy is better for savers, and worse for borrowers, relative to debt relief. It is always Pareto efficient, but it may not be Pareto improving relative to the equilibrium without policy. If borrowers are sufficiently impatient, they will be worse off with debt limits than in the equilibrium without policy: the cost of lower date 0 consumption outweighs the benefit of higher date 1 consumption.

Debt limits therefore appear to be more restrictive than debt-contingent transfers. However, if we allow the date 0 compensating transfers to depend on debt, debt limits implement the same set of allocations as date 1 debt-contingent transfers.

**Proposition 1.6.4.** *1. Every constrained efficient allocation can be implemented as an allocation with debt limits and date 0 debt-contingent transfers. The debt limit may be greater than  $\bar{d}_1$  if (ICS) binds.*

*2. A debt limit  $\phi_0 = \bar{d}_1$ , together with no transfer at date 0 ( $T_0(d) = 0$ ), implements constrained efficient allocations corresponding to a weight of  $\bar{\alpha}(\theta_B)$  in the social planner's problem.*

*3. There exists  $\bar{\theta}$  such that this allocation is not Pareto-improving, relative to the equilibrium without policy, if  $\theta_B > \bar{\theta}$ .*

*4. If the ZLB binds, the debt limit is always binding and equal to  $\bar{d}_1$ , unless (ZLB) and (ICS) both bind. In this case,  $d_1^B > \bar{d}_1$ , and there is underemployment at date 1.*

Part 1 of this Proposition states that date 0 transfers and date 1 transfers are equivalent: either policy defines a nonlinear mapping from date 0 consumption to date 1 consumption. Part 2 states that a debt limit without compensating transfers is Pareto efficient at the ZLB. However, part 3 states that this allocation is not Pareto improving if  $\theta_B$  is sufficiently large. A binding debt limit reduces borrowers' date 0 consumption, but increases date 1 consumption. If  $B$  is sufficiently impatient, a fall in date 0 consumption is very costly, and the binding debt limit reduces her welfare (although it increases  $S$ 's welfare). Similarly, part 4 notes that some Pareto efficient allocations in which the ZLB binds cannot be implemented with a debt limit equal to  $\bar{d}_1$ . These are the efficient allocations in which there is underemployment, and  $c_1^S + c_1^B < 2y^*$ .<sup>27</sup>

Date 0 transfers are equivalent to date 1 transfers, and can be used to induce exactly the same budget sets. In particular, we can construct date 0 transfer functions which induce exactly the same budget sets as debt relief transfer functions. These date 0 transfers can be interpreted as targeted loan support programs, combined with macroprudential taxes on excessive borrowing.

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<sup>27</sup>As discussed in Section 1.4, when (ICS) and (ZLB) both bind, some date 1 unemployment may be optimal.

**Definition 1.6.5.**  $T_0(d)$  is a *targeted loan support program* if it has the form

$$\begin{aligned} T^{LS}(d) &= -\bar{T} \text{ if } d < d^* \\ &= T^* - \tau d \text{ if } d \geq d^* \end{aligned}$$

where  $T^*, \bar{T} > 0, d^*, \tau$  are parameters.

There is a lump sum tax  $\bar{T}$  on all households who borrow less than a certain amount  $d^*$ . The government gives a subsidy of  $T^*$  to households who borrow exactly  $d^*$ . Above that point, further borrowing is penalized: the transfer falls by  $\tau$  dollars for each dollar of debt above the minimum level  $d^*$ .

Given that targeted loan support programs are isomorphic to debt relief transfer functions, the following result follows immediately from Propositions 1.4.4 and 1.4.7.

**Proposition 1.6.6.** 1. Targeted loan support implements the optimal allocation iff  $\alpha < \alpha(\theta_B)$ .

2. If (ICS) binds,  $\alpha < \alpha(\theta_B)$  and targeted loan support implements the optimal allocation.
3. If  $\theta_B \leq \theta^{ZLB}$ , the competitive equilibrium is Pareto optimal (it is the solution to the planner's problem with  $\alpha = \alpha(\theta_B)$ ). Targeted loan support is not Pareto improving.
4. If  $\theta_B > \theta^{ZLB}$ , the competitive equilibrium is Pareto inefficient. Targeted loan support is always Pareto improving.

Targeted loan support programs, like ex post debt relief with a cap, implement efficient allocations which are relatively favorable for borrowers. Unlike ex ante debt limits without compensating transfers, targeted loan support compensates borrowers for the reduction in their ability to borrow, while the tax on debt still discourages overborrowing.

#### 1.6.4 Debt limits with a distribution of types

Debt limits, like other forms of rationing, force agents who would otherwise borrow different amounts to borrow the same amount. This inefficiency is absent in an economy with only one type of borrower. To properly compare ex ante debt limits and ex post debt relief, I return to the economy with a distribution of types considered in Section 1.5, and ask whether a binding debt

limit without compensating transfers is constrained efficient.<sup>28</sup> The following proposition states that the debt limit is inefficient if some households are sufficiently impatient.

**Proposition 1.6.7.** *Take any allocation in which a debt constraint binds for agents in some interval  $[\theta^*, \bar{\theta}]$  at date 0. Suppose  $\frac{f(\theta)}{u'(c_0(\theta))}$  is increasing. Then there exists  $\theta^*$  such that if  $\bar{\theta} > \theta^*$ , the allocation is Pareto inefficient.*

Figure 1.17 illustrates a case where a debt limit is Pareto inefficient. The lightly shaded region shows the budget set with a debt limit, which binds for types  $\theta \in [\theta_M, \theta_H]$ . Construct a Pareto improving deviation as follows. Increase date 1 consumption for types  $\theta \leq \theta_M$ , rearranging their date 0 consumption so as to reduce it on average. If the date 1 resource constraint binds, pay for this increase in date 1 consumption by decreasing date 1 consumption for some high  $\theta$  types (such as  $\theta_H$ ), compensating them by increasing their date 0 consumption. If these households are impatient enough, even a small increase in date 0 consumption compensates for a large fall in date 1 consumption, and our deviation is feasible; thus the original allocation was Pareto inefficient. The dark shaded area shows one transfer function implementing this deviation.

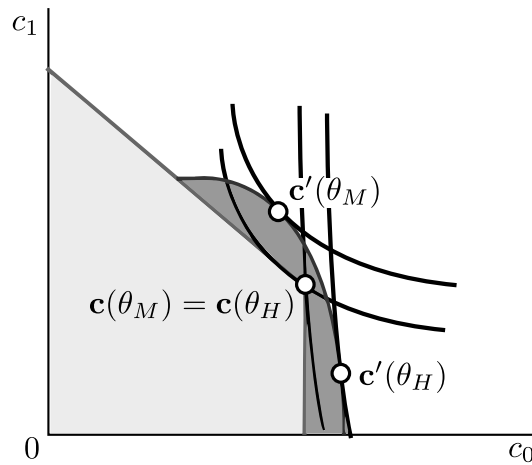


Figure 1.17: Inefficient debt limit and Pareto-improving deviation

While debt limits prevent overborrowing, they are a blunt instrument, preventing even the most impatient households from borrowing. This is unnecessary, since to prevent a debt-driven recession, it is only necessary to limit borrowers' aggregate debt. It may be better to allow im-

<sup>28</sup>Even with only two agents, we have already seen that a debt limit is not always Pareto improving, since it may make impatient borrowers worse off, by preventing them from borrowing.

patient households to borrow, but tax them at a high rate. This is exactly what targeted loan support programs do. Again, the next result follows immediately from Proposition (1.5.6), given the equivalence of targeted loan support programs and debt relief transfer functions.

**Proposition 1.6.8.** *If the ZLB binds in the equilibrium without policy:*

1. *The equilibrium is constrained inefficient.*
2. *Targeted loan support programs with  $\tau > 0$  implementing full employment allocations are constrained efficient.*

*If the ZLB is slack, the equilibrium is constrained efficient.*

## 1.7 Further questions

In this section, I consider three additional questions. First, how does optimal policy change if the crisis does not occur with probability one? Second, are the conclusions above robust to making borrowers and savers differ in their income, rather than their preferences? Third, are the conclusions robust to introducing endogenous labor supply?

Before considering these substantive extensions of the baseline model, I discuss how it can be reinterpreted. In the model presented above, borrowers and savers differ in their preferences. Consider the following three alternatives. First, suppose that individuals' date 0 income is  $y^* - \theta_i$ , where  $\theta_i \in \mathbb{R}$  is a transitory income shock, unobservable to the planner. Agents with a negative shock will want to borrow; agents with a positive shock will save. Second, suppose agents have initial debt  $\theta_i$ , which is unobservable to the planner. Again, agents with high debt will seek to roll over some of this debt, paying it off gradually over time. Finally, suppose that in addition to purchasing nondurable consumption, agents have inelastic demand for a certain quantity of a 'necessary' consumption good (which could be housing, healthcare, etc.). I now show that all these cases are isomorphic to the economy considered above.

In all three cases, households' date 0 consumption is  $c_0^i = y^* - \theta_i + \frac{d^i}{1+r_0}$ . The planner only observes households' debt and can no longer infer their period 0 consumption, as in the model with purely preference-based heterogeneity. Consequently, the planner cannot choose consumption allocations for workers. Instead, the planner chooses  $\hat{c}_0^i := c_0^i + \theta$ , which can be

inferred from a household's debt. Preferences over this object are  $u(\hat{c}_0^i - \theta_i)$ . This is equivalent to an economy in which  $\theta_i$  is a taste shock, and agents have preferences

$$u(c_0^i - \theta_i) + \sum_{t=1}^{\infty} \beta^t u(c_t^i),$$

which is a special case of the baseline model with  $U(c, \theta) = u(c - \theta)$ .

I now extend the baseline model with two types to answer the questions raised above.<sup>29</sup> Recall the main conclusions from Section 1.4: efficient allocations have the structure described in Proposition 1.4.1; debt-contingent transfers (in particular, debt relief) implement efficient allocations; and debt relief is Pareto-improving when the ZLB binds. In each of the three extensions below, I ask whether these results still hold.

### 1.7.1 Probability of crisis less than one

I now extend the results in Section 1.4 to the case where the probability of crisis is less than 1. With probability  $\pi$ , the borrowing constraint permanently falls to  $\phi$  at date 1, as before. With probability  $1 - \pi$ , the borrowing constraint never binds.<sup>30</sup> If the crisis does not occur, the economy immediately converges to steady state. Letting hats denote variables in the non-crisis state, households have preferences

$$\theta_i u(\hat{c}_0^i) + \pi \left[ \beta u(\hat{c}_1^i) + \frac{\beta^2}{1 - \beta} u(\hat{c}_2^i) \right] + (1 - \pi) \frac{\beta}{1 - \beta} u(\hat{c}_1^i)$$

Since there is now aggregate risk in the economy, it is necessary to specify the financial assets available to households. I consider two cases. In the **incomplete markets economy**, households trade a riskless bond, as before. In the **complete markets economy**, they trade in a complete set of Arrow-Debreu securities. Let  $q_0$  and  $\hat{q}_0$  denote the price of securities paying one unit of the consumption good at date 1 in the crisis state and in the non-crisis state, respectively, and let  $d_1^i$  and  $\hat{d}_1^i$  denote the securities of each type issued by household  $i$ .

At date 1, the transfers provided by the government now depend on the aggregate state, in addition to household borrowing. In the incomplete markets economy, the government offers

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<sup>29</sup>For simplicity, I focus on implementation with date 1 transfers; the results from Section 1.6 regarding implementation with macroprudential policy would carry over in the same way.

<sup>30</sup>I assume agents have the same, model-consistent expectations regarding the probability of crisis. Korinek and Simsek [2014] consider in detail the case where households have different expectations, and borrowers are more optimistic than savers.

two transfer functions,  $T(d)$  in the crisis state and  $\hat{T}(d)$  in the non-crisis state. In the complete markets economy, these transfers may depend on households' issuance of each security, so the functions have the form  $T(d, \hat{d}), \hat{T}(d, \hat{d})$ .

Naturally, since the crisis may not occur, debt relief distorts ex ante incentives less. The following proposition is elementary:

**Proposition 1.7.1.** *If  $\pi = 0$ , the full employment transfer is always incentive compatible.*

As before, I define a constrained efficient allocation as the solution a Pareto problem.<sup>31</sup> The following proposition states that solutions to the Pareto problem have the same structure as before, and can be implemented as equilibria with transfers.

**Proposition 1.7.2.** *In the economy with  $\pi < 1$ :*

1. *Constrained efficient allocations have the structure described in Proposition 1.4.1.*
2. *Every constrained efficient allocation can be implemented as an equilibrium with transfers in both the incomplete markets economy, and in the complete markets economy.*

As in Section 1.4.2, I now ask when debt relief implements optimal allocations. With  $\pi < 1$  it is no longer possible to implement optimal allocations with simple piecewise linear transfer functions (the debt relief transfer functions defined above), so a broader definition of debt relief is necessary. I will say that there is debt relief in the crisis state if  $d_1^B > 0 > d_1^S$  ( $B$  takes on debt while  $S$  saves) and  $T(d_1^B) > T(d_1^S)$  ( $B$  receives a larger transfer than  $S$  in equilibrium). With these redefinitions, Proposition 1.4.4 remains valid.

**Proposition 1.7.3.** *There exists  $\bar{\alpha}(\theta, \pi)$  such that*

1. *A transfer function with debt relief implements the optimal allocation if  $\alpha < \bar{\alpha}(\theta, \pi)$*
2. *If (ICS) binds,  $\alpha < \bar{\alpha}(\theta, \pi)$  and debt relief implements the optimal allocation.*

Finally, I ask whether debt relief is Pareto-improving, relative to the equilibrium without policy. The following Proposition confirms that Proposition 1.4.7 still holds in the complete markets economy: debt relief is generally purely redistributive, but is always Pareto-improving when the

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<sup>31</sup>The full Pareto problem with  $\pi < 1$  is presented in the Appendix.

ZLB binds. However, in the incomplete markets economy, debt relief is Pareto improving even when the ZLB does not bind, provided that the borrowing constraint binds.

**Proposition 1.7.4.** 1. If  $\theta_B \leq \theta^{BC}$ , the competitive equilibrium is Pareto optimal. Debt relief is not Pareto improving.

2. If  $\theta_B \in (\theta^{BC}, \theta^{ZLB}]$ , the competitive equilibrium is Pareto optimal under complete markets, and Pareto inefficient under incomplete markets. Debt relief is always Pareto improving in the incomplete markets economy.

3. If  $\theta_B > \theta^{ZLB}$ , the competitive equilibrium is Pareto inefficient. Debt relief is always Pareto improving.

When markets are incomplete, debt relief can be Pareto improving even if the ZLB does not bind. The incomplete markets economy has aggregate risk: with probability  $\pi$  the borrowing constraint will tighten, but agents can only invest in a riskless bond ex ante, and cannot insure against this shock. This creates an additional rationale for transfers to borrowers, to help them smooth consumption when credit markets do not allow them to do so. The first-best allocation smooths consumption across states of the world:  $c_1^i = \hat{c}_1^i$ ,  $i = S, B$ . When the borrowing constraint binds in the crisis state, this cannot be the equilibrium allocation, since for any level of debt  $d^1$ ,  $c_1^B = y^* + \phi - d_1$ , while  $\hat{c}_1^B = y^* - (1 - \beta)d_1 > c_1^B$ . In order to implement the first best, it is necessary to write off part of borrowers' debt. Debt relief, like bankruptcy, completes markets (Zame [1993]). In the complete markets economy this inefficiency disappears, and the competitive equilibrium is Pareto inefficient only when the ZLB binds.

Targeted transfers have two *distinct* macroeconomic roles (in addition to their purely redistributive role). First, when output is demand constrained, transfers can stimulate demand by redistributing resources to agents with a higher propensity to consume. Secondly, public transfers can substitute for private insurance opportunities (such as borrowing and lending in credit markets) in times when these opportunities are not available, helping agents to smooth consumption. There are many reasons why it may be harder for individuals to smooth consumption in recessions: household wealth is depleted, credit constraints are tighter, and lifetime income falls more after job loss (Davis and von Wachter [2011]). Public transfers (debt relief, unemployment

insurance, or stimulus payments) targeted at individuals who lack other insurance mechanisms can be Pareto improving, irrespective of whether they increase aggregate output.

## 1.7.2 Persistent types

So far, I have assumed that borrowers and savers only differ in their income or preferences at date 0. Borrowers are initially more impatient or have lower income at date 0, but starting at date 1 they are identical to savers. Alternatively, borrowers might borrow because they expect higher future income than savers, or because they have fewer necessary expenditures to make in the future and have less need to save. One might worry that in this case, redistribution from the poor savers to the rich borrowers might no longer increase aggregate demand.

To address this concern, I augment my baseline model to allow a household's type to have a persistent effect on preferences and income at all dates, not just date 0. I continue to assume the planner only observes households' debt, borrowing and lending, and not their income or other consumption needs. Let agents have preferences  $U(x, \theta) = \sum_{t=0}^{\infty} \beta^t u_t(x_t, \theta)$ , where  $x_t$  represents net financial plus public transfers:  $x_t = \frac{d_{t+1}}{1+r_t} - d_t + T(d_t)$ . This specification allows  $\theta$  to capture differences in preferences or income in any period  $t$ . Borrowers may be lucky individuals who borrow against their long-run income, unlucky individuals facing temporary income losses, or simply more impatient. The main conclusions of the model go through as long as preferences satisfy the following assumptions. First, preferences are concave and satisfy an Inada condition:

**Assumption 1.7.5.** For all  $t, \theta$ , there exists  $x_t(\theta)$  such that  $u_t(\cdot, \theta)$  is  $C^2$  on  $(x_t(\theta), \infty)$ , with  $u'_t > 0$ ,  $u''_t < 0$ ,  $\lim_{x \rightarrow x_t(\theta)} u'_t(x, \theta) = +\infty$ ,  $\lim_{x \rightarrow x_t(\theta)} u_t(x, \theta) = -\infty$ .

Second, preferences satisfy a standard Spence-Mirlees condition: higher  $\theta$  agents want to borrow more in period 0.

**Assumption 1.7.6.**  $\frac{u'_0(x_0, \theta)}{\beta u'_1(x_1, \theta)}$  is increasing in  $\theta$ .

Finally, I make the following assumption ensuring that the economy could reach steady state in period 1, were it not for the borrowing constraint.

**Assumption 1.7.7.**  $\frac{u'_{t+1}(x, \theta)}{u'_t(x, \theta)} = 1$ , for all  $\theta, x, t \geq 1$ .

Under these three assumptions, the main results from Section 1.4 and 1.4.2 go through. As before, I characterize constrained efficient allocations as the solution to a modified Pareto problem.<sup>32</sup>

**Proposition 1.7.8.** *Suppose Assumptions 1.7.5, 1.7.6, 1.7.7 are satisfied. Then:*

1. *Constrained efficient allocations have the structure described in Proposition 1.4.1.*
2. *Every constrained efficient allocation can be implemented as an equilibrium with transfers.*
3. *There exists  $\alpha(\theta_B)$  such that debt relief implements the optimal allocation if  $\alpha < \alpha(\theta_B)$ , and a savings subsidy implements the optimal allocation if  $\alpha > \alpha(\theta_B)$ . If (ICS) binds,  $\alpha < \alpha(\theta_B)$ ; if (ICB) binds,  $\alpha > \alpha(\theta_B)$ .*
4. *Debt relief is Pareto-improving relative to the equilibrium without policy if and only if  $\theta_B > \theta_{ZLB}$ .*

Above, I raised the concern that if  $B$ 's borrowing is motivated by higher future income (rather than low current income), transfers from  $S$  to  $B$  may not increase aggregate demand. This concern turns out to be unfounded. Even in this more general setting, household  $S$  is never liquidity constrained at date 1, whereas  $B$  may be constrained. Intuitively, type  $B$  households who borrow to consume more than their income at date 0, whatever their motive, must consume less than their income at date 1. Conversely, type  $S$  households who save at date 0 must consume more than their income at date 1. If any household is constrained at date 1, it must be  $B$ , since  $B$  is consuming less than her income (even if she earns more than  $S$ ).

In this simple model, the only difference in propensity to consume comes from binding liquidity constraints. Since  $B$  is sometimes constrained while  $S$  is never constrained, it follows that  $B$  must have a (weakly) higher propensity to consume than  $S$ ; the zero lower bound restricts  $S$ 's consumption, but not  $B$ 's; and redistribution from  $S$  to  $B$  can stimulate aggregate demand. The same result would go through if type  $B$  households were permanently more impatient than type  $S$  households: clearly, this would only increase type  $B$ 's propensity to consume.

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<sup>32</sup>The full Pareto problem is presented in the Appendix. There is one relatively minor difference, which only applies when the borrowing constraint does not bind in the planner's problem, and when one incentive constraint binds. Since I focus on optimal policy in the regime where the borrowing constraint and ZLB both bind, this makes no difference to any of the main results. It is described in the Appendix.

### 1.7.3 Endogenous labor supply

So far, I have discussed the optimality of debt relief in an endowment economy. I now modify the model to include endogenous labor supply in the simplest possible way.

Households have concave preferences  $U(c, h)$  over consumption and hours worked:

$$\theta_i U(c_0^i, h_0^i) + \sum_{t=1}^{\infty} \beta^t U(c_t^i, h_t^i)$$

Firms hire labor from households at a real wage  $w_t$ , and produce output using a linear technology,  $y = h$ . Each household receives an equal share of firms' real profit,  $\pi_t = (1 - w_t)h_t$ . In a Walrasian equilibrium,  $w_t = 1$  and labor supply is efficient. As we have seen, this equilibrium may imply a negative real interest rate. In order to introduce the zero lower bound in this economy, I assume that firms are demand constrained in the market for final goods, borrowing from the literature on non-Walrasian equilibria (Benassy [1993]). Let a firm's desired sales be  $y_t^* = \arg \max_{y \geq 0} (1 - w_t)y$ . We have  $y_t^* = 0$  for  $w_t > 1$ ,  $y_t^* = \infty$  if  $w_t < 1$ , and  $y_t^* = [0, \infty)$  if  $w_t = 1$ . Output is less than or equal to desired sales:  $y_t \leq y_t^*$ . As before, I assume that interest rates clear the goods market whenever this does not violate the ZLB:  $y_t = y_t^*$  if  $r_t > 0$ . When hiring labor, firms take into account the quantity constraints they face on the goods market as well as the real wage, so their demand for labor is  $h_t = y_t$  (not  $h_t = y_t^*$ ).<sup>33</sup> Households supply labor freely at the market-clearing real wage. In this economy, recessions occur when a fall in demand makes the real interest rate fall to zero. Firms become rationed in the goods market, and the real wage falls so that demand equals supply in the labor market.

I now consider optimal policy in this economy. I modify the definition of equilibrium with transfers to allow for a linear tax on labor income at date 1, which may depend on debt. Household  $i$ 's budget constraint at date 1 is

$$c_1^i + d_1^i = T(d_1^i) + (1 - \tau(d_1^i))w_1 h_1^i + \pi_1 + \frac{d_2^i}{1 + r_1}$$

where  $\tau(d_1^i)$  is the debt-contingent tax on labor income, and  $\pi_1$  is the representative firm's profit at date 1.

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<sup>33</sup>As emphasized by Benassy [1993], quantity constraints in the goods market can cause underemployment in the labor market, even if the labor market itself is flexible.

I modify the Pareto problem to allow for endogenous labor supply.<sup>34</sup> As before, I define a constrained efficient allocation as the solution to the Pareto problem: constrained efficient allocations solve this Pareto problem.

**Proposition 1.7.9.** *Any solution to the Pareto problem with endogenous labor supply can be implemented as an equilibrium with transfers.*

The following proposition characterizes constrained efficient allocations.

**Proposition 1.7.10.** *Suppose the borrowing constraint binds in the Pareto problem. Then in any optimal allocation:*

1. *For any  $\theta_B$ , there exist  $1 > \alpha_B > \alpha_S > 0$  such that ICB binds iff  $\alpha > \alpha_B$  and ICS binds iff  $\alpha < \alpha_S$ .*
2. *If the ZLB binds, S faces a positive labor wedge unless utility is quasilinear ( $U_{ch}/U_{cc} = U_h/U_c$ )*
3. *B faces a zero labor wedge unless ICS binds.*
4. *If preferences are separable ( $U_{ch} = 0$ ), B always faces a zero labor wedge.*

Finally, as in the endowment economy, debt relief is always optimal, and is Pareto improving when the ZLB binds in equilibrium.

**Proposition 1.7.11.** 1. *There exists  $\alpha(\theta_B) \in (\alpha_S, \alpha_B)$  such that the optimal allocation can be implemented with debt relief if  $\alpha < \alpha(\theta_B)$ , and the optimal allocation can be implemented with a savings subsidy if  $\alpha > \alpha(\theta_B)$ .*

2. *If the ZLB binds in equilibrium, the competitive equilibrium is Pareto inefficient. Debt relief is Pareto improving.*

## 1.8 Conclusion

I present a model in which both debt relief and macroprudential policy have costs and benefits. Debt relief redistributes towards households with a high propensity to consume, stimulating the economy at the zero lower bound, but encourages overborrowing ex ante. Macroprudential

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<sup>34</sup>The full Pareto problem with endogenous labor supply is presented in the Appendix.

policies prevent the overborrowing that leads to a recession, but can make borrowers worse off. Naive debt relief and macroprudential policies may not be Pareto improving, because the costs outweigh the benefits. However, it is possible to design sophisticated ex post or ex ante transfer policies which are Pareto improving, because the benefits outweigh the costs.

I conclude by comparing optimal debt relief and macroprudential policy to some practical policy proposals. I forbid the planner from relaxing the borrowing constraint. This rules out direct lending to households, deficit-financed lump sum transfers, (Bilbiie et al. [2013b]), and postponement of debt payments (a feature of most debt restructurings). More subtly, it rules out converting mortgages into shared appreciation mortgages (Caplin et al. [2008]), which give lenders a share of future house price appreciation. Such a policy compensates lenders for writing down principal, and prevents moral hazard, since applicants lose out if prices rise. It is ruled out in my model, since promising more payments from borrowers to lenders (if prices rise) violates the borrowing constraint. Relaxing the borrowing constraint would always be optimal, if possible. In order to evaluate credit policies, it is necessary to have a model in which the government can circumvent borrowing constraints, while the private sector cannot.<sup>35</sup>

In this model, households are identical except for their unobservable preferences and observable debt. In reality, households differ in many other observable characteristics; as in any optimal taxation problem, it will in general be optimal to target debt relief and macroprudential taxes based on all these observables. Governments should target transfers to high debt households only insofar as they are liquidity constrained, and have a higher MPC than low-debt households. For example, if MPCs depend on debt relative to income, debt relief (or macroprudential taxes) should be aimed at households with a high debt-income ratio, not high debt per se.<sup>36</sup>

Many debates concern households with an existing stock of debt. If lenders offer restructuring to households who miss payments, then even financially healthy households will have an incentive to miss payments (Mayer et al. [2014]). My model can be interpreted along these lines. Suppose all agents have some initial debt, and owe some payments at dates 0 and 1. Type S

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<sup>35</sup>Further, any credit policy must be relatively protracted. If the government was to lend to households for one period and then stop lending, that would merely postpone the liquidity trap.

<sup>36</sup>If low income households, rather than indebted households, have the highest propensity to consume, transfers should be targeted based on income, not wealth. In that case, optimal policy would balance the macroeconomic benefits from redistribution against the incentive effects of higher transfers.

households are financially healthy, and can make payments at date 0. Type B households face a negative shock at date 0, and must delay payments until date 1, when they repay everything and are liquidity constrained. Transfers to *B* would stimulate aggregate demand, but would encourage *S* to mimic *B* by delaying payments. My model suggests that optimal policy can avoid this problem in two ways. First, transfers should not be too large. Second, borrowers should only qualify for restructuring if they make some minimum level of payments. Such a policy is similar to [Hockett et al. \[2012\]](#)'s proposal for contingent principal reduction.

My model abstracts from housing, secured lending, and default. In reality, most household debt is mortgage debt, and most debates about debt relief focus on the mortgage market, where the benefits and cost of debt relief are subtly different from those considered here. As well as stimulating the overall economy, targeted mortgage debt relief could support the housing market, reducing fire sales and foreclosure externalities.<sup>37</sup> However, as discussed above, mortgage relief targeted to delinquent borrowers might induce even financially 'healthy' homeowners to delay payments. While these benefits and costs differ from those studied here, my main result - cleverly designed debt relief can be welfare improving - still applies. The core intuition is that in an equilibrium without policy, 'healthy' homeowners do not miss payments, as 'precarious' households do - presumably because they face some cost of delinquency (stigma, deterioration of their credit score, or risk of foreclosure). Therefore there is room to give *some* transfers to precarious households, without inducing healthy households to mimic them.

The model can also be reinterpreted to apply to sovereign debt relief. Outright debt relief is sometimes proposed as a solution to the European debt crisis; just as frequently, it is met with the criticism that debt relief represents a pure redistribution from creditor countries to debtor countries, and encourages overborrowing, sowing the seeds of future crises. My results suggest that one can design sovereign debt relief policies so that the benefits outweigh these costs. However, the model lacks several important characteristics of sovereign debt, especially default. A full extension of the model to cover sovereign debt relief is left to future work.

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<sup>37</sup>While the question is controversial, recent research tends to confirm that foreclosure reduces the value of nearby homes, and this externality seems to come from physical effects (foreclosed homes are not maintained, making the neighborhood less attractive) rather than a direct effect on prices ([Gerardi et al. \[2012\]](#), [Fisher et al. \[2014\]](#)).



*Optimal debt restructuring and lending policy in a monetary union*

## 2.1 Introduction

Europe seems set for a lost decade of low growth and high unemployment, driven in large part by public and private deleveraging. During the recession, highly indebted periphery countries lost access to financial markets, forcing them to pursue fiscal austerity and write down debt in order to reduce sovereign risk premia. Public deleveraging, combined with the private deleveraging associated with the global financial crisis, caused a slump in demand and a continent-wide recession, which conventional monetary policy seems unable to avert. A natural policy response to such crises is to prevent indebted countries from deleveraging in order to reduce their spreads, either by restructuring sovereign debt (as in the Greek restructuring of 2012) or by purchasing, or committing to purchase, sovereign debt (as in the ECB's Securities Markets Programme and Outright Monetary Transactions). However, such policies are controversial and the theoretical justification for them remains unclear.

I present a theoretical framework to understand the optimal policy response to episodes of international debt deleveraging in a monetary union. I build a model of a monetary union with nominal rigidities and defaultable debt, in which monetary policy is potentially constrained by the zero lower bound. The monetary union is a closed system with two groups of countries, borrowers (who initially have outstanding debt), and savers (who own the borrowers' debt). At date 1, it becomes common knowledge that at date 2, some borrower countries will have the option to default on their debt. Borrowers roll over their old debt by issuing new debt, internalizing the effect of their borrowing decision on their bond price (as in [Eaton and Gersovitz \[1981\]](#)). Consequently, borrowers have an incentive to reduce their consumption and reduce their debt, in order to raise the price at which they can issue their remaining debt. To maintain full employment, the monetary authority cuts the interest rate to raise demand in creditor countries.

However, the monetary authority is limited by the zero lower bound (ZLB) on nominal interest rates. When the ZLB binds, the monetary union enters a recession, and output falls below potential.

I characterize constrained efficient allocations in this economy, subject to the frictions imposed by the zero lower bound and default. I then ask whether optimal allocations can be implemented with three policies. The first policy I consider is debt relief, which writes off a portion of a country's short term debt. The second is lending policy, in which the union-wide authority offers to buy sovereign debt at above-market prices. This encompasses a range of policies, such as IMF crisis lending, the ECB's Securities Markets Programme (which involved directly purchases of sovereign debt) and the Outright Monetary Transactions (which involved a commitment to purchase sovereign debt). Finally, I consider debt postponement, which converts short term debt into long term debt.

First I consider an economy with no home bias, no long-term debt, and perfectly rigid prices. In this baseline economy, a transfer from creditor to debtor countries is Pareto improving. While transfers directly reduce creditors' income, they indirectly increase income throughout the monetary union, provided that borrowers spend the whole transfer, boosting aggregate demand. On net, creditors are no worse off, and borrowers are strictly better off. If borrowers are prevented from trading in bond markets after they receive a transfer, all three policies described above - debt relief, lending policy, and debt postponement - are equivalent, and implement constrained efficient allocations.

However, these policies are not equivalent if borrowers are permitted to 'retrade' after receiving a transfer. When retrading is permitted, borrowers will use some of their debt relief to issue less debt, rather than increasing consumption today. A much larger debt relief program is required to restore full employment, and such a program is not Pareto improving - it benefits borrowers, but makes savers strictly worse off. Lending policy, however, can still implement constrained efficient allocations. When the monetary authority buys bonds at below market prices, this encourages borrowers to issue more debt, inducing them to spend their transfer on date 1 consumption, rather than saving it for date 2. Intuitively, lending policy lowers borrowers' relative price of consumption at date 1, encouraging them to spend more at date 1 when their spending has a high social value.

Next I allow for long-term debt. If borrowers have some long-term debt outstanding, they over-issue new debt, diluting the value of their existing obligations. In normal times, when the ZLB does not bind, this incentive to over-issue debt renders a competitive equilibrium with long-term debt inefficient. But when the ZLB binds, borrowers with only short-term debt typically *under-issue* new debt, because they fail to internalize that by borrowing and spending, they boost union-wide aggregate demand. Optimal policy can use these two incentives to balance each other out. When the ZLB binds, converting short-term debt into long-term debt induces borrowers to dilute this long-term debt and issue the efficient amount of new debt. A recent literature has emphasized the benefits of long-term debt in insuring against risk and preventing self-fulfilling crises, and has studied how sovereigns trade off these benefits against the cost associated with debt dilution. I show that, in a liquidity trap, the ‘cost’ of debt dilution can actually be a benefit.

Transfers from creditors to debtors are Pareto improving at the ZLB because debtors spend the transfer, in part, on goods sold by creditor countries, increasing their income. One might worry that this result is not robust to the presence of home bias. If debtor countries spend most of the transfer on domestic goods and services, it would appear that creditor countries will no longer be better off. To address this concern, I allow for home bias. I show that the concern turns out to be unfounded: transfers are Pareto improving, even with home bias. If borrowers spend most of the transfer on their own goods and services, this increases their domestic income, which also increases their demand for foreign goods. Ultimately, budget constraints imply that a country must spend the whole of any transfer on buying either goods or assets from abroad. In fact, home bias increases the scope for Pareto improving policy: even when the ZLB does not bind, the competitive equilibrium is inefficient, and lending policy is Pareto improving.

A common argument against debt restructuring is that it gives countries an incentive to overborrow ex ante, knowing that they will be bailed out. To address this concern, I augment the model to include an ex ante stage in which countries decide how much to borrow and lend, taking into account that their debt may be restructured in the event of recession. Once the possibility of ex ante overborrowing is taken into account, it is necessary to combine ex post lending or debt restructuring policies with macroprudential capital controls or limits on borrowing ex ante. However, there remains a role for ex post lending and debt restructuring policies. In particular, some combination of ex ante debt limits and ex post lending policy or

debt restructuring is always ex ante Pareto improving.

The rest of the paper is structured as follows. Next I discuss the related literature. Section 2.2 presents the model. Section 2.3 shows that the equilibrium is inefficient when the ZLB binds, characterizes constrained efficient allocations, and shows how they can be implemented, in a baseline economy with no home bias, rigid prices, and short term debt. Section 2.4 extends this to allow for long-term debt. Section 2.5 allows for home bias. Section 2.6 discusses whether ex post debt restructuring is efficient ex ante, given that it may encourage overborrowing. Section 2.7 concludes.

### 2.1.1 Related literature

My paper explores how debt restructuring and lending policy can be used to correct a macroeconomic externality. As such, it is related to a wide literature on debt restructuring and lending policy, which mostly considers different motivations for these policies. It is also related to the recent literature on macroeconomic externalities, which largely considers other policy instruments which might correct these externalities.

The theoretical literature has emphasized three reasons why debt restructuring may be desirable. First, collective action problems (Wright [2012]), which can take many forms. Debt relief is a public good for creditors: if one creditor offers debt relief, the value of other creditors' claims increases. Holdout creditors have an incentive to delay agreeing to restructuring, in the hope that other creditors will settle first (Pitchford and Wright [2012]). A second strand of the literature (e.g. Krugman [1998]) emphasized debt overhang: writing down some debt may benefit creditors, if this increases the probability that the remaining debt will be repaid. This argument motivated 'market-based' debt reduction schemes, in which the debtor country buys back its own debt. These schemes were soon criticized by Bulow and Rogoff [1988, 1991] on the grounds that they mostly benefit creditors. Finally, in models with multiple equilibria, debt relief may prevent self-fulfilling crises (Cole and Kehoe [2000]). Most closely related to my paper, Roch and Uhlig [2012] show that a bailout guarantee can select the 'good' equilibrium in such a model. Relative to this whole literature, my contribution is to consider a different motivation for debt restructuring, namely to correct a macroeconomic externality.

More generally, my paper draws on the theoretical literature on sovereign debt (Eaton and

Gersovitz [1981] is the seminal contribution; Aguiar and Amador [2013b] provide a recent survey). In particular, recent contributions discuss the role of maturity. Aguiar and Amador [2013a] show that indebted sovereigns should write down their short-term debt, but not their long-term debt: writedowns of long-term debt are Pareto-improving, but cannot be implemented at equilibrium prices. Arellano and Ramanarayanan [2012] discuss the tradeoff between short and long term debt. Long term debt hedges against fluctuations in interest rate spreads, while short term debt provides better incentives to repay. Hatchondo et al. [2014] show that the inefficiency associated with debt dilution accounts for the bulk of default risk, and discuss how to design debt contracts that avoid dilution. Hatchondo et al. [2013] present a model in which voluntary debt exchanges can be Pareto improving for creditors and borrowers. Again, relative to this literature, I draw on standard models of defaultable debt to analyse how restructuring and lending policy can correct for a macroeconomic externality.

Another literature studies macroeconomic externalities associated with incomplete markets and/or fixed prices, and characterizes the macroprudential policies which correct for these externalities. Farhi and Werning [2013] provide a general theory of macroeconomic externalities. Farhi and Werning [2012] show that private insurance is inefficiently low for countries in a currency union, even if markets are complete. Individuals do not internalize that when they receive higher transfers, they increase their consumption of the nontradeable goods, which is desirable when employment is inefficiently low. The efficient allocation can be implemented with transfers within a fiscal union. However, transfers are not strictly necessary, as individual governments internalize the externality, and can implement the efficient allocation by trading in complete markets. In contrast, I study an economy with an *international* externality: sovereigns do not internalize that their borrowing decision affect demand in other markets, and supranational policy is necessary to implement efficient allocations. I also consider lending policies and debt restructuring, rather than a fiscal union.

Motivated by the European recession, a recent literature has considered sovereign debt crises in a monetary union and the role of policy. Most similar to my paper, Fornaro [2013b] presents a model in which debt relief is Pareto improving in a monetary union when the ZLB binds. In his model, indebted countries face a shock to their borrowing constraint, and are forced to deleverage. Since policy presumably cannot circumvent the borrowing constraint, there is no

scope in his model for the policies I consider, such as lending policy and debt rescheduling. Relative to his paper, my contribution is to consider alternative policies - debt relief, official lending, and debt rescheduling - and characterize optimal policy. One motivation for considering official lending and debt rescheduling is that these policies are more common (and arguably more politically feasible) than principal writedowns.<sup>1</sup> Forni and Pisani [2013] assess the effects of sovereign debt restructuring in a monetary union by simulating a medium-scale DSGE model. They assume that restructuring increases the spread faced by the sovereign, and this increase is fully transmitted to domestic households. I consider a relatively stylized model, but endogenize sovereign risk spreads, and analytically characterize optimal debt restructuring policy.

## 2.2 A model of a currency union with defaultable debt

In this section I present a model which embeds defaultable debt, as in Eaton and Gersovitz [1981], into a standard model of a currency union with nominal rigidities, drawing closely on Gali and Monacelli [2008].

The currency union is a closed system consisting of a continuum of small open economies indexed by  $i \in [0, 1]$ . Each economy is measure zero. Time is discrete,  $t = 1, 2, \dots$ . Countries with  $i \in [0, 1/2)$  are type  $S$  ('savers'); countries with  $i \in [1/2, 1]$  are type  $B$  ('borrowers'). These types differ only in their initial level of debt.

### 2.2.1 Households

Each economy contains a representative household with preferences

$$\sum_{t=1}^{\infty} \beta^{t-1} u(c_t^i - (1 - \beta)\chi^i \delta_t^i)$$

where  $u' > 0, u'' < 0$ .  $\delta_t^i = 1$  if country  $i$  has defaulted on or before date  $t$ , and  $(1 - \beta)\chi^i$  is country  $i$ 's cost of default. I describe how default works below.

$c_t^i$  is a consumption index defined by

$$c_t^i = \frac{(c_{H,t}^i)^{1-\alpha} (c_{F,t}^i)^\alpha}{(1 - \alpha)^{1-\alpha} \alpha^\alpha}$$

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<sup>1</sup>Trebesch et al. [2012] survey sovereign debt restructurings, and show that outright face value reductions are not common: most restructurings were pure rescheduling deals. In the European crisis, ECB lending policies have allowed indebted countries to borrow at below-market interest rates (Krishnamurthy et al. [2014]).

where  $c_{H,t}^i$  is an index of  $i$ 's consumption of goods produced at home, and  $c_{F,t}^i$  is an index of  $i$ 's consumption of foreign goods.  $\alpha$  measures the economy's openness: if  $\alpha = 1$ , there is no home bias in consumption. These consumption indices are defined as follows:

$$\begin{aligned} c_{H,t}^i &= \left( \int_0^1 c_{H,t}^i(j)^{\frac{\varepsilon-1}{\varepsilon}} dj \right)^{\frac{\varepsilon}{\varepsilon-1}} \\ c_{F,t}^i &= \exp \int_0^1 \log c_{f,t}^i df \\ c_{f,t}^i &= \left( \int_0^1 c_{f,t}^i(j)^{\frac{\varepsilon-1}{\varepsilon}} dj \right)^{\frac{\varepsilon}{\varepsilon-1}} \end{aligned}$$

$\varepsilon > 1$  is the elasticity of substitution between varieties produced within any given country.

Households do not themselves participate in financial markets. They receive wages, profits from the monopolistically competitive firms, and lump sum transfers (or taxes) from their governments, who borrow and lend in financial markets on their behalf.<sup>2</sup> The household's budget constraint is

$$\int_0^1 p_t^i(j) c_{H,t}^i(j) dj + \int_0^1 \int_0^1 p_t^f(j) c_{f,t}^i(j) dj df \leq W_t^i + T_t^i$$

where  $W_t^i$  denotes the nominal wage, and we combine profits and transfers into  $T_t^i$ . Each household inelastically supplies a single unit of labor.

As is standard, the household's optimization problem yields the demand functions

$$\begin{aligned} c_{H,t}^i(j) &= \left( \frac{p_t^i(j)}{p_t^i} \right)^{-\varepsilon} c_{H,t}^i \\ c_{f,t}^i(j) &= \left( \frac{p_t^f(j)}{p_t^f} \right)^{-\varepsilon} c_{f,t}^i \end{aligned}$$

for all  $i, f, j \in [0, 1]$ , where we denote country  $i$ 's domestic PPI by

$$p_t^i = \left( \int_0^1 p_t^i(j)^{1-\varepsilon} dj \right)^{\frac{1}{1-\varepsilon}}$$

Given that the law of one price holds, the price index for the bundle of goods imported from country  $f$  is identical to that country's domestic PPI:

$$p_t^f = \left( \int_0^1 p_t^f(j)^{1-\varepsilon} dj \right)^{\frac{1}{1-\varepsilon}}$$

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<sup>2</sup>Another interpretation of the model is that households participate in financial markets, and governments impose capital controls to induce households to make borrowing decisions that maximize domestic welfare (Na et al. [2014]).

As is standard, these demand functions satisfy  $\int_0^1 p_t^i(j) c_{H,t}^i(j) dj = p_t^i c_{H,t}^i$ ,  $\int_0^1 p_t^f(j) c_{f,t}^i(j) dj = p_t^f c_{f,t}^i$ .

Each household spends the same amount on products produced by each foreign country, so we have  $p_t^f c_{f,t}^i = p_t^* c_{F,t}^i$ , where  $p_t^* = \exp \int_0^1 \ln p_t^f df$  is the union-wide price index, and (for each country) the price of imported goods.

Finally,  $p_{C,t}^i = (p_t^i)^{1-\alpha} (p_t^*)^\alpha$  is the CPI in country  $i$ , and  $i$ 's optimal allocation of expenditure between domestic and imported goods is

$$p_t^i c_{H,t}^i = (1 - \alpha) p_{C,t}^i c_t^i, \quad p_t^* c_{F,t}^i = \alpha p_{C,t}^i c_t^i$$

## 2.2.2 Firms

Each country contains a continuum of monopolistically competitive firms indexed by  $j \in [0, 1]$ . Each firm combines labor and domestically produced intermediate inputs to produce output using the concave, constant returns to scale technology

$$x_t^i(j) = A_t^i m_t^i(j)^\phi n_t^i(j)^{1-\phi}$$

where  $\phi \in (0, 1)$  and  $A_t^i$  is a country-specific technology shock. The index of intermediate inputs,  $m_t^i(j)$ , is defined by

$$m_t^i(j) = \left( \int_0^1 m_t^i(j, k)^{\frac{\varepsilon-1}{\varepsilon}} dk \right)^{\frac{\varepsilon}{\varepsilon-1}}$$

where  $m_t^i(j, k)$  denotes the quantity of intermediate goods used by firm  $j$  in country  $i$ , and produced by firm  $k$  in country  $i$ .

Again, the firm's cost-minimization problem yields the standard demand function

$$m_t^i(j, k) = \left( \frac{p_t^i(k)}{p_t^i} \right)^{-\varepsilon} m_t^i(j)$$

Let firm  $j$ 's nominal total cost function be

$$S \left( \frac{x_t^i(j)}{A_t^i}, W_t^i, p_t^i \right) = \frac{x_t^i(j) (p_t^i)^\phi (W_t^i)^{1-\phi}}{A_t^i \phi^\phi (1-\phi)^{1-\phi}}$$

Nominal marginal cost is

$$\frac{1 (p_t^i)^\phi (W_t^i)^{1-\phi}}{A_t^i \phi^\phi (1-\phi)^{1-\phi}}$$

In symmetric equilibrium, each firm will employ one worker. Wages will be

$$W_t^i = p_t^i \frac{1-\phi}{\phi} \left( \frac{x_t^i}{A_t^i} \right)^{1/\phi}$$

Thus nominal marginal cost will be

$$p_t^i \frac{(x_t^i)^{\frac{1-\phi}{\phi}}}{\phi (A_t^i)^{1/\phi}}$$

Each firm  $j$  faces demand from three sets of customers. First, domestic consumers, with demand

$$c_{H,t}^i(j) = \left( \frac{p_t^i(j)}{p_t^i} \right)^{-\varepsilon} c_{H,t}^i$$

Second, foreign consumers in country  $f$ , with demand

$$c_{i,t}^f(j) = \left( \frac{p_t^i(j)}{p_t^i} \right)^{-\varepsilon} c_{i,t}^f$$

Third, domestic firm  $k$ , with demand

$$m_t^i(k, j) = \left( \frac{p_t^i(j)}{p_t^i} \right)^{-\varepsilon} m_t^i(k)$$

Thus the firm faces total demand

$$x_t^i(j) = X_t^i \left( \frac{p_t^i(j)}{p_t^i} \right)^{-\varepsilon}$$

where

$$X_t^i = c_{H,t}^i + \int_0^1 c_{i,t}^f df + \int_0^1 m_t^i(k) dk$$

### 2.2.3 Price setting

Firms face quadratic costs of price adjustment as in Rotemberg [1982]. Firm  $j$  in country  $i$  solves

$$\begin{aligned} & \max \sum_{t=1}^{\infty} Q_t^i \left\{ p_t^i(j) x_t^i(j) - (1-\tau) S \left( \frac{x_t^i(j)}{A_t^i}, W_t^i, p_t^i \right) - \frac{\varphi}{2} \left( \frac{p_t^i(j)}{p_{t-1}^i(j)} - 1 \right)^2 X_t^i \right\} \\ & \text{s.t. } x_t^i(j) = X_t^i \left( \frac{p_t^i(j)}{p_t^i} \right)^{-\varepsilon} \end{aligned}$$

where  $\tau = 1/\varepsilon$  is a subsidy that corrects the distortion induced by monopolistic competition,  $Q_t^i$  is the firm's nominal stochastic discount factor, with  $Q_1^i = 1$ .<sup>3</sup> Taking first order conditions and

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<sup>3</sup>I defer for now the question of who owns these firms; given the assumptions that will be made about monetary policy, this will not affect equilibrium in any way.

assuming a symmetric equilibrium with  $p_t^i(j) = p_t^i$  yields

$$\varphi \pi_t^i (\pi_t^i - 1) = (\varepsilon - 1)(MC_t^i - 1) + \varphi Q_{t,t+1}^i \pi_{t+1}^i \frac{X_{t+1}^i}{X_t^i} \pi_{t+1}^i (\pi_{t+1}^i - 1)$$

where we define inflation  $\pi_t^i = \frac{p_t^i}{p_{t-1}^i}$  and real marginal cost

$$MC_t^i = \frac{S_1 \left( \frac{x_t^i(j)}{A_t^i}, W_t^i, p_t^i \right)}{p_t^i A_t^i}$$

In any symmetric equilibrium, each firm employs 1 worker, and we have

$$MC_t^i = \frac{(x_t^i)^{\frac{1-\phi}{\phi}}}{\phi (A_t^i)^{1/\phi}}$$

Finally, in equilibrium we have  $x_t^i = X_t^i$ . So the aggregate supply equations become

$$\varphi \pi_t^i (\pi_t^i - 1) = (\varepsilon - 1) \left( \frac{(x_t^i)^{\frac{1-\phi}{\phi}}}{\phi (A_t^i)^{1/\phi}} - 1 \right) + \varphi Q_{t,t+1}^i \pi_{t+1}^i \frac{x_{t+1}^i}{x_t^i} \pi_{t+1}^i (\pi_{t+1}^i - 1)$$

## 2.2.4 Goods market clearing

Within each country  $i$ , each firm produces the same amount of gross output  $x_t^i$ , hires 1 worker, and uses the same amount of intermediate goods  $m_{f,t}^i$  from each country  $f$  (which is itself an aggregate, containing the same amount of the produce of each country  $f$  firm).

$$\begin{aligned} x_t^i &= c_{H,t}^i + \int c_{i,t}^f \, di + m_t^i \\ x_t^i &= (1 - \alpha) \left( \frac{p_t^*}{p_t^i} \right)^\alpha c_t^i + \alpha \left( \frac{p_t^*}{p_t^i} \right)^\alpha \int \left( \frac{p_t^f}{p_t^i} \right)^{1-\alpha} c_t^f \, di + m_t^i \end{aligned}$$

In equilibrium,  $m_t^i = \left( \frac{x_t^i}{A_t^i} \right)^{1/\phi}$ . So the complete set of equilibrium conditions are

$$x_t^i = (1 - \alpha) \left( \frac{p_t^*}{p_t^i} \right)^\alpha c_t^i + \alpha \frac{p_t^*}{p_t^i} \int \left( \frac{p_t^f}{p_t^*} \right) c_t^f \, df + \left( \frac{x_t^i}{A_t^i} \right)^{1/\phi}, \quad i \in [0, 1], t = 1, 2, \dots \quad (2.1)$$

$$\varphi \pi_t^i (\pi_t^i - 1) = (\varepsilon - 1) \left( \frac{(x_t^i)^{\frac{1-\phi}{\phi}}}{\phi (A_t^i)^{1/\phi}} - 1 \right) + \varphi Q_{t,t+1}^i \pi_{t+1}^i \frac{x_{t+1}^i}{x_t^i} \pi_{t+1}^i (\pi_{t+1}^i - 1), \quad t = 1, 2, \dots \quad (2.2)$$

where as before, we define  $\pi_t^i = \frac{p_t^i}{p_{t-1}^i}$ ,  $p_t^* = \exp \int_0^1 \ln p_t^i \, di$ . The following proposition shows how these conditions can be simplified further, in two useful special cases.<sup>4</sup>

<sup>4</sup>The proof of this Proposition (and all subsequent Propositions, unless stated otherwise) is in the Appendix.

**Proposition 2.2.1.** 1. If  $\alpha = 1$ ,  $A_t^i = A \forall i, t$ ,  $\pi_t = 1, \forall t \geq 2$ , then any  $\{c_t^i\}$  is consistent with (2.1), (2.2) and  $\pi_t^i = \pi_t^*$ ,  $x_t^i = x_t$  for all  $i$ , provided that

$$\int c_1^i \, di \leq y^*$$

$$\int c_t^i \, di = y^*, t \geq 2$$

where  $y^* = \arg \max_x x - \left(\frac{x}{A_t}\right)^{1/\phi}$ .

2. If  $\alpha < 1$ ,  $A_t^i = A$  and  $\varphi = \infty$  (prices are perfectly fixed), then any  $\{c_t^i, x_t^i\}$  is consistent with (2.1) and (2.2), provided that

$$y_t^i = (1 - \alpha)c_t^i + \alpha \int y_t^f \, df \leq y^*$$

where  $y_t^i = x_t^i - \left(\frac{x_t^i}{A}\right)^{1/\phi}$ .

## 2.2.5 Government, default, bond pricing, and monetary policy

Next, I describe how governments borrow, lend and default.

Governments seek to maximize the welfare of their representative household. They can issue two securities: a one period bond, which obliges the issuer to repay 1 unit of output next period, and a perpetuity, which obliges the issuer to repay  $1 - \beta$  units of output in each future period. I assume that a government cannot simultaneously issue debt and hold assets. At date 2, and only at date 2, a country with outstanding debt has the option to default on its debt. At the beginning of period 2, each country learns its utility cost of default,  $\chi^i$ . Each country's output cost  $\chi^i$  is identically and independently drawn from a distribution  $F(\chi)$ . As described above, if a country defaults, it pays a utility cost which is equivalent to losing  $\chi^i$  units of consumption in each period. In this economy, since there are no other shocks at date 2, whether a country defaults will depend solely on the level of  $\chi^i$  relative to its external debt. This default cost shock is a simple way to capture the fact that international investors face some uncertainty about whether a sovereign will default, even if they know its external debt position and other fundamentals.<sup>5</sup>

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<sup>5</sup>In standard models of defaultable international debt, default is driven by income shocks (Arellano [2008]). I follow a number of recent contributions which employ shocks to the utility cost of default as a tractable alternative (Roch and Uhlig [2012], Aguiar and Amador [2013a]).

The shock that causes a recession at date 1 is an increase in this uncertainty, which increases default risk and credit spreads.<sup>6</sup>

I now describe when a country defaults.

**Lemma 2.2.2.** *At date 2, after default, in any equilibrium with  $\pi_t = 1, t \geq 2$ , the economy enters a steady state. A country which did not default with short term debt  $d_{S,2}^i$  and long term debt  $d_{L,2}^i$  consumes  $c_t^i = y^* - (1 - \beta)d_2^i$  in every period  $t \geq 2$ , where we define  $d_2^i = d_{S,2}^i + d_{L,2}^i$ . Countries obtain utility*

$$V(-d_2^i) = \frac{u(y^* - (1 - \beta)d_2^i)}{1 - \beta}$$

*A country which defaulted and has default cost  $\chi^i$  obtains utility*

$$V(-\chi^i) = \frac{u(y^* - (1 - \beta)\chi^i)}{1 - \beta}$$

*Country  $i$  will default if  $d_2^i \geq \chi^i$ .*

It follows from this lemma that borrowers will be indifferent at date 1 between having  $d_2$  long term bonds outstanding at the end of date 1, and having  $d_2$  short term bonds outstanding at the end of date 1 and rolling them over each period. Without loss of generality, I restrict attention to equilibria in which borrowers only have long term debt outstanding at the end of date 2, and to save on notation I let  $d_2^i = d_{L,2}^i$ . The probability (as of date 1) that a country with debt  $d_2^i$  will default at date 2 is  $F(d_2^i)$ ; the probability that it will repay is  $p(d_2^i) := 1 - F(d_2^i)$ .

Having described equilibrium at date 2, given an amount of debt  $d_2^i$  issued at date 2, I now describe the price that borrower government obtains for this debt. Suppose government  $i$  starts date 1 owing  $\bar{d}_1^i - \bar{d}_2^i$  short term debt and  $\bar{d}_2^i$  long term debt, so the total amount it must repay at date 1 is  $\bar{d}_1^i$ . We will see that if the government ends period 1 owing  $d_2^i$ , it can sell its debt at price  $Q(d_2^i) = p(d_2^i) \frac{Q^{rf}}{1 - \beta}$ , where  $Q^{rf}$  is the price of a risk free bond, and which depends endogenously on  $d_2^i$ . The government internalizes that its bond price depends on its own borrowing decision.

In equilibrium, some debtor countries will default and some will not, but the fraction of countries who will default is known at time 1. Financial intermediaries hold defaultable short and long term debt issued by debtor countries, and issue short and long term bonds to creditor

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<sup>6</sup>One way to think about this model is that the fundamental shock that causes a recession is a *risk shock*, as in [Christiano et al. \[2014\]](#): an increase in idiosyncratic uncertainty about the cost of default. The shock can also be interpreted as a credit spread shock, as in [Curdia and Woodford \[2010\]](#), although here credit spreads are derived explicitly from a model of defaultable debt.

countries. The sole function of the financial intermediaries is to pool idiosyncratic country risk. Again, without loss of generality I assume that creditor countries only buy long term debt at date 2. Finally, savers also trade a risk free bond in zero net supply.

At the start of date 1, borrowers owe  $\bar{d}_1 > 0$  at date 1 and  $\bar{d}_2 \geq 0$  at date 2. Savers are initially owed  $\bar{d}_1$  at date 1 and  $\bar{d}_2$  at date 2. They can lend to borrowers, or sell back some of their bond holdings.

A borrower government's budget constraints are

$$\begin{aligned} Q(d_2^i)(d_2^i - \bar{d}_2) + T_1^i &= \bar{d}_1 \\ T_2^i &= d_2^i \end{aligned}$$

A saver government's budget constraints are

$$\begin{aligned} \bar{d}_1 + T_1^i &= Q_1(d_2^i - \bar{d}_2) + Q_1^{rf} a_2^i \\ T_2^i + p(d_2^i)d_2^i + a_2^i &= 0 \end{aligned}$$

Finally, monetary policy ensures that inflation is zero, except when constrained by the zero lower bound,  $Q^{rf} \leq 1$ . That is, we have

$$Q_t^{rf} \geq 1, \pi_t \leq 1, (1 - Q_t^{rf})(\pi_t - 1) = 0$$

## 2.2.6 Equilibrium

I now define equilibrium in the baseline economy with no home bias ( $\alpha = 1$ ), zero inflation after date 1 ( $\pi_t = 1, t \geq 2$ ), no productivity shocks ( $A_t^i = A$ ) and short term debt ( $\bar{d}_2 = 0$ ).

**Definition 2.2.3.** *An equilibrium in the baseline economy is a collection  $c_1^S, c_1^B, d_2, a_2, Q_1, Q^{rf}, y_1$  and a bond pricing function  $Q(\cdot)$  such that, given the initial debt level  $\bar{d}_1$ :*

1.  $c_1^S, d_2$  solve the saver country's problem:

$$\begin{aligned} \max_{c_1^S, d_2} & u(c_1^S) + \beta V(a_2 + p(d_2)d_2) \\ \text{s.t.} & c_1^S + Q_1 d_2 + Q^{rf} a_2 = y_1 + \bar{d}_1 \end{aligned}$$

2.  $c_1^B, d_2$  solve the borrower country's problem:

$$\begin{aligned} \max_{c_1^B, d_2} & u(c_1^B) + \beta \left[ \int_0^{d_2} V(-\chi) dF(\chi) + p(d)V(-d_2) \right] \\ \text{s.t.} & c_1^B + \bar{d}_1 = y_1 + Q(d_2)d_2 \end{aligned}$$

3. The bond pricing function is  $Q(d) = p(d)Q^{rf}$ , with  $Q(d_2) = Q_1$ .

4. The goods market clears:

$$c_1^S + c_1^B = 2y_1$$

5. The risk-free bond market clears:  $a_2 = 0$ .

6.  $Q^{rf} \leq 1, y_1 \leq y^*$ , with at least one strict equality.

## 2.3 Liquidity traps and optimal lending policy in the baseline economy

In this section, I consider a baseline economy with no home bias and no initial long-term debt. First I show that the equilibrium without policy is generally inefficient, due to a macroeconomic externality. When borrower countries have too much short term debt, their attempt to deleverage causes a union-wide recession. I then characterize constrained efficient allocations, and discuss how they can be implemented with debt restructuring and lending policies. Optimal allocations require a transfer to borrower countries, which can be implemented through outright debt relief, lending policy, or converting short term debt into long term debt. If borrowers can be prevented from re trading in secondary markets, these policies are equivalent; if re trading is possible, debt relief does not implement all constrained efficient allocations, whereas lending policy does.

### 2.3.1 International deleveraging and liquidity traps

I now describe equilibrium, given an initial level of debt  $\bar{d}_1 > 0$ , and assuming no home bias ( $\alpha = 1$ ) and no initial long term debt ( $\bar{d}_2 = 0$ ). I show that in equilibrium, the risk of default at date 2 increases the spreads faced by borrower countries. Borrowers attempt to pay down their short term debt to reduce these spreads, reducing demand throughout the monetary union.

The central bank reduces interest rates to keep output at its efficient level, whenever this is not prevented by the zero lower bound on nominal interest rates. When borrowers' external debt is sufficiently high, the zero lower bound binds, and output is below the efficient level throughout the monetary union.

