Growth Rate of 3-Manifold Homologies under Branched Covers

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ABSTRACT

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Over the last twenty years, a main focus of low-dimensional topology has been on categorified knot invariants such as knot homologies. This dissertation studies the case of two such homologies under the iteration of branched covering maps. In the first part, we find a spectral sequence on the sutured annular Khovanov homology of periodic links of period $r = 2^i$. In the second part, we study the asymptotic growth rate of Heegaard Floer homology of cyclic branched covers of a knot as the branching number increases.

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To My Loved Ones

Chapter 1

Introduction

Modern knot theory first arose as a mathematical endeavor based on a mistake. In the late nineteenth century Lord Kelvin theorized that atoms were formed by knotted vortices in the aether; while this speculation was entirely inaccurate it motivated mathematicians such as Peter Tait to attempt to study and tabulate knots [16]. In the years since then, knot theory has grown into an important sub-discipline of topology.

Since the dawn of the twenty-first century there has been a great deal of interest in 3-manifold and knot homologies — graded abelian groups associated to 3-manifolds or knots whose Euler characteristics are classical invariants, like ordinary first homology or the Alexander or Jones polynomial. One kind of these comes from studying certain classes of partial differential equations, called *Floer homology theories*; this includes Heegaard Floer homology, introduced by Ozsváth and Szabó in [48]. Another comes from the representation theory of quantum groups; one example is *Khovanov homology*, introduced by Khovanov in [32]. There has also been interest in finding combinatorial definitions of Floer invariants, and Floer-type definitions of Khovanov invariants. The latter includes *symplectic Khovanov homology*, defined by Seidel and Smith in [57].

1.1 Khovanov Homologies

1.1.1 Periodic Links

A link L is said to have *period* $r > 1$ if there is an auto-diffeomorphism ϕ of S^3 of order r such that the fixed set is an unknot disjoint from L and the diffeomorphism maps L onto itself. Notice that deleting a neighborhood of the fixed set places L into a thickened annulus. Let $\widetilde{L}$ be the quotient of L under ϕ. Note that the r-fold cyclic branched cover of the fixed set unknot has $\widetilde{L}$ as the image of L under the branched cover projection.

Murasugi found relations for the Alexander polynomial of a periodic knot [43]; he also discovered relations for the Jones polynomial of a periodic knot [44]. Using spectral sequences and Heegaard Floer theory Hendricks [28] categorified some of Murasugi's Alexander polynomial formulas.

Another homology theory of periodic links was provided by Chbili [13] and Politarczyk [51], who studied the Khovanov homology of such links, constructing what they call *equivariant Khovanov homology*. Another result for periodic links was proved by Seidel-Smith [58], in this case concerning symplectic Khovanov homology. Using a spectral sequence, much like what Hendricks would later do, they obtained a rank inequality [58, Corollary 3] between the symplectic Khovanov homology of a 2-periodic link K and its quotient $\widetilde{K}$.

1.1.2 Sutured Khovanov Homology

In 2004 Asaeda-Przytycki-Sikora [1] introduced an invariant of links in I-bundles over surfaces F which takes the form of a trigraded homology group that categorifies the Kauffman bracket skein module. In the special case that the surface F is an annulus this invariant is called *sutured annular Khovanov homology* and is denoted *SKh*. This case was further studied by Roberts [55], Grigsby-Wehrli [22, 23], and Auroux-Grigsby-Wehrli [2, 3].

1.1.3 Construction of Spectral Sequences

Using Hochschild homology, Lipshitz-Treumann [42] gave an approach to constructing spectral sequences. They show that if A is a dg algebra and M is a dg A-bimodule such that A fulfills certain technical conditions (namely that A is homologically smooth, homologically proper, and π-formal — see Chapter 2) then there is a spectral sequence from the Hochschild homology of the derived tensor product of M with itself that converges to the Hochschild homology of M. One may find the definitions of smoothness and properness in [42, Section 3.1.1] as well as Section 2.2; they are finiteness conditions to make the spectral sequence converge. See [42, Definition 3.15] and Section 2.3 for the definition of π-formality; it is a condition that requires certain operations d^{2i} vanish on Hochschild homology.

1.2 Heegaard Floer Homologies

1.2.1 Growth Rate of Cyclic Branched Covers

Let $\Sigma^n(K)$ be the nth cyclic branched cover of a knot K. It is natural to ask what happens to 3-manifold invariants as one takes the limit as n goes to infinity. For instance Silver and Williams [59], extending results of Riley [54] and González-Acuña and Short [20], studied the growth rate of $H_1(\Sigma^n(K))$. Let $M(f)$ be the Mahler measure of a polynomial f, where $M(f) := \prod_{\text{roots } \alpha \text{ of } f} \max\{1, |\alpha|\}$. Let Δ_K be the Alexander polynomial of a knot K, and $\operatorname{Tor}(G)$ be the torsion subgroup of a group G. It is shown in [59, Theorem 2.1] that

$$\lim_{n\to\infty} |\operatorname{Tor}(H_1(\Sigma^n(K)))|^{1/n} = M(\Delta_K). \tag{1.1}$$

1.2.2 Heegaard Floer Homology

An invariant related to H_1 and the Alexander polynomial is Heegaard Floer homology. Heegaard Floer homology comes in several varieties, but in this dissertation I will focus on the technically simplest variant, denoted $\widehat{HF}$.

This is an invariant of closed 3-manifolds which was defined by Ozsváth and Szabó in [48]. It is not hard to show that if Y is a rational homology sphere then the Euler characteristic of $\widehat{HF}(Y)$ is equal to $|H_1(Y)|$ as shown in [47, Proposition 5.1]; if $\operatorname{rk}\widehat{HF}(Y) = H_1(Y)$ than Y is said to be an *L-space*. This terminology comes from the fact that lens spaces are L-spaces. Heegaard Floer homology was extended to be an invariant of knots in [46, 53], denoted $\widehat{HFK}$, where it is proved that the Euler characteristic of $\widehat{HFK}$ of a knot K is the Alexander polynomial of K.

One important conjecture involving L-spaces was formalized by Boyer-Gordon-Watson [10, Conjecture 3]. A group G is *left-orderable* if there is a total order $<$ on G such that if $g_1 < g_2$ than $g * g_1 < g * g_2$ for all g in G.

Conjecture 1.2.1. *An irreducible rational homology 3-sphere is an L-space if and only if its fundamental group is not left-orderable.*

This conjecture is one of the most actively studied conjectures in low-dimensional topology; see [9, 8, 11, 25, 31, 45, 29, 24].

1.3 Structure of this Dissertation

This dissertation is organized as follows. Chapter 2 is about sutured annular Khovanov homology; it first begins with background in Section 2.1 before proving some preliminaries in Sections 2.2 and 2.3. The main theorem of this chapter is:

Theorem 2.4.1. *Let σ be a braid on $n+1$ strands, σ^2 denote its square as a braid, and $\widehat{\sigma}, \widehat{\sigma^2} \subset A \times I$ denote the annular closures of σ and σ^2 respectively. Then there is a spectral sequence starting at $SKh(\widehat{\sigma^2}; n-1)$ that converges to $SKh(\widehat{\sigma}; n-1)$.*

This theorem is proved in Section 2.4 while the decategorification version of this theorem along with some computations are in Section 2.5.

Chapter 3 is about the asymptotic behavior of Heegaard Floer homology under repeated branched covers; some preliminary results and theorems are provided in Section 3.1. This includes an upper bound on the growth rate of $\widehat{HFK}$ in Theorem 3.1.2 as well as a characterization of the asymptotic behavior of HF^+ of the mapping torus of a pseudo-Anosov diffeomorphism on a surface of genus g in the $2-g$ grading in Theorem 3.1.4. Thoughts about future directions for this research are provided in Section 3.2.

Chapter 2

Sutured Annular Khovanov Homology and Two Periodic Braids

This chapter first appeared on the arXiv in [14]. The goal of this chapter is to look at the relationship between the sutured annular Khovanov homology of a braid σ and the sutured annular Khovanov homology of σ^2. We find that there is a spectral sequence in a specific grading from the second invariant to the first invariant. This result also implies a relationship on the level of the Jones polynomial.

2.1 Background

We start by defining the algebra we shall be working over; we shall denote this algebra A_n and it is defined to be the quotient of a path algebra. The graph G_n of this path algebra is shown in Figure 2.1. The ring that the path algebra is defined over is $\mathbb{K} = \oplus_{v \in \mathrm{Vert}(G)} \mathbb{F}_2$. Thus in the path algebra the vertices are idempotents, with elements of the algebra being strings of edges so that the head of one edge is the tail of the next.

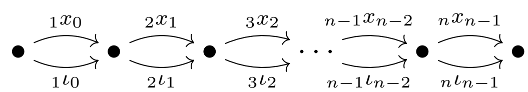

Figure 2.1: The graph G_n

Using composition order, i.e. multiplication should be thought of as concatenating paths starting from the rightmost element, we quotient out by the relations ${}_{i+2}\iota_{i+1} * {}_{i+1}x_i = {}_{i+2}x_{i+1} * {}_{i+1}\iota_i$ and ${}_{i+2}x_{i+1} * {}_{i+1}x_i = 0$ to form A_n.

A note about notation: Auroux-Grigsby-Wehrli use several different algebras in their papers. One of their algebras they denote B^{Kh} is what we are denoting A_n, with an isomorphism that sends our ${}_{i+1}x_i$ and ${}_{i+1}\iota_i$ to ${}_{i+1}x_i$ and ${}_{i+1}\mathbb{1}_i$ respectively. They denote another algebra A_n, which we are denoting $\bar{A}_n$. This algebra only appears in this dissertation as an intermediate step in the proof of Lemma 2.1.1.

Next we shall need another algebra B_n; we similarly define B_n to be the quotient of a path algebra, the graph for which is shown in Figure 2.2. The relations imposed are ${}_{i-1}w_i * {}_iy_{i+1} = {}_{i-1}y_i * {}_iw_{i+1}$ and ${}_{i-1}y_i * {}_iy_{i+1} = 0$.

