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Ambiguous Events and Maxmin Expected Utility

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We study the properties associated to various definitions of ambiguity ([8], [9], [18] and [23]) in the context of Maximin Expected Utility (MEU). We show that each definition of unambiguous events produces certain restrictions on the set of priors, and completely characterize each definition in terms of the properties it imposes on the MEU functional. We apply our results to two open problems. First, in the context of MEU, we show the existence of a fundamental incompatibility between the axiom of "Small unambiguous event continuity" ([8]) and the notions of unambiguous event due to Zhang [23] and Epstein-Zhang [8]. Second, we show that, in the context of MEU, the classes of unambiguous events according to either Zhang [23] or Epstein-Zhang [8] are always λ-systems. Finally, we reconsider the various definitions in light of our findings, and identify some new objects (Z-filters and EZ-filters) corresponding to properties which, while neglected in the current literature, seem relevant to us.

Key Words: ambiguous events, maxmin expected utility.
JEL classification number: D81

1. INTRODUCTION

The idea of Ambiguity has been central to the research in decision theory for several decades. The story is well-known. On one hand, Savage's theory [21] postulates that a decision maker be able to assign probabilities to all events. On the other hand, it is hard to dismiss the intuition that in many situations the information available to the decision maker might be insufficient for doing so. Roughly speaking, "Ambiguity" refers to these situations, and the classic experiments by Ellsberg [7] have convincingly demonstrated its empirical relevance.

The conflict featuring Savage's theory on one side and the idea of Ambiguity on the other, has generated spectacular theoretical developments: Choquet Expected Utility [22], Maxmin Expected Utility (henceforth, MEU) [11] and several generalizations of the latter ([9], [14], [17]). All these models allow for modes of behavior that are not necessarily consistent with Savage's Subjective Expected Utility theory. In particular, they can accommodate behavior of the type observed by Ellsberg. Yet, in most of this work, the idea of Ambiguity has remained in the background: more an inspirational muse rather than a central, fully spelled out concept.

[1] We are grateful to Massimo Marinacci and Klaus Nehring for useful discussions.

The formalization of the concept of Ambiguity is a more recent matter, mostly of the past decade. Since the idea refers to situations where not all events are assigned probabilities, the goal has been that of characterizing such events. These are called ambiguous, and all the others are termed unambiguous. As of today, several definitions have been proposed, amended, criticized and the literature, in spite of its recent start, is already quite sizeable. We refer the reader to [8] and [9] for some of the history of the problem.

This wealth of definitions as well as the richness of the debate sorrounding them (see, for instance, [8], [9], [10], [13], [15], [18], [19], [20], [23]) motivate the present work. Our goal is to contribute to the debate by providing a new way of looking at the problem. We do so by studying the properties associated to various definitions of ambiguity within the context of a familiar model like MEU. The novelty of our approach consists in the realization that the demand that a certain event be unambiguous is equivalent to the demand that the MEU functional – and, hence, the set of priors – display certain properties. As different definition of "ambiguous events" lead to different properties, one achieves a better understanding of those very definitions. All the more, the focus of the debate can shift toward the desiderabilty of these properties.

Our results admit a dual reading. Let x be a certain definition of Ambiguous events, and let T be an event. Our results say that T is x-unambiguous if and only if the MEU functional displays a certain property. Alternatively, one can view our results as follows. For x a definition of Ambiguous events, let $\{A_i\}$ be a certain collection of events that are termed at the outset as x-unambiguous. Then, our study shows (a) whether or not there exists a MEU model compatible with this situation; and, in the affirmative case, that (b) the set of priors defining the MEU functional must have a certain form. The latter case can be axiomatized by introducing an additional axiom in the familiar MEU setting. We notice this only occasionally.

In the final sections, we apply our results to study two open problems in the literature on ambiguous events. The first regards the axiom of "Small unambiguous event continuity" introduced in [8]. The desiderability of the axiom, which is crucial for the derivation of a unique probability measure on the classes of Zhang and Epstein-Zhang unambiguous events, has recently been questioned (see, in particular, [15] and [20]). In the context of MEU, our main result states the existence of a fundamental incompatibility between this axiom and the notions of unambiguous events due to Zhang and Epstein-Zhang (Section 6). The second problem regards the properties of the classes of unambiguous events according to [8] and [23], respectively. Both classes were originally claimed to be λ-systems. Later, Kopylov [15] showed that, in general, they are only mosaics (a notion he introduced). Yet, the problem of what conditions guarantee that those classes are, in addition, λ-systems has been left open. Here, we show that, in the context of MEU, those classes are indeed λ-systems.

Finally, we reconsider the various definitions of ambiguity in light of our findings, and identify some new objects (Z-filters and EZ-filters) corresponding to properties which, while neglected in the current literature, seem relevant to us.

2. SETTING

Our goal is to achieve a deeper understanding of the meaning of various definitions of ambiguous events proposed in the literature. To study the properties of unambiguous events in the context of a well-understood model like MEU seems to us one but an obvious step in this direction. Ultimately, differences across various definitions are better appreciated in terms of different properties they generate. In pursuing our goal, however, we need to avoid spurios differences which might emerge solely as a consequence of the model we are using. Put, in a different way, we should guarantee that (still within MEU) the setting we choose is as "neutral" as possible with respect to the various definitions. The crux of the matter, here, is that both Zhang [23] and Epstein-Zhang [8] impose, in their work, axioms leading to the existence of a countably additive, nonatomic probability measure on the class of unambiguous events. For other definitions, like Nehring's or Ghirardato-Maccheroni-Marinacci's, this is possibile – in the context of MEU – only if all the priors are both countably additive and nonatomic and only if the class of unambiguous events is sufficiently rich.

To guarantee that these properties can be satisfied, we need to impose three more axioms on the original Gilboa-Schmeidler's MEU axiomatization. Two of these have been recently identified by Chateauneuf-Maccheroni-Marinacci-Tallon [5] and ensure that all the priors in the representation are both countably additive and nonatomic. They also ensure that there exists a prior with respect to which all the others are absolutely continuous. We refer the reader to [5] (in particular, Lemma 3) for the full specification of the model and the proof of these facts.

The third axiom, which delivers the richness of the class of unambiguous events either in the sense of Nehring or Ghirardato-Maccheroni-Marinacci, requires a bit of notation. Let $\mathcal{C}$ be the set of priors in the representation. As noticed ([5], Lemma 3), there exists a prior in $\mathcal{C}$, say ν, such that all other priors are absolutely continuous with respect to ν. Hence, for each prior there exists a density (with respect to ν) and the set $\mathcal{C}$ is isometrically isomorphic to a set of densities $\mathcal{D} \subset \mathcal{L}^1(S, \Sigma, \nu)$.

Let $\mathcal{D}^\perp = \left\{\phi \in \mathcal{L}^\infty(S, \Sigma, \nu) \mid \int g\phi d\nu = 0 \text{ for all } g \in \mathcal{D}\right\}$, and let $\mathcal{D}^\perp(A)$ be the set of all ϕ's vanishing a.e. on the complement of the measurable set A.

Definition 1 (Kingman and Robertson [12]). $\mathcal{D}$ is said to be *thin* if and only if $\mathcal{D}^\perp(A)$ is different from the zero subspace whenever $\nu(A) > 0$.

We say that a set of priors $\mathcal{C}$ is thin if the corresponding set of densities is, and we restrict to preferences for which the set of priors is always thin. In the context of Chateauneuf-Maccheroni-Marinacci-Tallon's axiomatization, it was shown in [2] (Proposition 10) that there exists a countably additive, nonatomic probability measure on the class of Ghirardato-Maccheroni-Marinacci (equivalently, Nehring's) unambiguous events if and only if the assumption is satisfied.

We conclude this section by stating the notation that we use throughout the paper. S denotes the state space, Σ is a $\sigma-$algebra of events in S and Y is the prize space, which we assume is a mixture space ([11] and [5]). The set of acts is $\mathcal{F} = \{f : S \to Y \mid f \text{ is simple and } \Sigma-\text{measurable}\}$, with the generic elements $f, g, h, ...$etc. The decision maker has a preference relation, $\succcurlyeq$, on $\mathcal{F}$, which is represented by the functional $I : \mathcal{F} \to \mathbb{R}$ defined by

$$I(f) = \min_{P \in \mathcal{C}} \int_S u(f(s))dP(s) \tag{1}$$

where $\mathcal{C}$ is a convex and weak*-compact set of finitely additive probability measures on (S, Σ), and $u : Y \to \mathbb{R}$ is a linear utility on the prize space. As noted, the axioms introduced in [5] guarantee that these probabilities are, in addition, countably additive and nonatomic. The utility function u produces an embedding of the acts into the set $B_0(\Sigma)$ of bounded real-valued simple functions on S, and the functional I (and, hence, the preference relation) is extended to the whole $B_0(\Sigma)$ in the obvious way ([11]). Because of this, from now on we drop any reference to the utility function (that is, from now on an act f if viewed as a real-valued function).

3. DEFINITIONS OF AMBIGUOUS EVENTS

We distinguish between two groups of definitions of ambiguous events. One group features the definitions given by Nehring [18] and Ghirardato, Maccheroni and Marinacci [9], the other those given by Zhang [23] and Epstein and Zhang [8]. This classification is motivated solely by the structure displayed by the definitions, and not by other considerations.

In the context of Multiple Prior models, the definitions of ambiguous events given by Nehring and Ghirardato, Maccheroni and Marinacci are known to be equivalent, and correspond to the following

Definition 2. An event $T \in \Sigma$ is unambiguous if $\forall P, Q \in \mathcal{C}$, $P(T) = Q(T)$. Otherwise it is ambiguous.

Since we are going to be dealing with different notions of ambiguous events, we need a name to distinguish these unambiguous events from the others we will meet later. We borrow from Dubins and Margolis [6],[2] and we call *naturally measurable* all the events that are unambiguous in the sense of Definition 2. The class of naturally measurable events is denoted by $\mathcal{A}_{NM}$.

It is a straigtforward consequence of the definition that the class $\mathcal{A}_{NM}$ is closed under the operation of taking complements, and that it is, in fact, a λ-system ([9]). Under the *thinness* assumption of the previous section, it contains events of measure α, for any $\alpha \in [0, 1]$ (see [2]; this measure is univoquely defined because of the very nature of naturally measurable events).

Naturally measurable events admit the following behavioral characterization, which is proven in [9]. If f is a function on S and w is a real number (recall that acts have been identified to real-valued functions), we denoted by fTw the real-valued function on S which coincides with f on T and is identically equal to w on T^c (the complement of T).

Proposition 1 ([9]). *T is naturally measurable if and only if for any $f, g \in \mathcal{F}$, $w, w' \in \mathbb{R}$ and for any $\alpha \in (0, 1)$*

$$f \sim g \qquad \Longrightarrow \qquad \alpha f + (1-\alpha)(wTw') \sim \alpha g + (1-\alpha)(wTw')$$

The next two definitions are stated directly in behavioral terms. The definition proposed by Zhang [23] reads as follows

Definition 3. An event $T \in \Sigma$ is unambiguous if $\forall f, g \in \mathcal{F}$, $w \in \mathbb{R}$

$$fTw \succcurlyeq gTw \Longrightarrow fTw' \succcurlyeq gTw' \text{ for any } w' \in \mathbb{R} \tag{2}$$

and the same implication holds for T^c. Otherwise T is ambiguous.

[2] While introduced in a different context, the terminology seems especially suited to us (see [6]).

The definition given in Epstein-Zhang [8] displays a similar structure but it is weaker in that limits the comparison (2) only to a certain subset of $\mathcal{F} \times \mathcal{F}$. For $T \in \Sigma$, let A and B be two disjoint subsets of T. Consider an act f of the form

$$f = \begin{cases} y^* & if \quad s \in A \\ y & if \quad s \in B \\ z(s) & if \quad s \in S \setminus (A \cup B) \end{cases}$$

where y^* and y are real numbers and $z(s) = f\,|_{S \setminus (A \cup B)}$. An *EZ-conjugate* of f is an act $\bar{f}$ defined by

$$\bar{f} = \begin{cases} y & if \quad s \in A \\ y^* & if \quad s \in B \\ z(s) & if \quad s \in S \setminus (A \cup B) \end{cases}$$

that is, $\bar{f}$ is obtained from f by exchanging the prizes on A and B. The definition given in Epstein and Zhang reads as follows

Definition 4. An event $T \in \Sigma$ is unambiguous if $\forall f \in \mathcal{F}$, $w \in \mathbb{R}$ and for any conjugate $\bar{f}$ of f

$$fTw \succcurlyeq \bar{f}Tw \Longrightarrow fTw' \succcurlyeq \bar{f}Tw' \text{ for any } w' \in \mathbb{R}$$

and the same implication holds for T^c. Otherwise T is ambiguous.

The restriction to comparisons involving only conjugate acts is motivated by the intuition that an event T should be termed unambiguous if (and only if) the relative conditional likelyhood of any two of its subevents, A and B, is invariant with respect to changes in the prize on T^c (see [8]).

The classes of unambiguous events according to Zhang and Epstein-Zhang are denoted by $\mathcal{A}_Z$ and $\mathcal{A}_{EZ}$, respectively. These two notions of ambiguity have recently found a very elegant treatment in the work of Kopylov [15]. There, among other things, Kopylov shows that the two classes are mosaics and noticed that they are not necessarily λ-systems.

At any rate, in the setting we study, there is an obvious relation among the three notions, which we state in the next proposition.[3]

Proposition 2. $\mathcal{A}_{NM} \subseteq \mathcal{A}_Z \subseteq \mathcal{A}_{EZ}$.

As these inclusions come straight from the definitions, we do not provide a proof (however, many of the results we present later implicitly contain a proof of this fact).

It is well-known that, in the setting we consider, the inclusion $\mathcal{A}_{NM} \subseteq \mathcal{A}_Z$ may be strict. The following example is due to Nehring [20].

Example 1 (Nehring [20]). Fix an event $T \in \Sigma$. Let Π_1 and Π_2 denote two weak*-closed and convex sets of finitely additive probability measures supported by

[3] Obviously, the inclusion $\mathcal{A}_Z \subseteq \mathcal{A}_{EZ}$ is true in general.

T and T^c, respectively. Fix α, β such that $0 < \alpha < \beta < 1$. Define the weak*-closed and convex set Π as follows:

$$\Pi \equiv [\alpha, \beta]\Pi_1 + [1-\beta, 1-\alpha]\Pi_2 \equiv \{\gamma\pi_1 + (1-\gamma)\pi_2 \mid \alpha \leq \gamma \leq \beta,\ \pi_1 \in \Pi_1,\ \pi_2 \in \Pi_2\}$$

and let the preference $\succcurlyeq$ be the one induced by Π according to the MEU functional (1).

The reader can readily check that T is Z-unambiguous, and hence EZ-unambiguous, but not naturally measurable because $\alpha \neq \beta$.[4]

Nehring's example goes a long way beyond showing that the inclusion $\mathcal{A}_{NM} \subseteq \mathcal{A}_Z$ is strict. In fact, as we shall see (Theorem 4), it provides remarkable insights into the structure of the events which are unambiguous either according to Zhang or to Epstein and Zhang.

We study naturally measurable events in the next section, and Zhang and Epstein-Zhang unambiguous events in Section 5. The unifying theme is the realization that – for x a certain definition of unambiguous event – an event is x-unambiguous if and only if the set of priors (and hence the maxmin functional) displays a certain structure. These findings will clarify both the differences and the similarities existing across the various definitions we consider. In Section 6, we apply our results to address two issues. Namely, whether or not the classes $\mathcal{A}_Z$ and $\mathcal{A}_{EZ}$ are λ-systems and to which extent Zhang's and Epstein-Zhang's definitions are compatible with the axiom of "small unambiguous event continuity" ([8], Axiom 4). Section 7 contains some concluding remarks. Proofs are in Appendix. Section A.* in Appendix refers to material contained in Section * in the main text.

4. NATURALLY MEASURABLE EVENTS

The main result of this section (Theorem 1) states the existence of a certain relation between naturally measurable events and the structure of the set $\mathcal{C}$, which defines the MEU functional. Precisely, if $\Phi = \{\pi_i\}$ is a (finite) partition of S into naturally measurable events, then $\mathcal{C}$ can be written as a unique convex combination of a collection of sets $\{\mathcal{C}_{\pi_i}\}_{\pi_i \in \Pi}$, $\mathcal{C} = \sum \alpha_i \mathcal{C}_{\pi_i}$, and each measure in $\mathcal{C}_{\pi_i}$ is supported by π_i. Equivalently, given a partition into naturally measurable events, the set $\mathcal{C}$ can be decomposed into a (canonical) system of sets of conditional measures. Later in this section and more thoroughly in Section 7, we will elaborate on the interpretation attached to this type of decompositions. Notice, that in the special case $\Phi = \{T, T^c\}$, our result says that to each naturally measurable event there is associated a certain decomposition of the set of priors and, hence, a special form of the MEU functional I.

Let $\Phi = \{\pi_i\}$ be a finite partition of S with the property that each $\pi_i \in \mathcal{A}_{NM}$ (existence of such partitions is guaranteed by the thinness assumption of Section 2; see [2]).

