Local Regularity of the Complex Monge-Ampère Equation

Yu Wang

Submitted in partial fulfillment of the
requirements for the degree
of Doctor of Philosophy
in the Graduate School of Arts and Sciences

COLUMBIA UNIVERSITY

2013

ABSTRACT

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In this thesis, we study the local regularity of the complex Monge-Ampère equation,

$$\left(\sqrt{-1}\partial\bar{\partial}u\right)^n = f dx$$

where $\sqrt{-1}\partial\bar{\partial}u$ stands for the complex Hessian form and dx the Lebesgue measure.

The underline idea of our work is to consider this equation as a full-nonlinear equation and apply modern theory and techniques of elliptic PDEs. Our main results include

- A simplified viscosity theory on the solvability of the Dirichlet problem of the complex Monge-Ampère equation.
- A small perturbation result: if f is slightly better than Dini continuous and the solution u is L^∞-close to a quadratic polynomial whose complex Hessian has determinant 1, then u is C^2 at the points x on which $f(x) = 1$.
- A Liouville type theorem: if u solves $\left(\sqrt{-1}\partial\bar{\partial}u\right)^n = dx$ on entire $\mathbb{C}^n$ and $u - \frac{1}{2}|x|^2$ is of sub-quadratic growth at infinity, then u differs from $\frac{1}{2}|x|^2$ by a linear function.
- A converging theorem: Assume $f \geq \lambda > 0$, if a sequence of solutions u_k converging uniformly to a smooth solution φ, then u_k converges smoothly to φ.
- An $C^{2,\alpha}$-regularity theorem: if f is Hölder and the solution u is in $W^{2,p}$ for $p > n(n-1)$, then u is $C^{2,\alpha}$.

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Acknowledgments

First of all, I would like to express my great gratitude to my thesis advisor, Duong Hong Phong, for his guidance, support and constant encouragement over my Ph.D. years at Columbia University. His insightful discussion and broadness in literature has been a great impetus in my research; his precise and rigorous style of mathematics has set me a life-time example. Moreover, his valuable advise for me is not only within mathematics but also outreach to many aspects of my life.

I would also like to thank Prof. Ovidiu Savin for his inspirational discussions and his insightful PDE lectures. The PDE techniques that I learned from him have been a cornerstone of my research.

I would like to thank Prof. Nam Le, Prof. Valentino Tossati and many other post-docs in Columbia math department for their discussions and suggestions. I would like to thank my fellow graduate students Tristan Collins, Daniel Rubin, Thomas Nyvberg, Ye-Kai Wang and Mooney Conner. They have formed a friendly and constructive research group in geometric analysis from which I have learned many interesting topics.

Finally, I would like to thank my parents and my wife for all their love.

To My Parents and My Wife

Chapter 1

Introduction

The complex Monge-Ampère equation

$$(\sqrt{-1}\partial\bar{\partial}u)^n = f\ dx \tag{1.1}$$

is a central object in complex geometry, multi-variable complex analysis and stability theory. It has been undergoing intensive research in the past three decades. Well-known works include [BT76], [Yau78], [CKNS85] and many others.

On the global side, by the pioneer work of Yau [Yau78] and contribution from many other authors (see [Aub76, PS09, GL10] and reference therein), the regularity of solutions to (1.1) with smooth data are well-understood on compact Kähler manifold. In particular, one has

Theorem 1.1 (Yau, 1978)**.** *Let X be a compact Kähler manifold and u be a solution to* (1.1)*. If $f \in C^3(X)$, then $u \in C^{2,\alpha}(X)$*

$$\|u\|_{C^{2,\alpha}(X)} \leq C(X, \|f\|_{C^3(X)}). \tag{1.2}$$

In particular,

$$\|\nabla u\|_{L^\infty(X)}, \|D^2u\|_{L^\infty(X)} \leq C(X, \|f\|_{C^3(X)}, \|u\|_{L^\infty(X)}). \tag{1.3}$$