I also make the following technical assumptions:

**Assumption 2.3.1.** *Either  $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$  with  $\sigma > 1$ , or  $u(c) = \ln c$ .*

**Assumption 2.3.2.**  $\gamma(d) := \frac{f(d)d}{1-F(d)}$  *is nondecreasing in  $d$ .*

**Assumption 2.3.3.**  $u'(2y^*) < \beta u'(y^* + (1-\beta)p(d^*)d^*)$  *where  $d^* := \arg \max_d p(d)d$ .*

This assumption ensures that the ZLB will bind if the borrowers have enough external debt.

Borrowers attempt to deleverage, reducing their consumption to pay off debt and reduce their spreads. To see why, note that borrowers' Euler equation is

$$u'(c_1^B)[Q(d_2) + Q'(d_2)d_2] = \beta p(d_2)u'(c_2^B) \quad (2.3)$$

On the left hand side, a borrower's effective marginal price of debt - the funds it receives if it issues another bond - is  $Q(d_2) + Q'(d_2)d_2$ . This has two components. First, if the country issues another bond, it receives  $Q(d_2)$ , the price of the bond. Second, issuing another bond increases the probability of default, and reduces the value of the other bonds the country is issuing by  $Q'(d_2)d_2 < 0$ . On the right hand side, the cost of issuing another bond (lower utility in the steady state) is weighted by the probability that the borrower actually repays,  $p(d_2)$ .

A saver country's Euler equation is

$$Q(d_2)u'(c_1^S) = \beta P(d_2)u'(c_2^S) \quad (2.4)$$

Dividing (2.3) by (2.4) and rearranging, we have

$$\frac{u'(c_1^B)}{\beta u'(c_2^B)} \left[ 1 + \frac{Q'(d_2)d_2}{Q(d_2)} \right] = \frac{u'(c_1^S)}{\beta u'(c_2^S)}$$

$\frac{Q'(d_2)d_2}{Q(d_2)}$  is negative, so this expression means that borrowers reduce their consumption over time, relative to savers. Again, borrowers internalize that if their consumption is too low (their debt is too high) at date 2, this makes them very likely to default, and reduces the price they an

obtain for their bonds at date 1. They therefore have an incentive to pay back their debt. This captures the stylized fact that a global financial shock caused a compression in current account balances and a decline in gross capital flows (Lane and Milesi-Ferretti [2012]).

If debtor countries reduce their consumption at date 1, then in order to maintain efficient output and zero inflation, the monetary authority must cut interest rates to induce creditor countries to consume more. The more debt the borrowers must pay back, the more interest rates must fall (risk free bond prices must rise) to maintain full employment. Eventually, if debt is too large, the monetary authority would need a negative interest rate to maintain efficient output. This is not possible, because of the zero lower bound. So output will fall below potential output, and the monetary union will enter a recession at date 1. The following proposition formalizes this.

**Proposition 2.3.4.** *There exists  $\bar{d}_1^*$  such that:*

1. *If  $\bar{d}_1 < \bar{d}_1^*$ , then  $Q^{rf} < 1$  and  $y_1 = y^*$ .  $Q^{rf}$  is increasing in  $\bar{d}_1$ .*
2. *If  $\bar{d}_1 = \bar{d}_1^*$ , then  $Q^{rf} = 1$  and  $y_1 = y^*$ .*
3. *If  $\bar{d}_1 > \bar{d}_1^*$ , then  $Q^{rf} = 1$  and  $y_1 < y^*$ .  $y_1$  is decreasing in  $\bar{d}_1$ .*

*$c_1^S$  is increasing in  $\bar{d}_1$ .  $c_1^B$  is decreasing in  $\bar{d}_1$ .*

Figure 2.1 shows a numerical example, with  $\beta = 0.9, \sigma = 1, y^* = 1, F(d) = d$ . The figure plots each country's consumption, the union-wide level of output, and the risk free bond price, as functions of borrowers' initial level of external debt,  $\bar{d}_1$ . Going from left to right, as debt increases, borrower countries reduce their consumption in order to pay down debt. The risk free interest rate falls - i.e. the bond price rises - in order to induce savers to increase their consumption, keeping  $y_1 = y^*$ . Once  $\bar{d}_1 = \bar{d}_1^*$  (indicated by the black vertical line), the lower bound on interest rates binds -  $Q^{rf} = 1$  - and output can fall below potential output. In this region, the higher the borrowers' initial level of external debt, the lower is output.

This result suggests that the outcome is inefficient. Collectively, all countries could produce and consume more, while still satisfying resource constraints. But each individual saver country prefers to save at a zero real interest rate, and each borrower country prefers to write down its debt in order to reduce its spreads. Individual governments do not internalize that their

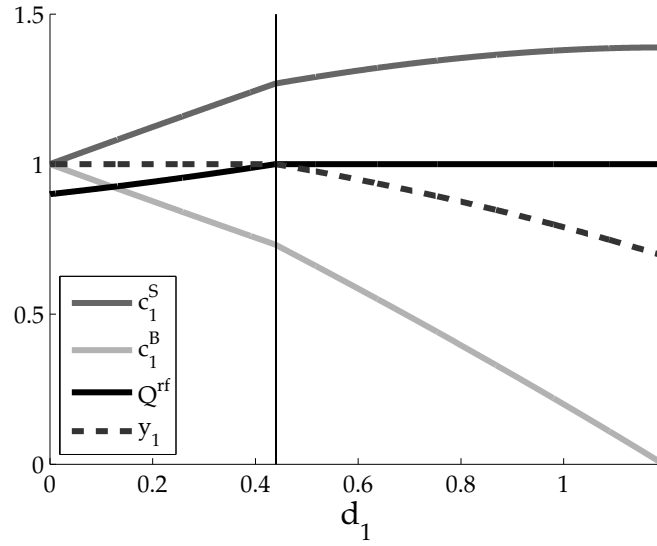


Figure 2.1: Equilibrium in the baseline economy

borrowing and lending decisions affect aggregate demand and output in other countries. This suggests that there is some scope for a Pareto improvement.

### 2.3.2 Constrained efficient allocations

I now characterize optimal policy, by considering a social planner's problem. The planner maximizes borrowers' utility, subject to three constraints: she must give the savers at least a certain level of utility, date 1 consumption cannot be greater than the full employment level of output, and - crucially - the savers' Euler equation must be satisfied with a non-negative risk free rate. That is, the planner solves

$$\max_{c_1^S, c_1^B, d_2} u(c_1^B) + \beta \left[ \int_0^d V(-\chi) dF(\chi) + p(d)V(-d) \right] \quad (\text{SP})$$

$$\text{s.t. } u(c_1^S) + \beta V(p(d_2)d_2) \geq U_S \quad (\text{US})$$

$$c_1^S + c_1^B \leq 2y^* \quad (\text{RC})$$

$$u'(c_1^S) \geq \beta u'(y^* + (1 - \beta)p(d_2)d_2) \quad (\text{ZLB})$$

There are two ways to interpret the zero lower bound constraint (ZLB). One interpretation is that the union-wide authority cannot prevent governments from lending to other governments, or holding risk free bonds. An alternative interpretation is that neither the union-wide authority

nor governments can prevent their citizens from saving at a zero interest rate (should they choose to do so). In either case, (ZLB) must hold.

Without loss of generality, we focus on allocations in which  $U_S \geq \frac{u(y^*)}{1-\beta}$ ; otherwise, type S countries would be borrowers and type B countries would be savers. Efficient allocations are solutions to (SP). The following proposition characterizes efficient allocations.

**Proposition 2.3.5.** 1. In every efficient allocation, there is full employment:  $c_1^S + c_1^B = 2y^*$

2. There exists  $U_S^* > \frac{u(y^*)}{1-\beta}$  such that the ZLB binds if  $U_S \geq U_S^*$ .

3.  $c_1^S$  and  $d_2$  are increasing in  $U_S$ ;  $c_1^B$  is decreasing in  $U_S$ .

4. The largest  $U_S$  for which a solution to this program exists is

$$\bar{U} := u(u'^{-1}(\beta u'(y^* + (1-\beta)p(d^*)d^*))) + \beta V(p(d^*)d^*)$$

where  $d^* := \arg \max_d p(d)d$ .

*Proof.* (1.) Putting Lagrange multipliers of  $\nu, \lambda, \mu$  on the constraints, the first order conditions are

$$\nu u'(c_1^S) - \lambda + \mu u''(c_1^S) = 0$$

$$u'(c_1^B) - \lambda = 0$$

$$\nu \beta V'(-p(d)d)[p'(d)d + p(d)] - \beta p(d)V'(-d) - \mu \beta u''(y^* + p(d)d)[p'(d)d + p(d)] = 0$$

Since  $u' > 0$ , we must have  $\lambda > 0$ : every efficient allocation has full employment at date 1.  $\square$

There are a range of Pareto efficient allocations, indexed by savers' utility  $U_S$ . As the utility promised to savers increases, the planner finds it optimal to give savers higher consumption at both dates 1 and 2. However, it is still optimal to make borrowers deleverage, consuming less at date 1 than at date 2. Further, the amount of deleveraging increases in  $U_S$ , as the planner gives higher and higher date 1 consumption to savers. From the savers' perspective, this means they must tolerate a larger and larger fall in consumption between dates 1 and 2. When the planner is required to deliver a sufficiently high utility to savers -  $U_S > U_S^*$  - she would like to give the savers such high date 1 consumption, and such a sharp fall in consumption between dates 1 and 2, that it violates the zero lower bound. It is still possible to increase the savers' utility beyond

this point, but it is no longer possible to increase the amount of deleveraging. Instead, if the planner wants to increase  $c_1^S$ , she must also increase  $d_2$  by more than she would if the ZLB was not a constraint, leading to a higher fraction of defaults than would otherwise be the case. This is still better for the saver, as long as  $d_2 < d^*$ . Once  $d_2 = d^*$ , debt is so high that requiring the borrowers to promise to repay more debt would actually decrease the amount received by savers, which cannot be Pareto optimal. Figure 2.2 provides a numerical example.

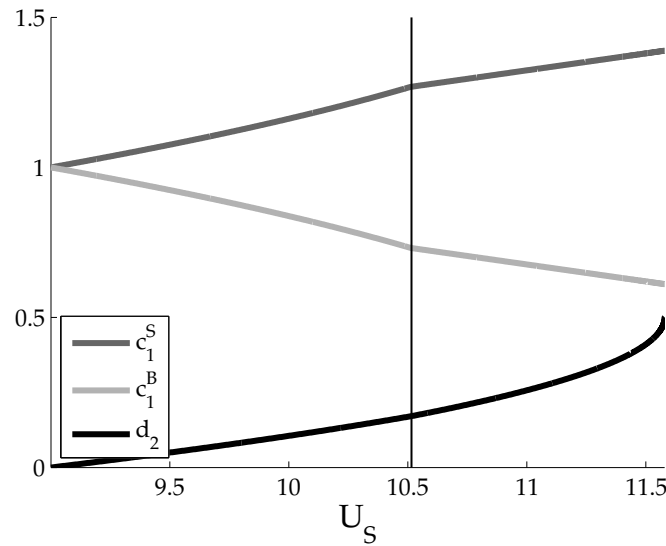


Figure 2.2: Constrained efficient allocations

Note that it is always efficient to have full employment at date 1. If savers' consumption is constrained by the zero lower bound, the planner can increase borrowers' consumption. Since we have seen that the equilibrium without policy has  $y_t < y^*$  when the ZLB binds, the following Proposition is immediate.

**Proposition 2.3.6.** *In the baseline economy, if the ZLB does not bind, then the equilibrium is Pareto efficient.*

*If the ZLB binds, the equilibrium is Pareto inefficient.*

### 2.3.3 Implementation without retrading: an equivalence result

Next, I discuss how optimal allocations can be implemented. I consider three policies. First, writedowns of short term debt. Second, lending policy, in which the union-wide authority buys

the debt of borrower countries directly, possibly offering a higher price for this debt than borrowers could have obtained in the private market. Third, debt postponement, in which borrowers' short term debt is converted into long term debt.

The crucial question is whether it is possible to prevent countries from re trading after a debt restructuring agreement. To make things concrete, consider the Pareto efficient allocation which gives savers the same utility as in the equilibrium without policy. It would appear that this allocation can be implemented with debt relief for the borrowers at date 1. If borrowers maintained the same level of date 2 debt, debt relief would increase their date 1 consumption, stimulating aggregate income and thus compensating saver countries for the writedown of their assets. However, if borrowers can retrade after receiving debt relief, since they are now richer at date 1 they would like to reduce their date 2 borrowing. This means that even more debt relief is required to increase their date 1 consumption enough to restore full employment. Moreover, this will not be Pareto improving: savers are strictly worse off than in the equilibrium without policy, as they consume the same amount at date 1, and less at date 2. But clearly, if it is possible to prevent re trading, there is no such problem.

In this example, if re trading is possible, it will be necessary to combine debt relief with a subsidy to borrowing, encouraging borrowers to consume more at date 1 (where their consumption has a high social marginal utility) and write down less of their debt. This can be interpreted as a bond price support program, in which the union-wide authorities purchase borrowers' debt at a guaranteed price which is lower than the market price of debt.

First, I assume it is possible to prevent re trading. I make the extreme assumption that the union-wide authority can prevent saver and borrower countries from interacting in the bond market. The union-wide authority issues risk-free debt to saver countries, and offers to buy a fixed amount of debt  $d_2$ , at a fixed price  $Q_1$ , from the borrowers. The union-wide authority can also impose taxes  $T_1^S, T_1^B$  on borrowers and savers at date 1. These taxes will typically be positive for savers, and negative for borrowers. Finally, I allow for the union wide authority to postpone the borrowers' debt, by giving them a positive amount of date 2 debt outstanding ( $\bar{d}_2 > 0$ ) and compensating them with a transfer at date 1 ( $T_1^B < 0$ ).

**Definition 2.3.7.** *An equilibrium without re trading in the baseline economy is a collection  $c_1^S, c_1^B, a_2, Q^{rf}, y_1$*

such that, given the initial debt level  $\bar{d}_1$  and policy  $d_2, \bar{d}_2, T_1^S, T_1^B, Q_1$ :

1.  $c_1^S, a_2$  solve the saver country's problem:

$$\begin{aligned} \max_{c_1^S, a_2} & u(c_1^S) + \beta V(a_2) \\ \text{s.t.} & c_1^S + Q^{rf} a_2 = y_1 + \bar{d}_1 - T_1^S \end{aligned}$$

2.  $c_1^B, d_2$  satisfy the borrower country's budget constraint:

$$c_1^B + \bar{d}_1 = y_1 + Q_1(d_2 - \bar{d}_2) - T_1^B$$

3. The government budget constraint is satisfied:

$$Q_1 d_2 = Q^{rf} a_2 + T_1^S + T_1^B$$

4. The goods market clears:

$$c_1^S + c_1^B = 2y_1$$

5. The risk-free bond market clears:

$$a_2 = p(d_2)d_2$$

6.  $Q^{rf} \leq 1, y_1 \leq y^*$ , with at least one strict equality.

Given this definition, I show that debt relief, lending policy, and postponement are equivalent policies when retrading is prevented.

**Proposition 2.3.8.** Any optimal allocation  $c_1^S, c_1^B, d_2$  can be implemented as an equilibrium without re-trading in three ways:

1. With debt relief ( $T_1^B < 0$ ) and fair market prices ( $Q_1 = Q^{rf} p(d_2)$ )
2. With a subsidized price for debt ( $Q_1 > Q^{rf} p(d_2)$ ) and no transfer to savers ( $T_1^B = 0$ )
3. With debt postponement ( $-T_1^B = \bar{d}_2 > 0$ ), and fair market prices for debt ( $Q_1 = Q^{rf} p(d_2)$ ).

These policies are related as follows:

$$-T_1^B = (Q_1 - Q^{rf} p(d_2))d_2 = \bar{d}_2 - Q^{rf} p(d_2)\bar{d}_2$$

*Proof.* Take any optimal  $c_1^B, d_2$ . To prove 1., let

$$T_1^B = y^* + Q^{rf} p(d_2) d_2 - \bar{d}_1 - c_1^B$$

To prove 2., choose  $Q_1$  so that

$$c_1^B = y^* + Q_1 d_2 - \bar{d}_1$$

To prove 3., choose  $\bar{d}_2$  so that

$$c_1^B = y^* + Q^{rf} p(d_2)(d_2 - \bar{d}_2) - \bar{d}_1 + \bar{d}_2$$

Comparing these three equations, we get the relation between the three policies stated above.  $\square$

Intuitively, in any optimal allocation, borrowers receive some amount at date 1, and are required to pay some amount at date 2. One way to implement this is to give borrowers a lump sum transfer at date 1, and require them to issue a certain amount of new debt (at market prices). An alternative way is to buy their debt at above-market prices. The difference between the actual price and the fair market price is an implicit transfer to borrowers, and plays exactly the same role as an explicit transfer (debt relief). A third way to implement this transfer is to turn short term debt into long term debt. Long term debt has a lower market value because of the possibility of default, so this postponement also acts as a transfer to borrowers.<sup>7</sup>

In this precise sense, debt relief, lending policy and debt postponement are all equivalent policies when it is possible to prevent retrading. Note that even when the ZLB binds, there are a range of optimal allocations, indexed by  $d_2$ . Allocations with higher  $d_2$ , higher  $c_1^S$  and lower  $c_1^B$  are better for savers and worse for borrowers. There are also a set of Pareto improving policies, relative to any equilibrium in which the ZLB binds. The Pareto improving allocation most favorable to borrowers keeps their debt level  $d_2$ , and the savers' consumption  $c_1^S$ , the same as in the equilibrium without policy.

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<sup>7</sup>This is reminiscent of the well known result that governments can smooth risk, replicating the complete markets allocation, by issuing debt of different maturities. (Angeletos [2002], Buera and Nicolini [2004]), although here long-term debt effects a transfer to indebted countries, rather than insuring against risk. A concern often raised in this literature is that the gross positions required to replicate complete markets may be implausibly large (Buera and Nicolini [2004]). This concern may apply here too.

### 2.3.4 Implementation with retrading: debt relief

The economy without retrading provides a useful benchmark result: debt relief, lending policy and debt postponement are different ways of providing essentially the same transfer to indebted countries. However, this result is arguably of little practical relevance. It is rare for creditors or international agencies to prevent a country from issuing *less* debt than the creditors require, and it is unclear how this could be enforced.<sup>8</sup> In the remainder of this paper, I assume that borrowers can freely decide how much debt to issue at date 1.

I now ask whether two of the policies considered so far - debt relief and bond price support programs - are still optimal when retrading is possible.<sup>9</sup> First I consider debt relief. We do not need a new equilibrium concept to think about debt relief with retrading. Instead, we can simply allow the union-wide authority to choose borrowers' initial level of debt,  $\bar{d}_1$ . The following result is immediate, given Propositions 3.4 and 3.6.

**Proposition 2.3.9.** *When retrading is permitted:*

1. *Debt relief only implements optimal allocations with  $U_S \leq U_S^*$ . These allocations can be implemented by writing down debt to a level below  $\bar{d}_1^*$ .*
2. *Debt relief is not Pareto improving. It makes savers worse off and makes borrowers better off, relative to an equilibrium with  $\bar{d}_1 > \bar{d}_1^*$ .*

*Proof.* The first part is immediate, since equilibria without policy are only optimal when  $\bar{d}_1 < \bar{d}_1^*$ . To prove the second part, note that  $c_1^S$  and  $p(d_2)d_2$  are increasing in  $\bar{d}_1$ . So reducing  $\bar{d}_1$  strictly reduces borrowers' utility. □

Debt relief, together with no lending policy (i.e. a bond price function which merely replicates market prices,  $\tilde{Q} = Q$ ) can implement efficient allocations in which the ZLB does not bind. The government can simply write off part of  $\bar{d}_1$  until the remaining debt,  $\bar{d}_1 + T_1^B$  (where  $T_1^B < 0$ ) is less than  $\bar{d}_1^*$ . We already know that the competitive equilibrium in this case is efficient. But it is not a Pareto improvement on the equilibrium without policy. Borrowers are better off, but savers

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<sup>8</sup>While official lending often comes with 'conditionalities', these usually require that the recipient country's debt must be *reduced* going forward, not that it should not be reduced too much.

<sup>9</sup>I discuss debt postponement in Section 4 below.

are strictly worse off. Because borrowers can retrade after receiving debt relief, since they are now richer at date 1 they would like to reduce their date 2 borrowing. This means that even more debt relief is required to increase their date 1 consumption enough to restore full employment. Moreover, this will not be Pareto improving: savers are strictly worse off than in the equilibrium without policy, as they consume the same amount at date 1, and less at date 2.

Figures 2.3 and 2.4 illustrate. Figure 2.3 shows how debt relief brings about a Pareto improvement when retrading is prevented. The black curves are borrowers' indifference curves, representing their preferences over date 1 consumption,  $c_1^B$ , and date 2 debt,  $d_2$ . The gray shaded area represents borrowers' budget set. If a borrower country takes out no debt, it consumes its income,  $y_1$ , minus its outstanding debt,  $\bar{d}_1$ . As the country issues more debt, it obtains more resources at date 1. But the price it can obtain for this debt decreases as debt increases, so consumption is a concave function of debt issued. Finally, the gray dashed line shows the combinations of  $c_1^B, d_2$  that satisfy the (ZLB) and (RC), i.e. that satisfy

$$u'(2y^* - c_1^B) = \beta u'(y^* + p(d_2)d_2)$$

Without policy,  $c_1^B, d_2$  lie below the ZLB curve, because output is below potential output. If the union-wide authority gives a transfer  $T$  to borrower countries (for example, by writing off debt), that shifts the borrower's budget set up, until a point on the ZLB curve is in the budget set.

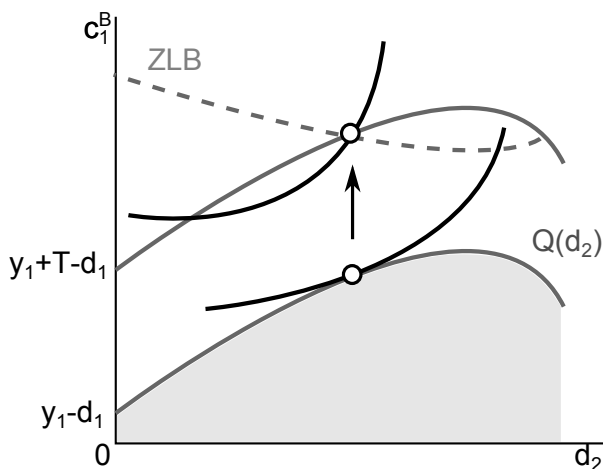


Figure 2.3: Debt relief without retrading

However, Figure 2.4 shows that this point is not optimal given the new budget set. Even if borrowers receive a large enough transfer that it is possible for them to consume enough to restore

the efficient level of output, they would not find it optimal to do so. Instead, they would prefer to use some of the transfer to issue less new debt  $d_2$ , reducing their borrowing costs. The resulting equilibrium will not have full employment (because consumption is below the ZLB curve) and it will not be a Pareto improvement on the equilibrium without policy (because  $d_2$  has fallen, and saver countries receive less in the steady state). In order for debt relief to restore full employment, it would be necessary to make an even larger transfer, reducing  $d_2$  further. Again, this will not be a Pareto improvement on the status quo: saver countries will be worse off.

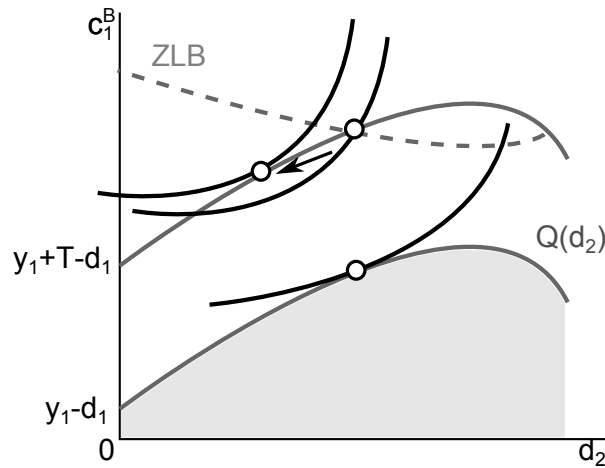


Figure 2.4: Debt relief with retrading

Another way to interpret this result is as follows. Borrower countries have a higher marginal propensity to consume than saver countries, because they do not have perfect access to capital markets. However, they are not completely liquidity constrained, so their MPC is strictly less than 1. A transfer to borrowers increases their spending, increasing aggregate demand and raising savers' income. But it does not raise savers' income one for one, so their income does not rise enough to compensate them for the fall in the value of their assets.

### 2.3.5 Implementation with retrading: lending policy

I maintain the assumption that retrading is possible, but now focus on the second policy considered above: lending policy, or a bond price support program. Again, to simplify matters I assume that the union-wide authority directly finances borrower governments, and prevents saver and borrower countries from interacting in bond markets in any other way. The union-wide author-

ity offers a bond pricing function  $\tilde{Q}(d)$ , which need not be the same as the market bond pricing function  $Q(d)$ . These loans are financed by issuing risk-free debt to saver countries with price  $Q^{rf}$ , and through lump sum taxes on savers and borrowers,  $T_1^S, T_1^B$ . (The tax on borrowers may be negative, i.e. it may be a subsidy.)

This definition of lending policy is intended to capture certain features of the ECB lending programs during the European crisis, which included both direct purchases of government debt (the Securities Markets Programme), commitments to purchase government bonds (Outright Monetary Transactions), and long term loans to banks, which could use these loans to purchase government debt (the Long-Term Refinancing Operations). The explicit motivation for these policies was that they would reduce sovereign risk, which would have beneficial macroeconomic spillovers, and there is some evidence for this proposition (Krishnamurthy et al. [2014]). More generally, Lane and Milesi-Ferretti [2012] find that official lending (IMF and EU loans, but mainly ECB liquidity funds) compensated for the exit of private capital flows from deficit countries with a pegged exchange rate, during the global financial crisis.

**Definition 2.3.10.** *An equilibrium with lending policy in the baseline economy is a collection  $c_1^S, c_1^B, d_2, Q^{rf}, y_1$  such that, given the initial debt level  $\bar{d}_1$  and given a bond pricing function  $\tilde{Q}(\cdot)$*

1.  $c_1^S, a_2$  solve the saver country's problem:

$$\begin{aligned} \max_{c_1^S, a_2, d_2} \quad & u(c_1^S) + \beta u(a_2) \\ \text{s.t.} \quad & c_1^S + Q^{rf} a_2 = y_1 + \bar{d}_1 - T_1^S \end{aligned}$$

2.  $c_1^B, d_2$  solve the borrower country's problem:

$$\begin{aligned} \max_{c_1^B, d_2} \quad & u(c_1^B) + \beta \left[ \int_0^d V(y^* - \chi) dF(\chi) + p(d) V(-d) \right] \\ \text{s.t.} \quad & c_1^B + \bar{d}_1 = y_1 + \tilde{Q}(d_2) d_2 - T_1^B \end{aligned}$$

3. The government budget constraint is satisfied:

$$\tilde{Q}(d_2) d_2 = Q^{rf} a_2 + T_1^S + T_1^B$$

4. The goods market clears:

$$c_1^S + c_1^B = 2y_1$$

5. The risk-free bond market clears:

$$a_2 = p(d_2)d_2$$

6.  $Q^{rf} \leq 1$ ,  $y_1 \leq y^*$ , with at least one strict equality.

The following proposition states that equilibria with lending policy implement efficient allocations.

**Proposition 2.3.11.** *Any efficient allocation can be implemented as an equilibrium with lending policy.*

Intuitively, bond pricing functions sketch out a nonlinear budget constraint for borrowers. By changing the slope of this budget constraint, we can induce borrower countries to accept any feasible allocation. Figure 2.5 illustrates. The union-wide authority offers a new bond price schedule,  $Q^*(d_2)$ , which gives borrower countries a higher average and (crucially) marginal price for their debt. This induces borrowers not to reduce their debt below  $d_2$ , and so savers are no worse off than in the equilibrium without policy, so this lending policy is Pareto improving.

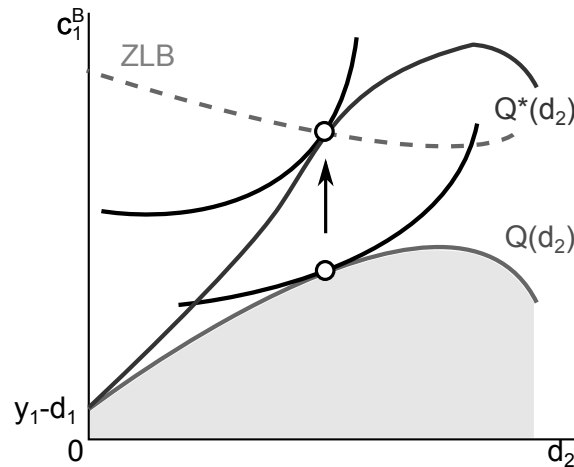


Figure 2.5: Lending policy

Why can lending policy implement all optimal allocations, while debt relief cannot? Both policies can be used to engineer a transfer to borrowers, as Proposition 3.8 states. The key difference is that lending policy can also affect the marginal price of debt, while debt relief cannot. By increasing the marginal price of debt, lending policy encourages borrowers to spend more of their wealth at date 1, boosting aggregate demand.

### 2.3.6 Optimal lending policy at the ZLB

Having shown that some lending policy implements optimal allocations, I now discuss what kind of lending policy does so.

We already know that when the ZLB binds in equilibrium, the equilibrium without policy is not efficient. The following Proposition describes how lending policies implement efficient allocations when the ZLB binds.

**Proposition 2.3.12.** *When  $U_S > U_S^*$ :*

1. *The solution to the planner's problem cannot be implemented as an equilibrium without policy.*
2. *The solution to the planner's problem can be implemented as an equilibrium with a lending policy. The marginal price of debt must be higher than in the equilibrium without policy: that is,*

$$\tilde{Q}(d_2) + \tilde{Q}'(d_2)d_2 > Q(d_2) + Q'(d_2)d_2$$

3. *When the ZLB binds in equilibrium, there always exists an equilibrium with lending policy which is Pareto superior to the equilibrium without policy.*
4. *Given  $\bar{d}_1$ , efficient allocations with higher  $U_S$  have higher  $a_2$  and lower  $T_1^S$ .*

*Proof.* (4.) When the ZLB binds,  $u'(y^* + \bar{d}_1 - T_1^S - a_2) = \beta u'(y^* + (1 - \beta)a_2)$ . Allocations which are better for savers have higher  $a_2$  and higher  $c_1^S = y^* + \bar{d}_1 - T_1^S - a_2$ , which means they must have lower  $T_1^S$ . □

The first part of the proposition follows from the result above that equilibria without policy are inefficient when the ZLB binds. Since there are some efficient allocations in which the ZLB binds, clearly these allocations cannot be implemented as an equilibrium without policy. The second part of the proposition states that lending policy can implement allocations in which the ZLB binds. Furthermore, lending policy must offer borrowers a higher marginal price of debt than in the equilibrium without policy. To see why, return to Figure 2.5, and note that the slope of the new bond pricing function is higher than the slope of the old function at the same level of debt. The third part of the proposition states that in particular, there are some equilibria with lending policy which are Pareto improving relative to a competitive equilibrium with a binding

ZLB and underemployment of resources. The last part of this proposition states that allocations which are relatively favorable for savers involve less debt relief and more lending.

Debt relief, together with no lending policy (i.e. a bond price function which merely replicates market prices,  $\tilde{Q} = Q$ ) can implement efficient allocations in which the ZLB does not bind. The government can simply write off part of  $\bar{d}_1$  until the remaining debt,  $\bar{d}_1 + T_1^B$  (where  $T_1^B < 0$ ) is less than  $\bar{d}_1^*$ . We already know that the competitive equilibrium in this case is efficient. But it is not a Pareto improvement on the equilibrium without policy. Borrowers are better off, but savers are strictly worse off.

## 2.4 Long term debt and debt postponement

In this section I consider equilibria in which borrower countries have some outstanding long-term debt,  $\bar{d}_2 > 0$ . This is of interest for two reasons. First, long-term debt introduces a new inefficiency, independent of the zero lower bound: borrowers have an incentive to over-issue new debt, to dilute existing debt. Second, in order to analyze debt postponement policy, which converts short term debt into long term debt, we need to characterize equilibria with long term debt.

### 2.4.1 Equilibrium with long term debt

I now define equilibrium with long-term debt, in the standard way. I let  $d_2$  denote the total face value of borrowers' obligations to savers at the start of date 2. New debt issued at date 1 is  $d_2 - \bar{d}_2$ . The probability of default, and the endogenous bond price, only depend on  $d_2$ .

**Definition 2.4.1.** *An equilibrium in the economy with long term debt is a collection  $c_1^S, c_1^B, d_2, a_2, Q_1, Q^{rf}, y_1$  and a bond pricing function  $Q(\cdot)$  such that, given initial debt levels  $\bar{d}_1, \bar{d}_2$ :*

1.  $c_1^S, d_2$  solve the saver country's problem:

$$\begin{aligned} & \max_{c_1^S, d_2} u(c_1^S) + \beta V(a_2 + p(d_2)d_2) \\ & \text{s.t. } c_1^S + Q_1(d_2 - \bar{d}_2) + Q^{rf} a_2 = y_1 + \bar{d}_1 \end{aligned}$$

2.  $c_1^B, d_2$  solve the borrower country's problem:

$$\begin{aligned} \max_{c_1^B, d_2} & u(c_1^B) + \beta \left[ \int_0^d V(-\chi) dF(\chi) + p(d)V(-d) \right] \\ \text{s.t.} & c_1^B + \bar{d}_1 = y_1 + Q(d_2)(d_2 - \bar{d}_2) \end{aligned}$$

3. The bond pricing function is  $Q(d) = p(d)Q^{rf}$ , with  $Q(d_2) = Q_1$ .

4. The goods market clears:

$$c_1^S + c_1^B = 2y_1$$

5. The risk-free bond market clears:  $a_2 = 0$ .

6.  $Q^{rf} \leq 1, y_1 \leq y^*$ , with at least one strict equality.

The following proposition characterizes equilibrium.

**Proposition 2.4.2.** For any  $\bar{d}_2$ , there exists  $\bar{d}_1^*(\bar{d}_2)$  such that:

1. If  $\bar{d}_1 < \bar{d}_1^*$ , then  $Q^{rf} < 1$  and  $y_1 = y^*$ .  $Q^{rf}$  is increasing in  $\bar{d}_1$ .
2. If  $\bar{d}_1 = \bar{d}_1^*$ , then  $Q^{rf} = 1$  and  $y_1 = y^*$ .
3. If  $\bar{d}_1 > \bar{d}_1^*$ , then  $Q^{rf} = 1$  and  $y_1 < y^*$ .  $y_1$  is decreasing in  $\bar{d}_1$ .

$c_1^S$  is increasing in  $\bar{d}_1$ .  $c_1^B$  is decreasing in  $\bar{d}_1$ .

As in the economy with only short term debt, borrower countries have an incentive to write down their short term debt at date 1. If their debt is sufficiently large, this depresses aggregate demand by so much that the market clearing risk free rate of interest is negative, and the monetary union enters a recession.

## 2.4.2 Debt dilution and inefficiency

The following proposition states that with outstanding long-term debt, even if the ZLB does not bind, the equilibrium is inefficient. This is for the standard reason that borrowers have an incentive to dilute long-term debt: issuing more debt reduces the value of their outstanding obligations.

**Proposition 2.4.3.** *In the economy with long term debt ( $\bar{d}_2 > 0$ ), if the ZLB does not bind, then the equilibrium is Pareto inefficient.*

When the ZLB does not bind, borrowers issue too much new debt in order to dilute the value of their existing debt. It can be Pareto improving to coordinate buy-backs of long-term debt, but this cannot be implemented at market prices (Aguiar and Amador [2013a]). It follows that a policy of replacing long term debt with short term debt is Pareto improving.

### 2.4.3 Postponement

Postponement is an important feature of debt restructurings in practice. Trebesch et al. [2012] find that out of 186 debt exchanges with foreign private creditors since 1950, 57 involved a cut in face value, while 129 were pure debt reschedulings, involving only a lengthening of maturities. Recently, the IMF has proposed modifying its lending framework to give a greater role for ‘re-profiling’, as an attractive alternative to outright debt forgiveness. Reprofiling was also proposed as a solution to the Greek debt crisis in 2011.

Recall that when the ZLB binds, borrower countries without long-term debt typically issue too little new debt, and reduce their consumption too much, because they do not internalize the effect of their consumption on union-wide aggregate demand. This suggests that when the ZLB binds, it may, perversely, be efficient for the borrowers to have long-term debt. As we will see, equilibria with a binding ZLB and long-term debt are only constrained efficient in a knife-edge case, when the dilution incentive to overborrow exactly outweighs the macroeconomic externality to underspend. However, optimal policy can use this idea to implement constrained efficient allocations. Suppose borrowers have only short-term debt, and the ZLB binds. The union-wide authority can postpone a portion of this short term debt, converting it into long-term debt. If the amount to be converted is chosen correctly, this implements an efficient allocation, as the following Proposition states.

**Proposition 2.4.4.** *Take any solution of the social planner’s problem when  $U_S > U_S^*$ . It can be implemented as an equilibrium with long term debt for some  $\bar{d}_1, \bar{d}_2 > 0$ .*

Another reason to favor long-term debt is that it is less vulnerable to self-fulfilling crises (Cole and Kehoe [2000]). This factor is absent here: the model has a unique equilibrium. Yet another

reason is that long-term debt helps hedge shocks (Angeletos [2002], Buera and Nicolini [2004]). This factor is also absent in the model so far, since there is no aggregate risk.

## 2.5 Home bias

Debt restructuring and lending policies which transfer resources from creditor to debtor countries can be Pareto improving only because debtors spend the transfer, in part, on goods sold by creditor countries, increasing their income. One might worry that if debtor countries spend most of the transfer on domestic goods and services, creditor countries will no longer be better off. To address this concern, I consider an economy with home bias ( $\alpha < 1$ ) but, for tractability, assume perfectly rigid prices ( $\varphi = \infty$ ) and no long-term debt ( $\bar{d}_2 = 0$ ). I show that the central result from the baseline economy goes through: transfers from creditors to debtors are still Pareto improving in a liquidity trap.

### 2.5.1 Equilibrium with home bias

With home bias and rigid prices, different countries will have different levels of income as well as different consumption. Recall that the market clearing condition with fixed prices and home bias is

$$y_t^i = (1 - \alpha)c_t^i + \alpha \int y_t^f df \leq y^*$$

Households in country  $i$  spend a fraction  $(1 - \alpha)$  of their total consumption expenditures on domestically produced goods. Since prices are constant and equal to unity, this means that the quantity of domestic goods they consume is a fixed proportion of their total consumption. Households in other countries spend a fraction  $\alpha$  of their total consumption (equivalently, of their income) on country  $i$ 's goods. If  $\alpha = 1$ , there is no home bias and every country's output is the same. If  $\alpha < 1$ , countries with lower domestic consumption will experience lower output.

When output is below potential in country  $i$ , the social value of higher consumption (from country  $i$ 's perspective) is higher than the private value. Suppose that country  $i$  receives a larger transfer from abroad (e.g. because it borrows more). Its citizens feel richer, and (since prices are fixed) increase consumption of both domestic and foreign goods. Since output is demand constrained, the increase in their consumption of domestic goods increases their income, making

them better off and leading to a second round effect on domestic spending. [Farhi and Werning \[2012\]](#) explore these within-country externalities at great length, and show that there is a role for government intervention in insurance markets, to correct the discrepancy between the private and national value of transfers. Since my goal is to study between-country externalities, I abstract from within-country externalities by assuming that the government borrows and lends on behalf of its citizens, internalizing the effect of its decisions on domestic output.<sup>10</sup>

To characterize equilibrium, start with date 2. Resource constraints are

$$\begin{aligned} y_2^S &= (1 - \alpha)c_2^S + \alpha\bar{y}_2 \leq y^* \\ y_2^B &= (1 - \alpha)c_2^B + \alpha\bar{y}_2 \leq y^* \\ y_2^D(\chi) &= (1 - \alpha)c_2^D(\chi) + \alpha\bar{y}_2 \leq y^* \\ \bar{y}_2 &= \frac{1}{2}y_2^S + \frac{p(d_2)}{2}y_2^B + \frac{1}{2}\int_0^{d_2} y_2^D(\chi) dF(\chi) \end{aligned}$$

where  $y_2^D(\chi)$ ,  $c_2^D(\chi)$  denote the income and consumption of a defaulting country with default cost  $\chi$ . We also have the budget constraints:

$$\begin{aligned} c_2^S &= y_2^S + p(d_2)d_2 \\ c_2^B &= y_2^B - d_2 \\ c_2^D(\chi) &= y_2^D(\chi) \end{aligned}$$

This implies that  $c_2^D(\chi) = y_2^D(\chi) = \bar{y}_2, \forall \chi$ .

I assume monetary policy does the best it can, which is to set  $y_2^S = y^*$ . This means that

$$\begin{aligned} \bar{y}_2 &= y^* - \frac{1 - \alpha}{\alpha}p(d_2)d_2 \\ y_2^B &= y^* - \frac{1 - \alpha}{\alpha}[1 + p(d_2)]d_2 \\ c_2^B &= y^* - \frac{1 - \alpha}{\alpha}[1 + p(d_2)]d_2 - d_2 \end{aligned}$$

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<sup>10</sup>Again, another interpretation of the model is that households participate in financial markets, and governments impose capital controls along the lines described in [Farhi and Werning \[2012\]](#) to induce households to make borrowing decisions that maximize domestic welfare. As noted above, even without home bias, if households participate in financial markets, it would be necessary for national governments to impose capital controls to correct the externalities associated with default and bond pricing, as described in [Na et al. \[2014\]](#).

A borrower will be indifferent between repaying and defaulting when

$$\begin{aligned}
 c_2^B &= c_2^D - \chi \\
 y^* - \frac{1-\alpha}{\alpha}[1 + p(d_2)]d_2 - d_2 &= y^* - \frac{1-\alpha}{\alpha}p(d_2)d_2 - \chi \\
 \frac{d_2}{\alpha} &= \chi
 \end{aligned}$$

The probability that a country repays debt  $d_2$  is  $p(d_2) = Pr(\chi > d_2/\alpha) = 1 - F\left(\frac{d_2}{\alpha}\right)$ , which is increasing in  $\alpha$ . With home bias and sticky prices, governments are more likely to default, because they internalize that repaying their debt would lead to a domestic recession.

Now consider equilibrium at date 1.

$$\begin{aligned}
 y_1^S &= (1-\alpha)c_1^S + \alpha\bar{y}_1 \leq y^* \\
 y_1^B &= (1-\alpha)c_1^B + \alpha\bar{y}_1 \leq y^* \\
 \bar{y}_1 &= \frac{y_1^S + y_1^B}{2} \\
 c_1^S &= y_1^S + \bar{d}_1 - Q(d_2)d_2 \\
 c_1^B &= y_1^B - \bar{d}_1 + Q(d_2)d_2
 \end{aligned}$$

Solving for all variables as a function of  $y_1^S$ ,

$$\begin{aligned}
 \bar{y}_1 &= y_1^S - \frac{1-\alpha}{\alpha}[\bar{d}_1 - Q(d_2)d_2] \\
 c_1^S &= y_1^S + \bar{d}_1 - Q(d_2)d_2 \\
 y_1^B &= y_1^S - 2\frac{1-\alpha}{\alpha}[\bar{d}_1 - Q(d_2)d_2] \\
 c_1^B &= y_1^S - \frac{2-\alpha}{\alpha}[\bar{d}_1 - Q(d_2)d_2]
 \end{aligned}$$

When the zero lower bound is slack, monetary policy sets  $y_1^S = y^*$ . But note that with home bias, even when the ZLB is slack, borrower countries experience a recession.

Governments internalize the effect of their borrowing and lending decisions on domestic output. For example, saver country governments perceive that they face the constraints

$$\begin{aligned}
 c_1^S &= y_1^S + \bar{d}_1 - Q(d_2)d_2 \\
 y_1^S &= (1-\alpha)c_1^S + \alpha\bar{y}_1
 \end{aligned}$$

and take  $\bar{y}_1$ , not  $y_1^S$ , as given. So they effectively face the constraint

$$c_1^S = \bar{y}_1 + \frac{\bar{d}_1 - Q(d_2)d_2}{\alpha}$$

Similarly, the remaining constraints are

$$\begin{aligned} c_2^S &= \bar{y}_2 + \frac{p(d_2)d_2}{\alpha} \\ c_1^B &= \bar{y}_1 - \frac{\bar{d}_1 - Q(d_2)d_2}{\alpha} \\ c_2^B &= \bar{y}_2 - \frac{d_2}{\alpha} \end{aligned}$$

**Definition 2.5.1.** *An equilibrium in the economy with home bias is a collection  $c_1^S, c_1^B, d_2, a_2, Q_1, Q^{rf}, y_1$  and a bond pricing function  $Q(\cdot)$  such that, given the initial debt level  $\bar{d}_1$ :*

1.  $c_1^S, d_2$  solve the saver country's problem:

$$\begin{aligned} \max_{c_1^S, d_2} u(c_1^S) + \beta V \left( \frac{\bar{y}_2 - y^*}{1 - \beta} + a_2 + \frac{p(d_2)d_2}{\alpha} \right) \\ \text{s.t. } c_1^S + \frac{Q_1 d_2 + Q^{rf} a_2}{\alpha} = \bar{y}_1 + \frac{\bar{d}_1}{\alpha} \end{aligned}$$

2.  $c_1^B, d_2$  solve the borrower country's problem:

$$\begin{aligned} \max_{c_1^B, d_2} u(c_1^B) + \beta \left[ \int_0^{d/\alpha} V \left( \frac{\bar{y}_2 - y^*}{1 - \beta} - \chi \right) dF(\chi) + p(d) V \left( \frac{\bar{y}_2 - y^*}{1 - \beta} - \frac{d}{\alpha} \right) \right] \\ \text{s.t. } c_1^S + \frac{\bar{d}_1}{\alpha} = y_1 + \frac{Q(d_2)d_2}{\alpha} \end{aligned}$$

3. The bond pricing function is  $Q(d) = p(d)Q^{rf}$ , with  $Q(d_2) = Q_1$ .

4. The goods markets clear:

$$\begin{aligned} y_1^S &= (1 - \alpha)c_1^S + \alpha\bar{y}_1 \\ y_1^B &= (1 - \alpha)c_1^B + \alpha\bar{y}_1 \\ \bar{y}_1 &= \frac{y_1^S + y_1^B}{2} \\ \bar{y}_2 &= y^* - \frac{1 - \alpha}{\alpha} p(d_2)d_2 \end{aligned}$$

5. The risk-free bond market clears:  $a_2 = 0$ .

6.  $Q^{rf} \leq 1, y_1^S \leq y^*$ , with at least one strict equality.

I now characterize the equilibrium. Equilibria have the same structure as before: if debt is sufficiently high, the ZLB binds. But, as we have seen, output is always below potential in borrower countries, even if the ZLB does not bind.

**Proposition 2.5.2.** *There exists  $\bar{d}_1^*$  such that:*

1. *If  $\bar{d}_1 < \bar{d}_1^*$ , then  $Q^{rf} < 1$  and  $y_1^S = y^*$ .  $Q^{rf}(\bar{d}_1)$  is increasing in  $\bar{d}_1$ .*
2. *If  $\bar{d}_1 = \bar{d}_1^*$ , then  $Q^{rf} = 1$  and  $y_1^S = y^*$ .*
3. *If  $\bar{d}_1 > \bar{d}_1^*$ , then  $Q^{rf} = 1$  and  $y_1^S < y^*$ .  $y^S(\bar{d}_1)$  is decreasing in  $\bar{d}_1$ .*

$c_1^S$  is increasing in  $\bar{d}_1$ .  $\bar{y}_1, y_1^B, \bar{y}_2, y_1^B$  and  $c_1^B$  are decreasing in  $\bar{d}_1$ .

## 2.5.2 Efficient allocations

I now characterize efficient allocations by solving a social planner's problem. It is convenient to define  $d = d_2/\alpha$ . The planner solves

$$\begin{aligned} & \max u(c_1^B) + \beta \left[ \int_0^d V(-(1-\alpha)p(d)d - \chi) dF(\chi) + p(d)V(-(1-\alpha)p(d)d - d) \right] \\ & \text{s.t. } u(c_1^S) + \beta V(\alpha p(d)d) \geq U_S \\ & u'(c_1^S) \geq \beta u'(y^* + (1-\beta)\alpha p(d)d) \\ & \left(1 - \frac{\alpha}{2}\right) c_1^S + \frac{\alpha}{2} c_1^B \leq y^* \\ & \left(1 - \frac{\alpha}{2}\right) c_1^B + \frac{\alpha}{2} c_1^S \leq y^* \end{aligned}$$

As before, without loss of generality we assume  $U_S > \frac{u(y^*)}{1-\beta}$ . The following Proposition characterizes efficient allocations. As before, it is never efficient for the zero lower bound to constrain output at date 1.

**Proposition 2.5.3.** *1. In every efficient allocation,  $y_1^S = y^*$ .*

2. *There exists  $U_S^* > (1+\beta)u(y^*)$  such that the ZLB binds if  $U_S \geq U_S^*$ .*
3.  *$c_1^S$  and  $d_2$  are increasing in  $U_S$ ;  $c_1^B$  is decreasing in  $U_S$ .*

### 2.5.3 Home bias and inefficiency

**Proposition 2.5.4.** *Any equilibrium in the economy with home bias is constrained inefficient.*

*Proof.* When the ZLB does not bind, a necessary condition for optimality is

$$\frac{u'(c_2^S)}{u'(c_1^S)} = \frac{u'(c_2^B)}{u'(c_1^B)} \frac{1}{1 - \gamma(d)} + \Omega$$

where

$$\Omega = \frac{1}{\phi'(d)} \frac{1 - \alpha}{2 - \alpha} \left[ \int_0^d \frac{u'(c_2^D(\chi)) - u'(c_2^B)}{u'(c_1^B)} dF(\chi) + p(d) \frac{u'(c_2^B)}{u'(c_1^B)} [\phi'(d) - 1] \right] < 0$$

But in equilibrium, we have

$$\frac{u'(c_2^S)}{u'(c_1^S)} = \frac{u'(c_2^B)}{u'(c_1^B)} \frac{1}{1 - \gamma(d)}$$

So the equilibrium allocation cannot be a solution to the planner's problem.

When the ZLB binds in equilibrium,  $y_1^S < y^*$  and so (by the above Proposition) the allocation is not constrained efficient.  $\square$

With home bias, equilibrium is typically inefficient, even when the ZLB does not bind. Borrowers deleverage too rapidly. When deciding how much new debt to issue, borrower governments trade off the benefit of debt - higher consumption at date 1 - against the cost - lower consumption at date 2, if they receive a high default cost  $\chi$  and have to repay the debt. They internalize the fact that higher consumption at date 1 boosts their own domestic output, and lower consumption at date 2 reduces their own domestic output. But they do not internalize that their consumption affects demand and output in other borrower countries. Those other countries benefit from higher consumption at date 1, but lose out from lower consumption at date 2. But the benefits outweigh the costs, because for every dollar of debt issued, less than one dollar will be repaid.

Another way to see this is that in this economy, an individual country's decision to default increases aggregate demand, because defaulting countries consume more than countries which repay their debt. In equilibrium, there are an inefficiently low number of defaults - at least from the perspective of borrower countries as a class. Creditors are hurt by default, but this is already priced into the cost of debt. So borrowers as a whole could strike a Pareto-improving deal with creditors where they take on more debt, reducing the price of debt to compensate creditors

for their losses, and making borrowers better off through the aggregate demand externalities from defaulting countries' higher consumption. In practice, there may be negative externalities associated with default (for example, the effect on the banking system in creditor countries) which are not priced into government debt. In this case, unsurprisingly, debt and default might be too high in equilibrium. I abstract from such externalities here, in order to focus on the Keynesian rationale for debt restructuring.

To summarize, transfers from creditors to debtors are Pareto improving at the ZLB because debtors spend the transfer, in part, on goods sold by creditor countries, increasing their income - even if debtor countries spend most of the transfer on domestic goods and services. If borrowers spend most of the transfer on their own goods and services, this increases their domestic income, which also increases their demand for foreign goods. Ultimately, budget constraints imply that a country must spend the whole of any transfer on buying either goods or assets from abroad.

As in the baseline economy, debt writedowns and lending policy are equivalent when re trading is prevented, but lending policy is preferable when re trading is possible.

**Proposition 2.5.5.** *In the economy with home bias, the solution to the planner's problem can be implemented as an equilibrium with lending policy. The marginal price of debt must be higher than in the equilibrium without policy. Given  $\bar{d}_1$ , efficient allocations with higher  $U_S$  have higher  $a_2$  and lower  $T_1^S$ .*

## 2.6 Ex ante policy and overborrowing

A common argument against debt restructuring is that it gives countries an incentive to overborrow, knowing that they will be bailed out. To address this argument, I augment the model to include an ex ante stage in which countries decide how much to borrow and lend, taking into account that the union-wide authority may offer bailouts or debt restructuring ex post.

### 2.6.1 Ex ante overborrowing

I now endogenize date 1 debt,  $\bar{d}_1$ , in the baseline economy with no home bias and rigid prices. At date 0, countries initially have no debt, and have preferences

$$U(c_0^i, \theta_i) + E_0\left[\sum_{t=1}^{\infty} \beta^t u(c_t^i)\right]$$

where  $U_c > 0, U_{cc} < 0$ .  $\theta_i$  measures country  $i$ 's demand for date 0 consumption, with  $U_{c\theta} > 0$ .  $\theta_B > \theta_S = 1$ : borrowers are more impatient and have a more urgent need for consumption at date 0. While I model  $\theta$  as a preference or discount factor shock, it can easily be reinterpreted in terms of income, so that type  $B$  countries borrow because they have temporarily low income at date 0: simply let  $U(c, \theta) = u(c - \theta)$ .

At date 1, with probability  $\pi$ , it becomes common knowledge that countries can default at date 2, with default costs  $\chi$  drawn from  $F(\chi)$ . The equilibrium, conditional on the endogenously chosen levels of debt, is as above. With probability  $1 - \pi$ , it becomes common knowledge that countries cannot default at date 2 (equivalently,  $\chi = \infty$  with probability 1).

I assume countries can only trade a short term bond. If the crisis does not occur at date 1, their budget constraints at dates  $t = 0, 1, \dots$  are

$$\begin{aligned} c_t^S &= y_t + d_t - Q_t^{rf} d_{t+1} \\ c_t^B &= y_t - d_t + Q_t^{rf} d_{t+1} \\ d_0 &= 0 \end{aligned}$$

In the non-crisis state,  $Q_1^{rf} = \beta$ , and indebted countries smooth debt repayments:  $c_t^S = c_1^S, c_t^B = c_1^B, \forall t \geq 1$ .

The following proposition states that if borrower countries are impatient enough, relative to savers, they choose  $d_1 > \bar{d}_1^*$ , triggering the ZLB in the crisis state.

**Proposition 2.6.1.** *For any  $\pi$ , there exists a function  $\theta_{ZLB}(\pi)$  such that  $d_1 > \bar{d}_1^*$  if  $\theta \geq \theta_{ZLB}(\pi)$ .*

This is essentially identical to the result of [Korinek and Simsek \[2014\]](#), in a similar model. Intuitively, countries want to take on more debt the more impatient they are.

## 2.6.2 Ex ante constrained efficient allocations under full information

To characterize ex ante efficient allocations, I again consider a social planner's problem. For now, I assume  $\pi = 1$ .

It is convenient to write the planner's problem in recursive form. Given that borrowers receive date 1 utility  $V_B$ , let  $V_S(V_B)$  be the maximum date 1 utility that can be given to savers. I restrict attention to the case where  $V_S(V_B) > (1 + \beta)u(y^*) > V_B$ : it is optimal to promise savers more

utility than borrowers at date 1, because savers are more patient.

$$V_S(V_B) = \max u(c_1^S) + \beta V(p(d_2)d_2) \quad (2.5)$$

$$\text{s.t. } u(c_1^B) + \beta \left[ \int_0^d V(-\chi) dF(\chi) + p(d)V(-d) \right] = V_B \quad (\text{VB})$$

$$c_1^S + c_1^B \leq 2y^* \quad (\text{RC1})$$

$$u'(c_1^S) \geq \beta u'(y^* + (1 - \beta)p(d_2)d_2) \quad (\text{ZLB})$$

The following Proposition characterizes the solutions to this date 1 Pareto problem.

**Proposition 2.6.2.**  $V_S(V_B)$  is weakly decreasing. There exist  $V_B^{UE} < V_B^{ZLB} < (1 + \beta)u(y^*)$  such that:

1. If  $V_B < V_B^{UE}$ , (ZLB) binds, (RC1) is slack,  $V'_S(V_B) = 0$
2. If  $V_B \in (V_B^{UE}, V_B^{ZLB})$ , (ZLB) and (RC1) both bind,  $-\frac{u'(c_1^S)}{u'(c_1^B)} < V'_S(V_B) < 0$
3. If  $V_B > V_B^{ZLB}$ , (ZLB) is slack, (RC1) binds, and  $V'_S(V_B) = -\frac{u'(c_1^S)}{u'(c_1^B)} < 0$

At date 0, the planner solves

$$\max \alpha [U(c_0^S, \theta_S) + \beta V_S(V_B)] + (1 - \alpha) [U(c_0^B, \theta_B) + \beta V_B] \quad (2.6)$$

$$\text{s.t. } c_0^S + c_0^B \leq 2y^* \quad (2.7)$$

I now characterize constrained efficient allocations under full information.

**Proposition 2.6.3.** Any ex ante constrained efficient allocation is ex post efficient. Define the wedge between borrowers' and savers' Euler equations:

$$\tau = \frac{U_c(c_0^B, \theta_B)}{\beta u'(c_1^B)} - \frac{U_c(c_0^S, \theta_S)}{\beta u'(c_1^S)}$$

If the ZLB binds at date 1,  $\tau > 0$ . If the ZLB is slack,  $\tau = 0$ .

If the union-wide authority knows which countries need to borrow and which do not, it is never ex ante optimal to commit to ex post inefficient outcomes, such as a recession.

### 2.6.3 Ex post efficient bailouts are not ex ante Pareto improving

It follows from the previous proposition that the ex post efficient policies discussed above - debt relief, lending policy, and postponement - are not ex ante Pareto improving, relative to an equilibrium without policy.

Consider the following experiment. Suppose it is common knowledge that, whatever level of debt  $\bar{d}_1$  borrower countries take out at date 0, at date 1 the union-wide authority will implement an ex post constrained efficient allocation  $c_1^S, c_1^B, d_2$ . This allocation may depend on the aggregate debt taken out by borrower countries  $\bar{d}_1$ , but individual borrower (saver) countries correctly perceive that their transfer (tax) does not depend on their own level of debt. In equilibrium, since there is no default at date 1, borrowers and savers face the same interest rate between dates 0 and 1, and we have  $\tau = 0$ . This is not constrained efficient.

To implement constrained efficient allocations, it is necessary to impose a macroprudential tax or limit on borrowing. Country  $i$ 's budget constraint at date 0 becomes

$$c_0^i = y_0 + Q_i^{rf} d_1 - T_0(d_1, \theta_i)$$

where  $T_0(d_1, \theta_i)$  is a nonlinear tax schedule that depends on a country's borrowing. This nests two simple special cases. First, borrowers can receive a tax on debt and a compensating lump sum transfer:  $T_0(d, \theta_B) = -\bar{T}(\theta_B) + \tau(\theta_B)d$ . Second, the union-wide authority can impose a hard limit on borrowing above a certain level, together with a compensating transfer:  $T_0(d, \theta_B) = -\bar{T}(\theta_B)$  if  $d \leq d^*$ ,  $T_0(d, \theta_B) = \infty$  if  $d > d^*$ . The tax on borrowing (alternatively, the borrowing constraint) prevent borrower countries from taking out more debt, in anticipation of the transfers they will receive at date 1.

### 2.6.4 Ex ante efficient allocations under private information

To characterize ex ante efficient allocations, I again consider a social planner's problem. In addition to the ZLB and resource constraints considered above, the planner faces incentive compatibility constraints, which state that no country's allocation can be so generous that another country prefers that allocation to its own. For now, I assume  $\pi = 1$ .

Again, we write the planner's problem in recursive form. The planner solves

$$\max_{\alpha} [U(c_0^S, \theta_S) + \beta V_S(V_B)] + (1 - \alpha)[U(c_0^B, \theta_B) + \beta V_B] \quad (2.8)$$

$$\text{s.t. } c_0^S + c_0^B \leq 2y^* \quad (2.9)$$

$$U(c_0^S, \theta_S) + \beta V_S(V_B) \geq U(c_0^B, \theta_S) + \beta V_B \quad (\text{ICS})$$

$$U(c_0^B, \theta_B) + \beta V_B \geq U(c_0^S, \theta_B) + \beta V_S(V_B) \quad (\text{ICB})$$

I now characterize constrained efficient allocations.

**Proposition 2.6.4.** 1. (2.9) always binds.

2. There exist  $\alpha_{ICS}, \alpha_{ICB}$  such that  $0 < \alpha_{ICS} < \alpha_{ICB} < 1$ , (ICS) binds if  $\alpha < \alpha_{ICS}$ , and (ICB) binds if  $\alpha > \alpha_{ICB}$ .
3.  $V_S'(V_B) < 0$  (that is, the allocation is ex post Pareto efficient) unless (ICS) binds. In this case, we may have  $V_S'(V_B) = 0$  (that is, the allocation may be ex post Pareto inefficient).

When (ICS) binds, so saver countries are tempted to mimic borrowers, it may be optimal to have a recession at date 1, even though this is ex post inefficient. This gives the borrowers lower date 1 consumption; but in return, they can enjoy higher date 0 consumption without inducing savers to mimic them.