THEOREM 1. *There exist (a) a unique collection of non-empty, weak* compact, and convex sets of priors $\{\mathcal{C}_{\pi_i}\}_{\pi_i \in \Phi}$, where for any $\pi_i \in \Phi$, $\mathcal{C}_{\pi_i} \subseteq \{P \mid P$ is a probability measure on $\pi_i\}$; and (b) a unique probability distribution q on S/Φ (the*

[4] In his original example [20], Nehring restricts to a prize space having two outcomes only. The reader can readily check that his reasoning extends without modifications to any number of prizes.

quotient of S by the partition Φ)[5] *such that for any $f \in \mathcal{F}$*

$$\min_{P \in \mathcal{C}} \int_S f dP = I(f) = \int_{S/\Phi} \left(\min_{P \in \mathcal{C}_{\pi_i}} \int_{\pi_i} f dP \right) dq(\pi_i)$$

We stress that the decomposition of Theorem 1 holds *for any* act $f \in \mathcal{F}$. With this in mind, Theorem 1 lends itself to an interesting interpretation. We can think of MEU decision maker as of someone who follows a two-step procedure. In the first step, acts are decomposed into a collection of subacts each defined on a naturally measurable event. No ambiguity is attached to any of these events and each subact is evaluated by means of a MEU functional. In the second step, all these evaluations are aggregated *linearly* by means of q.

We conclude the section by giving a behavioral counterpart to Theorem 1. This is achieved by introducing an additional axiom, which is, of course, intermediate between the c-independence axiom of Gilboa-Schmeidler [11] and the full independence axiom.

Let $\Phi = \{\pi_i\}$ be a finite partition into measurable events of (S, Σ), and let $\mathcal{F}^\Phi$ be the set of acts which are constant on each π_i. In other words, $\mathcal{F}^\Phi$ is the set of all step functions on the partition Φ. We say that a preference relation $\succcurlyeq$ on $\mathcal{F}$ satisfies the axiom of step-independence with respect to the partition Π if

Axiom (Step-independence) For any $f, g \in \mathcal{F}$, $a \in \mathcal{F}^\Phi$, and $\alpha \in [0, 1]$

$$f \sim g \qquad \Longrightarrow \qquad \alpha f + (1-\alpha)a \sim \alpha g + (1-\alpha)a$$

PROPOSITION 3. *Let a preference relation $\succcurlyeq$ on $\mathcal{F}$ satisfy axioms A1, A3 to A6 in [11] and the axiom of Step-independence with respect to a partition Φ. Then,*

(a) $\succcurlyeq$ has a MEU representation;

(b) The set of priors $\mathcal{C}$ can be decomposed with respect to Φ as in Theorem 1;

(c) $\Phi \subset \mathcal{A}_{NM}$.

While a full proof is in Appendix, it is nearly immediate to observe that the axiom of step-independence implies, along with the other axioms, the axiom of c-independence of Gilboa and Schmeidler. Hence, part (a). Then, one checks that events in Φ are naturally measurable (part (c)). Finally, that (c) implies (b) follows from Theorem 1.

5. ZHANG AND EPSTEIN-ZHANG UNAMBIGUOUS EVENTS

In this section, we study Zhang and Epstein-Zhang unambiguous events. We restrict attention to acts of the form fTw (and fT^cw) since both Zhang and Epstein-Zhang consider only acts having this form. Throughout the section, T is a fixed event and we use the notation $f(w)$ in the place of fTw. The rationale for this notational change will be clear in subsection 5.2, below. Finally, we restrict to events T such that $0 < \min P(T) \leq \max P(T) < 1$ because if T is such that either $\min P(T) = 0$ and $\max P(T) \neq 0$ or $\min P(T) \neq 1$ and $\max P(T) = 1$, then T is necessarily EZ-ambiguous (see Appendix, Proposition 8).

[5] The quotient is endowed with the finest measurable structure which makes the canonical projection measurable.

5.1. From naturally measurable events to Zhang unambiguous events

In light of the inclusion $\mathcal{A}_{NM} \subseteq \mathcal{A}_Z$ (Proposition 2), we are concerned with finding and characterizing those Zhang unambiguous events (if any) which are not naturally measurable. Our analysis takes off from the following considerations. Consider the two subsets of priors

$$\mathcal{C}_{MIN} = \{P \in \mathcal{C} \mid P(T) = \min\} \qquad \textit{and} \qquad \mathcal{C}_{MAX} = \{P \in \mathcal{C} \mid P(T) = \max\}$$

Clearly, both $\mathcal{C}_{MAX}$ and $\mathcal{C}_{MIN}$ are non-empty, convex and weak*-compact. Moreover, it is immediate to verify that either $\mathcal{C}_{MAX} \cap \mathcal{C}_{MIN} = \emptyset$ or $\mathcal{C}_{MAX} = \mathcal{C}_{MIN} = \mathcal{C}$.

By using the sets $\mathcal{C}_{MAX}$ and $\mathcal{C}_{MIN}$, we can define two functionals, $\tilde{I}$ and $\approx\!\!\!\!I$, by

$$\tilde{I}(f) = \min_{\mathcal{C}_{MAX}} \int f dP \qquad , \qquad \overset{\approx}{I}(f) = \min_{\mathcal{C}_{MIN}} \int f dP$$

The main reason for introducing $\tilde{I}$ and $\overset{\approx}{I}$ is that (as the reader can easily check) the ranking they induce on acts of the form $f(w)$ is independent of w. That is,

$$\tilde{I}(f(w)) \geq \tilde{I}(g(w)) \quad \text{for some } w \quad \Longrightarrow \quad \tilde{I}(f(w')) \geq \tilde{I}(g(w')) \quad \forall w' \in \mathbb{R}$$
$$\overset{\approx}{I}(f(w)) \geq \overset{\approx}{I}(g(w)) \quad \text{for some } w \quad \Longrightarrow \quad \overset{\approx}{I}(f(w')) \geq \overset{\approx}{I}(g(w')) \quad \forall w' \in \mathbb{R}$$

For brevity, we will refer to this property as to w-invariance. The functionals $\tilde{I}$ and $\overset{\approx}{I}$ provide us with another way to look at naturally measurable events. For if T is naturally measurable, then $\mathcal{C}_{MAX} = \mathcal{C}_{MIN} = \mathcal{C}$ and for any act f, we have

$$I(f) = \tilde{I}(f) = \overset{\approx}{I}(f)$$

Hence, the unambiguous nature of T follows at once from the observed w-invariance Incidentally, this also shows the inclusion $\mathcal{A}_{NM} \subseteq \mathcal{A}_Z$.

If T is not naturally measurable, the functional I which evaluates the acts is, in general, different from both $\tilde{I}$ and $\overset{\approx}{I}$. The next lemma, however, shows that for any act g there exists a $g(w^*)$ $[g(w^{**})]$ for which $I(g(w^*)) = \overset{\approx}{I}(g(w^*))$ $[I(g(w^{**})) = \tilde{I}(g(w^{**}))]$. Equivalently, such an act is evaluated by a prior in $\mathcal{C}_{MIN}$ $[\mathcal{C}_{MAX}]$.

Lemma 1. *For any act of the type $g(w)$,*
(a) $\exists w^*$ such that $I(g(w)) = \overset{\approx}{I}(g(w))$ for any $w \leq w^*$.
(b) $\exists w^{**}$ such that $I(g(w)) = \tilde{I}(g(w))$ for any $w \geq w^{**}$.

Lemma 1 is a basic result in our analysis as it immediately leads to uncover a number of properties associated with Zhang unambiguous events. In fact, by combining Lemma 1 with the w-invariance of $\tilde{I}$ and $\overset{\approx}{I}$, we obtain at once the following necessary condition for $T \in \mathcal{A}_Z$.

Proposition 4. *A necessary condition for $T \in \mathcal{A}_Z$ is that if $f(w') \succsim g(w')$ at some w', then both*

$$\begin{aligned} \overset{\approx}{I}(f(w')) &\geq \overset{\approx}{I}(g(w')) \\ \tilde{I}(f(w')) &\geq \tilde{I}(g(w')) \end{aligned}$$

While the proof is in the Appendix, the reader might want to defer its reading until next subsection, where we give a simple geometric explanation of the content of Proposition 4 (Figure 2).

5.2. A geometric analysis

The concepts we have seen so far lend themselves to a very simple geometric description. In fact, due to the restriction to acts of the form $f(w)$, the functionals I, $\tilde{I}$ and $\tilde{\tilde{I}}$ implicitly define certain families of real functions of real variable. For f of the type $f(w)$, define

$$I_f : \mathbb{R} \to \mathbb{R} \qquad \textit{by} \qquad I_f(w) = I(f(w))$$

Proposition 5. *For any $f = f(w)$, I_f is (a) increasing; (b) concave and (c) continuous.*

In a similar fashion, by using the functionals $\tilde{I}$ and $\tilde{\tilde{I}}$ we define

$$\begin{aligned} \tilde{I}_f \quad &: \quad \mathbb{R} \to \mathbb{R} \qquad \textit{by} \qquad \tilde{I}_f(w) = \tilde{I}(f(w)) \\ &\quad \textit{and} \\ \tilde{\tilde{I}}_f \quad &: \quad \mathbb{R} \to \mathbb{R} \qquad \textit{by} \qquad \tilde{\tilde{I}}_f(w) = \tilde{\tilde{I}}(f(w)) \end{aligned}$$

which are straight lines with slopes $1 - P_{\max}(T)$ and $1 - P_{\min}(T)$, respectively. Finally, by using $\tilde{I}_f$ and $\tilde{\tilde{I}}_f$, we define

$$\hat{I}_f(w) = \tilde{I}_f(w) \wedge \tilde{\tilde{I}}_f(w) = \min\left\{\tilde{I}_f(w), \tilde{\tilde{I}}_f(w)\right\}$$

The latter functions will play a major role in our analysis. Figure 1 below describes all the functions corresponding to a given f as well as the content of Lemma 1.

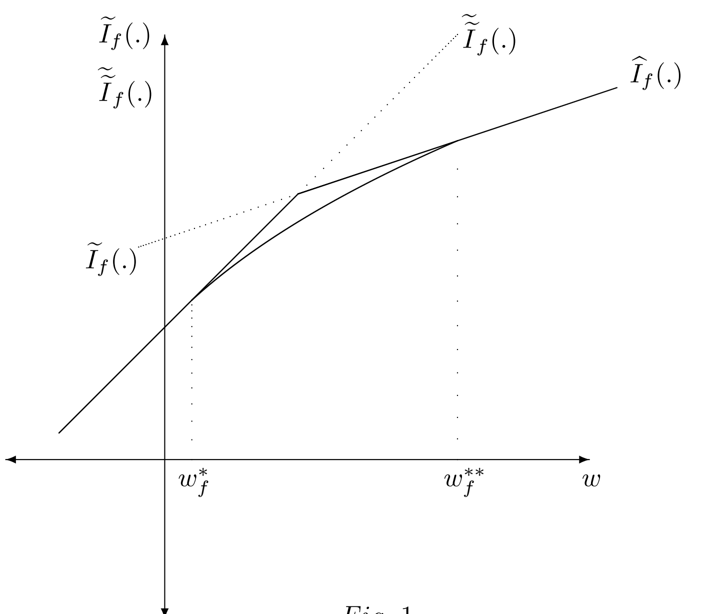


Fig. 1

For $w \leq w^*$, $I_f = \tilde{\tilde{I}}_f$ while for $w \geq w^{**}$, $I_f = \tilde{I}_f$. For $w \in (w^*, w^{**})$, I_f is a concave function between $\hat{I}_f = \tilde{I}_f \wedge \tilde{\tilde{I}}_f$ and the line joining $\tilde{\tilde{I}}_f(w^*)$ and $\tilde{I}_f(w^{**})$

By means of this type of diagrams, we can now give a simple illustration of the necessary condition found above. Recall that Proposition 4 states that a necessary condition for $T \in \mathcal{A}_Z$ is that there exists a w such that both

$$\begin{aligned} \tilde{\tilde{I}}(f(w)) &\geq \tilde{\tilde{I}}(g(w)) \\ \tilde{I}(f(w)) &\geq \tilde{I}(g(w)) \end{aligned}$$

We can concisely reformulate this as

Condition NC If $T \in \mathcal{A}_Z$, then $f(w) \succsim g(w)$ at some w implies

$$\hat{I}_f(w) \geq \hat{I}_g(w)$$

for any w.

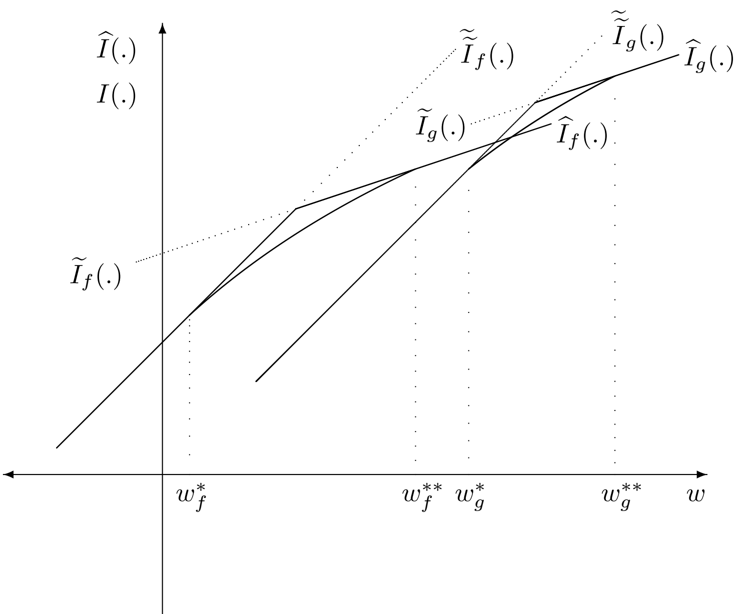


Fig. 2

If Cond NC is not satisfied, the $\tilde{I}$'s and $\tilde{\tilde{I}}$'s are as in the picture.
Any concave I_f and I_g intersect at some w leading to $T \notin \mathcal{A}_Z$.

5.2.1. Sufficient conditions for $T \in \mathcal{A}_Z$

Here, we state and geometrically illustrate some sufficient conditions for I – the functional that evaluates the acts – to be equal to $\hat{I}$. In such a situation, Condition NC becomes at once necessary and sufficient for $T \in \mathcal{A}_Z$. We stress that, in the context of a Multiple Prior model, conditions on the form of the functional are automatically conditions on the set of priors.

Below, we report only those conditions which we use in the rest of the paper, but additional sufficient conditions can be obtained by simple geometrical inspection.[6]

[6] For instance, one can easily see the following. Let $\bar{w}$ be the (unique) point st $\tilde{\tilde{I}}(f(\bar{w})) =$

For the sake of a more economical exposition, by a sufficient condition for $T \in \mathcal{A}_Z$ we mean only that T satisfies the first part of Zhang's definition. This is harmless. Since Zhang's definition consists of two separate parts, one gets a complete sufficient condition by simply requiring that the same condition be satisfied when one exchanges T with its complement T^c.

The first proposition below relies on the (obvious) fact that if $\mathcal{C} = co\{\mathcal{C}_{MIN}, \mathcal{C}_{MAX}\}$, then $I = \hat{I}$.

PROPOSITION 6. *If $\mathcal{C} = co\{\mathcal{C}_{MIN}, \mathcal{C}_{MAX}\}$ and Condition NC is satisfied, then $T \in \mathcal{A}_Z$.*

Next, we explicitly state a simple corollary to Proposition 6. We do so, because in Section 7 we will use Corollary 1 to give a geometric illustration to our comments on the notions of ambiguity we have been studying.

COROLLARY 1. *If $\mathcal{C}$ is generated by (at most) two priors, P_1 and P_2 (both nonatomic), then $T \in \mathcal{A}_Z$ iff P_1 and P_2 have the same conditionals both on T and on T^c.*

The next proposition exhibits a set of priors which is bigger than $co\{\mathcal{C}_{MIN}, \mathcal{C}_{MAX}\}$ but still guarantees that $I = \hat{I}$. Again, combined with Condition NC, this implies at once $T \in \mathcal{A}_Z$. Let

$$\alpha = \min_{\mathcal{C}} P(T) \qquad , \qquad \beta = \max_{\mathcal{C}} P(T)$$

For $\mathcal{P}$ an arbitrary set of priors, denote by

$$\mathcal{P}\mid_T = \{P(\cdot \mid T) \mid P \in \mathcal{P}\}$$

where $P(\cdot \mid T) = \frac{P(\cdot \cap T)}{P(T)}$. That is, $\mathcal{P}\mid_T$ is the set of conditional probabilities computed from probabilities in $\mathcal{P}$. Define

$$\begin{aligned} \mathcal{C}_1 &= [\{\mathcal{C}_{MIN}, \mathcal{C}_{MAX}\}]\mid_T \qquad , \qquad \mathcal{C}_2 = [\{\mathcal{C}_{MIN}, \mathcal{C}_{MAX}\}]\mid_{T^c} \\ &\quad \textit{and} \\ \mathcal{C}^* &= [\alpha, \beta]\mathcal{C}_1 + [1-\alpha, 1-\beta]\mathcal{C}_2 = \{\gamma\pi_1 + (1-\gamma)\pi_2 \mid \alpha \leq \gamma \leq \beta,\ \pi_1 \in \mathcal{C}_1,\ \pi_2 \in \mathcal{C}_2\} \end{aligned}$$

PROPOSITION 7. *If Condition NC is satisfied and $\mathcal{C} = \mathcal{C}^*$, then $T \in \mathcal{A}_Z$.*

$\tilde{I}(f(\bar{w}))$. The following are equivalent

(i) $\tilde{\tilde{I}}(f(\bar{w})) = \tilde{I}(f(\bar{w})) = I(f(\bar{w}))$;

(ii) $I(f(0)) = \hat{I}(f(0)) = \min\left\{\tilde{\tilde{I}}(f(0)), \tilde{I}(f(0))\right\}$

Moreover, if Condition NC is satisfied any of them implies $T \in \mathcal{A}_Z$.