By the well-known work of Caffarelli, Kohn, Nirenberg and Spruck [CKNS85], regularity of complex Monge-Ampère equation on a bounded domain $\Omega \subset \mathbb{C}^n$ with well-prescribed data is also well-understood. One has

Theorem 1.2 (Caffarelli, Kohn, Nirenberg, Spruck)**.** *Let Ω be in $\mathbb{C}^n$ and u be a plurisubharmonic function that solves the Dirichlet problem*

$$\begin{cases} (\sqrt{-1}\partial\bar{\partial}u)^n = f\ dx & \textit{in } \Omega \\ u = \varphi & \textit{on } \partial\Omega \end{cases} \tag{1.4}$$

If

$$\begin{aligned} &\lambda \le f \le \Lambda, \quad f \in C^3(\overline{\Omega}), \\ &\Omega \textit{ strictly pesudo-convex}, \quad \varphi \in C^4(\partial\Omega) \end{aligned} \tag{1.5}$$

then $u \in C^{2,\alpha}(\overline{\Omega})$ and

$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} \le C(\Omega, \|f\|_{C^2(\overline{\Omega})}, \|\varphi\|_{C^4(\partial\Omega)}). \tag{1.6}$$

On the local side, starting from foundational work of Bedford and Taylor [BT76], pluripotential theory has been applied to study the complex Monge-Ampère equation, specially with rough right-hand side. This direction was further developed by many authors. Related works include [BT82, Dem98, Ceg04, Kol98, Kol05]. One of the most important result along this direction is Kolodziej's L^∞-estimate.

Theorem 1.3 (Kolodziej 1998)**.** *Let Ω be a bounded pseudo convex domain in $\mathbb{C}^n$ and u satisfies* (1.1)*. If*

$$u = 0 \textit{ on } \partial\Omega \tag{1.7}$$

then, for any $p > 1$

$$-\inf_{\Omega} u \le C(n, p, \mathrm{diam}(\Omega))\|f\|_{L^p(\Omega)}^{1/n}. \tag{1.8}$$

However, the literature for the higher local regularity of the complex Monge-Ampère equation is extremely limited.

Similar to the real Monge-Ampère equation, the complex Monge-Ampère equation does not admit pure interior regularity. This is illustrated by an example due to Blocki [Blo97]: The function

$$u(x) = |z'|^{2-2/n}(1 + |z_n|^2), \quad z' = (z_1, ..., z_{n-1}) \tag{1.9}$$

satisfies

$$(i\partial\bar{\partial}u)^n = c_n(1 + |z_n|^2)^{n-2}\, dx, \quad \text{on } \mathbb{C}^n \tag{1.10}$$

in weak sense. However u is not C^2 near $\{z' = 0\}$.

The natural question is then to ask under what local condition, a solution u does not have singularity. Currently, very little is known in this direction.

One of the approaches has been considered is to study the Green's function of the complex Monge-Ampère equation, i.e., the solution of the Dirichlet problem

$$\begin{cases} (\sqrt{-1}\partial\bar{\partial}u)^n = \delta_0 \text{ the Dirac measure} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}. \tag{1.11}$$

Lempert [Lem83] gave an explicit form for solutions on convex domain in terms of Kobayashi distance and Guan [Gua98] proved $C^{1,\alpha}$-regularity of the solutions for any $\alpha \in (0, 1)$. However, due to the linearity, these results do not produce any interior regularity.

By our best knowledge, there are only three local regularity results available in the current literature.

The first one is a gradient estimate given by Blocki [Blo09]. It states that if f is Lipschitz and the solution u has a convex level set, then u has bounded gradient in the interior of this level set. The key point of this estimate is that the bound of ∇u does not depend on the regularity and curvature of the given level set.

The second result is given by Blocki and Dinew [BD11]. They have proven that if a solution u of (1.1) lies in $W^{2,p}$ for some $p > n(n-1)$ and the right-hand-side f is $C^{1,1}$, then u is indeed $C^{2,\alpha}$ in the interior.