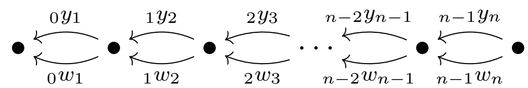

Figure 2.2: The graph for defining B_n

Recall that if a quadratic algebra A has a vector space of generators V and quadratic relations $I \subset V \otimes V$, then its *quadratic dual* has a vector space of generators V^* and quadratic relations $I^\perp \subset V^* \otimes V^*$ where $I^\perp$ is the orthogonal complement to I with respect to the natural pairing between $V \otimes V$ and $V^* \otimes V^*$. It is immediate from the definitions that B_n is the quadratic dual of A_n; we think of ${}_iw_{i+1}$ as ${}_{i+1}x_i^*$ and ${}_iy_{i+1}$ as ${}_{i+1}\iota_i^*$. Then the relations above for B_n are dual to the relations for A_n as required. Note that B_n is not directly related to B in [2, 3].

Let $\mathfrak{B}_{n+1}$ denote the braid group on $n+1$ strands. To a braid $\sigma \in \mathfrak{B}_{n+1}$ Auroux-Grigsby-Wehrli associate a dg bimodule M_σ over A_n in two steps. First they define bimodules $M_{\sigma_i^\pm}$ associated to the standard generators of the braid group σ_i and their inverses. The modules $M_{\sigma_i^\pm}$ are defined as the mapping cones of two-term complexes relating A_n and projective modules that depend on i; the direction of the arrow depends on the sign of $\sigma_i^\pm$. The exact definition can be found in [2, Propositions 3.17, 3.18]. Then if $\sigma = \sigma_{i_1}^\pm \cdots \sigma_{i_k}^\pm$ we have that $M_\sigma = M_{\sigma_{i_1}^\pm} \otimes_{A_n} \cdots \otimes_{A_n} M_{\sigma_{i_k}^\pm}$. The only properties of these chain complexes of bimodules that we need in this dissertation are that they are finite dimensional over $\mathbb{F}_2$, which is immediate from their definition, and that their Hochschild homologies agree with (part of) sutured annular Khovanov homology (see Lemma 2.1.1).

Note once again that our M_σ is denoted as M_σ^{Kh} in [2, 3]. We shall denote by $\bar{M}_\sigma$ what [2, 3] denote as M_σ. Again, this algebra only appears here as an intermediate step in the proof of Lemma 2.1.1.

Let $\sigma \in \mathfrak{B}_{n+1}$ and let $m(\widehat{\sigma}) \subset A \times I$ denote the mirror of the annular closure of σ. Let $SKh(\widehat{\sigma}; n-1)$ denote the sutured annular Khovanov homology of $\widehat{\sigma}$ with winding number grading $n-1$, where the winding number grading appropriately measures the amount of wrapping around the central hole of the annulus. Let n_- (respectively n_+) refer to the number of negative (positive) crossings of $\widehat{\sigma}$. Given a bigraded vector space V let $V[n]$ denote shifting the first (homological) grading down by n and let $V\{m\}$ denote shifting the second (quantum) grading up by m. Explicitly, $(V[n]\{m\})_{(i,j)} \cong V_{(i+n,j-m)}$. We get the following lemma:

Lemma 2.1.1. *There is an isomorphism of bigraded vector spaces*

$$SKh(\widehat{\sigma}; n-1) \cong HH(A_n, M_{m(\sigma)})[-n_-]\{(n-1) + n_+ - 2n_-\}.$$

Proof. By [3, Theorem 5.1], we know that as bigraded vector spaces we have $SKh(\widehat{\sigma}; n-1) \cong HH(\bar{A}_n, \bar{M}_{m(\sigma)})[-n_-]\{(n-1) + n_+ - 2n_-\}$. In the proof of [3, Theorem 6.1], we see that $HH(\bar{A}_n, \bar{M}_{m(\sigma)}) \cong HH(A_n, M_{m(\sigma)})$. □

2.2 Homologically Smooth and Proper

We follow [42, Section 3.1.1] to define homological smoothness and properness. This is a restatement from [33, Section 8]. A dg algebra A over $\mathbb{F}_2$ is *homologically proper* if the total homology of A is finite-dimensional. Recall that a *finite cell retract* is a subcomplex R of a bimodule C such that (1) C admits a filtration $C_1 \subset C_2 \subset \cdots \subset C_n$ such that C_{i+1}/C_i is isomorphic to a free bimodule of finite rank and (2) the inclusion $R \to C$ admits an (A, A)-bimodule retract $r : C \to R$. A *resolution* of a bimodule M by a finite cell retract is then a quasi-isomorphism $R \to M$ where R is a finite cell retract. A dg algebra A over $\mathbb{F}_2$ is *homologically smooth* if A admits a resolution by a finite cell retract as an (A, A) bimodule.

To construct our spectral sequence in Theorem 2.4.1, we shall use [42, Theorem 4] which states that if A is a dg algebra over $\mathbb{F}_2$ and M is an (A, A) dg bimodule such that (1) A is homologically smooth and proper, (2) M is bounded, i.e. supported in finitely many gradings, and (3) M is π-formal then there is a spectral sequence starting at $HH_*(A, M \otimes^L M)$, where $\otimes^L$ is the derived tensor product, and converging to $HH_*(A, M)$. Homological smoothness and properness are finiteness properties needed to make this spectral sequence converge. We shall prove our algebras A_n have these properties in this section. The fact that A_n is homologically proper is trivial.

Proposition 2.2.1. *The algebra A_n is homologically proper for all n.*

Proof. Each algebra A_n is itself finite. □

However to see that A_n is homologically smooth we shall first need a couple of definitions and a lemma. If A is a quadratic algebra and B is its quadratic dual, then A is a *Koszul algebra* if $\mathrm{Ext}^*_A(\mathbb{K}, \mathbb{K}) \simeq B$. A quadratic algebra generated by a vector space V is a *PBW algebra* if V admits an ordered basis which behaves in a nice way with respect to multiplication; see [52, Definition 5.1].

Lemma 2.2.2. *Each A_n is a homogeneous Koszul algebra.*

Proof. We will show that A_n is a PBW algebra and then by [52, Theorem 5.3] we will see that each A_n is a homogeneous Koszul algebra.

First, note that A_n is a quadratic algebra since it can be generated as an algebra by the elements of the form ${}_i\iota_{i+1}$ and ${}_ix_{i+1}$ with quadratic relations as above. These generators are also a PBW basis. To see this, first we lexicographically order the basis such that x's are greater than ι's. We then label the basis such that for any word in A_n the x, if there is one, is on the far left. Note that multiplying two elements of A_n preserves this labeling, is zero, or we increase the lexicographical ordering when rearranging to get back to this labeling. Moreover decomposing an element of A_n as a product in any fashion also preserves this labeling. Thus A_n is indeed PBW and thus Koszul. □

Proposition 2.2.3. *Each A_n is homologically smooth.*

Proof. Recall that B_n is the quadratic dual of A_n. Thus, by [50, Proposition 3.1] and Lemma 2.2.2 we have that $\mathrm{Tor}^{A_n}(\mathbb{K},\mathbb{K}) \simeq B_n^*$, so the Koszul resolution of A_n, as found in [52, Definition 3.7], is $A_n \otimes_{\mathbb{K}} B_n^* \otimes_{\mathbb{K}} A_n$, with differential

$$\partial(a_1 \otimes b_1^* \otimes a_2) = \sum_{(\xi,\xi')\in\{({}_i\iota_{i+1},{}_{i+1}y_i^*),({}_ix_{i+1},{}_{i+1}w_i^*)\}} a_1\xi \otimes \xi' b_1^* \otimes a_2 + a_1 \otimes b_1^*\xi' \otimes \xi a_2 \tag{2.1}$$

as derived from [52, Definition 3.5].

We claim that this is the required resolution by a finite cell retract. The Koszul resolution is a resolution due to [52, Theorem 3.8]; we claim that it is a finite cell retract of $A_n \otimes_{\mathbb{F}_2} B_n^* \otimes_{\mathbb{F}_2} A_n$, with differential the same as Equation (2.1). It is straightforward to check that $\partial^2 = 0$. Since $A_n \otimes_{\mathbb{F}_2} B_n^* \otimes_{\mathbb{F}_2} A_n$ is finite and free it is certainly a finite cell bimodule.

To see that $A_n \otimes_{\mathbb{K}} B_n^* \otimes_{\mathbb{K}} A_n$ is a cell retract of $A_n \otimes_{\mathbb{F}_2} B_n^* \otimes_{\mathbb{F}_2} A_n$ note that the retract map is the quotient map induced on the tensor product $(- \otimes_{\mathbb{K}} -) \to (- \otimes_{\mathbb{F}_2} -)$ by the ring inclusion $\mathbb{F}_2 \to \mathbb{K}$. To see the inclusion map, notice that for any element $\alpha \in A_n \otimes_{\mathbb{K}} B_n^* \otimes_{\mathbb{K}} A_n$ we have $\alpha = \sum_{i,j} a_i \otimes_{\mathbb{K}} {}_ib_j \otimes_{\mathbb{K}} {}_jc$ for some unique $a_i, {}_jc \in A_n$, ${}_ib_j \in B_n^*$ where $a_i k_{ii} b_j = a_{ii} b_j$ and ${}_ib_j k_{jj} c = {}_ib_{jj} c$ for some k_i and k_j that are primitive idempotents in $\mathbb{K}$. The inclusion map then sends α to $\sum_{i,j} a_i \otimes_{\mathbb{F}_2} {}_ib_j \otimes_{\mathbb{F}_2} {}_jc$. It is clear that the inclusion map is a chain map, and it is a quick check to see the retract map is as well. □

2.3 π-formality

The final condition we need on our algebras A_n is that they be π-formal, a condition that requires certain operations d^{2i} vanish on Hochschild homology.