Constrained efficient allocations under private information can be implemented with a combination of macroprudential taxes or limits on borrowing ex ante, and debt restructuring ex post. Unlike under full information, however, taxes can no longer be targeted at different countries directly. Country  $i$ 's budget constraint at date 0 becomes

$$c_0^i = y_0 + Q_t^{rf} d_1 - T_0(d_1)$$

where  $T_0(d_1)$  is a nonlinear tax schedule that now depends on only a country's borrowing, not on its type  $\theta_i$ .

## 2.7 Conclusion

In an international liquidity trap, sovereign debt restructuring and lending policy can be Pareto improving because they support aggregate demand in highly indebted countries, which in turn

supports output and incomes in these countries' trading partners. In order to obtain a Pareto improvement, optimal policy combines a transfer to indebted countries with a reduction in these countries' effective borrowing rates, inducing them to spend this transfer on imports, benefiting their trading partners, instead of buying back their debt. Lending policy and debt restructuring are superior to outright debt relief, since they combine a transfer with a reduction in effective borrowing rates.

A large existing literature has explored alternative motivations for sovereign debt restructuring and lending policy, in particular debt overhang, and the need to prevent self-fulfilling crises. In this paper I analyzed a model without these features, in order to focus on the Keynesian rationale for debt restructuring and lending policy. There may be important interactions between self-fulfilling crises and the macroeconomic externality considered in this paper. The risk of self-fulfilling crises in the future would encourage indebted countries to deleverage even more today. Equally, as indebted countries have lower income today because of the recession, they are more vulnerable to such crises. One direction for future research is to analyze this interaction between these channels and the aggregate demand channel explored in this paper.

Another direction is to compare debt policies to alternative monetary and fiscal policies. Higher inflation could potentially boost aggregate demand in two ways. First higher inflation dilutes the value of nominal debt and indirectly provides debt relief. Second, higher expected inflation decreases real interest rates, stimulating spending throughout the monetary union. However, these benefits must be traded off against the usual distortions associated with inflation. Finally, conventional fiscal stimulus - particularly in surplus economies - could potentially increase demand, and might involve less risk of moral hazard than debt relief. In future versions of this paper I will consider these policies.



### *Consumption Volatility, Liquidity Constraints and Household Welfare (with Olga Gorbachev)*

#### 3.1 Introduction

The increase in family income volatility in the United States since the 1970s has been widely documented. Most recently, [DeBacker et al. \[2013\]](#) use a confidential panel of tax returns from the IRS to show that family income volatility increased between 1987 and 2009. This reinforces the finding in earlier studies ([Dynan et al. \[2012\]](#), [Keys \[2008\]](#), [Gottschalk and Moffitt \[2009\]](#), and [Gorbachev \[2011\]](#)), based on PSID data, that household income volatility increased between 1970 and 2006.<sup>1</sup> Whether this increase in income volatility affected household welfare, however, depends on whether it led to a comparable increase in consumption volatility. A priori, households may have been able to use credit markets to smooth consumption, despite increasingly volatile income shocks.

We estimate whether consumption volatility increased for US households between 1980 and 2004. Building on [Gorbachev \[2011\]](#)'s work, which uses an Euler equation approach to separate unpredictable changes in consumption from predictable changes stemming from observable taste shifters, interest rates, and life-cycle factors; we estimate consumption volatility as the square of the Euler equation residuals. [Gorbachev \[2011\]](#) addressed liquidity constraints only by estimating Euler equations on a sample of households with positive net worth, assuming that these households are unconstrained while those with zero or negative net worth are constrained.<sup>2</sup>

In this paper, we use direct information on household access to credit from the Survey of

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<sup>1</sup>This increase in *family* income volatility contrasts with recent evidence, based on confidential administrative data from Social Security Administration and IRS, that male earnings volatility remained constant or even decreased since the 1980s. See [Dahl et al. \[2008\]](#), [Sabelhaus and Song \[2009\]](#), [Guvenen et al. \[2012\]](#), and [DeBacker et al. \[2013\]](#).

<sup>2</sup>Since Euler equations for constrained households contain an additional term, the Lagrange multiplier on the borrowing constraint, it is necessary to either have an estimate of the Lagrange multiplier, or to drop liquidity constrained households.

Consumer Finances (SCF) to identify the degree to which households are liquidity constrained, and predict the likelihood that individuals are constrained in our PSID sample. By using direct information about credit constraints, our approach improves on [Gorbachev \[2011\]](#) in three respects. First, it provides a more precise measure of liquidity constraints, since some positive net worth households may be liquidity constrained. Indeed, based on our SCF measure, 15 to 20 percent of positive net worth households are liquidity constrained. Second, it allows us to document how household access to credit changed over time, which is of independent interest. Third, it enables us to study how volatility of consumption evolved for the most vulnerable groups in society, liquidity constrained and wealth-poor households.

We find that consumption volatility increased by around 19 percent between 1980 and 2004, or by 3 volatility points for an average household, where as volatility of income went up by 14 volatility points, or 44 percent. Since unconstrained households can smooth temporary income shocks, this suggests that either a significant fraction of households were liquidity constrained, or that permanent income shocks became more volatile over this period, (or a combination of the two). In fact, consistent with our findings, a number of studies observe that the volatility of permanent shocks continued to increase into the 1990s.<sup>3</sup> Despite financial liberalisation and the near-tripling of household debt between 1983 and 2007, we find that the proportion of liquidity constrained households slightly *increased* during this period. In all years, poorer households and those headed by single parents, black or Hispanic individuals, or individuals with low education, were the most likely to be liquidity constrained. In fact, these gaps in access to credit (between rich and poor, white and black individuals, and so on) widened over time.

Households' inability to borrow and smooth consumption has a significant welfare cost. We find that the probability of being denied credit has an independent and strongly significant effect on consumption volatility beyond the effect of volatility of income. Consumption volatility was around 50 percent higher for the quarter of PSID households who were most likely to be constrained than for the quarter who were least likely to be constrained. Not surprisingly, households headed by black or Hispanic individuals, single parents or those with less than 13 years of education experienced the highest level of consumption volatility, and were the most

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<sup>3</sup>[Moffitt and Gottschalk \[2011\]](#) and [Jensen and Shore \[2008\]](#) find that the volatility of permanent shocks to men's labor income increased since the mid 1970s; [Keys \[2008\]](#) finds that the variance of permanent shocks to family income (but not individual income) increased between the 1980s and 1990s.

likely to be liquidity constrained and to have very low wealth holdings.

Our work relates to a large theoretical and empirical literature on the response of household consumption to income shocks. The standard incomplete markets model predicts that consumption should move almost one-for-one with shocks to permanent income, while shocks to transitory income should have only small effects on consumption. A substantial empirical literature, surveyed by [Jappelli and Pistaferri \[2010\]](#) and [Meghir and Pistaferri \[2011\]](#), finds evidence against both predictions: consumption reacts too much to transitory shocks, and not enough to permanent shocks. Consequently, research has explored a variety of mechanisms which allow households to partially insure against both permanent and transitory shocks. In particular, there is some evidence that borrowing constraints prevent certain households from smoothing consumption in response to transitory shocks.<sup>4</sup> Moreover, according to [Blundell et al. \[2008\]](#), households' ability to insure against transitory shocks did not change between 1980 and 1993. Our goal is to add to these findings by examine how the partial insurance mechanisms available to households have allowed them to smooth consumption in the face of increasingly volatile income shocks (both permanent and transitory) between 1980 and 2004.

Our results are also relevant to the literature on consumption inequality: income volatility is an important factor contributing to income inequality, and the same partial insurance mechanisms affecting the mapping from income to consumption inequality also affect the mapping from income to consumption volatility. Most recently, [Attanasio et al. \[2013\]](#) using Consumer Expenditure Survey and PSID data find that consumption inequality increased by almost as much as income inequality in the US between 1980 and 2010.<sup>5</sup> This is consistent with our finding that consumption volatility significantly increased between 1980 and 2004.<sup>6</sup>

The rest of the paper is organised as follows. In Section 3.2 we present a consumption model and explain how we proxy for the effect of liquidity constraints and precautionary savings on the Euler equation. In Section 3.3 we discuss our estimation of consumption volatility and results.

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<sup>4</sup>See for example [Zeldes \[1989a\]](#), [Jappelli \[1990\]](#), [Blundell et al. \[2008\]](#).

<sup>5</sup>See also [Aguar and Bils \[2011\]](#), [Browning and Crossley \[2009\]](#), [Primiceri and vanRens \[2009\]](#), [Davis and Kahn \[2008\]](#), [Blundell et al. \[2008\]](#), [Krueger and Perri \[2006\]](#) among many others.

<sup>6</sup>Evaluating the welfare cost of inequality depends on interpersonal comparisons of utility. [Sen \[1980\]](#) describes the difficulties this raises. The variance of unpredictable changes in consumption, however, has a direct welfare cost: risk-averse households would be willing to reduce their average consumption in return for a reduction in consumption volatility.

Section 3.4 concludes with estimates of the welfare cost of increased consumption volatility and unequal access to credit.

## 3.2 Consumption Model

In order to construct a measure of consumption volatility, we first estimate Euler equations, which give us an estimate of expected household consumption growth. We then compute unpredictable shocks to consumption as the difference between actual and expected consumption growth, the Euler equation residuals. We estimate consumption volatility as the square of these residuals. While the raw volatility of changes in household consumption would be easier to calculate, it is not an appropriate measure of welfare. Predictable changes in household consumption, (due, for example, to changes in household preferences) have different welfare implications from unpredictable changes in consumption, which arise due to households' inability to completely insure against shocks. An increase in the variance of predictable changes in consumption might not reduce household welfare, whereas an increase in the variance of unpredictable changes in consumption unambiguously reduces welfare, other things being equal.

We derive the household Euler equation from a relatively standard unitary, incomplete markets consumption model, building on [Gorbachev \[2011\]](#). In particular, we allow for endogenous income and binding liquidity constraints.

At time  $t$ , household  $h$  solves:

$$\begin{aligned} \max_{\{A_{h,s}, C_{h,s}, N_{h,s}^W, N_{h,s}^H\}_{s=t}^T} \mathbb{E}_t \left\{ \sum_{s=t}^T e^{\delta_h(s-t)} \frac{\exp\{\eta_W N_{h,s}^W + \eta_H N_{h,s}^H + \theta' Z_{h,s} + v_{h,s}\} C_{h,s}^{1-\gamma}}{1-\gamma} \right\} \\ \text{s.t. } A_{h,s+1} = (1 + r_{h,s,s+1})A_{h,s} + w_{h,s}^W N_{h,s}^W + w_{h,s}^H N_{h,s}^H + Y_{h,s} - C_{h,s} \\ A_{h,s+1} \geq -L_{h,s} \end{aligned}$$

where  $\delta_h$  is the household-specific annual discount rate,  $C_{h,t}$  non-durable consumption, and  $1/\gamma$  the intertemporal elasticity of substitution. Our specification of preferences follows [Attanasio \[1999\]](#):  $N_{h,t}^W$  and  $N_{h,t}^H$  are hours worked by the wife and husband respectively,  $w_{h,t}^W, w_{h,t}^H$  are their respective real wages,  $Z_{h,s}$  is a vector of other observable variables affecting preferences such as age, number of adults and children, marital status, and information on the household's housing

status, and  $v_{h,t}$  an unobservable preference shock.<sup>7</sup>  $A_{h,t+1}$  are household assets at the end of period  $t$ ,  $r_{h,t,t+1}$  is the household-specific risk free interest rate on loans taken out between  $t$  and  $t + 1$ , which depends on a household's marginal tax rate and on the local inflation rate;  $Y_{h,t}$  is non-labor income; and  $L_{h,t}$  is the household-specific and time-varying borrowing limit.

The PSID became biennial in 1997, so we have no data on annual changes in consumption after this date. To preserve the length of our sample, we consider two-year consumption growth rates. Throughout the paper,  $\Delta$  denotes two-year changes in a variable:  $\Delta X_t = X_t - X_{t-2}$ . After taking first order conditions, rearranging terms and using the assumption that households have rational expectations, we can rewrite the Euler equation between periods  $t - 2$  and  $t$  as:

$$\mathbb{E}_{t-2} \left[ (1 + r_{h,t}) e^{\Delta\theta_{h,t} - 2\delta_h} \left( \frac{C_{h,t}}{C_{h,t-2}} \right)^{-\gamma} \right] (1 + \lambda_{h,t-2}) = 1 \quad (3.1)$$

where for convenience, we define  $\theta_{h,s} = \eta_W N_{h,s}^W + \eta_H N_{h,s}^H + \theta' Z_{h,s} + v_{h,s}$ ;  $(1 + r_{h,t}) = (1 + r_{h,t-2,t-1})(1 + r_{h,t-1,t})$  is the ex post gross real interest rate on 2-year loans; and  $\lambda_{h,t-2}$  is the Lagrange multiplier on the household's borrowing constraint, normalised by a constant that is known at time  $t - 2$ .<sup>8</sup> For unconstrained households,  $\lambda_{h,t-2} = 0$ . Rational expectations implies that

$$(1 + r_{h,t}) e^{\Delta\theta_{h,t} - 2\delta_h} \left( \frac{C_{h,t}}{C_{h,t-2}} \right)^{-\gamma} (1 + \lambda_{h,t-2}) = 1 + e_{h,t} \quad (3.2)$$

where  $e_{h,t}$  is an expectational error with  $\mathbb{E}_{t-2} e_{h,t} = 0$ . Taking logs of both sides and rearranging:

$$\Delta \ln C_{h,t} = \frac{1}{\gamma} [\ln(1 + r_{h,t}) - 2\delta_h + \Delta\theta_{h,t} + \ln(1 + \lambda_{h,t-2}) - \ln(1 + e_{h,t})] \quad (3.3)$$

Since  $e_{h,t}$  has conditional mean zero,  $\ln(1 + e_{h,t})$  does not. Taking a Taylor expansion, we have

$$\ln(1 + e_{h,t}) = e_{h,t} - \frac{1}{2} e_{h,t}^2 + R_{h,t} \quad (3.4)$$

where  $R_{h,t}$  is a remainder containing third and higher order terms. We assume households never receive any news about third and higher order moments:<sup>9</sup>  $R_{h,t} = R_h + e_{h,t}^R$ , where  $\mathbb{E}_{t-2} e_{h,t}^R = 0$ .

<sup>7</sup>Note that hours worked is possibly an endogenous variable: households may change their labor supply in response to an unexpected income shock to partially insulate consumption. Since the Euler equation for consumption is an equilibrium relationship that holds at the optimal values of  $N_{h,t}^W, N_{h,t}^H$ , however they are determined, it is not necessary to model labor supply explicitly (Attanasio [1999]). We will control for the endogeneity of hours using instrumental variables, as we describe below.

<sup>8</sup>Following Zeldes [1989a], we define  $\lambda_{h,t-2}$  by  $\psi_{h,t-2} = \lambda_{h,t-2} \mathbb{E}_{t-2} \left\{ \left( (1 + r_{h,t}) e^{-\delta_h} \right) e^{\theta_{h,t}} C_{h,t}^{-\gamma} \right\}$ , where  $\psi_{h,t-2}$  is the actual Lagrange multiplier on the borrowing constraint.

<sup>9</sup>This is a standard assumption, see for example Attanasio [1999].

Let  $\sigma_{h,t-2}^2 = \mathbb{E}_{t-2}[e_{h,t}^2]$  be the year  $t - 2$  conditional variance of the year  $t$  expectational error, and let  $v_{h,t} = \frac{1}{2}(e_{h,t}^2 - \sigma_{h,t-2}^2)$  be the household's expectational error concerning  $e_{h,t}^2$ , which has conditional mean zero. Substituting back into the Euler equation, we have:

$$\begin{aligned} \Delta \ln C_{h,t} &= \frac{1}{\gamma} [\ln(1 + r_{h,t}) - 2\delta_h + \Delta\theta_{h,t}] \\ &+ \frac{1}{\gamma} [\ln(1 + \lambda_{h,t-2}) + \frac{1}{2}\sigma_{h,t-2}^2 - R_h - e_{h,t}^R + v_{h,t} - e_{h,t}] \end{aligned} \quad (3.5)$$

Because the PSID has a long panel on food consumption, we need an Euler equation for food consumption, not total non-durable consumption.<sup>10</sup> Following [Blundell et al. \[2008\]](#), we assume the demand for food consumption has the form

$$\ln F_{h,t} = \alpha_0 + \alpha_1 \ln p_t^F + \alpha_2 \ln p_t^O + \beta \ln C_{h,t} + \theta'_F Z_{h,t} + \iota_{h,t} \quad (3.6)$$

where  $p_t^F$  is the price of food,  $p_t^O$  is the price of other non-durables,  $Z_{h,t}$  is the vector of demographic variables discussed above, and  $\iota_{h,t}$  is an unobservable preference shock.<sup>11</sup> If  $\beta = 1$ , this is a standard homothetic demand function. If  $\beta \neq 1$ , it is a log-linear approximation to an arbitrary non-homothetic demand system, and  $\iota_{h,t}$  will also contain higher-order terms relating to the error in this approximation.<sup>12</sup> Our specification also allows for non-separability of preferences between consumption of food and other non-durables.<sup>13</sup> Taking two-year differences of this equation and substituting into equation (3.5), we obtain our estimating equation:

$$\begin{aligned} \Delta \ln F_{h,t} &= \frac{\beta}{\gamma} [\ln(1 + r_{h,t}) - 2\delta_h] + \alpha_1 \Delta \ln p_t^F + \alpha_2 \Delta \ln p_t^O + \mu \Delta Z_{h,t} \\ &+ \frac{\beta}{\gamma} [\eta_W \Delta N_{h,t}^W + \eta_H \Delta N_{h,t}^H + \ln(1 + \lambda_{h,t-2}) + \frac{1}{2}\sigma_{h,t-2}^2 - R_h] + \zeta_{h,t} \end{aligned} \quad (3.7)$$

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<sup>10</sup>Starting in 1968, the PSID collected information on food and housing expenditures, and also (somewhat inconsistently) on childcare expenses for those with working wives. In 1999, the PSID added information on the following non-durable (and services) categories: childcare (for working and non-working spouses), utilities, gasoline, transportation, home and auto insurance, and vehicle repair. In 2005, new categories were added: expenditure on clothing, home repair, furniture, trips, and other recreation activities. We use expenditure on food to estimate volatility of consumption in order to keep the length of the sample. We then use our estimate of the budget elasticity of food consumption parameter  $\hat{\beta}$  from the demand equation (3.6), estimated on the 2005 nondurable consumption data, to inform our welfare cost discussion in Section 3.4.

<sup>11</sup>Unlike [Blundell et al. \[2008\]](#), we restrict  $\beta$  to be the same across all households.

<sup>12</sup>[Crossley and Low \[2011\]](#) show that the assumption of a constant intertemporal elasticity of substitution imposes strong restrictions on within-period demand. In particular, it rules out the demand system specified here unless  $\beta = 1$ . We consider it worthwhile to allow for more general preferences in (3.6), even though they can only be an approximation to the true demand system, since it is well known that food is a necessary good. All our results go through if we assume  $\beta = 1$ .

<sup>13</sup>As pointed out by, for example, [Attanasio and Weber \[1995\]](#), [Meghir and Weber \[1996\]](#), [Banks et al. \[1997\]](#) it is important to control for non-separability of food consumption relative to consumption of other goods.

where  $\mu = \frac{\beta}{\gamma}\theta + \theta_F$ , and we combine the expectational errors and the two preference shocks into a single term:

$$\zeta_{h,t} = \frac{\beta}{\gamma}(v_{h,t} - e_{h,t} - e_{h,t}^R + \Delta v_{h,t}) + \Delta l_{h,t} \quad (3.8)$$

Euler equations such as (3.7) are typically estimated using instrumental variables techniques. We discuss our estimation strategy in Section 3.3; here it suffices to say that we assume that  $\mathbb{E}_{t-s}\zeta_{h,t} = 0$ , and use variables dated  $t - 4$  and earlier as our instruments. While the expectational errors which enter  $\zeta_{h,t}$  have mean zero conditional on year  $t - 2$  information,  $\Delta v_{h,t}$  and  $\Delta l_{h,t}$  will not, if  $v_{h,t}$  and  $l_{h,t}$  are i.i.d., but they will have mean zero conditional on year  $t - 4$  information. We consider *consumption volatility* to be the variance of household expectational errors,  $\mathbb{E}_{t-2}[e_{h,t}^2]$ . We will estimate volatility as the squared residuals,  $\hat{\zeta}_{h,t}^2$ .

It is well known that consumption is measured with error. Following Alan et al. [2009], we assume measurement error is stationary and independent of all the regressors, including lagged values of the measurement error and expectations error, consumption levels and interest rates.<sup>14</sup> Since the residual  $\zeta_{h,t}$  also contains this measurement error, our measure of consumption volatility,  $\hat{\zeta}_{h,t}^2$ , will mis-measure the *level* of the variance of household expectational errors. However, as long as the variance of measurement error and the variance of preference shocks are constant over time, we can accurately estimate the *changes* in the variance of household expectational errors.

We estimate the second-order approximation (3.7), rather than the non-linear Euler equation (3.2), since measurement error makes non-linear GMM estimation (but not linear estimators) inconsistent. Attanasio and Low [2004] perform Monte Carlo studies, and find that consistent estimates can be obtained with long panels (at least 30 quarters), given sufficient variation in real interest rates. Alan et al. [2012] find that even with a short panel (14 years), estimates of the EIS are relatively accurate, even though standard instrument exogeneity and relevance conditions are not satisfied. Both studies conclude that non-linear Euler equation estimation of (3.2) is more likely to be biased. We have data spanning 24 years, over a period which saw large variations in interest rates (1980 to 2004). Importantly, ex post real interest rates in our sample are household

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<sup>14</sup>If food consumption was subject to mean reverting error, as is true for income (see Section 3.2.2.1), our parameters would all be biased downward, and we would need to adjust for this mean reversion to recover the true parameters. However, since to our knowledge there are no studies suggesting mean-reverting measurement error in consumption, we prefer to follow current literature and assume classical measurement error.

specific, since they depend on a household's marginal tax rate and on the local inflation rate. Thus, we have substantially more variation in interest rates than in the Monte Carlo studies described above, which allows us to estimate the EIS more precisely.

Our ultimate goal is to estimate consumption volatility as the squared residuals in equation (3.7). In order to obtain these residuals, we need to consistently estimate all the parameters in this equation. This Euler equation, however, contains two unobserved terms. We now explain how we proxy for these two terms:  $\ln(1 + \lambda_{h,t-2})$ , the normalised Lagrange multiplier on the borrowing constraint, and  $\sigma_{h,t-2}^2$ , the precautionary savings term.

### 3.2.1 Liquidity Constraints

The normalised Lagrange multiplier on the household's borrowing constraint,  $\ln(1 + \lambda_{h,t-2})$ , is unobserved but enters the Euler equation. If we did not control for this term, it would enter the residual, potentially biasing our estimates. This term would also enter our measure of consumption volatility, the squared Euler equation residual: if the cross-sectional variance of  $\ln(1 + \lambda_{h,t-2})$  has been, e.g., increasing over time, we would wrongly identify this as an increase in consumption volatility. We control for  $\ln(1 + \lambda_{h,t-2})$  by using information on households' access to credit from the Survey of Consumer Finances. The SCF directly measures whether households have been denied credit or discouraged from applying. We regress this variable on explanatory variables common to both PSID and SCF, and use our estimates to compute fitted values for the PSID sample representing their estimated probability of being denied credit. We then proxy for  $\ln(1 + \lambda_{h,t-2})$  with a polynomial in the estimated probability of being denied credit in our Euler equation regressions.

Researchers have used several methods to identify liquidity constrained households.<sup>15</sup> Jappelli et al. [1998] regress the probability of being constrained, constructed using 1983 SCF data on variables common to both SCF and PSID, and use the coefficients to estimate the probability of being constrained for PSID households. We combine SCF and PSID data in a similar way, with two important differences. First, we combine SCF data from all the eight surveys between

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<sup>15</sup>Zeldes [1989b], Runkle [1991] and later Gorbachev [2011], used wealth information contained in PSID, Gross and Souleles [2002] credit card data, and Attanasio et al. [2008] data on auto loans; whereas Jappelli [1990] used direct data on liquidity constraints in SCF.

1983 and 2007 when regressing the liquidity constraints indicator on explanatory variables, thus allowing the relationship between household characteristics and access to credit to change over time. Second, while Jappelli et al. [1998] use the liquidity constraints variable to estimate switching regression models of the Euler equation, we use it to proxy for the Lagrange multiplier.

The SCF asks the following questions:

1. "In the past five years, has a particular lender or creditor turned down any request you (or your [husband/wife]) made for credit, or not given you as much credit as you applied for?"
2. "Were you later able to obtain the full amount you (or your husband/wife) requested by reapplying to the same institution or by applying elsewhere?"
3. "Was there any time in the past five years that you (or your [husband/wife]) thought of applying for credit at a particular place, but changed your mind because you thought you might be turned down?"

Following Jappelli et al. [1998], we count a household as liquidity constrained if the head answers "yes" to question 1 and "no" to question 2, or if she answers "yes" to question 3. That is, a household is constrained if it had an application for credit rejected, or if household members were discouraged from applying for credit because they thought they might be rejected.

Figure 3.1 plots the proportion of liquidity constrained, 'rejected' and 'discouraged' households between 1983 and 2007. This figure shows that the proportion of liquidity constrained households increased by 3 percentage points over this period, from 18 percent to 21 percent. This was driven by an increase in the proportion of discouraged households; the proportion of households with a loan application rejected stayed roughly constant.

### 3.2.1.1 Estimating Constraints in the PSID

We estimate a probit regression model using the SCF sample, with the liquidity constraints dummy as our dependent variable, using explanatory variables common to PSID and SCF. Our explanatory variables include demographics, income, information on home value and mortgages,

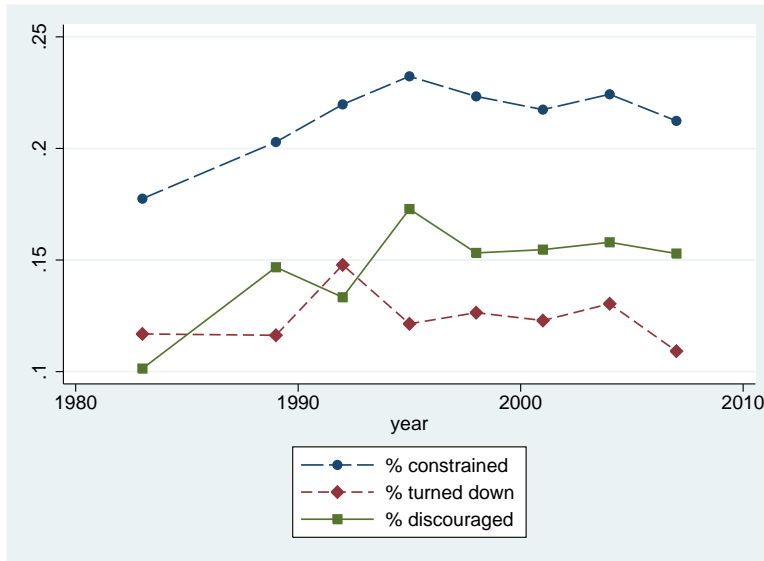


Figure 3.1: Proportion of liquidity constrained households

Note: ‘rejected’ households are those that had a request for credit turned down and were not later able to obtain the full amount; ‘discouraged’ households are those that thought of applying but did not because they thought they might be turned down; and ‘constrained’ households are those who were either ‘discouraged’ or ‘rejected’.

Source: Survey of Consumer Finances.

and a quadratic time trend.<sup>16</sup> A cubic in age is included to allow for variation in demand for debt over the life cycle. In addition to current income, the demand for debt should depend on a household’s permanent income; since the SCF has no panel dimension, we proxy for permanent income using dummies for the head of household’s years of education, interacted with the head’s race. We also include financial variables common to both PSID and SCF - a cubic in households’ house value, and quadratics in mortgage, annual mortgage payment, and asset income - since these variables are especially useful in predicting a household’s access to credit (having a mortgage is prima facie evidence that a household has had access to credit in the past). Household demand for debt is further proxied using number of children, number of adults, marital status, and a dummy for being a single parent. Finally, we allow for changes in access to credit over time. We use a quadratic trend, rather than a full set of time dummies, because we need to use the coefficient estimates to impute fitted values in our PSID sample, which contains different years of data than the SCF sample. We also interact income, mortgage, home value, and the

<sup>16</sup>See Appendix C.3 for a full description.

Table 3.1: Predicting liquidity constraints in SCF

Age	-0.007	(0.054)	ln(family income)	1.419	(1.312)
Age <sup>2</sup>	0.000	(0.001)	ln(family income) <sup>2</sup>	-0.118	(0.140)
Age <sup>3</sup>	-0.000	(0.000)	ln(family income) <sup>3</sup>	0.002	(0.005)
White, no HS	0.060	(0.049)	Asset income > 0	0.693***	(0.150)
White, college	-0.123***	(0.035)	ln(asset income+1)	-0.306***	(0.045)
Black, no HS	17.677***	(5.577)	ln(asset income+1) <sup>2</sup>	0.025***	(0.004)
Black, HS	17.722***	(5.572)	Black*ln(family income)	-6.269***	(2.053)
Black, college	17.665***	(5.580)	Black*ln(family income) <sup>2</sup>	0.707***	(0.248)
Hispanic, no HS	25.515**	(11.316)	Black*ln(family income) <sup>3</sup>	-0.025***	(0.010)
Hispanic, HS	25.532**	(11.310)	Hispanic*ln(family income)	-8.816**	(3.774)
Hispanic, college	25.336**	(11.323)	Hispanic*ln(family income) <sup>2</sup>	0.959**	(0.418)
1 child	-0.004	(0.043)	Hispanic*ln(family income) <sup>3</sup>	-0.033**	(0.015)
2 children	0.043	(0.044)	Black*ln(house value+1)	-0.257	(0.401)
3+ children	0.118**	(0.051)	Black*ln(house value+1) <sup>2</sup>	0.070	(0.076)
1 adult	-0.244***	(0.071)	Black*ln(house value+1) <sup>3</sup>	-0.004	(0.004)
2 adults	-0.162***	(0.062)	Hispanic*ln(house value+1)	0.208***	(0.075)
3 adults	-0.070	(0.069)	Hispanic*ln(house value+1) <sup>2</sup>	-0.017***	(0.007)
Single parent	0.187***	(0.054)	ln(mortgage+1)*t	-0.004	(0.004)
Divorced/separated	0.120**	(0.050)	ln(mortgage+1) <sup>2</sup> *t	0.000	(0.000)
Widow	0.068	(0.086)	ln(family income)*t	-0.098	(0.076)
Single	-0.008	(0.052)	ln(family income) <sup>2</sup> *t	0.011	(0.008)
Receive welfare	0.193***	(0.050)	ln(family income) <sup>3</sup> *t	-0.000	(0.000)
Have mortgage	-0.348	(1.592)	Homeowner*t	0.013	(0.017)
ln(annual mortgage payment + 1)	-0.117	(0.378)	ln(house value+1)*t	0.013	(0.018)
ln(annual mortgage payment + 1) <sup>2</sup>	0.017	(0.023)	ln(house value+1) <sup>2</sup> *t	-0.002	(0.003)
ln(mortgage+1)	0.060	(0.070)	ln(house value+1) <sup>3</sup> *t	0.000	(0.000)
ln(mortgage+1) <sup>2</sup>	-0.001	(0.006)	Asset income > 0 * t	-0.012***	(0.004)
Homeowner	-0.133	(0.287)	t	0.317	(0.234)
ln(house value+1)	0.205	(0.301)	t <sup>2</sup>	-0.001***	(0.000)
ln(house value+1) <sup>2</sup>	-0.045	(0.054)	Constant	-4.855	(4.217)
ln(house value+1) <sup>3</sup>	0.002	(0.003)			
Observations	21,607				

Robust standard errors in parentheses; \*\*\* p<0.01, \*\* p<0.05, \* p<0.1  
 Authors' calculations based on SCF data as described in the text.

dummy for positive asset income with a linear time trend.

Table 3.1 presents our estimation results. Since the model we estimate is only a reduced-form expression which does not distinguish factors affecting the demand and supply of credit, the estimated coefficients presented here do not have a straightforward interpretation: here we are more concerned with accurately predicting the probability of being constrained in the PSID. Nonetheless, our estimation results are broadly consistent with previous studies (Jappelli [1990]). Single parents and black or Hispanic, working heads of household with low education are more likely to be constrained. Individuals with only a high school degree were significantly more likely to be constrained than those with a college degree, whereas those with more than 16 years of education were much less likely to be constrained. Higher family income decreases the probability of being constrained.

Since our goal is to predict the probability that a household is constrained, and not to estimate

the causal effect of household characteristics on the probability of being constrained, the fact that several of our explanatory variables may be endogenous is irrelevant. What is important to us is that our prediction is as accurate as possible. We compute the accuracy of our predictions as follows. We label an SCF household as ‘constrained’ if their estimated probability of being constrained is greater than the average probability in that year, and label them ‘unconstrained’ otherwise. We find that we correctly classify 74 percent of constrained households and 66 percent of unconstrained households. That is, given that a household is truly constrained (respectively, unconstrained), we have a 74 percent (66 percent) chance of correctly identifying it as constrained (unconstrained).<sup>17</sup>

We then use our coefficient estimates to compute fitted values for observations in the PSID sample, and interpret these fitted values as the probability that PSID households are liquidity constrained. Starting in 1992, relative to SCF households, PSID households are about 2 percentage points less likely to be constrained, most likely because our SCF households include more welfare recipients and households headed by Black and Hispanic individuals, and have lower average income. As long as the relationship between the probability of being constrained and the explanatory variables is the same in both samples, these differences should not matter.

We use the predicted probability of being constrained to proxy for the normalised Lagrange multiplier  $\ln(1 + \lambda_{h,t-2})$  in the household’s Euler equation. We assume that

$$\ln(1 + \lambda_{h,t-2}) = F(\widehat{Pr(\text{denied credit})}_{h,t-2}) + u_{h,t-2} \quad (3.9)$$

where  $F(\cdot)$  is a polynomial function whose coefficients we estimate,  $\widehat{Pr(\text{denied credit})}_{h,t-2}$  is the predicted probability that the household is liquidity constrained, and  $u_{h,t-2}$  is orthogonal to our instrument set.

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<sup>17</sup>To assess the out-of-sample predictive accuracy of our model, we employ cross-validation (Stone [1974]). We randomly partition our sample 50 times into a “training sample”, which we use for estimation, and a “hold-out” sample. For each training sample, we estimate the model and use it to predict the probability of being denied credit for each observation in the associated hold-out sample. We then “predict” that a household is denied credit if  $\widehat{Pr(\text{denied credit})}$  is greater than the average probability of being denied credit in that year. Averaging across all 50 hold-out samples, we correctly predict 73 percent of constrained households, and 66 percent of unconstrained households.

### 3.2.2 Precautionary Savings

The variance of the household's expectational errors,  $\sigma_{h,t-2}^2$ , appears in the Euler equation because of the precautionary savings motive (Carroll [1992]). Precautionary savings will be higher for households who are more uncertain about future income, and for households with lower wealth (Browning and Lusardi [1996]). We therefore assume that the variance of household expectational errors can be approximated as a linear function of the variance of income volatility and the probability that the household has positive wealth:

$$\sigma_{h,t-2}^2 = \gamma_h + \gamma_1 \mathbb{E}_{t-2}[(\hat{\sigma}_{h,t}^Y)^2] + \gamma_2 \Pr(\widehat{W}_{h,t} > 0) + e_{h,t-2}^\sigma \quad (3.10)$$

We allow the constant term  $\gamma_h$  to vary across households, since some households face systematically higher uncertainty, regardless of their income volatility and wealth. In some specifications, we restrict  $\gamma_2$  to be zero. We assume the approximation error  $e_{h,t-2}^\sigma$  is orthogonal to our instruments set.

#### 3.2.2.1 Estimating Volatility of Family Income

To construct our measure of family income volatility, we estimate a standard model of the household income process, and measure volatility as the square of changes in residuals. Our model is:

$$\ln(Y_{h,t}) = X'_{h,t} \vartheta_t + u_{h,t} \quad (3.11)$$

$Y_{h,t}$  is real family income, and  $X_{h,t}$  is a set of household characteristics affecting income, which are observable, anticipated by consumers, and potentially time-varying.<sup>18</sup> Standard models of the income process (MaCurdy [1982]) assume that the residual  $u_{h,t}$  can be decomposed into a permanent and a transitory component;<sup>19</sup> we do not make any particular assumption about  $u_{h,t}$ , and do not attempt to distinguish between permanent and transitory shocks in our main specification. We define income volatility,  $\sigma_{\Delta u,t}^2$ , to be the variance of  $(u_{h,t} - u_{h,t-2})$ .

Income is measured with error, and this measurement error appears to be non-classical. While the textbook errors-in-variables model assumes that measurement error is independent of true

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<sup>18</sup>See Appendix C.3 for details.

<sup>19</sup>A recent literature, following Guvenen [2007], has examined models with more heterogeneous life-cycle income profiles and less persistent income shocks.

values, [Kim and Solon \[2005\]](#) find that measurement error in survey data on earnings is *mean-reverting* and is negatively correlated with true values. Following [Kim and Solon \[2005\]](#), we assume observed household income  $\ln Y_{h,t}^*$  is a function of true income  $\ln Y_{h,t}$ :

$$\ln Y_{h,t}^* = \alpha_h + \lambda \ln Y_{h,t} + \varphi_{h,t} \quad (3.12)$$

where  $\alpha_h$  is a household-specific fixed effect for reporting error,  $\varphi_{h,t}$  is white noise with variance  $\sigma_m^2$ , and  $0 < \lambda < 1$ . Substituting this into our model of the income process, observed family income is:

$$\ln(Y_{h,t}^*) = \alpha_h + X'_{h,t} \vartheta_t \lambda + u_{h,t} \lambda + \varphi_{h,t} \quad (3.13)$$

Squared residuals from this equation will be consistent estimates of  $\lambda^2 \sigma_{\Delta u,t}^2 + \sigma_m^2$ , rather than  $\sigma_{\Delta u,t}^2$ . We compute income volatility as  $\frac{(\Delta \hat{u}_{h,t})^2}{\hat{\lambda}^2}$ , where  $\hat{\lambda} = 0.67$  as estimated by [Bound et al. \[1994\]](#). Notice that dividing observed income volatility by  $0.67^2$  more than doubles the level of volatility and its change over time. The presence of  $\sigma_m^2$  biases our estimate of the level of income volatility, but as long as this variance does not vary over time, it does not bias our estimate of the trend.<sup>20</sup>

Figure 3.2 illustrates average household income volatility over the period 1980-2004 in deviations from its 1980 mean and after the effect of change in the survey methodology (i.e. the dummy for  $year > 1992$ ) is corrected for, and its linear trend. Volatility of family income increased significantly between 1980 and 2004 for an average household. This finding is consistent with the most recent study by [DeBacker et al. \[2013\]](#), who use a confidential panel of tax returns from the IRS to show that family income volatility increased between 1987 and 2009, as well as earlier studies ([Dynan et al. \[2012\]](#), [Keys \[2008\]](#), [Gottschalk and Moffitt \[2009\]](#), and [Gorbachev \[2011\]](#)), based on PSID data, who find that household income volatility increased between 1970 and 2006.

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<sup>20</sup>In 1993, the PSID converted the questionnaire to electronic form. [Kim and Stafford \[2000\]](#) describe the changes PSID underwent. We therefore allow  $\sigma_m^2$  to be different before and after the change in the survey. We remove the change in the variance of measurement error, by regressing income volatility on a time trend and a dummy for  $year > 1992$ , and subtracting out the effect of the dummy.

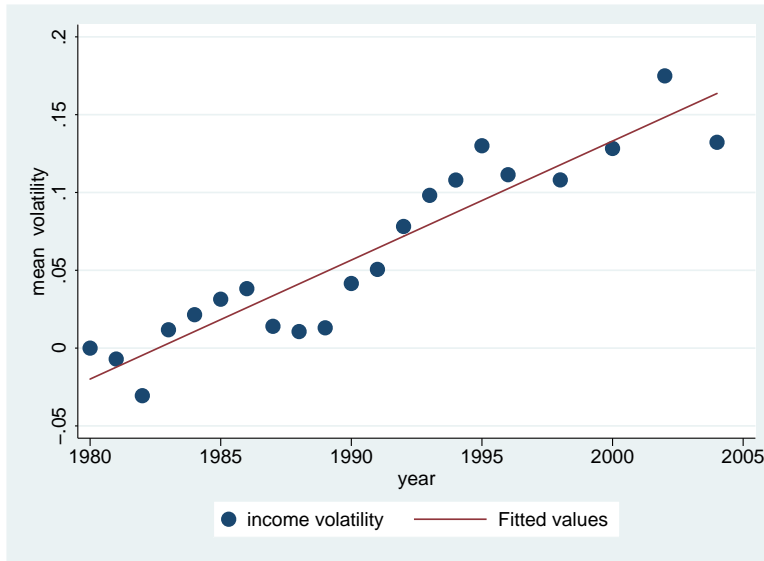


Figure 3.2: Mean Volatility of Household Income Shocks: deviations from 1980 mean.

Note: as  $(\Delta \hat{u}_{h,t})^2 / \hat{\lambda}^2$ , where  $\hat{u}_{h,t}^2$  is the squared residual from the family income regression (3.13) and  $\hat{\lambda}^2 = 0.67^2$  is the mean reversion correction as described in the text. Volatility is presented in deviations from the 1980 mean to correct for the presence of the measurement error,  $\sigma_m^2$ , and after the effect of  $year > 1992$  dummy is taken out, to correct for the change in the PSID survey methodology.

Source: Panel Study of Income Dynamics.

### 3.2.2.2 Net Wealth

To measure cash on hand, we use information on households' non-housing wealth. Information on wealth holdings in PSID is available for 1984, 1989, 1994, 1999, and biennially thereafter. To fill in for the missing years and to reduce mis-measurement, we estimate the probability that the household had positive net non-housing wealth,  $Pr(W_{h,t} > 0)$ , based on other variables available in all years.<sup>21</sup> We then predict the probability of having positive non-housing net wealth for this and 4 previous years, and use these predicted values as our proxy for cash on hand,  $Pr(\widehat{W}_{h,t} > 0)$ .

## 3.3 Estimating Consumption Volatility

We are now ready to estimate our Euler equation (3.7). Due to presence of second and higher order terms in the residual, it is typical to estimate the Euler equation using instrumental variable techniques or GMM. By rational expectations, any variables known at time  $t - 2$  will be

<sup>21</sup>See Appendix C.3 for details.

orthogonal to the expectational errors. However, since the second-differenced preference shocks may not be orthogonal to time  $t - 2$  variables, we use time  $t - s$  variables as instruments, where  $s \geq 4$ .

If  $X_{h,t-s}$  is our set of instruments, then our identifying assumptions are

$$\mathbb{E} \left[ \zeta_{h,t} \middle| X_{h,t-s} \right] = 0 \quad (3.14)$$

$$\mathbb{E} \left[ \ln(1 + \lambda_{h,t-2}) - F(\widehat{\text{Pr}(\text{denied credit})}_{h,t-2}) \middle| X_{h,t-s} \right] = 0 \quad (3.15)$$

$$\mathbb{E} \left[ \sigma_{h,t-2}^2 - \hat{\gamma}_h - \gamma_1(\hat{\sigma}_{h,t}^Y)^2 - \gamma_2 \widehat{\text{Pr}(W_{h,t} > 0)} \middle| X_{h,t-s} \right] = 0 \quad (3.16)$$

Restrictions (3.15) and (3.16) are necessary because we use proxy variables to estimate the effects of  $\ln(1 + \lambda_{h,t-2})$  and  $\sigma_{h,t-2}^2$ , which are unobserved, on the growth rate of consumption. Using proxy variables introduces approximation errors. The consistency of our estimates requires that the instruments we choose are orthogonal to these approximation errors. In practice, we may under- or over-predict the Lagrange multiplier or  $\sigma_{h,t-2}^2$ ; what is crucial is that this error is not correlated with the characteristics of the household  $s$  years ago.

Combining household fixed effects into a single term,  $\kappa_h = \frac{\beta}{\gamma}(\gamma_h - 2\delta_h - R_h)$ , the equation we estimate is:

$$\begin{aligned} \Delta \ln F_{h,t} = & \kappa_h + \frac{\beta}{\gamma} \ln(1 + r_{h,t}) + \alpha_1 \Delta \ln p_t^F + \alpha_2 \Delta \ln p_t^O \\ & + \frac{\beta}{\gamma} [\eta_W \Delta N_{h,t}^W + \eta_H \Delta N_{h,t}^H] + \mu \Delta Z_{h,t} \\ & + \frac{\beta}{\gamma} [F(\widehat{\text{Pr}(\text{denied credit})}_{h,t-2}) + \gamma_1(\hat{\sigma}_{h,t}^Y)^2 + \gamma_2 \widehat{\text{Pr}(W_{h,t} > 0)}] + \zeta_{h,t} \end{aligned} \quad (3.17)$$

The standard fixed effects estimator is inconsistent in a dynamic panel data model (Nickell [1981]). We therefore estimate (3.17) using the Arellano and Bover [1995] two-step system GMM estimator, after we removed fixed effects with forward orthogonal transformations. The forward orthogonal transformation subtracts the average of all future available observations of a variable, thus preserving the length of the sample.<sup>22</sup>

The observable variables affecting preferences,  $\Delta Z_{h,t}$ , include age, age squared, change in number of adults, change in number of children, change in marital status, and an indicator

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<sup>22</sup>If  $w_{h,t}$  is a variable, its forward orthogonal transform is  $\sqrt{\frac{T_{h,t}}{T_{h,t+1}}} (w_{h,t} - \frac{1}{T_{h,t}} \sum_{s>t} w_{h,s})$ .

variable for change in home ownership.<sup>23</sup> After testing for autocorrelation in the residuals we find that it is present up to the third lag. We therefore use variables dated  $t - 4$  and  $t - 5$  as instruments.<sup>24</sup> We limit the number of instruments to two lags and “collapse” our instruments to a single column to reduce the efficiency loss caused by too many instruments.<sup>25</sup> We allow for heteroskedasticity and intra-group correlation, and make the Windmeijer [2005] finite-sample correction to our standard errors.

Table 3.2 reports our estimation results. In column (1), we report results from our basic specification of the Euler equation in which we assume separable preferences between food, other non-durables, labor supply and housing. In Columns (2) to (4), we progressively relax these assumptions. Column (2) allows for non-separable preferences for food, other non-durables and labor supply, by including prices and the change in hours worked by both the spouse and head of the household.<sup>26</sup> In column (3), we also allow for non-separable preferences over housing by including the indicator variable for change in homeownership. In column (4) we add the probability that the household has positive wealth as a proxy for cash on hand in the precautionary savings term.

We report the results of the Hansen and Sargan tests for overidentifying restrictions. Unlike the Sargan test, the Hansen J test is robust to non-spherical errors but can be weakened by too many instruments. In all cases, we fail to reject the hypothesis that the overidentifying restrictions are valid. In addition, our set of explanatory variables is highly statistically significant in all specifications, according to the joint significance test.

In all specifications the coefficient on the interest rate is statistically significant at the 5 percent

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<sup>23</sup>This variable equals 1 when the household goes from renting to owning, equals 2 when the household moves from public housing to owning, is negative if the direction is reversed, and equals zero when there is no change.

<sup>24</sup>One might wonder how we can use variables dated  $t - 4$  and  $t - 5$  as instruments for biennial changes in consumption. For years before 1997, we generally have annual data on consumption, so we use overlapping years of data (i.e. we estimate two year changes between 1980 and 1982, between 1981 and 1983, and so forth). For years after 1997, we only have biennial data, and so variables dated  $t - 5$  are not available as instruments. It is still possible to estimate for these years, since our GMM estimator (via Stata’s `xtabond2` command) replaces missing values for instruments with zeros. (See Roodman [2009b] for details about the `xtabond2` estimator.)

<sup>25</sup>Roodman [2009a] describes the problems too many instruments could cause this type of GMM estimator.

<sup>26</sup>As discussed above, while hours worked are endogenous - in particular, because household members adjust labor supply to insure against shocks (Blundell et al. [2012]) - we deal with this endogeneity problem by using variables known at date  $t - 4$  or earlier as instruments. Our identifying assumption is that unexpected shocks to consumption growth are not correlated with predictable changes in hours worked. This is true under our maintained assumption that households have rational expectations.

Table 3.2: Euler Equation Estimation, biennial sample, 1980 to 2004.

	(1)	(2)	(3)	(4)
$\ln(1 + r_{h,t})$	0.570**	0.609**	0.704**	0.513*
	(0.266)	(0.257)	(0.289)	(0.293)
$(\hat{\sigma}_{h,t}^Y)^2$	0.054	0.039	0.093	0.069
	(0.076)	(0.051)	(0.062)	(0.063)
$\widehat{\text{Pr}}(\text{denied credit})$	2.389	2.145*	2.741*	1.467
	(1.769)	(1.277)	(1.642)	(1.597)
$\widehat{\text{Pr}}(\text{denied credit})^2$	-9.671*	-8.282*	-12.654*	-7.708
	(5.825)	(4.774)	(7.077)	(6.749)
$\widehat{\text{Pr}}(\text{denied credit})^3$	10.818*	9.243*	14.837*	9.938
	(6.367)	(5.473)	(8.045)	(7.706)
$\Delta \ln p^O$		0.266	0.208	0.397*
		(0.197)	(0.211)	(0.236)
$\Delta \ln p^F$		-0.334	-0.244	-0.890
		(0.458)	(0.484)	(0.640)
Age	-0.008	-0.008	-0.007	-0.010
	(0.009)	(0.006)	(0.007)	(0.013)
Age <sup>2</sup>	0.000	0.000	0.000	0.000
	(0.000)	(0.000)	(0.000)	(0.000)
Change in number of adults	0.127**	0.104**	0.065	0.114*
	(0.055)	(0.051)	(0.064)	(0.066)
Change in number of kids	0.052	0.066	0.108**	0.097
	(0.049)	(0.044)	(0.046)	(0.065)
Change in marital status	0.059	0.045	0.160	0.048
	(0.178)	(0.135)	(0.162)	(0.152)
Change in house ownership			-0.007	-0.001
			(0.101)	(0.109)
Change in number of hours worked, spouse		0.015*	0.018**	0.016
		(0.008)	(0.009)	(0.010)
Change in number of hours worked, head		-0.008	-0.025	0.023
		(0.045)	(0.047)	(0.050)
$\widehat{\text{Pr}}(W_h > 0)$				0.221
				(0.284)
Number of observations	34,002	34,002	34,002	30,524
Number of households	5,514	5,514	5,514	5,102
Number of Instruments	24	36	30	32
F-stat	32.13	30.14	25.31	20.72
Prob>F	0	0	0	0
Sargan test of overid	16.12	21.20	18.13	15.56
df	14	22	15	16
Prob> $\chi^2$	0.306	0.710	0.478	0.484
Hansen test of overid	13.06	17.93	14.63	13.49
df	14	22	15	16
Prob> $\chi^2$	0.522	0.508	0.256	0.637

Robust standard errors in parentheses; \*\*\* p<0.01, \*\* p<0.05, \* p<0.1

Note: we instrument:  $\ln(1 + r_{h,t})$ ,  $\widehat{\text{Pr}}(\text{denied credit})$  and its polynomial,  $\widehat{\text{Pr}}(W_h > 0)$ ,  $(\hat{\sigma}_{h,t}^Y)^2$ ,  $\Delta \ln p^O$ ,  $\Delta \ln p^F$ , change in the number of adults, change in the number of kids, change in marital status, change in the number of hours worked by head and spouse, with  $t - 4$  and  $t - 5$  lags of these variables, time dummies, and marginal tax rates.

Authors' calculations based on PSID and SCF data as described in the text.

level. If we assume preferences over food are homothetic ( $\beta = 1$ ), we estimate the intertemporal elasticity of substitution,  $\frac{1}{\gamma}$ , to be between 0.57 and 0.7. If we allow preferences to be non-homothetic ( $\beta \neq 1$ ), because we believe that food is a necessity, then this coefficient cannot be interpreted as the intertemporal elasticity of substitution; rather, it equals the IES multiplied by the budget elasticity of food consumption with respect to total non-durable expenditure,  $\beta$ . Using Consumer Expenditure Survey data for 1980-1992 period, [Blundell et al. \[2008\]](#) estimated  $\hat{\beta} = 0.85$ , and found that the budget elasticity fell during that time period. We re-estimate [Blundell et al. \[2008\]](#)-type regressions on PSID data using the newly available information on non-durable expenditure for 2005-2009 data. We find  $\hat{\beta} = 0.78$ .<sup>27</sup> Using this estimate, our results imply an IES of between 0.73 and 0.9, which is in line with evidence from other studies using microeconomic data ([Attanasio \[1999\]](#)).

Consumption growth should be larger for liquidity constrained households and households experiencing higher income volatility, who have a higher precautionary saving motive. Our coefficients on the polynomial in  $\widehat{\text{Pr}(\text{denied credit})}$  imply a non-monotonic relationship between the probability of being constrained and consumption growth. The effect of  $(\hat{\sigma}_h^Y)^2$  is always positive, as predicted by the theory, though it is never statistically significant.<sup>28</sup>

### 3.3.1 Evolution of Consumption Volatility

To compute volatility of household consumption, we first predict residuals,  $\hat{\xi}_{h,t}$ , from the above Euler equation, using our preferred specification, [Table 3.2](#) column 3. We then subtract out household fixed effects  $\hat{\kappa}_h = \frac{\hat{\beta}}{\hat{\gamma}}(\hat{\gamma}_h - 2\hat{\delta}_h - \hat{R}_h)$ , that are not directly computed by our estimator. Our measure of consumption volatility is  $(\hat{\xi}_{h,t} - \hat{\kappa}_h)^2$ . Recall that our measure of consumption volatility contains other terms, such as variances of measurement error, second and higher order terms and second-differenced preference shocks, which we are not directly interested in com-

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<sup>27</sup>We use data on detailed consumption categories kindly provided to us by Geng Li of Federal Reserve Board. Details of our estimation are described in the Appendix, section [C.4](#).

<sup>28</sup>Since we reject a joint significance test of the polynomial terms in  $\widehat{\text{Pr}(\text{denied credit})}$  at all conventional levels of significance, we perform several robustness checks of our estimation. In particular, we estimate one specification excluding the liquidity constraints proxy, and estimate another including interaction terms between the polynomial in the probability of being denied credit and income volatility. In these alternative specifications, the income volatility and liquidity constraint terms are statistically significant, but our results are otherwise unchanged. Most importantly, our results on evolution of volatility of consumption, described in the next section, are not sensitive to these different specifications of the Euler equation. The results for these additional regressions are available upon request.

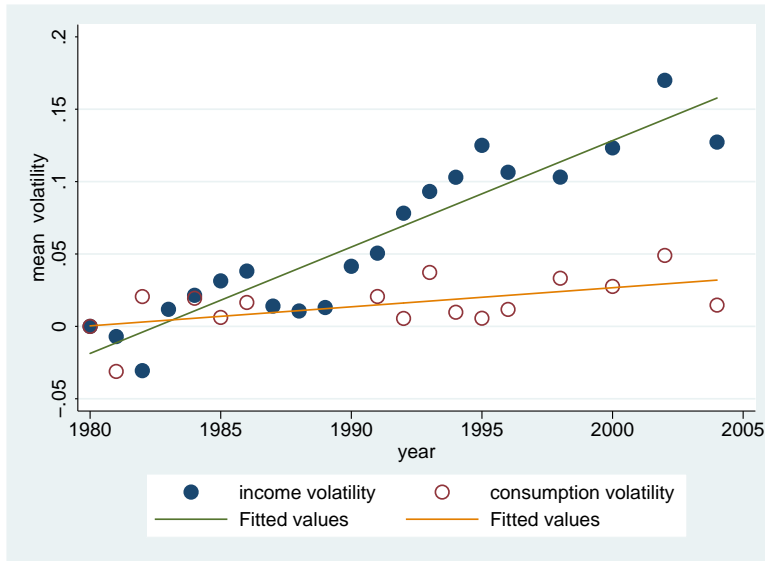


Figure 3.3: Mean income volatility and consumption volatility for 1980 to 2004

Note: Household income volatility is computed as  $(\Delta \hat{u}_{h,t})^2 / \hat{\lambda}^2$ , where  $\hat{u}_{h,t}^2$  is the squared residual from the family income regression (3.13) and  $\hat{\lambda}^2 = 0.67^2$  is the mean reversion correction as described in the text. Consumption volatility is computed as  $(\hat{\zeta}_{h,t} - \hat{\kappa}_h)^2$ , where  $\hat{\zeta}_{h,t}$  is the residual and  $\hat{\kappa}_h$  is the household fixed effect from the Euler equation (3.17). Volatility of consumption and of income are presented in deviations from their respective 1980 means to correct for the presence of measurement error, and after the effect of year > 1992 dummy was taken out, to correct for the change in the PSID survey methodology.  
Source: Panel Study of Income Dynamics.

puting, but cannot estimate separately. Since our goal is to measure *the change* in consumption volatility over time, this is not a problem, assuming these other terms do not vary over time.<sup>29</sup>

Figure 3.3 shows deviations from the 1980 mean for income and consumption volatilities between 1980 and 2004, and their respective linear trends.<sup>30</sup> Consumption volatility increased 3 volatility points, from an average of 14.6 in 1980-1984 to an average of 17.5 in 2000-2004, or by 19 percent. However, income volatility rose by 44 percent, or by 14 volatility points, over

<sup>29</sup>To allow for the possibility that the variance of measurement error in consumption also changed following the move to electronic surveys in 1992, we regress consumption volatility on a time trend and a dummy for year > 1992, and subtract the effect of the dummy from our estimate, following the strategy we used for correcting volatility of income, described earlier.

Moreover, our Euler residuals also contain the error  $u_t$ , the difference between the Lagrange multiplier and the polynomial in the predicted probability of being denied credit, equation (3.9). In principle, the variance of  $u_t$  could trend over time, if for example, our probit estimates become more or less accurate over time, biasing our estimates of the trend in consumption volatility. In practice, the variance of our probit errors (within the SCF sample) has only a moderate trend over time, increasing from 0.13 in 1983 to 0.14 in 2007. If we attempt to purge our consumption volatility estimates of these errors, the trend in volatility remains positive, significant, and of the same magnitude (results are available upon request).

<sup>30</sup>This figure is also corrected for the change in the survey methodology in 1992, as described above.

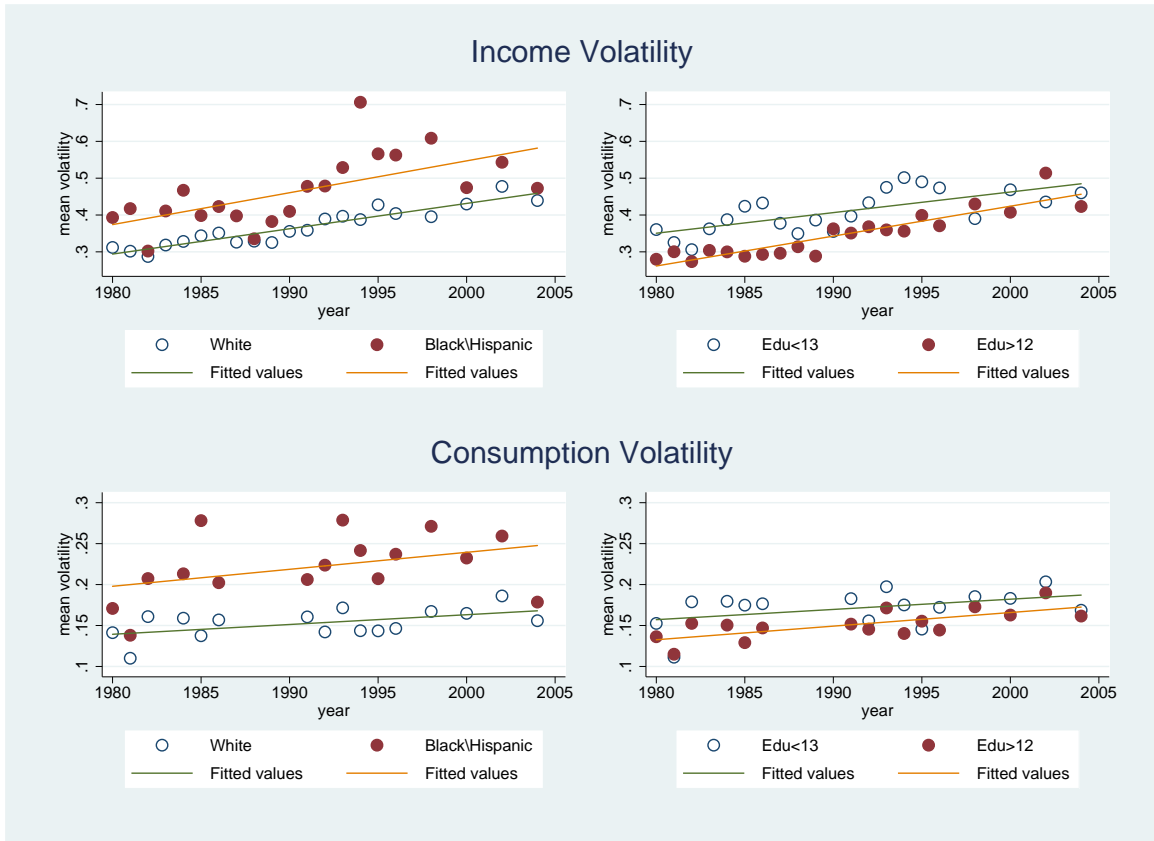


Figure 3.4: Mean income volatility and consumption volatility for 1980 to 2004

Note: Household income volatility is computed as  $(\Delta \hat{u}_{h,t})^2 / \hat{\lambda}^2$ , where  $\hat{u}_{h,t}^2$  is the squared residual from the family income regression (3.13) and  $\hat{\lambda}^2 = 0.67^2$  is the mean reversion correction as described in the text. Consumption volatility is computed as  $(\hat{\zeta}_{h,t} - \hat{\kappa}_h)^2$ , where  $\hat{\zeta}_{h,t}$  is the residual and  $\hat{\kappa}_h$  is the household fixed effect from the Euler equation (3.17).

