A geometric construction of a Calabi quasimorphism on projective space

Andre Carneiro

Submitted in partial fulfillment of the
requirements for the degree
of Doctor of Philosophy
in the Graduate School of Arts and Sciences

COLUMBIA UNIVERSITY

2013

ABSTRACT

A geometric construction of a Calabi quasimorphism on projective space

Andre Carneiro

We use the rotation numbers defined by Théret in [T] to construct a quasimorphism on the universal cover of the Hamiltonian group of $\mathbf{CP}^n$. We also show that this quasimorphism agrees with the Calabi invariant for isotopies that are supported in displaceable open subsets of $\mathbf{CP}^n$.

Table of Contents

List of Figures

Acknowledgments

I thank Dusa McDuff, both for her mathematical guidance and for constantly renewing my excitement in this research project. I also thank Melissa Liu, Mohammed Abouzaid, Eduardo Gonzalez and Jen Hom for being part of my defense committee and for making suggestions to improve the final draft. For many conversations that helped shape this text, I thank Claude Viterbo, Leonid Polterovich, Max Lipyanskiy, Adam Knapp, Peter Albers and Egor Shelukhin.

For constantly listening to my rants, for reminding me that a PhD program eventually ends, for always being there, for her amazing cooking and signature back sets, I thank my great friend, the unique Bianca Raglio Santoro.

I also thank Zachary Maddock, my roommate and friend, for his company, sense of humor and for a shared obsession with Mario Tennis 64.

For countless memorable moments and volleyball matches, I thank Michael S. Lam, Edgar K. P. Lazaro and Josefina Ramos.

For all the above, for being my love and (in)sanity, my family away from family and my partner in the greatest adventure there is, I thank Jesus Estrada.

Finally, I thank my parents for their unconditional support and encouragement.

I dedicate this dissertation to the math teachers that instilled in me the passion for the subject. In order of appearance, Robert Chaaya, Julio Rebelo, Marcos Alexandrino, Henri Anciaux and Dusa McDuff.

Chapter 1

Introduction

We consider complex projective space $M = \mathbf{CP}^n$, endowed with its standard symplectic form ω (see Definition 2.1.1). Diffeomorphisms of $\mathbf{CP}^n$ that are isotopic to the identity and preserve ω form a group $\mathrm{Ham}(M, \omega)$ called the Hamiltonian group of (M, ω).

Given a proper open subset $U \subset M$ on which ω is exact, the subgroup $\mathrm{Ham}^c(U, d\lambda)$ of Hamiltonians compactly supported in U admits an essentially unique homomorphism Cal_U to $\mathbf{R}$ known as the Calabi invariant. Since there exist numerous such open subsets U inside (M, ω), the Calabi homomorphism is well-defined on a large class of subgroups of $\mathrm{Ham}(M, \omega)$.

Due to a classical result of Banyaga [B], there is no real-valued homomorphism on $\mathrm{Ham}(M, \omega)$ that extends this system of Calabi homomorphisms. However, in [EP03], Entov and Polterovich constructed a Calabi quasimorphism

$$\mu_{\mathrm{EP}} : \mathrm{Ham}(\mathbf{CP}^n, \omega) \to \mathbf{R},$$

i.e. a map for which the quantity $\sup_{\phi,\psi} |\mu(\psi\phi) - \mu(\phi) - \mu(\psi)|$ is bounded and such that $\mu(\phi) = \mathrm{Cal}_U(\phi)$ whenever ϕ is compactly supported in a displaceable open subset U on which $\omega = d\lambda$ (see Definition 2.1.7). From the existence of such quasimorphism, they immediately deduce algebraic consequences for $\mathrm{Ham}(M, \omega)$ and interesting applications to the symplectic geometry of (M, ω). It is important to mention that their construction works for other closed symplectic manifolds, not only $\mathbf{CP}^n$.

For the particular case of $M = \mathbf{CP}^n$, an important contribution is that of Givental [G]. Even though it does not explicit mention "quasimorphisms", Ben Simon [Si] revisits his constructions and shows that the object there defined is indeed a Calabi quasimorphism.

The construction of Calabi quasimorphisms has been treated mainly with what Viterbo [V] calls the "Gromov-Floer approach", i.e. by using holomorphic curves or "hard" symplectic topology methods. In particular, the Entov-Polterovich construction relies on quantum homology calculations and properties of spectral invariants. This thesis is written with to motivate the alternative "Conley-Zehnder approach" to the problem of Calabi quasimorphisms. The techniques involved are more topologically flavored or "soft", such as the theory of generating functions, cohomological indices and Morse theory.

We rigorously construct a Calabi quasimorphism on $\mathbf{CP}^n$ using the rotation numbers that Théret [T] defines for Hamiltonian isotopies of complex projective space.

1.1 Outline

In Chapter 2, we review the necessary notions from symplectic geometry and we give a construction of the classical Calabi homomorphism for exact symplectic manifolds. It can be skipped by the more experienced reader or used as reference. In Chapter 3, we treat quasimorphisms on general groups and explain what we mean by a Calabi quasimorphism on a Hamiltonian group. We also include a brief survey of the theory of Calabi quasimorphisms in symplectic geometry. Chapter 4 starts by introducing the two main ingredients of our construction: generating functions and the cohomological index. It concludes with an outline of Théret's construction of his rotation numbers. Chapter 5 contains the main results of the text. It addresses the problem of how to track rotation numbers in a coherent fashion, defines the main invariant μ and contains the proofs of the quasimorphism and Calabi properties (Theorems 5.3.3 and 5.4.1).

Chapter 2

Symplectic geometry background

In this chapter, we describe the basic objects of interest and prove some of the elementary results in symplectic geometry. For a more comprehensive introduction to the subject, we refer to [MS]. We assume that the reader is familiar with smooth manifolds and differential forms.

2.1 Basic definitions and results

Definition 2.1.1. *A* **symplectic manifold** *is a pair* (M, ω)*, where* M *is a smooth manifold and* ω *is a closed non-degenerate 2-form on* M*. In other words,* $d\omega = 0$ *and the equation* $i_X\omega = \sigma$ *has exactly one vector-field solution* X *given a 1-form* σ*.*

The basic example of such structure is $\mathbf{R}^{2n}$ with its standard symplectic 2-form

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i,$$

where $x_1, \ldots, x_n, y_1, \cdots, y_n$ are the usual coordinates on Euclidean space.

Definition 2.1.2. *A diffeomorphism* $\phi : M \to M$ *is a* **symplectomorphism** *of* (M, ω) *if* $\phi^*\omega = \omega$*. The group of symplectomorphisms will be denoted* $\mathrm{Symp}(M, \omega)$*. We will use the symbol* $\mathrm{Symp}_0(M, \omega)$ *to denote the identity component of* $\mathrm{Symp}(M, \omega)$*, i.e. the group of symplectomorphisms that are isotopic to the identity through symplectomorphisms.*

It turns out that symplectomorphisms are abundant. Indeed, $\mathrm{Symp}(M, \omega)$ is infinite dimensional. To see this, let $H_t : M \to \mathbf{R}$ be a family of smooth functions, with $t \in [0, 1]$.

The family of 1-forms dH_t defines, by the non-degeneracy of ω, a time-dependent vector field X_t by the equation

$$dH_t = i_{X_t}\omega = \omega(X_t, \cdot). \tag{2.1}$$

Under some assumptions, e.g. compactness of M or compact support of the H_t, we can integrate X_t to a one-parameter family of diffeomorphisms ϕ_t such that ϕ_0 is the identity and

$$\frac{d}{dt}\phi_t(x) = X_t(\phi_t(x))$$

for all x in M and $t \in [0,1]$. We call H_t the **Hamiltonian function**, X_t the **Hamiltonian vector field** of H_t and $(\phi_t)_{t\in[0,1]}$ the **Hamiltonian isotopy** generated by H_t. Note that, in general, (ϕ_t) is not a one-parameter group of diffeomorphisms since the vector field that generates it is time-dependent.

We shall mostly deal with compact manifolds so integrability of vector fields will not be an issue. When this is not the case, we will mention it explicitly.

Definition 2.1.3. *A diffeomorphism $\phi : M \to M$ is called* **Hamiltonian** *if $\phi = \phi_1$ for some Hamiltonian isotopy (ϕ_t). The set of Hamiltonian diffeomorphisms is denoted* $\mathrm{Ham}(M,\omega)$.

We now check that Hamiltonian diffeomorphisms preserve ω and form a group.

Proposition 2.1.4. *Every Hamiltonian diffeomorphism is a symplectomorphism. In fact,* $\mathrm{Ham}(M,\omega)$ *is a subgroup of* $\mathrm{Symp}_0(M,\omega)$ *and these groups coincide if* $\pi_1 M = 0$.

Proof. Assume ϕ_t is a Hamiltonian isotopy generated by the vector field X_t and Hamiltonian function H_t and compute, using Cartan's formula:

$$\begin{aligned}\frac{d}{dt}\phi_t^*\omega &= \phi_t^*(\mathcal{L}_{X_t}\omega)\\ &= \phi_t^*(i_{X_t}d\omega + di_{X_t}\omega)\\ &= \phi_t^*(ddH_t)\\ &= 0,\end{aligned}$$

since ω is closed and by the definition of X_t. It follows that $\phi_t^*\omega = \phi_0^*\omega = \omega$ for every t since ϕ_0 is the identity. We conclude that $\mathrm{Ham}(M,\omega) \subset \mathrm{Symp}_0(M,\omega)$.

To see that $\mathrm{Ham}(M,\omega)$ is a group, consider two Hamiltonian isotopies (ϕ_t) and (ψ_t), generated by the Hamiltonian vector fields X_t and Y_t. By concatenating $(\phi_{2t})_{t\in[0,\frac{1}{2}]}$ with $(\psi_{2t-1}\circ\phi^1)_{t\in[\frac{1}{2},1]}$ we obtain an isotopy (γ_t) that ends at $\psi_1\circ\phi_1$. This can be made smooth by replacing the parameter t with $\alpha(t)$ a positive reparametrization of $[0,1]$ that is flat near 0 and near 1, but we leave the details to the interested reader. One obtains (γ_t) by first integrating X_{2t} for $t=[0,\frac{1}{2}]$ then integrating Y_{2t-1} for $t\in[\frac{1}{2},1]$. It is easy to "concatenate" the corresponding Hamiltonian functions in a smooth way to obtain the vector field we just integrated. We conclude that (γ_t) is a Hamiltonian isotopy and hence $\psi_1\circ\phi_1$ is Hamiltonian. To obtain the inverse of a Hamiltonian diffeomorphism, one replaces the Hamiltonian function H_t by $-H_{1-t}$, which generates the isotopy $(\phi_{1-t}\circ\phi_1^{-1})$ starting at the identity and ending at ϕ_1^{-1}.

Now assume $\pi_1 M=0$, let ϕ be a symplectomorphism and choose any smooth path (ϕ_t) in $\mathrm{Symp}_0(M,\omega)$ starting at the identity and with $\phi_1=\phi$. We define a vector field X_t by the formula

$$X_t(\phi_t(x))=\frac{d}{dt}\phi_t(x).$$

We compute $\frac{d}{dt}\phi_t^*\omega$ just like above. This quantity must be zero since ϕ_t is symplectic for every t. We conclude that $i_{X_t}\omega$ is closed for each t and, by our simply-connectedness assumption, exact:

$$i_{X_t}\omega=dH_t$$

for some family of functions H_t. This is exactly equation (2.1) so we have checked that the Hamiltonian H_t generates the path (ϕ_t), showing that ϕ_1 is Hamiltonian. □

Finally, we show that $(\mathbf{R}^{2n},\omega_0)$ is the local model for any symplectic manifold. We start with a simple lemma.

Lemma 2.1.5. *Given a finite dimensional vector space V and an skew-symmetric non-degenerate bilinear form ω on V, there exists a basis $\mathcal{B}=(e_1,\dots,e_n,f_1,\dots,f_n)$ such that $\omega=\sum_{i=1}^n e_i^*\wedge f_i^*$, where $(e_1^*,\dots,e_n^*,f_1^*,\dots,f_n^*)$ is the basis of V^* that is dual to $\mathcal{B}$.*

Proof. First note that if V were odd-dimensional, a matrix representing ω with respect to any given basis would be skew-symmetric and of odd size and hence have determinant zero.

This would contradicting the non-degeneracy of ω. We now assume the statement is true for $(2n-2)$-dimensional vector spaces and take V of dimension $2n$. Given a vector $e_n \neq 0$ in V, there exists f_n such that $\omega(e_n, f_n) = 1$, since ω is non-degenerate. We let W be the span of e_n and f_n and consider

$$W^\omega = \{v \in V \ : \ \omega(v, W) = 0\}.$$

The subspace W has dimension two since assuming otherwise would yield $\omega(e_n, e_n) = 1$, an absurd. And W^ω has dimension $2n-2$ since it is identified with the annihilator of W in V^* via the isomorphism $A_\omega : V \to V^*$ with formula $A_\omega(v)(w) = \omega(v, w)$. It is easy to check that ω is non-degenerate when restricted to W^ω. By the induction hypothesis, there exists a basis $(e_1, \ldots, e_{n-1}, f_1, \ldots, f_{n-1})$ of W^ω for which the restriction $\omega|_{W^\omega}$ has the desired form. It follows by construction that $(e_1, \ldots, e_n, f_1, \ldots f_n)$ is a basis for V satisfying the requirement. □

Theorem 2.1.6 (Darboux)**.** *Let (M, ω) be a symplectic manifold, not necessarily compact, and $p \in M$. There exist neighborhoods U of p in M and U' of the origin in $\mathbf{R}^{2n}$ and a diffeomorphism $\phi : U \to U'$ such that $\phi^*\omega = \omega_0$, where ω_0 is the standard symplectic form on Euclidean space described above.*

Proof. Since the statement is local, we take $M = \mathbf{R}^{2n}$ and $p = 0$. By Lemma 2.1.5, we assume (modulo a linear change of coordinates on M) that $\omega(0) = \omega_0(0)$. Consider the interpolation $\omega_t = t\omega + (1-t)\omega_0$ for $t \in [0, 1]$. The forms ω_t are closed and, since $\omega_t(0) = \omega_0(0)$ is non-degenerate, there exists a neighborhood U of 0 on which ω_t is non-degenerate. Finally,

$$\frac{d}{dt}\omega_t = \omega - \omega_0$$

is closed and vanishes at the origin. It follows that $\frac{d}{dt}\omega_t = d\sigma_t$ for some family of forms σ_t that vanishes at the origin. (This follows from the usual construction of a primitive of a closed form on a star-shaped domain by integrating along rays). Now define a vector field X_t by $i_{X_t}\omega = -\sigma_t$. Since $X_t(0) = 0$, we can assume, modulo shrinking U, that the isotopy (ϕ_t) that X_t generates is defined up to time 1 for any initial condition in U. We let

$U' = \phi_1(U)$ and compute

$$\begin{aligned}\frac{d}{dt}(\phi_t^*\omega_t) &= \phi_t^*(\mathcal{L}_{X_t}\omega_t) + \phi_t^* d\sigma_t) \\ &= \phi_t^*(i_{X_t}d\omega_t + di_{X_t}\omega_t + d\sigma_t) \\ &= 0,\end{aligned}$$

Since ϕ_0 is the identity, the computation above yields $\phi_1^*\omega_1 = \phi_0^*\omega_0 = \omega_0$, proving the theorem. □

Charts like the ones described in the theorem above are called **Darboux charts**. We meed a definition.

Definition 2.1.7. *A subset of U of (M, ω) is* **displaceable** *if $\phi(U) \cap U = \emptyset$ for some Hamiltonian diffeomorphism ϕ.*

Proposition 2.1.8. *Any small enough open set U is displaceable in (M, ω).*

Proof. In $(\mathbf{R}^{2n}, \omega_0)$, let V be a neighborhood of the origin that is contained in the ball $B(r)$ of radius r, also centered in the origin. We take the Hamiltonian

$$H_t(x_1, \ldots, x_n, y_1, \ldots, y_n) = Rx_1,$$

for some $R > 2r$. Its Hamiltonian vector field is $X_t = -R\frac{\partial}{\partial y_1}$ and the associated Hamiltonian ϕ_1 displaces $B(r)$ and hence V. We now take ρ a bump function on $\mathbf{R}^{2n}$ that equals one on $B(2R)$ and is zero outside $B(3R)$. The Hamiltonian ψ generated by $G_t = \rho H_t$ displaces V and is also supported in the ball $B(3R)$.