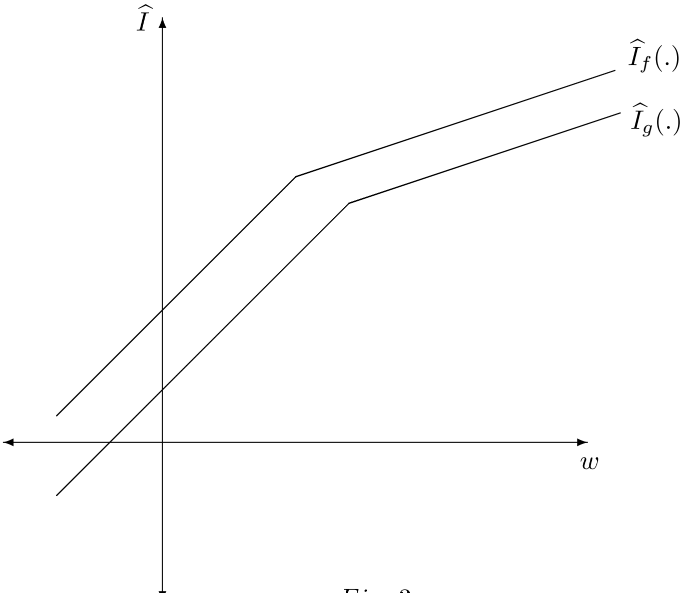


Fig. 3

The conditions in Prop 6 and Prop 7 both imply $I_f = \hat{I}_f$ for any f of the form $f(w)$. Hence, Cond NC is also sufficient.

5.3. Necessary and sufficient conditions

We are now ready to give a complete characterization of Zhang unambiguous events.

THEOREM 2. *$T \in \mathcal{A}_Z$ if and only if the following conditions are satisfied:*
(i) Condition NC: $\forall f, g$ of the form $f = fTw$ and $g = gTw$, either $\hat{I}_f(w) \geq \hat{I}_g(w)$ or $\hat{I}_f(w) \leq \hat{I}_g(w)$ for any $w \in \mathbb{R}$;
(ii) The functional I restricted to functions of the form fTw is equal to $\hat{I}$.

Sufficiency of the two conditions is immediate from what we said before. Necessity of (i) was shown above. Only the proof of necessity of (ii) requires a certain amount of work. Summarily, here is how it goes. Suppose that

$$\mathcal{C}_{MIN} \mid_T \cap \mathcal{C}_{MAX} \mid_T = \varnothing \tag{3}$$

and consider the act $0Tw$ (that is, the act that is identically equal to 0 on T and identically equal to w on T^c). A separation theorem due to [4] implies that there exist two disjoint subsets, A and B, of T, such that $P(A)-P(B) > 0 > Q(A)-Q(B)$ for any $P \in \mathcal{C}_{MIN}$ and any $Q \in \mathcal{C}_{MAX}$. This suggests that, for $\varepsilon > 0$, we consider the act εrTw, where r is the function $r = [\chi_A - \chi_B]$, and χ_A and χ_B are the indicator functions of A and B, respectively. The continuity properties of the MEU functional imply (see Lemma 5 in Appendix) that there exists an ε^* such that for any measure $R \in \left\{P \in \mathcal{C}_{MIN} \mid \int(\varepsilon^*[\chi_A - \chi_B]Tw)dP = \min\right\}$, we have that $R(A) - R(B) > 0$. Clearly, this implies that when both $\varepsilon^*[\chi_A - \chi_B]Tw$ and $0Tw$ are evaluated by measures in $\mathcal{C}_{MIN}$ (this certainly happens because of Lemma 1), we have $\varepsilon^*[\chi_A - \chi_B]Tw \succ 0Tw$. Analogously, when when both $\varepsilon^*[\chi_A - \chi_B]Tw$ and

$0Tw$ are evaluated by measures in $\mathcal{C}_{MAX}$, we have $\varepsilon^*[\chi_A - \chi_B]Tw \prec 0Tw$. But this means that T is Zhang ambiguous. In other words, if $T \in \mathcal{A}_Z$ then the intersection in (3) has to be non-empty. In the general case, for f an act of the type fTw, we consider the sets

$$\begin{aligned}
\tilde{\tilde{\Pi}}_f &= \left\{ P \in \mathcal{C}_{MIN} \mid \int f dP = \min \right\} \\
\tilde{\Pi}_f &= \left\{ P \in \mathcal{C}_{MAX} \mid \int f dP = \min \right\} \\
\Pi_f &= \left\{ P \in \mathcal{C} \mid \int f dP = \min \right\}
\end{aligned}$$

and prove the following characterization of Condition NC.

LEMMA 2. *Condition NC is equivalent to the condition that for any act f*

$$(\tilde{\tilde{\Pi}}_f \mid_T) \cap (\tilde{\Pi}_f \mid_T) \neq \emptyset$$

Next, we show (Lemma 6) that if for some f we have $I_f(w) < \hat{I}_f(w)$, then it must be the case that either $\Pi_f \mid_T \cap \tilde{\tilde{\Pi}}_f \mid_T = \varnothing$ or $\Pi_f \mid_T \cap \tilde{\Pi}_f \mid_T = \varnothing$. Then, in both cases, we can follow a procedure similar to the one outlined above to establish that T has to be Zhang ambiguous. That is if $T \in \mathcal{A}_Z$, then for any f, we must have $I_f(w) = \hat{I}_f(w)$.

5.4. Epstein-Zhang unambiguous events

As general matter (Section 3), Epstein-Zhang's definition is more permissive than Zhang's. In fact, conditions guaranteeing that the two are non-equivalent can be derived from Epstein-Zhang's paper, in particular Corollary 7.3. In principle, this might be the case in our setting as well. That is, an event T might be EZ-unambiguous without being Z-unambiguous. In such a case, however, at least one of the conditions in Theorem 2 must be violated, and (either case) there exist (a) acts fTw and fTw'; (b) priors P and P' with $I(fTw) = \int fTwdP$ and $I(fTw') = \int fTw'dP'$; and (c) events A and B, in T such that $P(A) - P(B) > 0 > P'(A) - P'(B)$. Notice that this means that the conditional relative likelihood of A versus B varies with the act which is evaluated. While this is certainly an undesirable feature, we cannot conclude at once that T must necessarily be EZ-ambiguous. The next theorem shows that this intuition is correct, nonetheless.

THEOREM 3. $\mathcal{A}_Z = \mathcal{A}_{EZ}$.

Details are in Appendix, but the intuition is simple. Given an $f(w)$ with properties (a), (b) and (c) above, we can construct two acts

$$\begin{aligned}
g_\varepsilon(w) &= f(w) + \varepsilon[\chi_B - \chi_A] \quad , \quad \varepsilon > 0 \\
h_\varepsilon(w) &= f(w) + \varepsilon[\chi_A - \chi_B]
\end{aligned}$$

and guarantee that there exist two values, w' and w'', such that

$$I(h_\varepsilon(w')) > I(f(w')) > I(g_\varepsilon(w')) \quad \textit{and} \quad I(h_\varepsilon(w'')) < I(f(w'')) < I(g_\varepsilon(w''))$$

Notice that if $f(w)$ is constant on T, we are done because $g_\varepsilon(w)$ and $h_\varepsilon(w)$ are EZ-conjugate. In the general case, a simple continuity argument would complete the proof.

6. APPLICATIONS

The results of the previous sections can be given an elegant formulation with the theorem below. In addition, the theorem provides a representation of the MEU functional restricted to acts of the form fTw and fT^cw for T an unambiguous event. Later, we use this representation to address two open questions concerning unambiguous events either in the sense of Zhang or Epstein-Zhang.

Let .

THEOREM 4. *$T \in \mathcal{A}_{EZ}$ iff there exists a set of priors $\mathcal{C}^*$ of the form*

$$\mathcal{C}^* = [\alpha, \beta]\Pi_1 + [1-\beta, 1-\alpha]\Pi_2 = \{\gamma\pi_1 + (1-\gamma)\pi_2 \mid \alpha \leq \gamma \leq \beta,\ \pi_1 \in \Pi_1,\ \pi_2 \in \Pi_2\}$$

where Π_1 and Π_2 are weak-closed, convex sets of priors supported by T and T^c, respectively, and $0 < \alpha \leq \beta < 1$, such that*

$$\min_{\mathcal{C}} \int f dP = \min_{\mathcal{C}^*} \int f dP$$

for all acts of type fTw and for all acts of type fT^cw. Moreover, $\mathcal{A}_Z = \mathcal{A}_{EZ}$. Finally, T is naturally measurable iff $\alpha = \beta$.

Next, we study the axiom of "small unambiguous event continuity" (see [8], axiom 4). Roughly, the axiom states that any unambiguous event contains unambiguous events of arbitrarily small probability. The axiom is important in the work of Zhang, Epstein-Zhang and Kopylov [15] in that it allows to derive a convex-ranged probability on the class of unambiguous events. Recently, Nehring [20] has questioned the desiderability of the assumption.

Our result is that, in the context of MEU, there exists a fundamental incompatibility between the definitions of unambiguous events given by Zhang and Epstein-Zhang and the axiom of "small unambiguous event continuity". Of course, the statement needs an obvious qualification. For naturally measurable events are both Z-unambiguous and EZ-unambiguous and, as we saw in Section 4, naturally measurable events can always be decomposed into smaller naturally measurable events. In other words, the incompatibility refers to EZ-unambiguous events which are not naturally measurable. Precisely, we have the following theorem.

THEOREM 5. *Let $T \in \mathcal{A}_{EZ} \backslash \mathcal{A}_{NM}$. Then, T contains no unambiguous events.*

The proof is constructive and, as such, displays a certain amount of detail. The strategy of proof is simple, nonetheless. By Theorem 4, an event T is EZ-unambiguous if and only if the set of priors can be decomposed in the way described above. Now, suppose that both T and $A \subset T$ are EZ-unambiguous. Then, we have two decompositions of the set of priors: one relative to T and one relative to A. Of course, these two decompositions cannot be unrelated. For instance, an act of the type fAw can be viewed both as an act which is constant outside A and as an act which is constant outside T. Clearly, these two views must lead to the same evaluation because the act is the same. This simple observation allows us to conclude that the two different decompositions must satisfy a number of restrictions. Finally, we use these to produce two acts, fTk and gTk, whose ranking is not invariant with respect to changes in the constant k, thus contradicting the assumption either $T \in \mathcal{A}_{EZ}$ or $A \in \mathcal{A}_{EZ}$.

The second question we address regards the properties of the classes $\mathcal{A}_Z$ and $\mathcal{A}_{EZ}$. These classes were originally believed to be λ-systems, but Kopylov [15] observed that, generally speaking, this is not the case. In [15], Kopylov provides an axiomatization guaranteeing that these classes are mosaics (a weaker property, see [15]), and, more recently [16], gave an example of a preference relation for which $\mathcal{A}_Z$ and $\mathcal{A}_{EZ}$ are not λ-systems. A rather immediate implication of Theorem 5 is that this cannot be the case in the setting we have been studying. In other words, we have

Corollary 2. *$\mathcal{A}_{EZ}$ (hence, $\mathcal{A}_Z$) is a λ-system.*

The reason is clear. By Theorem 5, any two events A and B in $\mathcal{A}_{EZ} \backslash \mathcal{A}_{NM}$ cannot be disjoint unless one is the complement of the other. Hence, the property of $\mathcal{A}_{EZ}$ to be a λ-system follows at once from the property of $\mathcal{A}_{NM}$. We remark that Corollary 2 does not say that $\mathcal{A}_{EZ} = \mathcal{A}_{NM}$. In fact, it is easy to give examples where $\mathcal{A}_{EZ}$ strictly contains $\mathcal{A}_{NM}$ (see next section).

7. COMMENTS

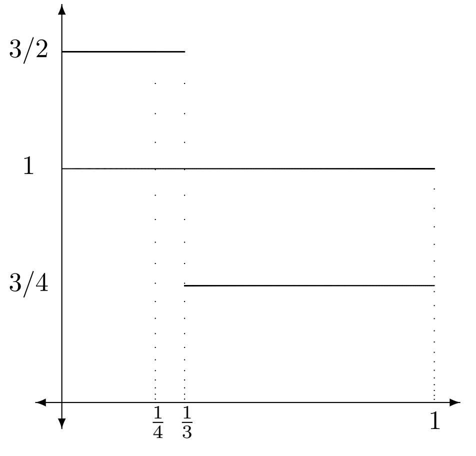
Theorem 4 makes it clear that there are unquestionable similarities across the various definitions of unambiguous events we have been examining. As a matter of fact, Theorem 4 shows that all the definitions convey the idea that if T is unambiguous in a MEU model, then the model itself can be thought of as consisting of two separate, but unambiguously defined, models: one which is defined on T and the other which is defined on T^c. The only difference is that these two models are aggregated "linearly" if T is naturally measurable, while this is not the case if T is Z/EZ-unambiguous. Clearly, scholars favoring the notion of naturally measurable events might argue that Zhang's and Epstein-Zhang's definitions are "too weak" [20] exactly because of this reason. To us, this difference does not seem substantial.

Let us elaborate on the point. For the sake of illustration, let us begin with a decision maker who conforms to the Subjective Expected Utility (SEU) criterion. Probably, every scholar would agree that any (measurable) subset T of S should be termed as unambiguous. Each subset is associated to a number of properties: the decision maker is "a SEU maximizer on T", he is "probabilistically sophisticated on T", he is "linear with respect to acts that are constant outside T", etc.. Now, suppose that we want to come up with a definition of unambiguous events. In a situation of complete (*a priori*) ignorance, we might adopt the following strategy. We abstract from the SEU model, and identify the unambiguous nature of an event with one of the properties displayed by the SEU example. For instance, one definition would term T unambiguous if the decision maker is probabilistically sophisticated on T, another if he is linear with respect to acts that are constant outside T, etc. One of the properties that each and every event T displays in the SEU example is that *knowledge of the conditional probabilities on T and T^c along with knowledge of the minimum probability assigned to T and T^c allows us to recover uniquely the entire model.* This is trivial, for if P is the probability which describes the decision maker, then P can be uniquely written as $P(\cdot) = P(T)P(\cdot \mid T) + P(T^c)P(\cdot \mid T^c)$. In a MEU model, Theorem 4 tells us that the unambiguous nature of an event according to Zhang or Epstein-Zhang is precisely identified to this property (because the conditions in Theorem 4 are necessary and sufficient). In this respect, these definitions appear to us as a natural extension of those proposed by Nehring and Ghirardato, Maccheroni and Marinacci.

A simple example will clarify the point further. Let $S = [0,1]$ be endowed with the usual Borel σ-algebra, and consider a MEU decision maker who is described by a set of priors $\mathcal{C} = co\{\mu, \lambda\}$. Assume further that μ has a density with respect to λ given by

$$f = \begin{cases} \frac{3}{2} & if\ x \in [0, \frac{1}{3}) \\ \frac{3}{4} & if\ x \in [\frac{1}{3}, 1] \end{cases}$$

and that λ is the Lebesgue measure on $[0,1]$ (we denote its density by g, $g = 1$ on $[0,1]$).

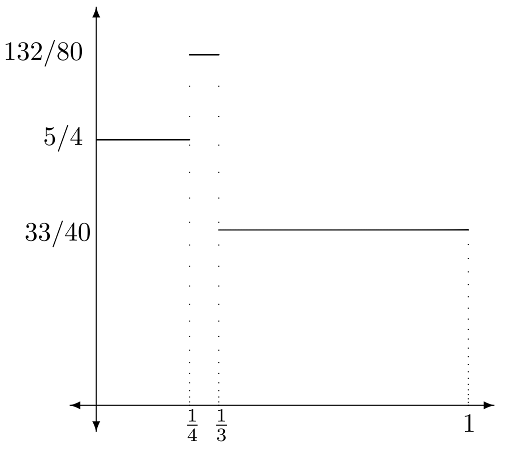


Let $T = [0, 1/3)$. It is easy to see that every non-null naturally measurable subset intersects both T and its complement. Moreover, T and its complement are EZ-unambiguous (Corollary 1) and, evidently, not naturally measurable. By Theorem 5, any subset of T is EZ-ambiguous and it is transparent that it cannot be naturally measurable.

In the notation of Theorem 4, it is immediate to check that for $T = [0, 1/3)$, Π_1 and Π_2 are defined by the densities $\{3\chi_T\}$ and $\{\frac{3}{2}\chi_{T^c}\}$, respectively. Moreover (in the same notation), $\alpha = 1/3$, $\beta = 1/2$ and $\mathcal{C} = [\alpha, \beta]\Pi_1 + [1-\beta, 1-\alpha]\Pi_2$. That is, knowledge of the conditional models, Π_1 and Π_2, and of α and β permits to reconstruct $\mathcal{C}$ uniquely. In contrast, suppose that we take $A = [0, 1/4)$ rather than T. In this case, the conditional models are defined by the two sets of densities

$$\{f_1 = 4\} \qquad and \qquad \left\{ f_2 = \begin{cases} \frac{12}{5} & if\ x \in [\frac{1}{4}, \frac{1}{3}) \\ \frac{6}{5} & if\ x \in [\frac{1}{3}, 1] \end{cases} ; g_2 = \frac{4}{3} \right\}$$

Let Π_1' and Π_2' be the corresponding sets of measures. The coefficients α and β are now $1/4$ and $3/8$, respectively. It is immediate to see that, with these choices, $\mathcal{C} \neq [\alpha, \beta]\Pi_1' + [1-\beta, 1-\alpha]\Pi_2'$. For instance, for $\gamma = 5/16$, the density $h_\upsilon = \gamma f_1 + (1-\gamma) f_2$ defines a measure $\upsilon \in [\alpha, \beta]\Pi_1' + [1-\beta, 1-\alpha]\Pi_2'$ but $\upsilon \notin \mathcal{C}$ (see figure below)



In other words, if A is not Z/EZ-unambiguous, knowledge of the conditional models is insufficient to uncover the decision maker's global behavior.