To define these operations let M be an A_n bimodule and let R be a biprojective resolution of A_n. Then the Hochschild chain complex $HC(M \otimes^L M)$, where $\otimes^L$ is the derived tensor product, is equal to $R \otimes_{A_n} (M \otimes_{A_n} R \otimes_{A_n} M)$ modulo the equivalence relation generated by $r \otimes (m \otimes r' \otimes m'a) \sim ar \otimes (m \otimes r' \otimes m')$. Let τ be the map on $HC(M \otimes^L M)$ that sends $r \otimes (m \otimes r' \otimes m')$ to $r' \otimes (m' \otimes r \otimes m)$. By definition τ commutes with $\partial_{HC(M \otimes^L M)}$ and $\tau^2 = 1$; thus since we are working over $\mathbb{F}_2$ we have $(1 + \tau)^2 = 0$.

We therefore define a bicomplex $HC^{\text{Tate}}_{*,*}(M \otimes^L M)$ where $HC^{\text{Tate}}_{p,q}(M \otimes^L M) = HC_q(M \otimes^L M)$ with the vertical differential being $\partial_{HC(M \otimes^L M)}$ and the horizontal differential being $(1 + \tau)$. We can induce a spectral sequence from this bicomplex by first taking homology with respect to the vertical differential and then the horizontal differential. Call the differential for this spectral sequence d^r. In [42, Proposition 3.10], it is proved that d^{2i+1} vanishes for all i. Note that the E^3-page of this spectral sequence is $HH(M) \otimes \mathbb{F}_2[\theta, \theta^{-1}]$. Thus if d^{2i} vanishes for $i \geq 2$ the spectral sequence collapses at this E^3 page. If the induced map d^{2i} on $HH(M)$ vanishes for $i \geq 2$ for a bimodule M we say that M is *π-formal*.

We shall be using the following theorems about π-formal algebras. Theorem 2.3.1 is [42, Theorem 5], while Theorem 2.3.2 is [42, Theorem 4].

Theorem 2.3.1. *Suppose that A is homologically smooth and proper. The following are equivalent:*

(1) Every dg bimodule over A is π-formal.

(2) The dg bimodule $A^!$ is π-formal (where $A^!$ is defined in Proposition 2.3.3).

(3) For each $i \geq 2$, the element $1 \in \mathrm{Hom}(A^!, A^!) \cong HH_0(A, A^!)$ is killed by $d^{2i} : HH_0(A, A^!) \dashrightarrow HH_{-i}(A, A^!)$.

Theorem 2.3.2. *Let A be a dg algebra over $\mathbb{F}_2$, let M be an (A, A) dg bimodule, and let $M \otimes^L M$ denote the derived tensor product, over A, of M with itself. Suppose that:*

(A-1) A is homologically smooth and proper.

(A-2) M is bounded.

(A-3) M is π-formal.

Then there is a spectral sequence starting at $HH_*(A, M \otimes^L M)$ and converging to $HH_*(A, M)$.

Proposition 2.3.3. *Each A_n is π-formal.*

Proof. We will drop the n from A_n and B_n for this proof. We need only check d^{2i} vanishes on $1 \in \mathrm{Hom}(A^!, A^!)$ due to Theorem 2.3.1. Here $A^!$ is the "inverse dualizing bimodule" of [33, Definition 8.1.6] and can be defined as $A^! := \mathrm{Hom}_{{}_A\mathrm{Mod}_A}(P, A^e)$ where P is a resolution of A. The bimodule $A^!$ is also characterized as the representing object for Hochschild homology as in [42, Proposition 3.3]. We have that $\bigoplus_k \mathrm{Hom}(A^![k], A) \cong HH_*(A, A^!) \cong H_*(A \otimes B \otimes A \otimes B / \sim)$ where $a_1 \otimes b_1 \otimes a_2 \otimes b_2 z \sim z a_1 \otimes b_1 \otimes a_2 \otimes b_2$. Note we get the first isomorphism due to $A^!$ being the representing object for HH, and the last isomorphism because the Koszul resolution $A \otimes B^* \otimes A$ gives us a model for $A^!$:

$$A^! = \mathrm{Hom}_{{}_A \mathrm{Mod}_A}(A \otimes B^* \otimes A, A^e) = A \otimes B \otimes A.$$

Note that the differential on $A \otimes B \otimes A \otimes B/\sim$ is induced from that in Equation (2.1); it is

$$\partial(a_1 \otimes b_1 \otimes a_2 \otimes b_2) = \sum_{(\xi,\xi') \in \{({}_i\iota_{i+1}, {}_{i+1}y_i), ({}_i x_{i+1}, {}_{i+1}w_i)\}} a_1\xi \otimes \xi' b_1 \otimes a_2 \otimes b_2 + a_1 \otimes b_1\xi' \otimes \xi a_2 \otimes b_2 \\ + a_1 \otimes b_1 \otimes a_2\xi \otimes \xi' b_2 + \xi a_1 \otimes b_1 \otimes a_2 \otimes b_2\xi'. \quad (2.2)$$

For notational convenience let x,y denote words in A, B that include one ${}_{i+1}x_i$ or ${}_iy_{i+1}$ respectively, while ι,w denote words with only ι's or w's. Then when $(\xi, \xi'), (\eta, \eta') \in \{(\iota, y), (x, w)\}$ with $\xi\eta$ and $\eta'\xi'$ both nonzero, we have $\xi\eta = x$ and $\eta'\xi' = y$. We will call this the xy phenomenon. Note that since there are exactly two ways to get $\xi\eta = x$ and $\eta'\xi' = y$, when summed over all $(\xi, \xi'), (\eta, \eta')$ these terms vanish.

The rest of the proof is a concrete computation. To keep notation under control, we will replace the tensor $\otimes$ with the vertical line $|$. Remember that $\tau(a_1 \otimes b_1 \otimes a_2 \otimes b_2) = a_2 \otimes b_2 \otimes a_1 \otimes b_1$.

We have that

$$\begin{aligned}\partial(1|1|1|1) &= \sum_{(\xi,\xi')} (\xi|\xi'|1|1) + (1|\xi'|\xi|1) + (1|1|\xi|\xi') + (\xi|1|1|\xi') \\ &= (1+\tau)\Big(\sum_{(\xi,\xi')} (\xi|\xi'|1|1) + (1|\xi'|\xi|1)\Big). \qquad (2.3)\end{aligned}$$

Let $(1+\tau)^{-1}(2.3)$ denote the expression obtained by dropping the $(1+\tau)$ from the right-hand side of Equation (2.3). Then

$$\begin{aligned}\partial \circ (1+\tau)^{-1}(2.3) = \sum_{(\xi,\xi'),(\eta,\eta')} & (\xi\eta|\eta'\xi'|1|1) + (\xi|\xi'\eta'|\eta|1) + (\xi|\xi'|\eta|\eta') \\ & + (\eta\xi|\xi'|1|\eta') + (\eta|\eta'\xi'|\xi|1) + (1|\xi'\eta'|\eta\xi|1) \\ & + (1|\xi'|\xi\eta|\eta') + (\eta|\xi'|\xi|\eta'). \qquad (2.4)\end{aligned}$$

Note the first term when summed up over all ξ, η is zero by the xy phenomenon, as is the sixth. The second and fifth terms cancel in the sum, while τ applied to the fourth term is the seventh over the sum of all ξ, η . Further:

$$\sum_{\xi,\eta}(\xi|\xi'|\eta|\eta') = \sum_i ({}_{i+1}x_i|{}_iw_{i+1}|{}_{i+1}\iota_i|{}_iy_{i+1}) + ({}_{i+1}\iota_i|{}_iy_{i+1}|{}_{i+1}x_i|{}_iw_{i+1})$$
$$+ ({}_{i+1}x_i|{}_iw_{i+1}|{}_{i+1}x_i|{}_iw_{i+1}) + ({}_{i+1}\iota_i|{}_iy_{i+1}|{}_{i+1}\iota_i|{}_iy_{i+1})$$
$$\sum_{\xi,\eta}(\eta|\xi'|\xi|\eta') = \sum_i ({}_{i+1}x_i|{}_iy_{i+1}|{}_{i+1}\iota_i|{}_iw_{i+1}) + ({}_{i+1}\iota_i|{}_iw_{i+1}|{}_{i+1}x_i|{}_iy_{i+1})$$
$$+ ({}_{i+1}x_i|{}_iw_{i+1}|{}_{i+1}x_i|{}_iw_{i+1}) + ({}_{i+1}\iota_i|{}_iy_{i+1}|{}_{i+1}\iota_i|{}_iy_{i+1})$$
$$\sum_{\xi,\eta}(\xi|\xi'|\eta|\eta') + (\eta|\xi'|\xi|\eta') = \sum_i ({}_{i+1}x_i|{}_iw_{i+1}|{}_{i+1}\iota_i|{}_iy_{i+1}) + ({}_{i+1}\iota_i|{}_iy_{i+1}|{}_{i+1}x_i|{}_iw_{i+1})$$
$$+ ({}_{i+1}x_i|{}_iy_{i+1}|{}_{i+1}\iota_i|{}_iw_{i+1}) + ({}_{i+1}\iota_i|{}_iw_{i+1}|{}_{i+1}x_i|{}_iy_{i+1}).$$

Substituting, we have

$$(2.4) = (1+\tau)\Big(\sum_{\xi,\eta}(\eta\xi|\xi'|1|\eta') + \sum_i ({}_{i+1}x_i|{}_iw_{i+1}|{}_{i+1}\iota_i|{}_iy_{i+1})$$
$$+ ({}_{i+1}x_i|{}_iy_{i+1}|{}_{i+1}\iota_i|{}_iw_{i+1})\Big). \qquad (2.5)$$

Differentiating again we have

$$\begin{aligned}\partial \circ (1+\tau)^{-1}(2.5) = \sum_{\xi,\eta,\nu} &((\eta\xi\nu|\nu'\xi'|1|\eta') + (\eta\xi|\xi'\nu'|\nu|\eta') + (\eta\xi|\xi'|\nu|\nu'\eta') \\ &+ (\nu\eta\xi|\xi'|1|\eta'\nu')) + \sum_{\nu,i}((_{i+1}x_i\nu|\nu'{}_iw_{i+1}|_{i+1}\iota_i|_iy_{i+1}) \\ &+ (_{i+1}x_i|_iw_{i+1}\nu'|\nu_{i+1}\iota_i|_iy_{i+1}) + (_{i+1}x_i|_iw_{i+1}|_{i+1}\iota_i\nu|\nu'{}_iy_{i+1}) \\ &+ (_{i+1}x_i|_iy_{i+1}\nu'|\nu_{i+1}\iota_i|_iw_{i+1}) + (_{i+1}x_i|_iy_{i+1}|_{i+1}\iota_i\nu|\nu'{}_iw_{i+1}) \\ &+ (\nu_{i+1}x_i|_iy_{i+1}|_{i+1}\iota_i|_iw_{i+1}\nu')), \qquad (2.6)\end{aligned}$$

where a couple of terms, namely $(\nu x|w|\iota|y\nu')$ and $(x\nu|\nu' y|\iota|w)$, have been left off due to being zero; for example one of νx or $y\nu'$ must be zero.