Given a point p in M, consider a Darboux chart around it. Any neighborhood U of p that is small enough when seen in this chart can be displaced using the method just described. Since the Hamiltonian used is compactly supported on the Darboux chart, it extends to the whole (M, ω). □

2.2 The Calabi homomorphism

In this section we consider an **exact** symplectic manifold (M, ω), i.e. one for which $\omega = d\lambda$. In this case, M cannot be closed, since Stokes's formula would yield

$$\int_M \omega^n = \int_{\partial M} \lambda \wedge \omega^{n-1} = 0,$$

contradicting the fact that ω^n is a volume form, by the non-degeneracy assumption. We shall assume that M is non-compact and has empty boundary. The superscript ^{c} will stand for "compactly supported". We follow the exposition of [MS], with some different sign conventions

Lemma 2.2.1. *Let ϕ be a compactly supported Hamiltonian diffeomorphism of (M, ω), where $\omega = d\lambda$. Then there exists a unique compactly supported function $F : M \to \mathbf{R}$ such that $\phi^*\lambda - \lambda = dF$.*

Proof. Let (ϕ_t) be a Hamiltonian isotopy with $\phi_1 = \phi$, with Hamiltonian H_t and Hamiltonian vector field X_t, all compactly supported. We compute

$$\begin{aligned} \phi^*\lambda - \lambda &= \int_0^1 \frac{d}{ds}\phi_s^*\lambda \, ds \\ &= \int_0^1 \phi_s^*(i_{X_s}d\lambda + di_{X_s}\lambda) \, ds \\ &= \int_0^1 \phi_s^*(dH_s + di_{X_s}\lambda) \, ds \\ &= d\left(\int_0^1 \phi_s^*(H_s + i_{X_s}\lambda) \, ds\right), \end{aligned}$$

showing that $\phi^*\lambda - \lambda$ is indeed exact. Since this form vanishes outside the support of ϕ, any primitive must be constant in that region. And since two primitives differ by a constant, there exists a unique F that is both a primitive for $\phi^*\lambda - \lambda$ and is compactly supported. $\square$

We wish to define

$$\mathrm{Cal}(\phi) = \frac{1}{n+1}\int_M F\omega^n,$$

but this depends, a priori, on the choice of the primitive λ for the symplectic form. We derive another formula for this quantity that does not depend on λ. This is the formula we shall use for the remainder of the text.

Lemma 2.2.2. *With the same notation as in the proof above,*

$$\mathrm{Cal}(\phi) = \int_0^1 \int_M H_t\omega^n \, dt \tag{2.2}$$

Proof. By Lemma 2.2.1, there is a unique compactly supported family F_t such that $\phi_t^*\lambda - \lambda = dF_t$, namely

$$F_t = \int_0^t \phi_s^*(H_s + i_{X_s}\lambda) \, ds.$$

It follows that

$$\begin{aligned}\int_M \frac{d}{dt} F_t \omega^n &= \int_M \phi_t^*(H_t + i_{X_t}\lambda)\omega^n \\ &= \int_M (H_t + i_{X_t}\lambda)\omega^n,\end{aligned}$$

since ϕ_t preserves ω. We compute

$$\begin{aligned}\int_0^1 \int_M H_t \omega^n \, dt &= \frac{1}{n+1}\int_0^1 \int_M (nH_t - i_{X_t}\lambda + H_t + i_{X_t}\lambda)\omega^n \, dt \\ &= \frac{1}{n+1}\int_0^1 \int_M \left(\frac{d}{dt}F_t + nH_t - i_{X_t}\lambda\right)\omega^n \, dt \\ &= \frac{1}{n+1}\int_M F_1\omega^n + \frac{1}{n+1}\int_0^1 \int_M (nH_t - i_{X_t}\lambda)\omega^n \, dt \\ &= \mathrm{Cal}(\phi) + \frac{1}{n+1}\int_0^1 \int_M (nH_t - i_{X_t}\lambda)\omega^n \, dt\end{aligned}$$

We now need to show that the last term in the equation above is zero. We consider the form $\omega^n \wedge \lambda$, which is zero since $\dim M = 2n$.

$$\begin{aligned}0 &= i_{X_t}(\lambda \wedge \omega^n) \\ &= (i_{X_t}\lambda)\omega^n - \lambda \wedge i_{X_t}(\omega^n) \\ &= (i_{X_t}\lambda)\omega^n - n\lambda \wedge i_{X_t}\omega \wedge \omega^{n-1} \\ &= (i_{X_t}\lambda)\omega^n + ndH_t \wedge \lambda \wedge \omega^{n-1} \\ &= (i_{X_t}\lambda)\omega^n + nd(H_t\lambda) \wedge \omega^{n-1} - nH_t d\lambda \wedge \omega^{n-1} \\ &= (i_{X_t}\lambda - nH_t)\omega^n + nd(H_t\lambda \wedge \omega^{n-1})\end{aligned}$$

This shows that, for every t,

$$\int_M (i_{X_t}\lambda - nH_t)\omega^n = 0,$$

finishing the proof. □

We now have a well-defined map $\mathrm{Cal} : \mathrm{Ham}^c(M,\omega) \to \mathbf{R}$ when $\omega = d\lambda$.

Theorem 2.2.3. Cal *is a homomorphism.*

Proof. Let ϕ and ψ be Hamiltonian diffeomorphisms. We have

$$\begin{aligned}(\psi \circ \phi)^*\lambda - \lambda &= \phi^*(\psi^*\lambda - \lambda) + (\phi^*\lambda - \lambda) \\ &= \phi^* dG + dF,\end{aligned}$$

by Lemma 2.2.1. It follows that

$$\begin{aligned}\mathrm{Cal}(\psi \circ \phi) &= \frac{1}{n+1}\int_M (G\circ\phi)\omega^n + \frac{1}{n+1}\int_M F\omega^n \\ &= \mathrm{Cal}(\psi) + \mathrm{Cal}(\phi),\end{aligned}$$

since ϕ preserves ω. □

The Calabi homomorphism is manifestly continuous and it is a celebrated result of Banyaga [B] that the kernel of Cal is **simple**, i.e. it has no proper normal subgroups except for the trivial one. We then conclude the following.

Theorem 2.2.4. *For M an exact manifold without boundary,* Cal *is the only continuous homomorphism from* $\mathrm{Ham}^c(M,\omega)$ *to the real numbers, up to rescaling.*

Proof. Let K be the kernel of Cal. It follows from the formula we derived in Lemma 2.2.2 that the Calabi homomorphism is surjective, so Cal induces an isomorphism $\overline{\mathrm{Cal}}$ from $\mathrm{Ham}^c(M,\omega)/K$ to the reals. Given another continuous homomorphism f, we have $K \cap \ker f$ a normal subgroup of K, which is hence trivial or equal to K itself, by Banyaga's theorem. In the former case,

$$f \oplus \mathrm{Cal} : \mathrm{Ham}^c(M,\omega) \to \mathbf{R} \oplus \mathbf{R}$$

would be injective, contradicting the fact that the Hamiltonian group is non-abelian. If $\ker f \supset K$ then f also factors through a homomorphism $\overline{f}$ on $\mathrm{Ham}^c(M,\omega)/K$. The composition $\overline{f} \circ \overline{\mathrm{Cal}}^{-1}$ is a continuous automorphism of $\mathbf{R}$, which one easily proves to be multiplication by a constant □

It is worth noting that, while the assumption $\omega = d\lambda$ simplifies the exposition, it can be avoided. We refer the reader to [MS] for the details and we only outline the results. For any non-compact symplectic manifold, it is possible to define

$$\widetilde{\mathrm{Cal}} : \widetilde{\mathrm{Ham}}^c(M,\omega) \to \mathbf{R}$$

on the universal cover of the Hamiltonian group, by equation (2.2) in Lemma 2.2.2. If we denote by Λ the image of $\pi_1 \mathrm{Ham}^c(M,\omega)$ under $\widetilde{\mathrm{Cal}}$, it follows that we have a well-defined map

$$\mathrm{Cal} : \mathrm{Ham}^c(M,\omega) \to \mathbf{R}/\Lambda.$$

Finally, the Calabi homomorphism cannot be defined when M is closed. In fact, Banyaga proves in [B] that the Hamiltonian group of a closed symplectic manifold is simple, hence not admitting any non-trivial homomorphisms to $\mathbf{R}$.

Chapter 3

Calabi quasimorphisms

3.1 Quasimorphisms on abstract groups

We start by discussing quasimorphisms in generality.

Definition 3.1.1. *A* **quasimorphism** *on a group G is a function $\mu : G \to \mathbf{R}$ such that the quantity*

$$C = C(\mu) := \sup_{g,h} |\mu(gh) - \mu(g) - \mu(h)| < \infty$$

The quantity C is called the **defect** *of μ.*

The simplest example of a quasimorphism is a bounded function $G \to \mathbf{R}$. It is natural to discard such examples and consider the quotient

$$\mathrm{QM}_h(G) = \frac{\mathrm{QM(G)}}{L^\infty(G)}$$

of the space of quasimorphisms on G by the subspace of bounded functions. While we will check that, for finite or Abelian groups, $\mathrm{QM}_h(G) = \mathrm{Hom}(G, \mathbf{R})$, this is certainly *not* the case for groups having a richer structure, such as free groups or fundamental groups of higher genus surfaces. Each equivalence class in $\mathrm{QM}_h(G)$ has a convenient homogeneous representative, as explained in the definition and lemma below.

Definition 3.1.2. *A quasimorphism $\mu : G \to \mathbf{R}$ is called* **homogeneous** *if, for any integer n, we have*

$$\mu(g^n) = n\mu(g).$$

Given any quasimorphism μ, one defines its homogenization

$$\bar{\mu}(g) = \lim_{k\to\infty} \frac{\mu(g^k)}{k}.$$

To understand this quantity, we will need the following classical lemma on subadditive sequences, due to Fekete.

Lemma 3.1.3. *Let a_k be a sequence of non-negative real numbers that is* **subadditive** *in the sense that*

$$a_{k+l} \leq a_k + a_l,$$

for all integers k, l. Then

$$\lim_{k\to\infty} \frac{a_k}{k} = \inf_k \frac{a_k}{k}.$$

Proof. We will denote by l the infimum of $\frac{a_k}{k}$ and we choose $\epsilon > 0$. Note that $l > -\infty$, since the a_k are non-negative. Let K be such that

$$\frac{a_K}{K} \leq l + \frac{\epsilon}{2}$$

and choose L such that

$$\frac{a_r}{KL} \leq \frac{\epsilon}{2}$$

for $r < K$. For $n \geq KL$, we write $n = Kq + r$ for some integers q, r such that $r < K$. It follows that $q \geq L$ and we have, by subadditivity and the choices just made,

$$\begin{aligned} \frac{a_n}{n} &\leq \frac{a_{Kq}}{Kq+r} + \frac{a_r}{Kq+r} \\ &\leq q\frac{a_K}{qK} + \frac{a_r}{Kq} \\ &\leq \frac{a_K}{K} + \frac{a_r}{KL} \\ &\leq l + \epsilon, \end{aligned}$$

showing that l is the limit of $\frac{a_k}{k}$. $\square$

We can now prove the following.

Proposition 3.1.4. *The limit $\bar{\mu}(g)$ exists for each g, $\bar{\mu}$ is a homogeneous quasimorphism and $\mu - \bar{\mu} \in L^\infty(G)$. Furthermore, if μ_1 and μ_2 are both homogeneous, their difference is either zero or unbounded.*

Proof. Assume without loss of generality that $\mu(g) \geq 0$ and define $a_k = \mu(g^k) + (k+1)C$, where C is the defect of μ. By the quasimorphism property,

$$a_k \geq k\mu(g) - (k-1)C + (k-1)C \geq 0$$

$$a_{k+l} = \mu(g^{k+l}) + (k+l+1)C \leq \mu(g^k) + \mu(g^l) + C + (k+l+1)C = a_k + a_l,$$

i.e. the sequence a_k is non-negative and subadditive. It follows from the lemma above that the sequence $\frac{a_k}{k}$ converges to its infimum and hence

$$\lim_{k\to\infty} \frac{\mu(g^k)}{k} = \lim_{k\to\infty} \frac{a_k}{k} - C = \inf \frac{a_k}{k} - C.$$

Now we compute

$$|\bar{\mu}(g) - \mu(g)| = \lim_{k\to\infty} \frac{1}{k}|\mu(g^k) - k\mu(g)| \leq \lim_{k\to\infty} \frac{1}{k}(k-1)C = C$$

and

$$|\bar{\mu}(g^n) - n\bar{\mu}(g)| = \lim_{k\to\infty} \frac{1}{k}|\mu(g^{kn}) - n\mu(g^k)| \leq \lim_{k\to\infty} \frac{1}{k}(n-1)C = 0.$$

Both inequalities make use of the quasimorphism property of μ. The first one shows that $\bar{\mu} - \mu$ is bounded and, hence, $\bar{\mu}$ is a quasimorphism with defect at most $4C$. The second proves the claimed homogeneity.

The last claim is straightforward since the difference $\nu = \mu_1 - \mu_2$ is also a homogeneous quasimorphism. If $\nu(g) \neq 0$ for some g, then $\nu(g^n) = n\nu(g)$ is unbounded as $|n|$ grows large. □

From this lemma it follows that $\mathrm{QM}_h(G)$ is (canonically isomorphic to) the space of homogeneous quasimorphisms on G. If G is finite, we clearly have $\mathrm{QM}(G) = L^\infty(G)$ so $\mathrm{QM}_h(G) = 0 = \mathrm{Hom}(G, \mathbf{R})$. We prove some simple facts about homogeneous quasimorphisms, for intuition.

Proposition 3.1.5. *A homogeneous quasimorphism μ is a class function, i.e.*

$$\mu(h^{-1}gh) = \mu(g)$$

for every $g, h \in G$.

Proof. By homogeneity,

$$\begin{aligned} \mu(h^{-1}g^n h) &= n\mu(h^{-1}gh) \\ \mu(g^n) &= n\mu(g) \\ \mu(h^{-1}) &= -\mu(h), \end{aligned}$$

so we have, for every n,

$$\begin{aligned} |\mu(h^{-1}gh) - \mu(g)| &= \frac{1}{n}|\mu(h^{-1}g^n h) - \mu(g^n)| \\ &= \frac{1}{n}|\mu(h^{-1}g^n h) - \mu(h^{-1}) - \mu(g^n) - \mu(h)| \\ &\leq \frac{2C}{n}. \end{aligned}$$

By taking the infimum of both sides over n, the result follows. □

Proposition 3.1.6. *If μ is homogeneous and g commutes with h then $\mu(gh) = \mu(g) + \mu(h)$.*

Proof. The proof is analogous to the one above. Compute

$$|\mu(gh) - \mu(g) - \mu(h)| = \frac{1}{n}|\mu(g^n h^n) - \mu(g^n) - \mu(h^n)| \leq \frac{2C}{n}$$

□

Corollary 3.1.7. *Every homogeneous quasimorphism on an Abelian group G is a homomorphism. Hence, a general quasimorphism on G is the sum of a homomorphism and a bounded function. In other words,* $\mathrm{QM}(G) \cong \mathrm{Hom}(G, \mathbf{R}) \oplus L^\infty(G)$.

For illustration, we wish to give two examples of quasimorphisms. The first one, due to Brooks [Br], is defined on a free group F. Fix a reduced word w in F and define $C_w(g)$ to be the number of times w appears as a subword of the reduced word representing g. We define

$$\mu_w(g) = C_w(g) - C_{w^{-1}}(g).$$

Given two elements g and h in F, the concatenation of the reduced words representing them is not, in general, reduced. In reducing this concatenated word, copies of w and w^{-1} might cancel each other, which is of no importance to the value of $\mu_w(gh)$. But, "new" copies of w could appear when the ending letters of g meet the initial letters of h, hence the existence of a defect, which is less than the length of w if we allow C_w to count overlapping copies of w or less than one if we restrict to counting non-overlapping copies of w. Note that if w consists of a single letter, μ_w is actually a homomorphism. Also, the family $(\mu_w)_{w\in F}$ has been used to show that $\mathrm{QM}_h(F)$ is infinite dimensional, while $\mathrm{Hom}(F, \mathbf{R})$ clearly has dimension equal to the rank of F.