We can now move to the differences. Transparently, the most striking difference is the lack of compatibility of Epstein-Zhang notion with the axiom of small unambiguous event continuity. An immediate implication of Theorem 5 is that partitions of S into EZ-unambiguous events that are not naturally measurable contain at most two elements. In contrast, one can exhibit partitions of S into naturally measurable events of any finite cardinality (Theorem 1) and even countable or uncountable partitions.[7] The reason for this is that the additional restriction imposed by Nehring and Ghirardato, Maccheroni and Marinacci ($\alpha = \beta$ in Theorem 4) guarantees that each conditional model is a MEU model of the same type as the unconditional one.[8] Somewhat technically, the "global" model is isomorphic to a product of conditional models (simply identify disjoint unions of sets with their products). Clearly, this is not true for Zhang and Epstein-Zhang: since $\alpha \neq \beta$, the value of the bet which pays 1 on T is not uniquely defined. This creates a link between the factors on T and T^c, thus making them non-independent.

We come now to our final observations. One of the main themes in the work of Zhang and Epstein-Zhang is the intuitive link between a notion of unambiguous events and Savage's Independence of Irrelevant Alternatives axiom. This intuition is transparently formulated in their definitions. Now, suppose that, in loose terms, one interprets their view as conveying that T should be termed unambiguous if *conditional on* T*, the decision maker is representable without reference to what happens outside* T. Consider the event $A = [0, 1/4)$ in the example above. There

[7] Under the assumption that the partition be measurable, one can easily extend Theorem 1 to the case of uncountable partitions.

[8] This feature provides an additional motivation for our assumption of non-atomicity. As we explained in the text, the notion of naturally measurable event delivers that the model, conditional on each and every naturally measurable event, is a mirror image of the global model. Clearly, the presence of atoms would break this symmetry. It is also clear, however, that no atom would have any special significance in the context of a general theory.

is no doubt that A satisfies the criterion. In fact, A satisfies the first part of Epstein-Zhang definition. Yet, A is ambiguous as we saw above and the reason is that its complement fails the first part of the definition. A natural question is, why should we demand that Zhang and Epstein-Zhang classes be closed under complementation? We do not have a clear answer. Yet, we would like to stress the legitimacy of our question: closure under complementation is a property which follows from the definition of naturally measurable events, while it is imposed in the definitions of Zhang and Epstein-Zhang. On one hand, Theorem 4 and the example of this section show that, if the classes are not closed under complementation, the decomposition property of Theorem 4 does not hold. Hence, if one believes that the unambiguous nature of an event should be identified to the property we described in the first paragraph of this section, then there is no choice but to impose closure under complementation. On the other hand, if one's intuition conforms to the less demanding interpretation we just gave, one should probably give up the requirement of closure under complementation.

Finally, let us observe that events like A in the example above have several special properties. In particular, A is such that all of its subsets satisfies the first part of Zhang definition (the decision maker is SEU conditional on A). While events like these have been completely neglected in the current literature, it seems indisputable to us that they deserve a place in a debate centered around the notion of ambiguity. Usually the idea of ambiguity is associated to that of coarse information. In contrast, SEU or probabilistic sophistication are associated to the idea of precise or fine information. It seems then natural to consider situations characterized by coarse information in some parts of the state space and by fine information in some other parts. To this end, depending on whether one leans toward SEU or probabilistic sophistication as a choice for a benchmark, one should identify those events A with the property that each and every event B in A satisfies the first part of Zhang (SEU) or Epstein-Zhang (probabilistic sophication) definition. Due to their properties, objects of this sort should be called Z-filters and EZ-filters, respectively. A little elaboration on the example we gave above would show that one can easily exhibit a decision maker who displays the following property: for almost every point in $[0, 1]$, there is a neighborhood of the point conditional on which the decision maker is SEU (probabilistically sophisticated), yet his global model is MEU with an infinite-dimensional set of priors. Such a decision maker is associated to a countable family of Z-filters (EZ-filters) none of which is either naturally measurable or Z/EZ-unambiguous.

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APPENDIX

A.4 NATURALLY MEASURABLE EVENTS

A.4(a) Proof of Theorem 1

Let E be an event. For $f \in \mathcal{F}$ denote by $f_E = f \mid_E$ the restriction of f to E, and let $\mathcal{F} \mid_E$ be set of all such restrictions. When E is endowed with the restriction of Σ to E, each element in $\mathcal{F} \mid_E$ is measurable.

Given E, define a preference relation $\succcurlyeq_E$ on $\mathcal{F} \mid_E$ by "$f_E \succcurlyeq_E g_E$ iff $fEw \succcurlyeq gEw$ for some $w \in \mathbb{R}$". The following lemma shows that if E is naturally measurable, then $\succcurlyeq_E$ is well-defined.

LEMMA 3. *Let $f, g \in \mathcal{F}$, $w \in \mathbb{R}$ and let $E \in \mathcal{A}_{NM}$. If $fEw \succcurlyeq gEw$ for some $w \in \mathbb{R}$, then for any $w' \in \mathbb{R}$, we have $fEw' \succcurlyeq gEw'$.*

Proof. If $fEw \succcurlyeq gEw$, then E naturally measurable gives

$$\begin{aligned}
I(fEw) &\geq I(gEw) \\
\min_{P \in \mathcal{C}} \left[\int_E f(s) dP(s) + wP(E^c) \right] &\geq \min_{P \in \mathcal{C}} \left[\int_E g(s) dP(s) + wP(E^c) \right] \\
\left[\min_{P \in \mathcal{C}} \int_E f(s) dP(s) \right] + wP(E^c) &\geq \left[\min_{P \in \mathcal{C}} \int_E g(s) dP(s) \right] + wP(E^c) \\
\left[\min_{P \in \mathcal{C}} \int_E f(s) dP(s) \right] + w'P(E^c) &\geq \left[\min_{P \in \mathcal{C}} \int_E g(s) dP(s) \right] + w'P(E^c) \\
I(fEw') &\geq I(fEw')
\end{aligned}$$

that is $fEw' \succcurlyeq gEw'$. ∎

We can now prove Theorem **1**.

*Proof of Theorem **1**.* Let $\Phi = \{\pi_i\}_{i=1,\ldots,n}$ be a finite partition of S with the property that each event in the partition is naturally measurable, i.e. $\pi_i \in \mathcal{A}_{NM}$ for each i. We are going to show that for any $f \in \mathcal{F}$ and any $g \in \mathcal{F}^{\Phi}$, we have

$$I(f+g) = I(f) + I(g) \tag{4}$$

$$\begin{aligned}
I(f+g) &= \min_{P \in \mathcal{C}} \int_S (f+g) dP \\
&= \min_{P \in \mathcal{C}} \left[\int_S f dP + \int_S g dP \right] \\
&= \min_{P \in \mathcal{C}} \left[\int_S f dP + \sum_{i=1}^{n} g_i P(\pi_i) \right] \qquad \text{(because } g \in \mathcal{F}^{\Phi}\text{)} \\
&= \left(\min_{P \in \mathcal{C}} \int_S f dP \right) + \sum_{i=1}^{n} g_i P(\pi_i) \qquad \text{(because each } \pi_i \in \mathcal{A}_{NM}\text{)} \\
&= I(f) + I(g)
\end{aligned}$$

Now, for each i, define the preference $\succcurlyeq_{\pi_i}$ by

$$f_{\pi_i} \succcurlyeq_{\pi_i} g_{\pi_i} \qquad iff \qquad f\pi_i w \succcurlyeq g\pi_i w \qquad \text{for some } w \in \mathbb{R}$$

By the previous lemma, $\succcurlyeq_{\pi_i}$ is well-defined. It is easy to see that $\succcurlyeq_{\pi_i}$ satisfies all the axioms of Gilboa-Schmeidler [11] (because $\succcurlyeq$ does). Hence, $\succcurlyeq_{\pi_i}$ has a MEU representation. That is, there exists a unique collection of non-empty, weak* compact, and convex set of priors $\mathcal{C}_{\pi_i}$, each supported by π_i such that $I_{\pi_i} : \mathcal{F}\mid_{\pi_i} \to \mathbb{R}$ defined by

$$I_{\pi_i}(f) = \min_{P\in\mathcal{C}_{\pi_i}} \int_{\pi_i} f dP$$

represents $\succcurlyeq_{\pi_i}$.

Now, define $z : \mathcal{F} \to \mathbb{R}^{|\Phi|}$ ($|\Phi|$ is the cardinality of Φ) by $f \longmapsto (I_{\pi_i}(f))_{\pi_i\in\Phi}$, and define $v : \mathbb{R}^{|\Phi|} \to \mathbb{R}$ as the unique mapping which makes the diagram below commute

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{z} & \mathbb{R}^{|\Phi|} \\ & I \searrow & \downarrow v \\ & & \mathbb{R} \end{array}$$

That is, $v((I_{\pi_i}(f))_{\pi_i\in\Phi}) = I(f)$.

To complete the proof, it suffices to show that $v(.)$ is a positive linear functional on $\mathbb{R}^{|\Phi|}$.

(a) $v(.)$ is homogeneous: Let $(w_i)_{i=1,\dots,n} \in (I_{\pi_i}(\mathcal{F}))_{\pi_i\in\Phi}$ and $\alpha \in \mathbb{R}$. Define an act

$$f(s) = \alpha w_i \text{ for } s \in \pi_i$$

then

$$v((\alpha w_i)_{i=1,\dots,n}) = v((I_{\pi_i}(f)) = I(f)$$

and

$$I(f) = \min_{C} \sum_{i=1}^{n} \alpha w_i P(\pi_i) = \sum_{i=1}^{n} \alpha w_i P(\pi_i)$$

for any P since $\pi_i \in \mathcal{A}_{NM}$. It follows that

$$v((\alpha w_i)_{i=1,\dots,n}) = \alpha \sum_{i=1}^{n} w_i P(\pi_i) = \alpha \min_{C} \sum_{i=1}^{n} w_i P(\pi_i) = \alpha v((w_i)_{i=1,\dots,n})$$

(b) $v(.)$ is additive: For any $(w_i)_{i=1,\dots,n}, (w_i')_{i=1,\dots,n} \in (I_{\pi_i}(\mathcal{F}))_{\pi_i\in\Phi}$, define a pair of acts f and g as

$$f(s) = w_i \qquad \text{and} \qquad g(s) = w_i' \quad \text{for } s \in \pi_i$$

then

$$\begin{aligned} v((w_i)_{i=1,\dots,n} + (w_i')_{i=1,\dots,n}) &= v((I_{\pi_i}(f) + I_{\pi_i}(g))_{i=1,\dots,n}) \\ &= v((I_{\pi_i}(f+g))_{i=1,\dots,n}) \\ &= I(f+g) \\ &= I(f) + I(g) \qquad \text{by (4)} \\ &= v((w_i)_{i=1,\dots,n}) + v((w_i')_{i=1,\dots,n}) \end{aligned}$$

(c) $v(.)$ is positive: Trivially because $I(.)$ is a positive function.

Finally, since $v(.)$ is a positive linear functional there is a positive measure q on S/Φ (the quotient of S by the partition Φ) such that

$$I(f) = v((I_{\pi_i}(f))_{\pi_i\in\Phi}) = \int_{S/\Phi} I_{\pi_i}(f) dq(\pi_i) = \int_{S/\Phi} \left(\min_{P\in\mathcal{C}_{\pi_i}} \int_{\pi_i} f dP \right) dq(\pi_i)$$

(q is a probability measure because $I(1) = 1$). ∎

A.4 (b) Proof of Proposition 3

Proof. Since a constant act $w \in \mathbb{R}$ is a step fuction with respect to Φ, step-independence (along with the Archimedean axiom in [11]) implies constant independence in [11], which proves part (a). Let $\pi \in \Phi$, and for any $w, w' \in \mathbb{R}$ define an act a that is a step function on Φ as

$$a(s) = \begin{cases} w & if \quad s \in \pi \\ w' & if \quad s \notin \pi \end{cases}$$

then, by step independence, for any $\alpha \in [0,1]$,and $f, g \in \mathcal{F}$ such that $f \sim g$, we have $\alpha f + (1-\alpha)a \sim \alpha g + (1-\alpha)a$. By Proposition 1, $\pi \in \mathcal{A}_{NM}$ and part (c) is proven. Finally, part (b) follows from Theorem 1. ∎

A.5 ZHANG AND EPSTEIN-ZHANG UNAMBIGUOUS EVENTS

Proposition 8. *Let T be such that either $\min_{\mathcal{C}} P(T) = 0$ or $\max_{\mathcal{C}} P(T) = 1$, then T is EZ-ambiguous.*

Proof. Let T be such that $0 = \min P(T) < \max P(T) \leq 1$. Let $\mathcal{C}_1 = \{P \in \mathcal{C} \mid P(T) = \max\}$. Since $\mathcal{C}_1$ is thin, there exists (by an easy application of Lyapunov's convexity Theorem) an event $A \subset T$ such that $P(A) = 2/3P(T)$ for any $P \in \mathcal{C}_1$.Consider the acts $a(w) = (\chi_A - \chi_{A^c})Tw$ and $\bar{a}(w) = (\chi_{A^c} - \chi_A)Tw$. It is immediate to see that there exists $n \in \mathbb{N}$ such that for $w \leq -n$ both acts are evaluated by P's such that $P(T) = 0$. Hence, $I(\bar{a}(w)) = w = I(a(w))$, that is $\bar{a}(w) \succsim a(w)$. Similarly, for $w' \geq n$, both acts are evaluated by P's in $\mathcal{C}_1$, and we have $I(a(w')) > I(\bar{a}(w'))$. The same argument with T^c in the place of T completes the proof. ∎

A.5.1 From naturally measurable events to Zhang unambiguous events

A.5.1.(a) Proof of Lemma 1

Proof. For each w, denote by Z_w a measure such that

$$\min_{\mathcal{C}} \int g(w)dP = \int g(w)dZ_w$$

Clearly, for any w, $\overset{\approx}{I}(g(w)) \geq I(g(w)) = \min_{\mathcal{C}} \int g(w)dP$.

Suppose that the inequality is strict for any w, that is[9]

$$\min_{\mathcal{C}_{MIN}} \int_T g(w)dP + wP_{MAX}(T^c) > \int_T g(w)dZ_w + wZ_w(T^c)$$

[9] $P \in \mathcal{C}_{MIN}$ implies $P(T^c) = \max$. Above, this is denoted by P_{MAX}. Of course, such a P need not be unique, but the reasoning in the proof uses its value on T^c only, which is independent of the choice we make.

which is equivalent to

$$\min_{\mathcal{C}_{MIN}} \int_T g(w)dP - \int_T g(w)dZ_w > w[Z_w(T^c) - P_{MAX}(T^c)] \tag{5}$$

Then, for any w there must exist a Z_w such that

(i) $Z_w(T^c) < P_{MAX}(T^c)$ (otherwise, we are done);

(ii) inequality (5) is true.

Consider the sequence $\{w_n = -n\}$ and the associated sequence $\{Z_{w_n}(T^c)\}$.[10]

The sequence of real numbers $\{Z_{w_n}(T^c)\} \subset [P_{MIN}(T^c), P_{MAX}(T^c)]$. Hence, there exists a subsequence $\left\{Z_{w_{n_k}}(T^c)\right\}$ such that $Z_{w_{n_k}}(T^c) \to x \in [P_{MIN}(T^c), P_{MAX}(T^c)]$. We have only two possibilities:

(a) if $x \neq P_{MAX}(T^c)$, then $\exists \varepsilon > 0$ such that $x + \varepsilon < P_{MAX}(T^c)$ and $\exists \bar{n}$ such that $\forall n \geq \bar{n}$

$$Z_{w_{n_k}}(T^c) < x + \varepsilon$$

which implies

$$Z_{w_{n_k}}(T^c) - P_{MAX}(T^c) < x + \varepsilon - P_{MAX}(T^c) = -\delta < 0$$

Hence,

$$w_{n_k}[Z_{w_{n_k}}(T^c) - P_{MAX}(T^c)] = -n_k[Z_{w_{n_k}}(T^c) - P_{MAX}(T^c)] > n_k\delta$$

that is the expression on the RHS of (5) is unbounded. Hence, inequality (5) must be violated because the LHS is bounded. It follows that the only possibility is

(b) $Z_{w_{n_k}}(T^c) \to P_{MAX}(T^c)$.

Since g is a simple function on T, g can be written as $g = \sum_{i=1}^{m} \alpha_i \chi_{A_i}$, $\{A_i\}_1^m$ a partition of T. Let P be such that $\int g(w)dP = \min_{\mathcal{C}_{MIN}} \int g(w)dP'$. Then,

$$\begin{aligned} &(\min_{\mathcal{C}_{MIN}} \int_T g(w)dP) - \int_T g(w)dZ_{w_{n_k}} \\ = \; & \sum^m \alpha_i P(A_i) - \sum^m \alpha_i Z_{w_{n_k}}(A_i) = \sum^m \alpha_i [P(A_i) - Z_{w_{n_k}}(A_i)] \end{aligned}$$

Define

$$\beta_i = \begin{cases} \alpha_i & if \quad sign(\alpha_i) = sign[P(A_i) - Z_{w_{n_k}}(A_i)] \\ -\alpha_i & otherwise \end{cases}$$

Then,

$$\sum^m \beta_i [P(A_i) - Z_{w_{n_k}}(A_i)] \geq \sum^m \alpha_i [P(A_i) - Z_{w_{n_k}}(A_i)]$$

Let $\beta = \sup_i \beta_i$. Then, from inequality (5) we have that $\forall n \in \mathbb{N}$

$$\beta \sum^m [P(A_i) - Z_{w_{n_k}}(A_i)] > -n_k[Z_{w_{n_k}}(T^c) - P_{MAX}(T^c)]$$

[10] Once again, Z_{w_n} need not be unique, but the reasoning in the proof is independent of the choice we make.