The third result given by Dinew, Zhang and Zhang considers the Hölder regularity of D^2u. They have shown that if f is α-Hölder and D^2u is bounded in L^∞, then D^2u is also α-Hölder.

On one hand, the local regularity results for the complex Monge-Ampère equations are very limited. On the other hand, it has gain increasing interests in the recent research, especially in the study of singular Kähler metrics (see [EGZ11] and reference therein). Besides its interests in complex geometry and analysis, complex Monge-Ampère is very interesting from a pure PDE-theoretical point of view. It is a nonlinear equation whose family of invariance is extremely large and complicated. The study of local regularity of the complex Monge-Ampère equation will be of great value in the development of the elliptic PDE theory.

There are two main purposes of this thesis: First, we give a self-contained account of the local regularity of the complex Monge-Ampère equation through the modern nonlinear PDE point of view. Second, we apply Savin's small perturbation theory [Sav07] to produce new regularity results of the complex Monge-Ampère equation.

The main results of this thesis are stated as follows.

Our main theorem, following from a generalization of Savin's small perturbation theorem (see Theorem 5.2), considers an estimate of D^2u for a solution u that is uniformly close to a quadratic polynomial. Denote the modulus of continuity of f by ω_f

Theorem 1.4. *Suppose that u is a viscosity solution of* (1.1) *in B_1 and f satisfies*

$$\begin{aligned} &f \in C^0(\overline{B}_1), \ \ \text{and } f(0) = 1, \\ &\int_0^1 \frac{\omega(r)\log r^{-1}}{r}\, dr < \infty. \end{aligned} \tag{1.12}$$

For any $\delta < 1/2$, there exists a constant μ only depending on n, ω_f, δ such that,if

$$\|u - |x|^2/2\| \le \mu, \tag{1.13}$$

then u has a Taylor expansion of order 2 *at origin and*

$$\|D^2u(0) - I\| \le \delta. \tag{1.14}$$

An important corollary of the above theorem is the following Liouville property.

Corollary 1.5. *If u satisfies*

$$\mathcal{M}[u] \equiv 1, \text{ on } \mathbb{C}^n \tag{1.15}$$

and

$$u = \frac{1}{2}|x|^2 + o(|x|^2), \quad x \to \infty \tag{1.16}$$

then

$$u = \frac{1}{2}|x|^2 + \text{constant}. \tag{1.17}$$

We have also obtained the following convergence property for the complex Monge-Ampère equation.

Corollary 1.6. *Let $\{u_k\}$ be a sequence of continuous function on $\overline{B}_1$ that satisfies*

$$\mathcal{M}[u_k) = f_k \in C^\alpha(B_1), \quad \Lambda \geq f_k \geq \lambda \quad k = 1, 2..., \tag{1.18}$$

and φ be a solution of

$$\mathcal{M}[\varphi] = f \in C^\alpha(B_1). \tag{1.19}$$

If $\varphi \in C^{2,\alpha}(B_1)$ and

$$\|u_k - \varphi\|_{L^\infty(B_1)} \to 0, \quad \|f_k - f\|_{C^\alpha(B_1)} \to 0, \tag{1.20}$$

then $u_k \in C^{2,\alpha}(B_{1/2})$ for k sufficiently large and

$$\|u_k - \varphi\|_{C^{2,\alpha}(B_{1/2})} \to 0 \tag{1.21}$$

Our last result considers an improvement of Blocki and Dinew's $C^{2,\alpha}$-regularity theorem.

Theorem 1.7. *Let u be a solution to* (1.1) *in B_1. Suppose that*

$$\|\Delta u\|_{L^p} \leq \Lambda, \text{ for some } p > n(n-1), \tag{1.22}$$

and

$$f \in C^{\alpha}(B_1), \quad f \geq \lambda. \tag{1.23}$$

Then $u \in C^{2,\alpha}(B_{1/2})$ *and*

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \leq C \tag{1.24}$$

where C *only depends on* $n, p, \lambda, \Lambda, \|f\|_{C^{\alpha}(B_1)}$.