The second example is more classical and is closer in spirit to the Calabi quasimorphism defined in later chapters. It can be found in [**?**]. Let $G = \mathrm{Homeo}^+(\mathbf{S}^1)$ be the group of orientation preserving homeomorphisms of the circle $\mathbf{S}^1 = \mathbf{R}/\mathbf{Z}$. The universal cover $\tilde{G}$ consists of strictly increasing homeomorphisms from $\mathbf{R}$ to itself that commute with the translation $x \mapsto x + 1$. We define a quasimorphism on $\tilde{G}$ by the formula

$$\tau(f) = \lim_{n\to\infty} \frac{f^n(x) - x}{n},$$

where the superscript ^{n} denotes n-fold composition. This limit always exists and is independent of the point x. If f and g commute then $\tau(gf) = \tau(g) + \tau(f)$ since

$$(gf)^n(x) - x = (g^n(f^n(x)) - f^n(x)) + (f^n(x) - x)\,.$$

In general, one uses the estimate $|gf(x) - fg(x)| \leq 1$ to conclude that τ is a quasimorphism.

3.2 Calabi quasimorphisms

Here we finally define our object of main interest. Let (M, ω) be a closed symplectic manifold and U a proper open subset of M. We have a map of groups

$$\widetilde{\mathrm{Ham}}^c(U, \omega) \to \widetilde{\mathrm{Ham}}(M, \omega) \tag{3.1}$$

induced by extending Hamiltonians by the identity outside U. As we mentioned in section 2.2, the group on the left carries a Calabi homomorphism Cal_U. Note that since we do not require ω to be exact on U, we need to use the "second formula" for the Calabi homomorphism, namely 2.2. The map in (3.1) need not be injective a priori, though no examples with non-trivial kernel are known. Furthermore, if a Hamiltonian isotopy (ϕ_t) is supported in an open $U' \subset U$, the value of Cal_U and $\mathrm{Cal}_{U'}$ on (ϕ_t) coincide since they are both expressed as

$$\int_0^1 \int_M F_t \omega^n \; dt,$$

where F_t is the unique Hamiltonian that generates (ϕ_t) and is compactly supported in U' and, hence, in U. We conclude that the quantity Cal is well-defined for a large class of Hamiltonian isotopies of (M, ω). The first question one can ask is whether there exists a homomorphism

$$\mathrm{Cal}_M : \widetilde{\mathrm{Ham}}(M, \omega) \to \mathbf{R}$$

such that $\mathrm{Cal}_M((\phi_t)) = \mathrm{Cal}_U((\phi_t))$ for every isotopy (ϕ_t) that is supported in a proper open subset U. The answer to this question is known to be no. This follows from the celebrated result of Banyaga [B] that $\mathrm{Ham}(M, \omega)$ is simple. We are led to the following.

Definition 3.2.1. *Let $\mathcal{D}$ be a class of proper open subsets of a closed symplecic manifold (M, ω). A Calabi quasimorphism for the class $\mathcal{D}$ is a quasimorphism*

$$\mu : \widetilde{\mathrm{Ham}}(M, \omega) \to \mathbf{R}$$

such that $\mu((\phi_t)) = \mathrm{Cal}_U((\phi_t))$ for every isotopy that is supported in an open $U \in \mathcal{D}$.

As we will see below, $\mathcal{D}$ is often taken to be the class of displaceable proper open subsets of M, see definition 2.1.7.

3.2.1 Some applications of Calabi quasimorphisms

The existence of a "non-trivial" quasimorphism, i.e. one that is both unbounded and not a homomorphism, has algebraic and geometric consequences. Note that a Calabi quasimorphism is always non-trivial in this sense.

3.2.1.1 Commutator norm

Let G be a group and h an element of $[G, G]$, the commutator subgroup. It is natural to define $||h||$ as the smallest number of simple commutators $aba^{-1}b^{-1}$ that need to be multiplied together to obtain h. It follows that, given a homogeneous quasimorphism μ on

G and $k = ||h||$,

$$\begin{aligned} |\mu(h)| &= |\mu(a_1 b_1 a_1^{-1} b^{-1} \cdots a_k b_k a_k^{-1} b_k^{-1})| \\ &= |\mu(a_1 b_1 a_1^{-1} b^{-1} \cdots a_k b_k a_k^{-1} b_k^{-1}) - \mu(a_1) + \mu(a_1) - \ldots - \mu(b_k) + \mu(b_k)| \\ &\leq (4||h|| - 1)C(\mu) \end{aligned}$$

by using the quasimorphism property. This yields a non-trivial lower bound for $||h||$ when $\mu(h)$ and $C(\mu)$ are non-zero. It is worth mentioning that Banyaga [B] actually proves that the Hamiltonian group of a closed symplectic manifold is **perfect**, i.e. it equals its commutator subgroup. Hence, since a Calabi quasimorphism is unbounded (and not a homomorphism), its existence implies that $\mathrm{Ham}(M, \omega)$ has unbounded commutator norm, an algebraic fact that is not obvious a priori.

3.2.1.2 Quasi-measures

For a more geometric application, a quasimorphism needs a stronger feature.

Definition 3.2.2. *A quasimorphism μ on $\widetilde{\mathrm{Ham}}(M, \omega)$ is* **stable** *is there exists a constant B such that, for every F_t and G_t we have*

$$\int_0^1 \min_M (F_t - G_t)\ dt \leq \frac{\mu(\phi^F) - \mu(\phi^G)}{B} \leq \int_0^1 \max_M (F_t - G_t)\ dt,$$

where ϕ^F is the Hamiltonian isotopy generated by F_t.

For a stable homogeneous quasimorphism μ, we can define $\zeta : C^\infty(M) \to \mathbf{R}$ by the formula

$$\zeta(H) = \frac{\int_M H\omega^n - \mu(\phi^H)}{\mathrm{Vol}(M)}.$$

The map ζ is called a **symplectic quasi-state** in that it has the following features.

1. **Quasi-linearity.** If the flows of H and K commute then ζ is linear on the span of H and K. This follows from the homogeneity of μ, see Proposition 3.1.6.

2. **Monotonicity.** If $H \leq K$ everywhere, then $\zeta(H) \leq \zeta(K)$. This is a direct consequence of stability.

3. **Normalization.** We have $\zeta(1) = 1$, as can be checked from the formula for ζ.

4. **Hamiltonian invariance.** For every Hamiltonian diffeomorphism ψ, $\zeta(F \circ \phi) = \zeta(F)$, since $F \circ \psi$ generates the isotopy $\psi^{-1}\phi^F\psi$ and homogeneous quasimorphisms are class functions, see Proposition 3.1.5.

Analogous to the Riesz representation theorem, a quasi-state ζ gives rise to a quasi-measure defined on the closed sets C of M via the formula

$$\tau(C) = \inf\{\zeta(F) \mid F : M \to [0,1],\ F \geq \mathbf{1}_C\}.$$

If μ has the Calabi property for the class $\mathcal{D}$ of displaceable sets then, by definition, ζ vanishes on functions that have displaceable support. (Recall definition 2.1.7.) Consequently, the quasi-measure of a displaceable set is zero. The Hamiltonian invariance of ζ shows that $\tau(\psi(C)) = \tau(C)$ for every Hamiltonian ψ. And the quasi-linearity of ζ shows that τ is additive on disjoint unions. Finally, τ is monotonic and the quasi-measure of M is 1.

We conclude that, if $\tau(C) > \frac{1}{2}$ then C is non-displaceable for that assuming otherwise would yield

$$\tau(C \cup \psi(C)) = \tau(C) + \tau(\psi(C)) = 2\tau(C) > 1$$

for any Hamiltonian ψ displacing C, a contradiction. In some cases, such as special toric fibers and the Clifford torus in $\mathbf{CP}^n$, one can use this technique to prove non-displaceability results. While these results are known, they require a lot of machinery, such as the Lagrangian Floer theory of [FOOO]. The use of quasi-measures might be a simpler alternative.

This observation also yields a tool for proving that two quasimorphisms μ_1 and μ_2 are distinct, a difficult question in general. For that, one can find disjoint closed sets C_1 and C_2 such that $\tau_1(C_1)$ and $\tau_2(C_2)$ are both greater that $\frac{1}{2}$, where τ_i is the quasi-measure induced by μ_i. This was done, for example, in [ElP].

3.2.2 A brief survey

In [EP03], Entov and Polterovich exhbited the first examples of Calabi quasimorphisms. They worked with a closed **spherically monotone** symplectic manifold (M, ω), i.e. one for which there exists a positive constant κ such that

$$\langle c_1(TM, \omega), A\rangle = \kappa\langle[\omega], A\rangle$$

for all homology classes A that can be represented by 2-spheres. When such manifolds have a semisimple quantum homology algebra - we shall not try to explain what that means here - their construction yields a Calabi quasimorphism for the class $\mathcal{D}$ of displaceable open subsets of M. Examples of manifolds satisfying their conditions include $\mathbf{CP}^n$, $\mathbf{S}^2 \times \mathbf{S}^2$, $\mathbf{CP}^2$ blown up at a point and complex Grassmannians. In the cases of $\mathbf{CP}^n$ and $\mathbf{S}^2 \times \mathbf{S}^2$, their Calabi quasimorphism vanishes on $\pi_1 \widetilde{\mathrm{Ham}}(M, \omega)$ and hence descends to $\mathrm{Ham}(M, \omega)$, with $\mathcal{D}$ replaced by the class $\mathcal{D}_{ex}$ of displaceable open subsets on which ω is exact so the Calabi homomorphism is well-defined. This is a highly non-trivial result for $M = \mathbf{CP}^n$ when $n \geq 3$ since the fundamental group in question is not known. They use tools from “hard” symplectic topology, i.e. the more analytical theories of J-holomorphic curves and quantum homology. In [U], Usher extended the applicability of the Entov-Polterovich construction of a Calabi quasimorphism to all one-point blow-ups of closed symplectic manifolds and also to all closed toric manifolds.

For a “soft” construction of Calabi quasimorphisms on surfaces of higher genus, one can look into Py [Py05]. In his case $\mathcal{D}$ is the class of open subsets that are diffeomorphic to a disk or annulus. In [Py06], he also constructs quasimorphims on $\mathrm{Symp}_0(\mathbf{T}^2, \omega)$ that coincide with the Calabi homomorphism for the class $\mathcal{D}$ of disks. These are easily calculated on time-independent Morse Hamiltonians by using the combinatorics of their level sets.

Finally, Givental [G] constructs a cycle in the quantomorphism group of the sphere $\mathbf{S}^{2n-1}$, i.e. the group of contactomorphisms that commute with the usual circle action. He explains how to coorient a subset of this cycle, called the train, and hence count the intersection points of a generic path of quantomorphisms with the train. He calls the invariant thus defined the **non-linear Maslov index** given the similarity to the construction of the usual (linear) Maslov index of a loop of symplectic matrices by counting intersections with the cycle $\det(A - I) = 0$ in the symplectic linear group $\mathrm{Sp}(2n)$. Later, Ben Simon showed in [Si] that the non-linear Maslov index yields a quasimorphism on the Hamiltonian group of complex projective space, a statement that was essentially contained in Givental, but not with such language. Ben Simon proceeds to show that the homogeneization of the non-linear Maslov index is a Calabi homomorphism for the class $\mathcal{D}$ of displaceable open subsets of $\mathbf{CP}^n$.

To the best of our knowledge, the quasimorphism we define is the same as the one in Givental, though we do not attempt to prove that statement. It is currently unknown whether the Entov-Polterovich quasimorphism on $\mathbf{CP}^n$ equals Givental's quasimorphism, but we hope that our construction may be a middle ground for comparing the two.

Chapter 4

Rotation numbers

The goal of this chapter is to associate rotation numbers to Hamiltonian isotopies of complex projective space, following Théret [T].

4.1 Preliminaries

We start by explaining the two main ingredients of the construction.

4.1.1 Generating functions

Any cotangent bundle T^*X has a canonical 1-form λ. Given a vector V tangent to T^*X at a point (x, p), where $x \in X$ and $p \in T^*_x X$, we have

$$\lambda(V) = p(\pi_* V),$$

π being the natural projection $T^*X \to X$ and π_* its derivative. The 2-form $\omega = d\lambda$ is a symplectic structure on T^*X. Non-degeneracy can be checked from the local expression

$$\lambda = \sum_{i=1}^{n} p_i dq_i,$$

where the q_i are any coordinates on X and the p_i are the coefficients of a cotangent vector in the basis dq_i. Most of our constructions will be conducted in $T^*\mathbf{C}^n$ endowed with the symplectic form just described.

Let $L \subset (T^*\mathbf{C}^n, \omega)$ be a **Lagrangian** submanifold, i.e. $\omega|_L \equiv 0$ and $\dim L = 2n$.

Definition 4.1.1. *A function $S : \mathbf{C}^n \times \mathbf{C}^k \to \mathbf{R}$ is a* **generating function** *for L (or simply* **generates** *L) if both conditions below are met.*

(i) Zero is a regular value of the map $(u;\xi) \mapsto \partial_\xi S(u;\xi)$, so the **fiber critical set**

$$\Sigma_S = \{(u;\xi) : \partial_\xi S(u;\xi) = 0\}$$

is a smooth submanifold of $\mathbf{C}^n \times \mathbf{C}^k$ of real dimension $2n$.

(ii) The map $i_S : \Sigma_S \to T^*\mathbf{C}^n$ defined by $i_S(u;\xi) = (u, \partial_u S(z;\xi))$ is an embedding with image L.

The variables $u \in \mathbf{C}^n$ in the above definition are called **principal variables** and the $\xi \in \mathbf{C}^k$ are referred to as **auxiliary variables**. When $k = 0$, any smooth $S : \mathbf{C}^n \to \mathbf{R}$ satisfies condition (i) vacuously and $\Sigma_S = \mathbf{C}^n$. In this case, the Lagrangian generated in $T^*\mathbf{C}^n$ is simply the graph of the 1-form dS. Conversely, any Lagrangian that projects diffeomorphically onto the zero section of T^*X admits a generating function that has no auxiliary variable. We will see below that these are the building blocks for functions that generate more complicated Lagrangians.

Remark 4.1.2. *In order to avoid heavy notation we will denote the partial derivatives of S with respect to the variables ξ by S_ξ instead of $\partial_\xi S$.*

As we have seen in Section 2.1, $\mathbf{C}^n \cong \mathbf{R}^{2n}$ has a standard symplectic form ω_0. If $\overline{\mathbf{C}}^n$ denotes $\mathbf{C}^n$ endowed with the opposite symplectic form $-\omega_0$, there exists a symplectic identification

$$\begin{aligned} I : \overline{\mathbf{C}}^n \times \mathbf{C}^n &\to T^*\mathbf{C}^n && (4.1)\\ (z, z') &\mapsto \left(\frac{z+z'}{2}, i(z-z')\right) && (4.2) \end{aligned}$$

If Φ is a symplectomorphism of $\mathbf{C}^n$, its graph $\Gamma_\Phi = \{(x, \Phi(x)) : x \in \mathbf{C}^n\}$ is a Lagrangian in $\overline{\mathbf{C}}^n \times \mathbf{C}^n$. We say that S generates Φ if it generates the Lagrangian $I(\Gamma_\Phi)$. Given the remark above, if Φ is sufficiently C^1-small then $I(\Gamma_\Phi)$ projects diffeomorphically onto the zero section of $T^*\mathbf{C}^n$ and hence admits a generating function with no auxiliary variables.

The following simple lemma will allow us to use Morse theory to study fixed points of symplectomorphisms. It follows from the simple observation that the isomorphism I maps the diagonal $\Delta = \{(z,z)\}$ onto the zero section of $T^*\mathbf{C}^n$.