$\Longleftrightarrow$
$$\beta[1 - P_{MAX}(T^c) - 1 + Z_{w_{n_k}}(T^c)] > -n[Z_{w_{n_k}}(T^c) - P_{MAX}(T^c)]$$
which implies
$$\beta < -n$$
a contradiction.

The preceding show that there exists a w^* such that $I(g(w^*)) = \widetilde{\widetilde{I}}(g(w^*))$. That is,
$$\min_{\mathcal{C}_{MIN}} \int_T g(w^*)dP + w^* P_{MAX}(T^c) \leq \int_T g(w^*)dZ + w^* Z(T^c)$$
for any $Z \notin \mathcal{C}_{MIN}$.

Now, we want to show that it is so for any $w \leq w^*$.[11] Let $\varepsilon > 0$, and suppose by the way of contradiction that there exist a measure $Z_{w^*-\varepsilon} \notin \mathcal{C}_{MIN}$ such that $\widetilde{\widetilde{I}}(g(w^* - \varepsilon)) > I(g(w^* - \varepsilon))$, that is
$$\min_{\mathcal{C}_{MIN}} \int_T g(w^*-\varepsilon)dP + (w^*-\varepsilon)P_{MAX}(T^c) > \int_T g(w^*-\varepsilon)dZ_{w^*-\varepsilon} + (w^*-\varepsilon)Z_{w^*-\varepsilon}(T^c)$$
The integral on T is not affected by changes in w (by the w-invariance of $\widetilde{\widetilde{I}}$). Hence, this is the same as
$$\min_{\mathcal{C}_{MIN}} \int_T g(w^*)dP + (w^* - \varepsilon)P_{MAX}(T^c) > \int_T g(w^*)dZ_{w^*-\varepsilon} + (w^* - \varepsilon)Z_{w^*-\varepsilon}(T^c)$$
which implies
$$\min_{\mathcal{C}_{MIN}} \int_T g(w^*)dP + w^* P_{MAX}(T^c) > \int_T g(w^*)dZ_{w^*-\varepsilon} + w^* Z_{w^*-\varepsilon}(T^c) + \varepsilon[P_{MAX}(T^c) - Z_{w^*-\varepsilon}(T^c)]$$
But, since $Z_{w^*-\varepsilon} \notin \mathcal{C}_{MIN}$, $\varepsilon[P_{MAX}(T^c) - Z_{w^*-\varepsilon}(T^c)] > 0$, and this contradicts $I(g(w^*)) = \widetilde{\widetilde{I}}(g(w^*))$.

Part (b) is proven in a similar way. ∎

A.5.1.(b) Proof of Proposition 4

We are going to show that if $T \in \mathcal{A}_Z$, then $f(w) \succsim g(w)$ at some w implies

(a) $\exists w^*$ such that $\widetilde{\widetilde{I}}(f(w^*)) \geq \widetilde{\widetilde{I}}(g(w^*))$;
(b) $\exists w^{**}$ such that $\tilde{I}(f(w^{**})) \geq \tilde{I}(g(w^{**}))$.

Immediately, this implies the statement in Proposition **4** because of the w-invariance of $\widetilde{\widetilde{I}}$ and $\tilde{I}$.

Proof. Suppose not. Then, $\forall w$
$$\widetilde{\widetilde{I}}(f(w)) < \widetilde{\widetilde{I}}(g(w))$$
Since $\forall w$, $I(f(w)) \leq \widetilde{\widetilde{I}}(f(w))$, we have $I(f(w)) < \widetilde{\widetilde{I}}(g(w))$. Hence, $\forall w$
$$I(f(w)) - I(g(w)) < \widetilde{\widetilde{I}}(g(w)) - I(g(w))$$

[11] This does not follow immediately. We have only shown that the strict inequality (5) has to be violated an infinite number of times along the subsequence we used.

By the previous lemma, $\exists \bar{w}$ such that $I(g(\bar{w})) = \tilde{\tilde{I}}(g(\bar{w}))$, and at such $\bar{w}$ we would have

$$I(f(\bar{w})) < I(g(\bar{w}))$$

contradicting $T \in \mathcal{A}_Z$. Similarly for part (b). ∎

A.5.2 A geometric analysis

A.5.2(a) Proof of Proposition 5

Proof. $I(f(w)) = \min_{\mathcal{C}}[\int_T fdP + wP(T^c)]$

(a) $w_1 \geq w_0$ implies that $\forall P$

$$\int_T fdP + w_1 P(T^c) \geq \int_T fdP + w_0 P(T^c)$$

Hence, $I(f(w_1)) \geq I(f(w_0))$.

(b) For any $\alpha \in [0, 1]$

$$\begin{aligned} \alpha I_f(w_1) + (1-\alpha) I_f(w_0) &= \alpha \min_{\mathcal{C}}[\int_T fdP + w_1 P(T^c)] + (1-\alpha)\min_{\mathcal{C}}[\int_T fdP + w_0 P(T^c)] \\ &\leq \min_{\mathcal{C}}[\int_T fdP + (\alpha w_1 + (1-\alpha) w_0) P(T^c)] = I_f(\alpha w_1 + (1-\alpha) w_0) \end{aligned}$$

(c) Let $\{w_n\} \subset \mathbb{R}$ be such that $w_n \to w$. Then, $f(w_n) \to f(w)$ in the supnorm topology. Continuity of I implies $I(f(w_n)) \to I(f(w))$. Hence, $I_f(w_n) \to I_f(w)$. ∎

A.5.2(b) Proof of Proposition 6

Proof. If $Q \in co\{\mathcal{C}_{MIN}, \mathcal{C}_{MAX}\}$, then for any $f(w)$, $\int fdQ = \gamma \int fdZ_1 + (1-\gamma)\int fdZ_0$, some $\gamma \in [0,1]$ and $Z_1 \in \mathcal{C}_{MIN}$, $Z_0 \in \mathcal{C}_{MAX}$. Hence,

$$\begin{aligned} \int fdQ &= \gamma \int fdZ_1 + (1-\gamma)\int fdZ_0 \\ &\geq \min\left\{\int fdZ_1, \int fdZ_0\right\} \geq \min\left\{\tilde{\tilde{I}}(f), \tilde{I}(f)\right\} = \hat{I}(f) \end{aligned}$$

That is, for any f, $I(f) = \hat{I}(f)$. By Condition NC, $f(w') \succsim g(w')$ at w' implies $I(f(w)) = \hat{I}(f(w)) \geq \hat{I}(g(w)) = I(g(w))$ for any w. That is, $T \in \mathcal{A}_Z$. ∎

A.5.2(c) Proof of Corollary 1

Proof. Under the hypothesis in the statement, the condition $\mathcal{C} = co\{\mathcal{C}_{MIN}, \mathcal{C}_{MAX}\}$ is automatically satisfied. In this case, Condition NC means that P_1 and P_2 order mappings of the type $f(w)$ in exactly the same way. Under nonatomicity, this is equivalent to saying that they have the same conditionals both on T and T^c (see [3]). ∎

A.5.2(d) Proof of Proposition 7

This is a corollary to Theorem 4, which we prove later. We refer the reader to that proof.

A.5.3 Necessary and sufficient conditions

A.5.3(a) Proof of Lemma 2

We begin by establishing a simple fact. Recall that for $\mathcal{C}$ a set of priors, the notation $\mathcal{C}\mid_T$ stands for the set of conditional probabilities computed from probabilities in $\mathcal{C}$.

LEMMA 4. *If $\mathcal{C}_{MIN}\mid_T = \mathcal{C}_{MAX}\mid_T$, then Condition NC in the main text is satisfied.*

Proof. Let f, g be of type $f(w)$ and $g(w)$, respectively. Observe that, if $\mathcal{C}_{MIN}\mid_T = \mathcal{C}_{MAX}\mid_T$, then,

$$\begin{array}{rcl}
\approx{I}(f(w)) & \geq & \approx{I}(g(w)) \\
& \Longleftrightarrow & \\
\min_{\mathcal{C}_{MIN}}[\int_T fdP + wP(T^c)] & \geq & \min_{\mathcal{C}_{MIN}}[\int_T gdP + wP(T^c)] \\
& \Longleftrightarrow & \\
\min_{\mathcal{C}_{MIN}}[P(T)\int_T fdP(\cdot & \mid & T) + w(1-P(T)] \geq \min_{\mathcal{C}_{MIN}}[P(T)\int_T gdP(\cdot \mid T) + w(1-P(T)] \\
& \Longleftrightarrow & \\
\min_{\mathcal{C}_{MIN}\mid_T}\int_T fdP(\cdot & \mid & T) \geq \min_{\mathcal{C}_{MIN}\mid_T}\int_T gdP(\cdot \mid T) \\
& \Longleftrightarrow & (by\ the\ assumption) \\
\min_{\mathcal{C}_{MAX}\mid_T}\int_T fdP(\cdot & \mid & T) \geq \min_{\mathcal{C}_{MAX}\mid_T}\int_T gdP(\cdot \mid T) \\
& \Longleftrightarrow & \\
\tilde{I}(f(w)) & \geq & \tilde{I}(g(w))
\end{array}$$

Hence,

$$\hat{I}_f(w) \geq \hat{I}_g(w) \qquad for\ any\ w \in R$$

because of the w-invariance of the functionals $\approx{I}$ and $\tilde{I}$.

Now, if $T \in \mathcal{A}_Z$ and $f(w) \succsim g(w)$ for some w, then $f(w') \succsim g(w')$ for any $w' \in \mathbb{R}$. By Lemma 1, $\exists w^*$ such that $\approx{I}(f(w^*)) \geq \approx{I}(g(w^*))$. Hence, the conclusion follows from the w-invariance of $\approx{I}$ and the previous observation. ∎

However, it is evident that, in general, the condition in the previous lemma is more than it is necessary. A weaker condition is obtained as follows. For f of the above type, define

$$\begin{array}{rcl}
\approx{\Pi}_f & = & \left\{P \in \mathcal{C}_{MIN} \mid \int fdP = \min\right\} \\
\tilde{\Pi}_f & = & \left\{P \in \mathcal{C}_{MAX} \mid \int fdP = \min\right\}
\end{array}$$

Both sets are nonempty (because $\mathcal{C}_{MIN}$ and $\mathcal{C}_{MAX}$ are closed (hence, weak*-compact) in the weak*-compact set $\mathcal{C}$). In general, neither set is a singleton. Note, however, that if P and Q are in, say, $\approx{\Pi}_f$, then we must have

$$\int_T fdP = \int_T fdQ \tag{6}$$

[from $\int f dP = \int_T f dP + wP(T^c) = \int_T f dQ + wQ(T^c)$ and $P, Q \in \tilde{\tilde{\Pi}}_f$]

Hence, the value $\min\limits_{\mathcal{C}_{MIN}} \int_T f dP$ is independent of the choice of the minimizer. Similarly, for $\tilde{I}$ in the place of $\tilde{\tilde{I}}$.

Lemma 2 Condition NC is equivalent to the condition that for any act f

$$(\tilde{\tilde{\Pi}}_f \mid_T) \cap (\tilde{\Pi}_f \mid_T) \neq \emptyset$$

Proof of Sufficiency. The proof is essentially the same as the proof of the previous lemma. The observation that the value

$$\min_{\mathcal{C}_{MIN}|_T} \int_T f dP(\cdot \mid T) = \frac{1}{\alpha} \min_{\mathcal{C}_{MIN}} \int_T f dP$$

($\alpha = \min\limits_{\mathcal{C}} P(T)$) is independent of the choice of the minimizer, allows us to select any measure in $\tilde{\tilde{\Pi}}_f$ to express such a value. Hence, the proof is completed by selecting exactly those measures in $\tilde{\tilde{\Pi}}.$ and $\tilde{\Pi}.$ whose conditionals on T coincide. ∎

In order to prove the necessity of the condition, we need two additional results. The first is a separation theorem proven in [4] (In [4], this is stated as Corollary 5).

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be weak*-compact subsets of $ba^1(\Sigma)$, and assume that both $\mathcal{C}_1$ and $\mathcal{C}_2$ consist of countably additive measures. Further, assume that every measure is nonatomic and that $\mathcal{C}_1 \cup \mathcal{C}_2$ is thin. All these assumptions are satisfied in our setting. We have

THEOREM 6 (Amarante and Maccheroni [4]). *$\mathcal{C}_1 \cap \mathcal{C}_2 = \varnothing$ iff there exist $A, B \in \Sigma$, $A \cap B = \emptyset$, such that $P(A) - P(B) > 0 > Q(A) - Q(B)$ for any $P \in \mathcal{C}_1$ and any $Q \in \mathcal{C}_2$.*

The second result is stated in the following lemma.

LEMMA 5. *For any $f \in B(\Sigma)$, the correspondence $f \longmapsto \tilde{\tilde{\Pi}}_f$ ($f \longmapsto \tilde{\Pi}_f$) is upper hemi-continuous.*

Proof. This is an immediate consequence of Berge's maximum theorem (see Aliprantis and Border [1]). ∎

Proof of Necessity. We are going to show that if the condition fails, then so does Condition NC.

Assume that there exists an f for which

$$(\tilde{\tilde{\Pi}}_f \mid_T) \cap (\tilde{\Pi}_f \mid_T) = \emptyset$$

f is simple. So let

$$f = \sum_i^n \lambda_i \chi_{H_i} + w\chi_{T^c}$$

Now, pick a $\tilde{\tilde{P}}_f \in \tilde{\tilde{\Pi}}_f \subset \mathcal{C}_{MIN}$ and a $\tilde{P}_f \in \tilde{\Pi}_f \subset \mathcal{C}_{MAX}$. Then,

$$\tilde{\tilde{P}}_f(\cdot \mid T) \neq \tilde{P}_f(\cdot \mid T)$$

Because of the assumption, this property does not depend on the choice we made.

By Theorem 6, there exist two disjoint sets $A, B \subset T$ $(A, B \in \Sigma)$ such that

$$\begin{aligned} \approx{P}_f(A \quad | \quad T) > \approx{P}_f(B \mid T) \qquad and \qquad \tilde{P}_f(A \mid T) < \tilde{P}_f(B \mid T) \\ \Longrightarrow \\ \approx{P}_f(A) \quad > \quad \approx{P}_f(B) \qquad and \qquad \tilde{P}_f(A) < \tilde{P}_f(B) \end{aligned} \tag{7}$$

for all $\approx{P}_f \in \mathcal{C}_{MIN}$ and all $\tilde{P}_f \in \mathcal{C}_{MAX}$.

The collection $\mathcal{H} = \{H_i\}$ is a partition of T. Consider the partition of T given by

$$\mathcal{Z} = \{A, B, T \backslash (A \cup B)\}$$

Further consider the partition of T given by

$$\mathcal{H} \wedge \mathcal{Z}$$

Then f is simple on $\mathcal{H} \wedge \mathcal{Z}$, and can be rewritten (in an obvious notation) as

$$f = \sum_i^m \gamma_i \chi_{\Gamma_i} + w\chi_{T^c}$$

Moreover (by construction), there exist two disjoint subsets $K, J \subset \{1, ..., m\}$ such that

$$\sum_K \chi_{\Gamma_i} = 1_A \qquad and \qquad \sum_J \chi_{\Gamma_i} = 1_B$$

Now, for $\varepsilon > 0$ consider the simple function defined by

$$h_\varepsilon = \sum_K (\gamma_i + \varepsilon)\chi_{\Gamma_i} + \sum_J (\gamma_i - \varepsilon)\chi_{\Gamma_i} + \sum_{i \notin K \cup J} \gamma_i \chi_{\Gamma_i} + w\chi_{T^c}$$

Notice that for every measure Q

$$\int h_\varepsilon dQ - \int f dQ = \varepsilon[Q(A) - Q(B)] \tag{8}$$

CLAIM 1: For any $w \in R$, we have

$$\varepsilon[\approx{P}_{h_\varepsilon}(A) - \approx{P}_{h_\varepsilon}(B)] \leq \approx{I}_{h_\varepsilon}(w) - \approx{I}_f(w) \leq \varepsilon[\approx{P}_f(A) - \approx{P}_f(B)] \tag{9}$$

and

$$\tilde{I}_{h_\varepsilon}(w) - \tilde{I}_f(w) \leq \varepsilon[\tilde{P}_f(A) - \tilde{P}_f(B)] \tag{10}$$

Proof of CLAIM 1: From equation 8, we have

$$\begin{aligned} \approx{I}(f(w)) \quad &= \quad \int f(w) d\approx{P}_f = \int h_\varepsilon(w) d\approx{P}_f - \varepsilon[\approx{P}_f(A) - \approx{P}_f(B)] \\ &\geq \quad \approx{I}(h_\varepsilon(w)) - \varepsilon[\approx{P}_f(A) - \approx{P}_f(B)] \end{aligned}$$

which proves one inequality in (9). As for the other,

$$\begin{aligned}\approx{I}(h_\varepsilon(w)) &= \int h_\varepsilon(w)d\approx{P}_{h_\varepsilon} = \int f(w)d\approx{P}_{h_\varepsilon} + \varepsilon[\approx{P}_{h_\varepsilon}(A) - \approx{P}_{h_\varepsilon}(B)] \\ &\geq \approx{I}(f(w)) + \varepsilon[\approx{P}_{h_\varepsilon}(A) - \approx{P}_{h_\varepsilon}(B)]\end{aligned}$$

Similarly for (10).