The first and fourth terms in Equation (2.6) must cancel in the sum by the xy phenomenon. For the rest of the sum there is exactly one ι or w in each nonzero tensor after multiplying. For example, in $(\eta\xi|\xi'\nu'|\nu|\eta')$, if $\eta\xi = \iota$, then $\eta' = \xi' = y$, so to be nonzero we need $\nu' = w$, so $\nu = x$ and $(\eta\xi|\xi'\nu'|\nu|\eta') = (w|y|x|y)$. There is less cancellation than one might expect however because writing $(w|y|x|y)$ in the last sentence hides the fact that w and the first y are words of length two, while the x and the second y are words of length one. In light of this, let x_j denote the (non-unique) word that is of length j and has one $_{i+1}x_i$; similarly for y_j,w_j,ι_j.

Nonetheless, since each tensor has exactly one ι or w, after differentiating each tensor will have zero ι's or w's. Thus we will see that even after factoring out a $(1+\tau)$, the next differential will be zero.

That is, in more detail,

$$\begin{aligned}
(2.6) = \sum_i &(\iota_2|y_2|x_1|y_1) + (x_2|w_2|x_1|y_1) + (x_2|y_2|x_1|w_1) + (\iota_2|y_1|x_1|y_2) \\
&+ (x_2|w_1|x_1|y_2) + (x_2|y_1|x_1|w_2) + (x_1|w_2|x_2|y_1) + (x_1|y_2|\iota_2|y_1) \\
&+ (x_1|w_1|x_2|y_2) + (x_1|y_2|x_2|w_1) + (x_1|y_1|x_2|w_2) + (x_1|y_1|\iota_2|y_2) \\
&= (1+\tau)\Big(\sum_i (\iota_2|y_2|x_1|y_1) + (x_2|w_2|x_1|y_1) + (x_2|y_2|x_1|w_1) \\
&+ (\iota_2|y_1|x_1|y_2) + (x_2|w_1|x_1|y_2) + (x_2|y_1|x_1|w_2)\Big),
\end{aligned}$$

where we left off the two terms that did cancel from Equation (2.6).

Differentiating, we have

$$\begin{aligned}
\partial \circ (1+\tau)^{-1}(2.6) = \sum_i &((x_3|y_3|x_1|y_1) + (x_3|y_2|x_1|y_2) + (x_3|y_3|x_1|y_1) \\
&+ (x_2|y_3|x_2|y_1) + (x_2|y_2|x_2|y_2) + (x_3|y_2|x_1|y_2) \\
&+ (x_3|y_2|x_1|y_2) + (x_3|y_1|x_1|y_3) + (x_3|y_2|x_1|y_2) \\
&+ (x_2|y_2|x_2|y_2) + (x_2|y_1|x_2|y_3) + (x_3|y_1|x_1|y_3)) \\
&= \sum_i (x_2|y_3|x_2|y_1) + (x_2|y_1|x_2|y_3).
\end{aligned}$$

Thus

$$\partial \circ (1+\tau)^{-1}(2.6) = (1+\tau)\Big(\sum_i (x_2|y_3|x_2|y_1)\Big). \tag{2.7}$$

Finally, we then have

$$\partial \circ (1+\tau)^{-1}(2.7) = 0. \qquad \square$$

2.4 Spectral Sequence

Theorem 2.4.1. *Let $\sigma \in \mathfrak{B}_{n+1}$, σ^2 denote its square in $\mathfrak{B}_{n+1}$, and let $m(\widehat{\sigma}), m(\widehat{\sigma^2}) \subset A \times I$ denote the mirror of the annular closures of σ and σ^2 respectively. Then there is a spectral sequence which has E^1 page isomorphic to $SKh(\widehat{\sigma^2}; n-1, q)$ where q is the quantum grading and E^∞ term isomorphic to the associated graded vector space of a filtration of $SKh(\widehat{\sigma}; n-1, (q+n-1)/2)$.*

Proof. By Lemma 2.1.1, we know that $SKh(\widehat{\sigma^2}; n-1) \cong HH(A_n, M_{m(\sigma^2)})$ as bigraded vector spaces. Now by Propositions 2.2.1 and 2.2.3, our algebras A_n are homologically smooth and proper. Since A_n is π-formal by Proposition 2.3.3, by Theorem 2.3.1 we know that $M_{m(\sigma)}$ is π-formal. Thus by Theorem 2.3.2, there is a spectral sequence from $HH(A_n, M_{m(\sigma^2)})$ to $HH(A_n, M_{m(\sigma)})$. Then by Lemma 2.1.1 again, we have that $HH(A_n, M_{m(\sigma)}) \cong SKh(\widehat{\sigma}; n-1)$.

To see the quantum grading shift, first note that following the grading shift of [3, Theorem 5.1], the isomorphism in Lemma 2.1.1 shifts the quantum grading of an element of $SKh(\widehat{\sigma^2}; n-1)$ down by $n-1$ when it sends that element to $HH(A_n, M_{m(\sigma^2)})$. Since both differentials in the bicomplex $HH(A_n, M_{m(\sigma^2)})$ preserve the quantum grading the spectral sequence then preserves the quantum grading. We then see that in the proof of [42, Theorem 4] the E^∞ page is isomorphic to the Hochschild homology of σ where every element has been tensored with itself and thus has had the quantum grading doubled. We finally use Lemma 2.1.1 again in reverse, which shifts the quantum grading back up by $n-1$ again.

Thus we shift the quantum grading down by $n-1$, divide it by two, and then shift it back up by $n-1$. Overall we have turned a quantum grading q into $(q+n-1)/2$ as stated. □

In fact using recent work by Beliakova, Putyra, and Wehrli [4] we can strengthen the ungraded version of the theorem to apply to all periodic links and obtain:

Theorem 2.4.2. *Let T be an $(n+1, n+1)$ tangle, and $\widehat{T}, \widehat{T^2} \subset A \times I$ denote the annular closures of T and T^2 respectively. Then there is a spectral sequence which has E^1 page isomorphic to $SKh(\widehat{T^2}; n-1)$ and E^∞ isomorphic to the associated graded algebra of a filtration of $SKh(\widehat{T}; n-1)$.*

Proof. In the proof of Lemma 2.1.1 use [4, Theorem C] instead of [3, Theorem 5.1]. One obtains the isomorphism

$$SKh(\widehat{T}; n-1) \cong HH(A_n, M_T). \tag{2.8}$$

Then follow the proof of Theorem 2.4.1 using Equation (2.8) instead of Lemma 2.1.1. □

Note that one should be able to use [4, Theorem C] to extend this result to apply to all winding number gradings of SKh, not just the next-to-top one. However one would have to redo Propositions 2.2.1, 2.2.3, and 2.3.3 to apply to larger algebras. While Propositions 2.2.1 and 2.2.3 shouldn't prove too difficult in general, the main issue would be Proposition 2.3.3; we needed to have the nice concrete description of the algebra involved, like we have with A_n, to do the required computation.

2.5 Decategorification and Computations

Theorem 2.4.1 has a decategorified statement. Following Roberts in [55], for a link L in the thickened annulus let $q_{L,k} = \sum_{i,j}(-1)^i y^j \mathrm{rk}(SKh^{i,j}(L,k))$. Then the graded Euler characteristic of sutured annular Khovanov homology is $\sum_k q_{L,k}x^k$; note that one can also define this from a skein relation as found in Section 2 of [55]. In addition $\sum_k q_{L,k}$ is the Jones polynomial of the link in S^3. Then we have the following proposition.

Proposition 2.5.1. *Let σ be a braid on $n+1$ strands. Then $y^{n-1}q_{\widehat{\sigma^2},n-1} \equiv q^2_{\widehat{\sigma},n-1} \pmod 2$.*

Proof. Theorem 2.4.1 gives us a spectral sequence from the sutured annular Khovanov homology in winding number grading $n-1$ of $\widetilde{K}$ to that of K, and thus we get a relation on the graded Euler characteristics. However, as above in Theorem 2.4.1, we need to shift the quantum grading for the spectral sequence; we therefore need to square the polynomial for $\widehat{\sigma}$ and multiply the polynomial for $\widehat{\sigma^2}$ by y^{n-1} as in the statement of the proposition. Note this relation is only true modulo 2 also due to grading issues; the spectral sequence in Theorem 2.4.1 is induced from a bicomplex where the horizontal differential has grading zero in $SKh^{i,j}(\widetilde{K}, n-1)$. Thus the differentials in the spectral sequence do not respect the homological grading of $SKh^{i,j}(\widetilde{K}, n-1)$. However the spectral sequence works by canceling generators in pairs, thus the Euler characteristic is preserved modulo 2. □

We conclude with a detailed computation as an example. Consider the Hopf link $\widetilde{L}$ inside the thickened annulus, thinking of the Hopf link as a braid on two strands with both strands being homologically nontrivial in the annulus. The Hopf link is a two-periodic link whose quotient is the unknot L with a single crossing; note L is also homologically nontrivial in the annulus. We have that $\widetilde{L}$ is the closure of σ_1^2 and L is the closure of σ_1.