Lemma 4.1.3. *If S generates Φ then critical points of S are in one-to-one correspondence with fixed points of Φ.*

Proof. If $\Phi(x) = x$, then $I(x, \Phi(x)) = (x, 0)$, by the formula above. It follows there exists $\xi_0 \in \mathbf{C}^k$ such that $S_\xi(x; \xi_0) = 0$ and $i_S(x, \xi_0) = I(x, \Phi(x)) = (x, 0)$. We conclude from this equation that $S_u(x; \xi_0)$ is also zero and, thus, $(x; \xi_0)$ is a critical point of S. Analogously, if $(u; \xi_0)$ is a critical point of S, then $(u; \xi_0) \in \Sigma_S$ and $i_S(u; \xi_0) = (u, 0)$ is a point of the form $I(x, \Phi(x))$ for some x, i.e.

$$\begin{aligned} x + \Phi(x) &= 2u \\ x - \Phi(x) &= 0, \end{aligned}$$

from which we conclude that $u = x$ and $\Phi(x) = x$. $\square$

It is known that a Lagrangian that is Hamiltonian isotopic to the zero section in $T^*\mathbf{C}^n$ and is "well behaved at infinity" admits a generating function. We shall not explain the necessary conditions since we will work in a slightly different setting. In addition, any two generating functions for the same Lagrangian can be related by a sequence of **stabilizations** and **fiberwise diffeomorphisms**. The former is the operation

$$S(z; \xi) \rightsquigarrow S \oplus Q(z; \xi, \xi') = S(z; \xi) + Q(\xi'),$$

where Q is a quadratic form, and the latter denotes

$$S(z; \xi) \rightsquigarrow S(z; F_z(\xi)),$$

where $F(z; \xi) = (z; F_z(\xi))$ is a diffeomorphism of $\mathbf{C}^n \times \mathbf{C}^k$. See [G] for the existence statement and [V] for "uniqueness".

Recall that a function $F : \mathbf{C}^N \to \mathbf{R}$ is **2-homogeneous** if $F(cz) = |c|^2 F(z)$ for every $c \in \mathbf{C}^*$. Any function $f : \mathbf{CP}^{N-1} \to \mathbf{R}$ can be pulled back to the unit sphere $\mathbf{S}^{2N-1} \subset \mathbf{C}^N$ and then extended to $\mathbf{C}^N$ in a 2-homogeneous fashion. Conversely, given a 2-homogeneous function F, its restriction to the unit sphere is invariant under the Hopf flow and hence induces a function f on projective space. We shall seamlessly move from one setting to the other. We will use lower-case letters for functions on $\mathbf{CP}^{N-1}$ and the corresponding upper-case letters for their 2-homogeneous lifts to $\mathbf{C}^N$.

We will say that $\Phi : \mathbf{C}^n \to \mathbf{C}^n$ is **equivariant** if it commutes with the diagonal action of $\mathbf{C}^*$ on $\mathbf{C}^n$.

When working in the usual setting of [V], generating functions can always be assumed smooth. But a 2-homogeneous function is badly behaved at the origin and it will be clear from Proposition 4.1.6 that this problem will "spread" to a finite collection of hyperplanes in $\mathbf{C}^N$. This forces one to relax the required differentiability of 2-homogeneous generating functions. Nonetheless, they will be differentiable enough (C^1 with Lipschitz differential) so that Morse-theoretical arguments apply. We will ignore this technicality and refer the interested reader to Théret's paper.

In Propositions 4.3 and 4.7 of [T], Théret proves the following result.

Theorem 4.1.4 (Théret)**.** *Every equivariant symplectomorphism of $\mathbf{C}^n$ admits a 2-homogeneous generating function. Furthermore, any two such generating functions are equivalent in the sense that they can be made equal through a sequence of stabilizations by 2-homogeneous quadratic forms and equivariant fiberwise diffeomorphisms.*

It is easy to see that any sufficiently C^1-small equivariant symplectomorphism Φ of $\mathbf{C}^n$ admits a 2-homogeneous generating function with no auxiliary variables. The subset $I(\Gamma_\Phi)$ of $T^*\mathbf{C}^n$ is $\mathbf{C}^*$-invariant and projects diffeomorphically onto the zero section. It follows that there exists a unique function S that vanishes at the origin and such that $I(\Gamma_\Phi)$ is the graph of dS. We just need the following.

Lemma 4.1.5. *If the function $S : \mathbf{C}^n \to \mathbf{R}$ generates a C^* equivariant symplectomorphism (or a $\mathbf{C}^*$-invariant Lagrangian submanifold) and $S(0) = 0$, then S is 2-homogeneous.*

Proof. By assumption, the graph of dS is a $\mathbf{C}^*$-invariant Lagrangian submanifold of $T^*\mathbf{C}^n$. In other words, we have $dS(cz) = c\, dS(z)$ for every non-zero complex number c. Note that here $dS(z)$ stands for the derivative 1-form at the point z. When we evaluate $dS(z)$ on the vector v we will write $dS(z) \cdot v$. So the 1-form $c\, dS(z)$ is defined by $(c\, dS(z)) \cdot v = dS(z) \cdot cv$.

The identification I (see 4.2) is $\mathbf{C}$-linear so it intertwines the $\mathbf{C}^*$ actions on $\overline{\mathbf{C}}^n \times \mathbf{C}^n$ and $T^*\mathbf{C}^n$. Recall that the different between $\overline{\mathbf{C}}^n$ and $\mathbf{C}^n$ is only in the symplectic form, not the $\mathbf{C}^*$-action. Hence, if a symplectomorphism is $\mathbf{C}^*$-equivariant, the image of its graph under I is $\mathbf{C}^*$-invariant

Let z be in $\mathbf{C}^n$ and t be a real number. Since $S(0) = 0$, the fundamental theorem of calculus reads

$$\begin{aligned} S(tz) &= \int_0^t dS(\tau z) \cdot z \; d\tau \\ &= \int_0^t (\tau \; dS(z)) \cdot z \; d\tau \\ &= \int_0^t dS(z) \cdot \tau z \; d\tau \\ &= \int_0^t (dS(z) \cdot z)\tau \; d\tau \\ &= \frac{t^2}{2} \; dS(z) \cdot z, \end{aligned}$$

showing that S is quadratic when restricted to the real line containing z. Note that the fourth equality only makes sense because τ is real, i.e. the 1-form $dS(z)$ is $\mathbf{R}$-linear, not $\mathbf{C}$-linear.

It remains to show that S is invariant under the $\mathbf{S}^1$-action. We compute

$$\int_0^{2\pi} dS(e^{i\theta} z) \cdot ie^{i\theta} z \; d\theta = S(z) - S(z) = 0.$$

Since both $dS(e^{i\theta} z)$ and $ie^{i\theta} z$ are invariant under the $\mathbf{S}^1$-action, the integrand is constant and hence must be zero. This shows that S is $\mathbf{S}^1$-invariant and finishes the proof. $\square$

For a general equivariant symplectomorphism Φ of $\mathbf{C}^n$, one writes $\Phi = \Phi_1 \circ \cdots \circ \Phi_\ell$ with C^1-small Φ_i and uses the composition formula below, which preserves 2-homogeneity.

Proposition 4.1.6. *If $R(u;\xi)$ generates Φ and $S(v;\eta)$ generates Ψ then the fixed points of $\Psi \circ \Phi$ are in bijection with the critical points of*

$$R\#S(u; v, w, \xi, \eta) = R(u + w; \xi) + S(v + w; \eta) + 2\langle u - v, iw\rangle.$$

Furthermore, if either R or S contain no auxiliary variables, then $R\#S$ is a generating function for $\Psi \circ \Phi$

Proof. The last statement is proved in [T]. We include a proof of the first statement for the reader's convenience.

Let $(u; v, w, \xi, \eta)$ be a critical point of $R\#S$. It follows that

$$\begin{array}{lllll}
(1) & (R\#S)_u & = & R_u + 2iw & = & 0 \\
(2) & (R\#S)_v & = & S_v - 2iw & = & 0 \\
(3) & (R\#S)_w & = & R_u + S_v + 2i(u-v) & = & 0 \\
(4) & (R\#S)_\xi & = & R_\xi & = & 0 \\
(5) & (R\#S)_\eta & = & S_\eta & = & 0
\end{array}$$

where we omitted the points where the partial derivatives are taken. From (1) and (2) it follows that

$$2iw = S_v = -R_u$$

Combined with (3) that yields $u = v$. Equation (4) says that $(u + w; \xi)$ is a fiber critical point for R and, hence, there exists an unique x such that

$$I(x, \Phi(x)) = \left(\frac{x + \Phi(x)}{2}, i(x - \Phi(x))\right) = (u + w, R_u) = i_R(u + w; \xi)$$

We conclude that

$$\begin{array}{rcl}
2iu + 2iw & = & i(x + \Phi(x)) \\
R_u & = & i(x - \Phi(x))
\end{array}$$

and one finally adds these to obtain $u = x$. By the same procedure applied to (5) one obtains a unique y such that

$$\begin{array}{rcl}
2iu + 2iw & = & i(y + \Psi(y)) \\
S_v & = & i(y - \Psi(y))
\end{array}$$

from which it follows that $u = \Psi(y)$. We also conclude that

$$\begin{array}{rcl}
x + \Phi(x) & = & y + \Psi(y) \\
\Phi(x) - x & = & y - \Psi(y),
\end{array}$$

the first equation being two ways to express $2u + 2w$ and the second being the relation $iR_u = -iS_v$. In other words, $\Psi \circ \Phi(u) = \Psi \circ \Phi(x) = \Psi(y) = u$, showing that u is the desired fixed point.

Conversely, if $\Psi \circ \Phi(u) = u$, there exist unique $(u'; \xi)$ and $(v'; \eta)$ such that

$$\begin{aligned} i_R(u'; \xi) &= I(u, \Phi(u)) \\ i_S(v'; \eta) &= I(\Phi(u), \Psi \circ \Phi(u)) = I(\Phi(u), u) \end{aligned}$$

One easily checks that $(u; u, \frac{\Phi(u)-u}{2}, \xi, \eta)$ is a critical point for $R\#S$. $\square$

The generating function obtained depends, of course, on the decomposition of Φ that we chose. They are, nonetheless, all equivalent in the sense of Theorem 4.1.4. See [T] for more details. We shall mention one lemma that goes into the proof of Theorem 4.1.4.

Lemma 4.1.7. *Regard $Q(u; \xi)$ as a family of functions on $\mathbf{C}^k$ paranetrized by $u \in \mathbf{C}^n$ and assume that, for u fixed, $Q(u; \xi)$ is 2-homogeneous non-degenerate quadratic form on the variables ξ. Then there exists a family $(A_s)_{s\in[0,1]}$ of equivariant fiberwise diffeomorphisms starting at the identity and such that $Q \circ A_1$ is independent of u.*

The proof of the lemma consists of applying Gram–Schmidt to each fiber, rendering each quadratic form standard.

We remark that the construction of generating functions also works for k-parameter families of equivariant symplectomorphisms of $\mathbf{C}^n$. The proofs are identical.

We now show the (much simpler) formula for the "inverse" generating function.

Proposition 4.1.8. *If S generates Φ then $-S$ generates Φ^{-1}.*

Proof. The fiber critical sets of S and $-S$ are the same. Given a point $(u; \xi) \in \Sigma_S$, we have, for some x

$$\begin{aligned} i_S(u; \xi) = (u, S_u(u; \xi)) &= \left(\frac{x + \Phi(x)}{2}, i(x - \Phi(x)) \right) \\ i_{-S}(u; \xi) = (u, -S_u(u; \xi)) &= \left(\frac{x + \Phi(x)}{2}, i(\Phi(x) - x) \right) \end{aligned}$$

showing that $-S$ generates the Lagrangian $L = \{(\Phi(x), x) : x \in \mathbf{C}^n\}$, which is exactly the graph of Φ^{-1}. $\square$

Finally, we compute an example which will be very useful below.

Proposition 4.1.9. *The quadratic form $q_t(z) = -\tan(\pi t)||z||^2$ generates the map $z \mapsto e^{-2\pi i t}z$ in the sense of Definition 4.1.1*

Proof. Since there is no auxiliary variable ξ, the first condition of Definition 4.1.1 is vacuous and $\Sigma_{q_t} = \mathbf{C}^n$. We have

$$i_{q_t}(u) = (u, -2u\tan(\pi u)),$$

which corresponds under the identification I^{-1} to the point

$$(z, \Phi(z)) = (u + iu\tan(\pi t), u - iu\tan(\pi t)).$$

One easily checks that

$$\frac{\Phi(z)}{z} = \frac{1 - i\tan(\pi t)}{1 + i\tan(\pi t)} = e^{-2\pi i t}$$

□

4.1.2 The cohomological index

Here we define and prove the basic properties of the cohomological index of Fadell-Rabinowitz [FR].

Definition 4.1.10. *The* **index** *of a subset $i : A \hookrightarrow \mathbf{CP}^N$ is the integer*

$$\text{ind } A = \dim\ \text{Image}\left(H^*(\mathbf{CP}^N; \mathbf{Q}) \to H^*(A; \mathbf{Q})\right) = 1 + \max\{k : i^*u^k \neq 0\},$$

where u is any non-zero element of $H^2(\mathbf{CP}^N; \mathbf{Q})$. If $f : \mathbf{CP}^N \to \mathbf{R}$ is a continuous function, we define ind f = ind $\{f \leq 0\}$. *We will often write* ind A *for a subset A of $\mathbf{C}^{N+1}$ that is invariant under the diagonal $\mathbf{C}^*$ action, in which case we mean* ind $(A - 0)/\mathbf{C}^*$. *As a particular case, the sublevel set $\{F \leq 0\}$ of a 2-homogeneous function F on $\mathbf{C}^N$ is $\mathbf{C}^*$-invariant. Its index will be denoted by* ind F.

Remark 4.1.11. *The usual definition of this index uses cohomology with integral coefficients. However, to avoid complications related to torsion, we shall use rational coefficients. The usual properties of the index also hold with coefficients in $\mathbf{Q}$ and we include short proofs.*

Example 4.1.12. *The index of $\mathbf{RP}^N$ inside $\mathbf{CP}^N$ is 1 since $H^2(\mathbf{RP}^N; \mathbf{Q}) = 0$. On the other hand the map $H^2(\mathbf{CP}^N; \mathbf{Z}) \to H^2(\mathbf{RP}^N; \mathbf{Z}) \cong \mathbf{Z}/2$ is non-trivial, as one can deduce* *from a careful examination of the spectral sequence of the fibration* $\mathbf{S}^1 \to \mathbf{RP}^{2N+1} \to \mathbf{CP}^N$*. This shows that the index of* $\mathbf{RP}^N$ *with integer coefficients would be greater than* 1*.*

Proposition 4.1.13. *If* $A \hookrightarrow B \hookrightarrow \mathbf{CP}^N$*, then* $\mathrm{ind}\ A \leq \mathrm{ind}\ B$*.*

Proof. If the class u^k restricts trivially to B then further restricting it to A also yields zero. □

Proposition 4.1.14. *If* $X = A \cup B$ *then* $\mathrm{ind}\ X \leq \mathrm{ind}\ A + \mathrm{ind}\ B$*.*

Proof. If u^a (resp. u^b) vanishes on A (resp. B), it defines a class in $H^*(X, A)$ (resp. $H^*(X, B)$). The conclusion now follows from the following diagram involving the relative cup product.

$$\begin{tikzcd}
H^*(X,A) \otimes H^*(X,B) \arrow[r, "\smile"] \arrow[d] & H^*(X, A \cup B) = 0 \arrow[d] \\
H^*(X) \otimes H^*(X) \arrow[r, "\smile"] & H^*(X)
\end{tikzcd}$$

□

In the absence of torsion, we can prove a stronger statement.

Proposition 4.1.15. *Let* $f : \mathbf{CP}^N \to \mathbf{R}$ *be a smooth function having zero as a regular value and denote by* A *and* B *the sets* $\{f \leq 0\}$ *and* $\{f \geq 0\}$ *respectively. It follows that*

$$\mathrm{ind}\ A + \mathrm{ind}\ B = \mathrm{ind}\ \mathbf{CP}^N = N + 1.$$

Proof. By Proposition 4.1.14, we already have the inequality

$$\mathrm{ind}\ A + \mathrm{ind}\ B \geq \mathrm{ind}\ \mathbf{CP}^N = N + 1.$$

Assume by contradiction that it is strict, i.e. $\mathrm{ind}\ A + \mathrm{ind}\ B > N + 1$. By Poincaré duality and the definition of the index, one can thus find cycles Z_A in A and Z_B in B such that:

(i) Z_A and Z_B have complementary dimensions $2a$ and $2b$. (One can take $a = -1 + \mathrm{ind}\ A$, for instance.)

(ii) When seen in $\mathbf{CP}^N$, these cycles pair nontrivially with u^a and u^b, respectively.