Source: Panel Study of Income Dynamics.

the same period. We should note that our estimate of the percentage point change in income volatility is highly sensitive to the mean-reversion correction,  $\hat{\lambda}^2 = 0.67^2$ , though the estimate of the *percentage change* in volatility of income is not affected by this correction. However, since measurement error causes us to overestimate the level of consumption and income volatilities, it biases downwards our estimates of their percentage change. Our estimate of a 19 percent increase in volatility of food consumption is therefore a lower bound.

Figure 3.4 shows consumption and income volatility for particular demographic groups. The levels of consumption and income volatility were around 7 and 10 volatility points higher, respectively, for black or Hispanic households relative to white households. Consistent with previ-

Table 3.3: Evolution of Food Consumption and Income Volatility, biennial sample, 1980 to 2004.

	(1)	(2)	(3)	(4)	(5)	(6)
	food	income	food	income	food	income
Year/1000	1.464*** (0.509)	6.971*** (1.490)	1.585*** (0.507)	7.419*** (1.494)	1.652*** (0.566)	8.510*** (1.742)
Year > 1992	-0.007 (0.007)	0.060*** (0.022)	-0.006 (0.007)	0.063*** (0.022)	-0.006 (0.007)	0.063*** (0.022)
Black/Hispanic			0.067*** (0.010)	0.095*** (0.027)	-2.276 (2.206)	-4.030 (6.038)
Education < 13			0.015*** (0.005)	0.046*** (0.016)	0.814 (1.215)	5.773* (3.488)
Black/Hispanic × year/1000					1.176 (1.107)	2.070 (3.035)
Education < 13 × year/1000					-0.401 (0.610)	-2.875 (1.753)
Age			-0.004** (0.002)	-0.028*** (0.006)	-0.004* (0.002)	-0.028*** (0.006)
Age <sup>2</sup>			0.000* (0.000)	0.000*** (0.000)	0.000* (0.000)	0.000*** (0.000)
Constant	-2.755*** (1.010)	-13.507*** (2.958)	-2.921*** (1.007)	-13.840*** (2.958)	-3.054*** (1.124)	-16.015*** (3.456)
Number of observations	33,652	33,652	33,652	33,652	33,652	33,652
Adj. R <sup>2</sup>	0.001	0.006	0.005	0.008	0.005	0.008

Robust, clustered at household level, standard errors in parentheses;

\*\*\* p < 0.01, \*\* p < 0.05, \* p < 0.1

Note: in columns (3) to (6) other controls, not shown to conserve space, include change in marital status, change in the number of kids and the number of adults.

Source: Authors' calculations based on PSID and SCF data as described in the text.

ous studies, income volatility was higher for less educated than for more educated households, though the trends in income volatility for these two groups have been converging over the sample period. Consumption volatility disaggregated by education, exhibited similar trends to those of income volatility, though the differences were not as pronounced.

To test whether the increase in volatility and the differences in the levels of volatility across households are statistically significant, we regress our squared Euler equation residuals on a time trend, the change in survey methodology correction term (year > 1992 dummy), and demographic controls. For comparison, we run the same regressions using income volatility. Table 3.3 reports these results. Columns (1) and (2) provide results from a regression on a linear time trend. Volatility of consumption increased by 1.5 points every 10 years, or by 3.5 points between 1980 and 2004. In contrast, volatility of income rose by 7 points every 10 years, or 17 percentage points between 1980 and 2004. Both trends are statistically significant at the 1 percent level. In columns (3) and (4) we allow for differential levels, and in columns (4) and (5) differential levels and trends in volatility by race and education, and a quadratic polynomial in age. These results confirm that the differences between demographic groups shown in Figure 3.4 are statistically significant: consumption volatility is 7 points higher for Black and Hispanic households than for

white households, and is 1.5 points higher for households whose head had less than 13 years of education. Unlike Gorbachev [2011], we do not find a statistically significant difference in the trends of income and consumption volatility for white and black or Hispanic households. We find support for previous findings that income volatility is U-shaped: it is high at a young age, falls during the mid-years, and rises again later in life. Consumption volatility follows a similar pattern. Nevertheless, controlling for age does not reduce the magnitude of the increase in income or consumption volatility, indicating that the increase in volatility is not explained by the ageing of our sample or by its life-cycle properties. In regressions not shown, but available in the Appendix, Table C.4, we also control for changes in marital status, and changes in the number of adults and children in the household. We find that marital status changes have a significant and positive impact on volatility of consumption and income (i.e. they increase volatility). However, changes in the size of the household appear to be unimportant. In addition, inclusion of cohort and state fixed effects, to account for compositional changes and for differential levels of volatility that are cohort and/or geography specific, do not change our main results.

Next we investigate the relation between liquidity constraints, income volatility and consumption volatility. Table 3.4 presents these results. Columns (1) and (2) illustrate the individual effects of income volatility or liquidity constraints, respectively, on volatility of consumption; column (3) includes both variables, and column (4) adds an interaction between them. Each of these variables, volatility of income and our proxy for liquidity constraints, has a positive and significant impact on consumption volatility. The interaction term has a negative sign, possibly due to a nonlinear relationship between income and consumption volatility.<sup>31</sup>

We find that the trend in consumption volatility is completely explained by the trend in liquidity constraints. In particular, when we control for income uncertainty, but not liquidity constraints, the time trend remains statistically significant and of similar magnitude as when income uncertainty was not accounted for (see Table 3 column 3 for comparison). Once we include our proxy for liquidity constraints, the significance of the time trend disappears and the magnitude of the coefficient is significantly reduced. This suggests that if the liquidity constraints weren't binding, changes in income uncertainty would not have been as costly to the households. Stated

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<sup>31</sup>These results are robust to including a third-order polynomial in the probability of being denied credit instead of a linear term, for consistency with our Euler equation. Results are available upon request.

Table 3.4: Effect of Liquidity Constraints and Income Uncertainty on Volatility of Consumption, biennial sample, 1980 to 2004.

	(1)	(2)	(3)	(4)	(5)	(6)
	OLS	OLS	OLS	OLS	IV	IV
Year/1000	1.145** (0.496)	0.532 (0.512)	0.323 (0.504)	0.266 (0.501)	0.259 (0.498)	0.159 (0.507)
Year > 1992	-0.010 (0.007)	-0.007 (0.007)	-0.010 (0.007)	-0.010 (0.007)	-0.011 (0.007)	-0.011 (0.007)
$(\hat{\sigma}_{h,t}^Y)^2$	0.052*** (0.004)		0.050*** (0.004)	0.062*** (0.008)	0.071*** (0.009)	0.088*** (0.019)
$\Pr(\widehat{\text{denied credit}}_{h,t-2})$		0.221*** (0.024)	0.186*** (0.024)	0.216*** (0.025)	0.167*** (0.050)	0.220*** (0.066)
$\Pr(\widehat{\text{denied credit}}_{h,t-2}) \times (\sigma_{h,t}^Y)^2$				-0.051** (0.024)		-0.081 (0.077)
Black/Hispanic	0.061*** (0.010)	0.024** (0.011)	0.026** (0.011)	0.025** (0.011)	0.028** (0.012)	0.025** (0.012)
Education < 13	0.012** (0.005)	-0.000 (0.005)	-0.000 (0.005)	-0.000 (0.005)	0.000 (0.005)	-0.000 (0.005)
Age	-0.002 (0.002)	0.000 (0.002)	0.001*** (0.000)	0.001*** (0.000)	0.001** (0.000)	0.001*** (0.000)
Change in Marital Status	0.092*** (0.011)	0.096*** (0.012)	0.085*** (0.011)	0.085*** (0.011)	0.081*** (0.011)	0.080*** (0.011)
Constant	-2.119** (0.985)	-0.987 (1.015)	-0.595 (0.999)	-0.488 (0.994)	-0.279 (0.999)	-1.310 (2.613)
Number of Observations	33,652	33,652	33,652	33,652	33,652	33,652
Adj. R <sup>2</sup>	0.036	0.015	0.040	0.040	0.035	0.036
Number of excluded instruments					3	3
Kleibergen-Paap rk LM statistic					135.4	69.97
Prob > $\chi^2$					0	0
weak id Kleibergen-Paap rk Wald F statistic					167.7	29.38
Hansen J statistic					1.147	-
p-value					0.284	-

Robust, clustered at household level, standard errors in parentheses;

\*\*\* p<0.01, \*\* p<0.05, \* p<0.1

Note: In columns (5) and (6) we instrument  $\Pr(\widehat{\text{denied credit}})$  with cohort-year-state averages of  $\Pr(\widehat{\text{denied credit}})$  and  $(\hat{\sigma}_{h,t}^Y)^2$  with cohort-year-industry averages; and their interaction with the cohort-year-state averages of their interaction.

Source: Authors' calculations based on PSID and SCF data as described in the text.

differently, consumption volatility did not increase because households were hit with many positive income shocks, but because liquidity constraints played a crucial role in propagating income shocks. The volatility increase had a significantly negative impact on household welfare.

To deal with potential mis-measurement of true liquidity constraints and income uncertainty, we instrument for income volatility and the probability of being denied credit. Table 3.4 columns (5) and (6) report these results. To instrument for income volatility we construct industry-time-cohort specific averages of income volatility. Although the choice of industry in itself is endogenous, the level of volatility for a specific year-cohort pair within an industry will be a good indicator of the level of volatility experienced by individuals working in that sector, and will

be subject to less variation than an individual specific measure, alleviating the problem of mis-measurement. To instrument for liquidity constraints, we construct state-time-cohort averages of the probability of being denied credit. This instrument will pick up state level lending differences as well as differences within state and over time for a specific cohort of individuals. We interact these averaged effects to instrument for the interaction of liquidity constraint and income volatility term. In column (5) we drop the interaction term and use all three instruments, and in column (6) we also instrument the interaction term. In both cases, the effect of income volatility and the probability of being denied credit remains positive, and of a similar size as in our OLS specification (columns (3) and (4)), although the effect of income volatility is slightly larger. When instrumenting, we lose the significance of the interaction term. In both specifications, the tests of the validity of our instruments are passed with high statistical significance. Moreover, our instruments do not suffer from the problems caused by weak identification, as the F-statistics are large for both regressions, especially for the overidentified case in column (5). These results remain unchanged with the inclusion of state and cohort fixed effects.

Households with higher transitory income volatility will find it harder to smooth consumption, and will be more likely to be denied credit. Thus although households who are more likely to be denied credit have higher consumption volatility, this might not be because they have less access to credit; instead, it might be because these households might have higher transitory income volatility. To address this possibility, we exploit the well known fact that individuals with less education face more volatile transitory income shocks, and less volatile permanent shocks (Gundersen and Ziliak [2008]). If we find that splitting the sample by education reduces the effect of liquidity constraints on consumption volatility, that would suggest that what we call an “access to credit” result merely reflects differences in transitory volatility. We do not find this to be true.<sup>32</sup> The coefficient on the proxy for liquidity constraints is larger for the less educated than for the better educated group, 0.24 vs. 0.18, but this difference is statistically insignificant. More importantly, the average effect of liquidity constraints is similar to the effect in our whole sample. This suggests that the endogeneity problem described here is not of great concern in practice.

Our results indicate that if we took an average household in the 25th percentile of the

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<sup>32</sup>These results are available in Appendix Table C.5.

$\widehat{\text{Pr}}(\text{denied credit})$  distribution, with a 9 percent probability of being liquidity constrained, and moved this household to the 75th percentile with a 31 percent probability of being constrained, while holding the size of their income shock constant, we would raise their consumption volatility by 5 points, or around 40 percent.

These results are important given the substantial inequality in households' access to credit. In our SCF sample, around 40 percent of households with a black or Hispanic head of household were liquidity constrained, compared to 20 percent of households headed by a white individual; around 30 percent of households whose head had less than 13 years of education were constrained, compared to 20 percent of households with at least 13 years of education. There is no evidence that these groups are more likely to apply for loans. In fact, according to SCF data, black individuals and less educated individuals were less likely to apply for loans than white or highly educated individuals between 1995 and 2007 (although they were significantly more likely to be discouraged from applying because they thought they would be denied). It remains unclear whether lenders are less willing to extend credit to these households because they have more volatile incomes and a higher risk of default, or because of other reasons, such as discrimination.

Wealth inequality can also explain differences in consumption volatility between households. According to SCF data, asset poverty (defined as households with net assets less than two months' income) among blacks and Hispanics fell by a quarter between 1983 and 2007, nevertheless, blacks and Hispanics remain twice as likely to be asset poor as whites. College educated households are significantly less likely to be asset poor. Although we do not attempt to identify the causal affect of asset poverty on consumption volatility, it seems likely that wealth inequality, in addition to inequalities in access to credit markets, contributed to disparities in households' ability to smooth consumption.<sup>33</sup>

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<sup>33</sup>Another partial insurance mechanism, which we do not explore in this paper, is risk-sharing within family networks. Our measure of income volatility is post-transfers (both private and public) but pre-tax. Thus, the income volatility process is smoother than it would have been if transfers were excluded. A proper investigation into the effect of household risk sharing on consumption volatility is left for future research.

### 3.4 Conclusion

The volatility of US household income increased by 44 percent between 1980 and 2004. Households were not able to completely smooth consumption in response to their increasingly volatile income shocks: over the same period, the volatility of unpredictable changes in household consumption increased by around 19 percent. A major factor limiting households' ability to smooth was limited access to credit. In fact, we find that if the liquidity constraints were not binding, the increase in income volatility would not have been enough to explain the increase in consumption volatility during this period. Between 1983 and 2007, around 1 in 5 households were denied a loan or were discouraged from applying for a loan in the past 5 years, and the proportion of liquidity constrained households increased slightly over time. While financial sector innovations such as credit scoring and risk-based pricing may have increased lenders' willingness to lend, it seems that households' demand for credit has increased by the same amount, so that the fraction of households unable to borrow as much as they would like remained unchanged. Differences in households' net wealth and access to credit led to significant differences in their ability to smooth consumption.

The increase in the volatility of household consumption has a significant welfare cost. Accurately estimating this welfare cost is beyond the scope of this paper. To show that the welfare cost is likely to be substantial, we use Lucas [1987]'s formula for the cost of consumption fluctuations to obtain a rough estimate. Lucas [1987] considered a representative consumer with isoelastic, time separable utility, and assumed that log consumption is normally distributed with variance  $\sigma_c^2$  around a linear trend. Under these assumptions, eliminating all consumption volatility would provide the same welfare benefit as increasing annual consumption by  $\mu = \frac{1}{2}\gamma\sigma_c^2$  percent. Unlike Lucas, we are considering the variance of unpredictable changes in household consumption, not the variance of deviations of aggregate consumption from a trend.<sup>34</sup> Since the elasticity of food consumption with respect to total non-durable consumption is  $\beta$ , shocks to food consumption growth will be  $\beta$  times as large as shocks to total consumption growth, and the variance of shocks to food consumption growth (our measure of consumption volatility) will be  $\beta^2$  times as

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<sup>34</sup>In our model consumption is close to a random walk with drift, rather than trend-stationary. Reis [2009] shows that this change in assumptions can increase the welfare cost of fluctuations by an order of 50. Our welfare cost estimate should therefore be considered a lower bound. Our results are of the same order of magnitude as Hai et al. [2013], who calculate the welfare cost of fluctuations in a structural model with memorable goods, fitted to CEX data.

large as the variance of shocks to total consumption growth. We therefore divide our measure of food consumption volatility by  $\beta^2$ , using our estimate of  $\hat{\beta} = 0.78$ . In our Euler equations, we estimate  $\hat{\beta}/\hat{\gamma} = 0.6$ ; we therefore take  $\hat{\gamma} = 0.78/0.6 = 1.3$ . Under these assumptions, reducing consumption volatility by 2.8 points from its 2000-2004 level to its 1980-1984 level would produce the same welfare gain as increasing annual non-durable consumption by 3 percent.<sup>35</sup> While this number is only a back of the envelope estimate, and is sensitive to assumptions, it is clear that the rise in consumption volatility is of first order importance for household welfare.

Similarly, since disadvantaged groups face higher levels of consumption volatility, driven in part by differences in access to credit, decreasing their consumption volatility would have a clear welfare benefit. Households headed by a black or Hispanic individual have on average 7 points higher consumption volatility than whites. Reducing consumption volatility for the average black or Hispanic household to the level experienced by the average white household would provide the same welfare gain as increasing annual consumption by around 7.3 percent. The difference between consumption volatility for the quarter of households most likely to be liquidity constrained and the quarter of households least likely to be constrained is of the same magnitude. This suggests that improving access to credit for disadvantaged groups, or providing them with other ways to smooth consumption, could significantly improve household welfare.

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<sup>35</sup>We compute welfare cost as  $\frac{1}{2}\hat{\gamma}(\hat{\sigma}_{c,2000-2004}^2 - \hat{\sigma}_{c,1980-1984}^2) = \frac{1}{2}\hat{\gamma}\frac{(\hat{\sigma}_{f,2000-2004}^2 - \hat{\sigma}_{f,1980-1984}^2)}{\hat{\beta}^2} = \frac{1}{2} * 1.3 * \frac{0.028}{0.78^2} = 0.03$ .

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## Appendix A

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### Appendix to Chapter 1

#### A.1 New Keynesian model

In this section, I present a relatively standard New Keynesian model and show that equilibria in this economy are isomorphic to ZLB-constrained equilibria in the limit as prices become perfectly sticky.

##### A.1.1 Households

Households  $i = S, B$  have period utility functions  $\tilde{U}(C_0^i, h_0^i, \theta_i)$  (at date 0) and  $\tilde{u}(C_t^i, h_t^i)$  (in subsequent periods), where  $C_t^i$  is consumption and  $h_t^i$  is hours worked. I will be interested in a special case where  $\tilde{U}(C, h) = U(C - v(h), \theta)$ ,  $\tilde{u}(C, h) = u(C - v(h))$  and we define  $c = C - v(h)$  to be net consumption. Each household's real income, excluding transfers, is  $\frac{W_t}{P_t} h_t^i + \pi_t - T_t$  where  $W_t$  is the nominal wage,  $P_t$  is a price index defined shortly, and  $\pi_t - T_t$  is total real profits from the monopolistically competitive firms, net of the lump sum transfer used to finance subsidies to the firms. Households trade a nominal bond:  $d_t^i$  is the nominal face value of debt which household  $i$  promises to repay in period  $t$ , and  $1 + i_t$  is the nominal interest rate between periods  $t$  and  $t + 1$ . (Since I consider perfect foresight equilibria in the baseline model, allowing households to trade a real bond would make no difference.) Household  $i$  solves:

$$\max U_0(C_0^i, h_0^i, \theta_i) + \sum_{t=1}^{\infty} \beta^t \tilde{U}(C_t^i, h_t^i) \quad (\text{A.1})$$

$$\frac{d_{t+1}^i}{1 + i_t} = d_t^i + P_t C_t^i - T_t - \Pi_t - W_t h_t^i$$

$$d_0^B = d_0^S = 0$$

$$d_{t+1}^i \leq P_{t+1} \phi_t, t = 1, \dots$$

where  $\tilde{U}, \tilde{u}$  are strictly concave, strictly increasing in  $C$  and decreasing in  $h$ , and satisfy  $\tilde{U}_{ch} > 0$ . Consumption  $C_t^i$  is a Dixit-Stiglitz aggregate:

$$C_t^i = \left[ \int_0^1 C_t^i(j)^{(\varepsilon-1)/\varepsilon} dj \right]^{\varepsilon/(\varepsilon-1)} \quad (\text{A.2})$$

with corresponding price index

$$P_t = \left[ \int_0^1 p_t(j)^{1-\varepsilon} dj \right]^{1/(1-\varepsilon)} \quad (\text{A.3})$$

Households' demand for variety  $j$  is given by

$$C_t^i(j) = C_t^i \left( \frac{p_t(j)}{P_t} \right)^{-\varepsilon} \quad (\text{A.4})$$

Real interest rates are defined by the Fisher equation:

$$(1 + i_t) = (1 + r_t) \frac{P_{t+1}}{P_t} \quad (\text{A.5})$$

Households' labor supply decision satisfies

$$\tilde{u}_c + \frac{W_t}{P_t} \tilde{u}_h = 0,$$

replacing  $\tilde{u}$  with  $\tilde{U}$  if  $t = 0$ .

## A.1.2 Firms

There is a continuum of monopolistically competitive firms indexed by  $j \in [0, 1]$  who hire labor and produce output using the linear technology  $y_t(j) = h_t(j)$ . They receive an employment subsidy  $\tau = 1/\varepsilon$ . In each period  $t$ , a fraction  $\alpha \in [0, 1]$  of firms are unable to change their price, while  $1 - \alpha$  can change their price. The probability of being able to change prices is independent of the firm's current price and the date on which it last adjusted prices. All firms who cannot set prices in period 0 have the same price,  $P_{-1}$ . I assume firms discount profits at the riskless nominal interest rate  $i_t$ .

Firms who can set their price in period  $t$  solve

$$\max_{p_t(j)} \sum_{s=t}^{\infty} \alpha^{s-t} Q_{t,s} (p_t(j) - W_s(1 - \tau)) Y_s \left( \frac{p_t(j)}{P_s} \right)^{-\varepsilon} \quad (\text{A.6})$$

where  $Q_{t,s} = \left( \prod_{k=t}^{s-1} \frac{1}{1+i_k} \right)$ . The firm's first order condition yields

$$p_t(j) = \frac{\sum_{s=t}^{\infty} \alpha^{s-t} Q_{t,s} Y_s \left( \frac{P_s}{P_t} \right)^{\varepsilon} W_s}{\sum_{s=t}^{\infty} \alpha^{s-t} Q_{t,s} Y_s \left( \frac{P_s}{P_t} \right)^{\varepsilon}}$$

$$\frac{p_t(j)}{P_t} = \frac{\sum_{s=t}^{\infty} \alpha^{s-t} Q_{t,s} Y_s \left( \frac{P_s}{P_t} \right)^{\varepsilon+1} \frac{W_s}{P_s}}{\sum_{s=t}^{\infty} \alpha^{s-t} Q_{t,s} Y_s \left( \frac{P_s}{P_t} \right)^{\varepsilon}}$$

$$\frac{p_t(j)}{P_t} = \frac{K_t}{F_t}$$

where we can define  $K_t$  and  $F_t$  recursively as

$$K_t = Y_t \frac{W_t}{P_t} + \frac{\alpha}{1+i_t} \Pi_{t+1}^{\varepsilon+1} K_{t+1} \quad (\text{A.7})$$

$$F_t = Y_t + \frac{\alpha}{1+i_t} \Pi_{t+1}^{\varepsilon} F_{t+1} \quad (\text{A.8})$$

The aggregate price level evolves according to

$$P_t^{1-\varepsilon} = \alpha P_{t-1}^{1-\varepsilon} + (1-\alpha)(p_t^*)^{1-\varepsilon}$$

$$1 = \alpha \Pi_t^{\varepsilon-1} + (1-\alpha) \left( \frac{p_t^*}{P_t} \right)^{1-\varepsilon}$$

$$1 = \alpha \Pi_t^{\varepsilon-1} + (1-\alpha) \left( \frac{K_t}{F_t} \right)^{1-\varepsilon} \quad (\text{A.9})$$

### A.1.3 Monetary policy

The monetary authority sets interest rates according to a Taylor rule, modified to take account of the zero lower bound:

$$1+i_t = \max \left\{ (1+r_t^n) \Pi_t^{\phi_{\pi}}, 1 \right\} \quad (\text{A.10})$$

where  $\phi_{\pi} > 1$  and  $r_t^n$  is the natural rate of interest, defined as the equilibrium real interest rate in the economy with  $\alpha = 0$  (perfectly flexible prices).

### A.1.4 Market clearing

Goods and labor markets clear (which ensures that the asset market also clears, by Walras' Law):

$$\begin{aligned}
C_t^S + C_t^B &= 2Y_t \\
&= 2 \left[ \int_0^1 y_t(j)^{(\varepsilon-1)/\varepsilon} dj \right]^{\varepsilon/(\varepsilon-1)} \\
&= 2 \left[ \int_0^1 h_t(j)^{(\varepsilon-1)/\varepsilon} dj \right]^{\varepsilon/(\varepsilon-1)} \\
2 \int_0^1 h_t(j) dj &= h_t^S + h_t^B
\end{aligned}$$

We can combine these conditions as

$$C_t^S + C_t^B = 2Y_t = \frac{h_t^S + h_t^B}{\Delta_t} \quad (\text{A.11})$$

where we define the measure of price dispersion

$$\Delta_t = \int_0^1 \left( \frac{p_t(j)}{P_t} \right)^{-\varepsilon} dj \geq 1$$

which evolves according to

$$\Delta_t = (1 - \alpha) \left( \frac{K_t}{F_t} \right)^{-\varepsilon} + \alpha \Delta_{t-1} \Pi_t^\varepsilon \quad (\text{A.12})$$

with initial condition  $\Delta_{-1} = 1$ .

### A.1.5 Equilibrium and isomorphism to ZLB-constrained equilibrium

I define equilibrium in the standard way.

**Definition A.1.1.** *An equilibrium is a sequence  $\{C_t^S, C_t^B, h_t^S, h_t^B, d_t^S, d_t^B, \frac{W_t}{P_t}, T_t, Y_t, \Delta_t, \Pi_t, F_t, K_t\}_{t=0}^\infty$  such that:*

1.  $\{C_t^i, h_t^i, d_t^i\}$  solves household  $i$ 's problem (A.1), for  $i = S, B$
2.  $\{\Delta_t, \Pi_t, F_t, K_t\}$  satisfy (A.7), (A.8), (A.12), (A.9)
3. Interest rates  $\{i_t\}$  satisfy the modified Taylor rule (A.10)
4. The market clearing conditions (A.11) is satisfied.

I now present the main result of this Appendix, which states that equilibria of the New Keynesian model are isomorphic to ZLB-constrained equilibrium when prices are fixed and there is no wealth effect on labor supply.

**Proposition A.1.2.** *Suppose preferences have the form  $\tilde{U}(C, h) = U(C - v(h), \theta)$ ,  $\tilde{u}(C, h) = u(C - v(h))$ , and suppose  $\alpha = 1$ . Then every equilibrium of the New Keynesian model is isomorphic to a ZLB-constrained equilibrium with  $c_t^i = C_t^i - v(h_t^i)$ ,  $\Pi_t = 1$ ,  $y_t = h_t - v(h_t)$ ,  $y^* = h^* - v(h^*)$ , where  $v'(h^*) = 1$ , and  $i_t = r_t$ .*

*Proof.* When  $\alpha = 1$ , firms never change their prices, so  $\Pi_t = \Delta_t = 1, \forall t$ . Since there is no inflation, real interest rates equal nominal interest rates, and are given by

$$r_t = \max\{r_t^n, 0\}$$

Using the definition of net consumption and the fact that  $\Delta_t = 1$ , we can write the resource constraint

$$c_t^S + c_t^B = 2(h_t - v(h_t)) = 2y_t$$

Note that by definition of  $y^*$ , we have  $y_t \leq y^*$ .

Real wages satisfy  $\frac{W_t}{P_t} = v'(h_t)$ , where  $h^t = h_t^S = h_t^B$  is the number of hours supplied by each household. In an economy with  $\alpha = 0$ , we would have  $\frac{W_t}{P_t} = v'(h_t) = 1$ , i.e.  $h_t = h^*$  and  $y_t = y^*$ . We know that whenever  $r_t^n \geq 0$ ,  $r_t = r_t^n$  and  $y_t = y^*$ . It follows that  $r_t(y^* - y_t) = 0$ , as required.  $\square$

Given our assumption that the central bank follows a Taylor rule, equilibrium in the New Keynesian model would almost be isomorphic to the ZLB-constrained equilibrium even if  $\alpha < 1$ . Any equilibrium of the New Keynesian model with zero inflation ( $\Pi_t = 1, \forall t$ ) is isomorphic to a ZLB-constrained equilibrium. However, whenever the ZLB binds at date 1, and output falls below potential, there is deflation. This causes relative prices to become dispersed. Under Calvo pricing, this dispersion in relative prices is only eliminated in the limit as  $t \rightarrow \infty$ . For this reason, if nothing else, output is below potential even at date 2, and the economy does not immediately reach steady state:  $y_2 < y^*$ ,  $y_t \rightarrow y^*$  as  $t \rightarrow \infty$ .

Note also that with  $\alpha = 1$  and the quasilinear preferences considered here, this Taylor rule is, trivially, optimal monetary policy. Since prices never adjust, the best that monetary policy can

do is to set real interest rates equal to the natural rate, ensuring  $y_t = y^*$ , whenever this does not violate the zero lower bound. There is no advantage to setting  $y_t < y^*$  for any  $t \geq 2$ : this only makes the ZLB tighter.

## A.2 Alternative microfoundations

In this section I present two alternative economies providing a microfoundation for this equilibrium concept. The first draws on the extensive literature on rationing or non-Walrasian equilibria. The second is an economy with downward nominal wage rigidity drawing on [Schmitt-Grohé and Uribe \[2011\]](#).

### A.2.1 Non-Walrasian equilibrium

I briefly show how the ZLB-constrained equilibrium presented in the main text can be interpreted as a rationing or non-Walrasian equilibria (see e.g. [Benassy \[1993\]](#) for a survey of this extensive literature).

As in the main text, households solve

$$\max U(c_0^i, \theta_i) + \sum_{t=1}^{\infty} \beta^t u(c_t^i) \quad (\text{A.13})$$

$$\text{s.t. } c_t^i = y_t^i - d_t^i + \frac{d_{t+1}^i}{1+r_t} \quad (\text{A.14})$$

$$d_0^i = 0, \forall i \quad (\text{A.15})$$

$$d_{t+1}^i \leq \phi_t, t = 1, \dots \quad (\text{A.16})$$

In a non-Walrasian equilibrium, prices are usually treated as exogenously fixed. Agents send quantity signals indicating how much they would like to supply and demand. They take as given not only prices, but also perceived quantity constraints operating on various markets. In equilibrium, the quantity of each good actually transacted is equal to the minimum of desired supply and desired demand. Agents' perceived quantity constraints are consistent with the amount actually transacted: the total amount of each good sold is rationed between all sellers according to an exogenously given rationing scheme (and the same goes for buyers).

In this economy, households always inelastically supply their total potential output  $y^*$ .<sup>1</sup> They receive quantity signals  $y_t^S, y_t^B$  indicating how much they are able to sell in each period; given these signals and the path of interest rates, they choose consumption, and send demand signals  $c_t^S, c_t^B$ . Output is the minimum of demand and supply:

$$2y_t = \min\{c_t^S + c_t^B, 2y^*\}$$

I assume a proportional rationing scheme: in equilibrium, the perceived quantity constraints are  $y_t^B = y_t^S := y_t$ . Finally, I assume that interest rates clear markets whenever this does not violate the ZLB. Formally:

**Definition A.2.1.** A *non-Walrasian equilibrium* is  $\{c_t^i, d_t^i, y_t, r_t\}$  such that

1. agents maximize (A.13) s.t. (A.14), (A.15), (A.16)
2. output is the minimum of demand and supply:

$$2y_t^B = 2y_t^S = 2y_t = \min\{c_t^S + c_t^B, 2y^*\}$$

3.  $r_t \geq 0$ . If  $r_t > 0$ ,  $c_t^S + c_t^B = 2y_t = 2y^*$ .

When interest rates can adjust to clear markets, they do, and agents sell all of their endowment. When the ZLB prevents interest rates from falling enough to clear markets, agents sell less than their total endowment, and income  $y_t$  is the variable that adjusts to clear markets. Clearly, this definition of equilibrium is isomorphic to the ZLB-constrained equilibrium defined in the main text.

## A.2.2 Economy with rigid wages

I now show how the ZLB-constrained equilibrium can be interpreted as an economy with downward nominal wage rigidity.

Households have preferences as in the main text, but now inelastically supply labor  $\bar{h}$ . Competitive firms hire labor and produce the consumption good using a linear technology,  $y_t = h_t$ .

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<sup>1</sup>I assume that each borrower (or saver) cannot consume her own output, but can consume the endowment of other borrowers and savers.

Nominal wages are downwardly rigid as in [Schmitt-Grohé and Uribe \[2011\]](#):

$$W_t \geq \gamma W_{t-1}, h_t \leq \bar{h}, (\bar{h} - h_t)(W_t - \gamma W_{t-1}) \quad (\text{A.17})$$

where  $\gamma > 0$  indexes the degree of nominal rigidity. When the wage rigidity constraint binds, households are equally rationed in the labor market:  $h_t^S = h_t^B = h_t$ . Since firms are competitive, they set  $P_t = W_t$ . We define  $\Pi_t = \frac{P_t}{P_{t-1}}$ . Households solve

$$\begin{aligned} \max U(c_0^i, \theta_i) + \sum_{t=1}^{\infty} \beta^t u(c_t^i) \quad (\text{A.18}) \\ \frac{d_{t+1}^i}{1 + i_t} &= d_t^i + P_t c_t^i - W_t h_t \\ d_0^B &= d_0^S = 0 \\ d_{t+1}^i &\leq P_{t+1} \phi_t, t = 1, \dots \end{aligned}$$

The monetary authority sets interest rates according to a Taylor rule, modified to take account of the zero lower bound:

$$1 + i_t = \max \left\{ (1 + r_t^n) \Pi_t^{\phi_\pi}, 1 \right\} \quad (\text{A.19})$$

where  $\phi_\pi > 1$  and  $r_t^n$  is the natural rate of interest, defined as the equilibrium real interest rate in the economy with  $\gamma = 0$  (no nominal wage rigidity).

The market clearing condition is

$$c_t^S + c_t^B = 2h_t \quad (\text{A.20})$$

**Definition A.2.2.** An equilibrium is a collection  $\{c_t^S, c_t^B, d_t^S, d_t^B, h_t, \Pi_t, \frac{W_t}{P_t}, i_t\}$  such that

1.  $\{C_t^i, h_t^i, d_t^i\}$  solves household  $i$ 's problem (A.18), for  $i = S, B$
2. Firms maximize profits:  $\frac{W_t}{P_t} = 1$
3. Wages satisfy (A.17)
4. Interest rates  $\{i_t\}$  satisfy the modified Taylor rule (A.19)
5. The market clearing condition (A.20) is satisfied.

**Proposition A.2.3.** When  $\gamma = 1$ , every ZLB-constrained equilibrium is an equilibrium of the rigid wage economy, with  $y^* = \bar{h}$ .

*Proof.* Take any ZLB-constrained equilibrium  $\{c_t^S, c_t^B, d_t^S, d_t^B, y_t, r_t\}$ . Set  $i_t = r_t, h_t = y_t, \Pi_t = 1, \frac{W_t}{P_t} = 1$ . Since there is no inflation, (A.17) is satisfied. Since  $i_t = r_t$ , either interest rates are zero, or they are at a level which ensures  $h_t = \bar{h}$ , as in the economy with  $\gamma = 0$ . So the modified Taylor rule (A.19) is satisfied. Finally, by definition of a ZLB constrained equilibrium, households optimize and markets clear.  $\square$

Note that unlike in the New Keynesian economy with  $\alpha = 1$ , here the Taylor rule is not optimal monetary policy. In the economy with rigid wages, there are no costs associated with inflation, and the monetary authority could costlessly circumvent the ZLB by setting  $\Pi_2 = \frac{1}{1 + r_1^n}$ .

The remainder of this appendix presents proofs of Propositions and Lemmas in the main text.

### A.3 Proof of Proposition 2.3.

Suppose  $d_1^B \leq \phi$ . I claim that the liquidity constraint never binds. In this case, equilibrium must satisfy each agent's Euler equations and market clearing:

$$\begin{aligned} u'(c_t^i) &= \beta(1 + r_t)u'(c_{t+1}^i), \quad i = S, B, t = 1, 2, \dots \\ c_t^S + c_t^B &= 2y^*, \quad t = 1, 2, \dots \end{aligned}$$

The proposed allocation satisfies all these constraints, and has  $d_t^i = d_1^i \leq \phi, \forall t > 1$ , so the liquidity constraint is indeed slack.

Even if  $d_1^B > \phi$ , then since  $d_2^B \leq \phi$ , an identical argument shows that there exists an equilibrium in which the economy enters steady state at date 2. Clearly, borrowers must be constrained at date 1: if not, we know they would attempt to keep debt constant,  $d_2^B = d_1^B$ , which violates the liquidity constraint since  $d_1^B > \phi$ . Substituting the binding liquidity constraint into borrowers' budget constraint, we have

$$c_1^B = y^* - d_1^B + \frac{\phi}{1 + r_1}$$

Market clearing means that

$$c_1^S = y^* + d_1^B - \frac{\phi}{1 + r_1}, c_2^S = y^* + \frac{r^*}{1 + r^*}\phi$$

Since savers are unconstrained, their Euler equation must hold with equality:

$$u'(c_1^S) = \beta(1 + r_1)u'(c_2^S)$$

$$u'\left(y^* + d_1^B - \frac{\phi}{1 + r_1}\right) = \beta(1 + r_1)u'(y^* + (1 - \beta)\phi)$$

This implicitly defines  $r_1$ , as claimed. So we are done.

#### A.4 Proof of Proposition 3.2.

Combining borrowers' and savers' date 0 Euler equations, we have

$$\frac{U_c(c_0^B, \theta_B)}{u'(c_1^B)} = \frac{U_c(c_0^S, \theta_S)}{u'(c_1^S)}$$

As  $\theta_B \rightarrow \infty$ ,  $c_1^B \rightarrow 0$ , and Assumption 3.1 guarantees that the ZLB binds.

#### A.5 Proof of Proposition 3.7.

Part 3. is immediate: if the ZLB does not bind in equilibrium, the transfer required to restore full employment is zero, which is (trivially) incentive compatible.

To prove parts 1 and 2, I show that if the transfer is not incentive compatible given  $\theta_B$ , it is not incentive compatible given  $\theta'_B > \theta_B$ . It is sufficient to show that  $c_0^B(\theta'_B) > c_0^B(\theta_B)$ . Suppose not, and  $c_0^S(\theta'_B) > c_0^S(\theta_B)$ ; then  $r_0(\theta'_B) < r_0(\theta_B)$ , since we know the ZLB binds in both regimes.

$$1 + r_0 = \frac{U_c(c_0^S, \theta_S)}{\beta u'(\bar{c}_1^S)}$$

Consequently,  $d_1(\theta'_B) = (1 + r_0(\theta'_B))(y^* - c_0^S(\theta'_B)) < (1 + r_0(\theta_B))(y^* - c_0^S(\theta_B)) = d_1(\theta_B)$ . In a ZLB-constrained equilibrium,  $c_1^B = \bar{c}_1^S + 2\phi - 2d_1$ , so  $c_1^B(\theta'_B) > c_1^B(\theta_B)$ . But this contradicts the equilibrium condition

$$\frac{U_c(c_0^B, \theta_B)}{u'(c_1^B)} = \frac{U_c(c_0^S, \theta_S)}{u'(c_1^S)}$$

So  $c_0^B$  is increasing in  $\theta_B$ .

$S$ 's gain from mimicking  $B$ , given that  $B$  receives a transfer which restores full employment and gives him consumption  $2y^* - \bar{c}_1^S$ , is

$$U(c_0^B, \theta_S) - U(2y^* - c_0^B, \theta_S) + \beta[u(2y^* - \bar{c}_1^S) - u(\bar{c}_1^S)] + \frac{\beta^2}{1 - \beta}[u(y^* - (1 - \beta)\phi) - u(y^* + (1 - \beta)\phi)]$$

which is increasing in  $c_0^B$ . It follows that if the transfer is not incentive compatible given  $\theta_B$ , it is not incentive compatible given  $\theta'_B > \theta_B$ .

Finally,

$$\lim_{\beta \rightarrow 1} \frac{u(y^* - (1 - \beta)\phi) - u(y^* + (1 - \beta)\phi)}{1 - \beta} = -2u'(y^*)\phi$$

so the last two terms in the above expression go to zero as  $\beta \rightarrow 1, \phi \rightarrow 0$ . By continuity, part 4 of the Proposition follows.<sup>5</sup>

## A.6 Proof of Proposition 4.1.

The Pareto problem is

$$\max \alpha \left\{ U(c_0^S, \theta_S) + \beta u(c_1^S) + \frac{\beta^2}{1 - \beta} u(c_2^S) \right\} + (1 - \alpha) \left\{ U(c_0^B, \theta_B) + \beta u(c_1^B) + \frac{\beta^2}{1 - \beta} u(c_2^B) \right\} \quad (\text{A.21})$$

$$\text{s.t. } c_0^S + c_0^B \leq 2y^* \quad (\text{RC0})$$

$$c_1^S + c_1^B \leq 2y^* \quad (\text{RC1})$$

$$c_2^S + c_2^B = 2y^* \quad (\text{RC2})$$

$$c_2^B \geq y^* - (1 - \beta)\phi \quad (\text{BC})$$

$$u'(c_1^S) \geq \beta u'(c_2^S) \quad (\text{ZLB})$$

$$U(c_0^S, \theta_S) + \beta u(c_1^S) + \frac{\beta^2}{1 - \beta} u(c_2^S) \geq U(c_0^B, \theta_S) + \beta u(c_1^B) + \frac{\beta^2}{1 - \beta} u(c_2^B) \quad (\text{ICS})$$

$$U(c_0^B, \theta_B) + \beta u(c_1^B) + \frac{\beta^2}{1 - \beta} u(c_2^B) \geq U(c_0^S, \theta_B) + \beta u(c_1^S) + \frac{\beta^2}{1 - \beta} u(c_2^S) \quad (\text{ICB})$$

**Lemma A.6.1.** (RC2) binds.

*Proof.* Suppose not: consider the following deviation. Increase both  $c_2^S$  and  $c_2^B$ , keeping  $u(c_2^S) - u(c_2^B)$  fixed; this satisfies all constraints, and increases utility, a contradiction. A corollary is that  $c_2^S \geq c_2^B$ .  $\square$

**Lemma A.6.2.**  $u'(c_1^i) > \beta u'(c_2^i)$  for at least one agent.

*Proof.* If not, then  $c_1^i > c_2^i$  for  $i = S, B$ ; summing, we have  $c_1^S + c_1^B > c_2^S + c_2^B = 2y^*$  (by the above result), which is infeasible.  $\square$

**Lemma A.6.3.** *If (ZLB) binds, (BC) binds.*

*Proof.* Suppose by contradiction that (ZLB) binds but (BC) does not. Consider the following deviation: increase  $c_2^S$  by  $\varepsilon > 0$  and reduce  $c_2^B$  by the same amount, and increase  $c_1^B$  by  $\delta$  and reduce  $c_2^B$  by the same amount. This deviation is feasible. Choose  $\varepsilon$  and  $\delta$  so that

$$u(c_1^S - \delta) + \frac{\beta}{1 - \beta} u(c_2^S + \varepsilon) - [u(c_1^B + \delta) + \frac{\beta}{1 - \beta} u(c_2^S - \varepsilon)] = u(c_1^S) + \frac{\beta}{1 - \beta} u(c_2^S) - [u(c_1^B) + \frac{\beta}{1 - \beta} u(c_2^S)]$$

By the Implicit Function Theorem, this defines  $\delta$  as an increasing function of  $\varepsilon$  in the neighborhood of  $(\delta, \varepsilon) = (0, 0)$ . To first order, we have

$$\delta \approx \frac{\beta}{1 - \beta} \frac{u'(c_2^S) + u'(c_2^B)}{u'(c_1^S) + u'(c_1^B)}$$

To first order, the effect on  $S$ 's utility is

$$\frac{\beta}{1 - \beta} u'(c_1^S) \left[ \frac{u'(c_2^S)}{u'(c_1^S)} - \frac{u'(c_2^S) + u'(c_2^B)}{u'(c_1^S) + u'(c_1^B)} \right]$$

By assumption,  $\frac{u'(c_2^S)}{u'(c_1^S)} = \frac{1}{\beta}$ , which implies that  $\frac{u'(c_2^B)}{u'(c_1^B)} < \frac{1}{\beta}$ . Thus the change in  $S$ 's utility is positive. Since by construction the difference between  $S$ 's utility and  $B$ 's utility is unchanged,  $B$ 's utility also increases. Thus the deviation yields a strictly higher value of the objective function, which contradicts the original allocation being optimal.  $\square$

**Lemma A.6.4.**  $c_1^B \leq c_1^S$ .

*Proof.* Suppose by contradiction that  $c_1^B > c_1^S$ . Then  $c_1^S < y^*$ ; since we know that  $c_2^S \geq y^*$ , (ZLB) cannot bind, and the following deviation is feasible.. Increase  $c_1^S$  by  $\delta$ , decreasing  $c_1^B$  by the same amount, and increase  $c_1^B$  by  $\varepsilon$ , decreasing  $c_1^S$  by the same amount. Choose  $\varepsilon$  and  $\delta$  as before. Again, this defines  $\delta$  as an increasing function of  $\varepsilon$  in the neighborhood of  $(\delta, \varepsilon) = (0, 0)$ . To first order, the effect on  $S$ 's utility is

$$\frac{\beta}{1 - \beta} u'(c_1^S) \left[ \frac{u'(c_2^S) + u'(c_2^B)}{u'(c_1^S) + u'(c_1^B)} - \frac{u'(c_2^S)}{u'(c_1^S)} \right]$$

Since  $c_1^S < c_2^S$  and  $c_1^B > c_2^B$ , this expression is positive. So utility increases for both  $S$  and  $B$ , which contradicts the original allocation being optimal.  $\square$

**Lemma A.6.5.** *At most one incentive constraint binds.*

*Proof.* If (by contradiction) **(ICS)** and **(ICB)** both hold with equality at an optimum, then subtracting one constraint from the other, we have

$$U(c_0^S, \theta_S) - U(c_0^S, \theta_B) = U(c_0^B, \theta_S) - U(c_0^B, \theta_B)$$

Since  $U_{c\theta} > 0$ , this implies  $c_0^S = c_0^B$ . Since  $c_1^B \leq c_1^S, c_2^B \leq c_2^S$ , we must have  $c_1^S = c_1^B$  and  $c_2^S = c_2^B$  (otherwise,  $B$  would clearly prefer  $S$ 's allocation). Thus **(ZLB)** is slack. To show that this allocation is not optimal, consider the following deviation: increase  $c_0^B$  by  $\varepsilon > 0$ , decreasing  $c_0^S$  by the same amount, and increase  $c_1^S$  by  $\delta > 0$ , decreasing  $c_1^B$  by the same amount. Choose

$$\frac{U_c(c_0^S, \theta_S)}{\beta u'(c_1^S)} < \frac{\delta}{\varepsilon} < \frac{U_c(c_0^B, \theta_B)}{\beta u'(c_1^B)}$$

This deviation increases utility for both agents, and is feasible, because it relaxes both incentive compatibility constraints. This contradicts the assumption that the original allocation was optimal.  $\square$

**Lemma A.6.6.** **(RC0)** binds.

*Proof.* Forming the Lagrangian, the first order necessary conditions for a maximum are

$$\begin{aligned} \alpha U_c(c_0^S, \theta_S) - \lambda_0 + \mu_S U_c(c_0^S, \theta_S) - \mu_B U_c(c_0^S, \theta_B) &= 0 \\ (1 - \alpha) U_c(c_0^B, \theta_B) - \lambda_0 - \mu_S U_c(c_0^B, \theta_S) + \mu_B U_c(c_0^B, \theta_B) &= 0 \\ \alpha u'(c_1^S) - \lambda_1 + \zeta u''(c_1^S) + (\mu_S - \mu_B) u'(c_1^S) &= 0 \\ (1 - \alpha) u'(c_1^B) - \lambda_1 - (\mu_S - \mu_B) u'(c_1^B) &= 0 \\ \alpha u'(c_2^S) - \lambda_2 - (1 - \beta) \zeta u''(c_2^S) + (\mu_S - \mu_B) u'(c_2^S) &= 0 \\ (1 - \alpha) u'(c_2^B) - \lambda_2 + \psi - (\mu_S - \mu_B) u'(c_2^B) &= 0 \end{aligned}$$

where  $\lambda_0, \beta \lambda_1, \frac{\beta^2}{1 - \beta} \lambda_2, \psi, \beta \zeta, \mu_S, \mu_B$  are the multipliers on **(RC0)**, **(A.21)**, **(RC2)**, **(BC)**, **(ZLB)**, **(ICS)**, **(ICB)** respectively.

Since at most one incentive constraint binds,  $\mu_S, \mu_B \geq 0$ , with at least one equality. It follows that either  $\alpha U_c(c_0^S, \theta_S) - \lambda_0 \geq 0$ , or  $(1 - \alpha) U_c(c_0^B, \theta_B) - \lambda_0 \geq 0$ , or both. Since  $U_c > 0$ , this implies  $\lambda_0 > 0$ . Thus **(RC0)** binds.  $\square$

**Lemma A.6.7.** If **(A.21)** is slack, **(ICS)** and **(ZLB)** both bind.

*Proof.* If (A.21) is slack, then  $\lambda_1 = 0$ , and

$$\begin{aligned}\alpha u'(c_1^S) + \zeta u''(c_1^S) + (\mu_S - \mu_B)u'(c_1^S) &= 0 \\ (1 - \alpha)u'(c_1^B) - (\mu_S - \mu_B)u'(c_1^S) &= 0\end{aligned}$$

From the second equation, we must have  $\mu_S > 0$ , so  $\mu_B = 0$ . From the first equation, since  $u''(c_1^S) < 0$ , we must have  $\zeta > 0$ . Thus (ICS) and (ZLB) bind.  $\square$

Suppose the incentive constraints do not bind. Define the functions  $s_t(\alpha)$ ,  $t = 0, 1, 2$  to solve

$$\begin{aligned}\alpha U_c(s_0(\alpha), \theta_S) &= (1 - \alpha)U_c(2y^* - s_0(\alpha), \theta_B) \\ \alpha u'(s_t(\alpha)) &= (1 - \alpha)u'(2y^* - s_t(\alpha)), t = 1, 2\end{aligned}$$

It is straightforward to show that in a relaxed problem without incentive constraints,

$$\begin{aligned}c_0^S &= 2y^* - c_0^B = g_0(\alpha) \\ c_1^S &= 2y^* - c_1^B = \min\{g_1(\alpha), \bar{c}_1^S\} \\ c_2^S &= 2y^* - c_2^B = \min\{g_2(\alpha), \bar{c}_2^S\}\end{aligned}$$

where  $\bar{c}_2^S = y^* + (1 - \beta)\phi$ ,  $u'(\bar{c}_1^S) = \beta u'(\bar{c}_2^S)$ . This defines  $c_0^S, c_1^S, c_2^S$  as increasing functions of  $\alpha$ , with  $c_0^S$  strictly increasing.  $S$ 's gain from mimicking  $B$  is therefore a decreasing function of  $\alpha$ . As  $\alpha \rightarrow 0$ ,  $U_c(c_0^S, \theta_S) \rightarrow \infty$  and  $U(c_0^S, \theta_S) \rightarrow -\infty$ . So there exists  $\alpha_S > 0$  such that (ICS) just holds and (ICB) is slack. For all  $\alpha < \alpha_S$ , this allocation would violate (ICS) but would satisfy (ICB). An identical argument shows that there exists  $\alpha_B < 1$  such that (ICB) just holds and (ICS) is slack.

## A.7 Proof of Proposition 4.2.

**Lemma A.7.1.** *Define the date 1 value function*

$$V(a_1^i) = \max_{\{c_t^i, d_{t+1}^i\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} u(c_t^i) \quad (\text{A.22})$$

$$\text{s.t. } c_1^i = y_1 + a_1^i + \frac{d_2^i}{1 + r_1} \quad (\text{A.23})$$

$$c_t^i = y_t - d_t^i + \frac{d_{t+1}^i}{1 + r_t}, t \geq 2 \quad (\text{A.24})$$

$$d_t^i \leq \phi, t \geq 2 \quad (\text{A.25})$$

$\{c_t^i, d_{t+1}^i\}_{t=0}^{\infty}$  solves  $i$ 's problem, given  $\{y_t, r_t\}$  and  $T(\cdot)$ , if and only if:

1.  $c_0^i, d_1^i$  solve

$$\begin{aligned} \max_{c_0^i, d_1^i} U(c_0^i, \theta_i) + \beta V(T(d_1^i) - d_1^i) \\ \text{s.t. } c_0^i = y_0 + \frac{d_1^i}{1 + r_0} \end{aligned}$$

2.  $\{c_t^i, d_{t+1}^i\}_{t=1}^\infty$  solve (A.22), given  $a_1^i = T(d_1^i) - d_1^i$ .

*Proof.* The proof is standard, and is therefore omitted.  $\square$

**Lemma A.7.2.** *In any equilibrium with transfers, for all  $t \geq 2$  and for all  $i$ ,  $r_t = r^* = \beta^{-1} - 1$ ,  $d_t^i = d_2^i$ ,  $c_t^i = c_2^i = y^* - (1 - \beta)d_2^i$ .*

*Proof.* First, suppose that households solve a relaxed problem in which  $\phi_t = \infty$  for all  $t \geq 3$ . In this case, household first order conditions yield

$$u'(c_t^i) = \beta(1 + r_t)u'(c_{t+1}^i)\phi \text{ for all } t \geq 2$$

I will show that the borrowing constraint does not bind, so households are indeed liquidity unconstrained after date 2.

It is straightforward to see that if  $r_t = r^*, \forall t \geq 2$ , the proposed allocation uniquely satisfies these first order conditions. Suppose by contradiction that there is also an equilibrium with  $r_t > r^*$  for some  $t \geq 2$ . Then for each household  $i$ ,  $c_t^i < c_{t+1}^i$ . Integrating, we have  $y_t = \int c_t^i di < \int c_{t+1}^i di = y^*$ . So  $y_t < y^*$ , which implies  $r_t = 0$  by the definition of ZLB-constrained equilibrium, a contradiction.

Suppose by contradiction that  $r_t < r^*$ . Then a similar argument implies that  $y_{t+1} = \int c_{t+1}^i di < y^*$  and  $r_{t+1} = 0$ . Iterating forward, we see that we must have  $r_{t+s} = 0, y_{t+s} < y^*$  for all  $s \geq 1$ . This deflationary equilibrium is clearly Pareto inferior to an equilibrium with  $y_t = y^*$ , so we can rule this equilibrium out when considering optimal policy.<sup>2</sup>

From the budget constraints, it follows that  $c_t^i = c_2^i = y^* - (1 - \beta)d_2^i$ ,  $d_{t+1}^i = d_t^i$ , for all  $t \geq 2$ . Since  $d_2^i \leq \phi$ , households' unconstrained borrowing decisions happen to satisfy the borrowing constraint, as claimed.  $\square$

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<sup>2</sup>Equivalently, we could append to our definition of equilibrium the condition that  $\lim_{t \rightarrow \infty} y_t = y^*$ .

**Lemma A.7.3.** *In the two-type economy,  $\{c_t^i\}$  can be implemented as an equilibrium with transfers if and only if there exists  $r_1$  such that*

$$c_0^S + c_0^B \leq 2y^* \quad (\text{A.26})$$

$$r_1 \geq 0, c_1^S + c_1^B \leq 2y^*, \text{ with at least one equality} \quad (\text{A.27})$$

$$c_2^S + c_2^B = 2y^* \quad (\text{A.28})$$

$$u'(c_1^i) \geq \beta(1 + r_1)u'(c_2^i), c_2^i \geq y^* - (1 - \beta), \text{ with at least one equality, } i = S, B \quad (\text{A.29})$$

$$U(c_0^S, \theta_S) + \beta u(c_1^S) + \frac{\beta^2}{1 - \beta} u(c_2^S) \geq U(c_0^B, \theta_S) + \beta u(c_1^B) + \frac{\beta^2}{1 - \beta} u(c_2^B) \quad (\text{A.30})$$

$$U(c_0^B, \theta_B) + \beta u(c_1^B) + \frac{\beta^2}{1 - \beta} u(c_2^B) \geq U(c_0^S, \theta_B) + \beta u(c_1^S) + \frac{\beta^2}{1 - \beta} u(c_2^S) \quad (\text{A.31})$$

*Proof.* First I show that these conditions are necessary for implementability. Suppose  $\{c_t^i, d_{t+1}^i, r_t, y_t\}$  is an equilibrium with transfers, given some policy  $T(\cdot)$ . (A.26) and (A.28) are satisfied by definition. By Lemma A.7.2, the economy enters a steady state at date 2 with full employment, thus (A.27) is satisfied. (A.29) describes necessary conditions for optimality in the household problem. Finally, the incentive compatibility constraints (A.30), (A.31) follow from a standard mimicking argument.  $S$ 's allocation,  $c^S$ , is feasible for  $B$ . If  $c^B$  is optimal for  $B$ , it must give at least as much utility to  $B$  as he would get from  $c^S$ , which is also feasible. The same argument applies for  $S$ .

Next, I show that conditions (A.26)-(A.31) are sufficient for implementability. Let  $\{c_t^i\}, r_1$  satisfy these conditions. Set  $2y_t = c_t^S + c_t^B$  for all  $t$  and set  $r_t = r^*$  for all  $t \geq 2$ . Set  $d_t^i = \frac{y^* - c_2^i}{1 - \beta}, \forall t \geq 2$ . If  $y_t < y^*$ , set  $r_0 = 0$ , otherwise choose any  $r_0 \geq 0$ .

It is clear that all equilibrium conditions are satisfied, except, possibly, the condition that for  $i = S, B$ ,  $c_0^i, d_1^i$  solve (A.22). Let  $U^i = U(c_0^i, \theta_i) + \beta u(c_1^i) + \frac{\beta^2}{1 - \beta} u(c_2^i)$  be the utility that each agent gets from her allocation. Define  $a_1^i = c_1^i - y_1 - \frac{d_2^i}{1 + r_1}$ . For each  $i = S, B$ , define the set

$$\mathcal{V}^i = \{(c, a) \in \mathbb{R}^2 : U(c, \theta_i) + \beta V(a) \leq U^i\}$$

By construction,  $\mathcal{V}^i$  is a closed set and  $c_0^i, a_1^i$  is contained in its boundary. Let

$$\mathcal{V} = \mathcal{V}^S \cap \mathcal{V}^B = \{(c, a) \in \mathbb{R}^2 : U(c, \theta_i) + \beta V(a) \leq U^i, i = S, B\}$$

be the set of allocations which both agents find weakly inferior to their equilibrium allocations. By (A.30) and (A.31), the boundary of  $\mathcal{V}$  contains  $c_0^S, a_1^S$  and  $c_0^B, a_1^B$ . To implement the desired

equilibrium, we can offer households any subset of  $\mathcal{V}$  which contains both their equilibrium allocations. Let  $a(c)$  be any function satisfying

$$(c, a(c)) \in \mathcal{V}, \forall c$$

$$a_1^i = a(c_0^i), i = S, B$$

It is immediate that

$$c_0^i \in \arg \max_c U(c, \theta_i) + \beta V(a(c))$$

Define  $T(d) = d + a\left(y_0 + \frac{d_1^i}{1+r_0}\right)$ . We have immediately that

$$c_0^i, d_1^i \in \arg \max_{c, a} U(c, \theta_i) + \beta V(T(d) - d)$$

$$\text{s.t. } c_0^i = y_0 + \frac{d_1^i}{1+r_0}$$

Since it is clear that these transfer functions satisfy the government budget constraint, we are done.  $\square$

**Lemma A.7.4.** *In any implementable allocation,  $c_0^B \geq c_0^S$ ,  $c_t^B \leq c_t^S$  for  $t \geq 1$ , and  $S$  is unconstrained at date 1.*

*Proof.* Combining the incentive constraints,

$$U(c_0^B, \theta_B) - U(c_0^B, \theta_S) \geq U(c_0^S, \theta_B) - U(c_0^S, \theta_S)$$

Since  $U_{c\theta} > 0$  and  $\theta_B > \theta_S$ , this implies  $c_0^B \geq c_0^S$ .

If no agent is constrained at date 1, then  $c_1^i = c_t^i$  for  $i = S, B, t \geq 1$ , and we cannot have  $c_1^B > c_1^S$ : otherwise  $S$  would strictly prefer  $B$ 's allocation. Suppose  $B$  is constrained at date 1. Then by Lemma (A.7.2),

$$c_2^B = y^* - (1 - \beta)\phi \leq y^* + (1 - \beta)\phi = c_2^S$$

Since  $B$  is constrained and  $S$  is not,

$$\frac{u'(c_1^B)}{u'(c_2^B)} > \frac{u'(c_1^S)}{u'(c_2^S)} = \beta(1 + r_1)$$

I claim that  $\beta(1+r_1) < 1$ . If not, then  $\frac{u'(c_1^B)}{u'(c_2^B)} > 1$ ,  $\frac{u'(c_1^S)}{u'(c_2^S)} \geq 1$ , which implies  $c_2^B > c_1^B$ ,  $c_2^S \geq c_1^S$ . Summing, we have  $y_1 < y_2 \leq y^*$ . But this is a contradiction, since  $r_1 > 0$  and we must have  $y_1 = y^*$ .

Since  $\beta(1+r_1) < 1$ , it follows that  $c_1^S > c_2^S \geq y^*$ . Since  $c_1^S + c_1^B \leq y^*$ , we have immediately that  $c_1^B < c_1^S$ .