CLAIM 2: There exist $\varepsilon^* > 0$ such that, for any $w \in \mathbb{R}$

$$\approx{I}_{h_{\varepsilon^*}}(w) > \approx{I}_f(w)$$

Proof of CLAIM 2: By the way of contradiction, suppose that there is no $\varepsilon > 0$ such that $\approx{I}_{h_\varepsilon}(w) > \approx{I}_f(w)$. Then, for all $\varepsilon > 0$, $\approx{I}_{h_\varepsilon}(w) \leq \approx{I}_f(w)$, and inequality (9) implies $[\approx{P}_{h_\varepsilon}(A) - \approx{P}_{h_\varepsilon}(B)] \leq 0$ for all $\approx{P}_{h_\varepsilon} \in \approx{\Pi}_{h_\varepsilon}$.

For a sequence $\{\varepsilon_n\} \subset R_{++}$ such that $\varepsilon_n \to 0$, h_{ε_n} converges (sup-norm) to f. Hence, the upper hemicontinuity of the correspondence $f \longmapsto \approx{\Pi}_f$ implies that if $\left\{\approx{P}_{h_{\varepsilon_n}}\right\}$ is a sequence such that $\approx{P}_{h_{\varepsilon_n}} \in \approx{\Pi}_{h_{\varepsilon_n}}$ and $\approx{P}_{h_{\varepsilon_n}}$ converges to $\bar{P}$, then $\bar{P} \in \approx{\Pi}_f$. Clearly, such convergent sequences exist. But, the limit measure $\bar{P}$ has the property that $\bar{P}(A) - \bar{P}(B) \leq 0$ and cannot be $\approx{\Pi}_f$ because of (7).

We can now complete the proof of the lemma by simply observing that

$$\approx{I}_{h_{\varepsilon^*}} > \approx{I}_f$$

as just shown. Moreover, by (7) and (10), $\tilde{I}_f > \tilde{I}_{h_\varepsilon}$ for all $\varepsilon > 0$. Hence, in particular

$$\tilde{I}_{h_{\varepsilon^*}} < \tilde{I}_f$$

By Lemma 1, $\exists w^* \leq \min\left\{w_f^*, w_{h_{\varepsilon^*}}^*\right\}$ such that f and h_{ε^*} are all evaluated by the $\approx{I}$ functionals, that is $I_z(w^*) = \approx{I}_z(w^*)$, for $z = f, h_{\varepsilon^*}$. By the same reason, $\exists w^{**} \geq \max\left\{w_f^{**}, w_{h_{\varepsilon^*}}^{**}\right\}$ such that f and h_{ε^*} are all evaluated by the $\tilde{I}$ functionals.

Therefore

$$\hat{I}_{h_{\varepsilon^*}}(w^*) > \hat{I}_f(w^*)$$

and

$$\hat{I}_{h_{\varepsilon^*}}(w^{**}) < \hat{I}_f(w^{**})$$

That is, if the condition in the statement fails, then there exists a pair of functions (f, h_{ε^*}) such that Condition NC fails. ∎

A.5.3(b) Proof of Theorem 2

Before proving Theorem 2, we need one more lemma. By virtue of the previous lemma, for any act f, there exist a $\approx{P}_f \in \approx{\Pi}_f$ and a $\tilde{P}_f \in \tilde{\Pi}_f$ whose conditionals on T coincide.

Lemma 6. *Assume that Condition NC holds. If there exist an f and a w such that $I_f(w) < \hat{I}_f(w)$, then*

$$\text{either} \qquad \Pi_f \mid_T \cap \overset{\approx}{\Pi}_f \mid_T = \varnothing \qquad \text{or} \qquad \Pi_f \mid_T \cap \tilde{\Pi}_f \mid_T = \varnothing$$

Proof. Suppose not, that is both sets are nonempty. Hence, there exist Q_1 and Q_2 in Π_f such that

$$\begin{aligned} Q_1(\cdot \mid T) &= \tilde{R}(\cdot \mid T) \qquad \text{for some } \tilde{R} \in \tilde{\Pi}_f \\ Q_2(\cdot \mid T) &= \overset{\approx}{R}(\cdot \mid T) \qquad \text{for some } \overset{\approx}{R} \in \overset{\approx}{\Pi}_f \end{aligned} \tag{11}$$

If there exist an f and a w such that $I_f(w) < \hat{I}_f(w)$, then at w we have the following inequalities

$$\begin{aligned} \int_T f(w) dQ_2 + wQ_2(T^c) &< \int_T f(w) d\overset{\approx}{R} + w\overset{\approx}{R}(T^c) \\ \int_T f(w) dQ_1 + wQ_1(T^c) &< \int_T f(w) d\tilde{R} + w\tilde{R}(T^c) \end{aligned}$$

because both Q_1 and Q_2 are in Π_f. These are equivalent to

$$\begin{aligned} Q_2(T) \int_T f(w) dQ_2(\cdot \mid T) - \overset{\approx}{R}(T) \int_T f(w) d\overset{\approx}{R}(\cdot \mid T) &< w[Q_2(T) - \overset{\approx}{R}(T)] \\ Q_1(T) \int_T f(w) dQ_1(\cdot \mid T) - \tilde{R}(T) \int_T f(w) d\tilde{R}(\cdot \mid T) &< w[Q_1(T) - \tilde{R}(T)] \end{aligned}$$

By (11), these imply

$$\begin{aligned} [Q_2(T) - \overset{\approx}{R}(T)] \int_T f(w) d\overset{\approx}{R}(\cdot \mid T) &< w[Q_2(T) - \overset{\approx}{R}(T)] \\ [Q_1(T) - \tilde{R}(T)] \int_T f(w) d\tilde{R}(\cdot \mid T) &< w[Q_1(T) - \tilde{R}(T)] \end{aligned}$$

Now, let $\overset{\approx}{P}_f$ and $\tilde{P}_f$ be two measures in $\overset{\approx}{\Pi}_f$ and $\tilde{\Pi}_f$ whose conditionals on T coincide. Such measures exist (by Lemma 2) because we assumed that Condition NC holds. By equation (6) (before the proof of Lemma 2), we have

$$\begin{aligned} \int_T f(w) d\overset{\approx}{R}(\cdot \mid T) &= \int_T f(w) d\overset{\approx}{P}_f(\cdot \mid T) \\ \int_T f(w) d\tilde{R}(\cdot \mid T) &= \int_T f(w) d\tilde{P}_f(\cdot \mid T) \end{aligned}$$

because both $\overset{\approx}{R}$ and $\overset{\approx}{P}_f$ [$\tilde{R}$ and $\tilde{P}_f$] are in $\overset{\approx}{\Pi}_f$ [$\tilde{\Pi}_f$]. Moreover, $\overset{\approx}{P}_f(\cdot \mid T) = \tilde{P}_f(\cdot \mid T)$. Hence, the previous inequalities can be rewritten as

$$\begin{aligned} [Q_2(T) - \overset{\approx}{P}_f(T)] \int_T f(w) d\overset{\approx}{P}_f(\cdot \mid T) &< w[Q_2(T) - \overset{\approx}{P}_f(T)] \\ [Q_1(T) - \tilde{P}_f(T)] \int_T f(w) d\overset{\approx}{P}_f(\cdot \mid T) &< w[Q_1(T) - \tilde{P}_f(T)] \end{aligned} \tag{12}$$

Since neither Q_1 nor Q_2 are in $\overset{\approx}{\Pi}_f \cup \tilde{\Pi}_f$ (because this would contradict $I_f(w) < \hat{I}_f(w)$), we have $[Q_2(T) - \overset{\approx}{P}_f(T)] > 0 > [Q_1(T) - \tilde{P}_f(T)]$. Hence, inequalities (12) imply

$$\int_T f(w) d\overset{\approx}{P}_f(\cdot \quad | \quad T) < w$$
$$\int_T f(w) d\overset{\approx}{P}_f(\cdot \quad | \quad T) > w$$

a contradiction. ∎

We are now ready to prove Theorem 2.

Proof of Theorem 2. Sufficiency is immediate. Necessity of (i) was shown in the first part. We have only to show the necessity of (ii). To this end, assume that (i) holds and suppose, by the way of contradiction, that there exists an $f(w) = fTw$ such that $I_f(w) < \hat{I}_f(w)$. By the previous lemma, it must be case that

$$\textit{either} \qquad \Pi_f\ |_T \cap \overset{\approx}{\Pi}_f\ |_T = \varnothing \qquad \textit{or} \qquad \Pi_f\ |_T \cap \tilde{\Pi}_f\ |_T = \varnothing$$

Without loss, assume that the first set is empty. By Theorem 6 (stated in the proof of Lemma 2, above), there exist two disjoint sets $A, B \subset T$ ($A, B \in \Sigma$) such that for any $\overset{\approx}{P}_f \in \overset{\approx}{\Pi}_f$ and any $P_f \in \Pi_f$, we have

$$\overset{\approx}{P}_f(A \quad | \quad T) > \overset{\approx}{P}_f(B \mid T) \qquad \textit{and} \qquad P_f(A \mid T) < P_f(B \mid T) \tag{13}$$
$$\Longrightarrow$$
$$\overset{\approx}{P}_f(A) \quad > \quad \overset{\approx}{P}_f(B) \qquad \textit{and} \qquad P_f(A) < P_f(B)$$

Now, we can proceed by defining a new function h_ε, $\varepsilon > 0$, exactly as in the proof of Lemma 2. Immediately, we have that for any $w' \in R$,

$$I_{h_\varepsilon}(w') - I_f(w') \leq \varepsilon[P_f(A) - P_f(B)] \tag{14}$$

The proof of this statement is exactly the same as the proof of CLAIM 1 in the proof of Lemma 2. For any $\varepsilon > 0$, there exists (by Lemma 1) a w^*_ε such that, $I_f(w^*_\varepsilon) = \overset{\approx}{I}_f(w^*_\varepsilon)$ and $I_{h_\varepsilon}(w^*_\varepsilon) = \overset{\approx}{I}_{h_\varepsilon}(w^*_\varepsilon)$. By CLAIM 2 in the proof of Lemma 2, we can find an ε^* (and an associated w^*_ε) such that

$$I_{h_{\varepsilon^*}}(w^*_\varepsilon) > I_f(w^*_\varepsilon)$$

At w, by assumption $I_f(w) < \hat{I}_f(w)$. It then follows from (13) and (14), that

$$I_f(w) > I_{h_\varepsilon}(w)$$

for all $\varepsilon > 0$. That is, the expression $[I_f(w') - I_h(w')]$ changes sign as w varies, and this contradict $T \in \mathcal{A}_Z$. ∎

A.5.4 Epstein-Zhang unambiguous events

Let f be such that

$$(\tilde{\tilde{\Pi}}_f \mid_T) \cap (\tilde{\Pi}_f \mid_T) = \emptyset$$

From Theorem 6, we know that there exist two disjoint sets $A, B \subset T$ $(A, B \in \Sigma)$ such that

$$\begin{array}{c} \tilde{\tilde{P}}_f(A \mid T) > \tilde{\tilde{P}}_f(B \mid T) \quad and \quad \tilde{P}_f(A \mid T) < \tilde{P}_f(B \mid T) \\ \Longrightarrow \\ \tilde{\tilde{P}}_f(A) > \tilde{\tilde{P}}_f(B) \quad and \quad \tilde{P}_f(A) < \tilde{P}_f(B) \end{array}$$

for all $\tilde{\tilde{P}}_f \in \mathcal{C}_{MIN}$ and a $\tilde{P}_f \in \mathcal{C}_{MAX}$. Define $r = \chi_A - \chi_B$ and $\bar{r} = \chi_B - \chi_A$, and, for $\varepsilon > 0$, consider the two functions

$$\begin{aligned} h_\varepsilon(w) &= f(w) + \varepsilon r(0) \\ g_\varepsilon(w) &= f(w) + \varepsilon \bar{r}(0) \end{aligned}$$

Lemma 7. *If there exists an f for which*

$$(\tilde{\tilde{\Pi}}_f \mid_T) \cap (\tilde{\Pi}_f \mid_T) = \emptyset$$

then, there exist $\varepsilon > 0$ and two points, w and w' such that

$$\hat{I}(h_\varepsilon(w)) > \hat{I}(f(w)) > \hat{I}(g_\varepsilon(w))$$

and

$$\hat{I}(h_\varepsilon(w')) < \hat{I}(f(w')) < \hat{I}(g_\varepsilon(w'))$$

Proof. The existence of an $\varepsilon^* > 0$ and of two points w and w' such that

$$\hat{I}(h_{\varepsilon^*}(w)) > \hat{I}(f(w))$$

and

$$\hat{I}(h_{\varepsilon^*}(w')) < \hat{I}(f(w'))$$

was already shown in Lemma 2. Moreover, the last inequality holds for any $\varepsilon > 0$ (see proof of Lemma 2). By a similar argument, one shows that there exists an $\varepsilon^{**} > 0$ such that (see proof of Lemma 2)

$$\hat{I}(h_{\varepsilon^*}(w)) > \hat{I}(f(w)) > \hat{I}(g_{\varepsilon^{**}}(w))$$

and

$$\hat{I}(h_{\varepsilon^*}(w')) < \hat{I}(f(w')) < \hat{I}(g_{\varepsilon^{**}}(w'))$$

With the inequality $\hat{I}(f(w)) > \hat{I}(g_\varepsilon(w))$ holding for any $\varepsilon > 0$. To complete the proof, we only need to show that we can choose the same ε for both h and g. To this end, it suffices to show that

$$\begin{aligned} \hat{I}(h_{\varepsilon^*}(w)) > \hat{I}(f(w)) \quad &\Longrightarrow \quad \hat{I}(h_\varepsilon(w)) > \hat{I}(f(w)) \quad for\ any\ 0 < \varepsilon < \varepsilon^* \\ \hat{I}(g_{\varepsilon^{**}}(w')) > \hat{I}(f(w')) \quad &\Longrightarrow \quad \hat{I}(g_\varepsilon(w')) > \hat{I}(f(w')) \quad for\ any\ 0 < \varepsilon < \varepsilon^{**} \end{aligned}$$

This follows straight from the concavity of $\hat{I}$. In fact, noticing that for any $0 < \varepsilon < \varepsilon^*$,

$$h_\varepsilon(w) = \alpha h_{\varepsilon^*}(w) + (1-\alpha) f(w) \qquad \textit{some } \alpha \in (0,1)$$

we have

$$\hat{I}(h_\varepsilon(w)) \geq \alpha \hat{I}(h_{\varepsilon^*}(w)) + (1-\alpha)\hat{I}(f(w)) > \hat{I}(f(w))$$

and similarly for $\hat{I}(g_\varepsilon(w'))$. ∎

Proof of Theorem 3. The inclusion $\mathcal{A}_Z \subset \mathcal{A}_{EZ}$ comes straight out of the definitions. We are going to show that $T \notin \mathcal{A}_Z \Longrightarrow T \notin \mathcal{A}_{EZ}$ thus showing that $\mathcal{A}_Z \supset \mathcal{A}_{EZ}$. First, suppose that condition (i) in Theorem 2 is violated. Then, there exist g_ε and h_ε like in the previous lemma so that (recall that, from Lemma 1, the points w and w' can be chosen so that $\hat{I}$ in the inequalities in Lemma 7 can be taken equal to I)

$$\begin{aligned} I(h_\varepsilon(w)) &> I(f(w)) > I(g_\varepsilon(w)) \\ I(h_\varepsilon(w')) &< I(f(w')) < I(g_\varepsilon(w')) \end{aligned}$$

Now, for $n \geq 1$, consider the functions $\left\{\frac{1}{n} f(w)\right\}_{n\geq 1}$. It is easy to check that there exists a $\underline{w}$ and an $\underline{n}$ such that $\arg\min_{\mathcal{C}} \int \frac{1}{n} f(\underline{w}) dP \in \mathcal{C}_{MIN}$ for all $n \geq \underline{n}$ (because eventually $\frac{1}{n} f$ is in a neighborhood of the function that is identically 0 on T). Similarly, there exist $\bar{w}$ and $\bar{n}$ such that $\arg\min_{\mathcal{C}} \int \frac{1}{n} f(\bar{w}) dP \in \mathcal{C}_{MAX}$ for all $n \geq \bar{n}$.

Next, observe that

$$\begin{aligned} \widetilde{\widetilde{\Pi}}_{f(w)} &= \left\{P \in \mathcal{C}_{MIN} \mid \int f(w) dP = \min\right\} = \widetilde{\widetilde{\Pi}}_{\frac{1}{n} f(w)} &= \left\{P \in \mathcal{C}_{MIN} \mid \int \frac{1}{n} f(w) dP = \min\right\} \\ \tilde{\Pi}_{f(w)} &= \left\{P \in \mathcal{C}_{MAX} \mid \int f(w) dP = \min\right\} = \tilde{\Pi}_{\frac{1}{n} f(w)} &= \left\{P \in \mathcal{C}_{MAX} \mid \int \frac{1}{n} f(w) dP = \min\right\} \end{aligned} \tag{15}$$

Define

$$\begin{aligned} g_{\varepsilon,n}(w) &= \frac{1}{n} f(0) + \varepsilon \bar{r}(w) \\ h_{\varepsilon,n}(w) &= \frac{1}{n} f(0) + \varepsilon r(w) \end{aligned}$$

For $n \geq \max\{\bar{n}, \underline{n}\}$, it follows from the previous lemma and the equalities (15) that

$$I(h_{\varepsilon,n}(w)) > I(\frac{1}{n} f(w)) > I(g_{\varepsilon,n}(w)) \qquad, \forall w \leq \underline{w} \tag{16a}$$

$$I(h_{\varepsilon,n}(w')) < I(\frac{1}{n} f(w')) < I(g_{\varepsilon,n}(w')) \qquad, \forall w' \geq \bar{w} \tag{17}$$

Notice that for $w \leq \underline{w}$, I in (16a) is equal to $\widetilde{\widetilde{I}}$. Similarly, for $w' \geq \bar{w}$, I in (17) is equal to $\tilde{I}$ (see proof of Lemma 2). Taking the limits for $n \to \infty$ of both (16a) and (17) and using the (sup-norm) continuity of I, we have

$$I(\varepsilon r(w)) \geq w \widetilde{\widetilde{P}}(T^c) \geq I(\varepsilon \bar{r}(w)) \qquad, \forall w \leq \underline{w} \tag{18a}$$

$$I(\varepsilon r(w)) \leq w\tilde{P}(T^c) \leq I(\varepsilon \bar{r}(w)) \qquad , \forall w' \geq \bar{w} \tag{19}$$

Now, we claim that at least one inequality in (18a) is strict. In fact, at $w \leq \underline{w}$

$$\begin{aligned} I(\varepsilon \bar{r}(w)) &= \min_{\mathcal{C}_{MIN}} \left\{ \varepsilon \int [(\chi_B - \chi_A) + w\chi_{T^c}] dP \right\} \\ &= \min_{\mathcal{C}_{MIN}} \varepsilon \int (\chi_B - \chi_A) dP + w\tilde{\tilde{P}}(T^c) < w\tilde{\tilde{P}}(T^c) \end{aligned}$$

because, for any n, $\tilde{\tilde{\Pi}}_{\frac{1}{n}f(w)} \subset \mathcal{C}_{MIN}$, and we know that $\tilde{\tilde{\Pi}}_{\frac{1}{n}f(w)}$ contains at least a measure P (in fact, all) such that $P(B) - P(A) < 0$. Since $\varepsilon r(w)$ and $\varepsilon \bar{r}(w)$ are EZ-conjugate, this shows that T is EZ-ambiguous.