The thesis is organized as follows: In the first chapter, we fix our notations and present basic preliminaries. In particular, we explain the identification of complex determinant as a function of real matrices. In the second chapter, we present a viscosity approach to the Dirichlet problem of the complex Monge-Ampère equation. Our approach utilize the specific relation between real and complex determinant and it is considerably simpler than the standard viscosity treatment [CIL92, EGZ11]. In the third chapter, we discuss the regularity results priori to this thesis and singular examples. In Chapter four, we give a slight generalization of Savin's small perturbation theorem [Sav07]. This will be the key tools in proving our main results. In the last Chapter, we present the proof of our main results.

Chapter 2

Preliminaries

2.1 Notations

In this section, we fix our basic notations and conventions

- Throughout this thesis, unless explicit mentioned, all constants are assumed to be positive.

- Let $\mathbb{R}^{2n}$ be the standard $2n$-dimensional Euclidean space. Denote its inner product by $< \cdot, \cdot >$.

- The coordinate system on $\mathbb{R}^{2n}$ is the coordinate system that given by the basis $\{e_i\}$ where

$$e_i = (0, ..1, ...0). \tag{2.1}$$

  We denote the coordinate functions by x_i.

- Let $\mathbb{C}^n$ be the standard n-dimensional complex space. Throughout this thesis, we shall identify $\mathbb{C}^n$ as $\mathbb{R}^{2n}$ by

$$z = x + iy \mapsto \begin{pmatrix} x \\ y \end{pmatrix} \tag{2.2}$$

- Let Ω be a domain in $\mathbb{R}^{2n}$. Denote its diameter by $\operatorname{diam}(\Omega)$.

- Let $\mathrm{Mat}(\mathbb{C}, n)$ be the space of $n \times n$ complex matrices and $\mathrm{Mat}(\mathbb{R}, 2n)$ be the space of $2n \times 2n$ real matrices. We identify $\mathrm{Mat}(\mathbb{C}, n)$ as a subspace of $\mathrm{Mat}(\mathbb{R}, 2n)$ by the following map:

$$\imath : N = A + iB \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix}. \tag{2.3}$$

- Let $\mathrm{Herm}(n)$ be the $n \times n$ be the space of Hermitian matrices and $\mathrm{Sym}(2n)$ be the space of $2n \times 2n$ symmetric matrices.

- Let $\mathrm{SU}(n)$ be the group of $n \times n$ unitary matrices and $\mathrm{SO}(2n)$ be the group of $2n \times 2n$ orthogonal matrices.

- Given $M \in \mathrm{Sym}(2n)$, we denote its spectral normal by $\|M\|$, i.e.,

$$\|M\| := \max_i \{|\lambda_i|\} \tag{2.4}$$

  where λ_i's are the eigenvalue of M.

- Denote the $n \times n$ identity matrix by I_n.

- Denote the standard determinant operator by det.

- A matrix $P \in \mathrm{Sym}(2n)$ is said to be positive, denoted by $P \geq 0$, if all its eigenvalue are positive. The notation $A \geq B$ means $A - B$ is positive. Similarly, we define strictly positive matrices and the notation $A > B$.

- The cone of positive matrices, denoted by $\mathcal{P}$, is the subset of $\mathrm{Sym}(2n)$ that consists all non-negative matrices $P \in \mathrm{Sym}(2n)$.

- Let $u, \varphi \in C^0(\Omega)$, we say φ touches u from below in Ω at x_0 if

$$u(x) \geq \varphi(x)\ \forall x \in \Omega, \text{ and } u(x_0) = \varphi(x_0). \tag{2.5}$$

  Similarly, we define φ touches $u$from above in Ω at x_0. We shall simply say φ touches u from above (below) if there is no confusion about the relevant domain Ω and point x_0.