Finally, suppose (by contradiction) that  $S$  is constrained at date 1. An identical argument shows that  $c_1^S < c_1^B$ ,  $c_t^S \leq c_t^B$  for  $t \geq 2$ . Since  $c_0^B \geq c_0^S$ , this is a contradiction, because  $S$  would strictly prefer  $B$ 's allocation.  $\square$

**Corollary A.7.5.** *In the two-type economy,  $\{c_t^i\}$  can be implemented as an equilibrium with transfers if and only if (A.26), (A.28), (A.30), (ICB) are satisfied, together with*

$$u'(c_1^S) \geq \beta u'(c_2^S), c_1^S + c_1^B \leq 2y^*, \text{ with at least one equality} \quad (\text{A.32})$$

$$\frac{u'(c_1^B)}{\beta u'(c_2^B)} \geq \frac{u'(c_1^S)}{\beta u'(c_2^S)}, c_2^B \geq y^* - (1-\beta)\phi, \text{ with at least one equality} \quad (\text{A.33})$$

*Proof.* Since  $S$  is always unconstrained,  $r_1 = \frac{u'(c_1^S)}{\beta u'(c_2^S)}$ , and (A.32) is equivalent to (A.27). For the same reason, (A.33) is equivalent to (A.29) holding for  $B$ . Clearly if the allocation is implementable, then  $S$  is unconstrained and  $c_2^S \geq c_2^B \geq y^* - (1-\beta)\phi$ . Take any allocation satisfying the equations: it remains to show that (A.29) holds for  $S$ . By construction, the Euler equation inequality is satisfied, so it is only necessary to show that  $c_2^S \geq y^* - (1-\beta)\phi$ . Suppose not: then  $c_2^B > y^* + (1-\beta)\phi$ , and so  $\frac{u'(c_1^B)}{\beta u'(c_2^B)} = \frac{u'(c_1^S)}{\beta u'(c_2^S)}$  by (A.33). Thus  $c_2^B > c_2^S$  implies  $c_1^B > c_1^S$ . But incentive compatibility implies that  $c_0^B, c_0^S$ , so the allocation cannot satisfy (A.30), a contradiction.  $\square$

I now show that we can neglect the first and last parts of (A.33), and the complementary slackness condition in (A.32), in the Pareto problem.

**Lemma A.7.6.** *Suppose  $\{c_t^i\}$  solves the relaxed Pareto problem (A.21). Then it satisfies (A.33).*

*Proof.* First, I show that we can never have a solution to (A.21) with  $\frac{u'(c_1^B)}{\beta u'(c_2^B)} < \frac{u'(c_1^S)}{\beta u'(c_2^S)}$ . Suppose by contradiction that we have a solution in which this inequality holds. Then by Lemma (A.7.4), (ZLB) cannot hold with equality, and  $\zeta = 0$  in the first order conditions for a maximum.

Combining these conditions, we have

$$\frac{u'(c_1^B)}{u'(c_2^B)} = \frac{\lambda_1}{\lambda_2 - \psi} \geq \frac{\lambda_1}{\lambda_2} = \frac{u'(c_1^S)}{u'(c_2^S)},$$

a contradiction.

Next, I show that we can never have  $\frac{u'(c_1^B)}{\beta u'(c_2^B)} > \frac{u'(c_1^S)}{\beta u'(c_2^S)}$  and  $c_2^B > y^* - (1 - \beta)\phi$ . If we had such a solution, then by Lemma (A.6.3), (BC) and (ZLB) will both be slack, and the first order conditions imply  $\frac{u'(c_1^S)}{u'(c_2^S)} = \frac{u'(c_1^B)}{u'(c_2^B)}$ .

Finally, we know by Lemma (A.6.7) that if (A.21) is slack, (ZLB) must bind, thus the complementary slackness condition in (A.32) is satisfied.  $\square$

Proposition 4.2 follows.

## A.8 Proof of Proposition 4.4.

**Lemma A.8.1.** *Suppose  $\{c_t^i\}$  solves (A.21). Define  $a_1^i = c_1^i - y_1 - \frac{d_2^i}{1 + r_1}$ .*

*Take any transfer function  $T$  and interest rate  $r_0 \geq 0$ . Define the associated net wealth function*

$$a(c) := T((1 + r_0)(c - y^*)) - (1 + r_0)(c - y^*)$$

*Sufficient conditions for  $T, r_0$  to implement  $\{c_t^i\}$  are that:*

1.  $a(c_0^i) = c_1^i - y_1 + \frac{c_2^i - y^*}{(1 + r_1)(1 - \beta)}$  for  $i = S, B$ , and
2. for all  $c$ ,

$$(c, a(c)) \in \mathcal{V} = \mathcal{V}^S \cap \mathcal{V}^B = \{(c, a) \in \mathbb{R}^2 : U(c, \theta_i) + \beta V(a) \leq U^i, i = S, B\}$$

*Proof.* Suppose  $\{c_t^i\}$  solves (A.21). Then by Proposition 4.2, for some transfer function  $T^*(\cdot)$  and some  $\{r_t^*, y_t, d_t^i\}$ ,  $T^*, \{c_t^i, d_{t+1}^i, r_t^*, y_t\}$  is an equilibrium with transfers. Let  $T, r_0$  satisfy the conditions in the Lemma. I will show that if we replace  $T^*$  with  $T$  and replace  $r_0^*$  with  $r_0$ , keeping all other variables the same, we have an equilibrium with transfers.

If the conditions in the Lemma are satisfied, then for each  $i$ ,

$$c_0^i \in \arg \max_c U(c, \theta_i) + \beta V(a(c))$$

In other words,

$$\begin{aligned} c_0^i, d_1^i &\in \arg \max_{c, a} U(c, \theta_i) + \beta V(T(d) - d) \\ \text{s.t. } c_0^i &= y_0 + \frac{d_1^i}{1 + r_0} \end{aligned}$$

Defining  $d_1^i = (1 + r_0)(c_0^i - y^*)$ , we have

$$\begin{aligned} \sum_{i=S,B} T(d_1^i) &= \sum_{i=S,B} a_1^i + \sum_{i=S,B} d_1^i \\ &= \sum_{i=S,B} \left( c_1^i - y_1 + \frac{c_2^i - y^*}{(1 + r_1)(1 - \beta)} \right) + \sum_{i=S,B} (1 + r_0)(c_0^i - y^*) \\ &= 0 \end{aligned}$$

So the government budget constraint is satisfied. The remaining conditions are satisfied by assumption.  $\square$

**Lemma A.8.2.** *Let  $(c, a)$ ,  $(c', a')$  be two allocations with  $c' > c$ . If  $U(c, \theta_S) + \beta V(a) \geq U(c', \theta_S) + \beta V(a')$ , then  $U(c, \theta_B) + \beta V(a) > U(c', \theta_B) + \beta V(a')$ .*

*Proof.* This is immediate, since  $U_{c\theta} > 0$  and  $\theta_B > \theta_S$ .  $\square$

**Lemma A.8.3.** *If (ICS) binds, the solution to (A.21) can be implemented with a debt relief transfer function.*

*Proof.* The  $a(c)$  function associated with a debt relief transfer function has the form

$$\begin{aligned} a(c) &= (1 + r_0)(c - y^*) - \bar{T} \text{ if } c \leq \underline{c} \\ &= (1 + r_0)(\underline{c} - y^*) - \bar{T} \text{ if } c \in [\underline{c}, \bar{c}] \\ &= (1 + r_0)(\underline{c} - y^*) - \bar{T} - (1 + \tau)(1 + r_0)(c - \bar{c}) \text{ if } c > \bar{c} \end{aligned}$$

for some  $\bar{T} > 0$ ,  $\bar{c} > \underline{c}$ .

Let  $\{c_t^i\}$  be a solution to (A.21) in which (ICS) binds. Set  $y_1 = \frac{1}{2}(c_1^S + c_1^B)$ ,  $1 + r_1 = \frac{u'(c_1^S)}{\beta u'(c_2^S)}$  and

$$\begin{aligned} \bar{c} &= c_0^B \\ r_0 &= \frac{U_c(c_0^S, \theta_S)}{\beta u'(c_1^S)} - 1 \\ \tau &= \frac{U_c(c_0^B, \theta_B)}{\beta(1 + r_0)u'(c_1^B)} - 1 \\ \bar{T} &= (1 + r_0)(y^* - c_0^S) + y_1 - c_1^S + \frac{y^* - c_2^S}{(1 + r_1)(1 - \beta)} \end{aligned}$$

Clearly neither agent will ever choose  $c \in (\underline{c}, \bar{c})$ .  $S$  prefers  $c_0^S$  to any other point  $c \leq \underline{c}$ , since the budget set is linear in this range and the objective function is concave. By the same arguments,  $B$

prefers  $c_0^B$  to any other  $c \geq \bar{c}$ . Since  $S$  is indifferent between  $c_0^S$  and  $c_0^B$ ,  $S$  prefers  $c_0^B$  (and therefore  $c_0^S$ ) to any  $c > \bar{c}$ , by Lemma A.8.2. Since  $S$  is indifferent between these points,  $B$  strictly prefers  $c_0^B$  to  $c_0^S$ , and therefore to any  $c < \underline{c}$ . □

The following assumption is sufficient to ensure that the competitive equilibrium is unique.

**Assumption A.8.4.**  $-\frac{u''(c)}{u'(c)}$  is nonincreasing in  $c$ . If  $c_0^B > c_0^S$ , then  $-\frac{U_{cc}(c_0^B, \theta_B)}{U_c(c_0^B, \theta_B)} < -\frac{U_{cc}(c_0^S, \theta_S)}{U_c(c_0^S, \theta_S)}$ .

**Lemma A.8.5.**  $R(\alpha) := \frac{U_c(c_0^S(\alpha), \theta_S)}{\beta u'(c_1^S)}$  is decreasing in  $\alpha$  on  $[\alpha_S, \alpha_B]$ .  $T(\alpha) := R(\alpha)(y^* - c_0^S(\alpha)) - a_1^S(\alpha)$  is decreasing in  $\alpha$  on  $[\alpha_S, \alpha_B]$ .  $T(\alpha_S) > 0 > T(\alpha_B)$ . There exists  $\bar{\alpha} \in (\alpha_S, \alpha_B)$  such that  $T(\bar{\alpha}) = 0$ .

*Proof.* I will show that  $r = \ln R$  is decreasing in  $a = \ln \alpha - \ln(1 - \alpha)$ .  $r(a)$  is defined by

$$r(a) = \ln U_c(c_0^S(a), \theta_S) - \ln u'(c_1^S(a))$$

$c_0^S, c_1^S$  are defined by

$$\begin{aligned} a + \ln U_c(c_0^S(a), \theta_S) &= \ln U_c(2y^* - c_0^S(a), \theta_B) \\ a + \ln u'(g_1(a)) &= \ln u'(2y^* - g_1(a)) \\ c_1^S(a) &= \min\{g_1(a), \bar{c}_1^S\} \end{aligned}$$

Define  $\Gamma(c, \theta) = -\frac{U_{cc}(c, \theta)}{U_c(c, \theta)}$  and  $\gamma(c) = -\frac{u''(c)}{u'(c)}$ .

$$r'(a) = \frac{\gamma(c_1^S)}{\gamma(c_1^S) + \gamma(c_1^B)} \mathbb{1}(g_1(a) < \bar{c}_1^S) - \frac{\Gamma(c_0^S, \theta_S)}{\Gamma(c_0^S, \theta_S) + \Gamma(c_0^B, \theta_B)}$$

Under Assumption A.8.4,  $r'(a) \leq 0$ , and  $R$  is decreasing in  $\alpha$ . Since  $a_1^S$  and  $c_1^S$  are increasing in  $\alpha$ , and  $c_0^S < y^*$ ,  $T(\alpha)$  is increasing in  $\alpha$ .

When  $\alpha = \alpha_S$ ,  $U(c_0^S, \theta_S) + \beta V(a_1^S) = U(c_0^B, \theta_S) + \beta V(a_1^B)$ . Since these functions are concave,

$$\begin{aligned} U_c(c_0^S, \theta_S)(c_0^B - c_0^S) + \beta V'(a_1^S)(a_1^B - a_1^S) &> 0 \\ \frac{U_c(c_0^S, \theta_S)}{\beta u'(c_1^S)}(y^* - c_0^S) - a_1^S &> 0 \\ T(\alpha_S) &> 0 \end{aligned}$$

An analogous argument establishes that  $T(\alpha_B) < 0$ . Finally, since  $T$  is clearly continuous, there exists  $\bar{\alpha}$  such that  $T(\bar{\alpha}) = 0$ . □

**Lemma A.8.6.** *If neither incentive constraint binds and  $T(\alpha) > 0$ , the solution to (A.21) can be implemented with a debt relief transfer function.*

*Proof.* The proof proceeds exactly as for Lemma A.8.3, noting that  $\bar{T} = T > 0$ . □

This concludes the proof of Proposition 4.4. The proof of Proposition 4.6 is essentially identical, and is therefore omitted.

## A.9 Proof of Proposition 4.7.

Suppose the ZLB does not bind in competitive equilibrium. Then we have full employment in all periods and

$$\begin{aligned}\frac{U_c(c_0^S, \theta_S)}{\beta u'(c_1^S)} &= \frac{U_c(c_0^B, \theta_B)}{\beta u'(c_1^B)} = 1 + r_0 \\ u'(c_1^S) &= \beta(1 + r_1)u'(c_2^S) \\ u'(c_1^B) &\geq \beta(1 + r_1)u'(c_2^B)\end{aligned}$$

Choose  $\alpha$  so that  $\frac{\alpha}{1 - \alpha} = \frac{U_c(c_0^B, \theta_B)}{U_c(c_0^S, \theta_S)}$ . It follows that

$$\begin{aligned}\alpha U_c(c_0^S, \theta_S) &= (1 - \alpha)U_c(c_0^B, \theta_B) \\ \alpha u'(c_1^S) &= (1 - \alpha)u'(c_1^B) \\ \alpha u'(c_1^S) &= (1 - \alpha)u'(c_1^B) + \psi\end{aligned}$$

for some  $\psi \geq 0$ . So the allocation satisfies the first order sufficient conditions in (A.21), and is Pareto optimal.

If  $\theta_B > \theta^{ZLB}$ , we know the ZLB binds and there is underemployment in the non-Walrasian equilibrium. We also know that neither incentive constraint binds in the non-Walrasian equilibrium allocation. Each agent has strictly concave preferences, and strictly prefers her chosen allocation to any other allocation in the budget set. In particular,  $S$  strictly prefers  $c^S$  to  $c^B$ . We know from Proposition 4.1 that underemployment can only be optimal if (ICS) binds. So this allocation cannot be Pareto optimal.

To show that debt relief is Pareto improving, consider the following deviation. Increase  $c_1^B$  until either the resource constraint binds at date 1,  $c_1^B = \bar{c}_1^B$ , or (ICS) binds. In the first case,

this leads to a Pareto optimal allocation, because any full employment, incentive compatible allocation with  $c_1^S = \bar{c}_1^S$  is Pareto optimal. We clearly have  $T(\alpha) > 0$ , so the allocation can be implemented with debt relief. In the second case, (ICS) binds, so the allocation can be implemented with debt relief.

## A.10 Proof of Proposition 4.9.

The borrower-optimal allocation solves

$$\max_{c_0^S, c_1^S, c_2^S, c_0^B, c_1^B, c_2^B} U(c_0^B, \theta_B) + \beta u(c_1^B) + \frac{\beta^2}{1-\beta} u(c_2^B) \quad (\text{A.34})$$

$$\text{s.t. } U(c_0^S, \theta_S) + \beta u(c_1^S) + \frac{\beta^2}{1-\beta} u(c_2^S) \geq \bar{U}(\theta_S, \theta_B, \phi) \quad (\text{US})$$

$$(\text{RC0}), (\text{A.21}), (\text{RC2}), (\text{BC}), (\text{ZLB}), (\text{ICS})$$

1. Obvious, since the equilibrium is constrained efficient.

2. The allocation is clearly feasible: by construction it satisfies (A.21), by assumption (ICS), and clearly it satisfies the remaining constraints because the savers' consumption, and everyone's date 0 and 2 consumption is the same as in equilibrium, so it is feasible. Any increase in  $c_1^S$  is not feasible: it violates (ZLB). Decreasing  $c_0^B$  and increasing  $c_1^B$  while satisfying resource constraints and (US) would decrease the borrower's utility: already in equilibrium the agents' marginal rates of substitution between dates 0 and 1 were equal, and now the borrower has more date 1 consumption, so he does not want to increase it further.

3,4,5. Consider the relaxed problem in which we ignore (A.21), (US). (ICS) must bind in the relaxed program (since it corresponds to our regular Pareto problem with  $\alpha = 0$ ). Substituting constraints into the objective function, we have

$$\max_{c_0^B} U(c_0^B, \theta_B) - U(c_0^B, \theta_S) + U(2y^* - c_0^B, \theta_S) + \text{constants}$$

By Assumption 4.8, this function is concave and the first order condition

$$U_c(\hat{c}, \theta_B) = U_c(\hat{c}, \theta_S) + U_c(2y^* - \hat{c}, \theta_S) \quad (\text{A.35})$$

is necessary and sufficient for a solution. So the solution to this program is  $\hat{c}(\theta_B)$ . If  $\hat{c}(\theta_B) < \underline{c}(\phi)$ , this solution violates (A.21), so (A.21) must bind in the true program. If  $\hat{c}(\theta_B) > c_0^B$ , this solution violates (US), so (US) must bind. In the intermediate range, neither constraint binds.

## A.11 Proof of Proposition 5.1.

The Walrasian equilibrium allocation  $\{c_0^i(\theta_N), c_1^i(\theta_N)\}_{i \in [0,1]}$ , given a function  $\theta_N$  mapping individuals to types, satisfies:

$$\frac{\theta_N(i)u'(c_0^i(\theta_N))}{u'(c_1^i(\theta_N))} = \frac{u'(c_0^0(\theta_N))}{u'(c_1^0(\theta_N))}, \forall i \in (0, 1] \quad (\text{A.36})$$

$$\int_0^1 c_0^i(\theta_N) \, di = y^* \quad (\text{A.37})$$

$$\int_0^1 c_1^i(\theta_N) \, di = y^* \quad (\text{A.38})$$

We will show that we must have  $c_1^0(\theta_N) \rightarrow \infty$  as  $N \rightarrow \infty$ . The proof is by contradiction. Suppose  $c_1^0(\theta_N)$  does not converge to  $\infty$ . Then  $\liminf_{N \rightarrow \infty} c_1^0(\theta_N) = c^* < \infty$ , and the denominator of the right hand side of (A.36) does not converge to 0.

Suppose the numerator converges to  $\infty$ : then  $c_0^0(\theta_N) \rightarrow 0$ . Consider the lifetime utility of household 0:

$$v^0(\theta_N) = u(c_0^0(\theta_N)) + \beta u(c_1^0(\theta_N))$$

Since  $\liminf c_0^0(\theta_N) = 0$  and  $\liminf c_1^0(\theta_N) = c^* < \infty$ , it follows that  $\liminf v^0(\theta_N) < u(y^*) + \beta u(y^*)$ : for infinitely many  $N$ , household 0 gets lower utility than he would under autarky. This cannot be the case in any competitive equilibrium, since autarky is in his budget set. This contradicts the supposition that the numerator  $u'(c_0^0(\theta_N))$  converges to  $\infty$ . So we have shown that, under the assumption that  $c_1^0(\theta_N)$  does not converge to  $\infty$ , the right hand side of (A.36) does not converge to  $\infty$ .

Since  $\theta_N(i) \rightarrow \infty$  for every  $i \in (0, 1]$ , it follows that  $\frac{u'(c_0^i(\theta_N))}{u'(c_1^i(\theta_N))} \rightarrow 0$ . There are two possibilities: either  $c_0^i(\theta_N) \rightarrow \infty$ , or  $c_1^i(\theta_N) \rightarrow 0$  (or both). If we define

$$S_0 = \{i \in (0, 1] : c_0^i(\theta_N) \rightarrow \infty\}, S_1 = \{i \in (0, 1] : c_1^i(\theta_N) \rightarrow 0\},$$

we must have  $S_0 \cup S_1 = (0, 1]$ .  $S_0$  must have measure zero: otherwise the resource constraint (A.37) cannot be satisfied, given that consumption must be nonnegative. So  $S_1$  must have measure 1. Given the date 1 resource constraint (A.38), this implies that  $c_1^0(\theta_N) \rightarrow \infty$ .<sup>3</sup> This contradicts our original assumption. So we must have  $c_1^0(\theta_N) \rightarrow \infty$ .

---

<sup>3</sup>Note that since  $c_1^i(\theta_N)$  is decreasing in  $i$  for any  $\theta_N$  - lower types consume more at date 1 - if  $c_1^i(\theta_N) \rightarrow \infty$  for any type  $i$ , then  $c_1^0(\theta_N) \rightarrow \infty$ .

This implies that for large enough  $N$ , the date 1 interest rate satisfying the most patient household's Euler equation,  $r_1 = \frac{u'(c_1^0(\theta_N))}{\beta u'(y^*)} - 1$ , is negative.

## A.12 Proof of Lemma 5.2

1, 2 and 3 are standard results. 4 results from differentiating  $C_t(v_1, r_1) = X_t(E(v_1, r_1), r_1)$  and using the Envelope Theorem. To prove 5, note that  $C_1, C_2$  are increasing in  $v_1$ : thus there exists  $v_1$  low enough that if we considered a relaxed problem without the borrowing constraint, it would be optimal to set  $c_2 < \underline{c}_2$ , thus the constraint must bind. The second half of 5 then results from duality. To prove 6, note that the first order conditions yield the Euler equation  $c_1^{-\sigma} = \beta(1 + r_1)c_2^{-\sigma}$  when the borrowing constraint does not bind; substituting this into the budget constraint yields the desired result. To prove 7, note that if  $X_1$  is concave,  $X_2$  is convex.  $C_2(v_1, r_1) = X_2(E(v_1, r_1), r_1)$  is the composition of two increasing, convex functions and is convex.

## A.13 Proof of Proposition 5.3.

Before proving Proposition 5.3, I verify that it is possible to express incentive compatibility as an integral condition.

**Lemma A.13.1.**  $u_0, v_1$  satisfies

$$\theta u(c_0(\theta)) + \beta v_1(\theta) \geq \theta u(c_0(\hat{\theta})) + \beta v_1(\hat{\theta}), \forall \theta, \hat{\theta} \quad (\text{A.39})$$

if and only if

$$v(\theta) = v(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} u_0(z) dz \quad (\text{A.40})$$

and  $u_0$  is nondecreasing.

*Proof.* To show that (A.39) implies (A.40), we use Theorem 2 in [Milgrom and Segal \[2002\]](#). Define

$$W(\theta, \hat{\theta}) = \theta u_0(\hat{\theta}) + \beta v_1(\hat{\theta})$$

and suppose  $W(\theta, \theta) = v(\theta) = \max_{\hat{\theta}} W(\theta, \hat{\theta})$ . For any fixed  $\hat{\theta}$ ,  $W(\theta, \hat{\theta})$  is linear in  $\theta$ , and therefore differentiable and absolutely continuous. We must also show that there exists an integrable function  $b : [1, \bar{\theta}] \rightarrow \mathbb{R}$  such that  $|W_{\theta}(\theta, \hat{\theta})| \leq b(\theta)$  for all  $\theta \in \Theta$  and almost all  $\hat{\theta} \in \Theta$ . Since

$W_\theta(\theta, \hat{\theta}) = u_0(\hat{\theta})$  is increasing in  $\hat{\theta}$ , we can set  $b(\theta) = \max\{|u_0(\underline{\theta})|, |u_0(\bar{\theta})|\}$ , and this condition is satisfied. By Theorem 2 in [Milgrom and Segal \[2002\]](#), we have that

$$v(\theta) = v(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} u_0(z) dz$$

as required.

Next, we show that (A.40) and  $u_0$  nondecreasing imply (A.39). (The proof follows [Mirrlees \[1986\]](#) Lemma 6.3.) Take any  $\hat{\theta}, \theta$ : we want to show that

$$v(\theta) := \theta u_0(\theta) + \beta v_1(\theta) \geq \theta u_0(\hat{\theta}) + \beta v_1(\hat{\theta})$$

Since  $u_0(\theta)$  is nondecreasing, we have

$$\begin{aligned} v(\theta) - v(\hat{\theta}) &= \int_{\hat{\theta}}^{\theta} u_0(z) dz \\ &\geq \int_{\hat{\theta}}^{\theta} u_0(\hat{\theta}) dz = (\theta - \hat{\theta})u_0(\hat{\theta}) \\ \theta u_0(\theta) + \beta v_1(\theta) &\geq \theta u_0(\hat{\theta}) + \beta v_1(\hat{\theta}) \end{aligned}$$

This completes the proof. □

I now proceed to prove Proposition 5.3. Recall that the social planner's problem is

$$\mathcal{W}^* = \max_{u_0 \in \Omega, \underline{v}, r_1} \underline{v} + \int (1 - A(\theta))u_0(\theta) d\theta \quad (\text{PP}')$$

$$\text{s.t } \int C_0(u_0(\theta))f(\theta) d\theta \leq y^* \quad (\text{RC0}')$$

$$\int C_1\left(\beta^{-1}\left[\underline{v} + \int_{\underline{\theta}}^{\theta} u_0(z) dz - \theta u_0(\theta)\right], r_1\right) f(\theta) d\theta \leq y^* \quad (\text{RC1}')$$

$$\int C_2\left(\beta^{-1}\left[\underline{v} + \int_{\underline{\theta}}^{\theta} u_0(z) dz - \theta u_0(\theta)\right], r_1\right) f(\theta) d\theta \leq y^* \quad (\text{RC2}')$$

$$r_1 \geq 0 \quad (\text{ZLB})$$

I need to show that  $u_0, \underline{v}, r_1$  solves (PP') if and only if there exist Lagrange multipliers  $\lambda_0, \lambda_1, \lambda_2$  such that  $u_0, \underline{v}$  solve

$$\begin{aligned} \mathcal{W}^* = \max_{u_0 \in \Omega, \underline{v}} \underline{v} + \int (1 - A(\theta))u_0(\theta) d\theta &- \lambda_0 \int C_0(u_0(\theta))f(\theta) d\theta \\ &- \int M\left(\beta^{-1}\left[\underline{v} + \int_{\underline{\theta}}^{\theta} u_0(z) dz - \theta u_0(\theta)\right] \mid \lambda_1, \lambda_2, r_1\right) f(\theta) d\theta \end{aligned} \quad (\text{A.41})$$

where

$$M(v_1|\lambda_1, \lambda_2, r_1) := \lambda_1 C_1(v_1, r_1) + \lambda_2 C_2(v_1, r_1)$$

and the Lagrange multipliers satisfy the following conditions. If the ZLB is slack,  $\frac{\lambda_1}{\lambda_2} = (1 + r_1)(1 - \beta)$ . If the ZLB binds,  $\frac{\lambda_1}{\lambda_2} < 1 - \beta$ .

**Overview.** The proof has six steps. First we show that solutions to (PP') also solve a modified problem (PP'') in which we replace the date 1 resource constraint with the aggregate expenditure function. Second, the modified problem can be solved in two stages: first maximize social welfare given  $r_1$ , yielding welfare  $\mathcal{W}(r)$ , and then choose  $r$  to maximize  $\mathcal{W}(r)$  subject to the ZLB. Third, the first stage of this problem is concave, and Lagrangian theorems apply. Fourth, we can also express the expenditure functions as maximized sub-Lagrangians. Substituting these sub-Lagrangians into the main Lagrangian, we see that  $W(r)$  is also the maximum of an expanded Lagrangian. Fifth, returning to our two stage problem, we can switch the order of maximization, first choosing  $r$  to minimize a certain function, subject to the ZLB, and then choosing utilities to maximize social welfare. Sixth, and finally, I show that when the ZLB is slack, one constraint in the planner's problem becomes slack, and the expanded Lagrangian is equivalent to (A.41), with  $\frac{\lambda_1}{\lambda_2} = (1 + r_1)(1 - \beta)$ . When the ZLB binds, we have  $\frac{\lambda_1}{\lambda_2} < 1 - \beta$ .

**1. Modified problem.**  $u_0, \bar{v}, r_1$  solves (PP') if and only if it solves the modified function (PP''):

$$\mathcal{W}^* = \max_{u_0 \in \Omega, \bar{v}, r_1} \bar{v} + \int (1 - A(\theta)) u_0(\theta) d\theta \quad (\text{PP}'')$$

$$\text{s.t. } \int C_0(u_0(\theta)) f(\theta) d\theta \leq y^* \quad (\text{A.42})$$

$$E \left( \beta^{-1} \left[ \bar{v} + \int_{\theta}^{\theta} u_0(z) dz - \theta u_0(\theta) \right], r_1 \right) f(\theta) d\theta \leq y^* + \frac{y^*}{(1 + r_1)(1 - \beta)} \quad (\text{EC})$$

$$\int C_2(v_1(\theta), r_1) f(\theta) d\theta \leq y^* \quad (\text{RC2})$$

$$r_1 \geq 0 \quad (\text{A.43})$$

This is true because the constraint sets are the same in (PP') and (PP'') are the same. By definition,

$$E(v_1, r_1) = C_1(v_1, r_1) + \frac{C_2(v_1, r_1)}{(1 + r_1)(1 - \beta)}$$

So (EC) is equivalent to

$$\int \left[ C_1(v_1, r_1) + \frac{C_2(v_1, r_1)}{(1 + r_1)(1 - \beta)} \right] f(\theta) d\theta \leq y^* + \frac{y^*}{(1 + r_1)(1 - \beta)}$$

This is a weighted sum of the constraints (RC1') and (RC2'). Clearly then,  $v_1, r_1 \geq 0$  satisfies (EC) and (RC2) if and only if it satisfies (A.21).

**2. Two-stage problem.**  $\mathcal{W}^* = \max_{r_1 \geq 0} \mathcal{W}(r_1)$ , where

$$\mathcal{W}(r) = \max_{u_0 \in \Omega, \bar{v}} \bar{v} + \int (1 - A(\theta)) u_0(\theta) \, d\theta \quad (\text{A.44})$$

$$\text{s.t. } \int C_0(u_0(\theta)) f(\theta) \, d\theta \leq y^* \quad (\text{A.45})$$

$$E \left( \beta^{-1} \left[ \bar{v} + \int_{\underline{\theta}}^{\theta} u_0(z) \, dz - \theta u_0(\theta) \right], r_1 \right) f(\theta) \, d\theta \leq y^* + \frac{y^*}{(1+r_1)(1-\beta)}$$

$$\int C_2(v_1(\theta), r_1) f(\theta) \, d\theta \leq y^*$$

**3. Lagrangian.** There exist Lagrange multipliers  $\lambda_0, \lambda_E, \lambda_C$  such that  $u_0, \bar{v}$  solve (A.44) if and only if they solve

$$\mathcal{W}(r) = \max_{u_0 \in \Omega, \bar{v}} \bar{v} + \int (1 - A(\theta)) u_0(\theta) \, d\theta - \lambda_0 \int C_0(u_0(\theta)) f(\theta) \, d\theta \quad (\text{A.46})$$

$$- \lambda_E \int E \left( \beta^{-1} \left[ \bar{v} + \int_{\underline{\theta}}^{\theta} u_0(z) \, dz - \theta u_0(\theta) \right], r_1 \right) f(\theta) \, d\theta$$

$$- \lambda_C \int C_2 \left( \beta^{-1} \left[ \bar{v} + \int_{\underline{\theta}}^{\theta} u_0(z) \, dz - \theta u_0(\theta) \right], r_1 \right) f(\theta) \, d\theta$$

Before proving this statement, note for future reference that since by definition  $E = C_1 + \frac{C_2}{(1+r_1)(1-\beta)}$ , we could equivalently write

$$\mathcal{W}(r) = \max_{u_0 \in \Omega, \bar{v}} \bar{v} + \int (1 - A(\theta)) u_0(\theta) \, d\theta - \lambda_0 \int C_0(u_0(\theta)) f(\theta) \, d\theta \quad (\text{A.47})$$

$$- \lambda_1 \int C_1 \left( \beta^{-1} \left[ \bar{v} + \int_{\underline{\theta}}^{\theta} u_0(z) \, dz - \theta u_0(\theta) \right], r_1 \right) f(\theta) \, d\theta$$

$$- \lambda_2 \int C_2 \left( \beta^{-1} \left[ \bar{v} + \int_{\underline{\theta}}^{\theta} u_0(z) \, dz - \theta u_0(\theta) \right], r_1 \right) f(\theta) \, d\theta$$

where we define  $\lambda_1 = \lambda_E, \lambda_2 = \frac{\lambda_E}{(1+r_1)(1-\beta)}$ . That is, showing necessity and sufficiency for (A.46) is equivalent to showing necessity and sufficiency for (A.47).

(A.44) is a special case of the general problem considered in Luenberger [1969], Sections 8.3-8.4:

$$\begin{aligned} & \inf_{x \in \Omega} f(x) \\ \text{s.t. } & G(x) \leq 0 \end{aligned}$$

where  $X$  is a linear vector space,  $Z$  a normed space,  $\Omega$  a convex subset of  $X$ ,  $P$  the positive cone in  $Z$ ,  $f$  a real valued convex functional on  $\Omega$ ,  $G$  a convex mapping from  $\Omega$  into  $Z$ . We have

$$\begin{aligned} X &= \{\underline{v}, u_0 \mid \underline{v} \in V(\mathbb{R}_+), u_0 : \Theta \rightarrow \mathbb{R}\} \\ \Omega &= \{\underline{v}, u_0 \mid \underline{v} \in V, u_0 : \Theta \rightarrow U(\mathbb{R}_+), u_0 \text{ non-decreasing}\} \\ Z &= \mathbb{R}^3, P = \mathbb{R}_+^3 \\ f(\underline{v}, u_0) &= -\underline{v} - \int (1 - A(\theta))u_0(\theta) d\theta \\ G(\underline{v}, u_0) &= \begin{bmatrix} \int C_0(u_0(\theta))f(\theta) d\theta - y^* \\ E(v_1(\theta), r_1) f(\theta) d\theta - y^* - \frac{y^*}{(1+r_1)(1-\beta)} \\ \int C_2(v_1(\theta), r_1)f(\theta) d\theta - y^* \end{bmatrix} \end{aligned}$$

$\Omega$  is convex,  $P$  contains an interior point, and  $f$  is convex (since it is linear).  $C_0$  is convex, and  $E$  and  $C_2$  are convex in  $v_1$  by Lemma 5.2; since  $v_1$  is a linear function of  $\underline{v}, u_0$ , and  $G$  contains weighted sums of  $C_0, E, C_2$ , it follows that  $G$  is convex. There exists a point  $\underline{v}, u_0 \in \Omega$  such that  $G(\underline{v}, u_0) \leq 0$ : choose  $v_1$  small enough that  $E(v_1, r_1) < y^* - \frac{y^*}{(1+r_1)(1-\beta)}$ ,  $C_2(v_1, r_1) < y^*$ , and set  $u_0(\theta) = u(y^*/2)$ ,  $\underline{v} = \theta u(y^*/2) + \beta v_1$ . Then since the hypotheses of Theorem 1 in Luenberger [1969] p217 are met, it follows that if  $u_0, \underline{v}$  solve (A.44), they solve (A.46). Conversely, by Theorem 1 in Luenberger [1969] p220, if  $u_0, \underline{v}$  solve (A.46), they solve (A.44). Finally, as noted above, showing necessity and sufficiency of (A.46) is equivalent to showing necessity and sufficiency of (A.47). This completes the proof of the first part of Proposition 5.3. It remains to derive the condition on the Lagrange multipliers.

**4. Expenditure minimization sub-Lagrangians.** Applying the same theorems to the expenditure minimization problem (EMP), we have that, for each type  $\theta$ , solutions to the expenditure minimization problem also minimize an appropriately defined Lagrangian:

$$\begin{aligned} E(v_1(\theta), r_1) &= \min_{c_1(\theta), c_2(\theta)} c_1(\theta) + \frac{c_2(\theta)}{(1+r_1)(1-\beta)} + \psi(\theta)[c_2(\theta) - \underline{c}_2] \\ &\quad + \mu(\theta) \left[ u(c_1(\theta)) + \frac{\beta}{1-\beta} u(c_2(\theta)) - v_1(\theta) \right] \end{aligned}$$

Integrating across all types,

$$\begin{aligned}
& - \int E \left( \beta^{-1} \left[ \bar{v} + \int_{\underline{\theta}}^{\theta} u_0(z) dz - \theta u_0(\theta) \right], r_1 \right) f(\theta) d\theta \\
& = - \int \left\{ \min_{c_1(\theta), c_2(\theta)} c_1(\theta) + \frac{c_2(\theta)}{(1+r_1)(1-\beta)} + \psi(\theta)[c_2(\theta) - y^* + (1-\beta)\psi] \right. \\
& \left. \mu(\theta) \left[ u(c_1(\theta)) + \frac{\beta}{1-\beta} u(c_2(\theta)) - \beta^{-1} \left( \bar{v} + \int_{\underline{\theta}}^{\theta} u_0(z) dz - \theta u_0(\theta) \right) \right] \right\} f(\theta) d\theta \\
& = \max_{c_1, c_2} \left\{ \int -c_1(\theta) - \frac{c_2(\theta)}{(1+r_1)(1-\beta)} - \psi(\theta)[c_2(\theta) - \underline{c}_2] \right. \\
& \left. - \mu(\theta) \left[ u(c_1(\theta)) + \frac{\beta}{1-\beta} u(c_2(\theta)) - \beta^{-1} \left( \bar{v} + \int_{\underline{\theta}}^{\theta} u_0(z) dz - \theta u_0(\theta) \right) \right] \right\} f(\theta) d\theta
\end{aligned}$$

Substituting into the main Lagrangian, we see that if  $u_0, \bar{v}$  solve (A.44), they also solve

$$\begin{aligned}
\mathcal{W}(r) = & \max_{u_0 \in \Omega, \bar{v}} \bar{v} + \int (1 - A(\theta)) u_0(\theta) d\theta - \lambda_0 \int C_0(u_0(\theta)) f(\theta) d\theta \\
& - \lambda_E \max_{c_1, c_2} \left\{ \int -c_1(\theta) - \frac{c_2(\theta)}{(1+r_1)(1-\beta)} - \psi(\theta)[c_2(\theta) - \underline{c}_2] \right. \\
& \left. - \mu(\theta) \left[ u(c_1(\theta)) + \frac{\beta}{1-\beta} u(c_2(\theta)) - \beta^{-1} \left( \bar{v} + \int_{\underline{\theta}}^{\theta} u_0(z) dz - \theta u_0(\theta) \right) \right] \right\} f(\theta) d\theta \\
& - \lambda_C \int C_2 \left( \beta^{-1} \left[ \bar{v} + \int_{\underline{\theta}}^{\theta} u_0(z) dz - \theta u_0(\theta) \right], r_1 \right) f(\theta) d\theta
\end{aligned}$$

This will be true if and only if  $u_0, \bar{v}, c_1, c_2$  solve

$$\begin{aligned}
\mathcal{W}(r) = & \max_{u_0 \in \Omega, \bar{v}, c_1, c_2} \bar{v} + \int (1 - A(\theta)) u_0(\theta) d\theta - \lambda_0 \int C_0(u_0(\theta)) f(\theta) d\theta \\
& - \lambda_E \left\{ \int -c_1(\theta) - \frac{c_2(\theta)}{(1+r_1)(1-\beta)} - \psi(\theta)[c_2(\theta) - \underline{c}_2] \right. \\
& \left. - \mu(\theta) \left[ u(c_1(\theta)) + \frac{\beta}{1-\beta} u(c_2(\theta)) - \beta^{-1} \left( \bar{v} + \int_{\underline{\theta}}^{\theta} u_0(z) dz - \theta u_0(\theta) \right) \right] \right\} f(\theta) d\theta \\
& - \lambda_C \int C_2 \left( \beta^{-1} \left[ \bar{v} + \int_{\underline{\theta}}^{\theta} u_0(z) dz - \theta u_0(\theta) \right], r_1 \right) f(\theta) d\theta
\end{aligned}$$

The economic meaning of this result is that when the ZLB is slack, private and social valuations of date 1 and date 2 consumption coincide, and the planner would not want to distort date 1 or date 2 consumption away from their equilibrium levels, even if it were possible to do so. Since households share the same preferences over date 1 and date 2 consumption, distorting these allocations would not help relax incentive constraints. Instead, it is always optimal to deliver utility in an ex post efficient way at dates 1 and 2.

**5. Reversing the order of maximization.** Next, note that  $\mathcal{W}(r)$  has the form  $\mathcal{W}(r) = \max_x f(x) + g(x, r)$ , where  $x = (u_0, \underline{v}, c_1, c_2)$ . Then we have

$$\begin{aligned}\mathcal{W}^* &= \max_{r \geq 0} \mathcal{W}(r) \\ &= \max_{r \geq 0} \max_x \{f(x) + g(x, r)\} \\ &= \max_x \max_{r \geq 0} \{f(x) + g(x, r)\} \\ &= \max_x f(x) + \max_r \{g(x, r)\}\end{aligned}$$

Applying this result to the Lagrangian

$$\begin{aligned}\mathcal{W}^* &= \max_{u_0 \in \Omega, \underline{v}, c_1, c_2} \underline{v} + \int (1 - A(\theta)) u_0(\theta) \, d\theta - \lambda_0 \int C_0(u_0(\theta)) f(\theta) \, d\theta \\ &\quad - \lambda_E \left\{ \int -c_1(\theta) - \frac{c_2(\theta)}{(1+r_1)(1-\beta)} - \psi(\theta)[c_2(\theta) - \underline{c}_2] \right. \\ &\quad \left. - \mu(\theta) \left[ u(c_1(\theta)) + \frac{\beta}{1-\beta} u(c_2(\theta)) - \beta^{-1} \left( \underline{v} + \int_{\underline{\theta}}^{\theta} u_0(z) \, dz - \theta u_0(\theta) \right) \right] \right\} f(\theta) \, d\theta \\ &\quad - \min_{r_1 \geq 0} \left\{ \lambda_C \int C_2 \left( \beta^{-1} \left[ \underline{v} + \int_{\underline{\theta}}^{\theta} u_0(z) \, dz - \theta u_0(\theta) \right], r_1 \right) f(\theta) \, d\theta \right\}\end{aligned}$$

**6. ZLB.** When the ZLB is slack,  $\min_{r_1 \geq 0} \lambda_C \int C_2 \, dF(\theta) = \min_{r_1 \in \mathbb{R}} \lambda_C \int C_2 \, dF(\theta)$ . Since  $C_2$  is increasing in  $r_1$ ,  $r_1 > 0$  can only attain the minimum in this problem when  $\lambda_C = 0$ . That is, if the ZLB is slack,  $\lambda_C = 0$ . However, if the ZLB binds at an optimum,  $\lambda_C > 0$ .

In the Lagrangian (A.47), we had  $\lambda_1 = \lambda_E, \lambda_2 = \frac{\lambda_E}{(1+r_1)(1-\beta)} + \lambda_C$ . Thus when the ZLB is slack,  $\lambda_2 = (1+r_1)(1-\beta)\lambda_1$ . When the ZLB binds,  $\lambda_2 = (1-\beta)\lambda_1 + \lambda_C > (1-\beta)\lambda_1$ . This establishes the desired result.

## A.14 Proof of Lemma 5.5.

**Definition A.14.1.** Let  $f$  be a real valued functional defined on a vector space  $X$ . Define the (one sided) Gateaux differential of  $f$  at  $x$  with increment  $h$  to be

$$\delta f(x; h) = \lim_{\alpha \downarrow 0} \frac{1}{\alpha} [f(x + \alpha h) - f(x)]$$

If this limit exists for each  $h \in X$ ,  $f$  is Gateaux differentiable at  $x$ .

**Lemma A.14.2.** Let  $\Omega$  be a subset of the space of functions mapping  $\Theta = [\underline{\theta}, \bar{\theta}] \subset \mathbb{R}_+$  into  $\mathbb{R}$ . Let  $T : \Omega \rightarrow \mathbb{R}$  be defined by  $T(x) = \int_{\Theta} \psi(x(\theta))f(\theta) d\theta$ . Suppose that  $\psi$  is convex, its left and right hand side derivatives  $\psi'_-(x(\theta)), \psi'_+(x(\theta))$  exist and are continuous in  $x(\theta)$ , and there exists  $\varepsilon > 0$  such that  $x + \alpha h \in \Omega$ . Then

$$\delta T(x; h) = \int_{\{\theta \in \Theta: h(\theta) > 0\}} \psi'_+(x(\theta))h(\theta)f(\theta) d\theta + \int_{\{\theta \in \Theta: h(\theta) < 0\}} \psi'_-(x(\theta))h(\theta)f(\theta) d\theta$$

Note that we can equivalently write this as

$$\delta T(x; h) = \int_{\Theta} [\psi'_+(x(\theta))h_+(\theta) + \psi'_-(x(\theta))h_-(\theta)]f(\theta) d\theta$$

where  $h_+(\theta) = \max\{h(\theta), 0\}$  and  $h_-(\theta) = \min\{h(\theta), 0\}$ .

*Proof.* The proof is essentially identical to the proof of Lemma A.1 in [Amador et al. \[2006\]](#). The only reason this result is not a special case of theirs is that  $\psi$  may not be differentiable, although its left and right hand derivatives exist.

Define  $\Theta_+ = \{\theta \in \Theta : h(\theta) > 0\}$ ,  $\Theta_- = \{\theta \in \Theta : h(\theta) < 0\}$ . From the definition of the Gateaux differential,

$$\begin{aligned} \delta T(x; h) &= \lim_{\alpha \downarrow 0} \frac{1}{\alpha} [T(x + \alpha h) - T(x)] \\ &= \lim_{\alpha \downarrow 0} \int_{\Theta} \frac{1}{\alpha} [\psi(x(\theta) + \alpha h(x)) - \psi(x(\theta))]f(\theta) d\theta \\ &= \lim_{\alpha \downarrow 0} \int_{\Theta_+} \frac{1}{\alpha} [\psi(x(\theta) + \alpha h(x)) - \psi(x(\theta))]f(\theta) d\theta + \lim_{\alpha \downarrow 0} \int_{\Theta_-} \frac{1}{\alpha} [\psi(x(\theta) + \alpha h(x)) - \psi(x(\theta))]f(\theta) d\theta \end{aligned}$$

Adding and subtracting  $\int_{\Theta_+} \psi'_+(x(\theta))h_+(\theta)f(\theta) d\theta$  from the first term,

$$\begin{aligned} &\lim_{\alpha \downarrow 0} \int_{\Theta_+} \frac{1}{\alpha} [\psi(x(\theta) + \alpha h(x)) - \psi(x(\theta))]f(\theta) d\theta \\ &= \int_{\Theta_+} \psi'_+(x(\theta))h_+(\theta)f(\theta) d\theta + \lim_{\alpha \downarrow 0} \int_{\Theta_+} \left[ \frac{1}{\alpha} [\psi(x(\theta) + \alpha h(x)) - \psi(x(\theta))] - \psi'_+(x(\theta))h_+(\theta) \right] f(\theta) d\theta \end{aligned}$$

I will show that the last term vanishes. For  $\alpha < \varepsilon$ , we have

$$\left| \frac{1}{\alpha} [\psi(x(\theta) + \alpha h(x)) - \psi(x(\theta))] - \psi'_+(x(\theta))h_+(\theta) \right| < \left| \frac{1}{\varepsilon} [\psi(x(\theta) + \varepsilon h(x)) - \psi(x(\theta))] - \psi'_+(x(\theta))h_+(\theta) \right|$$

for all  $\theta \in \Theta_+$ , since  $\psi$  is convex. As in the proof of Lemma A.1 in [Amador et al. \[2006\]](#), this provides the required integrable bound to apply Lebesgue's Dominated Convergence Theorem,

so we have

$$\begin{aligned} & \lim_{\alpha \downarrow 0} \int_{\Theta_+} \left[ \frac{1}{\alpha} [\psi(x(\theta) + \alpha h(x)) - \psi(x(\theta))] - \psi'_+(x(\theta))h_+(\theta) \right] f(\theta) \, d\theta \\ &= \int_{\Theta_+} \lim_{\alpha \downarrow 0} \left[ \frac{1}{\alpha} [\psi(x(\theta) + \alpha h(x)) - \psi(x(\theta))] - \psi'_+(x(\theta))h_+(\theta) \right] f(\theta) \, d\theta = 0 \end{aligned}$$

by definition of  $\psi'_+(x(\theta))$ , noting that for  $\theta \in \Theta_+$ , and for  $\alpha > 0$ ,  $x(\theta) + \alpha h(x) > x(\theta)$ . It follows that the first term is equal to  $\int_{\Theta_+} \psi'_+(x(\theta))h(\theta)f(\theta) \, d\theta$ , as required. An identical argument applies to the second term. So we have

$$\delta T(x; h) = \int_{\Theta_+} \psi'_+(x(\theta))h(\theta)f(\theta) \, d\theta + \int_{\Theta_-} \psi'_-(x(\theta))h(\theta)f(\theta) \, d\theta$$

as required. □

**Lemma A.14.3.** *Define*

$$M(v_1 | \lambda_1, \lambda_2, r_1) := \lambda_1 C_1(v_1, r_1) + \lambda_2 C_2(v_1, r_1)$$

where  $\lambda_1, \lambda_2 \geq 0$  and  $\frac{\lambda_1}{\lambda_2} \leq (1 + r_1)(1 - \beta)$ .

1.  $M$  is convex in  $v_1$ .
2. If  $\frac{\lambda_1}{\lambda_2} = (1 + r_1)(1 - \beta)$ ,  $M(v_1 | \lambda_1, \lambda_2, r_1) = \lambda_1 E(v_1, r_1)$ , and  $M$  is differentiable.
3. If  $r_1 = 0$  and  $\frac{\lambda_1}{\lambda_2} < (1 - \beta)$ ,  $M$  is left and right differentiable, and it is differentiable except at  $v_1 = \bar{v}_1(0)$ .

*Proof.*  $M = \lambda_1 E_1(v_1, r_1) + \left( \lambda_2 - \frac{\lambda_1}{(1 + r_1)(1 - \beta)} \right) C_2(v_1, r_1)$  is the non-negative-weighted sum of convex functions and is therefore convex. It is immediate that if  $\frac{\lambda_1}{\lambda_2} = (1 + r_1)(1 - \beta)$ , and  $M$  is differentiable. If  $r_1 = 0$  and  $\lambda_2 > \frac{\lambda_1}{(1 - \beta)}$ ,  $M$  is the weighted sum of a differentiable function  $E$ , and a function  $C_2$  which is left- and right-differentiable everywhere, and differentiable except at  $\bar{v}_1(0)$ . The desired result follows. □

**Lemma A.14.4.** *Define the Lagrangian*

$$\begin{aligned} \mathcal{L}(u_0, \bar{v}) = & \bar{v} + \int (1 - A(\theta))u_0(\theta) \, d\theta - \lambda_0 \int C_0(u_0(\theta))f(\theta) \, d\theta \\ & \int M \left( \beta^{-1} \left[ \bar{v} + \int_{\theta}^{\theta} u_0(z) \, dz - \theta u_0(\theta) \right] | \lambda_1, \lambda_2, r_1 \right) f(\theta) \, d\theta \end{aligned}$$

where

$$M(v_1(\theta)|\lambda_1, \lambda_2, r_1) = \lambda_1 C_1(v_1(\theta), r_1) + \lambda_2 C_2(v_1(\theta), r_1)$$

The Gateaux differential of the Lagrangian is

$$\begin{aligned} \delta \mathcal{L}(u_0, \underline{v}; \Delta_0, \underline{\Delta}) &= \underline{\Delta} + \int (1 - A(\theta)) \Delta_0(\theta) \, d\theta - \lambda_0 \int C'_0(u_0(\theta)) \Delta_0(\theta) f(\theta) \, d\theta \\ &\quad - \int_{\Theta_+} M'_+(v_1(\theta)|\lambda_1, \lambda_2, r_1) \beta^{-1} \left[ \underline{\Delta} + \int_{\underline{\theta}}^{\theta} \Delta_0(z) \, dz - \theta \Delta_0(\theta) \right] f(\theta) \, d\theta \\ &\quad - \int_{\Theta_-} M'_-(v_1(\theta)|\lambda_1, \lambda_2, r_1) \beta^{-1} \left[ \underline{\Delta} + \int_{\underline{\theta}}^{\theta} \Delta_0(z) \, dz - \theta \Delta_0(\theta) \right] f(\theta) \, d\theta \end{aligned}$$

where  $v_1(\theta) = \beta^{-1} \left[ \underline{v} + \int_{\underline{\theta}}^{\theta} u_0(z) \, dz - \theta u_0(\theta) \right]$ , and

$$\begin{aligned} \Theta_+ &= \left\{ \theta \in \Theta : \underline{\Delta} + \int_{\underline{\theta}}^{\theta} \Delta_0(z) \, dz - \theta \Delta_0(\theta) > 0 \right\} \\ \Theta_- &= \left\{ \theta \in \Theta : \underline{\Delta} + \int_{\underline{\theta}}^{\theta} \Delta_0(z) \, dz - \theta \Delta_0(\theta) < 0 \right\} \end{aligned}$$

*Proof.* By Lemma A.14.2, the Gateaux differential of

$$\underline{v} + \int (1 - A(\theta)) u_0(\theta) \, d\theta - \lambda_0 \int C_0(u_0(\theta)) f(\theta) \, d\theta$$

with increment  $\Delta_0, \underline{\Delta}$  is

$$\underline{\Delta} + \int (1 - A(\theta)) \Delta_0(\theta) \, d\theta - \lambda_0 \int C'_0(u_0(\theta)) \Delta_0(\theta) f(\theta) \, d\theta$$

since  $C_0$  is convex and differentiable. The Gateaux differential of  $\int M(v_1(\theta)|\lambda_1, \lambda_2, r_1) f(\theta) \, d\theta$  with increment  $\Delta_1(\theta)$  is

$$\int_{\Theta_+} M'_+(v_1(\theta)|\lambda_1, \lambda_2, r_1) \Delta_1(\theta) f(\theta) \, d\theta + \int_{\Theta_-} M'_-(v_1(\theta)|\lambda_1, \lambda_2, r_1) \Delta_1(\theta) f(\theta) \, d\theta$$

where  $\Theta_+ = \{\theta \in \Theta : \Delta_1(\theta) > 0\}$ ,  $\Theta_- = \{\theta \in \Theta : \Delta_1(\theta) < 0\}$ . This follows because  $M$  is convex and both left- and right- differentiable. Defining

$$\Delta_1(\theta) = \beta^{-1} \left[ \underline{\Delta} + \int_{\underline{\theta}}^{\theta} \Delta_0(z) \, dz - \theta \Delta_0(\theta) \right],$$

it follows that the Gateaux differential of

$$\int M \left( \beta^{-1} \left[ \underline{v} + \int_{\underline{\theta}}^{\theta} u_0(z) \, dz - \theta u_0(\theta) \right] | \lambda_1, \lambda_2, r_1 \right) f(\theta) \, d\theta$$

is

$$\begin{aligned} & \int_{\Theta_+} M'_+(v_1(\theta)|\lambda_1, \lambda_2, r_1)\beta^{-1} \left[ \underline{\Delta} + \int_{\underline{\theta}}^{\theta} \Delta_0(z) dz - \theta\Delta_0(\theta) \right] f(\theta) d\theta \\ & + \int_{\Theta_-} M'_-(v_1(\theta)|\lambda_1, \lambda_2, r_1)\beta^{-1} \left[ \underline{\Delta} + \int_{\underline{\theta}}^{\theta} \Delta_0(z) dz - \theta\Delta_0(\theta) \right] f(\theta) d\theta \end{aligned}$$

The result then follows. □

## A.15 Proof of Lemma 5.5.

**Lemma A.15.1.** *Let  $f$  be a concave, real valued functional defined on a vector space  $X$ . Suppose that  $f$  is (one-sided) Gateaux differentiable. Then  $x_0$  maximizes  $f$  on the convex set  $\Omega \subset X$  if and only if*

$$\delta f(x_0; x - x_0) \leq 0$$

for all  $x \in \Omega$ .

*Proof.* The proof of necessity is essentially identical to [Luenberger \[1969\]](#) Theorem 1, p178, except that I use the one-sided Gateaux differential. Suppose  $x_0$  maximizes  $f$ . Since  $\Omega$  is convex,  $x_0 + \alpha(x - x_0) \in \Omega$  for  $0 \leq \alpha \leq 1$ . So for any  $x \in \Omega$ ,

$$\begin{aligned} f(x_0 + \alpha(x - x_0)) - f(x_0) &\leq 0 \\ \frac{1}{\alpha}[f(x_0 + \alpha(x - x_0)) - f(x_0)] &\leq 0 \\ \lim_{\alpha \downarrow 0} \frac{1}{\alpha}[f(x_0 + \alpha(x - x_0)) - f(x_0)] &\leq 0. \end{aligned}$$

The proof of sufficiency follows [Luenberger \[1969\]](#) Lemma 1, p227. Suppose there exists  $x_0 \in \Omega$  such that  $\delta f(x_0; x - x_0) \leq 0, \forall x \in \Omega$ . Take any  $x \in \Omega$ . Since  $f$  is concave, for any  $\alpha \in [0, 1]$ ,

$$\begin{aligned} f(x_0 + \alpha(x - x_0)) &\geq f(x_0) + \alpha[f(x) - f(x_0)] \\ f(x) - f(x_0) &\leq \frac{1}{\alpha}[f(x_0 + \alpha(x - x_0)) - f(x_0)] \end{aligned}$$

Taking limits, and using the definition of the Gateaux differential,

$$f(x) - f(x_0) \leq \delta f(x_0; x - x_0) \leq 0$$

by assumption. So  $f(x_0) \geq f(x)$  for any  $x \in \Omega$ , and  $x_0$  is a maximum. □

Lemma 5.5 then follows from applying Lemma [A.15.1](#) to Proposition 5.3, and using the definition of the Gateaux differential in [A.14.2](#). Note that the Lagrangian is concave.

## A.16 Proof of Proposition 5.6.

**Lemma A.16.1.** *If  $r_1 > 0$  in equilibrium, the equilibrium is constrained efficient.*

*Proof.* Suppose  $c_0, c_1, c_2, r_0, r_1 > 0$  is an equilibrium. Set

$$\begin{aligned}\frac{1}{\lambda_1} &= \int \frac{f(\theta)}{\beta u'(c_1(\theta))} d\theta \\ \frac{\lambda_0}{\lambda_1} &= 1 + r_0 \\ g(\theta) &= \lambda_1, \forall \theta\end{aligned}$$

In a competitive equilibrium, we have  $\frac{\beta u'(c_1(\theta))}{\theta u'(c_0(\theta))} = \frac{1}{1 + r_0}$  for all  $\theta$ . Since the ZLB does not bind,  $\lambda(\theta) = \lambda_1$  for all  $\theta$ . Thus the first order sufficient conditions for optimality are satisfied.  $\square$

**Lemma A.16.2.** *Suppose  $u_0(\theta)$  is continuous and strictly increasing on  $(\theta_1, \theta_2)$ , and  $\alpha(\theta_1)$  is continuous on  $(\theta_1, \theta_2)$ . Suppose also that  $\int_{\Theta} \alpha(\theta) \Delta_0(\theta) d\theta \leq 0$  for all functions  $\Delta_0 : \Theta \rightarrow \mathbb{R}$  such that  $u_0 + \Delta_0$  is increasing. Then  $\alpha(\theta) = 0$  for all  $\theta \in (\theta_1, \theta_2)$ .*

*Proof.* Suppose by contradiction that  $\alpha(\theta) > 0$  for some  $\theta \in [\theta_1, \theta_2]$ . Then by continuity,  $\alpha(\theta) > 0$  on some interval  $[\theta'_1, \theta'_2] \subset [\theta_1, \theta_2]$ . Set

$$\Delta_0(\theta) = \frac{(u_0(\theta) - u_0(\theta_1))(u_0(\theta_2) - u_0(\theta))}{(u_0(\theta_2) - u_0(\theta_1))^2} \text{ for } \theta \in [\theta'_1, \theta'_2],$$

$\Delta_0(\theta) = 0$  everywhere else. It can be verified that  $u_0(\theta) + \Delta_0(\theta)$  is increasing, is an admissible deviation, and is positive for  $\theta \in (\theta'_1, \theta'_2)$ . Then we have

$$\int \alpha(\theta) \Delta_0(\theta) d\theta > 0,$$

a contradiction.  $\square$

**Lemma A.16.3.** *If  $u_0(\theta)$  is continuous and strictly increasing and  $u_0, \bar{v}, r_1$  solves (PP'), then there exist Lagrange multipliers  $\lambda_0, \lambda_1, \lambda_2$  such that*

$$1 = \int M'(v_1(\theta) | \lambda_1, \lambda_2, r_1) f(\theta) d\theta \tag{A.48}$$

$$\lambda_0 C'(u_0(\theta)) f(\theta) - \beta^{-1} \theta M'(v_1(\theta) | \lambda_1, \lambda_2, r_1) f(\theta) = \int_{\theta}^{\bar{\theta}} [a(z) - \beta^{-1} M'(v_1(z) | \lambda_1, \lambda_2, r_1) f(z)] dz \tag{A.49}$$

for all  $\theta \in \Theta$ , unless  $r_1 = 0$  and  $\bar{v}_1(0) = v_1(\theta)$ . If the ZLB is slack,  $\frac{\lambda_1}{\lambda_2} = (1 + r_1)(1 - \beta)$ . If the ZLB binds,  $\frac{\lambda_1}{\lambda_2} < 1 - \beta$ .

Conversely, if the ZLB is slack and (A.48), (A.49) hold for all  $\theta \in \Theta$  with  $u_0$  continuous and strictly increasing, and if  $u_0, \underline{v}, r_1$  satisfy the constraints in (PP'), then  $u_0, \underline{v}, r_1$  solve (PP').