Finally, a proof that T is EZ-ambiguous if condition (i) in Theorem 2 is satisfied but condition (ii) is violated is obtained exactly along the same lines by taking into account Lemma 6 (see proof of Theorem 2). ∎

A.6 APPLICATIONS

A.6 (a) Proof of Theorem 4

Proof. We prove the statement only for acts of the form fTw. The proof for acts of the form fT^cw follows at once by exchanging T and T^c.

(Necessity) Fix an MEU model with set of priors $\mathcal{C}$. Since $T \in \mathcal{A}_Z$, we have $I = \hat{I}$ for all acts of type fTw (Theorem 2). Hence,

$$I(f(w)) = \min \left\{ \tilde{\tilde{I}}(f(w)), \tilde{I}(f(w)) \right\}$$

$\tilde{\tilde{I}}(f(w))$ and $\tilde{I}(f(w))$ can be written as

$$\begin{aligned} \tilde{I}(f(w)) &= \beta \min_{\mathcal{C}_{MAX}|_T} \int_T f dP(\cdot \mid T) + w(1-\beta) \\ \tilde{\tilde{I}}(f(w)) &= \alpha \min_{\mathcal{C}_{MIN}|_T} \int_T f dP(\cdot \mid T) + w(1-\alpha) \end{aligned}$$

Define

$$\begin{aligned} \Pi_1 &= co\left\{ \mathcal{C}_{MAX} \mid_T \cup \mathcal{C}_{MIN} \mid_T \right\} \\ \Pi_2 &= co\left\{ \mathcal{C}_{MAX} \mid_{T^c} \cup \mathcal{C}_{MIN} \mid_{T^c} \right\} \end{aligned}$$

and

$$\mathcal{C}^* = [\alpha, \beta]\Pi_1 \times [1-\beta, 1-\alpha]\Pi_2$$

For any $f(w)$, it can be readily checked that

$$\min_{\mathcal{C}^*} \int f(w) dP = \min \left\{ \alpha \min_{\Pi_1} \int_T f dP(\cdot \mid T) + w(1-\alpha), \beta \min_{\Pi_1} \int_T f dP(\cdot \mid T) + w(1-\beta) \right\}$$

Since $\Pi_1 \supset \mathcal{C}_{MAX} \mid_T \cup \mathcal{C}_{MIN} \mid_T$, we have

$$\min_{\mathcal{C}^*} \int f(w) dP \leq \min_{\mathcal{C}} \int f(w) dP \tag{20}$$

We are going to show that equality holds.

By definition of Π_1, any minimizer of $\int_T f dP(\cdot \mid T)$ is either in $\mathcal{C}_{MAX} \mid_T$ or in $\mathcal{C}_{MIN} \mid_T$, Hence, let $\bar{P} \in \mathcal{C}_{MAX} \mid_T \cup \mathcal{C}_{MIN} \mid_T$ be such that

$$\int_T f d\bar{P} = \min_{\mathcal{C}} \int f dP$$

Suppose that $\bar{P} \in \mathcal{C}_{MAX} \mid_T$. By definition, there exists a $\bar{\bar{P}} \in \mathcal{C}_{MAX}$ whose conditional on T coincides with $\bar{P}$.

CLAIM: if $Q \in \tilde{\Pi}_f$, then $\int_T f dQ(\cdot \mid T) \leq \int_T f d\bar{P}$.

In fact, $Q \in \tilde{\Pi}_f$ implies that for any measure in $\mathcal{C}_{MAX}$, hence for $\bar{\bar{P}}$, we have

$$\begin{aligned} \min_{\mathcal{C}_{MAX}} \int f dP &= \int f dQ \\ &= Q(T) \int_T f dQ(\cdot \mid T) + wQ(T^c) \\ &\leq \bar{\bar{P}}(T) \int_T f d\bar{P}(\cdot \mid T) + w\bar{\bar{P}}(T^c) \end{aligned}$$

which implies $\int_T f dQ(\cdot \mid T) \leq \int_T f d\bar{P}$.

By definition, $Q(\cdot \mid T) \in \mathcal{C}_{MAX} \mid_T$ (because $\tilde{\Pi}_f \subset \mathcal{C}_{MAX}$). Hence, $\int_T f dQ(\cdot \mid T) = \int_T f d\bar{P}$.

Since $T \in \mathcal{A}_Z$, Condition NC holds and by Lemma 2

$$(\tilde{\tilde{\Pi}}_f \mid_T) \cap (\tilde{\Pi}_f \mid_T) \neq \emptyset$$

That is, there exists a $\tilde{\tilde{P}}_f(\cdot \mid T) \in (\tilde{\tilde{\Pi}}_f \mid_T)$ such that

$$\int_T f d\bar{P} = \int_T f d\tilde{\tilde{P}}_f(\cdot \mid T)$$

By definition, $\tilde{\tilde{P}}_f(\cdot \mid T) \in \mathcal{C}_{MIN} \mid_T$. Summing up, for any $f(w)$

$$\begin{aligned} \min_{\Pi_1} \int_T f(w) dP &= \min_{\mathcal{C}_{MAX}\mid_T \cup \mathcal{C}_{MIN}\mid_T} \int_T f(w) dP = \int_T f d\bar{P} = \int_T f dQ(\cdot \mid T) \quad (21) \\ &= \min_{\mathcal{C}_{MAX}\mid_T} \int_T f(w) dP = \min_{\mathcal{C}_{MIN}\mid_T} \int_T f(w) dP \end{aligned}$$

Similarly, we reach the same conclusion if we start by assuming that $\bar{P} \in \mathcal{C}_{MIN} \mid_T$.

Now, (21) show that equality holds in (20), that is

$$\min_{\mathcal{C}^*} \int f dP = \hat{I} = \min_{\mathcal{C}} \int f dP$$

(Sufficiency) Immediate. ∎

A.6(b) Proof of Theorem 5

We divide the proof into several claims. To begin, let $T \in \mathcal{A}_Z \backslash \mathcal{A}_{NM}$ and let $A \subset T$. Define

$\mathcal{C}^T_{MAX} = \{P \in \mathcal{C} | P \in \arg\max_{P\in\mathcal{C}} P(T)\}$

$\mathcal{C}^T_{MIN} = \{P \in \mathcal{C} | P \in \arg\min_{P\in\mathcal{C}} P(T)\}$

$\mathcal{C}^A_{MAX} = \{P \in \mathcal{C} | P \in \arg\max_{P\in\mathcal{C}} P(A)\}$

$\mathcal{C}^A_{MIN} = \{P \in \mathcal{C} | P \in \arg\min_{P\in\mathcal{C}} P(A)\}$

Let $\alpha = \min P(T)$, $\beta = \max P(T)$, $\overline{\alpha} = \min P(A)$, and $\overline{\beta} = \max P(A)$. Assume $\alpha \neq \overline{\alpha}$ and $\beta \neq \overline{\beta}$.

CLAIM 1 $\mathcal{C}^T_{MAX} \cap \mathcal{C}^A_{MAX} \neq \varnothing$, and $\mathcal{C}^T_{MIN} \cap \mathcal{C}^A_{MIN} \neq \varnothing$.

Proof. Let $f(w) = 0Aw$. Observe that $f(w)$ is an act which is constan both on A^c and on T^c. Therefore, by Theorems 4 and 2

$$I(f(w)) = \widehat{I}^T(f(w)) = \widehat{I}^A(f(w)),$$

where $\widehat{I}^T(f(w)) = \min\{\min_{P\in\mathcal{C}^T_{MIN}} \int f(w)dP, \min_{P\in\mathcal{C}^T_{MAX}} \int f(w)dP\}$. We have,

$$\begin{aligned}
\widehat{I}^T(f(w)) &= \min\left\{\alpha(\min_{P\in\mathcal{C}^T_{MIN}} \int_T f dP(.|T) - w) + w, \beta(\min_{P\in\mathcal{C}^T_{MAX}} \int_T f dP(.|T) - w) + w\right\} \\
&= \min\left\{\alpha(\min_{P\in\mathcal{C}^T_{MIN}} (wP(T\backslash A|T) - w)) + w, \beta(\min_{P\in\mathcal{C}^T_{MAX}} (wP(T\backslash A|T) - w)) + w\right\}
\end{aligned}$$

By $(\stackrel{\approx}{\Pi}_f |_T) \cap (\tilde{\Pi}_f |_T) \neq \emptyset$ (Lemma 2), $\min_{P\in\mathcal{C}^T_{MIN}} (wP(T\backslash A|T) - w)) = \min_{P\in\mathcal{C}^T_{MAX}} (wP(T\backslash A|T) - w))$. Since $w > 0$, we have

$$\begin{aligned}
\widehat{I}^T(f(w) &= \beta(\min_{P\in\mathcal{C}^T_{MAX}} (wP(T\backslash A|T) - w)) + w \\
&= \beta(wP^*(T\backslash A|T) - w)) + w \qquad for\ some\ P^* \in \mathcal{C}^T_{MAX} \\
&= \widehat{I}^A(f(w)) \\
&= \min_{\overline{\alpha},\overline{\beta}}\{w(1-\overline{\alpha}), w(1-\overline{\beta})\} \\
&= w(1-\overline{\beta})
\end{aligned}$$

Hence,

$$\beta(wP^*(T\backslash A|T) - w)) + w = w(1-\overline{\beta})$$

which implies

$$P^*(T\backslash A|T) = \frac{\beta - \overline{\beta}}{\beta}$$

From which we conclude (since $P^* \in \mathcal{C}^T_{MAX}$) that

$$P^*(T\backslash A) = \beta - \overline{\beta} \Longrightarrow P^*(A) = \overline{\beta} \Longrightarrow P^* \in \mathcal{C}^A_{MAX} \Longrightarrow \mathcal{C}^A_{MAX} \cap \mathcal{C}^T_{MAX} \neq \varnothing$$

Similarly, one shows that $C^T_{MIN} \cap C^A_{MIN} \neq \varnothing$ by taking $w < 0$. ∎

Before tackling the proof of Theorem 5, we need one more observation. Let $f \in \mathcal{F}$ be a three step act:

$$f = \begin{cases} k \text{ if } s \in A \\ k' \text{ if } s \in T \backslash A \\ k'' \text{ if } s \in T^c \end{cases}$$

Such an act can be thought of in two different ways. As an act of the form fTk'' and as an act of the form $fA^c k$. For $T \in \mathcal{A}_{EZ}$, from Theorem 4 we have that $I(f(k'')) = \widehat{I}^T(f(k''))$. Hence,

$$\begin{aligned} & \widehat{I}^T(f(k'')) \\ = \quad & \min \left\{ \alpha(\min_{P \in \mathcal{C}^T_{MIN}} kP(A|T) + k'P(T \backslash A|T) - k'') + k'', \beta(\min_{P \in \mathcal{C}^T_{MAX}} kP(A|T) + k'P(T \backslash A|T) - k'') \right. \\ = \quad & \min \left\{ \alpha(\min_{P \in \mathcal{C}^T_{MIN}} (k-k')P(A|T) + k' - k'') + k'', \beta(\min_{P \in \mathcal{C}^T_{MAX}} (k-k')P(A|T) + k' - k'') + k'' \right\} \end{aligned}$$

By choosing k'' small enough, we can guarantee that α and some $P \in \mathcal{C}^T_{MIN}$ attain the minimum. Moreover, for $k > k'$ we can guarantee that the P in $\mathcal{C}^T_{MIN}$ is such that $P(A) = \overline{\alpha}$ (existence of such a P in $\mathcal{C}^T_{MIN}$ was shown in CLAIM 1 above).

In the same fashion, by choosing k'' big enough, we can guarantee that the minimum obtains for β and some P' in $\mathcal{C}^T_{MAX}$. Moreover, for $k > k'$ such a P' would be such that $P'(A) = \mu = \min_{P \in \mathcal{C}^T_{MAX}} P(A)$ is used. Summarizing, for $k > k'$ the evaluation of a three step act like ours is either

$$\widehat{I}^T(f(k'')) = \overline{\alpha}k + (\alpha - \overline{\alpha})k' + (1-\alpha)k'' \tag{22}$$

or

$$\widehat{I}^T(f(k'')) = \mu k + (\beta - \mu)k' + (1-\beta)k'' \tag{23}$$

By definition of $\overline{\alpha}$, and $\overline{\beta}$, we have $\overline{\alpha} \leq \mu \leq \overline{\beta}$.

We can prove our first result. The proof is restricted to the case $\overline{\alpha} < \mu$. The case $\overline{\alpha} = \mu$ will be dealt with in the proof of Proposition 10, below.

Proposition 9. *If $T \in \mathcal{A}_{EZ} \backslash \mathcal{A}_{NM}$, then there is no $A \subset T$ such that $A \in \mathcal{A}_{EZ} \backslash \mathcal{A}_{NM}$.*

Proof. We proceed by contradiction. Let $f(k'')$, $g(k'')$, $f(\overline{k})$ and $g(\overline{k})$ be four three step acts

$$f(k'') = \begin{cases} k \text{ if } s \in A \\ k' \text{ if } s \in T \backslash A \\ k'' \text{ if } s \in T^c \end{cases} \quad \text{and} \quad g(k'') = \begin{cases} t \text{ if } s \in A \\ t' \text{ if } s \in T \backslash A \\ k'' \text{ if } s \in T^c \end{cases}$$

$$f(\overline{k}) = \begin{cases} k \text{ if } s \in A \\ k' \text{ if } s \in T \backslash A \\ \overline{k} \text{ if } s \in T^c \end{cases} \quad \text{and} \quad g(\overline{k}) = \begin{cases} t \text{ if } s \in A \\ t' \text{ if } s \in T \backslash A \\ \overline{k} \text{ if } s \in T^c \end{cases}$$

where we choose $k > k'$ and $t > t'$.

For k'' small enough we can guarantee that $f(k'')$ and $g(k'')$ are evaluated as in (22), i.e.

$$I(f(k'')) = \widehat{I}^T(f(k'')) = \overline{\alpha}k + (\alpha - \overline{\alpha})k' + (1-\alpha)k''$$

$$I(g(k'')) = \widehat{I}^T(g(k'')) = \overline{\alpha}t + (\alpha - \overline{\alpha})t' + (1-\alpha)k''$$

while for $\overline{k}$ big enough we can guarantee that $f(\overline{k})$ and $g(\overline{k})$ are evaluated as in (23), i.e.

$$I(f(\overline{k})) = \widehat{I}^T(f(\overline{k})) = \mu k + (\beta - \mu)k' + (1-\beta)\overline{k}$$

$$I(g(\overline{k})) = \widehat{I}^T(g(\overline{k})) = \mu t + (\beta - \mu)t' + (1-\beta)\overline{k}$$

Let us rename variables by setting

$$x = k - t; \qquad y = k' - t'; \qquad a = \overline{\alpha}; \qquad b = \alpha - \overline{\alpha}; \qquad c = \mu; \qquad d = \beta - \mu; \quad (24)$$

Case 1: $\frac{a}{b} < \frac{c}{d}$. We have

$$I(f(k'')) - I(g(k'')) = \overline{\alpha}(k-t) + (\alpha - \overline{\alpha})(k' - t') = ax + by$$

and

$$I(f(\overline{k})) - I(g(\overline{k})) = \mu(k-t) + (\beta - \mu)(k' - t') = cx + dy$$

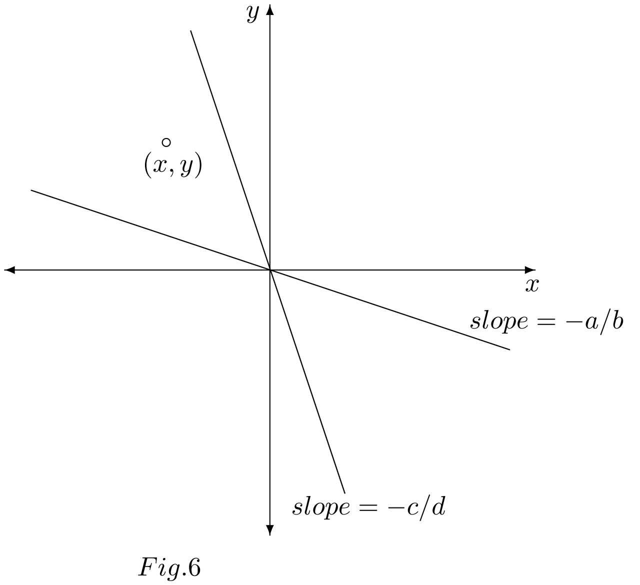

Fig.6

Bt choosing x and y as in Figure 6, we get $I(f(k'')) - I(g(k'')) > 0$ and $I(f(\overline{k})) - I(g(\overline{k})) < 0$, thus contradicting $T \in \mathcal{A}_{EZ}$.