- We shall denote the complex Monge-Ampère operator by $\mathcal{M}$,i.e., for a C^2-function u

$$\mathcal{M}[u] := \det(2u_{\bar{k}j}). \tag{2.6}$$

- A function is called a paraboloid of opening K if it is of the form

$$P(z) = \frac{K}{2}|z|^2 + l \tag{2.7}$$

where l is an affine function.

2.2 Basic linear algebra

In this section, we summarize some basic linear algebra for our investigation. A large part considers complex matrices and their real embeddings, which are standard but less well-known.

2.2.1 Complex structure

Definition 2.1. *A complex structure on $\mathbb{R}^n$ is a linear map $J : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$J^2 = -id. \tag{2.8}$$

The canonical complex structure J on $\mathbb{R}^{2n}$ is a complex structure whose matrix form with respect to the canonical coordinate is

$$\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \tag{2.9}$$

where I_n is the $n \times n$ identity matrix.

By reordering coordinate system on $\mathbb{R}^n$ as $\{x_1, x_{n+1}, x_2, x_{n+2}, ...x_n, x_{2n}\}$, the canonical complex structure J takes the form

$$J = \begin{pmatrix} J_2 & \cdots & 0 \\ \vdots & \ddots & 0 \\ 0 & 0 & J_2 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{2.10}$$

The following lemma summarizes the important relations between complex and real matrices. The proof is straight forward and we shall omit it.

Lemma 2.2. *Under the identification* (2.3)*, we have*

$$\begin{aligned} \mathrm{Herm}(n) &= \{M \in \mathrm{Sym}(2n) : [M, J] = 0\} \subset \mathrm{Sym}(2n) \\ SU(n) &= \{O \in \mathrm{SO}(2n) : OJ = JO\} \subset \mathrm{SO}(2n). \end{aligned} \tag{2.11}$$

Moreover, if $\mathfrak{p} : \mathrm{Mat}(\mathbb{R}, 2n) \to \mathrm{Mat}(\mathbb{R}, 2n)$ *be defined by*

$$\mathfrak{p} : M \mapsto \frac{M + J^T M J}{2}, \tag{2.12}$$

then the following diagram commute

$$\begin{array}{ccc} \mathrm{Mat}(n, \mathbb{C}) & \xrightarrow{\imath} & \mathrm{Mat}(\mathbb{R}, 2n) \\ & \searrow^{\sim} & \downarrow \mathfrak{p} \\ & & \{M \in \mathrm{Mat}(\mathbb{R}, 2n) : [M, J] = MJ - JM = 0\}. \end{array} \tag{2.13}$$

The above diagram remains commutative when restrict to $\mathrm{Sym}(2n)$

2.2.2 Complex determinant

Definition 2.3. *We define the complex determinant operator* $\det_{\mathbb{C}}$ *on* $\mathrm{Sym}(2n)$ *by*

$$\det_{\mathbb{C}}(M) := \det^{1/2}\left(\frac{M + J^T M T}{2}\right), \quad M \in \mathrm{Sym}(2n). \tag{2.14}$$

The following lemma justify the terminology.

Lemma 2.4. *Let* $H \in \mathrm{Herm}(n)$*, then*

$$\det(H) = \det_{\mathbb{C}}(\imath(H)). \tag{2.15}$$

Proof. If H is diagonal, then the identify follows from direct calculation. The identity for general H then follows from diagonalization. □

The following inequality connecting real and complex determinant is important for our later discussion. Recall $\mathcal{P}$ stands the cone of positive matrices in $\mathrm{Sym}(2n)$.

Lemma 2.5. *If $M \in \mathcal{P}$, then*

$$\det_{\mathbb{C}}^{1/n}(M) \geq \det^{1/2n}(M). \tag{2.16}$$

Proof. This follows immediately from the fact that $\det^{1/2n}$ is concave in the cone of positive matrices. $\square$

2.2.3 Complex Hessian

Let φ be a C^2-function on $\mathbb{C}^n$. Its complex Hessian $\sqrt{-1}\partial\bar{\partial}\varphi$ is a Hermitian matrices given by

$$(\sqrt{-1}\partial\bar{\partial}\varphi)_{\bar{k}i} = 2\partial_{z_i}\partial_{z_{\bar{k}}}\varphi. \tag{2.17}$$