*Proof.* Suppose  $u_0$  is continuous and strictly increasing. Then in any incentive compatible allocation,  $v_1$  is continuous and strictly decreasing. It follows that there exists at most one type  $\theta^*$  such that  $v_1(\theta) = \bar{v}_1(r_1)$  (that is,  $\theta$  is 'just' liquidity constrained). By Lemma A.14.3,  $M$  is differentiable everywhere except at  $\bar{v}_1(r_1)$  (and if the ZLB is slack, it is differentiable at this point too). Thus the first-order condition necessary condition for a maximum, stated in Lemma 5.5, can be rewritten as

$$\begin{aligned} \delta \mathcal{L}(u_0, \underline{v}; \Delta_0, \underline{\Delta}) = & \underline{\Delta} + \int (1 - A(\theta)) \Delta_0(\theta) d\theta - \lambda_0 \int C'_0(u_0(\theta)) \Delta_0(\theta) f(\theta) d\theta \\ & - \int M'(v_1(\theta) | \lambda_1, \lambda_2, r_1) \beta^{-1} \left[ \underline{\Delta} + \int_{\theta}^{\bar{\theta}} \Delta_0(z) dz - \theta \Delta_0(\theta) \right] f(\theta) d\theta \leq 0 \end{aligned}$$

for all  $\underline{\Delta}, \Delta_0$  such that  $u_0 + \Delta_0$  is increasing. Applying Fubini's Theorem to reverse the order of integration, and rearranging terms, we can rewrite this as

$$\begin{aligned} \int \Delta_0(\theta) \left\{ -\lambda_0 C'_0(u_0(\theta)) f(\theta) + \beta^{-1} \theta M'(v_1(\theta)) f(\theta) - \int_{\theta}^{\bar{\theta}} [a(z) - \beta^{-1} M'(v_1(z)) f(z)] dz \right\} d\theta \\ + \underline{\Delta} \left\{ 1 - \int M'(v_1(\theta)) f(\theta) d\theta \right\} \leq 0 \text{ for all } \underline{\Delta}, \Delta_0 \text{ such that } u_0 + \Delta_0 \in \Omega \end{aligned}$$

where we suppress the dependence of  $M'$  on  $\lambda_1, \lambda_2, r_1$  to save notation. Clearly, we must have  $1 - \int M'(v_1(\theta)) f(\theta) d\theta = 0$ , so (A.48) holds. To show that (A.49) holds, we use Lemma A.16.2, first setting  $[\theta_1, \theta_2] = [\theta, \theta^*]$ , and then setting  $[\theta_1, \theta_2] = [\theta^*, \bar{\theta}]$ . Since

$$\alpha(\theta) := -\lambda_0 C'_0(u_0(\theta)) f(\theta) + \beta^{-1} \theta M'(v_1(\theta)) f(\theta) + \int_{\theta}^{\bar{\theta}} [a(z) - \beta^{-1} M'(v_1(z)) f(z)] dz$$

is continuous on  $[\theta, \theta^*)$  and on  $(\theta^*, \bar{\theta}]$ , it follows that  $\alpha(\theta) = 0$  for any  $\theta \neq \theta^*$ . If the ZLB does not bind,  $M'$  exists and is continuous everywhere, and we also have  $\alpha(\theta^*) = 0$ . Then (A.49) holds.

Finally, if the ZLB is slack, and if  $\alpha(\theta) = 0$  everywhere, then clearly  $\int \alpha(\theta) \Delta_0(\theta) d\theta = 0$  for all  $\Delta_0$ , and by Lemma 5.5,  $u_0, \underline{v}, r_1$  solve (PP').  $\square$

**Lemma A.16.4.** *If  $r_1 > 0$ ,  $\int c_1(\theta) f(\theta) d\theta < y^*$  in equilibrium, the equilibrium is constrained inefficient.*

*Proof.* Suppose by contradiction that such an allocation  $u_0, \bar{v}$  solves (PP') for some non-negative Pareto weights  $a(\theta)$  (equivalently, for some differentiable, nondecreasing  $A(\theta) = \int_{\theta}^{\bar{\theta}} a(z) dz$ ). In any competitive equilibrium, allocations and utilities are continuous in  $\theta$ . Consequently, by Lemma A.16.3, there must exist Lagrange multipliers  $\lambda_0 > 0, \lambda_1 = 0, \lambda_2 > 0$  such that

$$\lambda_0 C'(u_0(\theta))f(\theta) - \beta^{-1}\theta M'(v_1(\theta)|\lambda_1, \lambda_2, r_1)f(\theta) = 1 - A(\theta) - \int_{\theta}^{\bar{\theta}} \beta^{-1}M'(v_1(z)|\lambda_1, \lambda_2, r_1)f(z) dz$$

Let  $\theta^*$  be the type who is just liquidity constrained. For  $\theta < \theta^*$ , this condition states that

$$1 - A(\theta) = \lambda_0 \frac{f(\theta)}{u'(c_0(\theta))} - \frac{\theta f(\theta)\mu\lambda_2}{\beta u'(c_1(\theta))} + \int_{\theta}^{\theta^*} \frac{f(z)\mu\lambda_2}{\beta u'(c_1(z))} dz$$

where  $\mu := \frac{(1-\beta)\beta^{1/\sigma}}{1-\beta+\beta^{1/\sigma}}$ , the date 2 MPC of unconstrained households. As  $\theta \rightarrow \theta^*$  from below, since all right hand side terms are continuous,

$$1 - A(\theta) \rightarrow \lambda_0 \frac{f(\theta^*)}{u'(c_0(\theta^*))} - \frac{\theta^* f(\theta^*)\mu\lambda_2}{\beta u'(c_1(\theta^*))}$$

For  $\theta > \theta^*$ ,

$$1 - A(\theta) = \lambda_0 \frac{f(\theta)}{u'(c_0(\theta))} \rightarrow \lambda_0 \frac{f(\theta^*)}{u'(c_0(\theta^*))}$$

as  $\theta \rightarrow \theta^*$  from above. These two limits are not the same, which contradicts the hypothesis that  $A(\theta)$  was continuous. So the allocation cannot be a solution to (PP') for any welfare weights, and is not constrained efficient.  $\square$

An alternative proof is as follows.

*Proof.* Consider a deviation, relative to a competitive equilibrium allocation in which the ZLB binds, in which we add the point  $(\bar{c}, \bar{a}_1)$  to the budget set. In utility space, this point is  $\bar{u}, \bar{v}_1$ . If  $\bar{u} = u_0(\theta^*)$ , there is no deviation. For any  $\bar{u} > u_0(\theta^*)$ , a set of types  $(\theta_1(\bar{u}), \theta_2(\bar{u}))$  (containing  $\theta^*$ ) will be attracted to the deviation, where

$$\theta_1(\bar{u})\bar{u} + \beta\bar{v}_1 = v(\theta_1(\bar{u}))$$

$$\theta_2(\bar{u})\bar{u} + \beta\bar{v}_1 = v(\theta_2(\bar{u}))$$

Their derivatives are

$$\theta'_i(\bar{u}) = -\frac{\theta_i(\bar{u})}{\bar{u} - u_0(\theta_i(\bar{u}))}, i = 1, 2$$

The change in date 0 consumption induced by this deviation is

$$\Delta C_0(\bar{u}, \theta^*) = \int_{\theta_1(\bar{u})}^{\theta^*} [c_0(\bar{u}) - c_0(\theta)] f(\theta) d\theta - \int_{\theta^*}^{\theta_2(\bar{u})} [c_0(\theta) - c_0(\bar{u})] f(\theta) d\theta$$

Taking derivatives,

$$\Delta C'_0(\bar{u}, \theta^*) = c'_0(\bar{u}) [F(\theta_2(\bar{u})) - F(\theta_1(\bar{u}))] + f(\theta_1(\bar{u})) \theta_1(\bar{u}) \frac{c_0(\bar{u}) - c_0(\theta_1(\bar{u}))}{\bar{u} - u_0(\theta_1(\bar{u}))} - \theta_2(\bar{u}) \frac{c_0(\bar{u}) - c_0(\theta_2(\bar{u}))}{\bar{u} - u_0(\theta_2(\bar{u}))}$$

All these terms vanish as  $\bar{u} \rightarrow u(\theta^*)$ , so a small deviation has no first order effect on date 0 consumption.

The change in date 2 consumption induced by this deviation is

$$\Delta C_2(\bar{u}, \theta^*) = \int_{\theta_1(\bar{u})}^{\theta^*} [c_2 - c_2(\theta)] f(\theta) d\theta$$

The derivative of the change in date 2 consumption is

$$\begin{aligned} \Delta C'_2(\bar{u}, \theta^*) &= f(\theta_1(\bar{u})) \theta_1(\bar{u}) \frac{c_2 - c_2(\theta_1(\bar{u}))}{\bar{u} - u_0(\theta_1(\bar{u}))} \\ &= f(\theta_1(\bar{u})) \theta_1(\bar{u}) \frac{c_2 - c_2(\theta_1(\bar{u}))}{c_0(\theta^*) - c_0(\theta_1(\bar{u}))} \frac{c_0(\theta^*) - c_0(\theta_1(\bar{u}))}{c_0(\bar{u}) - c_0(\theta_1(\bar{u}))} \frac{c_0(\bar{u}) - c_0(\theta_1(\bar{u}))}{\bar{u} - u_0(\theta_1(\bar{u}))} \end{aligned}$$

The first fraction is equal to a negative constant, and the third fraction converges to  $C'_0(u(\theta^*)) >$

0. Consider the middle term,

$$\frac{g(\bar{u})}{h(\bar{u})} := \frac{c_0(\theta^*) - c_0(\theta_1(\bar{u}))}{c_0(\bar{u}) - c_0(\theta_1(\bar{u}))}$$

. Both  $g(\bar{u})$  and  $h(\bar{u})$  converge to zero as  $\bar{u} \rightarrow u_0(\theta^*)$ .

$$\begin{aligned} g'(\bar{u}) &= c'_0(\theta_1(\bar{u})) \frac{\theta_1(\bar{u})}{\bar{u} - u_0(\theta_1(\bar{u}))} \\ h'(\bar{u}) &= c'_0(\bar{u}) - c'_0(\theta_1(\bar{u})) \frac{\theta_1(\bar{u})}{\bar{u} - u_0(\theta_1(\bar{u}))} \end{aligned}$$

So we have

$$\frac{g'(\bar{u})}{h'(\bar{u})} = \frac{-c'_0(\theta_1(\bar{u})) \frac{\theta_1(\bar{u})}{\bar{u} - u_0(\theta_1(\bar{u}))}}{c'_0(\bar{u}) - c'_0(\theta_1(\bar{u})) \frac{\theta_1(\bar{u})}{\bar{u} - u_0(\theta_1(\bar{u}))}} \rightarrow 1 \text{ as } \bar{u} \rightarrow u(\theta^*).$$

Thus  $\Delta C'_2(\bar{u}, \theta^*)$  converges to a positive number as  $\bar{u} \rightarrow u_0(\theta^*)$ .

Putting this all together, we see that a small deviation has no first order effect on  $C_0$ , and causes a first order reduction in  $C_2$ . It must therefore decrease the value of the Lagrangian. So the original allocation was inefficient.

The only assumption we used here was that  $u_0(\theta)$  is continuous and strictly increasing at  $\theta^*$ . The above argument thus shows that such an allocation can never be optimal when the date 1 resource constraint is slack.  $\square$

### A.16.1 Efficiency of piecewise linear equilibria

**Definition A.16.5.**  $c_0, c_1, c_2, r_1 = 0$  is a full employment piecewise linear equilibrium (FEPL) if  $\int c_t(\theta)f(\theta) = y^*$ ,  $t = 0, 1, 2$ , and there exist  $R_B \geq R_S, \theta_B > \theta_S$  such that

1. for  $\theta > \theta_B$ :

$$R_B = \frac{\theta u'(c_0(\theta))}{\beta u'(c_1(\theta))}$$

$$c_2(\theta) = y^* - (1 - \beta)\phi$$

$$u'(c_1(\theta)) \geq \beta u'(c_2(\theta))$$

2. for  $\theta < \theta_S$ :

$$R_S = \frac{\theta u'(c_0(\theta))}{\beta u'(c_1(\theta))}$$

$$c_2(\theta) \geq y^* - (1 - \beta)\phi$$

$$u'(c_1(\theta)) \geq \beta u'(c_2(\theta))$$

3. for  $\theta \in [\theta_S, \theta_B]$ :

$$c_0(\theta) = c_0 \text{ constant}$$

$$\frac{\theta u'(c_0(\theta))}{\beta u'(c_1(\theta))} \in (R_S, R_B)$$

$$c_2(\theta) = y^* - (1 - \beta)\phi$$

$$u'(c_1(\theta)) = \beta u'(c_2(\theta))$$

**Proposition A.16.6.** *Every FEPL is constrained efficient.*

*Proof.* Suppose  $c_0, c_1, c_2, r_0$  satisfies the conditions in Definition A.16.5, and let  $u_0 = u(c_0(\theta))$ ,  $\bar{v} = \theta u(c_0(\theta)) + \beta u(c_1(\theta)) + \frac{\beta^2}{1 - \beta} u(c_2(\theta))$  be the associated utilities. The proof proceeds by constructing Lagrange multipliers  $\lambda_0, \lambda_1, \lambda_2$  and Pareto weights  $a(\theta)$  so that the sufficient conditions in Lemma 5.5 are satisfied.

Note that the proposed allocation has a discontinuity at  $\theta_S$ . Furthermore, there is a set of types  $[\theta_S, \theta_B]$  with positive measure who have  $v_1(\theta) = \bar{v}_1(0)$ , and for whom  $M(v_1(\theta))$  is not differentiable.

$$\begin{aligned}\lambda_0 &= \left[ \int \frac{C'_0(u_0(\theta))f(\theta)}{\theta} d\theta \right]^{-1} \\ a(\theta) &= \frac{\lambda_0 C'_0(u_0(\theta))f(\theta)}{\theta} \\ \lambda_1 &= \frac{\lambda_0}{R_B} \\ \lambda_2 &= \beta^{-1/\sigma} \left[ \frac{1-\beta+\beta^{1/\sigma}}{R_S(1-\beta)} \lambda_0 - \lambda_1 \right]\end{aligned}$$

Note that since  $R_S < R_B$ ,  $\lambda_1 < (1-\beta)\lambda_2$ , as required.

Next, note that by construction, for  $\theta < \theta_S$ ,  $\lambda(\theta) = \frac{\lambda_0}{R_S}$ , and

$$M'(v_1(\theta)) = \frac{\lambda_0}{R_S u'(c_1(\theta))} = \frac{\lambda_0 \beta}{\theta u'(c_0(\theta))} = \frac{\beta a(\theta)}{f(\theta)} \quad (\text{A.50})$$

where the second equality uses the definition of a FEPL, and the third equality uses the construction of the Pareto weights. Similarly, for  $\theta > \theta_B$ ,  $\lambda(\theta) = \frac{\lambda_0}{R_B}$  and

$$M'(v_1(\theta)) = \frac{\lambda_0}{R_B u'(c_1(\theta))} = \frac{\lambda_0 \beta}{\theta u'(c_0(\theta))} = \frac{\beta a(\theta)}{f(\theta)} \quad (\text{A.51})$$

Finally, for  $\theta \in [\theta_S, \theta_B]$ , we have  $v_1(\theta) = \bar{v}_1$  and

$$\frac{\beta a(\theta)}{f(\theta)} = \frac{\lambda_0 \beta}{\theta u'(c_0(\theta))} \in \left[ \frac{\lambda_0}{R_B u'(c_1(\theta))}, \frac{\lambda_0}{R_B u'(c_1(\theta))} \right] = [M'_-(\bar{v}_1), M'_+(\bar{v}_1)] \quad (\text{A.52})$$

Define  $M'(v_1(\theta)|\Delta_1(\theta)) = M'_+(v_1(\theta))$  if  $\Delta_1(\theta) > 0$ ,  $M'_-(v_1(\theta))$  if  $\Delta_1(\theta) < 0$ . Note that since

$$\underline{\Delta} + \int_{\underline{\theta}}^{\theta} \Delta_0(z) dz = \theta \Delta_0(\theta) + \beta \Delta_1(\theta)$$

we can write the Gateaux differential of the Lagrangian as follows:

$$\begin{aligned}\delta \mathcal{L}(u_0, \bar{v}; \Delta_0, \underline{\Delta}) &= \int_{\Theta} a(\theta) [\theta \Delta_0(\theta) + \beta \Delta_1(\theta)] - \lambda_0 \int C'_0(u_0(\theta)) \Delta_0(\theta) f(\theta) d\theta \\ &\quad - \int M'(v_1(\theta)|\Delta_1(\theta)) f(\theta) d\theta \\ &= \int_{\Theta} \Delta_0(\theta) \{ \theta a(\theta) - \lambda_0 C'_0(u_0(\theta)) f(\theta) \} d\theta + \int_{\Theta} \Delta_1(\theta) \{ \beta a(\theta) - M'(v_1(\theta)|\Delta_1(\theta)) f(\theta) \} d\theta\end{aligned}$$

The first term in curly brackets is zero, by definition of  $a(\theta)$ . The second term is zero except for  $\theta \in [\theta_S, \theta_B]$ , by (A.50) and (A.51). So we have

$$\delta \mathcal{L}(u_0, \bar{v}; \Delta_0, \underline{\Delta}) = \int_{\theta_S}^{\theta_B} \Delta_1(\theta) \{ \beta a(\theta) - M'(\bar{v}_1|\Delta_1(\theta)) f(\theta) \} d\theta$$

By (A.52),  $\beta a(\theta) - M'(\bar{v}_1|\Delta_1(\theta))f(\theta)$  is positive when  $\Delta_1(\theta) < 0$  and negative when  $\Delta_1(\theta) > 0$ . It follows that

$$[\beta a(\theta) - M'(\bar{v}_1|\Delta_1(\theta))f(\theta)]\Delta_1(\theta) \leq 0, \forall \theta \in [\theta_S, \theta_B]$$

So we must have  $\delta\mathcal{L}(u_0, v; \Delta_0, \underline{\Delta}) \leq 0$ , as required.  $\square$

Alternatively, the Proposition can be proved by showing that a FEPL solves a relaxed Pareto problem without incentive constraints. Since it also satisfies the incentive constraints, it must solve the restricted Pareto problem (PP').

*Proof.* Consider the relaxed problem

$$\begin{aligned} & \max_{u_0, v_1, r_1} \int a(\theta) [\theta u_0(\theta) + \beta v_1(\theta)] d\theta \\ & \text{s.t. } \int C_0(u_0(\theta))f(\theta) d\theta \leq y^* \\ & \int C_1(v_1(\theta), r_1)f(\theta) d\theta \leq y^* \\ & \int C_2(v_1(\theta), r_1)f(\theta) d\theta \leq y^* \\ & r_1 \geq 0 \end{aligned}$$

and suppose that the ZLB binds at an optimum. The same arguments as above demonstrate that solutions to this problem solve a Lagrangian

$$\int a(\theta) [\theta u_0(\theta) + \beta v_1(\theta)] - \lambda_0 \int C(u_0(\theta))f(\theta) d\theta - \int M(v_1(\theta))$$

The first order necessary and sufficient conditions for optimality are

$$\begin{aligned} & a(\theta)\theta - \lambda_0 C'(u_0(\theta))f(\theta) = 0 \\ & a(\theta)\beta - M'(v_1(\theta))f(\theta) = 0 \text{ if } v_1(\theta) \neq \bar{v}_1(0) \\ & a(\theta)\beta - M'_+(v_1(\theta))f(\theta) \leq 0, \text{ if } v_1(\theta) = \bar{v}_1(0) \\ & a(\theta)\beta - M'_-(v_1(\theta))f(\theta) \geq 0 \end{aligned}$$

As in the previous proof, define

$$\lambda_0 = \left[ \int \frac{C'_0(u_0(\theta))f(\theta)}{\theta} d\theta \right]^{-1}$$

$$a(\theta) = \frac{\lambda_0 C'_0(u_0(\theta))f(\theta)}{\theta}$$

$$\lambda_1 = \frac{\lambda_0}{R_B}$$

$$\lambda_2 = \beta^{-1/\sigma} \left[ \frac{1 - \beta + \beta^{1/\sigma}}{R_S(1 - \beta)} \lambda_0 - \lambda_1 \right]$$

By construction, a FEPLS satisfies the first order conditions with these multipliers. So a FEPLS solves a relaxed Pareto problem. As in the previous proof, any FEPLS is incentive compatible. It follows that a FEPLS solves the restricted problem (PP').  $\square$

### A.16.2 Existence of a FEPLS

Let  $R_S = (1 + r_0)$ . Define  $c_0(y_S, y_B, R_B; \theta), a_1(y_S, y_B, R_B; \theta)$  as the solutions to

$$\max_{c_0, a_1} \theta u(c_0) + \beta V(a_1) \tag{A.53}$$

$$\text{s.t. } a_1 = \bar{a}_1 + R_S(y_S - c_0) \text{ if } c_0 \leq y_S \tag{A.54}$$

$$a_1 = \bar{a}_1 \text{ if } c_0 \in [y_S, y_B] \tag{A.55}$$

$$a_1 = \bar{a}_1 - R_B(c_0 - y_B) \text{ if } c_0 \geq y_B \tag{A.56}$$

where from now on we suppress dependence on  $r_1$ , since this will always equal zero (i.e. we write  $V(a_1)$  for  $V(a_1, 0)$ ,  $\bar{a}_1$  for  $\bar{a}_1(0)$ , etc.) We also suppress dependence on  $R_S$ , since this will always be equal to  $1 + r_0$ . Define  $c_t(y_S, y_B, R_B; \theta) = X_t(a_1(y_S, y_B, R_B; \theta))$  for  $t = 1, 2$ . Define the aggregate excess demand functions

$$Z_t(y_S, y_B, R_B) = \int c_t(y_S, y_B, R_B; \theta) f(\theta) d\theta - y^*$$

If the ZLB binds in equilibrium, there exist  $\bar{y}, R_S = 1 + r_0$  such that  $Z_0(\bar{y}, \bar{y}, R_S) = Z_2(\bar{y}, \bar{y}, R_S) = 0$ ,  $Z_1(\bar{y}, \bar{y}, R_S) < 0$ .

**Lemma A.16.7.**  $Z_t(y_S, y_B, R_B)$  is  $C^1$  in all its arguments for  $t = 0, 1, 2$  on the set  $\{y_S, y_B, R_B : y_B \geq y_S, R_B \geq R_S\}$ .  $Z_2$  is increasing in  $y_S$ , decreasing in  $y_B$ , and does not depend on  $R_B$ .

*Proof.* Let  $s_0, s_1, s_2$  be the solution to (A.53) subject to the constraint that  $c_0 \leq y_S$ , and define the associated value function  $V_S(y_S; \theta)$ . Let  $b_0, b_1, b_2$  be the solution subject to the constraint that  $c_0 \geq y_B$ , and define the associated value function  $V_B(y_B, R_B; \theta)$ . These programs have continuous, differentiable, concave objective functions and linear constraints; they therefore give rise to continuous policy functions and continuous, differentiable value functions. In addition, the policy functions are differentiable when the inequality constraints  $c_0 \leq y_S$ ,  $c_0 \geq y_B$  are slack.

When  $c_0 = y_B$ , the solution is  $y_B, \underline{c}_1, \underline{c}_2$ , where  $u'(\underline{c}_1) = \beta u'(\underline{c}_2)$ . Let  $\theta_S(y_S, y_B)$  be the type who is just indifferent between choosing some  $c_0 \leq y_S$  and  $c_0 = y_B$ , and let  $\theta_B(y_B, R_B)$  be the highest type who chooses  $c_0 = y_B$ .  $\theta_S$  is implicitly defined by

$$V_S(y_S; \theta_S) - \theta_S u(y_B) - \beta V(\bar{a}_1) = 0$$

By the Envelope Theorem, the derivative of this expression with respect to  $\theta_S$  is  $u(s_0(y_S; \theta_S)) - u(y_B) \leq 0$ , which is nonzero provided that  $s_0(y_S; \theta_S) \neq y_B$ , which will be true if  $y_B > y_S$ . Then by the Implicit Function Theorem, this defines  $\theta_S$  as a  $C^1$  function of  $y_S, y_B$ . It can be verified that  $\theta_S$  is increasing in  $y_S$  and decreasing in  $y_B$ .

I now show that  $Z_t$  is right-differentiable with respect to  $y_B$  when  $y_B = y_S$ . Fix  $y_S$ . Define  $\hat{\theta}$  by  $s_0(\hat{\theta}) = y_S$ . Define

$$X(y_B) = \int_{\theta(y_B)}^{\hat{\theta}} [y_B - s_0(\theta)] f(\theta) d\theta$$

where  $\theta(y_B)$  is defined by

$$V_S(\theta) = \theta u(y_B) + \beta V(\bar{a}_1)$$

Since  $\theta(y_B)$  is continuous,  $X(y_B)$  is continuous. For  $y_B > y_S$ ,

$$\theta'(y_B) = \frac{-\theta(y_B) u'(y_B)}{u(y_B) - u(s_0(\theta(y_B)))}$$

by the Implicit Function Theorem. Applying Leibniz's Theorem to  $X$ , for  $y_B > y_S$  we have

$$\begin{aligned} X'(y_B) &= -\theta'(y_B)[y_B - s_0(\theta(y_B))]f(\theta(y_B)) + \int_{\theta(y_B)}^{\hat{\theta}} f(\theta) d\theta \\ &= \theta(y_B) u'(y_B) f(\theta(y_B)) \frac{y_B - s_0(\theta(y_B))}{u(y_B) - u(s_0(\theta(y_B)))} + \int_{\theta(y_B)}^{\hat{\theta}} f(\theta) d\theta \end{aligned}$$

$$\lim_{y_B \downarrow y_S} X'(y_B) = \hat{\theta} f(\hat{\theta})$$

Thus since  $X$  is continuous,  $X'_+(y_S) = \hat{\theta} f(\hat{\theta})$ .

$\theta_B$  is explicitly defined by

$$\theta_B = \frac{\beta R_B u'(\bar{c}_1)}{u'(y_B)}$$

Then we have

$$\begin{aligned} Z_t(y_S, y_B, R_B) &= \int_{\underline{\theta}}^{\theta_S(y_S, y_B)} s_t(y_S; \theta) f(\theta) d\theta + [F(\theta_B(y_B, R_B)) - F(\theta_S(y_S, y_B))] c_t^B \\ &\quad + \int_{\theta_B(y_B, R_B)}^{\bar{\theta}} b_t(y_B, R_B; \theta) f(\theta) d\theta - y^* \end{aligned}$$

Provided that  $s_0(y_S; \theta_S) < y_B$ , we can apply Leibniz's formula to show that  $Z_t$  is  $C^1$ . If  $s_0(y_S; \theta_S) = y_B$ , then we must have  $y_B = y_S$ .

Next we show that  $Z_2$  is increasing in  $y_S$ , decreasing in  $y_B$ , and does not depend on  $R_B$ . All types with  $a_1 \leq \bar{a}_1$  are liquidity constrained, and consume  $\underline{c}_2$  at date 2. Thus we have

$$Z_2(y_S, y_B, R_B) = \int_{\underline{\theta}}^{\theta_S(y_S, y_B)} s_2(y_S; \theta) f(\theta) d\theta + [1 - F(\theta_S(y_S, y_B))] \underline{c}_2 - y^* \quad (\text{A.57})$$

which clearly does not depend on  $R_B$ .  $\theta_S$  is increasing in  $y_S$  and decreasing in  $y_B$ . Since  $s_2(y_S; \theta_S) \geq \underline{c}_2$ , it follows that  $Z_2$  is increasing in  $y_S$  and decreasing in  $y_B$ .  $\square$

Since  $Z_2$  does not depend on  $R_B$ , we henceforth write  $Z_2(y_S, y_B)$ .

**Lemma A.16.8.**  $Z_2(y_S, y_B) = 0$  defines  $y_S = \varphi_S(y_B)$  as a  $C^1$ , increasing function of  $y_B$ , with  $\varphi_S(\bar{y}) = \bar{y}$ ,  $\varphi_S(y_B) \rightarrow \infty$  as  $y_B \rightarrow \infty$ .

*Proof.* Since  $Z_2$  is  $C^1$ , increasing in  $y_S$  and decreasing in  $y_B$ , the first part follows from the Implicit Function Theorem. Given our assumption that the ZLB binds in equilibrium, there exists  $\bar{y}$  such that  $Z_2(\bar{y}, \bar{y}) = 0$ . To prove the last part, note that since  $\bar{c}_2 < y^*$ , we must have  $\theta_S > \underline{\theta}$  if  $Z_2 = 0$ . That is, we must have  $V_S(y_S; \underline{\theta}) - \theta_S u(y_B) - \beta V(\bar{a}_1) > 0$ . As  $y_B \rightarrow \infty$ , this can only be satisfied if  $y_S \rightarrow \infty$ .  $\square$

**Lemma A.16.9.**  $\Phi_0(y_B, R_B) := Z_0(\varphi(y_B), y_B, R_B)$  is  $C^1$ , increasing in  $y_B$  and decreasing in  $R_B$ .

*Proof.*  $\Phi_0$  is the composition of  $C_1$  functions and is therefore  $C_1$ . It is decreasing in  $R_B$  because  $b_0$  is decreasing in  $R_B$ .

To see that  $\Phi_0$  is increasing in  $y_B$ , suppose that  $y'_B > y_B$  and  $Z_2(y_S, y_B) = Z_2(y'_S, y'_B) = 0$ . Then  $y'_S > y_S$ . Inspecting (A.57), we see that we must have  $\theta'_S < \theta_S$ , since  $s_2$  is increasing in  $y_S$ .

There are then two effects on  $Z_0$ . The increase in  $y_S$  and  $y_B$  increases date 0 consumption for all types. And the fall in  $\theta_S$  means that some households switch from a low to a high level of date 0 consumption. The overall effect is to increase  $Z_0$ .  $\square$

**Corollary A.16.10.**  $\Phi_0(y_B, R_B) = 0$  defines  $y_B = \varphi_B(R_B)$  as a continuous, increasing function of  $R_B$ .

*Proof.* This follows immediately by applying the Implicit Function Theorem to the above result.  $\square$

**Lemma A.16.11.**  $\Phi_1(R_B) = Z_1(\varphi_S(\varphi_B(R_B)), \varphi_B(R_B), R_B)$  is a continuous function of  $R_B$  with  $\Phi_1(R_S) < 0$ ,  $\lim_{R_B \rightarrow \infty} \Phi_1(R_B) > 0$ .

*Proof.*  $\Phi_1$  is the composition of continuous functions, and is therefore continuous. Under the assumption that the ZLB binds in equilibrium,  $\varphi_S(\varphi_B(R_S)) = \varphi_B(R_S) = \bar{y}$  and  $Z_1(\varphi_S(\varphi_B(R_S)), \varphi_B(R_S), R_S) < 0$ . For  $R_B$  sufficiently high, all types bunch at  $y_B$  and consume  $\underline{c}_1 > y^*$  at date 1; it therefore follows that  $\Phi_1 > 0$ .  $\square$

**Lemma A.16.12.** There exist  $y_S > \bar{y}$ ,  $y_B > \bar{y}$ ,  $R_B > R_S$  such that  $Z_t(y_S, y_B, R_B) = 0$ ,  $t = 0, 1, 2$ .

*Proof.*  $\Phi_1(R_B)$  is continuous, negative for  $R_B = R_S$  and positive for high enough  $R_B$ . By the Intermediate Value Theorem, there exists  $R_B$  such that  $\Phi_1(R_B) = 0$ . Define  $y_S = \varphi_S(\varphi_B(R_B))$ ,  $y_B = \varphi_B(R_B)$ : the result then follows.  $\square$

The following proposition is now immediate.

**Proposition A.16.13.** Suppose the ZLB binds in equilibrium. Then there exists a FEPL.

## A.17 Proof of Proposition 6.2.

1. Take any solution to the relaxed Pareto problem, and set

$$\begin{aligned}\tau(\theta_S) &= 0 \\ 1 + \tau(\theta_B) &= \frac{u'(c_1^S)}{u'(c_1^B)} \frac{U_c(c_0^B, \theta_B)}{U_c(c_0^S, \theta_S)} \\ 1 + r_0 &= \frac{U_c(c_0^S, \theta_S)}{\beta u'(c_1^S)} \\ 1 + r_1 &= \frac{u'(c_1^S)}{\beta u'(c_2^S)} \\ T_1(\theta_S) = -T_1(\theta_B) &= (1 + r_0)(c_0^S - y^*) + c_1^S - y^* + \frac{c_2^S - y^*}{(1 + r_1)(1 - \beta)}\end{aligned}$$

It is straightforward to show that this implements the allocation as an equilibrium with macroprudential taxes. To see that  $\tau(\theta_B) = 0$  when the ZLB is slack and  $> 0$  when the ZLB binds note that the first order conditions in the planner's problem have

$$\begin{aligned}\alpha U_c(c_0^S, \theta_S) - \lambda_0 &= 0 \\ (1 - \alpha) U_c(c_0^B, \theta_B) - \lambda_0 &= 0 \\ \alpha u'(c_1^S) - \lambda_1 + \zeta u''(c_1^S) &= 0 \\ (1 - \alpha) u'(c_1^B) - \lambda_1 &= 0\end{aligned}$$

2. Take any solution to the relaxed Pareto problem. Set

$$\begin{aligned}\phi_0 &= \bar{c}_1^S - y^* + \phi \\ 1 + r_0 &= \frac{U_c(c_0^S, \theta_S)}{\beta u'(c_1^S)} \\ 1 + r_1 &= \frac{u'(c_1^S)}{\beta u'(c_2^S)} \\ T_0(\theta_S) = -T_0(\theta_B) &= c_0^S - y^* + \frac{c_1^S - y^*}{1 + r_0} + \frac{c_2^S - y^*}{(1 + r_0)(1 + r_1)(1 - \beta)}\end{aligned}$$

If the ZLB is slack, then  $c_1^S < \bar{c}_1^S$ , and the date 0 debt limit does not bind. If the ZLB binds, then in any solution to the planner's problem,  $c_1^S = 2y^* - c_1^B = \bar{c}_1^S$ , as required.

## A.18 Proof of Proposition 6.3.

The relevant first order conditions in the planner's problem are

$$\begin{aligned}\alpha U_c(c_0^S, \theta_S) - \lambda_0 + \mu_S U_c(c_0^S, \theta_S) - \mu_B U_c(c_0^S, \theta_B) &= 0 \\ (1 - \alpha) U_c(c_0^B, \theta_B) - \lambda_0 - \mu_S U_c(c_0^B, \theta_S) + \mu_B U_c(c_0^B, \theta_B) &= 0 \\ \alpha u'(c_1^S) - \lambda_1 + \zeta u''(c_1^S) + (\mu_S - \mu_B) u'(c_1^S) &= 0 \\ (1 - \alpha) u'(c_1^B) - \lambda_1 - (\mu_S - \mu_B) u'(c_1^B) &= 0\end{aligned}$$

Combining,

$$\begin{aligned}\frac{\alpha U_c(c_0^S, \theta_S) + \mu_S U_c(c_0^S, \theta_S) - \mu_B U_c(c_0^S, \theta_B)}{(1 - \alpha) U_c(c_0^B, \theta_B) - \lambda_0 - \mu_S U_c(c_0^B, \theta_S) + \mu_B U_c(c_0^B, \theta_B)} &= \frac{\alpha u'(c_1^S) + \zeta u''(c_1^S) + (\mu_S - \mu_B) u'(c_1^S)}{(1 - \alpha) u'(c_1^B) - (\mu_S - \mu_B) u'(c_1^B)} \\ \frac{U_c(c_0^S, \theta_S)}{U_c(c_0^B, \theta_B)} \frac{1 + \frac{\mu_S}{\alpha} - \frac{\mu_B U_c(c_0^S, \theta_B)}{\alpha U_c(c_0^S, \theta_S)}}{1 - \frac{\mu_S U_c(c_0^B, \theta_S)}{(1 - \alpha) U_c(c_0^B, \theta_B)} + \frac{\mu_B}{1 - \alpha}} &= \frac{u'(c_1^S)}{u'(c_1^B)} \frac{1 + \frac{\zeta u''(c_1^S)}{\alpha u'(c_1^S)} + \frac{\mu_S - \mu_B}{\alpha}}{1 - \frac{\mu_S - \mu_B}{\alpha}}\end{aligned}$$

We have

$$\frac{1 - T'(d_1^B)}{1 - T'(d_1^S)} = \frac{u'(c_1^S)}{u'(c_1^B)} \frac{U_c(c_0^B, \theta_B)}{U_c(c_0^S, \theta_S)} = \frac{1 + \frac{\mu_S}{\alpha} - \frac{\mu_B U_c(c_0^S, \theta_B)}{\alpha U_c(c_0^S, \theta_S)}}{1 + \frac{\zeta u''(c_1^S)}{\alpha u'(c_1^S)} + \frac{\mu_S - \mu_B}{\alpha}} \frac{1 - \frac{\mu_S - \mu_B}{\alpha}}{1 - \frac{\mu_S U_c(c_0^B, \theta_S)}{(1 - \alpha) U_c(c_0^B, \theta_B)} + \frac{\mu_B}{1 - \alpha}}$$

The Proposition follows immediately.

## A.19 Proof of Proposition 6.4.

The proof of Lemma A.7.3 shows that any incentive compatible allocation can be implemented with a continuous, strictly decreasing function  $a(c)$  (and corresponding date 0 interest rate  $r_0$ ) which gives an agent's date 1 cash on hand as a function of her date 0 consumption. This function is therefore invertible. Define  $T_0(d) = a^{-1}(-d) - y^* - \frac{d}{1 + r_0}$ . The budget set with date 0 transfers is then identical to the budget set with date 1 debt contingent transfers. Since date 1 debt contingent transfers implement efficient allocations, it follows that date 0 transfers also implement efficient allocations.

Consider the equilibrium induced by a debt limit  $\phi_0 = \bar{d}_1 = \bar{c}_1^S - y^* - \phi$ . If  $d_1 < \bar{d}_1$  in equilibrium, the ZLB does not bind, and the equilibrium is constrained efficient. If the debt limit

binds, the equilibrium satisfies

$$U_c(c_0^S, \theta_S) = \beta(1 + r_0)u'(\bar{c}_1^S)$$

$$U_c(c_0^B, \theta_S) > \beta(1 + r_0)u'(c_1^B)$$

$$c_t^S + c_t^B = 2y^*, t = 0, 1$$

$$\mathcal{U}(\mathbf{c}^S, \theta_S) > \mathcal{U}(\mathbf{c}^B, \theta_S)$$

$$\mathcal{U}(\mathbf{c}^B, \theta_B) > \mathcal{U}(\mathbf{c}^S, \theta_B)$$

Set  $\frac{\alpha}{1 - \alpha} = \frac{U_c(c_0^B, \theta_B)}{U_c(c_0^S, \theta_S)}$ ,  $\lambda_0 = \alpha U_c(c_0^S, \theta_S)$ ,  $\lambda_1 = (1 - \alpha)u'(c_1^B)$ ,  $\zeta = \frac{\lambda_1 - \alpha u'(c_1^S)}{u''(c^S)} > 0$ . Then the allocation satisfies the first order sufficient conditions for a solution to the Pareto problem.

For high enough  $\theta_B$ , the borrowing constraint binds (in both the equilibrium without policy and under a debt limit).  $B$ 's gain from the equilibrium with a debt limit is

$$U(c_0^{B'}, \theta_B) - U(c_0^B, \theta_B) + \beta[u(2y^* - \bar{c}_1^S) - u(c_1^B)]$$

where  $c_0^{B'}$  denotes consumption with a debt limit. In the limit as  $\theta_B \rightarrow \infty$  and  $U_c \rightarrow \infty$ , borrowers only derive utility from date 0 consumption. Since the debt limit necessarily reduces date 0 consumption, it makes them worse off.

Part 4 of the Proposition follows immediately from our earlier characterization of constrained efficient allocations.

## A.20 Proof of Proposition 7.2.

The Pareto problem is

$$\max \alpha \mathcal{U}(\mathbf{c}^S, \theta_S) + (1 - \alpha) \mathcal{U}(\mathbf{c}^B, \theta_B) \quad (\text{A.58})$$

$$\text{s.t. } c_0^S + c_0^B \leq 2y^* \quad (\text{RC0})$$

$$c_1^S + c_1^B \leq 2y^* \quad (\text{RC1})$$

$$\hat{c}_1^S + \hat{c}_1^B \leq 2y^* \quad (\text{RC1}')$$

$$c_2^S + c_2^B = 2y^* \quad (\text{RC2})$$

$$c_2^B \geq y^* - (1 - \beta)\phi \quad (\text{BC})$$

$$u'(c_1^S) \geq \beta u'(c_2^S) \quad (\text{ZLB})$$

$$\mathcal{U}(\mathbf{c}^S, \theta_S) \geq \mathcal{U}(\mathbf{c}^B, \theta_S) \quad (\text{ICS})$$

$$\mathcal{U}(\mathbf{c}^B, \theta_B) \geq \mathcal{U}(\mathbf{c}^S, \theta_B) \quad (\text{ICB})$$

where

$$\mathcal{U}(\mathbf{c}^i, \theta) := U(c_0^i, \theta) + \pi \left\{ \beta u(c_1^i) + \frac{\beta^2}{1 - \beta} u(c_2^i) \right\} + (1 - \pi) \frac{\beta}{1 - \beta} u(\hat{c}_1^i)$$

**Lemma A.20.1.** (RC2) and (RC1') bind.  $u'(c_1^i) > \beta u'(c_2^S)$  for at least one agent. If (ZLB) binds, (BC) binds.  $c_1^B \leq c_1^S$ .

*Proof.* Identical to the proofs of Lemmas 4.1-4.4. □

**Lemma A.20.2.** At most one incentive constraint binds.

*Proof.* Suppose by contradiction that  $\mathcal{U}(\mathbf{c}^S, \theta_S) = \mathcal{U}(\mathbf{c}^B, \theta_S)$ ,  $\mathcal{U}(\mathbf{c}^S, \theta_S) = \mathcal{U}(\mathbf{c}^B, \theta_S)$ . By the same argument as in the proof of Lemma 4.5,  $c_0^S = c_0^B$ .

Suppose first that  $\hat{c}_1^S = \hat{c}_1^B$ ,  $c_1^S = c_1^B$ ,  $c_2^S = c_2^B$ . Then the argument in the proof of Lemma 4.5 applies, and we have a contradiction.

If  $\mathbf{c}^S \neq \mathbf{c}^B$ , then set  $c_1^i = \frac{c_1^S + c_1^B}{2}$ ,  $c_2^i = \frac{c_2^S + c_2^B}{2} = y^*$ ,  $\hat{c}_1^i = \frac{\hat{c}_1^S + \hat{c}_1^B}{2} = y^*$ . This deviation satisfies all the constraints. Since preferences are strictly concave, this increases utility, contradicting the assumption that the original allocation was optimal. □

**Lemma A.20.3.** (RC0) binds.

*Proof.* The proof is essentially identical to the proof of Lemma A.6.6. Forming the Lagrangian, the first order necessary conditions for a maximum are

$$\begin{aligned}
\alpha U_c(c_0^S, \theta_S) - \lambda_0 + \mu_S U_c(c_0^S, \theta_S) - \mu_B U_c(c_0^S, \theta_B) &= 0 \\
(1 - \alpha) U_c(c_0^B, \theta_B) - \lambda_0 - \mu_S U_c(c_0^B, \theta_S) + \mu_B U_c(c_0^B, \theta_B) &= 0 \\
\alpha u'(c_1^S) - \lambda_1 + \zeta u''(c_1^S) + (\mu_S - \mu_B) u'(c_1^S) &= 0 \\
(1 - \alpha) u'(c_1^B) - \lambda_1 - (\mu_S - \mu_B) u'(c_1^B) &= 0 \\
\alpha u'(\hat{c}_1^S) - \hat{\lambda}_1 + (\mu_S - \mu_B) u'(\hat{c}_1^S) &= 0 \\
(1 - \alpha) u'(\hat{c}_1^B) - \hat{\lambda}_1 - (\mu_S - \mu_B) u'(\hat{c}_1^B) &= 0 \\
\alpha u'(c_2^S) - \lambda_2 - (1 - \beta) \zeta u''(c_2^S) + (\mu_S - \mu_B) u'(c_2^S) &= 0 \\
(1 - \alpha) u'(c_2^B) - \lambda_2 + \psi - (\mu_S - \mu_B) u'(c_2^B) &= 0
\end{aligned}$$

where  $\lambda_0, \beta\pi\lambda_1, \beta(1 - \pi)\hat{\lambda}_1, \frac{\beta^2}{1 - \beta}\pi\lambda_2, \psi, \beta\zeta, \mu_S, \mu_B$  are the multipliers on (RC0), (RC1), (RC1'), (RC2), (BC), (ZLB), (ICS), (ICB) respectively.

Since at most one incentive constraint binds,  $\mu_S, \mu_B \geq 0$ , with at least one equality. It follows that either  $\alpha U_c(c_0^S, \theta_S) - \lambda_0 \geq 0$ , or  $(1 - \alpha) U_c(c_0^B, \theta_B) - \lambda_0 \geq 0$ , or both. Since  $U_c > 0$ , this implies  $\lambda_0 > 0$ . Thus (RC0) binds.  $\square$

**Lemma A.20.4.** *If (RC1) is slack, (ICS) and (ZLB) both bind.*

*Proof.* Again, this follows directly from the proof of Lemma A.6.7.  $\square$

This concludes the proof of part 1 of Proposition 7.2. Next, I show that ever constrained efficient allocation can be implemented as an equilibrium with transfers.

**Lemma A.20.5.** *If an allocation can be implemented as an equilibrium with transfers in the incomplete markets economy, it can be implemented as an equilibrium with transfers in the complete markets economy.*

*Proof.* Suppose  $T(d), \hat{T}(d), r_0$  implement an allocation  $\mathbf{c}^S, \mathbf{c}^B$  in the incomplete markets economy.

In the incomplete markets economy, consider the transfer functions

$$T(d, \hat{d}) = T(d) \text{ if } d = \hat{d}$$

$$= -\infty \text{ if } d \neq \hat{d}$$

$$\hat{T}(d, \hat{d}) = \hat{T}(d) \text{ if } d = \hat{d}$$

$$= -\infty \text{ if } d \neq \hat{d}$$

together with the interest rates  $r_0, \hat{r}_0 = r_0$ . Clearly it is feasible for all households to choose the same allocation as they would in the incomplete markets economy, by setting  $\hat{d} = d$ . And it can never be optimal for them to do anything else, since this would incur an infinitely large consumption loss.  $\square$

With this Lemma in hand, I focus on implementation in the incomplete markets economy, without loss of generality.

**Lemma A.20.6.** Define the date 1 value function  $V(a_1^i)$  as in Lemma A.7.1, and define

$$\hat{V}(\hat{a}_1^i) = \max_{\{c_t^i, d_{t+1}^i\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} u(c_t^i) \quad (\text{A.59})$$

$$\text{s.t. } c_1^i = y_1 + \hat{a}_1^i + \frac{d_2^i}{1+r_1} \quad (\text{A.60})$$

$$c_t^i = y_t - d_t^i + \frac{d_{t+1}^i}{1+r_t}, t \geq 2 \quad (\text{A.61})$$

$\{c_t^i, d_{t+1}^i\}_{t=0}^{\infty}$  solves  $i$ 's problem, given  $\{y_t, r_t\}$  and  $T(\cdot), \hat{T}(\cdot)$ , if and only if:

1.  $c_0^i, d_1^i$  solve

$$\max_{c_0^i, d_1^i} U(c_0^i, \theta_i) + \beta \pi V(T(d_1^i) - d_1^i) + \beta(1 - \pi) \hat{V}(\hat{T}(d_1^i) - d_1^i)$$

$$\text{s.t. } c_0^i = y_0 + \frac{d_1^i}{1+r_0}$$

2.  $\{c_t^i, d_{t+1}^i\}_{t=1}^{\infty}$  solve (A.22), given  $a_1^i = T(d_1^i) - d_1^i$ .

3.  $\{\hat{c}_t^i, \hat{d}_{t+1}^i\}_{t=1}^{\infty}$  solve (A.59), given  $\hat{a}_1^i = \hat{T}(d_1^i) - d_1^i$ .

*Proof.* Again, the proof is standard and is therefore omitted.  $\square$

**Lemma A.20.7.** In any equilibrium with transfers:

1. for all  $t \geq 2$  and for all  $i$ ,  $r_t = r^* = \beta^{-1} - 1$ ,  $d_t^i = d_2^i$ ,  $c_t^i = c_2^i = y^* - (1 - \beta)d_2^i$ .

2. for all  $t \geq 1$  and for all  $i$ ,  $\hat{r}_t = r^* = \beta^{-1} - 1$ ,  $\hat{d}_t^i = \hat{d}_1^i$ ,  $\hat{c}_t^i = \hat{c}_1^i = y^* - (1 - \beta)\hat{d}_1^i$

*Proof.* The proof is identical to that of Lemma A.7.2.  $\square$

**Lemma A.20.8.** *In the two-type economy,  $\{c_t^i\}$  can be implemented as an equilibrium with transfers if and only if there exists  $r_1$  such that*

$$c_0^S + c_0^B \leq 2y^* \quad (\text{A.62})$$

$$r_1 \geq 0, c_1^S + c_1^B \leq 2y^*, \text{ with at least one equality} \quad (\text{A.63})$$

$$\hat{c}_1^S + \hat{c}_1^B = 2y^* \quad (\text{A.64})$$

$$c_2^S + c_2^B = 2y^* \quad (\text{A.65})$$

$$u'(c_1^i) \geq \beta(1 + r_1)u'(c_2^i), c_2^i \geq y^* - (1 - \beta), \text{ with at least one equality, } i = S, B \quad (\text{A.66})$$

$$\mathcal{U}(c^S, \theta_S) \geq \mathcal{U}(c^B, \theta_S) \quad (\text{A.67})$$

$$\mathcal{U}(c^B, \theta_B) \geq \mathcal{U}(c^S, \theta_B) \quad (\text{A.68})$$

*Proof.* As in the proof of Lemma A.7.3, it is straightforward to show that these conditions are necessary for implementability, and that if an allocation satisfies these conditions, then all equilibrium conditions are satisfied - except, possibly, the condition that for  $i = S, B$ ,  $c_0^i, d_1^i$  solve (A.22). Let  $U^i = \mathcal{U}(c^i, \theta_i)$  be the utility that each agent gets from her allocation. Define  $a_1^i = c_1^i - y_1 - \frac{d_2^i}{1 + r_1}$ ,  $\hat{a}_1^i = \hat{c}_1^i - y^* - \frac{\hat{d}_2^i}{1 + \hat{r}_1}$ . For each  $i = S, B$ , define the set

$$\mathcal{V}^i = \{(c, a, \hat{a}) \in \mathbb{R}^3 : U(c, \theta_i) + \beta\pi V(a) + \beta(1 - \pi)\hat{V}(\hat{a}) \leq U^i\}$$

By construction,  $\mathcal{V}^i$  is a closed set and  $c_0^i, a_1^i, \hat{a}_1^i$  is contained in its boundary. Let

$$\mathcal{V} = \mathcal{V}^S \cap \mathcal{V}^B = \{(c, a, \hat{a}) \in \mathbb{R}^3 : U(c, \theta_i) + \beta\pi V(a) + \beta(1 - \pi)\hat{V}(\hat{a}) \leq U^i, i = S, B\}$$

be the set of allocations which both agents find weakly inferior to their equilibrium allocations. By (A.67) and (A.68), the boundary of  $\mathcal{V}$  contains  $c_0^S, a_1^S, \hat{a}_1^S$  and  $c_0^B, a_1^B, \hat{a}_1^B$ . To implement the desired equilibrium, we can offer households any subset of  $\mathcal{V}$  which contains both their equilibrium allocations. Let  $a(c), \hat{a}(c)$  be any functions satisfying

$$(c, a(c), \hat{a}(c)) \in \mathcal{V}, \forall c$$

$$a_1^i = a(c_0^i), \hat{a}_1^i = \hat{a}(c_0^i), i = S, B$$

It is immediate that

$$c_0^i \in \arg \max_c U(c, \theta_i) + \beta \pi V(a(c)) + \beta(1 - \pi) \hat{V}(\hat{a}(c))$$

Define  $T(d) = d + a\left(y_0 + \frac{d_1^i}{1 + r_0}\right)$ ,  $\hat{T}(d) = d + \hat{a}\left(y_0 + \frac{d_1^i}{1 + r_0}\right)$ . We have immediately that

$$c_0^i, d_1^i \in \arg \max_{c, d} U(c, \theta_i) + \beta \pi V(T(d) - d) + \beta(1 - \pi) \hat{V}(\hat{T}(d) - d)$$

$$\text{s.t. } c_0^i = y_0 + \frac{d_1^i}{1 + r_0}$$

Since it is clear that these transfer functions satisfy the government budget constraint, we are done.  $\square$

**Lemma A.20.9.** *In any constrained efficient allocation,  $S$  is unconstrained at date 1.*

*Proof.* Suppose by contradiction that  $S$  is constrained at date 1 in the crisis state: then  $c_1^S < c_1^B$ ,  $c_t^S \leq c_t^B$  for all  $t \geq 2$ . Since  $c_0^S < c_0^B$ , we must have  $\hat{c}_1^S > \hat{c}_1^B$ , otherwise  $B$  consumes more at all dates and states, which cannot be incentive compatible. Suppose then that  $\hat{c}_1^S > \hat{c}_1^B$ : I will show that this cannot be optimal. Consider the following deviation. Increase  $c_1^S$  and decrease  $c_1^B$  by  $\varepsilon > 0$ , and increase  $\hat{c}_1^B$  and decrease  $\hat{c}_1^S$  by  $\hat{\varepsilon} > 0$ , where  $\varepsilon, \hat{\varepsilon}$  are chosen so that

$$\pi[u(c_1^S + \varepsilon) - u(c_1^B - \varepsilon)] + \frac{1 - \pi}{1 - \beta}[u(\hat{c}_1^S - \hat{\varepsilon}) + u(\hat{c}_1^B + \hat{\varepsilon})] = \pi[u(c_1^S) - u(c_1^B)] + \frac{1 - \pi}{1 - \beta}[u(\hat{c}_1^S) + u(\hat{c}_1^B)]$$

By construction, resource and incentive compatibility constraints are satisfied. For small enough  $\varepsilon$ ,  $S$  is still borrowing constrained. To first order,

$$\hat{\varepsilon} = \frac{\pi(1 - \beta) u'(c_1^S) + u'(c_1^B)}{1 - \pi u'(\hat{c}_1^S) + u'(\hat{c}_1^B)}$$

and the change in each agent's utility is

$$\Delta U^B = -\beta \pi u'(c_1^B) \varepsilon + \beta \pi u'(\hat{c}_1^B) \frac{u'(c_1^S) + u'(c_1^B)}{u'(\hat{c}_1^S) + u'(\hat{c}_1^B)} \varepsilon > 0$$

$$\Delta U^S = \beta \pi u'(c_1^S) \varepsilon - \beta \pi u'(\hat{c}_1^S) \frac{u'(c_1^S) + u'(c_1^B)}{u'(\hat{c}_1^S) + u'(\hat{c}_1^B)} \varepsilon > 0$$

where the inequalities hold because

$$\frac{u'(c_1^S)}{u'(c_1^B)} > \frac{u'(c_1^S) + u'(c_1^B)}{u'(\hat{c}_1^S) + u'(\hat{c}_1^B)} > \frac{u'(\hat{c}_1^S)}{u'(\hat{c}_1^B)}$$

So for some  $\varepsilon > 0$ , the deviation increases both agents' utilities and satisfies all the constraints.

Thus the original allocation cannot have been optimal.  $\square$

**Corollary A.20.10.** *In the two-type economy,  $\{c_t^i\}$  can be implemented as an equilibrium with transfers if and only if (A.62), (A.64), (A.65), (A.67), (A.68) are satisfied, together with*

$$u'(c_1^S) \geq \beta u'(c_2^S), c_1^S + c_1^B \leq 2y^*, \text{ with at least one equality} \quad (\text{A.69})$$

$$\frac{u'(c_1^B)}{\beta u'(c_2^B)} \geq \frac{u'(c_1^S)}{\beta u'(c_2^S)}, c_2^B \geq y^* - (1 - \beta)\phi, \text{ with at least one equality} \quad (\text{A.70})$$

*Proof.* Identical to the proof of Corollary A.7.5. □

## A.21 Proof of Proposition 7.3.

**Lemma A.21.1.** *Suppose  $\{c_t^i, \hat{c}_t^i\}$  solves (A.58). Define  $a_1^i = c_1^i - y_1 - \frac{d_2^i}{1 + r_1}$ ,  $\hat{a}_1^i = \hat{c}_1^i - y^* - \frac{\hat{d}_2^i}{1 + \hat{r}_1}$ .*

*Take any transfer functions  $T, \hat{T}$  and interest rate  $r_0 \geq 0$ . Define the associated net wealth functions*

$$a(c) := T((1 + r_0)(c - y^*)) - (1 + r_0)(c - y^*), \hat{a}(c) = \hat{T}((1 + r_0)(c - y^*)) - (1 + r_0)(c - y^*)$$

*Sufficient conditions for  $T, \hat{T}, r_0$  to implement  $\{c_t^i, \hat{c}_t^i\}$  are that:*

1.  $a(c_0^i) = c_1^i - y_1 + \frac{c_2^i - y^*}{(1 + r_1)(1 - \beta)}$  for  $i = S, B$ , and
2.  $\hat{a}(c_0^i) = \frac{\hat{c}_1^i - y^*}{1 - \beta}$  for  $i = S, B$ ,
3. for all  $c$ ,

$$(c, a(c), \hat{a}(c)) \in \mathcal{V} = \mathcal{V}^S \cap \mathcal{V}^B = \{(c, a, \hat{a}) \in \mathbb{R}^3 : U(c, \theta_i) + \beta\pi V(a) + \beta(1 - \pi)\hat{V}(\hat{a}) \leq U^i, i = S, B\}$$

*Proof.* Suppose  $\{c_t^i, \hat{c}_t^i\}$  solves (A.58), and is therefore implementable. Let  $T, \hat{T}, r_0$  satisfy the conditions in the Lemma; I show that  $T, \hat{T}, r_0$  implement this allocation.

If the conditions in the Lemma are satisfied, then for each  $i$ ,

$$c_0^i \in \arg \max_c U(c, \theta_i) + \beta\pi V(a(c)) + \beta(1 - \pi)\hat{V}(\hat{a}(c))$$

That is,

$$\begin{aligned} c_0^i, d_1^i &\in \arg \max_{c, d} U(c, \theta_i) + \beta\pi V(T(d) - d) + \beta(1 - \pi)\hat{V}(\hat{T}(d) - d) \\ \text{s.t. } c_0^i &= y_0 + \frac{d_1^i}{1 + r_0} \end{aligned}$$

Defining  $d_1^i = (1 + r_0)(c_0^i - y^*)$ , we have

$$\begin{aligned}
\sum_{i=S,B} T(d_1^i) &= \sum_{i=S,B} a_1^i + \sum_{i=S,B} d_1^i \\
&= \sum_{i=S,B} \left( c_1^i - y_1 + \frac{c_2^i - y^*}{(1 + r_1)(1 - \beta)} \right) + \sum_{i=S,B} (1 + r_0)(c_0^i - y^*) \\
&= 0 \\
\sum_{i=S,B} \hat{T}(d_1^i) &= \sum_{i=S,B} \hat{a}_1^i + \sum_{i=S,B} d_1^i \\
&= \sum_{i=S,B} \frac{\hat{c}_1^i - y^*}{1 - \beta} + \sum_{i=S,B} (1 + r_0)(c_0^i - y^*) \\
&= 0
\end{aligned}$$

So the government budget constraints are satisfied. The remaining conditions are satisfied by assumption.  $\square$

**Assumption A.21.2.**  $u'(y^* + (1 - \beta)\phi)\phi$  is increasing in  $\phi$ .

**Lemma A.21.3.** In any constrained efficient allocation,  $a_1^B \geq \hat{a}_1^B$ ,  $a_1^S \leq \hat{a}_1^S$ .

Recall the necessary conditions for a maximum:

*Proof.*

$$\begin{aligned}
[\alpha + \mu_S - \mu_B]u'(c_1^S) + \zeta u''(c_1^S) &= \lambda_1 = [1 - \alpha - \mu_S + \mu_B]u'(c_1^B) \\
[\alpha + \mu_S - \mu_B]u'(\hat{c}_1^S) &= \hat{\lambda}_1 = [1 - \alpha - \mu_S + \mu_B]u'(\hat{c}_1^B) \\
[\alpha + \mu_S - \mu_B]u'(c_2^S) - (1 - \beta)\zeta u''(c_2^S) &= \lambda_2 = [1 - \alpha - \mu_S + \mu_B]u'(c_1^B) + \psi
\end{aligned}$$

It is clear that if the ZLB is slack and  $\zeta = 0$ ,  $c_1^S = \hat{c}_1^S$ ,  $c_1^B = \hat{c}_1^B$ , while if the ZLB binds,  $c_1^B \geq \hat{c}_1^B$ ,  $c_1^S \leq \hat{c}_1^S$ . Finally, if the borrowing constraint

If the borrowing constraint is slack, clearly optimal allocations are the same in the two states, and  $a_1^i = \hat{a}_1^i$ ,  $i = S, B$ . If the borrowing constraint binds but the ZLB is slack, then if (by contradiction)  $a_1^i = \hat{a}_1^i$  for  $i = S, B$  in the optimal allocation, we would have  $c_1^B < \hat{c}_1^B$ . To see this, note that borrowers' consumption when the constraint binds is

$$c^B(\phi) = y^* + a + \beta \frac{u'(y^* + (1 - \beta)\phi)\phi}{u'(c_1^B)}$$

By Assumption A.21.2,  $c_1^B(\phi)$  is increasing in  $\phi$ . In the non-crisis state, borrowers roll over their debt, and  $\hat{c}_1^B = c(-\hat{a}_1^B)$ . In the crisis state,  $\phi < -\hat{a}_1^B$ , and  $c_1^B = c(\phi) < \hat{c}_1^B$ . We know it is optimal to smooth consumption across states. To implement this, we must have  $a_1^B > \hat{a}_1^B$ , and thus by market clearing  $a_1^S < \hat{a}_1^S$ .