Note that here we need coefficients in a field. In general, the class $u^a|_A$ could be torsion and hence pair trivially with every cycle in $H_{2a}(A)$.

Since zero is a regular value of f, the sublevel set A is a manifold with boundary and we can use collar neighborhoods to "push" Z_A away from ∂A, i.e. Z_A is the image of $H_*(A - \partial A)$ in $H_*(A)$. Arguing similarly for B, we conclude that $Z_A \frown Z_B = 0$ (since $A \cap B = \partial A = \partial B$). On the other hand, $u^a \smile u^b$ is a generator of $H^{2N}(\mathbf{CP}^N)$ and, by duality, $Z_A \frown Z_B \neq 0$. $\square$

Recall that the **projective join** $A * B$ of $A \subset \mathbf{CP}^M$ and $B \subset \mathbf{CP}^N$ is the union of all projective lines in $\mathbf{CP}^{M+N+1}$ passing through a point in A and a point in B, where we regard $\mathbf{CP}^M$ and $\mathbf{CP}^N$ as subspaces of $\mathbf{CP}^{M+N+1}$ in general position, i.e. disjoint.

Proposition 4.1.16. *If F_i is a 2-homogeneous function on $\mathbf{C}^{N_i+1}$, then*

$$\text{ind } F_1 \oplus F_2 = \text{ind } F_1 + \text{ind } F_2,$$

where $F_1 \oplus F_2(z, z') = F_1(z) + F_2(z')$.

Proof. This proof is sketched in Givental [G]. We fill in the details.

We consider the three sublevel sets $M = \{F_1 \oplus F_2 \leq 0\}$ and $M_i = \{F_i \leq 0\}$, where we identify the $\mathbf{C}^{N_i+1}$ with the obvious subspaces of $\mathbf{C}^{N_1+1} \oplus \mathbf{C}^{N_2+1}$. In view of homogeneity, this allows us to regard M and the M_i as subsets of $\mathbf{CP}^{N_1+N_2+1}$, the latter being a subset of $\mathbf{CP}^{N_i} \subset \mathbf{CP}^{N_1+N_2+1}$. We note that, since $F_1 \oplus F_2$ is determined by its values on $\mathbf{C}^{N_i+1}$, we have

$$M_i = M \cap \mathbf{CP}^{N_i}.$$

Given a point x in $\mathbf{CP}^{N_1+N_2+1}$ not lying in either $\mathbf{CP}^{N_i}$, there exists a unique line L_x through x that intersects both $\mathbf{CP}^{N_i}$. In other words, the join $\mathbf{CP}^{N_1} * \mathbf{CP}^{N_2}$ equals $\mathbf{CP}^{N_1+N_2+1}$. To see this, note that x represents the complex line $\mathbf{C}z$, for some non-zero z in $\mathbf{C}^{N_1+1} \oplus \mathbf{C}^{N_2+1}$. We decompose z as $z_1 + z_2$, where $z_i \in \mathbf{C}^{N_i+1}$. The line L_x is the image of the complex plane $\mathbf{C}z_1 \oplus \mathbf{C}z_2$ in projective space. We will denote by x_i the intersection of L_x with $\mathbf{CP}^{N_i}$. Note that the value $F_i(x_i)$ is not well defined but, by homogeneity, its sign is:

$$F_i(\lambda z) = |\lambda|^2 F_i(z) \leq 0 \Leftrightarrow F_i(z) \leq 0.$$

We will abuse notation and write $F_i(x_i)$ for the value of F_i on a lift of x_i lying on the unit sphere.

1. $F_1(x_1) \leq 0$ and $F_2(x_2) \leq 0$. In this case, by homogeneity,

$$F(\lambda x_1 + \mu x_2) = |\lambda|^2 F_1(x_1) + |\mu|^2 F_2(x_2) \leq 0$$

   so the line through x_1 and x_2 is contained in the join $M_1 * M_2$.

2. $F_1(x_1) \leq 0$ and $F_2(x_2) > 0$. Here, only part of the line connecting x_1 and x_2 is in M. Namely,

$$F(\lambda x_1 + \mu x_2) \leq 0 \Leftrightarrow \left|\frac{\mu}{\lambda}\right|^2 \leq -\frac{F_1(x_1)}{F_2(x_2)}$$

   so this line intersects M in a closed disk centered in M_1. The case corresponding to $F_1(x_1) < 0$ and $F_2(x_2) \leq 0$ is analogous.

3. $F_1(x_1) > 0$ and $F_2(x_2) > 0$. When this happens, the line connecting x_1 and x_2 is disjoint from M.

This shows that the complement of the join $M_1 * M_2$ in M is foliated by disks that are centered on the M_i. Note that, indeed, a pair of distinct disks can only intersect in their common center, by the uniqueness of the line L_x above given x. Naively, one would say that shrinking each disk to its center gives a deformation retraction of M onto $M_1 * M_2$. This would certainly be discontinuous. To see that, consider a line L_ϵ connecting points x_i with $F_1(x_1) \leq 0$ and $F_2(x_2) = \epsilon > 0$ very small. By shrinking ϵ, one can make L_ϵ arbitrarily close to a line L' contained in the join $M_1 * M_2$. Since L' remains fixed during the retraction, we cannot fully retract all disks $M \cap L_\epsilon$ to their centers in M_1. We describe a more careful retraction.

Given $\epsilon > 0$, we define M_i^ϵ by intersecting $\{F_i \leq \epsilon\}$ with the unit sphere and then taking its image in projective space. Choose a continuous real-valued ρ_ϵ that is zero in the interval $[0, \frac{\epsilon}{2}]$ and one in $[\epsilon, \infty)$. Given a point x in M, we can assume without loss that $F_1(x_1) \leq 0$, since we cannot be in case 3 above. If $F_2(x_2) \leq 0$, then the line L_x is in the join $M_1 * M_2$ and our retraction will leave it fixed. If $F_2(x_2) = \delta > 0$, we will retract the disk $L_x \cap M$ according to the value $\rho(\delta)$. If the latter is zero, we do not retract at all. If the value is

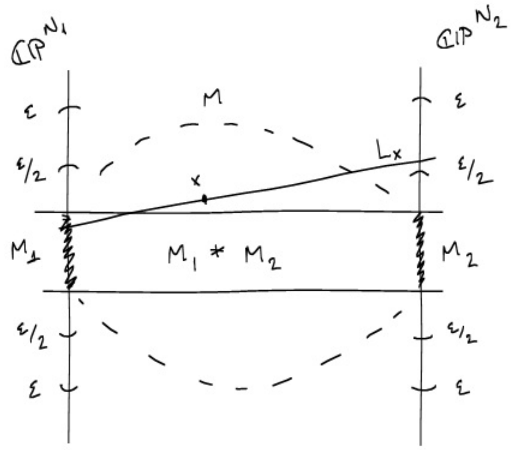

Figure 4.1: The geometry of a projective join.

one, we fully retract the given disk and so on. To be precise, the line L_x is identified with $\mathbf{C}$ once the point x_2 is removed and the retraction of the disk $M \cap L_x$ is done radially. For small enough ϵ, the retract of M we obtained retracts in turn onto $M_1 * M_2$. This follows because $M_1 * M_2$ is locally contractible and thus every small enough neighborhood retracts onto it. See Hatcher [H], Theorem A.7 for the details. See the Figure 4.1 for geometric intuition.

The final step is to show that the index is additive under joins. This is carried out in detail in [G] so we only give an outline. First, instead of working with subsets and joins in projective space, one considers their lifts to the sphere under the Hopf map. One must use $\mathbf{S}^1$-equivariant cohomology to calculate the index in this setting. Then one uses suspensions to relate the join operation to the simpler smash product. And finally, the equivariant cohomology of a smash product can be computed via an equivariant version of Kunneth's theorem, from which additivity follows. $\square$

The last property of the cohomological index will be very useful in the sequel. It is also outlined in [G].

Proposition 4.1.17. *Assume $A \subset \mathbf{CP}^N$ is a deformation retract of an open neighborhood*

U. If A' is the intersection of A with a hyperplane, then

$$-1 + \text{ind } A \leq \text{ind } A' \leq \text{ind } A.$$

In other words, intersecting with a hyperplane reduces the index by at most 1.

Proof. The second inequality follows from the monotonicity of the index, Proposition 4.1.13. Now let i and j be the inclusions of A and A' into $M = \mathbf{CP}^N$, respectively. We shall omit the coefficient group $\mathbf{Q}$ from the notation. Let $u \in H^2(M)$ be nonzero and assume $i^* u^k \neq 0$. It follows that there exists a class α in $H_{2k}(A)$ such that $i_* \alpha = [\mathbf{CP}^k]$, a generator of $H_{2k}(M)$. Given H a hyperplane, the relative cup product gives a map

$$H^{2N-2k}(M, M - A) \times H^2(M, M - H) \to H^{2N-2k+2}(M, M - A').$$

By applying Alexander duality, we obtain a map

$$\eta : H_{2k}(A) \times H_{2N-2}(H) \to H_{2k-2}(A').$$

The class u above has as Poincaré dual the hyperplane class, i.e. $\mathrm{PD}(u) = [H]$ in $H_{2N-2}(M) \cong H_{2N-2}(H)$. We define $\alpha' = \eta(\alpha, \mathrm{PD}(u))$. The map η is just a refinement of the usual intersection pairing in $H_*(M)$. By naturality, we have

$$j_* \alpha' = j_* \eta(\alpha, \mathrm{PD}(u)) = i_* \alpha \cap \mathrm{PD}(u) = [\mathbf{CP}^k] \cap [H] = [\mathbf{CP}^{k-1}],$$

a generator of $H_{2k-2}(M)$. It follows that

$$\langle j^* u^{k-1}, \alpha' \rangle = \langle u^{k-1}, j_* \alpha' \rangle = \langle u^{k-1}, [\mathbf{CP}^{k-1}] \rangle \neq 0.$$

We conclude that the indices of A and A' differ by at most 1, finishing the proof. □

We will be particularly interested in the following.

Corollary 4.1.18. *If $R : \mathbf{C}^n \times \mathbf{C}^k \to \mathbf{R}$ and $S : \mathbf{C}^n \times \mathbf{C}^l \to \mathbf{R}$ are 2-homogeneous, then*

$$0 \leq \text{ind } R\#S - \text{ind } R - \text{ind } S \leq n.$$

Proof. Note that the functions $R\#S$ (defined in the previous section) and

$$R \oplus S \oplus Q(u, \xi, v, \eta, w) = R(u; \xi) + S(v; \eta) + |w|^2$$

coincide on the subspace $w = 0$. In other words, after n hyperplane sections, these functions have the same index. Since we know the index of $R \oplus S \oplus Q$ to be ind R+ind S, the inequality follows by repeated application of the proposition above (note that $Q(w) = |w|^2$ is positive definite and thus has index zero). We remark that we can apply Proposition 4.1.17 since a sublevel set of the form $\{F \leq 0\}$ is the deformation retract of a neighborhood of the form $\{F < \epsilon\}$ for small $\epsilon > 0$. □

4.2 Théret's construction

Let $(\phi_t)_{t \in [0,1]}$ be a smooth path in the group of Hamiltonian diffeomorphisms of $\mathbf{CP}^{n-1}$, starting at the identity. By definition, this isotopy is generated by some (time-dependent) Hamiltonian function $h_t : \mathbf{CP}^{n-1} \to \mathbf{R}$. Given a fixed point x of ϕ_1, we can consider the (contractible) loop $\phi_t(x)$ and a disk D that caps it. The **action** of x is the $\mathbf{R}/\mathbf{Z}$-valued quantity

$$c(x) = -\frac{1}{\pi} \left(\int_D \omega + \int_0^1 h_t(\phi_t(x)) \, dt \right) \mod 1. \tag{4.3}$$

Note that this formula appears with different signs in different texts, since it depends directly on the choice of signs made in equation 2.1. Since the symplectic area of any sphere in $\mathbf{CP}^{n-1}$ is π, $c(x)$ does not depend on D. It depends, though, on the choice of Hamiltonian function in a very precise way. Given another Hamiltonian h'_t generating the isotopy, we have

$$h'_t = h_t + c(t)$$

since we must have $dh'_t = dh_t$ for every t in $[0,1]$. It follows that the action $c(x)$ would change by $\int_0^1 c(t)\, dt$ if we replaced h_t by h'_t.

Definition 4.2.1. *A time-dependent Hamiltonian function h_t is* **mean-normalized** *if, for every $t \in [0,1]$, we have*

$$\int_{\mathbf{CP}^{n-1}} h_t \omega^{n-1} = 0.$$

To avoid the ambiguity in the value of the action mentioned above, we will only deal with mean-normalized Hamiltonians on $\mathbf{CP}^{n-1}$. The main result in [T] is the following.

Theorem 4.2.2 (Théret)**.** *Given $(\phi_t)_{t\in[0,1]}$, there exist n rotation numbers $0 < \lambda_1 \leq \cdots \leq \lambda_n \leq 1$ with the following properties.*

(a) The set of values $\{\lambda_1, \ldots, \lambda_n\}$, when viewed in $\mathbf{S}^1 \cong \mathbf{R}/\mathbf{Z}$, is continuous with respect to (ϕ_t) if we endow the space of isotopies with the C^1 topology.

(b) If $\lambda_{i-1} < \lambda_i < \lambda_{i+1}$ for some i (with the convention that $\lambda_{-1} = \lambda_n$ and $\lambda_{n+1} = \lambda_1$) then ϕ_1 has a fixed point with action λ_i.

(c) If $\lambda_i = \lambda_{i+1}$ for some i then ϕ_1 has an infinite number of fixed points with action λ_i.

(d) If $\lambda_1 = \cdots = \lambda_n$ then ϕ_1 is the identity map of $\mathbf{CP}^{n-1}$

(e) The rotation numbers do not change (up to what is said in (a)) through a homotopy with fixed ends of the isotopy (ϕ_t).

We shall outline Théret's construction of the rotation numbers λ_i.

Remark 4.2.3. *In [T] the rotation numbers are only well-defined up to a global rotation of $\mathbf{S}^1$. It should be clear from the exposition below that, if one requires h_t to be mean-normalized as we do, then the rotation numbers λ_i become well-defined in $\mathbf{R}/\mathbf{Z}$ so one can remove the words "up to rotation" from Théret's original statement.*

We lift the mean-normalized Hamiltonian h_t that genereates (ϕ_t) to a function $H_t : \mathbf{S}^{2n-1} \to \mathbf{R}$ using the standard projection $\pi : \mathbf{S}^{2n-1} \to \mathbf{CP}^{n-1}$ and extend it to the whole of $\mathbf{C}^n$ by enforcing

$$H_t(\lambda z) = |\lambda|^2 H_t(z)$$

for all $\lambda \in \mathbf{C}$.

The Hamiltonian H_t generates an $\mathbf{C}^*$-equivariant isotopy Φ_t that covers ϕ_t. A fixed point of ϕ_1 corresponds to a complex line on which Φ_1 acts by rotation. We will detect n such complex lines $C_1, \ldots, C_n$. The rotation numbers λ_j in the theorem above will have the geometric meaning that $\Phi_1|_{C_j}$ is given by multiplication by $e^{2\pi i\lambda_j}$.

We are thus interested in finding fixed points of $e^{-2\pi i\tau}\Phi_1$ for any $0 \leq \tau \leq 1$. This is accomplished with the aid of generating functions and the cohomological index described in the previous sections.

Proposition 4.2.4. *There exists a family $S_\tau : \mathbf{C}^n \times \mathbf{C}^k \to \mathbf{R}$ of functions satisfying:*

1. *S_τ is 2-homogeneous, i.e. $S_\tau(\lambda z) = |\lambda|^2 S_\tau(z)$. (Hence, the critical points of S_τ come in complex lines.)*
2. *S_τ generates $e^{-2\pi i\tau}\Phi_1$ so that critical lines of S_τ are in bijection with the fixed points of $e^{-2\pi i\tau}\Phi_1$ (see Lemma 4.1.3).*
3. *Zero is a regular value of the total map $(\tau, z) \mapsto S_\tau(z)$.*
4. *The family $\tau \mapsto S_\tau$ is non-increasing.*

Proof. The idea is to take a 2-homogeneous generating function S for the symplectomorphism Φ_1 and let $S_\tau = S\#Q_\tau$, where Q_τ is a family of quadratic forms generating the rotation $z \mapsto e^{-2\pi i\tau}z$. Note that the q_τ defined in Proposition 4.1.9 could be used but are only defined for $|\tau| < \frac{1}{2}$. One can allow for larger values of τ at the price of adding more auxiliary variables:

$$Q_\tau = q_{\tau/3}\#q_{\tau/3}\#q_{\tau/3}.$$

It can be checked that S_τ has the desired properties, see [T]. □

When working with 2-homogeneous functions, one only needs to look for critical points at the zero level.