Label the x_{n+k} coordinate function as y_k and recall

$$\partial_{z_i} = \frac{1}{2}\left(\partial_{x_i} - \sqrt{-1}\partial_{y_i}\right). \tag{2.18}$$

The entries of $\sqrt{-1}\partial\bar{\partial}\varphi$ in real coordinates is given by

$$\left(i\partial\bar{\partial}\varphi\right)_{\bar{k}i} = \frac{1}{2}\left[\left(\frac{\partial\varphi}{\partial x_i\partial x_k} + \frac{\partial\varphi}{\partial y_i\partial y_k}\right) + \sqrt{-1}\left(\frac{\partial\varphi}{\partial x_k\partial y_i} - \frac{\partial\varphi}{\partial y_k\partial x_i}\right)\right]. \tag{2.19}$$

Then it is easy to verify that

$$\imath\left(\sqrt{-1}\partial\bar{\partial}\varphi\right) = \frac{1}{2}\left(D^2u + J^T D^2 u J\right) \tag{2.20}$$

where D^2u stands for the real Hessian of u.

Now, Lemma 2.4 implies that for any C^2-function,

$$\mathcal{M}[\varphi] = \det^{1/2}\left(\frac{D^2u + J^T D^2 u T}{2}\right). \tag{2.21}$$

2.3 Semi-concave functions

We recall the basic definition and properties of semi-concave functions.

Definition 2.6. *Let w be a continuous function on a domain $\Omega \subset \mathbb{R}^{2n}$. A function w is said to be K-semi-concave in Z with respect to Ω, if for any point $x \in Z$, there exists a paraboloid of opening K touch w from above in Ω. We shall simply say w is semi-concave if $Z = \Omega$. Similarly, one define semi-convexity.*

Remark 2.7. *Notice that we do not require Z to be open.*

In order to state a fundamental property of semi-concave functions, we shall need the following terminology.

Definition 2.8. *A continuous function u on Ω is said to be C^2 at point x_0 if there exists a quadratic polynomial P such that*

$$u = P + o(|x - x_0|^2), \qquad as\ x \to x_0. \tag{2.22}$$

If u is C^2 at x_0, we define

$$D^2 u(x_0) := DP(x_0), \quad \nabla u(x_0) := \nabla P(x_0) \tag{2.23}$$

Clearly, if u is C^2 at x_0, then the polynomial P is unique. Therefore the above definition is well-defined.

We recall the Alexandroff's theorem on second order differentiability.

Theorem 2.9. *If u is semi-concave (convex) in B_1, then u is C^2 at almost all points (with respect to the Lebesgue measure) in B_1.*

For following lemma will be used in the later of this thesis.

Lemma 2.10. *Let w be a continuous function on a domain Ω and A be a subset of Ω. If w is a-semi-concave on A with respect to Ω and b-semi-convex on A with respect to Ω, then w is differentiable at every point $x \in A$ and the map*

$$T(x) := Dw(x) \tag{2.24}$$

is Lipschitz on A and

$$|T(x) - T(y)| \leq C\,|x - y|\,, \forall x, y \in A \tag{2.25}$$

with C only depending on a, b.

Proof. Fix a point $x_0 \in A$, by the definition of semi-concavity, there exists two paraboloids P_1, P_2 such that

$$\begin{aligned} &P_1(x) \le w(x) \le P_2(x), \quad \forall x \in \Omega \\ &w(x_0) = P_1(x_0) = P_2(x_0). \end{aligned} \tag{2.26}$$

It follows then, for every direction e

$$\lim_{h\to 0} \frac{P_1(x_0+he) - P_1(x_0)}{h} \le \lim_{h\to 0} \frac{w(x_0+he) - w(x_0)}{h} \le \lim_{h\to 0} \frac{P_2(x_0+he) - P_2(x_0)}{h} \tag{2.27}$$

Since P_1, P_2 are quadratic polynomial, the limits on both sides exists. Moreover, by the maximum principle,

$$\nabla P_1(x_0) = \nabla P_2(x_0). \tag{2.28}$$

Therefore the limit for w also exists and equal to $\nabla P_1(x_0) \cdot e$. This proves the differentiability and the map T is well-defined.