By Lemma 2.1.1, [42, Theorem 4] and Proposition 2.2.3 , the zeroth page E^0 of the spectral sequence of Theorem 2.4.1 from $SKh(\widetilde{L}; 1)$ to $SKh(L; 1)$ is $M_{\sigma_1^-} \otimes_{A_1} A_1 \otimes_{\mathbb{K}} B_1^* \otimes_{\mathbb{K}} A_1 \otimes_{A_1} M_{\sigma_1^-} \otimes_{A_1} A_1 \otimes_{\mathbb{K}} B_1^* \otimes_{\mathbb{K}} A_1 / \sim$ which is isomorphic to $M_{\sigma_1^-} \otimes_{\mathbb{K}} B_1^* \otimes_{\mathbb{K}} M_{\sigma_1^-} \otimes_{\mathbb{K}} B_1^* \otimes_{\mathbb{K}} / \sim$, where note the second $\sim$ is the equivalence relation generated by $m_1 \otimes b_1^* \otimes m_2 \otimes b_2^* k_i \sim k_i m_1 \otimes b_1^* \otimes m_2 \otimes b_2^*$ for $k_i \in \mathbb{K}$. Note that to have the first page E^1 isomorphic to $SKh(\widetilde{L}; 1)$ as bigraded vector spaces as in [3, Theorem 5.1] we need to shift $HH(A_1, M_{\sigma_1^-})$ by $[-n_-]\{(n-1) + n_+ - 2n_-\}$ where n_- and n_+ refer to the positive and negative crossings in $\widetilde{L}$ respectively, the square brackets denote shifting the homological grading down, and the curly brackets denote shifting the quantum grading up. We thus need to shift all quantum gradings up by two; below we have already done this shift.

Now in this spectral sequence E^0 has 34 elements; listing first the homological and then the quantum gradings, there are 7 elements in bigrading (0,2), 10 elements in bigrading (1,2), 4 elements in bigrading (2,2), 6 elements in bigrading (1,4), 6 elements in bigrading (2,4), and one element in bigrading (2,6). We show this complex in the diagrams below. Note the solid arrows are the differential in the Hochschild complex while the dashed arrows denote the map τ. There are only two idempotents in A_1 and B_1; below the vertex at the tail of ${}_1x_0$ is denoted 0 while the vertex at the head of ${}_1x_0$ is denoted 1. Since there is only one x edge in A_1, ${}_1x_0$ is denoted as x; similarly ι, y, and w correspond to the obvious edges in A_1 and B_1^*.

Finally, note that in $M_{\sigma_1^-}$ there is a term that [2] calls $P_1^{Kh} \otimes {}_1P^{Kh}$ with four elements $u^* \otimes u$, $v^* \otimes v$, $v^* \otimes u$, and $u^* \otimes v$ that are denoted below as u, v, t, and s respectively. Note that the bigradings of these elements, after a shift of $[-1]\{2\}$ in the $M_{\sigma_1^-}$ complex but before the 2 shift in the entire Hochschild complex, are $(1,0)$, $(1,0)$, $(2,1)$, and $(0,-1)$ respectively. The bigradings of 0, 1, ι, x, w, and y are $(0,0)$, $(0,0)$, $(0,1)$, $(-1,-1)$, $(-2,-1)$ and $(-1,1)$ respectively before the 2 shift in the entire Hochschild complex.

First we look at the subcomplex of the E^0 page in quantum grading two.

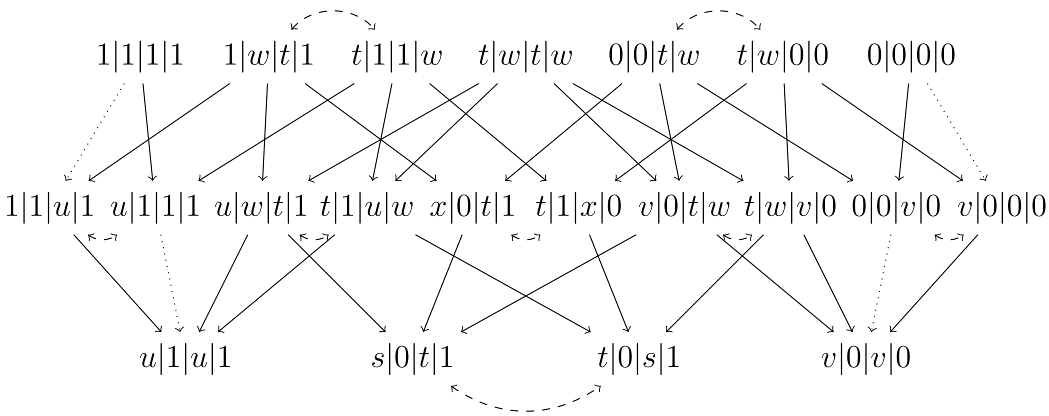

Figure 2.3: Subcomplex of the E^0 page in quantum grading two

To find the first page E^1 we cancel the vertical arrows; one way to do this is to cancel first the dotted arrows on the top row and then the dotted arrows on the bottom row of Figure 2.3. One obtains the following complex:

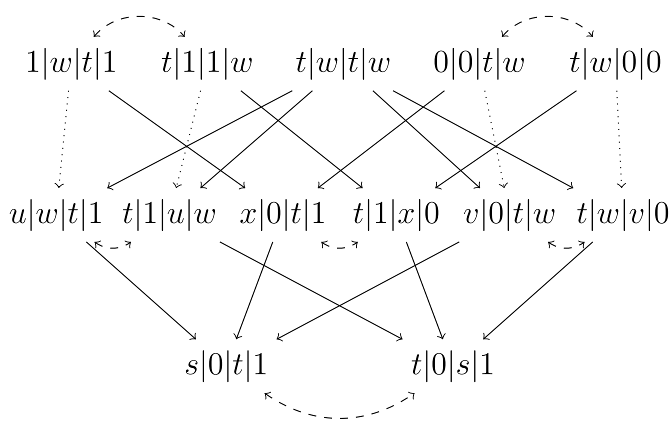

Figure 2.4: Quantum grading two subcomplex after canceling some arrows

We then cancel the dotted arrows in this complex to obtain:

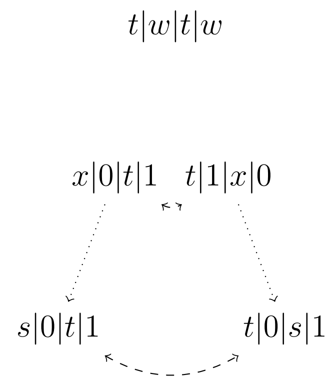

Figure 2.5: Quantum grading two subcomplex after canceling more arrows

One then cancels the final two dotted arrows to discover that only one element in bigrading $(0, 2)$ survives to the first page from this subcomplex.

Below is the subcomplex of the E^0 page in quantum grading four.

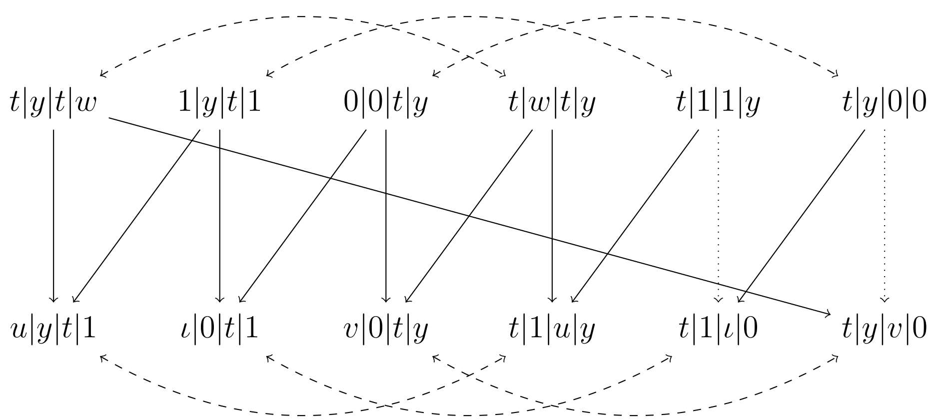

Figure 2.6: Subcomplex of the E^0 page in quantum grading four

We cancel the two dotted arrows starting from the left to obtain the following complex:

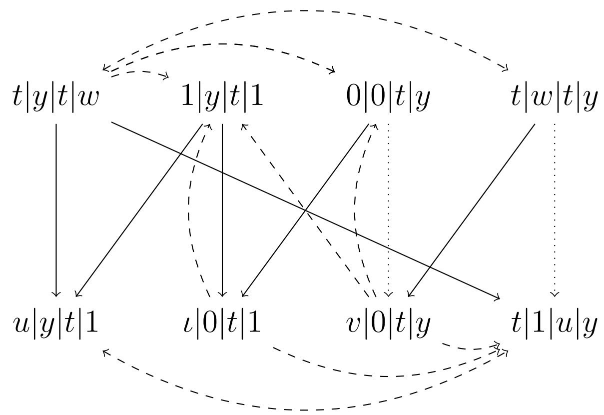

Figure 2.7: Quantum grading four subcomplex after canceling some arrows

We again cancel the two dotted arrows starting from the left to obtain:

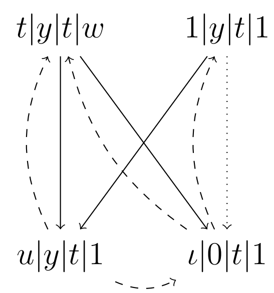

Figure 2.8: Quantum grading four subcomplex after canceling more arrows

We finally cancel the dotted arrow to obtain:

$$\begin{array}{c} t|y|t|w \\ \uparrow \\ u|y|t|1 \end{array}$$

Figure 2.9: Subcomplex of the E^1 page in quantum grading four

Notice that this dashed arrow is therefore the only map that survives to the E^1 page; we already saw that the quantum grading two subcomplex was reduced to one element in the E^1 page while the quantum grading six subcomplex only had one element to begin with. Thus the E^1 page has four elements, of bigradings $(0,2)$, $(1,4)$, $(2,4)$, and $(2,6)$, which is isomorphic to the sutured annular Khovanov homology of the Hopf link, as it should be.