Finally, when the ZLB binds, it is optimal to give the borrowers even higher consumption than in the non-crisis state, but their pre-transfer income is (weakly) lower, because we may have  $y_1 < y^*$ . Thus again, we must have  $a_1^B > \hat{a}_1^B$  and  $a_1^S < \hat{a}_1^S$ .  $\square$

**Lemma A.21.4.**  $R(\alpha) := \frac{U_c(c_0^S(\alpha), \theta_S)}{\beta[\pi u'(c_1^S(\alpha)) + (1 - \pi)u'(\hat{c}_1^S(\alpha))]}$  is decreasing in  $\alpha$  on  $[\alpha_S, \alpha_B]$ .  $T(\alpha) := R(\alpha)(y^* - c_0^S(\alpha)) - a_1^S(\alpha)$  is decreasing in  $\alpha$  on  $[\alpha_S, \alpha_B]$ . There exists  $\bar{\alpha} \in (\alpha_S, \alpha_B)$  such that  $T(\bar{\alpha}) = 0$ .

*Proof.*  $R(\alpha)$  is defined by

$$\begin{aligned} R(\alpha) &= \frac{U_c(c_0^S(\alpha), \theta_S)}{\beta[\pi u'(c_1^S(\alpha)) + (1 - \pi)u'(\hat{c}_1^S(\alpha))]} \\ \alpha U_c(c_0^S(\alpha), \theta_S) &= (1 - \alpha)U_c(2y^* - c_0^S(\alpha), \theta_S) \\ \alpha u'(g_1(\alpha)) &= (1 - \alpha)u'(2y^* - g_1(\alpha)) \\ \alpha u'(\hat{c}_1^S(\alpha)) &= (1 - \alpha)u'(2y^* - \hat{c}_1^S(\alpha)) \\ c_1^S(\alpha) &= \min\{g_1(\alpha), \bar{c}_1^S\} \end{aligned}$$

Under Assumption A.8.4,  $R$  is decreasing in  $\alpha$ . Since  $a_1^S$  and  $c_1^S$  are increasing in  $\alpha$ , and  $c_0^S < y^*$ ,  $T(\alpha)$  is increasing in  $\alpha$ .

When  $\alpha = \alpha_S$ ,  $U(c_0^S, \theta_S) + \beta\pi V(a_1^S) + \beta(1 - \pi)\hat{V}(\hat{a}_1^S) = U(c_0^B, \theta_S) + \beta\pi V(a_1^B) + \beta(1 - \pi)\hat{V}(\hat{a}_1^B)$ .

Since these functions are concave,

$$\begin{aligned} U_c(c_0^S, \theta_S)(c_0^B - c_0^S) + \beta\pi V'(a_1^S)(a_1^B - a_1^S) + \beta(1 - \pi)\hat{V}'(\hat{a}_1^S)(\hat{a}_1^B - \hat{a}_1^S) &> 0 \\ R(\alpha)(c_0^B - c_0^S) + \frac{\pi V'(a_1^S)}{\pi V'(a_1^S) + (1 - \pi)\hat{V}'(\hat{a}_1^S)}(a_1^B - a_1^S) + \frac{(1 - \pi)\hat{V}'(\hat{a}_1^S)}{\pi V'(a_1^S) + (1 - \pi)\hat{V}'(\hat{a}_1^S)}(\hat{a}_1^B - \hat{a}_1^S) &> 0 \\ R(\alpha)(c_0^B - c_0^S) + a_1^S - a_1^B &> 0 \\ T(\alpha) &> 0 \end{aligned}$$

where the third line uses Lemma A.21.3. An analogous argument establishes that  $T(\alpha_B) < 0$ .

Finally, since  $T$  is clearly continuous, there exists  $\bar{\alpha}$  such that  $T(\bar{\alpha}) = 0$ .  $\square$

## A.22 Proof of Proposition 7.4.

If the borrowing constraint does not bind in equilibrium,  $y_t = y^*$  in all periods and the economy enters steady state at date 1,  $c_1^i = \hat{c}_1^i, i = S, B$ , and

$$\frac{U_c(c_0^i, \theta_i)}{\beta u'(c_1^i)} = \frac{U_c(c_0^i, \theta_i)}{\beta u'(\hat{c}_1^i)} = 1 + r_0 = 1 + \hat{r}_0, i = S, B$$

$$u'(c_1^i) = \beta(1 + r_1)u(c_2^i), u'(\hat{c}_1^i) = \beta(1 + \hat{r}_1)u(\hat{c}_2^i), i = S, B$$

Choose  $\alpha$  so that  $\frac{\alpha}{1 - \alpha} = \frac{U_c(c_1^S, \theta_S)}{U_c(c_0^B, \theta_B)}$ . It follows that the allocation satisfies the first order sufficient conditions for an optimum.

Suppose the borrowing constraint binds, but the ZLB is slack. In the complete markets economy, we have

$$\frac{U_c(c_0^i, \theta_i)}{\beta u'(c_1^i)} = 1 + r_0, i = S, B$$

$$\frac{U_c(c_0^i, \theta_i)}{\beta u'(\hat{c}_1^i)} = 1 + \hat{r}_0, i = S, B$$

$$u'(c_1^i) \geq \beta(1 + r_1)u(c_2^i), i = S, B$$

Choose  $\alpha$  so that  $\frac{\alpha}{1 - \alpha} = \frac{U_c(c_1^S, \theta_S)}{U_c(c_0^B, \theta_B)}$ . It follows that the allocation satisfies the first order sufficient conditions for an optimum.

Choose  $\alpha$  so that  $\frac{\alpha}{1 - \alpha} = \frac{U_c(c_0^B, \theta_B)}{U_c(c_0^S, \theta_S)}$ . It follows that

$$\alpha U_c(c_0^S, \theta_S) = (1 - \alpha)U_c(c_0^B, \theta_B)$$

$$\alpha u'(c_1^S) = (1 - \alpha)u'(c_1^B) + \psi \alpha u'(\hat{c}_1^S) = (1 - \alpha)u'(\hat{c}_1^B)$$

for some  $\psi \geq 0$ . So the allocation satisfies the first order sufficient conditions in (A.58), and is Pareto optimal.

In the incomplete markets economy,  $c_1^B < \hat{c}_1^B$ . But in any solution to the planner's problem,  $c_1^B = \hat{c}_1^B$ . So the incomplete markets equilibrium cannot be efficient. To see that debt relief is Pareto improving, take an equilibrium without policy and increase  $c_1^B$  while decreasing  $c_1^S$ . This clearly leads to a Pareto improvement, and can be implemented with debt relief.

Finally, if the ZLB binds in equilibrium (in either economy),  $y_1 < y^*$ , which cannot be optimal since neither incentive constraint binds.

## A.23 Proof of Proposition 7.8.

The Pareto problem is

$$\max_{x^S, x^B} \alpha \mathcal{U}(x^S, \theta_S) + (1 - \alpha) \mathcal{U}(x^B, \theta_B) \quad (\text{A.71})$$

$$\text{s.t } x_0^S + x_0^B \leq 0 \quad (\text{A.72})$$

$$x_1^S + x_1^B \leq 0 \quad (\text{A.73})$$

$$x_2^S + x_2^B = 0 \quad (\text{A.74})$$

$$u'_1(x_1^S, \theta_S) \geq \beta u'_2(x_2^S, \theta_S) \quad (\text{A.75})$$

$$x_2^B \geq -(1 - \beta)\phi \quad (\text{A.76})$$

$$\frac{u'_1(x_1^B, \theta_B)}{\beta u'_2(x_1^B)} \geq \frac{u'_1(x_1^S, \theta_S)}{\beta u'_2(x_1^S)} \quad (\text{A.77})$$

$$(x_2^B + (1 - \beta)\phi) \left( \frac{u'_1(x_1^B, \theta_B)}{\beta u'_2(x_1^B)} - \frac{u'_1(x_1^S, \theta_S)}{\beta u'_2(x_1^S)} \right) = 0 \quad (\text{A.78})$$

$$\mathcal{U}(x^B, \theta_B) \geq \mathcal{U}(x^S, \theta_B) \quad (\text{A.79})$$

$$\mathcal{U}(x^S, \theta_S) \geq \mathcal{U}(x^B, \theta_S) \quad (\text{A.80})$$

where

$$\mathcal{U}(x, \theta) := u_0(x_0, \theta) + \beta u_1(x_1, \theta) + \frac{\beta^2}{1 - \beta} u_2(x_2, \theta)$$

There are two new constraints, (A.77) and (A.78). These constraints impose that if the borrowing constraint does not bind, agents must have the same marginal rate of substitution between date 1 and date 2 consumption. If neither incentive constraint binds, these new constraints do not bind for the planner, since it is already optimal to give agents the same MRS. However, if one incentive constraint binds, the planner might want to distort allocations away from the first best, giving agents different MRSs, in order to make incentive compatibility hold.<sup>4</sup> Then these new constraints, which restrict MRSs to be the same, will bind.

**Lemma A.23.1.**  $u'_1(x_1^i, \theta_i) > \beta u'_2(x_2^i, \theta_i)$  for at least one agent.

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<sup>4</sup>In the baseline model considered throughout the paper, there was no difference between agents' preferences between dates 1 and 2. Thus the planner had no motive to distort the MRS between these two dates.

*Proof.* If not, then  $x_1^i > x_2^i$  for  $i = S, B$ ; summing, we have  $x_1^S + x_1^B > x_2^S + x_2^B = 0$ , which is infeasible.  $\square$

**Lemma A.23.2.** *If (A.75) binds, (A.76) binds.*

*Proof.* Suppose by contradiction that (A.75) binds but (A.76) does not. By (A.77), we must have  $\frac{u'_1(x_1^B, \theta_B)}{\beta u'_2(x_1^B)} = \frac{u'_1(x_1^S, \theta_S)}{\beta u'_2(x_1^S)} = 1$  (since (A.75) binds), contradicting Lemma A.23.1.  $\square$

**Lemma A.23.3.** *At most one incentive constraint binds.*

*Proof.* First I show that if both incentive compatibility constraints hold, the allocation is weakly inferior to the autarchic allocation  $x_t^i = 0, \forall i, t$ . Then I show that this allocation itself cannot be optimal.

If both (A.80) and (A.79) bind, then for  $i = S, B$ ,

$$\begin{aligned} \mathcal{U}(\mathbf{x}^S, \theta_i) &= \mathcal{U}(\mathbf{x}^B, \theta_i) \\ \mathcal{U}(\mathbf{0}, \theta_i) &\geq \mathcal{U}\left(\frac{1}{2}(\mathbf{x}^S + \mathbf{x}^B), \theta_i\right) \geq \mathcal{U}(\mathbf{x}^S, \theta_i) \end{aligned}$$

To show that autarky is not optimal, consider the following deviation: set  $x_0^B = -x_0^S = \varepsilon_0 > 0$ ,  $x_1^S = -x_0^B = \varepsilon_1 > 0$ ,  $x_2^S = -x_2^B = \varepsilon_2$ , choosing  $\varepsilon_0, \varepsilon_1$  so that

$$\frac{u'_0(0, \theta_S)}{\beta u'_1(0, \theta_S)} < \frac{\delta}{\varepsilon} < \frac{u'_0(0, \theta_B)}{\beta u'_1(0, \theta_B)}$$

and choosing  $\varepsilon_2$  to satisfy the agents' Euler equations. This deviation increases utility for both agents, and is feasible, because it relaxes both incentive compatibility constraints.  $\square$

**Lemma A.23.4.** *(RC0) binds.*

*Proof.* Identical to the proof of Lemma A.6.6.  $\square$

**Lemma A.23.5.** *If (A.21) is slack, (ICS) and (ZLB) both bind.*

*Proof.* Identical to the proof of Lemma A.6.7.  $\square$

The proof that every solution to this Pareto problem can be implemented as an equilibrium with transfers has the same structure as the corresponding proof in the baseline model. I show that household optimality conditions can be expressed in recursive form, show that the economy enters steady state at date 2, show that we can represent optimality conditions using incentive compatibility constraints, and then show that certain constraints do not bind.

**Definition A.23.6.** An *equilibrium with transfers* is a collection  $\{x_t^i, d_{t+1}^i, z_t^i, r_t\}$  such that, given a policy  $T(\cdot)$ :

1. for each  $i = S, B$ ,  $\{x_t^i, d_{t+1}^i\}$  solves agent  $i$ 's problem, given  $\{z_t^i, r_t\}$  and given policy  $T(\cdot)$ :

$$\max u_0(x_0^i, \theta_i) + \sum_{t=1}^{\infty} \beta^t u_t(x_t^i, \theta_i) \quad (\text{A.81})$$

$$\text{s.t. } \frac{d_2^i}{1+r_1} = d_1^i + x_1^i + z_1^i - T(d_1^i) \quad (\text{A.82})$$

$$\frac{d_{t+1}^i}{1+r_t} = d_t^i + x_t^i + z_t^i, \forall t \neq 1 \quad (\text{A.83})$$

$$d_t^i \leq \phi, t \geq 2 \quad (\text{A.84})$$

$$d_0^i = 0 \quad (\text{A.85})$$

2. for all  $t$ ,

$$x_t^S + x_t^B + 2z_t = 0 \quad (\text{A.86})$$

$$r_t \geq 0, z_t \geq 0, r_t z_t = 0 \quad (\text{A.87})$$

3. the government budget constraint is satisfied:

$$T(d_1^S) + T(d_1^B) = 0 \quad (\text{A.88})$$

**Lemma A.23.7.** Define the date 1 value function

$$V_i(a_1^i, \theta_i) = \max_{\{x_t^i, d_{t+1}^i\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} u_t(x_t^i, \theta_i) \quad (\text{A.89})$$

$$\text{s.t. } x_1^i = a_1^i - z_1 + \frac{d_2^i}{1+r_1} \quad (\text{A.90})$$

$$x_t^i = -d_t^i - z_t + \frac{d_{t+1}^i}{1+r_t}, t \geq 2 \quad (\text{A.91})$$

$$d_t^i \leq \phi, t \geq 2 \quad (\text{A.92})$$

$\{x_t^i, d_{t+1}^i\}_{t=0}^{\infty}$  solves  $i$ 's problem, given  $\{z_t, r_t\}$  and  $T(\cdot)$ , if and only if:

1.  $x_0^i, d_1^i$  solve

$$\max_{c_0^i, d_1^i} u(x_0^i, \theta_i) + \beta V_i(T(d_1^i) - d_1^i)$$

$$\text{s.t. } x_0^i = \frac{d_1^i}{1+r_0} - z_0$$

2.  $\{x_t^i, d_{t+1}^i\}_{t=1}^\infty$  solve (A.89), given  $a_1^i = T(d_1^i) - d_1^i$ .

*Proof.* Again, the proof is standard and is therefore omitted.  $\square$

Recall the following assumptions:

**Assumption A.23.8.** For all  $t, \theta$ , there exists  $x_t(\theta)$  such that  $u_t(\cdot, \theta)$  is  $C^2$  on  $(x_t(\theta), \infty)$ , with  $u_t' > 0$ ,  $u_t'' < 0$ ,  $\lim_{x \rightarrow x_t(\theta)} u_t'(x, \theta) = +\infty$ ,  $\lim_{x \rightarrow x_t(\theta)} u_t(x, \theta) = -\infty$ .

**Assumption A.23.9.**  $\frac{u_0'(x_0, \theta)}{\beta u_1'(x_1, \theta)}$  is increasing in  $\theta$ .

**Assumption A.23.10.**  $\frac{u_{t+1}'(x, \theta)}{u_t'(x, \theta)} = 1$ , for all  $\theta, x, t \geq 1$ .

**Lemma A.23.11.** In any equilibrium with transfers, for all  $t \geq 2$  and for all  $i$ ,  $r_t = r^* = \beta^{-1} - 1$ ,  $d_t^i = d_2^i$ ,  $x_t^i = x_2^i = -(1 - \beta)d_2^i$ .

*Proof.* First, suppose that households solve a relaxed problem in which  $\phi_t = \infty$  for all  $t \geq 3$ . In this case, household first order conditions yield

$$u_t'(x_t^i, \theta_i) = \beta(1 + r_t)u_{t+1}'(x_{t+1}^i)\phi \text{ for all } t \geq 2$$

I will show that the borrowing constraint does not bind, so households are indeed liquidity unconstrained after date 2.

If  $r_t = r^*, \forall t \geq 2$ , the proposed allocation uniquely satisfies these first order conditions, by assumption A.23.10. Suppose by contradiction that there is also an equilibrium with  $r_t > r^*$  for some  $t \geq 2$ . Then for each household  $i$ ,  $x_t^i < x_{t+1}^i$ . Integrating, we have  $z_t = -\int x_t^i di > -\int x_{t+1}^i di = 0$ . So  $z_t > 0$ , which implies  $r_t = 0$  by the definition of equilibrium, a contradiction.

Suppose by contradiction that  $r_t < r^*$ . Then a similar argument implies that  $z_{t+1} = -\int x_{t+1}^i di > 0$  and  $r_{t+1} = 0$ . Iterating forward, we see that we must have  $r_{t+s} = 0, z_{t+s} > 0$  for all  $s \geq 1$ . This deflationary equilibrium is clearly Pareto inferior to an equilibrium with  $z_t = 0$ , so we can rule this equilibrium out when considering optimal policy.

From the budget constraints, it follows that  $x_t^i = x_2^i = -(1 - \beta)d_2^i$ ,  $d_{t+1}^i = d_1^i$ , for all  $t \geq 2$ . Since  $d_2^i \leq \phi$ , households' unconstrained borrowing decisions happen to satisfy the borrowing constraint, as claimed.  $\square$

**Lemma A.23.12.** If  $i$  is constrained at date 1,  $x_t^i \leq x_1^i, \forall t \geq 1$ . If  $\phi > 0$ , the inequality is strict.

*Proof.* It follows immediately from Lemma A.23.11 that if  $i$  is constrained, he consumes less than  $j$  in steady state:

$$x_2^i = -(1 - \beta)\phi \leq (1 - \beta)\phi = x_2^j$$

with strict inequality if  $\phi > 0$ . Since  $i$  is constrained and  $j$  is not, we have

$$\frac{u'_1(x_1^i, \theta_i)}{u'_2(x_2^i, \theta_i)} > \frac{u'_1(x_1^j, \theta_i)}{u'_2(x_2^j, \theta_i)} = \beta(1 + r_1)$$

I claim that  $\beta(1 + r_1) < 1$ . If not, then  $\frac{u'_1(x_1^j, \theta_i)}{u'_2(x_2^j, \theta_i)}, \frac{u'_1(x_1^i, \theta_i)}{u'_2(x_2^i, \theta_i)} \geq 0$ , which implies  $x_2^i > x_1^i, x_2^j > x_1^j$  by Assumption A.23.10. Summing, we have  $x_1^i + x_1^j < x_2^i + x_2^j \leq 0$ . But this is a contradiction, since  $r_1 > 0$  and we must have full employment.

Since  $\beta(1 + r_1) < 1$ , it follows that  $x_1^i > x_2^i \geq 0$ . Since  $x_1^i + x_1^j \leq 0$ , we have immediately that  $x_1^i < x_1^j$ . □

**Lemma A.23.13.** *In any equilibrium with transfers:*

1. If  $x_0^i > x_0^j$ , then  $x_t^i \leq x_t^j, \forall t \geq 0$ , with at least one strict inequality
2. If  $x_0^i = x_0^j$ , then  $x_t^i = x_t^j, \forall t$

*Proof.* Suppose  $x_0^i > x_0^j$ . If no agent is constrained at date 1, then the economy enters steady state and  $x_1^i = x_1^j$ , for all  $t \geq 1$ . We must have  $x_1^i < x_1^j$ , otherwise  $j$  would strictly prefer  $i$ 's allocation. If  $j$  is constrained, we know from Lemma A.23.12 that  $i$  must consume more than  $j$  at every date  $t \geq 1$ . So  $j$  consumes less in every period, which is impossible, since then  $j$  would prefer  $i$ 's allocation. If  $i$  is constrained, then  $j$  consumes more than  $i$  at every date  $t \geq 1$ , which is what we wanted to show.

Suppose  $x_0^i = x_0^j$ . If either agent is liquidity constrained at date 1, that agent consumes less in every subsequent period, and would rather choose the other agent's allocation. If neither agent is constrained, the economy enters steady state and  $x_t^i = x_1^i = x_1^j = x_t^j$  in every period. Since there is full employment in steady state,  $x_t^i = x_t^j = 0$ . □

**Lemma A.23.14.** *In the two-type economy,  $\{c_t^i\}$  can be implemented as an equilibrium with transfers if and only if there exists  $r_1$  such that*

$$x_0^S + x_0^B \leq 0 \quad (\text{A.93})$$

$$r_1 \geq 0, x_1^S + x_1^B \leq 0, \text{ with at least one equality} \quad (\text{A.94})$$

$$x_2^S + x_2^B = 0 \quad (\text{A.95})$$

$$u_1'(x_1^i, \theta_1) \geq \beta(1 + r_1)u_2'(x_2^i, \theta_2), x_2^i \geq -(1 - \beta)\phi, \text{ with at least one equality, } i = S, B \quad (\text{A.96})$$

$$\mathcal{U}(x^B, \theta_B) \geq \mathcal{U}(x^S, \theta_B) \quad (\text{A.97})$$

$$\mathcal{U}(x^S, \theta_S) \geq \mathcal{U}(x^B, \theta_S) \quad (\text{A.98})$$

*Proof.* First I show that these conditions are necessary for implementability. Suppose  $\{x_t^i, d_{t+1}^i, r_t, z_t\}$  is an equilibrium with transfers, given some policy  $T(\cdot)$ . (A.93) and (A.95) are satisfied by definition. By Lemma A.23.11, the economy enters a steady state at date 2 with full employment, thus (A.94) is satisfied. (A.29) describes necessary conditions for optimality in the household problem. Finally, the incentive compatibility constraints (A.98), (A.97) follow from a standard mimicking argument.

Next, I show that conditions (A.93)-(A.97) are sufficient for implementability. Let  $\{x_t^i\}, r_1$  satisfy these conditions. Set  $2z_t = -x_t^S - x_t^B$  for all  $t$  and set  $r_t = r^*$  for all  $t \geq 2$ . Set  $d_t^i = \frac{-x_2^i}{1 - \beta}, \forall t \geq 2$ . If  $z_t > 0$ , set  $r_0 = 0$ , otherwise choose any  $r_0 \geq 0$ .

It is clear that all equilibrium conditions are satisfied, except, possibly, the condition that for  $i = S, B$ ,  $x_0^i, d_1^i$  solve (A.89). Let  $U^i = \mathcal{U}(x^i, \theta_i)$  be the utility that each agent gets from her allocation. Define  $a_1^i = x_1^i + z_1 - \frac{d_2^i}{1 + r_1}$ . For each  $i = S, B$ , define the set

$$\mathcal{V}^i = \{(x, a) \in \mathbb{R}^2 : u_0(x, \theta_i) + \beta V_i(a) \leq U^i\}$$

By construction,  $\mathcal{V}^i$  is a closed set and  $x_0^i, a_1^i$  is contained in its boundary. Let

$$\mathcal{V} = \mathcal{V}^S \cap \mathcal{V}^B = \{(x, a) \in \mathbb{R}^2 : u_0(x, \theta_i) + \beta V_i(a) \leq U^i, i = S, B\}$$

be the set of allocations which both agents find weakly inferior to their equilibrium allocations. By (A.98) and (A.97), the boundary of  $\mathcal{V}$  contains  $x_0^S, a_1^S$  and  $x_0^B, a_1^B$ . To implement the desired equilibrium, we can offer households any subset of  $\mathcal{V}$  which contains both their equilibrium

allocations. Let  $a(x)$  be any function satisfying

$$(x, a(x)) \in \mathcal{V}, \forall x$$

$$a_1^i = a(x_0^i), i = S, B$$

It is immediate that

$$x_0^i \in \arg \max_x u_0(x, \theta_i) + \beta V_i(a(x))$$

Define  $T(d) = d + a\left(\frac{d_1^i}{1+r_0} - z_0\right)$ . We have immediately that

$$x_0^i, d_1^i \in \arg \max_{x,a} u_0(x, \theta_i) + \beta V_i(T(d) - d)$$

$$\text{s.t. } x_0^i = \frac{d_1^i}{1+r_0} - z_0$$

Since it is clear that these transfer functions satisfy the government budget constraint, we are done.  $\square$

**Lemma A.23.15.** *Suppose  $\theta_B > \theta_S$ . Let  $\mathbf{b}, \mathbf{s} = \{b_t\}_{t=0}^\infty, \{s_t\}_{t=0}^\infty$  be two allocations with  $b_0 > s_0$ ,  $b_t \leq s_t, \forall t \geq 1$ , with strict inequality for  $t = 1$ . If  $U(\mathbf{b}, \theta_S) \geq U(\mathbf{s}, \theta_S)$ , then  $U(\mathbf{b}, \theta_B) > U(\mathbf{s}, \theta_B)$ .*

*Proof.* Suppose not, and  $U(\mathbf{b}, \theta_B) \leq U(\mathbf{s}, \theta_B)$ . By continuity, there exists  $\bar{\theta} \in [\theta_S, \theta_B)$  such that  $U(\mathbf{b}, \bar{\theta}) = U(\mathbf{s}, \bar{\theta})$ . There exists some isoutility curve  $\{(x_0(\tau), x_1(\tau), \dots) | \tau \in [0, 1]\}$  linking  $\mathbf{s}$  and  $\mathbf{b}$ , with  $x(0) = \mathbf{s}, x(1) = \mathbf{b}, x'_0(\tau) > 0, x'_t(\tau) \leq 0, \forall t \geq 1$ , with at least one strict inequality, such that  $U(x(\tau), \bar{\theta})$  is constant for all  $\tau \in [0, 1]$ . That is, for all  $\tau \in [0, 1]$ , we have

$$\frac{d}{d\tau} U(x(\tau), \bar{\theta}) = 0$$

$$u'_0(x_0(\tau), \bar{\theta})x'_0(\tau) + \sum_{t=1}^{\infty} \beta^t u'_t(x_t(\tau), \bar{\theta})x'_t(\tau) = 0$$

$$x'_0(\tau) + \sum_{t=1}^{\infty} \beta^t \frac{u'_t(x_t(\tau), \bar{\theta})}{u'_0(x_0(\tau), \bar{\theta})} x'_t(\tau) = 0$$

$$x'_0(\tau) + \sum_{t=1}^{\infty} \beta^t \frac{u'_t(x_t(\tau), \theta_B)}{u'_0(x_0(\tau), \theta_B)} x'_t(\tau) > 0$$

where the last line uses Assumption A.23.9. Multiplying by  $u'_0(x_0(\tau), \theta_B)$  and integrating, we have

$$\begin{aligned} u'_0(x_0(\tau), \theta_B)x'_0(\tau) + \sum_{t=1}^{\infty} \beta^t u'_t(x_t(\tau), \theta_B)x'_t(\tau) &> 0 \\ \int_0^1 u'_0(x_0(\tau), \theta_B)x'_0(\tau) d\tau + \sum_{t=1}^{\infty} \beta^t \int_0^1 u'_t(x_t(\tau), \theta_B)x'_t(\tau) d\tau &> 0 \\ U(\mathbf{b}, \theta_B) - U(\mathbf{s}, \theta_B) &> 0 \end{aligned}$$

a contradiction. So we must have  $U(\mathbf{b}, \theta_B) > U(\mathbf{s}, \theta_B)$ .  $\square$

**Lemma A.23.16.** *In any implementable allocation,  $x_0^B \geq x_0^S$ . If  $x_0^B > x_0^S$ , then  $x_t^B \leq x_t^S$ , for all  $t \geq 1$ , with at least one strict inequality. If  $x_0^B = x_0^S$ , then  $x_t^B = x_t^S$  for all  $t$ .  $S$  is unconstrained at date 1.*

*Proof.* The first part follows from Lemma A.23.13 and Lemma A.23.15. Suppose by contradiction that there exists an implementable allocation in which  $S$  is constrained. Then by Lemma A.23.12,  $x_t^S \leq x_t^B$ , for all  $t \geq 1$ . We know that  $x_0^B \geq x_0^S$ . If any one of these inequalities is strict,  $S$  prefers  $B$ 's allocation, which contradicts the assumption that the allocation is implementable. If  $x_t^S = x_t^B$  for all  $t$ ,  $S$  is unconstrained.  $\square$

**Corollary A.23.17.**  $\{x_t^i\}$  can be implemented as an equilibrium with transfers if and only if (A.93), (A.95), (A.98), (A.79) are satisfied, together with

$$u'_1(x_1^S, \theta_S) \geq \beta u'_2(x_2^S, \theta_S), x_1^S + x_1^B \leq 0, \text{ with at least one equality} \quad (\text{A.99})$$

$$\frac{u'_1(x_1^B, \theta_B)}{\beta u'_2(x_2^B, \theta_B)} \geq \frac{u'_1(x_1^S, \theta_S)}{\beta u'_2(x_2^S, \theta_S)}, x_2^B \geq -(1 - \beta)\phi, \text{ with at least one equality} \quad (\text{A.100})$$

*Proof.* The proof follows exactly the proof of Corollary A.7.5.  $\square$

**Lemma A.23.18.** *Suppose  $\{x_t^i\}$  solves the Pareto problem (A.71). Then (A.99) and (A.100) are satisfied.*

*Proof.* (A.100) is satisfied by construction. (A.99) is satisfied by Lemma A.23.5.  $\square$

Next, I show that debt relief implements some constrained efficient allocations.

**Assumption A.23.19.** *The economy has a unique equilibrium.*

Fix  $\theta_S, \theta_B$ . Let  $\tilde{x}_t^i(\alpha)$  denote allocations which solve a relaxed Pareto problem without incentive constraints and without the ZLB. Define the net transfer from borrowers to savers in this relaxed Pareto problem as

$$\tilde{T}(\alpha) = \frac{u'_0(\tilde{x}_0^S(\alpha), \theta_S)}{\beta u'_1(\tilde{x}_1^S(\alpha), \theta_S)} \tilde{x}_0^B(\alpha) + \tilde{a}_1^B(\alpha)$$

Since the ZLB binds in the original Pareto problem if  $x_1^S$  is large enough, and  $x_1^S$  is increasing in  $\alpha$ , there exists  $\alpha_{ZLB}$  (which may equal 1) such that the ZLB binds if  $\alpha > \alpha_{ZLB}$ . Assumption A.23.19 implies that  $\tilde{T}(\alpha) = 0$  has at most one solution in  $[0, \alpha_{ZLB}]$  (otherwise both solutions would be competitive equilibria).

When the ZLB binds, the solution to the relaxed Pareto problem is  $\tilde{x}_t^i(\alpha), \tilde{x}_1^i, \tilde{x}_2^i, i = S, B$ , where  $\tilde{x}_2^S = (1 - \beta)\phi$ ,  $u'_1(\tilde{x}_1^S, \theta_S) = \beta u'_2(\tilde{x}_2^S, \theta_S)$ ,  $\tilde{x}_t^B = -\tilde{x}_t^S, t = 1, 2$ . Define

$$T_{ZLB}(\alpha) = \frac{u'_0(\tilde{x}_0^S(\alpha), \theta_S)}{\beta u'_1(\tilde{x}_1^S, \theta_S)} \tilde{x}_0^B(\alpha) + \tilde{a}_1^B;$$

it follows that  $T_{ZLB}$  is strictly decreasing in  $\alpha$ . Define  $T(\alpha) = T_{ZLB}(\alpha)$  if  $\alpha \leq \alpha_{ZLB}$ ,  $T(\alpha) = \tilde{T}(\alpha)$  if  $\alpha > \alpha_{ZLB}$ . An identical argument to that in the proof of Lemma A.8.5 shows that  $T(\alpha_S) > 0$ ,  $T(\alpha_B) < 0$ . It follows that there exists  $\bar{\alpha}$  such that  $T(\bar{\alpha}) = 0$ ,  $T(\alpha) > 0$  for  $\alpha < \bar{\alpha}$ ,  $T(\alpha) < 0$  for  $\alpha > \bar{\alpha}$ .

**Lemma A.23.20.** *Constrained efficient allocations with  $T(\alpha) > 0$  can be implemented with debt relief. Constrained efficient allocations with  $T(\alpha) < 0$  can be implemented with a savings subsidy.*

*Proof.* As in Lemma A.8.3. □

Part 3 of the Proposition follows.

Finally, the proof of part 4 is essentially identical to the proof of Proposition 4.7 presented above, and is omitted.

## A.24 Proof of Proposition 7.9.

Let

$$U(c^i, h^i, \theta_i) := \theta_i U(c_0^i, h_0^i) + \beta U(c_1^i, h_1^i) + \frac{\beta^2}{1 - \beta} U(c_2^i, h_2^i)$$

We have a Pareto problem

$$\max \alpha \mathcal{U}(\mathbf{c}^S, \mathbf{h}^S, \theta_S) + (1 - \alpha) \mathcal{U}(\mathbf{c}^B, \mathbf{h}^B, \theta_B) \quad (\text{A.101})$$

$$c_t^S + c_t^B = h_t^S + h_t^B, t = 0, 1, 2 \quad (\text{A.102})$$

$$U_c(c_t^S, h_t^S) + U_h(c_t^S, h_t^S) = 0, t = 0, 2 \quad (\text{A.103})$$

$$U_c(c_t^B, h_t^B) + U_h(c_t^B, h_t^B) = 0, t = 0, 2 \quad (\text{A.104})$$

$$U_c(c_1^S, h_1^S) \geq \beta U_c(c_2^S, h_2^S) \quad (\text{A.105})$$

$$\frac{U_c(c_1^B, h_1^B)}{\beta U_c(c_2^B, h_2^B)} \geq \frac{U_c(c_1^S, h_1^S)}{\beta U_c(c_2^S, h_2^S)} \quad (\text{A.106})$$

$$\left( \frac{U_c(c_1^B, h_1^B)}{\beta U_c(c_2^B, h_2^B)} - \frac{U_c(c_1^S, h_1^S)}{\beta U_c(c_2^S, h_2^S)} \right) (c_2^B - h_2^B - \frac{r^*}{1 + r^*} \phi) = 0 \quad (\text{A.107})$$

$$c_2^B \geq h_2^B - (1 - \beta) \phi \quad (\text{A.108})$$

$$\mathcal{U}(c^S, h^S, \theta_S) \geq \mathcal{U}(c^B, h^B, \theta_S) \quad (\text{A.109})$$

$$\mathcal{U}(c^B, h^B, \theta_B) \geq \mathcal{U}(c^S, h^S, \theta_B) \quad (\text{A.110})$$

I will confine attention to the case in which the borrowing constraint binds. Note that I impose that labor supply must be efficient at date 0, as well as in the steady state.

**Definition A.24.1.** An equilibrium with transfers is a collection  $\{c_t^i, d_t^i, h_t^i, r_t, w_t, \pi_t\}_{t=0}^{\infty}$  such that, given a policy  $T(d), \tau(d)$ ,

1. for each  $i$ , given  $\{r_t, w_t\}$  and given policy,  $\{c_t^i, d_t^i, h_t^i\}$  solves

$$\begin{aligned} & \max \theta_i U(c_0^i, h_0^i) + \sum_{t=1}^{\infty} \beta^t U(c_t^i, h_t^i) \\ & \text{s.t. } c_t^i + d_t^i = w_t h_t^i + \pi_t + \frac{d_{t+1}^i}{1 + r_t}, t \neq 1 \\ & c_1^i + d_1^i = T(d_1^i) + (1 - \tau(d_1^i)) w_1 h_1^i + \pi_1 + \frac{d_2^i}{1 + r_1} \\ & d_t^i \leq \phi, t \geq 2 \\ & d_0^i = 0 \end{aligned}$$

2. firms' profits are  $2\pi_t = (1 - w_t)(h_t^S + h_t^B), \forall t$

3. markets clear:

$$c_t^S + c_t^B = h_t^S + h_t^B, \forall t$$

$$4. w_t \leq 1, r_t \geq 0, r_t(1 - w_t) = 0.$$

**Lemma A.24.2.** Define the date 1 value function

$$V(a_1^i) = \max_{\{c_t^i, h_t^i, d_{t+1}^i\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} U(c_t^i, h_t^i) \quad (\text{A.111})$$

$$\text{s.t. } c_1^i = a_1^i + (1 - \tau(a_1^i))w_1 h_1 + \pi_1 + \frac{d_2^i}{1 + r_1} \quad (\text{A.112})$$

$$c_t^i = w_t h_t + \pi_t - d_t^i + \frac{d_{t+1}^i}{1 + r_t}, t \geq 2 \quad (\text{A.113})$$

$$d_t^i \leq \phi, t \geq 2 \quad (\text{A.114})$$

$\{c_t^i, h_t^i, d_{t+1}^i\}_{t=0}^{\infty}$  solves  $i$ 's problem, given  $\{w_t, r_t\}$  and  $T(\cdot), \tau(\cdot)$ , if and only if:

1.  $c_0^i, h_0^i, d_1^i$  solve

$$\begin{aligned} \max_{c_0^i, h_0^i, d_1^i} \theta_i U(c_0^i, h_0^i) + \beta V(T(d_1^i) - d_1^i) \\ \text{s.t. } c_0^i = w_0 h_0 + \frac{d_1^i}{1 + r_0} \end{aligned}$$

2.  $\{c_t^i, h_t^i, d_{t+1}^i\}_{t=1}^{\infty}$  solve (A.111), given  $a_1^i = T(d_1^i) - d_1^i$ .

*Proof.* Again, the proof is standard and is therefore omitted.  $\square$

**Lemma A.24.3.** In any equilibrium with transfers, for all  $t \geq 2$  and for all  $i$ ,  $r_t = r^* = \beta^{-1} - 1$ ,  $w_t = 1$ ,  $d_t^i = d_2^i$ ,  $c_t^i = c_2^i = h_2 - (1 - \beta)d_2^i$ .

*Proof.* First, suppose that households solve a relaxed problem in which  $\phi_t = \infty$  for all  $t \geq 3$ . In this case, household first order conditions yield

$$U_c(c_t^i, h_t^i) = \beta(1 + r_t)U_c(c_{t+1}^i, h_{t+1}^i) \text{ for all } t \geq 2$$

I will show that the borrowing constraint does not bind, so households are indeed liquidity unconstrained after date 2.

If  $r_t = r^*, \forall t \geq 2$ , then  $w_t = 1, \forall t \geq 2$ , and  $-U_h = -U_c$ , which defines  $h$  as a function  $h(c)$  of  $c$ . Since  $\beta(1 + r_t) = 1$ , marginal utility must be constant, so consumption and labor supply must also be constant. Then budget constraints impose that  $c_t^i = c_2^i = h_2 - (1 - \beta)d_2^i$ , as claimed.

Suppose by contradiction that there is also an equilibrium with  $r_t > r^*$  for some  $t \geq 2$ . Then for each household  $i$ ,  $U_c(c_t^i, h_t^i) > c_{t+1}^i$ . Integrating, we have  $y_t = \int c_t^i di < \int c_{t+1}^i di = y^*$ . So  $y_t < y^*$ , which implies  $r_t = 0$  by the definition of ZLB-constrained equilibrium, a contradiction.

Suppose by contradiction that  $r_t < r^*$ . Then a similar argument implies that  $y_{t+1} = \int c_{t+1}^i di < y^*$  and  $r_{t+1} = 0$ . Iterating forward, we see that we must have  $r_{t+s} = 0, y_{t+s} < y^*$  for all  $s \geq 1$ . This deflationary equilibrium is clearly Pareto inferior to an equilibrium with  $y_t = y^*$ , so we can rule this equilibrium out when considering optimal policy.<sup>5</sup>

From the budget constraints, it follows that  $c_t^i = c_2^i = y^* - (1 - \beta)d_2^i, d_{t+1}^i = d_t^i$ , for all  $t \geq 2$ . Since  $d_2^i \leq \phi$ , households' unconstrained borrowing decisions happen to satisfy the borrowing constraint, as claimed.  $\square$

**Lemma A.24.4.**  $\{c_t, h_t\}$  can be implemented as an equilibrium with transfers if and only if there exists  $r_1 \geq 0$  such that

$$c_t^i = c_2^i, h_t^i = h_2^i, t > 2 \quad (\text{A.115})$$

$$c_t^S + c_t^B = h_t^S + h_t^B, \forall t \quad (\text{A.116})$$

$$U_c(c_2^i, h_2^i) + U_h(c_2^i, h_2^i) = 0, i = S, B \quad (\text{A.117})$$

$$-\frac{U_h(c_0^S, h_0^S)}{U_c(c_0^S, h_0^S)} = -\frac{U_h(c_0^B, h_0^B)}{U_c(c_0^B, h_0^B)} \quad (\text{A.118})$$

$$U_c(c_1^i, h_1^i) \geq \beta(1 + r_1)U_c(c_2^i, h_2^i), c_2^i \geq h_2^i - (1 - \beta)\phi, \text{ with at least one equality, } i = S, B \quad (\text{A.119})$$

$$U(c^S, h^S, \theta_S) \geq U(c^B, h^B, \theta_S) \quad (\text{A.120})$$

$$U(c^B, h^B, \theta_B) \geq U(c^S, h^S, \theta_B) \quad (\text{A.121})$$

*Proof.* Take any equilibrium with transfers. From Lemma A.24.3, we know the first three conditions hold. (A.118) follows from households' date 0 first order conditions, given that they face the same wage. (A.119) follows from households' date 1 problem. Finally, the incentive compatibility conditions (A.120), (A.121) hold by a standard mimicking argument.

Next, we show that these conditions are sufficient for the allocation to be implementable. Suppose we have an allocation  $\{c_t, h_t\}$  and associated  $r_1 \geq 0$  such that these conditions hold. Set  $r_t = r^*, w_t = 1$  for all  $t \geq 2$ . Set  $w_0 = -\frac{U_h(c_0^S, h_0^S)}{U_c(c_0^S, h_0^S)}$ . If  $w_0 < 1$ , set  $r_0 = 0$ ; otherwise, choose any

<sup>5</sup>Equivalently, we could append to our definition of equilibrium the condition that  $\lim_{t \rightarrow \infty} y_t = y^*$ .

$r_0 \geq 0$ . Set  $w_1 = 1$  and let  $\tau(a)$  be any continuous function such that  $\tau(a_1^i) = 1 + \frac{U_h(c_1^i, h_1^i)}{U_c(c_1^i, h_1^i)}$ ,  $i = S, B$ , where for each  $i$ , we define

$$a_1^i = c_1^i - h_1^i + \frac{c_2^i - h_2^i}{(1+r_1)(1-\beta)}$$

It remains to show that each household solves (A.111). Given prices, and given the transfer function  $\tau$ , the argument in Lemma A.7.3 applies directly, and the incentive compatibility conditions (A.120), (A.121) imply this. So we are done.  $\square$

As in the previous sections, it is straightforward to show that any solution to the Pareto problem satisfies the conditions in Lemma A.24.4.

## A.25 Proof of Proposition 7.10.

If the borrowing constraint binds,  $c_2^S, c_2^B, h_2^S, h_2^B$  are pinned down by (A.104), (A.102) and (A.108). Let  $W(x)$  solve

$$\begin{aligned} W(x) &= U(c, h) \\ U_c(c, h) + U_h(c, h) &= 0c - h = x \end{aligned}$$

Then we can write the Pareto problem as

$$\begin{aligned} \max_{x_0^S, c_1^S, h_1^S, x_0^B, c_1^B, h_1^B} \quad & \alpha \theta_S W(x_0^S) + \alpha \beta U(c_1^S, h_1^S) + (1-\alpha) \theta_B W(x_0^B) + (1-\alpha) \beta U(c_1^B, h_1^B) \\ & x_0^S + x_0^B = 0 \\ & c_1^S + c_1^B = h_1^S + h_1^B \\ & U_c(c_1^S, h_1^S) \geq \beta U_c(c_2^S, h_2^S) \\ & \theta_S W(x_0^S) + \beta U(c_1^S, h_1^S) + \Delta \geq \theta_S W(x_0^B) + \beta U(c_1^B, h_1^B) \\ & \theta_B W(x_0^B) + \beta U(c_1^B, h_1^B) \geq \theta_B W(x_0^S) + \beta U(c_1^S, h_1^S) + \Delta \end{aligned}$$

where  $\Delta := \frac{\beta^2}{1-\beta}[U(c_2^S, h_2^S) - U(c_2^B, h_2^B)]$  is fixed. First order necessary and sufficient conditions for an optimum are

$$\begin{aligned}(\alpha\theta_S + \mu_S - \mu_B)W'(x_0^S) &= \lambda_0 \\ ((1-\alpha)\theta_B - \mu_S + \mu_B)W'(x_0^B) &= \lambda_0 \\ (\alpha + \mu_S - \mu_B)U_c(c_1^S, h_1^S) + \zeta U_{cc}(c_1^S, h_1^S) &= \lambda_1 \\ -(\alpha + \mu_S - \mu_B)U_h(c_1^S, h_1^S) + \zeta U_{ch}(c_1^S, h_1^S) &= \lambda_1 \\ (1-\alpha - \mu_S + \mu_B)U_c(c_1^B, h_1^B) &= \lambda_1 \\ -(1-\alpha - \mu_S + \mu_B)U_h(c_1^B, h_1^B) &= \lambda_1\end{aligned}$$

First, suppose no incentive constraints bind,  $\mu_S = \mu_B = 0$ . The first two equations then define  $x_0^S, x_0^B$  as (respectively) strictly increasing and strictly decreasing functions of  $\alpha$ .  $S$ 's date 1 utility is also weakly increasing in  $\alpha$ .  $S$ 's net gain from choosing his own allocation is

$$\theta_S[W(x_0^S(\alpha)) - W(x_0^B(\alpha))] + \beta[U_1^S(\alpha) - U_1^B(\alpha)] + \Delta$$

which is increasing in  $\alpha$ , and is positive for sufficiently small  $\alpha$ , so there exists  $\alpha_S > 0$  such that ICS binds if  $\alpha < \alpha_S$ . An analogous argument shows that there exists  $\alpha_B$  such that ICB binds if  $\alpha > \alpha_B$ .

If  $\lambda_1 > 0$ , we have

$$(1 + \tau(d_1^S)) = -\frac{U_h(c_1^S, h_1^S)}{U_c(c_1^S, h_1^S)} = \frac{\alpha + \mu_S - \mu_B + \zeta \frac{U_{cc}(c_1^S, h_1^S)}{U_c(c_1^S, h_1^S)}}{\alpha + \mu_S - \mu_B + \zeta \frac{U_{ch}(c_1^S, h_1^S)}{U_h(c_1^S, h_1^S)}}$$

Under the regularity condition  $\frac{U_{ch}}{U_h} > \frac{U_{cc}}{U_c}$ , the labor wedge  $\tau(d_1^S) > 0$ . With quasilinear preferences,  $\frac{U_{ch}}{U_h} = \frac{U_{cc}}{U_c}$ , and the labor wedge is zero.

If  $\lambda_1 = 0$ ,  $\alpha + \mu_S - \mu_B = 1$  and  $U_c(c_1^S, h_1^S) + \zeta U_{cc}(c_1^S, h_1^S) = -U_h(c_1^S, h_1^S) - \zeta U_{ch}(c_1^S, h_1^S) = 0$ , and so

$$(1 + \tau(d_1^S)) = \frac{-U_h}{U_c} = 1 + \zeta \frac{U_{cc} + U_{ch}}{U_c}$$

Under the regularity condition that  $U_{cc} + U_{ch} < 0$ , again the labor wedge is positive.

To prove part 3, note that if  $B$  faces a positive labor wedge, it must be that  $\lambda_1 = 0$ , which in turn can only be the case if  $\mu_S > 0$  and ICS binds. (Specifically, it can only be the case if

$\alpha + \mu_S - \mu_B = 1$ .) To prove part 4, note that when  $U_{ch} = 0$ ,  $\lambda_1 = -(\alpha + \mu_S - \mu_B)U_h(c_1^S, h_1^S)$ , which is positive when  $\alpha + \mu_S - \mu_B = 1$ . So we cannot have  $\lambda_1 = 0$ .

## A.26 Proof of Proposition 7.11.

Again, it simplifies matters to directly assume uniqueness of equilibrium in the absence of policy.

**Assumption A.26.1.** *The economy has a unique equilibrium.*

Fix  $\theta_S, \theta_B$ . First consider a relaxed Pareto problem without incentive constraints in which the ZLB never binds. In this case, labor supply is always efficient ( $U_c + U_h = 0, \forall i, t$ ) and each household obtains utility  $W(x) = \max_h U(x + h, h)$  in each period. Thus when the ZLB does not bind, the economy with endogenous labor supply is isomorphic to a special case of the economy with persistent types, and the argument presented there establishes that  $\tilde{T}(\alpha) = 0$  has at most one solution in  $[0, \alpha_{ZLB}]$ , where

$$\tilde{T}(\alpha) = \frac{\theta_S U_c(\tilde{c}_0^S(\alpha), \tilde{h}_0^S(\alpha))}{\beta U_c(\tilde{c}_1^S(\alpha), \tilde{h}_1^S(\alpha))} \tilde{x}_0^B(\alpha) + \tilde{a}_1^B(\alpha)$$

and where a tilde denotes the solution to the relaxed Pareto problem. When the ZLB binds,  $U_c(c_1^S, h_1^S)$  is fixed by the ZLB constraint, and the same argument above shows that  $T_{ZLB}(\alpha)$  (defined in the obvious way) is decreasing. Define  $T(\alpha) = T_{ZLB}(\alpha)$  if  $\alpha \leq \alpha_{ZLB}$ ,  $T(\alpha) = \tilde{T}(\alpha)$  if  $\alpha > \alpha_{ZLB}$ . Again, an identical argument to that in the proof of Lemma A.8.5 shows that  $T(\alpha_S) > 0$ ,  $T(\alpha_B) < 0$ . It follows that there exists  $\bar{\alpha}$  such that  $T(\bar{\alpha}) = 0$ ,  $T(\alpha) > 0$  for  $\alpha < \bar{\alpha}$ ,  $T(\alpha) < 0$  for  $\alpha > \bar{\alpha}$ .

**Lemma A.26.2.** *Constrained efficient allocations with  $T(\alpha) > 0$  can be implemented with debt relief. Constrained efficient allocations with  $T(\alpha) < 0$  can be implemented with a savings subsidy.*

*Proof.* As in Lemma A.8.3. □

Again, the proof that debt relief is Pareto improving at the ZLB is essentially identical to the proof of Proposition 4.7 presented above, and is omitted.



Appendix to Chapter 2

**B.1 Proof of Proposition 2.1**

1. In this case the equilibrium conditions become

$$x_t^i - \left(\frac{x_t^i}{A_t}\right)^{1/\phi} = \frac{p_t^*}{p_t^i} \int c_t^f \, df, \forall i, t \quad (\text{B.1})$$

$$\varphi \pi_t^i (\pi_t^i - 1) = (\varepsilon - 1) \left( \frac{(x_t^i)^{1-\phi}}{\phi A_t^{1/\phi}} - 1 \right) + \varphi Q_{t,t+1}^i \pi_{t+1}^i \frac{x_{t+1}^i}{x_t^i} \pi_{t+1}^i (\pi_{t+1}^i - 1), \forall i, t \quad (\text{B.2})$$

Take any  $\{c_t^i\}$  satisfying

$$\int c_1^i \, di \leq y^*$$

$$\int c_t^i \, di = y^*, t \geq 2$$

Set  $y_t = \int c_t^i \, di$ , and let  $x_t$  satisfy  $y_t = x_t - \left(\frac{x_t}{A}\right)^{1/\phi}$ . Let  $\pi_1$  satisfy

$$\varphi \pi_1 (\pi_1 - 1) = (\varepsilon - 1) \left( \frac{x_1^{1-\phi}}{\phi A^{1/\phi}} - 1 \right)$$

and set  $\pi_t = 1$  for  $t > 1$ . Set  $p_t^i = p_t^*$ ,  $x_t^i = x_t$ ,  $\pi_t^i = \pi_t$ . It is clear that this satisfies (B.1) and (B.2).

2. With  $\varphi = \infty$ , prices never adjust, and any  $x_t^i$  is consistent with firm optimality. The first equilibrium condition gives us

$$y_t^i := x_t^i - \left(\frac{x_t^i}{A_t}\right)^{1/\phi} = (1 - \alpha)c_t^i + \alpha \int c_t^f \, di \leq y^*$$

Since  $\int c_t^f \, di = \int y_t^f \, di$ , we can equivalently write

$$y_t^i := x_t^i - \left(\frac{x_t^i}{A_t}\right)^{1/\phi} = (1 - \alpha)c_t^i + \alpha \int y_t^f \, di \leq y^*$$

as stated in the Proposition. □

## B.2 Proof of Lemma 2.2

By assumption, there is no default, and no uncertainty, after date 2. From Proposition 2.1, aggregate income is constant and equals  $y^*$ . A standard argument establishes that the one period bond price  $Q_t^{rf} = \beta, \forall t \geq 2$ , and each country's consumption is constant. It follows that a country with debt  $d_2^i$  consumes  $y^* - (1 - \beta)d_2^i$  in every period. The remainder of the Lemma follows immediately.  $\square$

## B.3 Proof of Proposition 3.4

I restate the assumptions made in the main text:

**Assumption B.3.1.** Either  $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$  with  $\sigma > 1$ , or  $u(c) = \ln c$ .

**Assumption B.3.2.**  $\gamma(d) := \frac{f(d)d}{1-F(d)}$  is nondecreasing in  $d$ .

**Lemma B.3.3.** Let  $Y(d_2, Q^{rf})$  solve the following equations:

$$\begin{aligned} Q^{rf} u'(c_1^S) &= \beta u'(y^* + (1 - F(d_2))d_2) \\ Q^{rf} u'(c_1^B) &= \beta u'(y^* - d_2) \frac{1}{1 - \gamma(d_2)} \\ Y(d_2) &= c_1^S + c_1^B \end{aligned}$$

Then  $Y(d_2)$  is decreasing in  $d_2$  and increasing in  $Q^{rf}$ .

*Proof.* First, note that

$$\frac{\partial}{\partial d_2} (1 - F(d_2))d_2 = (1 - \gamma(d_2))(1 - F(d_2)).$$

Given our functional form assumptions, we have

$$\begin{aligned} c_1^S &= (Q^{rf}/\beta)^{1/\sigma} (y^* + (1 - F(d_2))d_2) \\ c_1^B &= (Q^{rf}/\beta)^{1/\sigma} (y^* - d_2)(1 - \gamma(d_2))^{1/\sigma} \\ Y(d_2, Q^{rf}) &= (Q^{rf}/\beta)^{1/\sigma} \left[ y^* + (1 - F(d_2))d_2 + (y^* - d_2)(1 - \gamma(d_2))^{1/\sigma} \right] \end{aligned}$$

Taking derivatives with respect to  $d_2$ ,

$$\begin{aligned} \frac{\partial Y(d_2)}{\partial d_2} &= (Q^{rf}/\beta)^{1/\sigma} \left[ (1 - \gamma(d_2))(1 - F(d_2)) - (1 - \gamma(d_2))^{1/\sigma} - \frac{\gamma'(d_2)}{\sigma} (y^* - d_2)(1 - \gamma(d_2))^{1/\sigma-1} \right] \\ &= -(Q^{rf}/\beta)^{1/\sigma} \left[ (1 - \gamma(d_2))^{1/\sigma-1} - (1 - F(d_2)) + \frac{\gamma'(d_2)}{\sigma} (y^* - d_2)(1 - \gamma(d_2))^{1/\sigma-1} \right] \end{aligned}$$

The first term inside the square brackets is positive, since  $1 - \gamma(d_2) < 1$ ,  $1/\sigma - 1 < 0$ , and  $1 - F(d_2) < 1$ . The second term is also positive, since  $\gamma'(d_2) \geq 0$ . Thus  $\frac{\partial Y(d_2)}{\partial d_2} < 0$ .

Since the whole term in square brackets is positive, it is clear that  $Y$  is increasing in  $Q^{rf}$ . So we are done.  $\square$

Note that  $c_1^S$  is increasing in  $d_2$ . But  $c_1^B$  is decreasing in  $d_2$ , and at a faster rate. Both are increasing in  $Q^{rf}$ .

**Corollary B.3.4.** *There exists a unique  $d_2^*$  such that the ZLB just binds - that is, we have  $Q^{rf} = 1$  and  $y_1 = y^*$  in equilibrium - if and only if  $d_2 = d_2^*$  in equilibrium.*

*Proof.* The ZLB just binds when  $Y(d_2, 1) = 2y^*$ . Since  $Y(d_2, 1)$  is decreasing in  $d_2$ , there is a unique  $d_2^*$  such that this is the case.  $\square$

Another corollary is that the ZLB binds if and only if  $d_2 \geq d_2^*$  in equilibrium:

**Corollary B.3.5.** *In any equilibrium in which  $Y \leq Y^*$  and  $Q^{rf} \leq 1$ :*

1. *If  $d_2 < d_2^*$ , then  $Q^{rf} < 1$  and  $Y = Y^*$ .  $Q^{rf}(d_2)$  is increasing in  $d_2$ .*
2. *If  $d_2 > d_2^*$ , then  $Q^{rf} = 1$  and  $Y < Y^*$ .  $Y(d_2)$  is decreasing in  $d_2$ .*

**Lemma B.3.6.**  *$c_1^S(d_2)$  is increasing in  $d_2$ .*

*Proof.* When  $d_2 < d_2^*$ , the ZLB does not bind and

$$\left( \frac{c_1^S}{2y^* - c_1^S} \right)^\sigma = \left( \frac{1 + \phi(d_2)}{1 - d_2} \right)^\sigma \frac{1}{1 - \gamma(d_2)}$$

where  $\phi(d) := (1 - F(d))d$ . Clearly since  $\gamma(d)$  and  $\phi(d)$  are nondecreasing in  $d$ , this defines an increasing relation between  $c_1^S$  and  $d_2$ , call it  $c_{FB}(d_2)$ .

When  $d_2 > d_2^*$ , the ZLB binds and

$$c_1^S = \beta^{-1/\sigma} [1 + \phi(d_2)]$$

which also defines an increasing relation between  $c_1^S$  and  $d_2$ , call it  $c_{ZLB}(d_2)$ . The two functions intersect only at  $d_2^*$ :  $c_{FB}(d_2^*) = c_{ZLB}(d_2^*)$ . When  $d_2 < d_2^*$ ,  $c_{ZLB} > c_{FB}$ ; when  $d_2 > d_2^*$ ,  $c_{FB} > c_{ZLB}$ .  $\square$

**Lemma B.3.7.**  *$\bar{d}_1(d_2)$  is increasing in  $d_2$ .*

*Proof.* When the ZLB does not bind,

$$\bar{d}_1 = c_1^S(d_2) + Q^{rf}(d_2)\phi(d_2) - y^*$$

which is increasing in  $d_2$ . When the ZLB binds,

$$\bar{d}_1 = c_1^S(d_2) + \phi(d_2) - \frac{1}{2}Y(d_2)$$

which is also increasing in  $d_2$ . □

As a consequence of this lemma, we can invert the function  $\bar{d}_1(d_2)$  and express all equilibrium variables as functions of  $\bar{d}_1$ . So we have:

**Proposition B.3.8.** *There exists  $\bar{d}_1^*$  such that:*

1. *If  $\bar{d}_1 < \bar{d}_1^*$ , then  $Q^{rf} < 1$  and  $Y = Y^*$ .  $Q^{rf}(\bar{d}_1)$  is increasing in  $\bar{d}_1$ .*
2. *If  $\bar{d}_1 = \bar{d}_1^*$ , then  $Q^{rf} = 1$  and  $Y = Y^*$ .*
3. *If  $\bar{d}_1 > \bar{d}_1^*$ , then  $Q^{rf} = 1$  and  $Y < Y^*$ .  $Y(\bar{d}_1)$  is decreasing in  $\bar{d}_1$ .*

$c_1^S$  is increasing in  $\bar{d}_1$ .  $c_1^B$  is decreasing in  $\bar{d}_1$ .

## B.4 Proof of Propositions 3.5 and 3.6

It is convenient to prove Proposition 3.6 first. Since  $y_1 < y^*$  when the ZLB binds in equilibrium, it follows from part 1. of Proposition 3.5 (proved in the main text) that the equilibrium cannot be a solution to the Pareto problem in this case. When the ZLB does not bind in equilibrium, set  $\lambda = u'(c_1^B)$ ,  $\nu = \frac{u'(c_1^B)}{u'(c_1^S)}$ ,  $\mu = 0$ . The equilibrium satisfies the first order necessary and sufficient conditions in the planner's problem.

To prove the rest of Proposition 3.5, I first construct a solution to the planner's problem, and then show that it satisfies the necessary and sufficient conditions for optimality.

Define  $U_S(d_2) = u(c_1^S(d_2)) + \beta V(\phi(d_2))$  for  $d_2 \leq d_2^*$ , where  $\phi(d_2) := p(d_2)d_2$  is the debt that is repaid to savers when the face value of debt is  $d_2$ . Since  $c_1^S$  and  $p(d_2)d_2$  are increasing in  $d_2$  over this range,  $U_S(d_2)$  is increasing. For  $d_2 \in (d_2^*, d^*)$ , where  $d^* = \arg \max_d \phi(d)$ , define  $U_S(d_2) = \beta^{1/\sigma}(y^* + (1 - \beta)\phi(d_2)) + \beta V(\phi(d_2))$ . Clearly  $U_S$  is also increasing over this range.

Set  $\bar{U} = U_S(d^*)$  and  $U_S^* = U_S(d_2^*)$ . Since  $U_S(d_2)$  is increasing, we can invert it to obtain an increasing function  $d_2(U_S)$ . With some abuse of notation, define  $c_1^S(U_S) = c_1^S(d_2(U_S))$  for  $U_S \leq U_S^*$ ,  $c_1^S(U_S) = \beta^{1/\sigma}(y^* + (1 - \beta)\phi(d_2(U_S)))$  for  $U_S \in [U_S^*, U^*]$ . Define  $c_1^B(U_1) = 2y^* - C_1^S(U_2)$ . By construction, this solution satisfies properties 2, 3 and 4 in Proposition 3.5. It only remains to show that this is in fact the solution to the planner's problem.