Lemma 4.2.5. *If $S : \mathbf{C}^N \to \mathbf{R}$ is 2-homogeneous and $z \in \mathbf{C}^N$ is a critical point for S, then $S(z) = 0$*

Proof. We simply compute the derivative of S at the point z evaluated at the vector z.

$$\begin{aligned} dS(z)\cdot z &= \lim_{\lambda\to 0}\frac{S(z+\lambda z)-S(z)}{\lambda} \\ &= \lim_{\lambda\to 0}\frac{[(1+\lambda)^2-1]S(z)}{\lambda} \\ &= \lim_{\lambda\to 0}(2+\lambda)S(z) \\ &= 2S(z) \end{aligned}$$

The result follows □

From Lemma 4.2.5, the only possible critical value for each S_τ is zero. One can thus detect critical lines by detecting changes in the topology of the sublevel sets $\{S_\tau \leq 0\}$ as τ varies from zero to one. For that purpose, we examine the function $l(\tau) = \text{ind } S_\tau$, where ind is the cohomological index defined in Section 4.1.2.

In summary, if $l(\tau)$ jumps at $\tau = \lambda$, then the topology of $\{S_\tau \leq 0\}$ changes as τ crosses the value λ. By Morse theory, S_λ must have a critical point, which corresponds to a fixed point of $e^{-2\pi i\lambda}\Phi_1$. Since this symplectomorphism is equivariant, we have found a complex line that Φ_1 rotates by an angle of $2\pi\lambda$. This number λ would then be one of the rotation numbers announced by Theorem 4.2.2. The proposition below shows that this procedure will yield exactly n roation numbers.

Proposition 4.2.6. *The function $l(\tau)$ is increasing and $l(1) - l(0) = n$*

Proof. For $\tau < \tau'$, item 4 in Proposition 4.2.4 above shows that we have the inclusion

$$\{S_\tau \leq 0\} \subset \{S_{\tau'} \leq 0\}.$$

Hence, by monotonicity of the index (Proposition 4.1.13), $l(\tau)$ is increasing. Now we compare the indices of $S\#Q_1$ and $S\#Q_0$.

Both Q_1 and Q_0 are quadratic forms (on the auxiliary variables) that generate the identity. By Lemma 4.1.7, they are fiberwise diffeomorphic to quadratic forms $\overline{Q}_1$ and $\overline{Q}_0$ that have no principal variable. It follows that

$$\begin{aligned} l(1) - l(0) &= \text{ind } S\#Q_1 - \text{ind } S\#Q_0 \\ &= \text{ind } S\#\overline{Q}_1 - \text{ind } S\#\overline{Q}_0 \\ &= \text{ind } \overline{Q}_1 - \text{ind } \overline{Q}_0. \end{aligned}$$

The last inequality follows since the composition formula in Proposition 4.1.6 reads

$$S\#\overline{Q}_i(u; w, \xi, \eta) = S(u + w; \xi) + Q_i(\eta) + 2\langle u, iw\rangle,$$

given that $\overline{Q}_i$ has no principal variable. We conclude that $S\#\overline{Q}_i = S' \oplus \overline{Q}_i$, where

$$S' = S + 2\langle u, iw\rangle$$

and the desired equality is a consequence of Proposition 4.1.16.

Finally, note that the index of a quadratic form is half the number of its nonpositive eigenvalues. So we are interested in the number of eigenvalues of $\overline{Q}_\tau$ that change sign as τ ranges from 0 to 1, which is exactly half the Maslov index of the loop $\left(e^{-2\pi i\tau}\right)_{0\leq\tau\leq 1}$ it generates. Hence, $l(1) - l(0) = n$. □

One now define

$$\lambda_i = \inf\{\tau : l(\tau) \geq l(0) + i\}.$$

The rotation numbers do not depend on the particular choice of generating function S because index differences are robust under stabilization and fiberwise diffeomorphism. For more details and the proof of the announced properties of the numbers λ_i, the reader is referred to [T].

Chapter 5

Main results

5.1 The spectrum bundle

Let G be the Hamiltonian group of $\mathbf{CP}^{n-1}$ and denote by $\mathcal{P}$ the space of smooth paths in G starting at the identity, endowed with the C^1 topology. We note that any such path is indeed a Hamiltonian isotopy.

Much inspired by [S], we construct the following "spectrum bundle":

$$\Lambda = \{(x, \lambda) \in \mathcal{P} \times \mathbf{S}^1 |\ \lambda \text{ is a rotation number for } x\}.$$

By Theorem 4.2.2, each fiber of the projection $\pi : \Lambda \to \mathcal{P}$ consists of n points, counted with multiplicity.

5.1.1 A remark on notation

Throughout this section, our notation will include many parameters and parentheses, so we set a couple of conventions. The symbol ϕ_t is reserved for a single diffeomorphism, while (ϕ_t) denotes an isotopy with parameter $t \in [0, 1]$. And when we write (ϕ_{st}), one should read $(\phi_{st})_{t\in[0,1]}$, i.e. it will be understood that t is the isotopy parameter, ranging in $[0, 1]$, and $s \in [0, 1]$ is fixed.

Since we only deal with 2-homogeneous generating functions, we will commit the abuse of regarding them as functions on $\mathbf{CP}^N$.

5.1.2 Existence of sections

Analogous to [S], we construct sections $\mathcal{P} \to \Lambda$. We will do this by showing that the spectrum bundle is trivial over a large subset of $\mathcal{P}$. This will allow us to number the rotation numbers of each isotopy in a consistent manner. We make a brief digression into singularity theory. For more details, we refer the reader to [Gib].

Let p be a critical point (or singularity) of a smooth function $f : X \to \mathbf{R}$, i.e. $df(p) = 0$. It is called **non-degenerate** if $d^2f(p)$, which is well-defined since p is a critical point, is non-degenerate as a bilinear form. In suitable coordinates, it has the form

$$(x_1, \ldots, x_n) \mapsto Q(x_1, \ldots, x_n),$$

where Q is a non-degenerate quadratic form and the origin $x_1 = \ldots = x_n = 0$ corresponds to p. Similarly, p is a **birth-death** or **fold** critical point if the null space N of $d^2f(p)$ has dimension 1 and $d^3f(p)$, which is well-defined on N, is non-zero. It follows that, in this case, f has local form

$$(x_1, \ldots, x_n) \mapsto x_1^3 + Q(x_2, \ldots, x_n),$$

where Q is as before. Finally, p is a **cusp** if it is conjugated to $x_1^4 + Q(x_2, \ldots, x_n)$, which is equivalent to N being 1-dimensional as above, $d^3f(p)$ being zero on N, but $d^4(p) \neq 0$ on N. Non-degenerate, birth-death and cusp critical points are collectively referred to as **singularities of codimension** ≤ 2.

Remark 5.1.1. *From their local forms, it is immediate that such critical points are isolated.*

If X has a symplectic structure, a fixed point p of the symplectomorphism $\phi : X \to X$ will be called **non-degenerate/birth-death/cusp/of codimension** ≤ 2 according to the nature of the critical point of a generating function for ϕ that corresponds to it. These notions do not depend on the generating function chosen since they are invariant under the operations of stabilization and fiberwise diffeomorphism, as defined in Section 4.1.1

We define $\mathcal{P}^*$ to be the subset of $\mathcal{P}$ consisting of isotopies (ϕ_t) such that the endpoint ϕ_1 only has fixed points of codimension ≤ 2. The justification of the nomenclature comes from the result below. See [Gib] for the general definition of codimension of a singularity and the classification of low-codimension critical points.

Theorem 5.1.2. *Let $\Xi : \mathbf{D}^2 \times \mathbf{CP}^n \to \mathbf{R}$ be a (smooth) family of real-valued functions on projective space, parametrized by a two-dimensional disk $\mathbf{D}^2$. Then there exists a family Ξ' arbitrarily close to Ξ in the C^k topology (for any k) such that $\Xi'(T)$ only has singularities of codimension ≤ 2. Furthermore, if Ξ is a family of real-valued functions parametrized by a three-dimensional ball $\mathbf{D}^3$, then it is approximated by a family Ξ' such that $\Xi'(T)$ only has codimension ≤ 2 singularities whenever T is on the boundary of $\mathbf{D}^3$.*

We now study the basic topological properties of $\mathcal{P}^*$.

Lemma 5.1.3. *The subset $\mathcal{P}^*$ is dense in $\mathcal{P}$.*

Proof. We will approximate a given isotopy (ϕ_t) by elements in $\mathcal{P}^*$. First we choose a family $r_t : \mathbf{CP}^N \to \mathbf{R}$ of generating functions for (ϕ_t). Instead of using the theorem above, we outline the classical result of Cerf's that states that a "generic" one-parameter of real-valued functions only has non-degenerate and birth-death singularities. Similar techniques lie in the proof of Theorem 5.1.2

In order to understand the critical points of r_t, we consider the 2-jets $j^2 r_t$, a family of sections of the 2-jet bundle $\mathcal{J}^2 \to M$ parametrized by t. We are interested in intersections of $j^2 r_t$ with the subset $Z \subset \mathcal{J}^2$ consisting of jets with critical points. Note that Z is not smooth. It is indeed stratified by the rank of the second derivative at the critical point. While $\mathcal{J}^2$ has dimension $2N + 1 + 2N + (2N)^2$, the top stratum Z^0 of Z has dimension $2N + 1 + (2N)^2$, since it is defined by the equation $\nabla j(p) = 0$. The next stratum, Z^1, consists of jets whose Hessian has rank $2N - 1$. It has dimension $2N + (2N)^2$ since matrices of rank $(\ell - 1)$ are codimension 1 inside the space of $\ell \times \ell$ matrices. The Thom transversality theorem (for jets) now says that, given any k, we can find arbitrarily small perturbations r'_t of r_t such that $j^2 r'_t$ is transverse to every stratum of Z. By the dimension count above, this means that $j^2 r'_t$ only meets Z^0 and Z^1. It is easy to check that transverse intersections with Z^1 are exactly values of t for which r'_t has a birth-death singularity.

Since r'_0 need not generate the identity, we choose a smooth cutoff function $\alpha : [0,1] \to [0,1]$ that equals zero in a neighborhood of $t = 0$ and one in a neighborhood of $t = 1$. We define

$$r''_t = \alpha(t) r'_t + (1 - \alpha(t)) r_t.$$

and let (ϕ'_t) be the isotopy it generates. By construction, ϕ'_0 is the identity and ϕ'_1 only has non-degenerate and birth death fixed points, so $(\phi'_t) \in \mathcal{P}^*$.

Finally, we compute

$$\begin{aligned} \sup_t ||r''_t - r_t|| &= \sup_t \alpha(t)||r'_t - r_t|| \\ &\leq \sup_t ||r'_t - r_t||. \end{aligned}$$

This shows that the family r''_t we constructed can be chosen as close to r_t as desired, since Thom transversality guarantees that r'_t can be chosen arbitrarily close to r_t. $\square$

Lemma 5.1.4. *$\mathcal{P}^*$ is path-connected.*

Proof. Let (ϕ^i_t) be an element of $\mathcal{P}^*$ for $i = 0, 1$. We choose a family r^i_t that generates (ϕ^i_t). The space $\mathcal{P}$, as any pathspace, is contractible. In particular, $\mathcal{P}$ is connected, so there exists a path $\gamma(s) = (\phi^s_t)$ in $\mathcal{P}$ starting at (ϕ^0_t) and ending at (ϕ^1_t). We think of γ as a map

$$\begin{aligned} \Gamma : [0,1]^2 &\rightarrow G \\ (s,t) &\mapsto \phi^s_t \end{aligned}$$

We choose a two-parameter family of generating functions $r_{s,t} : \mathbf{CP}^N \rightarrow \mathbf{R}$ for (ϕ^s, t). Using Theorem 5.1.2, we perturb the functions $r_{s,t}$ supported in $(s,t) \in C$ sketched in Figure 5.1. We obtain a family $r'_{s,t}$ that only has singularities of codimension ≤ 2 in the interior of C. Since that was already the case for $r_{0,1}$ and $r_{1,1}$, this property is true of $r_{s,t}$ for all s when $t = 1$.

If the perturbation is sufficiently small in the C^2 topology, the Lagrangians generated by $r'_{s,t}$ are C^1-close to the ones generated by $r_{s,t}$. In particular, they will also be graphs of symplectomorphisms, which will denote by ψ^s_t. By construction, for any given $s \in [0,1]$, the isotopy (ψ^s_t) is in $\mathcal{P}^*$. Also, since we did not alter $r_{s,t}$ outside C, we have $(\psi^i_t) = (\phi^i_t)$ when $i = 0, 1$. We conclude that $\gamma'(s) = (\psi^s_t)$ is a path from (ϕ^0_t) to (ϕ^1_t) lying in $\mathcal{P}^*$, which proves the lemma. $\square$

The next proof is very similar, but has one extra parameter.

Lemma 5.1.5. *$\mathcal{P}^*$ is simply connected.*

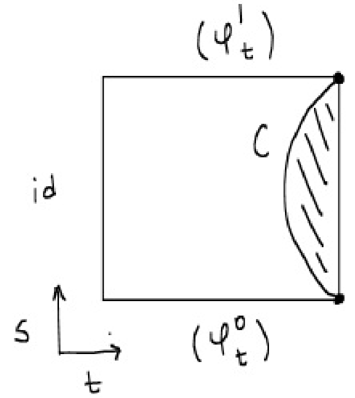

Figure 5.1: The region C.

Proof. Fix an element x of $\mathcal{P}^*$ and let $\gamma(s) = (\phi_{s,t})$ be a loop in $\mathcal{P}^*$ that starts and ends at x. Since $\mathcal{P}$ is contractible, we have $\pi_1(\mathcal{P}, x) = 0$ so that there exists a family of loops $\gamma_\epsilon(s) = (\phi_{\epsilon,s,t})$ based at x such that $\gamma_0 = \gamma$ and γ_1 is the constant loop at x. We think of this as a map

$$\begin{aligned} \Gamma : [0,1]^3 &\to G \\ (\epsilon, s, t) &\mapsto \phi_{\epsilon,s,t} \end{aligned}$$

We choose a 3-parameter family of generating functions $r_{\epsilon,s,t}$ and we perturb it using Theorem 5.1.2. The support of the perturbation in $[0,1]^3$ will be a closed 3-ball D. Part of the boundary of D is the face $t = 1$ and the other part is in the interior of the 3-cube. In particular, $r_{\epsilon,s,t}$ is left unperturbed on the other five faces of $[0,1]^3$. A sketch of D can be found in Figure 5.2.

We denote the perturbed family by $r'_{\epsilon,s,t}$. As in the proof above, if we take the perturbation to be small enough, the function $r'_{\epsilon,s,t}$ generates a symplectomorphism $\psi_{\epsilon,s,t}$. While some of these symplectomorphisms can have fixed points that are not of codimension ≤ 2, Theorem 5.1.2 guarantees that this is not the case on the boundary component $t = 1$. Consequently, for each ϵ and s, the isotopy $(\psi_{\epsilon,s,t})$ is in $\mathcal{P}^*$. And since we left the other faces unperturbed, $\gamma'_\epsilon(s) = (\psi_{\epsilon,s,t})$ is also a homotopy between the original γ and the constant loop at x, lying entirely in $\mathcal{P}^*$. We conclude that $\pi_1(\mathcal{P}^*, x) = 0$. $\square$

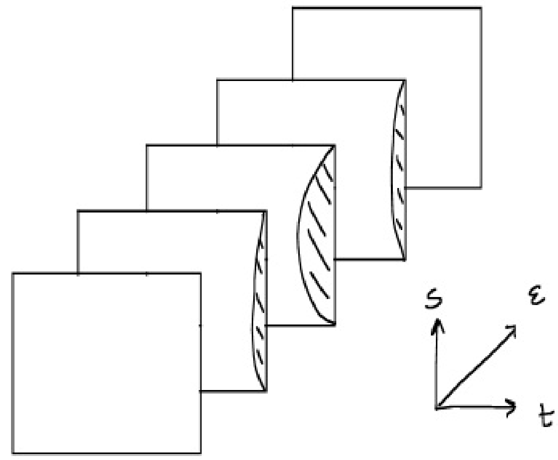

Figure 5.2: Region D, slice by slice.