Note that the dashed arrow above goes against the Hochschild homology grading by one while it goes with the $1+\tau$ grading by two; it is thus canceled at the E^2 page and the spectral sequence therefore stabilizes at the E^3 page. Thus E^∞ has two generators of bigradings $(0,2)$ and $(2,6)$. Notice that as in the proof of Proposition 2.5.1 the isomorphism of this page to the sutured annular Khovanov homology of the unknot with one crossing doubles the quantum gradings. We thus obtain a spectral sequence from $SKh(\widetilde{L};1)$ to $SKh(L;1)$ as in Theorem 2.4.1.

Remark 2.5.2. As stated in Section 1.1.1 Seidel and Smith [58] also show there is a spectral sequence for two-periodic links to their quotients, although this spectral sequence is on symplectic Khovanov homology. Note that, although it has not been shown that a quantum grading can be well defined for symplectic Khovanov homology, the spectral sequence of Seidel-Smith cannot preserve any quantum grading. For example, the unlink with three components has period two with quotient the unlink with two components; however the Jones polynomial of the unlink with three components is $q^3 + 3q + 3q^{-1} + q^{-3}$ while the Jones polynomial of the unlink with two components is $q^2 + 2 + q^{-2}$ and these polynomials are not congruent modulo 2.

Chapter 3

Growth of Heegaard Floer Homology in Branched Covers

The goal of this chapter is to study the growth rate of Heegaard Floer homology under branched covers. There is a lower bound based on the growth rate of H_1; this lower bound as well as what examples are known in the literature are discussed in Section 3.1.1. We then give an upper bound for fibered hyperbolic knots in Section 3.1.2. Because there is a surgery exact sequence relating $\widehat{HF}(\Sigma^n(K))$ to $\widehat{HF}(T_{\phi^n})$ where ϕ is the monodromy of K and T_{ϕ^n} is the mapping torus of ϕ^n, in Section 3.1.3 we turn to the Floer homology of T_ϕ, giving an exact answer to the growth rate of $\widehat{HF}(T_{\phi^n})$ in certain gradings. In Section 3.1.4, we show that a natural conjecture for the growth rate of $\widehat{HFK}(\Sigma^n(K), \widetilde{K})$ is false. Possible directions for future research are discussed in Section 3.2.

3.1 Results

3.1.1 L-spaces and Branched Covers

As a first step towards investigating the growth rate of Heegaard Floer homology of cyclic branched covers, consider the following definitions. Let K be a knot, and let $\Sigma^n(K)$ be the n-fold cyclic branched cover of K. If $\Sigma^n(K)$ is an L-space for n sufficiently large, we say that K is *covered by L-spaces*. If

$$\lim_{n\to\infty} \frac{\operatorname{rk} \widehat{HF}(\Sigma^n(K))}{|H_1(\Sigma^n(K))|} = 1$$

then we say that K is *asymptotically covered by L-spaces*. If the above limit is constant, then we say that K is *approximately asymptotically covered by L-spaces*.

Theorem 3.1.1. *There are hyperbolic knots K which are covered by L-spaces. There are hyperbolic knots K which are not covered by L-spaces. There are knots K which are not approximately asymptotically covered by L-spaces.*

Proof. An example of the first is the figure-eight knot, all of whose branched covers are L-spaces; see Peters [49]. Examples of the second come from Hedden and Mark [26], who show that if K is a fibered knot with nonzero fractional Dehn twist then K is not covered by L-spaces; note their work does not rule out that these knots are asymptotically covered by L-spaces. Finally, Tweedy [65] provides calculations that show that not only are the torus knots $T(2,5)$ and $T(2,7)$ not covered by L-spaces, they are not even approximately asymptotically covered by L-spaces. □

3.1.2 Upper Bound for Fibered Hyperbolic Knots

If K is a fibered hyperbolic knot, then the monodromy ϕ on the Seifert surface of the knot is pseudo-Anosov. Note that ϕ^n can be used to construct the n-fold cyclic branched cover of K; the exterior of K is diffeomorphic to the mapping torus of ϕ, and thus the mapping torus of ϕ^n is the n-fold cyclic cover of the exterior of K. The pseudo-Anosov constant λ of a surface diffeomorphism is a description of how that diffeomorphism stretches the surface. We first show there is an upper bound on $\widehat{HFK}$.

Theorem 3.1.2. *If g is the genus of a Seifert surface of a fibered hyperbolic knot K, then $c\lambda^{gn} \geq \operatorname{rk} \widehat{HFK}(\Sigma^n(K), \widetilde{K})$ for some constant c where $\widetilde{K}$ is the preimage of K in $\Sigma^n(K)$.*

Corollary 3.1.3. *If g is the genus of a Seifert surface of a fibered hyperbolic knot K, then $c\lambda^{gn} \geq \operatorname{rk} \widehat{HF}(\Sigma^n(K))$ for some constant c.*

Proof. This immediately follows from Theorem 3.1.2 and the existence of a spectral sequence from $\widehat{HFK}(\Sigma^n(K), \widetilde{K})$ to $\widehat{HF}(\Sigma^n(K))$, as can be found in [46, Lemma 3.6]. □

Note that Theorem 3.1.2 and Corollary 3.1.3 are the first upper bounds on the growth rate of $\widehat{HFK}(\Sigma^n(K), \widetilde{K})$ and $\widehat{HF}(\Sigma^n(K))$ respectively.

Proof of Theorem 3.1.2. We follow the proof of [40, Corollary 4.2]. Let ϕ be the pseudo-Anosov monodromy associated to K with dilatation λ. Take the canonical bordered Heegaard diagram for ϕ [41, Definition 5.35] which closes up to a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ for K. Note that the Seifert surface F of K sits as a subspace of Σ and that the generators of $\widehat{CFK}(\Sigma^n(K), \widetilde{K}, m-g)$, where $m-g$ is the Alexander grading, are in bijection with m-tuples of points $(x_1, ..., x_m)$ where for each i there are distinct j_i, k_i such that $x_i \in \alpha_{j_i} \cap \phi^n(\beta_{k_i})$. Now [18, Theorem 12.2] shows that the intersection numbers in pseudo-Anosov maps grow exponentially with the iteration; more specifically,

$$\lim_{n\to\infty} \frac{i(\phi^n(\beta_j), \alpha_k)}{\lambda^n} = \mu_s(\beta_j)\mu_u(\alpha_k),$$

where μ_s refers to the transverse measure of the stable foliation of ϕ and μ_u refers to the transverse measure of the unstable foliation of ϕ. Now $\mu_s(\beta_j)\mu_u(\alpha_k) \neq 0$; if $\mu_s(\beta_j) = 0$ then we could get a reducing curve by connecting the endpoints of β_j along the boundary of our Heegaard surface (and similarly for $\mu_u(\alpha_k)$). We thus have that $\lim_{n\to\infty}(\mathrm{rk}\,\widehat{CFK}(\Sigma^n(K), \widetilde{K}, m-g))^{1/n} = c\lambda^m$ for all m for some constant c depending on K. There is a symmetry between Alexander gradings j and $-j$, so the maximal of these growth rates is when $m = g$. We then take homology to get the desired result. $\square$

3.1.3 Ordinary Covers of Mapping Tori

One can prove a sharper theorem for ordinary covers of mapping tori of pseudo-Anosov maps when you restrict to particular spin^c structures.

Theorem 3.1.4. *Let $\phi : \Sigma_g \to \Sigma_g$ be a pseudo-Anosov mapping class of a surface Σ_g of genus g, with dilatation λ. Let T_ϕ denote the mapping torus of ϕ and $[\Sigma] \in H_2(T_\phi)$ the homology class of a fiber. Let*

$$\widehat{HF}(T_\phi; 2-g) := \bigoplus_{\{\mathfrak{s} \in \mathrm{spin}^c(T_\phi) | \langle c_1(\mathfrak{s}), [\Sigma] \rangle = 4-2g\}} \widehat{HF}(T_\phi; \mathfrak{s}).$$

Then

$$\lim_{n \to \infty} \left(\mathrm{rk}\, \widehat{HF}(T_{\phi^n}; 2-g) \right)^{1/n} = \lambda.$$

To prove this theorem, we use a result of Cotton-Clay [15, Theorem 5.1] to establish a version of this theorem for fixed point Floer cohomology. We then combine this result with a string of isomorphisms to prove the theorem. First, we cover the isomorphisms needed.

Let Σ_g be a closed surface of genus g, equipped with an area form ω. Given a diffeomorphism ϕ from Σ_g to itself, let T_ϕ denote the mapping torus of ϕ. If ϕ is area-preserving then ω induces a class $w_f \in H^2(Y; \mathbb{R})$. Let $T_v T_\phi$ be the vertical tangent bundle to T_ϕ, i.e. the tangent distribution to the surfaces Σ. Following Seidel [56], call an area-preserving diffeomorphism ϕ as above *monotone* if w_f is a multiple of the Euler class $e(T_v T_\phi)$, i.e. $w_f = -\kappa e(T_v T_\phi)$. By [56, Section 2], every diffeomorphism is isotopic to a monotone one.

Proposition 3.1.5. *Let Σ_g be a closed surface of genus $g \geq 2$ and let $\phi\colon \Sigma_g \to \Sigma_g$ be a monotone symplectomorphism. Let T_ϕ denote the mapping torus of ϕ. Then the following $\mathbb{F}_2[U]$-modules are isomorphic:*

(F1) The fixed point Floer cohomology $HF^*(\phi)$ of ϕ.

(F2) The direct sum of periodic Floer cohomology groups

$$\bigoplus_{\Gamma\cdot[\Sigma]=1} HP^*(\phi,\Gamma),$$

where the sum is over monotone homology classes with intersection number 1 with the fiber.