When  $U_S < U_S^*$ , set  $\lambda = u'(c_1^B)$ ,  $\nu = \frac{u'(c_1^B)}{u'(c_1^S)}$ ,  $\mu = 0$ . The solution satisfies the first order necessary and sufficient conditions in the planner's problem. When  $U_S \in [U_S^*, U^*]$ , we can construct positive  $\nu, \mu$  that satisfy the first order conditions. □

## B.5 Proof of Proposition 3.11

When the ZLB does not bind, the result is trivial. When the ZLB binds in the planner's problem, let  $c_1^B, c_1^S, d_2$  be the optimal allocation. A borrower country's Euler equation is

$$u'(c_1^B)[\tilde{Q}(d_2) + \tilde{Q}'(d_2)d_2] = \beta p(d_2)u'(c_2^B)$$

Set  $Q^{rf} = 1, y_1 = y^*$ , and  $\tilde{Q}(d_2) = \Gamma p(d_2)$ ; clearly we can always choose  $\Gamma$  such that the Euler equation is satisfied. Set  $T_1^B$  to satisfy the borrower country's budget constraint, with  $y^* = 1$ . Choose  $T_1^S$  to satisfy the government budget constraint; by Walras's law, the saver country's budget constraint is also satisfied, and their choice  $c_1^S, a_2 = \phi(d_2)$  satisfies their Euler equation with  $Q^{rf} = 1$  and is optimal. Thus all equilibrium conditions are satisfied. □

## B.6 Proof of Proposition 3.12

Part 1 follows immediately from Proposition 3.6. To prove part 2, note that when  $U_S > U_S^*$ , the ZLB binds,  $Q^{rf} = 1$ , and so the marginal price of debt in the equilibrium without policy is  $p(d_2) + p'(d_2)d_2$ . From the borrower's Euler equation, we have

$$\tilde{Q}(d_2) + \tilde{Q}'(d_2)d_2 = \frac{\beta p(d_2)u'(c_2^B)}{u'(c_1^B)}$$

From the planner's optimality condition, we have

$$\nu \beta u'(c_2^S)[p(d_2) + p'(d_2)d_2] = \beta p(d_2)u'(c_2^B) + \mu \beta u''(c_1^S)[p(d_2) + p'(d_2)d_2]$$

Using  $v\beta u'(c_2^S) = vu'(c_1^S) = u'(c_1^B) - \mu u''(c_1^S)$ , we have

$$(u'(c_1^B) - \mu u''(c_1^S))[p(d_2) + p'(d_2)d_2] = \beta p(d_2)u'(c_2^B) + \mu\beta u''(c_2^S)[p(d_2) + p'(d_2)d_2]$$

$$p(d_2) + p'(d_2)d_2 = \frac{\beta p(d_2)u'(c_2^B)}{u'(c_1^B)} + \frac{\mu(u''(c_1^S) + \beta u''(c_2^S))[p(d_2) + p'(d_2)d_2]}{u'(c_1^B)}$$

$$p(d_2) + p'(d_2)d_2 < \frac{\beta p(d_2)u'(c_2^B)}{u'(c_1^B)}$$

where the last inequality follows because  $u'' < 0, \mu > 0$ . Using the borrower's Euler equation, we have the stated result.

Finally, Part 3 is trivial. When the ZLB binds, the equilibrium is Pareto inefficient, and there exists some solution to the planner's problem that is Pareto superior to the equilibrium. By Proposition 3.11, this Pareto superior allocation can be implemented as an equilibrium with lending policy.  $\square$

## B.7 Proof of Proposition 4.2

Define  $\gamma(d, \bar{d}) = \frac{f(d)(d - \bar{d})}{1 - F(d)}$ . Since  $\gamma(d, 0)$  is increasing in  $d$ , so is  $\gamma(d, \bar{d})$  when  $\bar{d} > 0$ .

**Lemma B.7.1.** *Let  $Y(d_2, Q^{rf}, \bar{d}_2)$  solve the following equations:*

$$Q^{rf} u'(c_1^S) = \beta u'(y^* + (1 - F(d_2))d_2)$$

$$Q^{rf} u'(c_1^B) = \beta u'(y^* - d_2) \frac{1}{1 - \gamma(d_2, \bar{d}_2)}$$

$$Y(d_2, Q^{rf}, \bar{d}_2) = c_1^S + c_1^B$$

*Then  $Y(d_2)$  is decreasing in  $d_2$ , increasing in  $Q^{rf}$ , and increasing in  $\bar{d}_2$  when  $d_2 > \bar{d}_2$ .*

**Corollary B.7.2.** *Given  $\bar{d}_2$ , there exists an increasing function  $d_2^*(\bar{d}_2) > \bar{d}_2$  such that the ZLB just binds if and only if  $d_2 = d_2^*(\bar{d}_2)$  in equilibrium.*

*Proof.* The ZLB just binds when  $Y(d_2, 1, \bar{d}_2) = 2y^*$ . By the previous lemma, this defines  $d_2$  as an increasing function of  $\bar{d}_2$ .  $\square$

**Corollary B.7.3.** *In any equilibrium in which  $Y \leq Y^*$  and  $Q^{rf} \leq 1$ :*

1. *If  $d_2 < d_2^*(\bar{d}_2)$ , then  $Q^{rf} < 1$  and  $Y = Y^*$ .  $Q^{rf}(d_2, \bar{d}_2)$  is increasing in  $d_2$  and decreasing in  $\bar{d}_2$  when  $d_2 > \bar{d}_2$ .*

2. If  $d_2 > d_2^*(\bar{d}_2)$ , then  $Q^{rf} = 1$  and  $Y < Y^*$ .  $Y(d_2, \bar{d}_2)$  is decreasing in  $d_2$  and increasing in  $\bar{d}_2$  when  $d_2 > \bar{d}_2$ .

**Lemma B.7.4.**  $\bar{d}_1(d_2, \bar{d}_2)$  is increasing in  $d_2$  and decreasing in  $\bar{d}_2$  when  $d_2 > \bar{d}_2$ .

*Proof.* When the ZLB does not bind,

$$\bar{d}_1 = c_1^S(d_2) + Q^{rf}(d_2, \bar{d}_2)\phi(d_2) - y^*$$

which is increasing in  $d_2$  and decreasing in  $\bar{d}_2$ . When the ZLB binds,

$$\bar{d}_1 = c_1^S(d_2) + \phi(d_2) - \frac{1}{2}Y(d_2, \bar{d}_2)$$

which is also increasing in  $d_2$  and decreasing in  $\bar{d}_2$ . □

Given this lemma, we can invert the function to write  $d_2 = d_2(\bar{d}_1, \bar{d}_2)$ , which is increasing in both arguments.

So we have:

**Proposition B.7.5.** *There exists  $\bar{d}_1^*(\bar{d}_2)$  such that:*

1. If  $\bar{d}_1 < \bar{d}_1^*$ , then  $Q^{rf} < 1$  and  $Y = Y^*$ .  $Q^{rf}(\bar{d}_1)$  is increasing in  $\bar{d}_1$  and decreases
2. If  $\bar{d}_1 = \bar{d}_1^*$ , then  $Q^{rf} = 1$  and  $Y = Y^*$ .
3. If  $\bar{d}_1 > \bar{d}_1^*$ , then  $Q^{rf} = 1$  and  $Y < Y^*$ .  $Y(\bar{d}_1)$  is decreasing in  $\bar{d}_1$ .

$c_1^S$  is increasing in  $\bar{d}_1$ .  $c_1^B$  is decreasing in  $\bar{d}_1$ .

## B.8 Proof of Proposition 4.3

The borrower and saver countries' Euler equations imply that in equilibrium

$$\frac{u'(c_1^S)}{u'(c_1^B)} = \frac{u'(c_2^S)}{u'(c_2^B)} \frac{1 - f(d)(d - \bar{d})}{1 - F(d)}$$

In any Pareto efficient allocation where the ZLB is slack, we have

$$\frac{u'(c_1^S)}{u'(c_1^B)} = \frac{u'(c_2^S)}{u'(c_2^B)} \frac{1 - f(d)d}{1 - F(d)}$$

Whenever  $\bar{d} > 0$ , the equilibrium cannot satisfy this optimality condition, and must be Pareto inefficient. □

## B.9 Proof of Proposition 4.4

Take any Pareto efficient allocation in which  $U_S > U_S^*$ , so that the ZLB binds. We have

$$\frac{u'(c_1^S)}{u'(c_1^B)} > \frac{u'(c_2^S)}{u'(c_2^B)} \frac{1 - f(d)d}{1 - F(d)}$$

Thus we can also choose  $\bar{d}_2$  large enough that

$$\frac{u'(c_1^S)}{u'(c_1^B)} = \frac{u'(c_2^S)}{u'(c_2^B)} \frac{1 - f(d)(d - \bar{d}_2)}{1 - F(d)}$$

Set  $Q^{rf} = 1, y_1 = y^*, \bar{d}_1 = y^* + Q(d_2)(d_2 - \bar{d}_2 - c_1^B)$ . It is straightforward to verify that this equilibrium satisfies all the conditions in Proposition 4.1.  $\square$

## B.10 Proof of Proposition 5.2

**Assumption B.10.1.**  $u(c) = \ln c$ .

This assumption is not necessary, but makes it easier to prove the following proposition.

**Lemma B.10.2.** Let  $y^S(d_2, Q^{rf})$  solve the following equations:

$$\begin{aligned} Q^{rf} u'(c_1^S) &= \beta u'(y^* + (1 - F(d_2/\alpha))d_2) \\ Q^{rf} u'(c_1^B) &= \beta u'(y^* - d_2/\alpha) \frac{1}{1 - \gamma(d_2/\alpha)} \\ y^S(d_2, Q^{rf}) &= (1 - \alpha/2)c_1^S + (\alpha/2)c_1^B \end{aligned}$$

Then  $Y(d_2, Q^{rf})$  is decreasing in  $d_2$  and increasing in  $Q^{rf}$ .

**Corollary B.10.3.** There exists a unique  $d_2^*$  such that the ZLB just binds - that is, we have  $Q^{rf} = 1$  and  $y_1^S = y^*$  in equilibrium - if and only if  $d_2 = d_2^*$  in equilibrium.

**Corollary B.10.4.** In any equilibrium in which  $Y \leq Y^*$  and  $Q^{rf} \leq 1$ :

1. If  $d_2 < d_2^*$ , then  $Q^{rf} < 1$  and  $y_1^S = y^*$ .  $Q^{rf}(d_2)$  is increasing in  $d_2$ .
2. If  $d_2 > d_2^*$ , then  $Q^{rf} = 1$  and  $y_1^S < y^*$ .  $y^S(d_2)$  is decreasing in  $d_2$ .

**Lemma B.10.5.**  $c_1^S(d_2)$  is increasing in  $d_2$ .

*Proof.* When  $d_2 < d_2^*$ , the ZLB does not bind and

$$\left( \frac{c_1^S}{\frac{2y^* - (2-\alpha)c_1^S}{\alpha}} \right)^\sigma = \left( \frac{y^* + \phi(d_2)}{y^* - \frac{1-\alpha}{\alpha}[1 + p(d_2)]d_2 - d_2} \right)^\sigma \frac{1}{1 - \gamma(d_2/\alpha)}$$

Clearly since  $\gamma(d)$  and  $\phi(d)$  are nondecreasing in  $d$ , this defines an increasing relation between  $c_1^S$  and  $d_2$ , call it  $c_{FB}(d_2)$ .

When  $d_2 > d_2^*$ , the ZLB binds and

$$c_1^S = \beta^{-1/\sigma}[1 + \phi(d_2)]$$

which also defines an increasing relation between  $c_1^S$  and  $d_2$ , call it  $c_{ZLB}(d_2)$ . The two functions intersect only at  $d_2^*$ :  $c_{FB}(d_2^*) = c_{ZLB}(d_2^*)$ . When  $d_2 < d_2^*$ ,  $c_{ZLB} > c_{FB}$ ; when  $d_2 > d_2^*$ ,  $c_{FB} > c_{ZLB}$ .  $\square$

**Lemma B.10.6.**  $\bar{d}_1(d_2)$  is increasing in  $d_2$ .

*Proof.* When the ZLB does not bind,

$$\bar{d}_1 = \frac{\alpha}{2-\alpha}(c_1^S(d_2) - y^*) + Q^{rf}(d_2)\phi(d_2)$$

which is increasing in  $d_2$ . When the ZLB binds,

$$\bar{d}_1 = \frac{\alpha}{2-\alpha}(c_1^S(d_2) - y^S(d_2)) + \phi(d_2)$$

which is also increasing in  $d_2$ .  $\square$

As a consequence of this lemma, we can invert the function  $\bar{d}_1(d_2)$  and express all equilibrium variables as functions of  $\bar{d}_1$ . So we have:

**Proposition B.10.7.** *There exists  $\bar{d}_1^*$  such that:*

1. If  $\bar{d}_1 < \bar{d}_1^*$ , then  $Q^{rf} < 1$  and  $y_1^S = y^*$ .  $Q^{rf}(\bar{d}_1)$  is increasing in  $\bar{d}_1$ .
2. If  $\bar{d}_1 = \bar{d}_1^*$ , then  $Q^{rf} = 1$  and  $y_1^S = y^*$ .
3. If  $\bar{d}_1 > \bar{d}_1^*$ , then  $Q^{rf} = 1$  and  $y_1^S < y^*$ .  $y^S(\bar{d}_1)$  is decreasing in  $\bar{d}_1$ .

$c_1^S$  is increasing in  $\bar{d}_1$ .  $c_1^B$  is decreasing in  $\bar{d}_1$ .

The proof of Propositions 5.3 and 5.5, respectively, follow exactly the proofs of Proposition 3.5 and 3.11, and are therefore omitted.

## B.11 Proof of Proposition 6.1

Combining borrowers' and savers' Euler equations, we have

$$\frac{U_c(c_0^S, \theta_S)}{U_c(c_0^B, \theta_B)} = \frac{\pi u'(c_1^S(d_1)) + (1 - \pi)u'(y^* + (1 - \beta)d_1)}{\pi u'(c_1^B(d_1)) + (1 - \beta)u'(y^* - (1 - \beta)d_1)}$$

$c_1^S$  is increasing in  $d_1$ , and  $c_1^B$  is decreasing. Therefore the right hand side is decreasing in  $d_1$ , while the left hand side is decreasing in  $d_1$ . As  $\theta_B \rightarrow \infty$ ,  $c_1^B(d_1) \rightarrow 0$ . Thus for high enough  $\theta_B$  and  $d_1$ , the zero lower bound binds in equilibrium.  $\square$

## B.12 Proof of Propositions 6.2 and 6.3

The highest possible value of  $V_S$  is  $u(c_1^S(d^*)) + \beta V(\phi(d^*))$ . Let  $V_B^{UE} = u(c_1^B(d^*)) + \beta \left[ \int_0^{d^*} V(-\chi) dF(\chi) + p(d^*)V(-d^*) \right]$ . Then if  $V_B < V_B^{UE}$ , there exists  $c_1^B < 2y^* - c_1^B(d^*)$  such that

$$u(c_1^B) + \beta \left[ \int_0^{d^*} V(-\chi) dF(\chi) + p(d^*)V(-d^*) \right] = V_B$$

It is immediate that when  $V_B < V_B^{UE}$ ,  $V_S'(V_B) = 0$ . Define  $V_B^{ZLB}$  so that  $U_S^* = V_S(V_B^{ZLB})$ .

Putting Lagrange multipliers  $\xi, \lambda, \mu$  on the constraints, the Envelope condition and first order conditions for  $c_1^B$  and  $c_1^S$  are

$$\begin{aligned} V_S'(V_B) &= -\xi \\ \xi u'(c_1^B) &= \lambda \\ u'(c_1^S) + \mu u''(c_1^S) &= \lambda \end{aligned}$$

Combining, we have

$$V_S'(V_B) = -\frac{u'(c_1^S) + \mu u''(c_1^S)}{u'(c_1^B)}$$

When  $V_B \in (V_B^{UE}, V_B^{ZLB})$ , the ZLB binds,  $\mu > 0$ , and  $V_S'(V_B) > \frac{-u'(c_1^S)}{u'(c_1^B)}$ . When  $V_B \geq V_B^{ZLB}$ , the ZLB is slack,  $\mu = 0$ , and  $V_S'(V_B) = \frac{-u'(c_1^S)}{u'(c_1^B)}$ .

The first order conditions in the planner's date 0 problem yield

$$\frac{U_c(c_0^S, \theta_S)}{U_c(c_0^B, \theta_B)} = V_S'(V_B)$$

Proposition 6.3 follows immediately.  $\square$

### B.13 Proof of Proposition 6.4

At most one incentive constraint binds. Suppose not: subtracting the two incentive constraints from each other, we have

$$U(c_0^S, \theta_S) - U(c_0^S, \theta_B) = U(c_0^B, \theta_S) - U(c_0^B, \theta_B)$$

which, since  $U_{c\theta} > 0$ , implies  $c_0^S = c_0^B = c_0$  and  $V_S(V_B) = V_B$ . Consider the following deviation: increase  $c_0^B$  by  $\varepsilon > 0$ , decreasing  $c_0^S$  by the same amount, and decrease  $V_B$  by  $\delta > 0$ . Each country's utility changes by:

$$\begin{aligned}\Delta U_S &= -U_c(c_0, \theta_S)\varepsilon - V_S'(V_B)\delta \\ \Delta U_B &= U_c(c_0, \theta_B)\varepsilon - \delta\end{aligned}$$

Choose  $\varepsilon, \delta$  such that both expressions are positive. This deviation increases social welfare, and satisfies incentive compatibility and resource constraints, which contradicts the original allocation being optimal.

Putting Lagrange multipliers of  $\lambda_0, \mu_S, \mu_B$  on the constraints, the first order conditions for a maximum are

$$\begin{aligned}(\alpha + \mu_S)U_c(c_0^S, \theta_S) - \mu_B U_c(c_0^S, \theta_B) - \lambda_0 &= 0 \\ (1 - \alpha + \mu_B)U_c(c_0^B, \theta_B) - \mu_S U_c(c_0^B, \theta_S) - \lambda_0 &= 0 \\ (\alpha + \mu_S - \mu_B)V_S'(V_B) + (1 - \alpha + \mu_B - \mu_S) &= 0\end{aligned}$$

Since at most one incentive constraint binds, at least one of  $\mu_S, \mu_B$  equals zero. It follows that  $\lambda_0 > 0$ , and the resource constraint binds.

Suppose the incentive constraints do not bind. Define  $c_0^S(\alpha), V_B(\alpha)$  to solve

$$\begin{aligned}\alpha U_c(c_0^S(\alpha), \theta_S) &= (1 - \alpha)U_c(2y^* - c_0^S(\alpha)) \\ \alpha V_S'(V_B(\alpha)) + (1 - \alpha) &= 0\end{aligned}$$

$c_0^S(\alpha)$  is increasing and  $V_B(\alpha)$  is decreasing. It follows that there exist  $\alpha_S < \alpha_B$  such that ICS must bind if  $\alpha < \alpha_S$ , and ICB must bind if  $\alpha > \alpha_B$ .

Finally, by examining the last first order condition, we see that  $V_S'(V_B) = 0$  implies  $\mu_S > 0$ , so ICS must bind. □



*Appendix to Chapter 3*

## **C.1 Data Sample: Panel Study of Income Dynamics**

The Panel Study of Income Dynamics (PSID), which began in 1968, is a longitudinal study of a representative sample of U.S. individuals (men, women, and children) and the family units in which they reside, and is conducted by the University of Michigan. The PSID's sample size has grown from 4,800 families in 1968 to more than 7,000 families (and over 60,000 individuals) in 2001. Some families are followed for as much as 36 consecutive years.

Consumption data in PSID are limited to food and shelter. We compute all the consumption volatility measures on food consumption calculated as a sum of food consumed at home plus away from home plus food stamps received. The core sample contains data from 1968 to 2005, and consists of heads of households (both female and male) who are not students and are not retired. We keep households whose head is at least 25 years old but less than 65. We drop all the households that belonged to the Latino or Immigrant samples, and those that were drawn from the Survey of Economic Opportunity (SEO). Households that report negative or zero total food consumption levels are also eliminated. In order to minimise effects of outliers on the results, we follow the literature by dropping households who report more than 500 percent change in family income or food consumption over a one year period as well as those whose income or consumption fall by more than 95 percent (see for example [Zeldes \[1989b\]](#) or [Blundell et al. \[2008\]](#)).

The most important issue to note regarding the data is that it became biennial after 1997. We construct a hypothetical biennial sample to study the evolution of consumption volatility up to 2004. Since income and consumption data is collected for the previous year, the biennial sample has data for *even* years from 1976 to 2004. In addition, food consumption data was not collected in 1973, 1988 and 1989. We do not impute for the missing years in order to keep measurement

error and misidentification to a minimum.

At the time of the interview, the respondent is asked questions about income, transfers, wealth and expenditures on food and shelter. The families are asked to report income and transfers received during the previous year. We use total family income to compute income uncertainty. We adjust income data by one period to correspond to the appropriate demographic characteristics for each household. The timing of consumption data is more ambiguous. We follow [Blundell et al. \[2008\]](#), among many others, and assume that the respondent provided information on food expenditures for the previous year. We use interest rates on two-year constant maturity Treasury bills.

All the income, expenditure, wealth, and interest rate data are expressed in real terms. Nominal data are converted into real using item specific regional not seasonally adjusted all urban Consumers Consumer Price Index (CPI-U) with base period of 1982-1984=100. Thus, food expenditures are deflated using the Food and Beverages CPI; housing expenditures, using the Housing CPI; and all income, wealth and interest rate series, using All-Items CPI.

## **C.2 Data Sample: Survey of Consumer Finances**

The 1983, 1989, 1992, 1995, 1998, 2001, 2004 and 2007 Surveys of Consumer Finances (SCF), sponsored by the Board of Governors of the Federal Reserve System, are cross-sectional surveys of the balance sheet, pension, income, and other demographic characteristics of U.S. families. The SCF collects data from two samples: a standard multi-stage area-probability sample selected from the 48 contiguous US states, and a list sample designed to disproportionately sample wealthy families. For example, 3,007 of the 4,522 interviews for the 2004 SCF were from the area probability sample, and 1,515 were from the list sample, therefore the total sample is not representative of US households. The SCF provides probability weights which account for the sample design, and also for differential patterns of non-response among families with different characteristics; unless otherwise noted, all SCF data presented here is weighted.

Over 1989-2007, the SCF uses a multiple imputation method to account for missing data. For each piece of missing data, the SCF provides 5, possibly different, responses (referred to as “implicates”), resulting in a data set with 5 times the actual number of households. We average

across all five imputates to reduce the likelihood of biasing our results. Lindamood et al. [2007] report that using only one impute may bias results; ideally, all imputates should be used according to the “repeated-imputation inference” method.

### C.3 Variables Used in Regressions

In our probit regression model for the probability of being denied credit in SCF data, our explanatory variables are: a cubic in age, categorical variables for a female head of household, a single parent, marital status, receiving welfare payments, having positive asset income; cubics in log income and house value, quadratic functions of log annual mortgage payments, mortgage, and asset income; and interactions between race, education (no high school, high school, or college), and the cubic in log house value. We test for coefficient stability by interacting these variables with linear and quadratic time trends, and checking whether the coefficients on these interactions are significant. In our final specification, we include a quadratic time trend, and interact several variables, which we found to have time-varying coefficients, with a linear trend.

Next, we describe the explanatory variables used in our income regressions, which we use to estimate income volatility. In individual labour income models, these variables usually include age, age squared, dummy variables for education, occupation and industry, sex, race, cohort dummies, time dummies (to control for aggregate shocks), and various interaction. Since we model the family income process, we redefine these variables as those pertaining to the head of household, and include additional variables, such as head’s marital status, number of hours worked by head and his partner, and the number of children and adults in the household.

Finally, we describe our probit regression model for the probability of having positive cash on hand. For each year wealth data is available, we estimate the probability of having positive net non-housing wealth as a function of observable characteristics such as age, age squared, cohort, race, gender, education, real house value, real rental and mortgage cost, home ownership, marital status, number of children and adults, real family income, real asset income, information on welfare, public transfers, and state of residence.

## C.4 Estimating the Demand for Food in PSID.

To estimate our  $\beta$  coefficient for the food demand equation 3.6, we follow methodology outlined in the [Blundell et al. \[2008\]](#). In particular, we estimate:

$$\ln F_{h,t} = \alpha_0 + \alpha_1 \ln p_t^F + \alpha_2 \ln p_t^O + \beta \ln C_{h,t} + \theta_F' Z_{h,t} + \iota_{h,t}$$

where we measure  $\ln C_{h,t}$  using additional information available in the PSID starting 2005. In particular, in addition to food, starting 1999, the PSID added information on the following non-durable and service categories: expenditure on childcare (for working and non working spouses), utilities, gasoline, transportation, home and auto insurance, and vehicle repair. Moreover, starting 2005, additional expenditure on non-durable (and semi-durables) categories include clothing, home repair, furniture, trips, and other recreation activities. To construct our measure of total nondurable consumption, we use the sum of the above listed items. We then estimate food demand equations using 2005-2009 data following the methodology outlined in [Blundell et al. \[2008\]](#).<sup>1</sup>

Table C.6 provides the results of our estimation. This table reports IV estimates of the demand equation for (the logarithm of) food spending in PSID. We instrument the log of total nondurable expenditure, defined above, (and its interactions with time, education, and kids dummies) with the cohort-education-year specific average of the log of the husband's hourly wage and the cohort-education-year specific average of the log of the wife's hourly wage (and their interactions with time, education, and kids dummies). Other controls in this regression include a polynomial in age, education and cohort dummies, dummies for the number of kids, race, gender, marital status, region, and (the logarithm of) prices for food, tobacco and alcohol, prices for transportation, and prices for utilities. The estimates of the coefficients on these controls are not shown in the table to preserve space, but are available upon request.

The first 5 columns of the table present results of the estimation for a narrower definition of nondurables, that based on the available categories as of 1999 interview. In column 5, we restrict the time span to 2005-2009, to be comparable to the regression results in columns (6)-(8) that are estimated using 2005 definitions. In the first column of the table, we allow the elasticity

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<sup>1</sup>For these regressions we use data kindly provided to us by Geng Li of Federal Reserve Board.

to vary over time, by education, and with the number of kids. This specification does not pass the validity of instruments tests. Moreover, we find that the elasticity does not vary with the number of kids. The next columns restrict the elasticity to be invariant in time, by education, and/or the number of kids. Finally in column (4), we find that the best specification using 1999 definition, gives us an estimate of  $\hat{\beta} = 0.73$ . Looking at the wider set of categories available from 2005 onward, results shown in column (6), we again allow the elasticity to vary with time, education and the number of kids. We again reject that the elasticity varies across all of these dimensions. Finally, in our preferred specification, results in column (8), we allow the elasticity to vary with the number of kids, but not over time or with education. This regression passes all the over- and under-identification tests. We estimate  $\hat{\beta} = 0.78$ . Not surprisingly, if we compare beta estimate from column (5) and that of column (8), we see that the beta on a narrower definition of non-durables (1999) is larger ( $\hat{\beta} = 0.85$ ) than beta using 2005 definition ( $\hat{\beta} = 0.78$ ).

## C.5 Supplementary Tables and Figures

Table C.1: Regression of the liquidity constraint dummy on demographic variables and current income, interacted with a time trend.

	Coefficient		Standard errors
age	0.024	**	(0.012)
age <sup>2</sup>	-0.000	**	(0.000)
female	0.099		(0.091)
white, no HS	0.403	***	(0.100)
white, college	-0.389	***	(0.081)
black, no HS	0.037		(0.212)
black, HS	0.138		(0.175)
black, college	-0.417		(0.297)
lowest income quartile	-0.220	*	(0.121)
second income quartile	-0.002		(0.104)
highest income quartile	-0.118		(0.088)
single parent	0.070		(0.161)
on welfare	0.063		(0.163)
time trend	-0.207		(0.320)
Observations	30,152		
R-squared	0.161		

Coefficients on interactions with the trend, coefficients are multiplied by 100  
 Robust standard errors in parentheses: \*\*\* p<0.01, \*\* p<0.05, \* p<0.1

Table C.2: Summary statistics: PSID vs. SCF samples

Year	Age		percent Black		percent Hispanic		Family size	
	PSID	SCF	PSID	SCF	PSID	SCF	PSID	SCF
1983	39.17	41.60	0.08	0.13	0.02	0.04	3.02	3.05
1989	39.71	41.40	0.08	0.12	0.02	0.10	2.97	3.05
1992	40.34	41.53	0.08	0.13	0.02	0.09	2.98	2.98
1995	40.83	41.66	0.08	0.14	0.02	0.06	2.93	2.89
1998	41.41	42.18	0.08	0.12	0.03	0.09	2.87	2.91
2000	41.53		0.08		0.03		2.86	
2001		42.85		0.14		0.09		2.91
2002	41.92		0.08		0.03		2.80	
2004	41.96	43.23	0.09	0.14	0.03	0.11	2.80	2.88

Year	Education		percent on welfare		percent unemployed		Family income	
	PSID	SCF	PSID	SCF	PSID	SCF	PSID	SCF
1983	13.49	12.86	0.07	0.08	0.04	0.07	34,218	31,551
1989	13.64	13.20	0.04	0.08	0.03	0.04	39,442	37,869
1992	13.69	13.52	0.05	0.07	0.05	0.06	39,714	33,623
1995	13.73	13.47	0.05	0.08	0.02	0.03	40,416	32,550
1998	13.73	13.49	0.03	0.05	0.03	0.04	43,035	37,180
2000	13.69		0.04		0.04		45,805	
2001		13.58		0.04		0.03		43,809
2002	13.69		0.04		0.05		42,391	
2004	13.75	13.63	0.06	0.06	0.03	0.04	46,071	41,929

Source: SCF and PSID

Table C.3: Summary statistics: SCF

Year	Age		percent female		percent black		Family size		Single parent	
	C	UC	C	UC	C	UC	C	UC	C	UC
1983	36.34	42.91	0.33	0.19	0.26	0.11	3.04	3.02	0.22	0.09
1989	37.9	42.72	0.28	0.2	0.21	0.1	3.15	2.93	0.14	0.07
1992	37.75	42.89	0.29	0.19	0.26	0.11	2.89	2.89	0.21	0.06
1995	38.42	43.04	0.3	0.22	0.27	0.11	2.91	2.8	0.25	0.11
1998	39.08	43.59	0.3	0.19	0.24	0.1	3.02	2.78	0.25	0.1
2001	39.28	44.27	0.33	0.19	0.28	0.11	2.95	2.81	0.27	0.11
2004	39.83	45.08	0.34	0.2	0.3	0.11	2.81	2.78	0.25	0.1
2007	40.47	45.73	0.34	0.2	0.28	0.12	2.97	2.79	0.3	0.1

Year	Education		percent homeowners		percent have credit card		percent on welfare		percent unemployed	
	C	UC	C	UC	C	UC	C	UC	C	UC
1983	12.69	13	0.4	0.73	0.5	0.79	0.19	0.05	0.12	0.05
1989	13.08	13.53	0.51	0.74	0.58	0.82	0.14	0.05	0.08	0.02
1992	13.48	13.94	0.48	0.75	0.59	0.86	0.13	0.04	0.07	0.05
1995	13.3	13.72	0.51	0.75	0.6	0.86	0.16	0.05	0.06	0.03
1998	13.23	13.92	0.48	0.78	0.63	0.85	0.1	0.03	0.06	0.03
2001	13.18	14	0.49	0.8	0.68	0.88	0.08	0.02	0.04	0.03
2004	13.18	14.27	0.51	0.83	0.6	0.88	0.12	0.03	0.06	0.03
2007	13.02	14.19	0.54	0.8	0.53	0.85	0.16	0.04	0.07	0.03

Year	Family income		Assets		Net worth		Debt		percent with some debt	
	C	UC	C	UC	C	UC	C	UC	C	UC
1983	20,919	34,508	74,902	213,678	42,503	143,480	19,650	25,502	0.8	0.84
1989	26,719	43,144	95,525	215,014	71,876	183,503	23,649	31,511	0.89	0.86
1992	22,670	39,024	79,271	202,323	58,387	167,366	20,884	34,957	0.87	0.86
1995	20,873	37,485	69,617	212,156	47,039	175,727	22,577	36,429	0.88	0.88
1998	24,631	43,681	91,898	262,861	64,021	219,662	27,877	43,199	0.89	0.88
2001	23,827	52,648	74,747	331,351	47,245	284,408	27,502	46,944	0.88	0.88
2004	22,881	51,944	83,913	387,465	49,587	322,327	34,326	65,138	0.85	0.89
2007	23,753	56,236	99,431	422,507	61,275	353,158	38,157	69,349	0.86	0.89

Source: Survey of Consumer Finances

Table C.4: Evolution of Food Consumption and Income Volatility, allowing for state fixed effects and cohort dummies, biennial sample, 1980 to 2004.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)
	food	income	food	food	income	food	income
Year/1000	1.525*** (0.504)	7.295*** (1.486)	1.520*** (0.505)	1.448*** (0.505)	6.867*** (1.482)	2.876* (1.696)	13.948*** (5.157)
Year > 1992	-0.006 (0.007)	0.064*** (0.022)	-0.006 (0.007)	-0.006 (0.007)	0.065*** (0.022)	-0.010 (0.007)	0.062*** (0.022)
Black/Hispanic	0.066*** (0.010)	0.092*** (0.027)	0.066*** (0.010)	0.064*** (0.010)	0.086*** (0.027)	0.065*** (0.010)	0.085*** (0.027)
Education < 13	0.014*** (0.005)	0.044*** (0.016)	0.014*** (0.005)	0.014*** (0.005)	0.045*** (0.016)	0.017*** (0.005)	0.050*** (0.016)
Age	-0.003 (0.002)	-0.027*** (0.006)	-0.003 (0.002)	-0.003 (0.002)	-0.026*** (0.006)	-0.009*** (0.003)	-0.040*** (0.009)
Age <sup>2</sup>	0.000 (0.000)	0.000*** (0.000)	0.000 (0.000)	0.000 (0.000)	0.000*** (0.000)	0.000*** (0.000)	0.000*** (0.000)
Change in marital status	0.104*** (0.012)	0.240*** (0.030)	0.104*** (0.012)	0.104*** (0.012)	0.234*** (0.030)	0.104*** (0.012)	0.234*** (0.030)
Change in number of adults	-0.005 (0.005)	-0.012 (0.011)					
Change in number of kids	-0.001 (0.004)	-0.007 (0.011)					
Constant	-2.830*** (1.003)	-13.646*** (2.943)	-2.820*** (1.003)	-2.679*** (1.003)	-12.821*** (2.933)	-5.598* (3.353)	-26.797*** (10.191)
Observations	33,652	33,652	33,652	33,594	33,594	33,594	33,594
State FE	No	No	No	Yes	Yes	Yes	Yes
Cohort dummies	No	No	No	No	No	Yes	Yes
Adjusted R <sup>2</sup>	0.0104	0.0108	0.0104	0.0126	0.0142	0.0152	0.0148

Robust, clustered at household level, standard errors in parentheses;

\*\*\* p<0.01, \*\* p<0.05, \* p<0.1

Source: Authors' calculations based on PSID and SCF data as described in the text.

Table C.5: Effect of Liquidity Constraints and Income Uncertainty on Volatility of Consumption, by education, biennial sample, 1980 to 2004.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
	all	ed < 13	ed ≥ 13	all	ed < 13	ed ≥ 13	all	ed < 13	ed ≥ 13
Year/1000	1.128** (0.496)	1.174 (0.798)	1.039* (0.628)	0.514 (0.512)	-0.101 (0.830)	0.934 (0.650)	0.268 (0.501)	-0.135 (0.820)	0.522 (0.630)
Year > 1992	-0.010 (0.007)	-0.012 (0.011)	-0.008 (0.009)	-0.007 (0.007)	-0.008 (0.011)	-0.006 (0.009)	-0.010 (0.007)	-0.011 (0.011)	-0.008 (0.009)
$(\sigma_{h,t}^Y)^2$	0.052*** (0.004)	0.046*** (0.005)	0.056*** (0.007)				0.062*** (0.008)	0.051*** (0.009)	0.068*** (0.011)
Pr(denied credit)				0.222*** (0.024)	0.265*** (0.036)	0.181*** (0.032)	0.215*** (0.025)	0.242*** (0.036)	0.187*** (0.034)
Pr(denied credit) × $(\sigma_{h,t}^Y)^2$							-0.051** (0.024)	-0.023 (0.029)	-0.075** (0.035)
Black/Hispanic	0.061*** (0.010)	0.066*** (0.014)	0.055*** (0.012)	0.024** (0.011)	0.025 (0.016)	0.019 (0.015)	0.025** (0.011)	0.025 (0.015)	0.023 (0.014)
Education < 13	0.012** (0.005)			0.000 (0.005)			-0.000 (0.005)		
Constant	-2.116** (0.985)	-2.178 (1.585)	-1.952 (1.246)	-0.977 (1.016)	0.238 (1.644)	-1.804 (1.290)	-0.493 (0.994)	0.303 (1.625)	-0.991 (1.249)
Observations	33,652	14,683	18,969	33,652	14,683	18,969	33,652	14,683	18,969
Adjusted R <sup>2</sup>	0.036	0.029	0.043	0.015	0.016	0.013	0.040	0.034	0.046

The effect of  $(\sigma_{h,t}^Y)^2$  is the same across groups (p-value)

0.229

The effect of Pr(denied credit) is the same across groups (p-value)

0.269

The effect of Pr(denied credit) ×  $(\sigma_{h,t}^Y)^2$  is the same across groups (p-value)

0.249

Robust, clustered at household level, standard errors in parentheses;

\*\*\* p<0.01, \*\* p<0.05, \* p<0.1

Source: Authors' calculations based on PSID and SCF data as described in the text.

Table C.6: Estimating the Demand for Food in the PSID: The budget elasticity of Food with respect to total Nondurable Consumption.

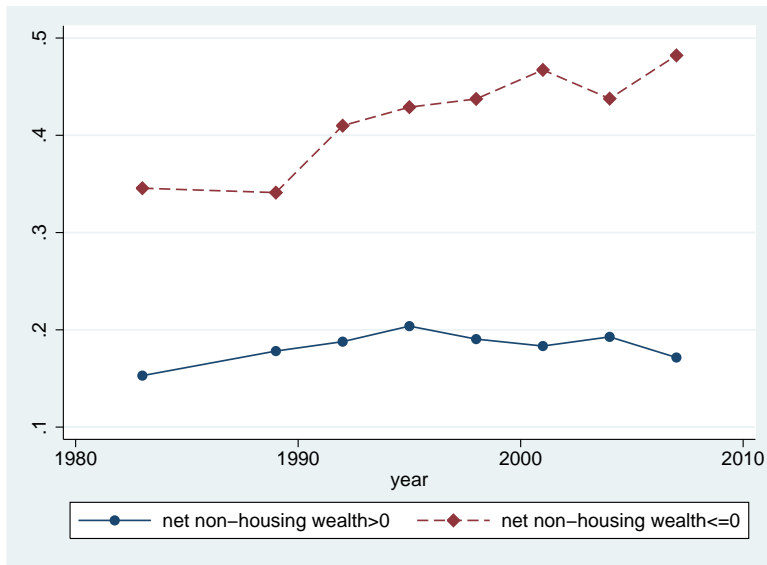
years of data used:	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
	1999-2009	1999-2009	1999-2009	1999-2009	2005-2009	2005-2009	2005-2009	2005-2009
$\ln(ND_{1999})$	0.741*** (0.260)	0.681** (0.280)	0.810*** (0.254)	0.734*** (0.236)	0.850*** (0.201)			
$\ln(ND_{2005})$						0.516*** (0.155)	0.863*** (0.268)	0.783*** (0.255)
Observations	20,853	20,853	20,853	20,853	10,718	10,718	10,718	10,718
R-squared	0.606	0.595	0.614	0.613	0.609	0.477	0.415	0.436
Underid test	8.351	7.654	9.697	9.075	16.72	15.69	9.481	9.843
p-value	0.820	0.663	0.406	0.028	0.001	0.109	0.009	0.020
Hansen J overid test	10.80	6.262	7.221	2.207	2.227	9.830	0.00484	1.688
p-value	0.546	0.713	0.287	0.332	0.893	0.364	0.945	0.430
weak id Cragg-Donald Wald F statistic	0.471	0.612	1.115	4.461	4.689	0.754	5.384	2.768
elasticity is time invariant (p-value)	0.0796	0.0639	0.0917	-	-	0.592	-	-
elasticity is edu invariant (p-value)	0.0423	0.0456	-	-	-	0.117	-	-
elasticity is kid invariant (p-value)	0.215	-	-	-	-	0.408	-	-
time varying elasticity	Yes	Yes	Yes	No	No	Yes	No	No
elasticity varies with education	Yes	Yes	No	Yes	Yes	Yes	No	No
elasticity varies with number of children	Yes	No	No	No	No	Yes	No	Yes

Note 1: This table reports IV estimates of the demand equation for (the logarithm of) food spending in PSID. We instrument the log of total nondurable expenditure, defined below, (and its interactions with time, education, and kids dummies) with the cohort-education-year specific average of the log of the husband's hourly wage and the cohort-education-year specific average of the log of the wife's hourly wage (and their interactions with time, education, and kids dummies). Robust, clustered at household level, standard errors in parentheses; \*\*\*  $p < 0.01$ , \*\*  $p < 0.05$ , \*  $p < 0.1$ .

Note 2:  $\ln(ND_{1999})$  is defined as NID expenditure, given available expenditure data as of 1999 forward, on the following categories: expenditure on food at home and away from home, food stamps, childcare, utilities, gasoline, transportation, home and auto insurance, and vehicle repair.

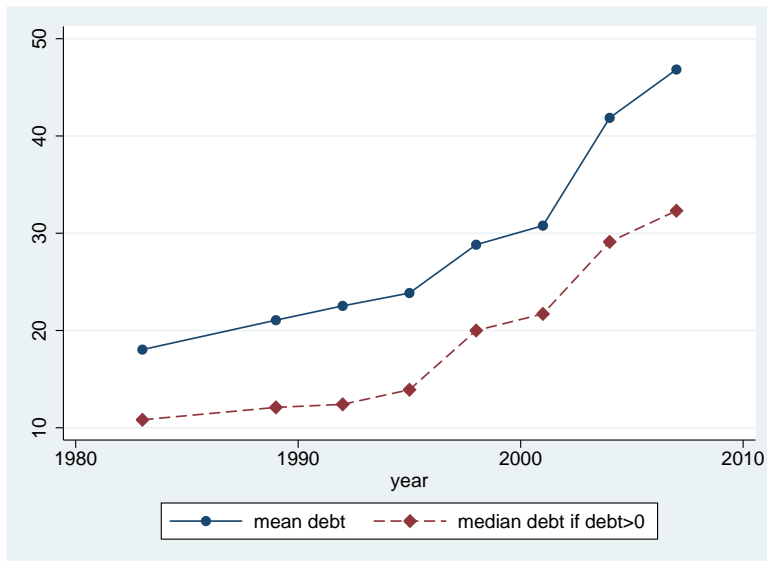
$\ln(ND_{2005})$  includes ND expenditure given available expenditure data as of 2005 forward. It includes categories in 1999 definition, plus expenditure on clothing, home repair, furniture, trips, and other recreation activities.

Figure C.1: Proportion of liquidity constrained households, by wealth



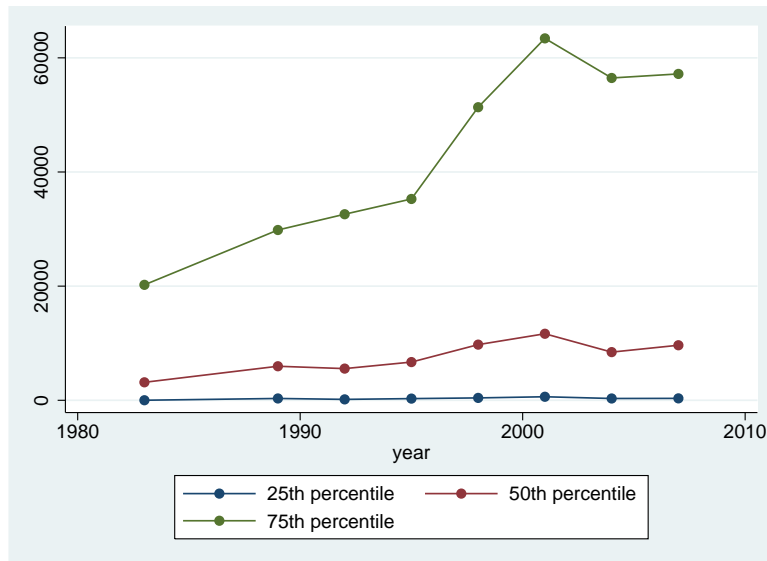
Source: Survey of Consumer Finances.

Figure C.2: Increase in average and median debt, in thousands, 1983 dollars



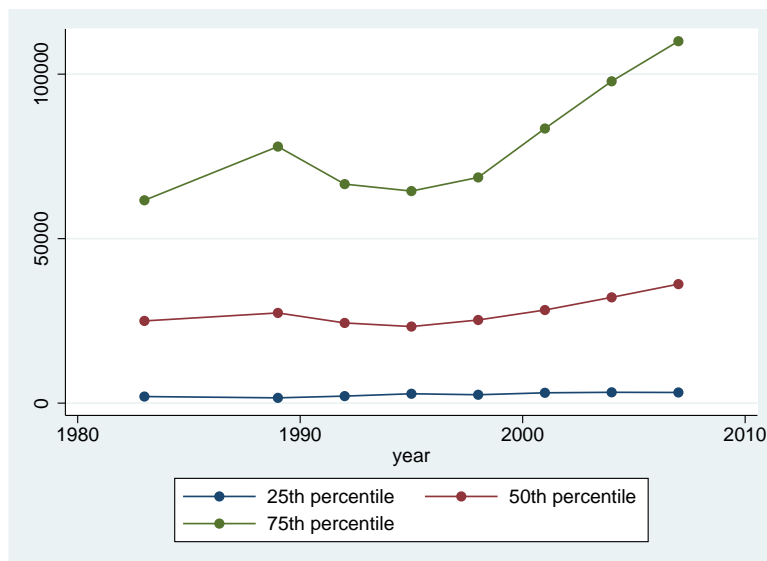
Source: Survey of Consumer Finances.

Figure C.3: Evolution of Financial Net Worth, in thousands, 1983 dollars



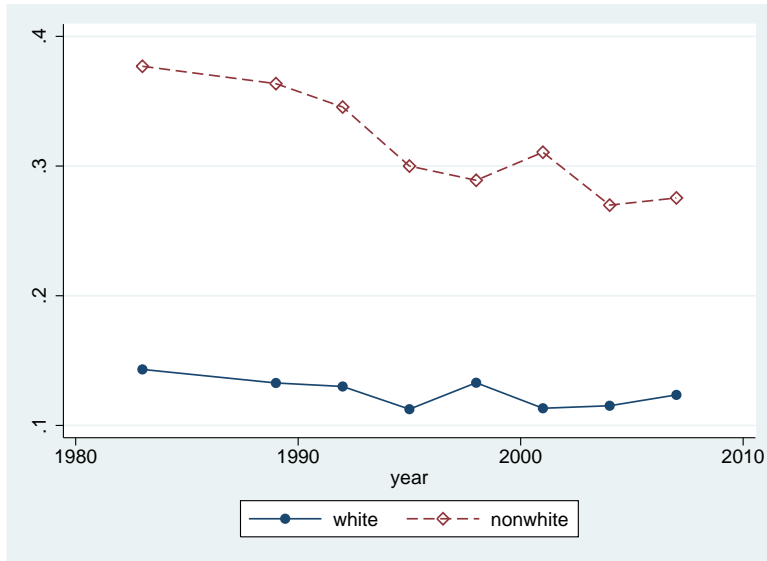
Source: Survey of Consumer Finances.

Figure C.4: Evolution of Nonfinancial Net Worth, in thousands, 1983 dollars



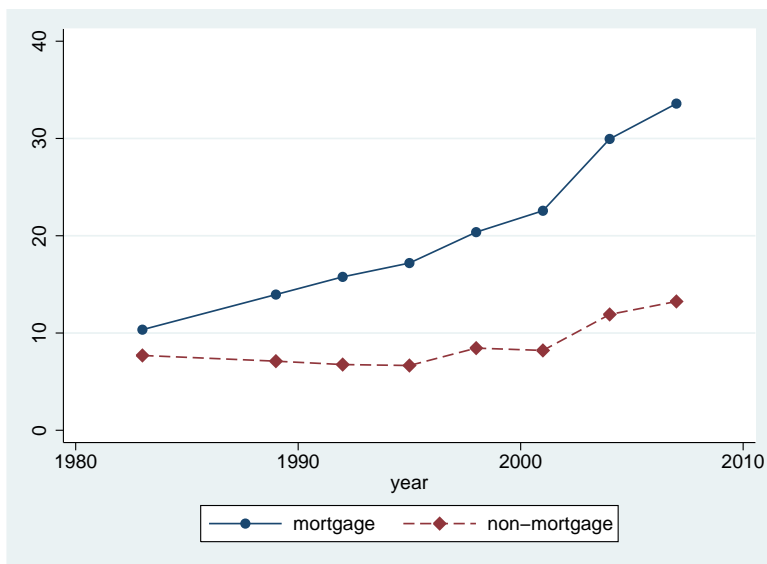
Source: Survey of Consumer Finances.

Figure C.5: Percent of Households with net assets less than two months of income.



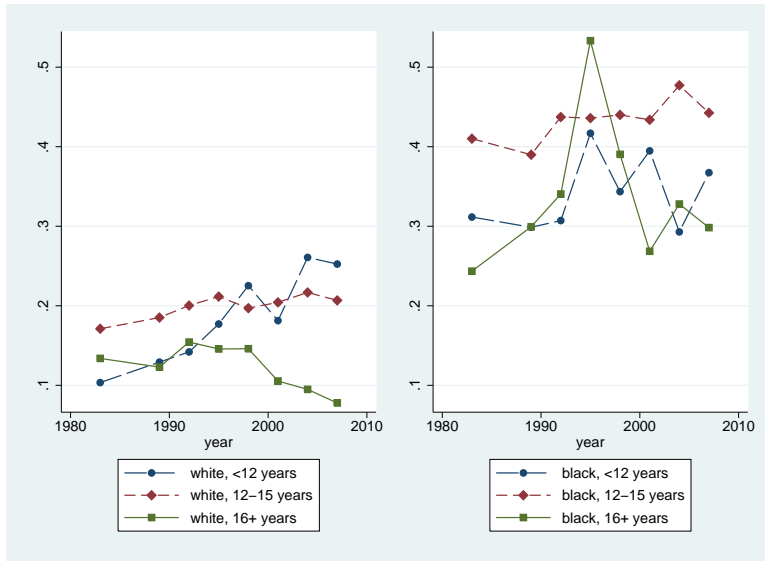
Source: Survey of Consumer Finances.

Figure C.6: Average mortgage vs. non-mortgage debt, in 1983 dollars



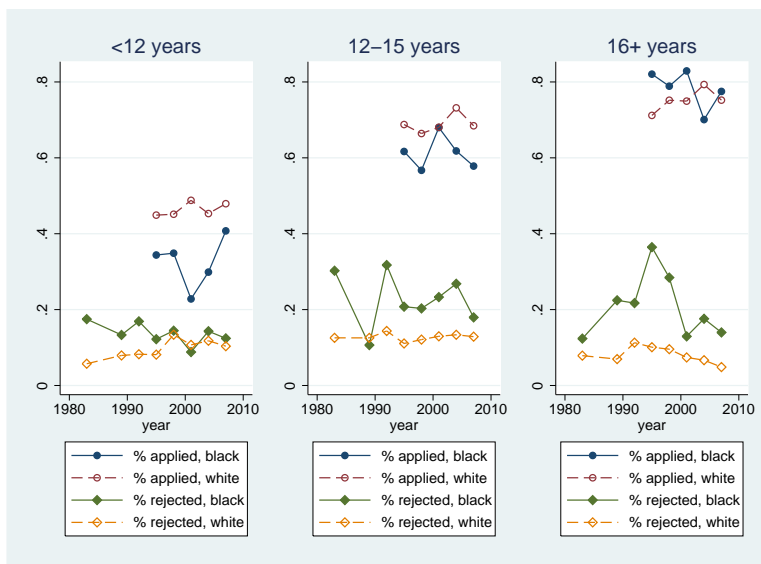
Source: Survey of Consumer Finances.

Figure C.7: Proportion of liquidity constrained households, by race and education



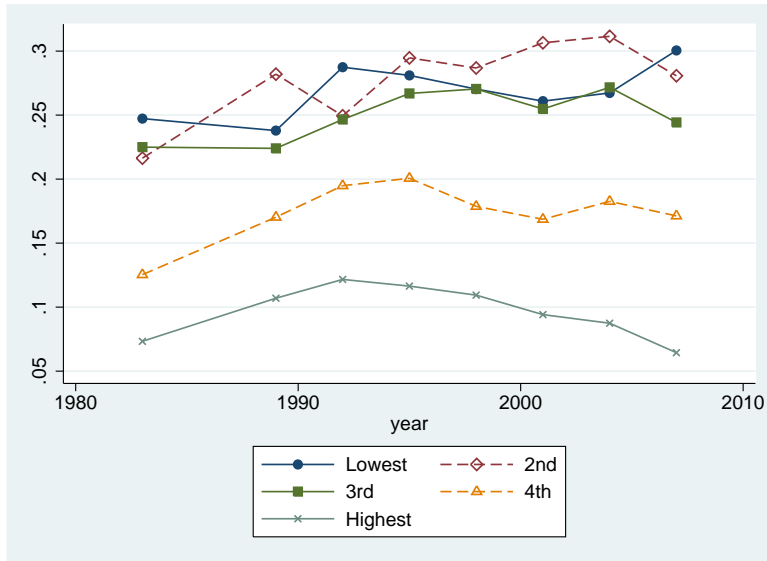
Source: Survey of Consumer Finances.

Figure C.8: Proportion of households applying for credit and rejected, by race and education



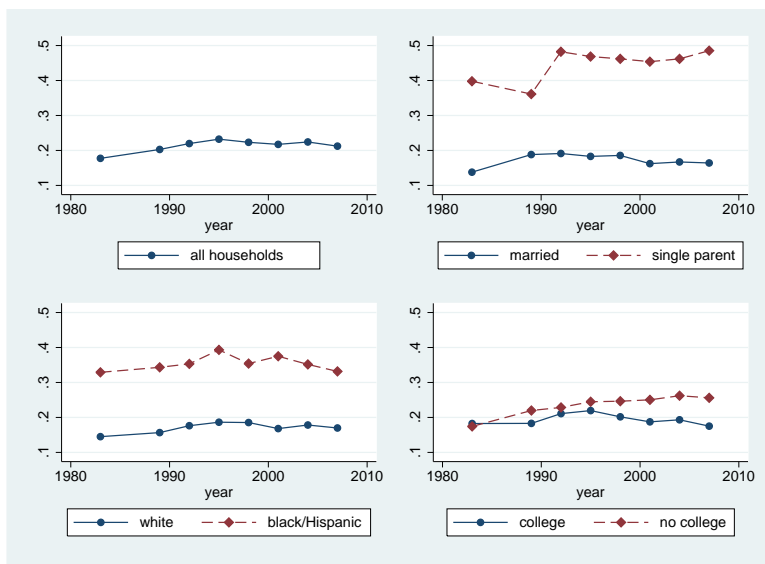
Source: Survey of Consumer Finances.

Figure C.9: Proportion of constrained household, by income quintile



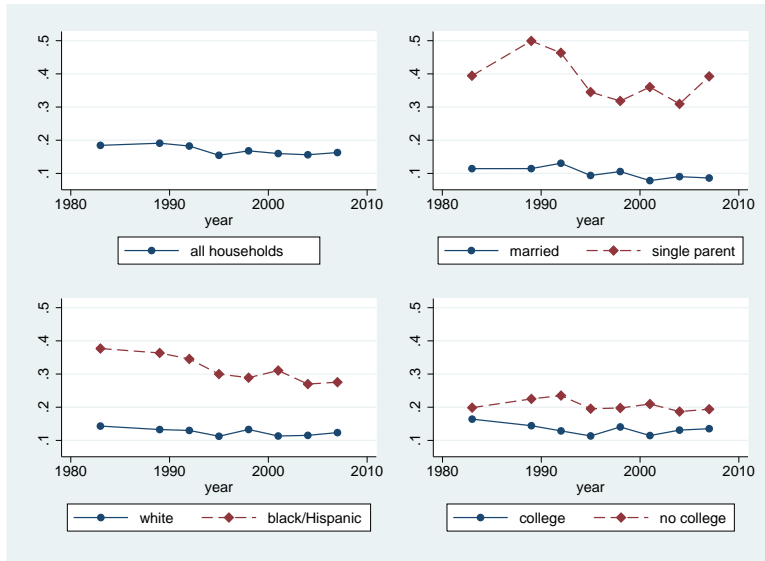
Source: Survey of Consumer Finances.

Figure C.10: Proportion of constrained households, by demographic group



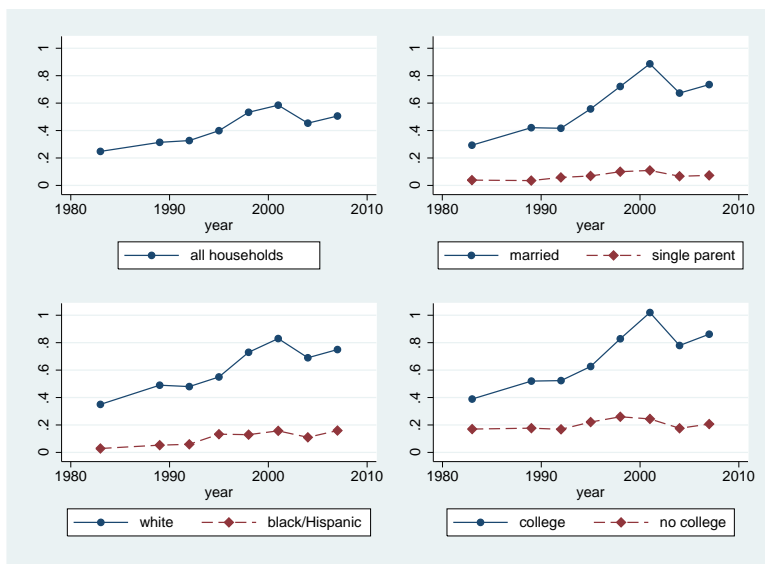
Source: Survey of Consumer Finances.

Figure C.11: Percentage with net assets less than two months' income, by demographic group



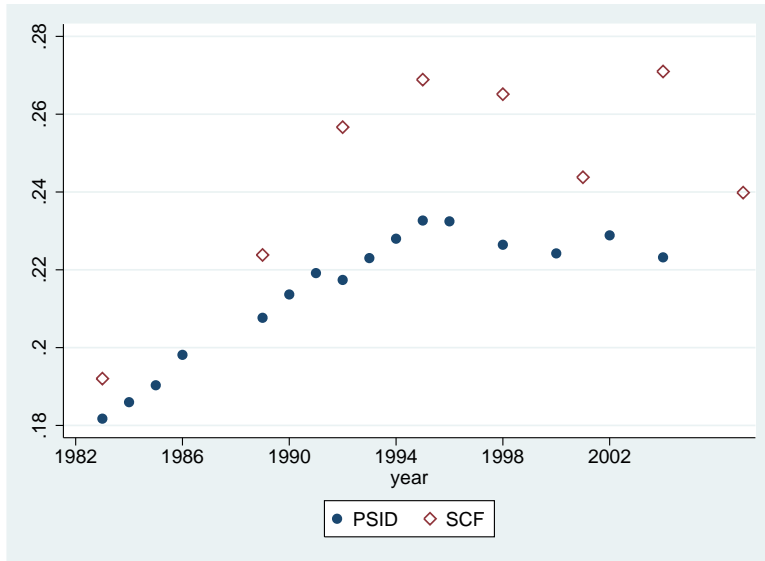
Source: Survey of Consumer Finances.

Figure C.12: Median net assets to income ratio, by demographic group



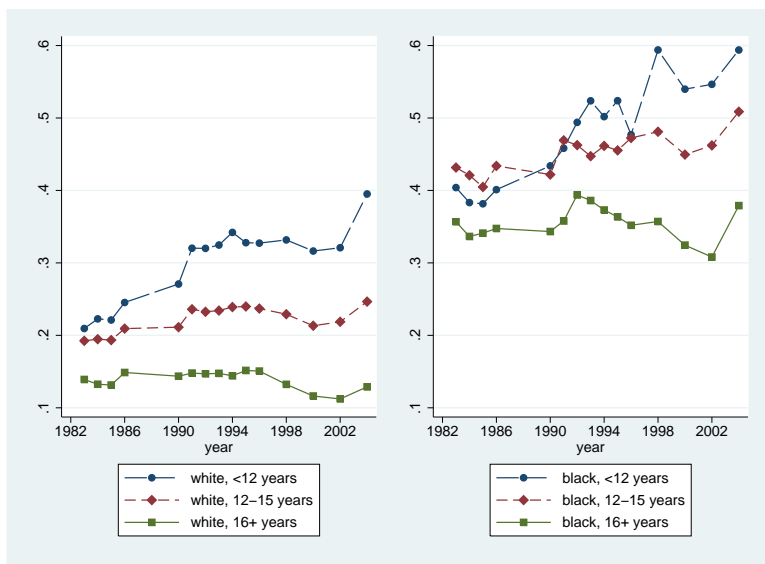
Source: Survey of Consumer Finances.

Figure C.13: Mean Estimated Probabilities in the SCF and the PSID



Source: Author estimates from Survey of the Consumer Finances and the Panel Study of Income Dynamics.

Figure C.14: Proportion of constrained household, by demographic group in PSID



Source: Panel Study of Income Dynamics.