The next result is needed to track the rotation numbers continuously. The reader can compare this to Schwarz's construction of spectral invariants on aspherical symplectic manifolds, see [S]. Since Λ is a subset of $\mathcal{P} \times \mathbf{S}^1$ we will abuse terminology and call a function $\sigma : \mathcal{P} \to \mathbf{S}^1$ a **section** of $\pi : \Lambda \to \mathcal{P}$ if the function $x \mapsto (x, \sigma(x))$ is a section in the usual sense.

Proposition 5.1.6. *There exist n sections $\sigma_1, \ldots, \sigma_n : \mathcal{P} \to \mathbf{S}^1$ of π such that*

$$\pi^{-1}(x) = \{\sigma_1(x), \ldots, \sigma_n(x)\}$$

for each x in $\mathcal{P}$, the multiplicity of each element being the same in both sets. Furthermore, these functions are uniquely defined up to a renumbering of their indices.

Proof. We define $\Lambda^* = \pi^{-1}(\mathcal{P}^*)$. By Theorem 4.2.2, since each isotopy in $\mathcal{P}^*$ has an endpoint with isolated fixed points, each fiber of $\pi^* : \Lambda^* \to \mathcal{P}^*$ consists of n distinct points. By the continuity statement of Theorem 4.2.2, π^* is a locally trivial fiber bundle and, hence, a n-sheeted covering map. And since $\mathcal{P}^*$ is simply connected by Lemma 5.1.5, it must be the trivial n-sheeted cover. We can thus define $\sigma_1^*, \cdots, \sigma_n^* : \mathcal{P}^* \to \mathbf{S}^1$ having the desired property. The only choice here is the numbering of the sheets of π^*.

We now extend the σ_i^* to the whole space $\mathcal{P}$. Given $x \in \mathcal{P}$, we choose a path γ from the identity to x. By the exact same argument as in Lemma 5.1.4, we can assume modulo a small perturbation relative to endpoints that $\gamma(s) \in \mathcal{P}^*$ for $s \in (0,1)$. We set

$$\sigma_i(x) = \lim_{s \to 1} \sigma_i^*(\gamma(s)).$$

This limit exists and is a rotation number of x due to the continuity statement in Theorem 4.2.2 (a). Furthermore, if x is already in $\mathcal{P}^*$ then $\sigma_i(x)$ defined in this fashion agrees with the previously defined $\sigma_i^*(x)$.

We only need to check that this definition of $\sigma_i(x)$ does not depend on the path γ chosen. To see this, let γ' be another path from the identity to x, lying in $\mathcal{P}^*$ except for the endpoints. Choose a homotopy h between the two paths that keeps the endpoints fixed. By the argument in the proof of Lemma 5.1.5, we can assume modulo a small perturbation that $h(\epsilon, s) \in \mathcal{P}^*$ for every ϵ (the homotopy parameter) and for $s \in (0,1)$. We define, for each ϵ,

$$\sigma_{i,\epsilon}(x) = \lim_{s \to 1} \sigma_i^*(h(\epsilon, s)).$$

For $\epsilon = 0$ we recover the construction above using γ and $\epsilon = 1$ corresponds to using γ' instead. But since $\sigma_{i,\epsilon}(x)$ is a rotation number of x (again by Theorem 4.2.2 (a)) and such set of rotation numbers is discrete, we conclude that

$$\sigma_{i,0}(x) = \sigma_{i,1}(x).$$

This finishes the proof. □

We fix a numbering of the sections given by the proposition above once and for all.

Definition 5.1.7. *Given an isotopy (ϕ_t), we define*

$$\rho_i(s, (\phi_t)) = \sigma_i(\phi_{st}),$$

where the σ_i are the sections of the spectrum bundle constructed in Proposition 5.1.6. If the underlying isotopy (ϕ_t) is understood, we will simply write $\rho_i(s)$.

5.1.3 Ad hoc numbering of a single isotopy

Given an isotopy, it is easy to determine, modulo permuting indices, what is the “correct” way to number the rotation numbers of (ϕ_{st}) for each s, i.e. the numbering induced by

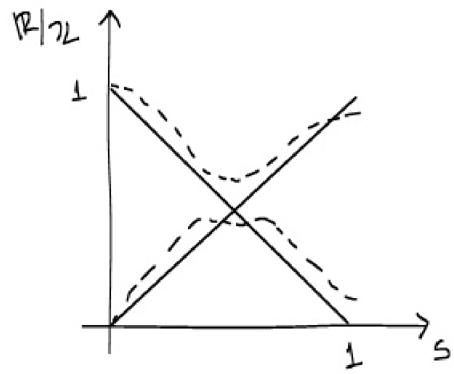

Figure 5.3: Rotation numbers of a nearby isotopy.

the sections of Proposition 5.1.6. As an example, assume that (ϕ_{st}) has $\{-s, s\}$ as rotation numbers. Modulo permuting the labels, there are two numbering options, namely:

$$\alpha_1(s) = \begin{cases} s, & s \leq \frac{1}{2} \\ -s, & s \geq \frac{1}{2} \end{cases}$$
$$\alpha_2(s) = \begin{cases} -s, & s \leq \frac{1}{2} \\ s, & s \geq \frac{1}{2} \end{cases}$$

$$\begin{aligned} \beta_1(s) &= s \\ \beta_2(s) &= -s \end{aligned}$$

The ambiguity arises since the two rotation numbers are equal when $s = \frac{1}{2}$. If (ψ_t) is close to (ϕ_t) and $(\psi_{st}) \in \mathcal{P}^*$ for $s \in (0, 1)$, then there is no ambiguity on the numbering of the rotation numbers of (ψ_{st}) for each $s \in [0, 1]$. Figure 5.2.4 shows the rotation numbers of (ϕ_{st}) and, with a dotted line, those of the nearby "generic" (ψ_{st}).

It is clear that the second numbering $\beta_i(s)$, which has a "crossing" at $s = \frac{1}{2}$, is not close to $\rho_i(s, (\psi_t))$. As (ψ_{st}) approaches (ϕ_{st}), we recover the numbering $\alpha_i(s)$, which must be the same as the numbering $\rho_i(s, (\phi_{st}))$ given by Proposition 5.1.6.

Figure 5.4 shows how the rule of "no crossings" can be used to figure out an allowable numbering for a complicated diagram of rotation numbers.

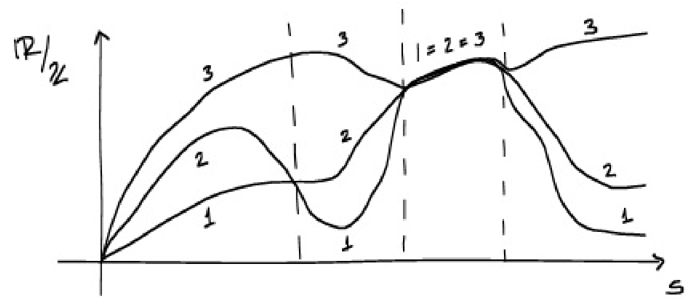

Figure 5.4: Another numbering example.

5.2 Definition and first properties of the invariant

Definition 5.2.1. *Given (ϕ_t), let $\tilde{\rho}_i : [0,1] \to \mathbf{R}$ be the lift of $\rho_i : [0,1] \to \mathbf{R}/\mathbf{Z}$ that satisfies $\tilde{\rho}_i(0) = 0$, see Definition 5.1.7. Note that this is possible since, when $s = 0$, the isotopy (ϕ_{st}) is constant at the identity and hence $\rho_i(0) = 0 \in \mathbf{R}/\mathbf{Z}$. We define*

$$\mu_i((\phi_t)) = \tilde{\rho}_i(1),$$

called the **winding** *of the i-th rotation number of (ϕ_t). We also define $\rho(s)$ to be the sum of the $\rho_i(s)$ and*

$$\mu((\phi_t)) = \frac{\tilde{\rho}(1)}{n},$$

where $\tilde{\rho}$ is, analogously, the lift of ρ to a $\mathbf{R}$-valued function that is zero at zero.

We start with a simple observation.

Proposition 5.2.2. *For each i, j we have*

$$\begin{aligned} |\tilde{\rho}_i(s) - \tilde{\rho}_j(s)| &\leq 1 \\ |\tilde{\rho}(s) - n\tilde{\rho}_i(s)| &\leq n - 1 \end{aligned}$$

Proof. The first inequality follows from the discussion in Section 5.1.3: since rotation numbers are not allowed to "cross" one another, the cyclic order of the set $\big(\rho_1(s), \ldots, \rho_n(s)\big)$ remains unchanged as s varies. This shows that, up to an error of 1, the rotation numbers ρ_i have the same winding.

By definition, the ρ defined above is the sum of the ρ_i and, by uniqueness of lifts, $\tilde{\rho} = \tilde{\rho}_1 + \cdots \tilde{\rho}_n$. We conclude that

$$|\tilde{\rho}(s) - n\tilde{\rho}_i(s)| \leq |\tilde{\rho}_1(s) - \tilde{\rho}_i(s)| + \cdots + |\tilde{\rho}_n(s) - \tilde{\rho}_i(s)| \leq n - 1$$

by using the first inequality □

Corollary 5.2.3. *For every i and every isotopy (ϕ_t), we have*

$$|\mu(\phi_t) - \mu_i(\phi_t)| \leq 1$$

Proof. We compute

$$|\mu - \mu_i| = \frac{1}{n}|\tilde{\rho}(1) - n\tilde{\rho}_i(1)| \leq \frac{n-1}{n} \leq 1$$

□

We study a simple example.

Example 5.2.4. Consider the loop $\phi_t([z_0 : z_1]) = [e^{2\pi i t} z_0 : z_1]$ of Hamiltonian diffeomorphisms of $\mathbf{CP}^1$. It is generated by the normalized Hamiltonian

$$h_t([z_0 : z_1]) = \frac{\pi |z_0|^2}{|z_0|^2 + |z_1|^2} - \frac{\pi}{2}.$$

Seen as a function on $\mathbf{S}^{2n-1}$, we can write

$$h_t(z_0, z_1) = \pi |z_0|^2 - \frac{\pi}{2}$$

and it follows that

$$H_t(z_0, z_1) = \pi|z_0|^2 - \frac{\pi}{2}\left(|z_0|^2 + |z_1|^2\right) = \frac{\pi}{2}|z_0|^2 - \frac{\pi}{2}|z_1|^2.$$

The corresponding Hamiltonian vector field is

$$X_t(z_0, z_1) = i\nabla H_t(z_0, z_1) = (\pi i z_0, -\pi i z_1),$$

with flow

$$\Phi_t(z_0, z_1) = \left(e^{\pi i t} z_0, e^{-\pi i t} z_1\right).$$

Each Φ_t fixes exactly two complex lines, namely the coordinate axes $z_0 = 0$ and $z_1 = 0$. The rotation numbers of (ϕ_{st}) must then be ($\frac{1}{2\pi i}$ times the log of) the corresponding eigenvalues, i.e. $\rho_1(s) = \frac{s}{2}$ and $\rho(s) = -\frac{s}{2}$, modulo renumbering. (Recall that λ being a rotation number of (ϕ_{st}) implies that there exists a complex line on which Φ_s acts by multiplication by $e^{2\pi\lambda}$, see Section 4.2.2.) We conclude that $\rho(s) = \frac{s}{2} - \frac{s}{2} \equiv 0$ and $\mu(\phi_t) = 0$. On the other hand, $\mu_1(\phi_t) = -\frac{1}{2}$ and $\mu_2(\phi_t) = \frac{1}{2}$. If we consider instead the squared isotopy (ϕ_t^2), we obtain rotation numbers $\{s, -s\}$. As in the example above, an allowable parametrization would be

$$\rho_1(s,(\phi_t^2)) = \begin{cases} s, & s \leq \frac{1}{2} \\ -s, & s \geq \frac{1}{2} \end{cases}$$

$$\rho_2(s,(\phi_t^2)) = \begin{cases} -s, & s \leq \frac{1}{2} \\ s, & s \geq \frac{1}{2} \end{cases}$$

It follows that $\mu(\phi_t^2) = \mu_1(\phi_t^2) = \mu_2(\phi_t^2) = 0$. One can easily extend this computation to other powers (ϕ_t^n): for odd n we obtain $\mu = 0$, $\mu_1 = -\frac{1}{2}$ and $\mu_2 = \frac{1}{2}$ and for even n all three quantities are zero.

Even though we constructed the invariant on the space $\mathcal{P}$ of paths in the Hamiltonian group of $\mathbf{CP}^{n-1}$, its natural domain of definition is the universal cover of the same group.

Proposition 5.2.5. *The functions $\mu_i : \mathcal{P} \to \mathbf{R}$ are invariant under homotopies fixing endpoints. In other words, each μ_i descends to the universal cover $\widetilde{\mathrm{Ham}}(\mathbf{CP}^{n-1}, \omega)$. The same applies to μ.*

Proof. We prove it for μ. The proof for μ_i is analogous. We consider a continuous family $(\phi_t^\lambda)_{\lambda\in[0,1]}$ of isotopies, all having the same endpoints. Denote by $\mu(\lambda)$ and ρ^λ the corresponding values of μ and ρ. It follows from Theorem 4.2.2 item (e) and Remark 4.2.3 that $\rho^\lambda(1)$ does not depend on λ. Consequently, $\mu(\lambda) = \tilde{\rho}^\lambda(1)$ is congruent mod 1 to $\mu(0) = \tilde{\rho}^0(1)$. Since μ is manifestly continuous, $\mu(1) = \mu(0)$. □

Now we point out a surprising property of the invariant μ. A priori, this property is only expected to hold after one homogenizes μ, see Definition 3.1.2.

Proposition 5.2.6. $\mu(\phi_t^{-1}) = -\mu(\phi_t)$

Proof. In order to understand the rotation numbers of Φ^{-1}, we wish to keep track of the index of the subset $\{(-S)\#Q_t \leq 0\}$, where S generates Φ. Using Propositions 4.1.6 and

4.1.8 and the equivariance of Φ, we notice that both $(-S)\#Q_t$ and $-(S\#Q_{-t})$ generate the same diffeomorphism $\Phi^{-1}\circ e^{-2\pi it}$. It follows that one can instead study the index of $\{-(S\#Q_{-t})\leq 0\}$. Similarly, for index computations, one can replace Q_{-t} by $Q_{-1}\#Q_{1-t}$. One obtains, for almost all t and t' - those for which the relevant sublevel sets are manifolds with boundary,

$$\begin{aligned} & \text{ind}\ \{(-S)\#Q_t\leq 0\}-\text{ind}\ \{(-S)\#Q_{t'}\leq 0\} \\ = & \text{ind}\ \{S\#Q_{-t}\geq 0\}-\text{ind}\ \{S\#Q_{-t'}\geq 0\} \\ = & \text{ind}\ \{S\#Q_{-t'}\leq 0\}-\text{ind}\ \{S\#Q_{-t}\leq 0\} \\ = & \text{ind}\ \{S\#Q_{-1}\#Q_{1-t'}\leq 0\}-\text{ind}\ \{S\#Q_{-1}\#Q_{1-t}\leq 0\}, \end{aligned}$$

where we also used Proposition 4.1.15. Now note that $S\#Q_{-1}$ is a generating function for Φ and conclude that the rotation numbers for Φ^{-1} are obtained from those for Φ by changing signs. Applying this to the isotopy $(\phi_{st}^{-1})_{0\leq t\leq 1}$ for each s proves the result. □

5.3 Proof of the quasimorphism property

We start with a simple result.