Case 2: $\frac{a}{b} > \frac{c}{d}$. By choosing x and y as in Figure 7, we get $I(f(k''))-I(g(k'')) > 0$ and $I(f(\overline{k})) - I(g(\overline{k})) < 0$, again contradicting $T \in \mathcal{A}_{EZ}$.

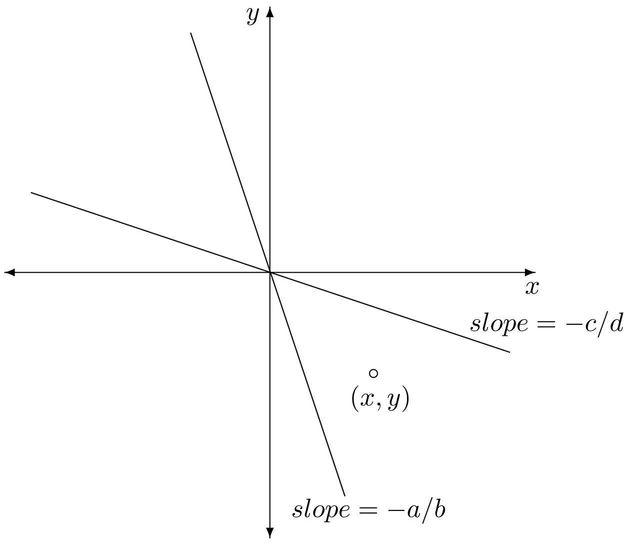

Fig.7

Case 3: $\frac{a}{b} = \frac{c}{d}$. We distinguish between two subcases.
Case 3.1: $b > d$. Observe that

$$b > d \quad \Longleftrightarrow \quad \alpha - \overline{\alpha} > \beta - \mu \quad \Longleftrightarrow \quad \frac{2\overline{\alpha}}{\alpha - \overline{\alpha}} < \frac{2\overline{\alpha}}{\beta - \mu} \quad \Longrightarrow \frac{2\overline{\alpha}}{\alpha - \overline{\alpha}} < \frac{\overline{\alpha} + \mu}{\beta - \mu}$$

Now, take an $\varepsilon > 0$ and choose δ so that $0 < \left(\frac{2\overline{\alpha}}{\alpha - \overline{\alpha}}\right)\varepsilon < \delta < \left(\frac{\overline{\alpha} + \mu}{\beta - \mu}\right)\varepsilon$. Next, choose $\overline{k} > 0$ big enough so that $-\frac{(\beta - \alpha)}{b - d}\overline{k} + \delta < -\varepsilon$. Define

$$k = \varepsilon; \qquad t = -\varepsilon; \qquad k' = -\frac{(\beta - \alpha)}{b - d}\overline{k}; \qquad t' = k' + \delta; \tag{25}$$

By choosing k'' small enough so that both $f(k'')$ and $g(k'')$ are evaluated as in (22), we have

$$\begin{aligned} I(f(k'')) - I(g(k'')) &= \overline{\alpha}(k - t) + (\alpha - \overline{\alpha})(k' - t') \\ &= \overline{\alpha}2\varepsilon + (\alpha - \overline{\alpha})(-\delta) \end{aligned}$$

Hence, because of our choice of δ

$$I(f(k'')) - I(g(k'')) < 0 \tag{27}$$

On the other hand, by definition of a and c, we have $a < c$. Hence,

$$\begin{aligned} (a - c)\varepsilon &< 0 \\ &\Longleftrightarrow \\ (a - c)k + (b - d)k' + (\beta - \alpha)\overline{k} &< 0 \qquad \text{(by (25))} \\ &\Longleftrightarrow \\ \overline{\alpha}k + (\alpha - \overline{\alpha})k' + (1 - \alpha)\overline{k} &< \mu k + (\beta - \mu)k' + (1 - \beta)\overline{k} \qquad \text{(by (24))} \end{aligned}$$

That is, $I(f(\overline{k})) = \overline{\alpha}k + (\alpha - \overline{\alpha})k' + (1-\alpha)\overline{k}$. Next, observe that $c > a$, $b > d$ and $\delta > 0$ imply

$$\begin{aligned} (c-a)(-\varepsilon) + (d-b)\delta &< 0 \\ &\Longleftrightarrow \\ (c-a)t + (d-b)t' + (\alpha-\beta)\overline{k} &< 0 \qquad \text{(because } (d-b)t' + (\alpha-\beta)\overline{k} = (d-b)\delta \text{ by (25))} \\ &\Longleftrightarrow \\ \mu t + (\beta-\mu)t' + (1-\beta)\overline{k} &< \overline{\alpha}t + (\alpha-\overline{\alpha})t' + (1-\alpha)\overline{k} \qquad \text{(by (24))} \end{aligned}$$

That is, $I(g(\overline{k})) = \mu t + (\beta-\mu)t' + (1-\beta)\overline{k}$. Summing up,

$$\begin{aligned} I(f(\overline{k})) - I(g(\overline{k})) &= \left[\overline{\alpha}k + (\alpha-\overline{\alpha})k' + (1-\alpha)\overline{k}\right] - \left[\mu t + (\beta-\mu)t' + (1-\beta)\overline{k}\right] \\ &= (\overline{\alpha}+\mu)\varepsilon + (b-d)k' - (\beta-\mu)\delta + (\beta-\alpha)\overline{k} \\ &= (\overline{\alpha}+\mu)\varepsilon - (\beta-\mu)\delta \end{aligned}$$

Hence,

$$I(f(\overline{k})) - I(g(\overline{k})) > 0 \tag{28}$$

Now, inequality (27) and inequality (28) contradict the assumption $T \in \mathcal{A}_{EZ}$.

Case 3.2: $b < d$

Here, we consider acts which are constant on A, and we will produce a contradiction with the hypothesis that $A \in \mathcal{A}_{EZ}$ ($\Longleftrightarrow A^c \in \mathcal{A}_{EZ}$).

Let $f(k)$ and $g(k)$ be defined by:

$$f(k) = \begin{cases} k \text{ if } s \in A \\ k' \text{ if } s \in T\backslash A \\ k'' \text{ if } s \in T^c \end{cases} ; \qquad g(k) = \begin{cases} k \text{ if } s \in A \\ t' \text{ if } s \in T\backslash A \\ t'' \text{ if } s \in T^c \end{cases}$$

where we choose $k > k'$ and $t > t'$, and let $f(\overline{k})$ and $g(\overline{k})$ be defined by

$$f(\overline{k}) = \begin{cases} \overline{k} \text{ if } s \in A \\ k' \text{ if } s \in T\backslash A \\ k'' \text{ if } s \in T^c \end{cases} ; \qquad g(\overline{k}) = \begin{cases} \overline{k} \text{ if } s \in A \\ t' \text{ if } s \in T\backslash A \\ t'' \text{ if } s \in T^c \end{cases}$$

where we choose $\overline{k} > k'$ and $\overline{k} > t'$.

Set

$$\overline{k} = -\varepsilon; \qquad t' = -2\varepsilon; \qquad k' = t' - \frac{\varepsilon}{\alpha - \overline{\alpha}}; \qquad t'' = \varepsilon; \qquad k'' = t'' + \frac{\varepsilon}{1-\alpha}$$

Choose k big enough so that both $f(k)$ and $g(k)$ are evaluated as in (22), we have

$$\begin{aligned} I(f(k)) - I(g(k)) &= (\alpha-\overline{\alpha})(k'-t') + (1-\alpha)(k''-t'') \\ &= (\alpha-\overline{\alpha})\left(-\frac{\varepsilon}{\alpha-\overline{\alpha}}\right) + (1-\alpha)\left(\frac{\varepsilon}{1-\alpha}\right) \end{aligned}$$

That is,

$$I(f(k)) - I(g(k)) = 0 \tag{29}$$

Next, observe that $c > a$, $d > b$, $\beta > \alpha$ (see (24); $d > b$ by the assumption) imply

$$\begin{aligned} (c-a)(-\varepsilon) + (d-b)\left(-2\varepsilon - \frac{\varepsilon}{\alpha-\overline{\alpha}}\right) + (\alpha-\beta)\left(\varepsilon + \frac{\varepsilon}{1-\alpha}\right) &< 0 \\ &\Longleftrightarrow \\ (c-a)\overline{k} + (d-b)k' + (\alpha-\beta)k'' &< 0 \end{aligned}$$

That is

$$\mu\overline{k} + (\beta - \mu)k' + (1-\beta)k'' < \overline{\alpha}\overline{k} + (\alpha - \overline{\alpha})k' + (1-\alpha)k'' \qquad (see\ (24))$$

which implies $I(f(\overline{k})) = \mu\overline{k} + (\beta - \mu)k' + (1-\beta)k''$.

On the other hand, $c > a$, $d > b$ and $\beta > \alpha$ also impy

$$\begin{array}{rcl} (c-a)(-\varepsilon) + (d-b)(-2\varepsilon) + (\alpha - \beta)\varepsilon & < & 0 \\ & \Longleftrightarrow & \\ (c-a)\overline{k} + (d-b)t' + (\alpha-\beta)t'' & < & 0 \\ & \Longleftrightarrow & \\ \mu\overline{k} + (\beta-\mu)t' + (1-\beta)t'' & < & \overline{\alpha}\overline{k} + (\alpha - \overline{\alpha})t' + (1-\alpha)t'' \qquad (see\ (24)) \end{array}$$

That is, $I(g(\overline{k})) = \mu\overline{k} + (\beta-\mu)t' + (1-\beta)t''$. Combining the last two findings,

$$\begin{array}{rcl} I(f(\overline{k})) - I(g(\overline{k})) & = & \left[\mu\overline{k} + (\beta-\mu)k' + (1-\beta)k''\right] - \left[\mu\overline{k} + (\beta-\mu)t' + (1-\beta)t''\right] \\ & = & (\beta-\mu)(k'-t') + (1-\beta)(k''-t'') \\ & = & (\beta-\mu)(-\frac{\varepsilon}{\alpha-\overline{\alpha}}) + (1-\beta)(\frac{\varepsilon}{1-\alpha}) \\ & = & -\frac{\varepsilon}{(\alpha-\overline{\alpha})(1-\alpha)}\left[(\beta-\mu)(1-\alpha) - (1-\beta)(\alpha-\overline{\alpha})\right] \\ & = & -\frac{\varepsilon}{(\alpha-\overline{\alpha})(1-\alpha)}\left[d-b\right] \qquad (\text{since } \frac{a}{b} = \frac{c}{d}) \end{array}$$

Hence,

$$I(f(\overline{k})) - I(g(\overline{k})) < 0 \tag{30}$$

Now, equality (29) and inequality (30) contradict $A^c \in \mathcal{A}_{EZ}$. To complete the proof, the only case left to consider is $\overline{\alpha} = \mu$. This can be dealt with by using the same construction as in the proof of Proposition 10, below. ∎

PROPOSITION 10. *If $T \in \mathcal{A}_{EZ} \backslash \mathcal{A}_{NM}$, then there is no $A \subset T$ such that $A \in \mathcal{A}_{NM}$.*

Proof. By the way of contradiction, assume that such an A exists. Recall that $A \in \mathcal{A}_{NM}$ means $\overline{\alpha} = \overline{\beta} = \mu$. For any three step act,

$$f(k'') = \begin{cases} k \text{ if } s \in A \\ k' \text{ if } s \in T\backslash A \\ k'' \text{ if } s \in T^c \end{cases}$$

we have

$$\begin{array}{rcl} I(f(k'')) & = & \widehat{I}^T(f(k'')) \\ & = & \min\left\{\overline{\alpha}k + (\alpha-\overline{\alpha})k' + (1-\alpha)k'', \overline{\alpha}k + (\beta-\overline{\alpha})k' + (1-\beta)k''\right\} \end{array}$$

Hence,

$$k' > k'' \quad \Longrightarrow \quad I(f(k'')) = \overline{\alpha}k + (\alpha-\overline{\alpha})k' + (1-\alpha)k'' \tag{31}$$

$$k' < k'' \quad \Longrightarrow \quad I(f(k'')) = \overline{\alpha}k + (\beta-\overline{\alpha})k' + (1-\beta)k'' \tag{32}$$

Now, let $f(k'')$ and $g(k'')$ be defined by

$$f(k'') = \begin{cases} k \text{ if } s \in A \\ k' \text{ if } s \in T\backslash A \\ k'' \text{ if } s \in T^c \end{cases} ; \qquad g(k'') = \begin{cases} t \text{ if } s \in A \\ t' \text{ if } s \in T\backslash A \\ k'' \text{ if } s \in T^c \end{cases}$$

for some $k' > k''$ and $t' > k''$; and let $f(\overline{k})$ and $g(\overline{k})$ be defined by

$$f(\overline{k}) = \begin{cases} k \text{ if } s \in A \\ k' \text{ if } s \in T \backslash A \\ \overline{k} \text{ if } s \in T^c \end{cases} ; \qquad g(\overline{k}) = \begin{cases} t \text{ if } s \in A \\ t' \text{ if } s \in T \backslash A \\ \overline{k} \text{ if } s \in T^c \end{cases}$$

for some $\overline{k} > k'$ and $t' > \overline{k}$. Notice that all these acts are constant on T^c. Let $\varepsilon > 0$, and define

$$k' = 0; \qquad \overline{k} = \varepsilon; \qquad t' = 2\varepsilon; \qquad t = 0; \qquad k = \frac{(\alpha - \overline{\alpha})2\varepsilon}{\overline{\alpha}} + \frac{\tau}{\overline{\alpha}}$$

where τ is chosen so that $0 < \tau < (\beta - \alpha)\varepsilon$.

For k'' small enough, both $f(k'')$ and $g(k'')$ are evaluated as in (31). Thus,

$$\begin{aligned} I(f(k'')) - I(g(k'')) &= \overline{\alpha}(k - t) + (\alpha - \overline{\alpha})(k' - t') \\ &= \overline{\alpha}\left(\frac{(\alpha - \overline{\alpha})2\varepsilon}{\overline{\alpha}} + \frac{\tau}{\overline{\alpha}}\right) + (\alpha - \overline{\alpha})(-2\varepsilon) \\ &= \tau \end{aligned}$$

Hence,

$$I(f(k'')) - I(g(k'')) > 0 \tag{33}$$

Next, observe that by construction $t' > \overline{k} > k'$, which implies that $f(\overline{k})$ is evaluated as in (32) while $g(\overline{k})$ is evaluated as in (31). Thus,

$$\begin{aligned} &I(f(\overline{k})) - I(g(\overline{k})) \\ &= \left[\overline{\alpha}k + (\beta - \overline{\alpha})k' + (1 - \beta)\overline{k}\right] - \left[\overline{\alpha}t + (\alpha - \overline{\alpha})t' + (1 - \alpha)\overline{k}\right] \qquad (34) \\ &= \left[\overline{\alpha}\left(\frac{(\alpha - \overline{\alpha})2\varepsilon}{\overline{\alpha}} + \frac{\tau}{\overline{\alpha}}\right) + (1 - \beta)\varepsilon\right] - [(\alpha - \overline{\alpha})(2\varepsilon) + (1 - \alpha)\varepsilon] \\ &= \tau + (\alpha - \beta)\varepsilon \end{aligned}$$

Hence, by our choice of τ

$$I(f(\overline{k})) - I(g(\overline{k})) < 0 \tag{35}$$

Now, inequality (33) and inequality (35) contradict $T \in \mathcal{A}_{EZ}$. ∎

This, along with Proposition 9 completes the proof of Theorem 5.

A.6(c) Proof of Corollary 2

Proof. It is immediate that $\varnothing$ and S are in $\mathcal{A}_{EZ}$. Moreover, by definition, $\mathcal{A}_{EZ}$ is closed under complementation. Now, we need to show that if A, B are in $\mathcal{A}_{EZ}$ and $A \cap B = \varnothing$, then $A \cup B \in \mathcal{A}_{EZ}$. We are going to show that if A, B are in $\mathcal{A}_{EZ}$ and $A \cap B = \varnothing$, $A \neq B^c$, then both A and B are necessarily naturally measurable. Then, the property follows from the fact that naturally measurable events make up a λ-system.

To begin, suppose that $B \in \mathcal{A}_{EZ} \backslash \mathcal{A}_{NM}$. Then, $B^c \in \mathcal{A}_{EZ}$ because $\mathcal{A}_{EZ}$ is closed under complementation, and $B^c \notin \mathcal{A}_{NM}$ (because otherwise one would contradict $B \in \mathcal{A}_{EZ} \backslash \mathcal{A}_{NM}$). That is, $B^c \in \mathcal{A}_{EZ} \backslash \mathcal{A}_{NM}$. By assumption, $A \cap B = \varnothing$ which implies $B^c \supset A$ (strictly). By Theorem 5, this contradicts $A \in \mathcal{A}_{EZ}$. That is, $B \in \mathcal{A}_{NM}$. By reversing the role of A and B, one shows $A \in \mathcal{A}_{NM}$.

Finally, the property that $\mathcal{A}_{EZ}$ is closed under countable disjoint unions follows from the fact that $\mathcal{A}_{NM}$ is. ∎