(F3) The direct sum of embedded contact cohomology groups

$$\bigoplus_{\{\mathfrak{s}\in \mathrm{spin}^c(T_\phi) \mid \langle c_1(\mathfrak{s}),[\Sigma]\rangle = 4-2g\}} ECH^*(T_\phi,\mathfrak{s}).$$

(F4) The direct sum of monopole Floer homology groups [35]

$$\bigoplus_{\{\mathfrak{s}\in \mathrm{spin}^c(T_\phi) \mid \langle c_1(\mathfrak{s}),[\Sigma]\rangle = 4-2g\}} \widecheck{HM}(-T_\phi,\mathfrak{s}).$$

(F5) The direct sum of Heegaard Floer homology groups

$$\bigoplus_{\{\mathfrak{s}\in \mathrm{spin}^c(T_\phi) \mid \langle c_1(\mathfrak{s}),[\Sigma]\rangle = 4-2g\}} HF^+(-T_\phi,\mathfrak{s}).$$

Proof. For a similar list of isomorphisms, see Kotelskiy [34, Section 1.2].

Kutluhan-Lee-Taubes show [37, Main Theorem] that

$$HF^+(Y, \mathfrak{s}) \cong \widecheck{HM}(Y, \mathfrak{s}, c_b),$$

where $\widecheck{HM}(Y, \mathfrak{s}, c_b)$ denotes monopole Floer homology with a balanced perturbation [35, Section 30]. In fact, for the variant $\widecheck{HM}$,

$$\widecheck{HM}(Y, \mathfrak{s}, c_b) \cong \widecheck{HM}(Y, \mathfrak{s})$$

[35, Section 31.1] (the same holds for $\widehat{HM}$ if $\mathfrak{s}$ is non-torsion). Thus,

$$\bigoplus_{\{\mathfrak{s} \in \mathrm{spin}^c(T_\phi) | \langle c_1(\mathfrak{s}), [\Sigma] \rangle = 4-2g\}} HF^+(-T_\phi, \mathfrak{s}) \cong \bigoplus_{\{\mathfrak{s} \in \mathrm{spin}^c(T_\phi) | \langle c_1(\mathfrak{s}), [\Sigma] \rangle = 4-2g\}} \widecheck{HM}(-T_\phi, \mathfrak{s}). \tag{3.1}$$

Lee-Taubes [39] give an isomorphism between periodic Floer homology and monopole Floer homology, and deduce an isomorphism between periodic Floer homology and embedded contact homology. They call a class $\Gamma \in H_1(T_\phi; \mathbb{Z})$ *positively monotone* if

$$[w_f] = -\lambda\Big(2\,\mathrm{PD}(\Gamma) + e(T_v T_\phi)\Big)$$

for some $\lambda > 0$. Since ϕ is monotone, this is equivalent to $\mathrm{PD}(\Gamma)$ being a multiple of $[w_f]$ and $\Gamma \cdot [\Sigma] < g - 1$.

Then, for $\Gamma \in H_1(T_\phi;\mathbb{Z})$ a positive monotone class with respect to w_f,

$$ECH_*(T_\phi,\xi;\Gamma - h_\xi) \simeq HP_*(\phi,\Gamma) \tag{3.2}$$

where ξ is a positively-oriented contact structure on T_ϕ and

$$h_\xi = \mathfrak{s}(\xi) - \mathfrak{s}(T_v T_\phi) \in H^2(Y) = H_1(Y)$$

is the difference between the spin^c-structure associated to ξ and the spin^c-structure associated to the vertical tangent bundle [39, Corollary 1.4].

Lee-Taubes also observe that if ϕ is monotone then

$$HF(\phi) \cong \bigoplus_{\Gamma\cdot[\Sigma]=1} HP(\phi,\Gamma) \tag{3.3}$$

[39, Appendix B], where the sum is over monotone classes. They also prove that for $\Gamma\cdot[\Sigma]=1$,

$$HP(\phi,\Gamma) \simeq \widehat{HM}^{-*}(Y,\mathfrak{s}_\Gamma) \tag{3.4}$$

where $\mathfrak{s}_\Gamma = \mathfrak{s}(T_v T_\phi) + \mathrm{PD}(\Gamma)$ [39, Proof of Corollary 6.7]. Taking a direct sum, this gives

$$\bigoplus_{\Gamma\cdot[\Sigma]=1} HP(\phi,\Gamma) \simeq \bigoplus_{\langle c_1(\mathfrak{s}),[\Sigma]\rangle = 4-2g} \widehat{HM}^{-*}(Y,\mathfrak{s}_\Gamma). \tag{3.5}$$

(Lee-Taubes do not explicitly show that the isomorphisms (3.2) and (3.4) respect the U-action, but it follows from the same argument [36, 38]. The fact that the isomorphism (3.3) respects the U-action is immediate from the definitions.)

Taubes' [60, 61, 62, 63, 64] show that:

$$ECH_*(Y,\mathfrak{s}) \cong \widehat{HM}^{-*}(Y,\mathfrak{s}) \tag{3.6}$$

(see also [39, p. 95]), where the right hand side is monopole Floer cohomology. There is a duality result

$$\widehat{CM}^{-*}(Y,\mathfrak{s}) \cong \mathrm{Hom}(\widecheck{CM}_*(-Y,\mathfrak{s}),\mathbb{F}_2),$$

where these are the chain complexes computing monopole Floer homology [35, Section 22.5]. Thus, Equation (3.6) dualizes to

$$ECH^*(T_\phi,\mathfrak{s}) \cong \widecheck{HM}_*(-T_\phi,\mathfrak{s}). \tag{3.7}$$

To summarize, (F1) is isomorphic to (F2) by (dualizing) Equation 3.3, (F2) is isomorphic to (F4) by (dualizing) Equation (3.5), (F4) is isomorphic to (F3) by Equation (3.7), and (F4) is isomorphic to (F5) by Equation (3.1). □

Corollary 3.1.6. *With notation as in Proposition 3.1.5, let*

$$HF^+(T_\phi, 2-g) := \bigoplus_{\{\mathfrak{s}\in\mathrm{spin}^c(T_\phi)\mid\langle c_1(\mathfrak{s}),[\Sigma]\rangle=4-2g\}} HF^+(T_\phi,\mathfrak{s}).$$

If ϕ is not isotopic to the identity map then the action of U on $HF^+(T_\phi, 2-g)$

is trivial (i.e., $U\mathbf{x} = 0$ for all $\mathbf{x} \in HF^+(T_\phi, 2-g)$).

Proof. Seidel proved this result for $HF_*(\phi)$ in [56, Theorem 1]. By Proposition 3.1.5 then, it is also true for $HF^+(T_\phi, 2-g)$. □

We can give a more direct proof of a somewhat weaker statement. Note that one can see this lemma as something like the adjunction inequality from [47].

Lemma 3.1.7. *With notation as in Corollary 3.1.6 the action of U^2 on $HF^+(T_\phi, 2-g)$ is trivial.*

Proof. Jabuka and Mark [30, Theorem 4.4] proved this in the case that ϕ is the identity map. For the case when ϕ is nontrivial, we will use Zemke's work on graph cobordism maps [66]. A graph cobordism is the data of a smooth 4-manifold W that is a cobordism from Y_1 to Y_2 with a certain type of embedded graph G called a *ribbon graph* in W and spin^c structure $\mathfrak{s}$ on W. To this data, along with a coloring $(\sigma, \mathfrak{P})$ of G, Zemke associates a map from the Heegaard Floer homology of Y_1 with $\mathfrak{s}$ restricted to Y_1 to the Heegaard Floer homology of Y_2 with $\mathfrak{s}$ restricted to Y_2. Moreover, [66, Theorem E] shows that graph cobordism maps compose when gluing graph cobordisms together, in the following sense:

Theorem E. *Suppose that (W, G) is a graph cobordism with coloring $(\sigma, \mathfrak{P})$. Suppose that (W, G) decomposes as $(W, G) = (W_2, G_2) \cup (W_1, G_1)$. If $\mathfrak{s}_i$ is a* spin^c *structure on* W_i*, then we have the following:*

$$F^{\circ}_{W_2,G_2,\sigma|_{G_2},\mathfrak{P},\mathfrak{s}_2} \circ F^{\circ}_{W_1,G_1,\sigma|_{G_1},\mathfrak{P},\mathfrak{s}_1} = \sum_{\substack{\mathfrak{s}\in \mathrm{spin}^c(W)\\ \mathfrak{s}|_{W_2}=\mathfrak{s}_2\\ \mathfrak{s}|_{W_1}=\mathfrak{s}_1}} F^{\circ}_{W,G,\sigma,\mathfrak{P},\mathfrak{s}}.$$

For us, the set of colors $\mathfrak{P}$ consists of a single element, and so the map σ is trivial, hence we will suppress these from the rest of the lemma.

Now consider the 4-manifold $W = (T_\phi \times [0,1]) \setminus nb(\Sigma_g \times \{1/2\})$. Take the following as the graph G embedded in W: pick $x \in T_\phi$ and let $G = (x \times [0,1]) \bigcup \{\text{line segment from } (x,1/2) \text{ to } y \in S^1 \times \Sigma_g\}$. While there are different choices of line segment for this, any choice suffices for the rest of the proof. Consider this as graph cobordism from $T_\phi \bigsqcup (S^1 \times \Sigma_g)$ to T_ϕ. Let us then fill in $S^1 \times \Sigma_g$ in W with $D^2 \times \Sigma_g$ to form W'. Take G' to be the graph for W' where $G' = G$. Note $W' = T_\phi \times [0,1]$ as a manifold, but as a graph cobordism it is not quite trivial; it has an extra "leg" on its graph from the filled in $S^1 \times \Sigma_g$ factor in W. Let G'' be the trivial graph $x \times [0,1]$, and W'' the graph cobordism $(T_\phi \times [0,1], G'')$.

For our spin^c class, let $\mathfrak{s} \in \mathrm{spin}^c(T_\phi)$ be such that $\langle c_1(\mathfrak{s}), [\Sigma_g]\rangle = 4 - 2g$. This uniquely extends to a spin^c class in $T_\phi \times [0,1]$ and thus in W', which we shall therefore also call $\mathfrak{s}$; take $\mathfrak{s}|_W$ for our spin^c class in W. Since $\langle c_1(\mathfrak{s}), [\Sigma_g]\rangle = 4 - 2g$, we know that $\mathfrak{s}|_{S^1\times\Sigma_g}$ is in the second to lowest spin^c structures.