Lemma 5.3.1. *Let $F:(a,b)\times M\to \mathbf{R}$, be a smooth map having zero as a regular value and $X=F^{-1}(0)$. Denote $F(s,x)$ by $f_s(x)$ and by $\pi: X\to (a,b)$ the projection onto the first factor. Then c is a critical value of π if and only if zero is a critical value of f_c.*

Proof. Assume c is a critical value of π. It follows that there exists p such that $\pi_*: T_{(c,p)}X\to T_c\mathbf{R}$ is the zero map. In other words, the vectors in $T_{(c,p)}X$ all have a zero component in the $\frac{\partial}{\partial s}$ direction so that $T_{(c,p)}X=\{0\}\oplus T_pM$. But since $T_{(c,p)}X$ is the kernel of F_*,

$$\{0\}=F_*(T_{(c,p)}X)=(\frac{df_s}{ds}\oplus (f_c)_*)(0\oplus T_pM)=0+(f_c)_*(T_pM)$$

so zero is a critical value of f_c. The same argument worked backwards proves that c is a critical value of π whenever $f_c^{-1}(0)$ is singular. □

Now we relate the winding of rotation numbers to the change of index of a family of generating functions.

Proposition 5.3.2. *Let (ϕ_t) be an isotopy of $\mathbf{CP}^{n-1}$, Φ_s the equivariant symplectomorphism of $\mathbf{C}^n$ obtained by lifting (ϕ_{st}) as before and R_s a family of (2-homogeneous) generating functions for Φ_s. Assume the following properties:*

1. *Φ_{s_0} has a fixed point.*
2. *There exist $\epsilon, \delta > 0$ such that, for each $s \in I = [s_0 - \epsilon, s_0 + \epsilon]$, the equation $\Phi_s(x) = e^{2\pi i t}x$ has exactly one solution $t(s)$ satisfying $|t(s)| \leq \delta$.*
3. *The function $t(s)$ thus defined is smooth near $s = s_0$ and satisfies $t'(s_0) < 0$.*

Then, the index of R_s is locally constant on $I - \{s_0\}$. Furthermore, if zero is not a rotation number of Φ_{s_0} then $\operatorname{ind} R_s$ *is constant throughout I. Otherwise,* $\operatorname{ind} R_{s_0+\epsilon} = \operatorname{ind} R_{s_0-\epsilon} + 1$.

Proof. By assumption 3, we can shrink ϵ to guarantee that the only zero of the function $t(s)$ in I is s_0. It follows that Φ_s has no fixed points for $s \in I - \{s_0\}$ since those would yield new solutions of $\Phi_s(x) = e^{2\pi i t}x$, which are not allowed by assumption 2. We conclude that R_s has no critical points for $s \in I - \{s_0\}$ (since these critical points correspond to fixed points of Φ_s). By the lemma above, the projection $\{(s, x) : R_s(x) = 0\} \mapsto s$ only has s_0 as a critical value in I. By the usual Morse theory argument, the index of R_s is locally constant in $I - \{s_0\}$ Recall the family Q_τ of quadratic forms generating $z \mapsto e^{-2\pi i\tau}z$. Instead of R_s, we examine the index change of the family of generating functions obtained by concatenating the following families:

(i) $R_{s_0-\epsilon}\#Q_\tau$, where τ goes from zero to $-\delta$,

(ii) $R_s\#Q_{-\delta}$, with s moving form $s_0 - \epsilon$ to s_0,

(iii) $R_{s_0}\#Q_\tau$, for τ ranging from $-\delta$ to δ,

(iv) $R_s\#Q_\delta$, where s goes from s_0 to $s_0 + \epsilon$ and

(v) $R_{s_0+\epsilon}\#Q_\tau$ with τ moving from δ to zero.

The index of this family remains constant throughout segment (i) since $t(s_0 - \epsilon)$ is positive and, hence, $R_{s_0-\epsilon}$ cannot have rotation numbers in the interval $[-\delta, 0]$, by assumption 2. Analogously, the index cannot change throughout segment (v). In segment (ii), a change of index imply the existence of a critical point of $R_s\#Q_{-\delta}$ for some $s \in [s_0 - \epsilon, s_0]$, which in turn would yield a solution of $\Phi_s(x) = e^{-2\pi i\delta}x$ that is not $t(s)$ (since $t(s) \geq 0$ in this interval), contradicting assumption 2 once again. Respectively, there is no index variation in segment (iv). Finally, we conclude that the total change of index of R_s equals the change of index in segment (iii) of the family above. And by construction, the index "jumps" by 1 iff $t(s_0) = 0$ is a rotation number of Φ_{s_0}, concluding the proof □

We remark that, if we have instead a positive derivative in the assumption 3 above, the index change of the generating family in the interval I will be -1. The proof is absolutely analogous. We are now ready to prove the main property of the map μ we constructed.

Theorem 5.3.3. *For any pair of isotopies (ϕ_t) and (ψ_t), we have $|\mu(\psi_t\phi_t) - \mu(\phi_t) - \mu(\psi_t)| \leq 4$.*

Proof. Let R_s be a family that generates Φ_s and T_s one that generates Ψ_s. We define

$$\begin{aligned} \Delta_R &= \text{ind } R_1 - \text{ind } R_0 \\ \Delta_T &= \text{ind } T_1 - \text{ind } T_0 \end{aligned}$$

We claim that $|n\mu(\phi_t) + \Delta_R| \leq n$ and $|n\mu(\psi_t) - \Delta_T| \leq n$. By the proposition above, $-\Delta_R$ is equal to the net number of times the rotation numbers $\rho_i(s, (\phi_t))$ cross zero. We conclude that, by definition,

$$-\Delta_R = \sum_i \lfloor \mu_i(\phi_t) \rfloor,$$

where $\lfloor x \rfloor$ is the integer part of x. Since $|x - \lfloor x \rfloor| \leq 1$ for any x, it follows that

$$\begin{aligned} |n\mu(\phi_t) + \Delta_R| &= \Big|\sum_i \mu_i(\phi_t) + \Delta_R\Big| \\ &\leq \sum_i |\mu_i(\phi_t) - \lfloor \mu_i(\phi_t) \rfloor| \\ &\leq n \end{aligned}$$

This proves the claim. We now consider the family $R_s\#T_s$ that generates the composition $\Psi_s \circ \Phi_s$ and compare the following quantities:

$$\begin{aligned} \Delta_\# &= \text{ind } R_1\#T_1 - \text{ind } R_0\#T_0 \\ \Delta_\oplus &= (\text{ind } R_1 - \text{ind } R_0) + (\text{ind } T_1 - \text{ind } T_0) = \Delta_R + \Delta_T \end{aligned}$$

Note that, by the above argument, $|n\mu(\psi_t\phi_t) + \Delta_\#| \leq n$. Also, by Lemma 4.1.16, $\Delta_\oplus = \operatorname{ind} R_1 \oplus T_1 - \operatorname{ind} R_0 \oplus T_0$. Now by Lemma 4.1.18 both quantities $\operatorname{ind} R_1 \# T_1 - \operatorname{ind} R_1 \oplus T_1$ and $\operatorname{ind} R_0 \# T_0 - \operatorname{ind} R_0 \oplus T_0$ lie between zero and n so their difference satisties

$$|\Delta_\# - \Delta_\oplus| \leq n$$

Finally, we compute

$$\begin{aligned}
& |n\mu(\psi_t\phi_t) - n\mu(\phi_t) - n\mu(\psi_t)| \\
= \;& |n\mu(\psi_t\phi_t) + \Delta_\# - n\mu(\phi_t) - \Delta_R - n\mu(\psi_t) - \Delta_T - \Delta_\# + \Delta_R + \Delta_T| \\
\leq \;& |n\mu(\psi_t\phi_t) + \Delta_\#| + |n\mu(\phi_t) + \Delta_R| + |n\mu(\psi_t) - \Delta_T| + |\Delta_\# - \Delta_\oplus| \\
\leq \;& 4n
\end{aligned}$$

□

Together with Corollary 5.2.3, we immediately obtain the following.

Corollary 5.3.4. *The functions μ_i from Definition 5.2.1 are all quasimorphisms.*

5.4 Proof of the Calabi property

Let $\bar{\mu}$ be the homogenization of μ and $\bar{\mu}_i$ that of μ_i, as in Section 3.1. It follows from Corollary 5.2.3 that $\bar{\mu} = \bar{\mu}_i$ for every i.

We now show that $\bar{\mu}$ has the Calabi property. In particular, it follows that $\bar{\mu}$ is not identically zero.

Theorem 5.4.1. *Let $U \subset M$ be a displaceable open set and (ϕ_t) an isotopy compactly supported in U. If h_t is compactly supported in U and generates (ϕ_t), then*

$$\bar{\mu}(\phi_t) = \frac{1}{\pi V}\operatorname{Cal}(\phi_t) = \frac{1}{\pi V}\int_0^1 \int_M h_t \, \omega^{n-1} \, dt,$$

where V is the volume of $\mathbf{CP}^{n-1}$ with respect to ω^{n-1}

Proof. Let g_t be a mean normalized Hamiltonian that generates an isotopy (ψ_t) with the property $\psi_1(U) \cap U = \emptyset$. We divide the proof into four claims.

Claim 1. If x is a fixed point of $\phi_1^k \psi_1$ for some integer k, then $\psi_1(x) = x$. Here, ϕ_1^k denotes the k-fold composition $\phi_1 \circ \cdots \circ \phi_1$.

Proof of Claim 1: We look for solutions of $\psi_1(x) = \phi_1^{-k}(x)$. If $x \in U$, the right hand side is in U while the left hand side is not, so no solutions exists. If x is not in U, the equation reduces to $\psi_1(x) = x$.

Claim 2. The Hamiltonian $K_t(z) = c(t)|z|^2$ on $\mathbf{C}^n$ generates the isotopy $\Theta_t(z) = e^{-2i\theta(t)}z$, where $\theta(t) = \int_0^t c(\tau)\, d\tau$.

Proof of Claim 2: The vector field Y_t generating Θ_t satisfies

$$Y_t(\Theta_t(z)) = \frac{d}{dt}\Theta_t(z) = -2ic(t)e^{-2i\theta(t)}z = -2ic(t)\Theta_t(z),$$

from which we conclude that $Y_t(z) = -2ic(t)z$ and

$$\Omega(Y_t, z') = \langle iY_t, z' \rangle = \langle 2c(t)z, z' \rangle = \langle \nabla K_t, z' \rangle = dK_t(z'),$$

proving the claim.

Claim 3. $\bar{\mu}(\phi_t) = \lim_{k\to\infty} \frac{1}{k}\mu(\phi_t^k \psi_t)$.

Proof of Claim 3: From the quasimorphism property of μ,

$$|\mu(\phi_t^k \psi_t) - \mu(\phi_t^k) - \mu(\psi_t)| \leq C.$$

Dividing this inequality by k and letting $k \to \infty$ proves the claim.

Claim 4. $lim_{k\to\infty} \frac{1}{k}\mu(\phi_t^k \psi_t) = \frac{1}{\pi}\mathrm{Cal}(\phi_t)$.

Proof of Claim 4: We define

$$c(t) = -\frac{1}{V}\int_{\mathbf{CP}^{n-1}} h_t\ \omega^{n-1},$$

where V is the volume of $\mathbf{CP}^{n-1}$ with respect to the volume form ω^{n-1}. The Hamiltonian $h_t+c(t)$ is mean normalized, generates ϕ_t and equals $c(t)$ outside the open set U. We consider the isotopy (δ_t) obtained by concatenating the paths (ψ_t) and $(\phi_t^k\psi_1)$. By Proposition 5.2.5, $\mu(\delta_t)=\mu(\psi_t\phi_t^k)$. And by Claim 2, the lift (Δ_t) of (δ_t) to $\mathbf{C}^n$ coincides with the concatenation of (Ψ_t) and $(\Theta_t^k\Psi_1)$ on points that do not lie over U. Now let x be a fixed point of $\phi_1^k\psi_1$. By Claim 1, $x\neq U$ and $\psi_1(x)=x$. Thus, if z lies above x, then

$$\Psi_1(z)=e^{2\pi i\alpha}z$$

for some $\alpha\in[0,1)$ and

$$\Theta_t^k\Psi_1(z)=e^{2\pi i(\alpha-\frac{k}{\pi}\theta(t))}z.$$

We conclude that

$$\begin{aligned}
\lim_{k\to\infty}\frac{1}{k}\mu(\phi_t^k\psi_1) &= -\frac{1}{\pi}(\theta(1)-\theta(0))\\
&= -\frac{1}{\pi}\int_0^1 c(\tau)\ d\tau\\
&= \frac{1}{\pi V}\int_0^1\int_{\mathbf{CP}^{n-1}} h_t\ \omega^{n-1}\ dt\\
&= \frac{1}{\pi V}\mathrm{Cal}(\phi_t).
\end{aligned}$$

Note that we have been deliberately careless with the contribution of the path (Ψ_t) since it will not matter after we divide by k and take the limit. This finishes the proof of Claim 4 and, together with Claim 3, proves the theorem. $\square$

Bibliography

[B] Banyaga, A. *Sur la structure du groupe des diffomorphismes qui prservent une forme symplectique.* (French) Comment. Math. Helv. 53 (1978), no. 2, 174227.

[Br] Brooks, R. *Some remarks on bounded cohomology.* Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978), pp. 5363, Ann. of Math. Stud., 97, Princeton Univ. Press, Princeton, N.J., 1981.

[EP03] Entov, M.; Polterovich, L. *Calabi quasimorphism and quantum homology.* Int. Math. Res. Not. 2003, no. 30, 16351676.

[EP09] Entov, M.; Polterovich, L. *Rigid subsets of symplectic manifolds.* Compos. Math. 145 (2009), no. 3, 773826.

[ElP] Eliashberg, Ya.; Polterovich L. *Symplectic quasi-states on the quadric surface and Lagrangian submanifolds.* arXiv:1006.2501 [math.SG]

[FR] Fadell, E. R.; Rabinowitz, P. H. *Generalized cohomological index theories for Lie group actions with an application to bifurcation questions for Hamiltonian systems.* Invent. Math. 45 (1978), no. 2, 139174.

[FOOO] Fukaya, K.; Oh, Y.-G.; Ohta, H.; Ono, K. *Lagrangian intersection Floer theory: anomaly and obstruction.* Parts I and II. AMS/IP Studies in Advanced Mathematics, 46.1 and 46.2. American Mathematical Society, Providence, RI; International Press, Somerville, MA, 2009.

[G] Givental, A. B. *Nonlinear generalization of the Maslov index.* Theory of singularities and its applications, 71103, Adv. Soviet Math., 1, Amer. Math. Soc., Providence, RI, 1990.

[Gib] Gibson, C. G. *Singular points of smooth mappings.* Research Notes in Mathematics, 25. Pitman (Advanced Publishing Program), Boston, Mass.-London, 1979.

[H] Hatcher, A. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

[LS] Laudenbach, F.; Sikorav, J.-C. *Persistance d'intersection avec la section nulle au cours d'une isotopie hamiltonienne dans un fibr cotangent.* Invent. Math. 82 (1985), no. 2, 349357.

[MS] McDuff, D.; Salamon, D. *Introduction to symplectic topology.* Second edition. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1998.

[Py06] Py, P. *Quasi-morphismes de Calabi et graphe de Reeb sur le tore.* (French) [Calabi quasi-morphisms and Reeb graph on the torus] C. R. Math. Acad. Sci. Paris 343 (2006), no. 5, 323328.

[Py05] Py, P. *Quasi-morphisme de Calabi sur les surfaces de genre suprieur.* (French) [Calabi quasi-morphism on higher-genus surfaces] C. R. Math. Acad. Sci. Paris 341 (2005), no. 1, 2934.

[S] Schwarz, M. *On the action spectrum for closed symplectically aspherical manifolds.* Pacific J. Math. 193 (2000), no. 2, 419461.

[Sh] Shelukhin, E. *The Action homomorphism, quasimorphisms and moment maps on the space of compatible almost complex structures.* arXiv:1105.5814v2 [math.SG]

[Si] Simon, G, B. *The nonlinear Maslov index and the Calabi homomorphism.* Commun. Contemp. Math. 9 (2007), no. 6, 769780.

[T] Théret, D. *Rotation numbers of Hamiltonian isotopies in complex projective spaces.* Duke Math. J. 94 (1998), no. 1, 1327.

[U] Usher, M. *Deformed Hamiltonian Floer theory, capacity estimates and Calabi quasi-morphisms.* Geom. Topol. 15 (2011), no. 3, 13131417.

[V] Viterbo, C. *Symplectic topology as the geometry of generating functions.* Math. Ann. 292 (1992), no. 4, 685710.