Now by [66, Theorem C] graph cobordisms induce maps on Heegaard

Floer homology. As the underlying manifold of W' and W'' is $T_\phi \times [0,1]$, by [66, Lemma 12.3] these maps for W' and W'' depend only on their underlying graphs G' and G''; these maps are called $A_{G'}$ and $A_{G''}$ respectively by Zemke. It is easy to show that $A_{G''}$ is the identity. Moreover, by [66, Lemma 7.11] $A_{G''} = A_{G'} \circ S_v^-$, where S_v^- is a map associated to adding the vertex needed to get from G'' to G'; therefore since $A_{G''}$ is clearly surjective so is $A_{G'}$.

But as $\mathfrak{s} \in \operatorname{spin}^c(T_\phi)$ extends uniquely to $T_\phi \times [0,1]$, [66, Theorem E] says that graph cobordisms compose when gluing graph cobordisms together; thus as the map associated to W' is surjective so is the map associated to W. But by Jabuka and Mark's result [30, Theorem 4.4] we have that U^2 acts by 0 on $HF^+(S^1 \times \Sigma_g, \mathfrak{s}|_{S^1\times\Sigma_g})$ due to $\mathfrak{s}|_{S^1\times\Sigma_g}$ being in the second to lowest spin^c structure. Since Zemke's cobordism map is U-equivariant we are done. □

Proof of Theorem 3.1.4. Let $HF^*(\phi)$ be the fixed point Floer cohomology of a pseudo-Anosov map ϕ, let τ be the invariant train track for ϕ from [5], and let M be the transition matrix (called the incidence matrix in [15]). Note that M is the matrix whose entry $M_{i,j}$ is the number of times that ϕ of the ith branch of τ runs over the jth branch of τ. This matrix is a Perron-Frobenius matrix with dominant eigenvalue λ, the dilatation of ϕ, as seen in [6].

Then [15, Theorem 5.1] states that the rank of $HF^*(\phi)$ is equal to $\operatorname{Tr}(M)+S$, where S is a correction term which can be bounded based on the invariant train track τ. Moreover, it is clear from the definition that the transition matrix of ϕ^n is M^n. As λ is the eigenvalue of M with the largest absolute value, we thus get that $\lim_{n\to\infty}(\operatorname{rk} HF^*(\phi^n))^{1/n} = \lambda$ (see also Fel'shtyn [19]).

We then use Proposition 3.1.5 to get the theorem for $HF^+(T_\phi, 2-g)$. To deduce the result for $\widehat{HF}$, by Corollary 3.1.6 the U-action on $HF^+(T_\phi, 2-g)$ is trivial. There is a long exact sequence [48]

$$\ldots \longrightarrow \widehat{HF}(Y) \longrightarrow HF^+(Y) \xrightarrow{U} HF^+(Y) \longrightarrow \ldots$$

which splits into short exact sequences here as U acts by 0, thus showing that $2\,\mathrm{rk}\, HF^+(T_\phi, 2-g) = \mathrm{rk}\, \widehat{HF}(T_\phi, 2-g)$. We are therefore done. □

3.1.4 The Symplectic Polynomial

Birman-Brinkman-Kawamuro [6] define a polynomial invariant of pseudo-Anosov maps (and thus of fibered hyperbolic knots) that they call the *symplectic polynomial*. The pseudo-Anosov constant λ is always a root of this polynomial. This could lead one to conjecture that the growth rate of $\widehat{HFK}$ for a fibered hyperbolic knot K would be the Mahler measure of its symplectic polynomial, in the manner of the growth rate of H_1 being the Mahler measure of the Alexander polynomial as stated in Equation (1.1). If s_K is the symplectic polynomial of a fibered hyperbolic knot K however, by Equation (1.1) we would need $M(s_K) \geq M(\Delta_K)$ for all knots K. This is false; for example the knot 7_7 has $M(\Delta_{7_7}) \approx 5.613$ and $M(s_{7_7}) \approx 2.966$. (Here, $M(s_{7_7})$ was computed used the software package XTrain [12] as well as results from [6]. Specifically, XTrain can calculate the characteristic polynomial f_K of the transition matrix of the monodromy associated to a knot K, then by [6, Theorem 1.1] we have $M(f_k) = M(s_K)$.)

We can however prove such a proposition when substituting in Δ_K for s_K in the case where the genus of the Seifert surface of the knot is 1. Note that this isn't quite as trivial as it seems, as we are not limiting our knots to live in S^3.

Proposition 3.1.8. *Let K be a hyperbolic fibered knot with fiber F of genus 1 that has pseudo-Anosov monodromy ϕ. Let ϕ have pseudo-Anosov constant λ. Let $\widetilde{K}$ be the preimage of K in $\Sigma^n(K)$. Then*

$$\lim_{n\to\infty}(\mathrm{rk}(\widehat{HFK}(\Sigma^n(K)))^{1/n} = M(\Delta_K) = \lambda.$$

Proof. By Theorem 3.1.2 we have that $c\lambda^n \geq \mathrm{rk}(\widehat{HFK}(\Sigma^n(K), \widetilde{K}))$ for some constant c. As in Equation (1.1), by [59] we have that for all $\epsilon > 0$, and large enough n, $|\operatorname{Tor}(H_1(\Sigma^n(K)))|^{1/n} \geq M(\Delta_K) - \epsilon$. Finally, by [17, Chapter 14], for genus 1 knots we have that λ is a root of Δ_K and as Δ_K is a monic degree 2 polynomial we have $M(\Delta_K) = \lambda$. Putting this all together, we have

$$\begin{aligned}\lambda c^{1/n} &\geq (\mathrm{rk}(\widehat{HFK}(\Sigma^n(K), \widetilde{K})))^{1/n} \\ &\geq |\operatorname{Tor}(H_1(\Sigma^n(K)))|^{1/n} \geq M(\Delta_K) - \epsilon = \lambda - \epsilon.\end{aligned}$$

By taking the limit as n goes to infinity and then the limit as ϵ goes to zero, we get the desired result. □

Note that for knots with a special kind of train track called an *orientable* *train track*, by [6, Corollary 4.2] we have that $s_K(t) = \Delta_K(t)$ or $s_K(t) = \Delta_K(-t)$; thus $M(s_K) = M(\Delta_K)$ and so $\widehat{HFK}$ does grow like $M(s_K)$ for genus 1 hyperbolic knots whose monodromy has an orientable train track.

3.2 Possible Directions for Future Research

3.2.1 Extending the SKh Spectral Sequence

The spectral sequence for the sutured annular Khovanov homology of periodic links proved in Chapter 2 and [14], and the spectral sequence proven by Zhang in [67], only applies to links of period 2. It should be possible to prove such spectral sequences for links that are n-periodic, or at least for n of the form p^r where p is prime.

Conjecture 3.2.1. *Let T be an $(m+1, m+1)$ tangle, and $\widehat{T}, \widehat{T^n} \subset A \times I$ denote the annular closures of T and T^n respectively. Then there is a spectral sequence which has E^1 page isomorphic to $SKh(\widehat{T^n})$ and E^∞ page isomorphic to the associated graded algebra of a filtration of $SKh(\widehat{T})$.*

Zhang also conjectured [67, Conjecture 1] that there should be a spectral sequence from the ordinary Khovanov homology of a 2-periodic link L to the sutured Khovanov homology of its quotient $\widetilde{L}$. If such a spectral sequence exists it would prove a cascade of rank inequalities

$$\operatorname{rk} SKh(L) \geq \operatorname{rk} Kh(L) \geq \operatorname{rk} SKh(\widetilde{L}) \geq \operatorname{rk} Kh(\widetilde{L}).$$

In particular, this would be stronger than Seidel-Smith's rank inequality.

3.2.2 Growth Rate of SKh

Take a link L and an unknot disjoint from L. Take the n-fold branched cover of the unknot, and call the preimage of L under this map L^n. Notice that L^n has period n and quotient L. If $n = 2$ then by the spectral sequence of Zhang in [67] we know that $\text{rk}\, SKh(L^n) \geq \text{rk}\, SKh(L)$. If one proves Conjecture 3.2.1 then it would follow that $\text{rk}\, SKh(L^n)$ is at least non-decreasing as n increases. However an interesting question would be whether one could say anything about the growth rate of $\text{rk}\, SKh(L^n)$.

3.2.3 Floer Homology Growth Rates

Using an exact sequence found in Hedden and Mark [26] it should be possible to extend Theorem 3.1.4 to branched covers rather than ordinary covers. Additionally from Hedden and Mark [26] we can get lower bounds for the growth rate of fibered knots. However all known upper bounds are far higher than this lower bound. This relates to questions arrising from Theorem 3.1.1:

Question 3.2.2. *Are there examples of hyperbolic knots which are not asymptotically covered by L-spaces? Are there examples of knots which are not covered by L-spaces but are asymptotically or approximately asymptotically covered by L-spaces? Are there examples of knots which are not asymptotically covered by L-spaces but are approximately asymptotically covered by*

L-spaces?

3.2.4 L-space Conjecture

The L-space conjecture, Conjecture 1.2.1, is one of the motivations for the growth rate problems stated above. Gordon showed in [21] that if K is a 2-bridge knot with nonzero signature then $\Sigma^n(K)$ has a left-orderable fundamental group if n is sufficiently large. If the L-space conjecture is true $\Sigma^n(K)$ cannot be an L-space for these n. Knowing more about the growth rate of $\widehat{HF}$ for such knots is one way to prove this.

Other questions that could possibly be answered using growth rate include: if $\Sigma^n(K)$ is not an L-space, can $\Sigma^{n+1}(K)$ be an L-space? Indeed, there are no known knots K where $\Sigma^6(K)$ is not an L-space but some $\Sigma^n(K)$ for $n > 6$ is an L-space; are there any such knots? Some evidence that this is true can be found in Boileau-Boyer-Gordon [7]. Showing that there is a spectral sequence from $\widehat{HF}(\Sigma^{mn})$ to $\widehat{HF}(\Sigma^n)$, analogous to the spectral sequence in Hendricks [27], would be a way to start answering these questions, as this would show that if Σ^n is not an L-space then Σ^{mn} is not an L-space.

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