Now, we estimate the Lipschitz constant of T. Let the constant C be fixed and be specified later in the proof. Suppose that T has Lipschitz constant strictly greater than C. Then, up to scaling and subtract a plane from w, there exist two points $x, y \in$ such that

$$\begin{aligned} &|x-y| = 1, \nabla w(x) = 0 \\ &\nabla w(y) \cdot (y-x) > C \end{aligned} \tag{2.29}$$

Let

$$P(x) = \frac{a}{2}|x-y|^2 + \nabla w(y) \cdot (x-y) + c \tag{2.30}$$

be the paraboloid that touches w from above at y. Since w is touched from above by a paraboloid of opening a, we have

$$P(y) = c = w(y) \le \frac{a}{2} \tag{2.31}$$

Thus by take C large enough according to a, b, we have

$$w(x) \le P(x) < 0. \tag{2.32}$$

This contradicts to the fact that w is a graph. □

2.4 Pointwise $C^{2,\alpha}$-regularity

When dealing with viscosity solutions of an elliptic PDE, it is more convenient to work with point-wise regularity.

Definition 2.11. *A continuous function u on $\mathbb{R}^2$ is said to be $C^{2,\alpha}$ at point x_0 if there exists a quadratic polynomial P and constants C, ρ such that*

$$\|u - P\|_{L^\infty(B_r)} \leq Cr^{2+\alpha}, \quad \forall r \leq \rho. \tag{2.33}$$

We define point-wise $C^{2,\alpha}$-norm by

$$[D^2u]_{\alpha,\rho}(x_0) := \min\{C \mid \|u - P\|_{L^\infty(B_r)} \leq Cr^\alpha, \forall r < \rho\}. \tag{2.34}$$

The following lemma justifies the relation between point-wise $C^{2,\alpha}$-regularity and the classical $C^{2,\alpha}$-regularity.

Lemma 2.12. *A function $u \in C^{2,\alpha}(B_1)$ if and only if u is $C^{2,\alpha}$ at all point $x \in B_1$ and there exist constant C, ρ independent of x such that*

$$\left|D^2u(x_0)\right| + [D^2u]_{\alpha,\rho}(x) \leq C. \tag{2.35}$$

Proof. The necessity follows immediately from the definition. The sufficiency follows from a similar argument in the proof of Lemma 2.10. □

2.5 Continuous plurisubharmonic functions

Recall the definition of plurisubharmonic function.

Definition 2.13. *A function $u \in L^1(\Omega)$ is said to be subharmonic in Ω if for every complex line*

$$\{a + bz \mid z \in \mathbb{C}\} \tag{2.36}$$

the function $z \mapsto u(a + bz)$ is a subharmonic function on the set

$$\{z \in \mathbb{C} \mid a + bz \in \Omega\}. \tag{2.37}$$

For our purpose, the following characterization of plurisubharmonic function is convenient.

Lemma 2.14. *$u \in C(\Omega)$ is plurisubharmonic if and only if the following statement holds: for every C^2-function φ if φ touches u from above at $x \in \Omega$, then*

$$\sqrt{-1}\partial\bar{\partial}\varphi(x) \geq 0.$$

Proof. To show the necessity, assume that u is plurisubharmonic. Let $L := \{a + bz : z \in \mathbb{C}\}$ be an eigenspace of $\sqrt{-1}\partial\bar{\partial}\varphi(x)$. Since φ touches u from above, φ touches $u|_L$ from above. By standard viscosity theory, φ is subharmonic on $L \cap \Omega$. This holds for all eigenspace of $\sqrt{-1}\partial\bar{\partial}\varphi(x)$, thus $\sqrt{-1}\partial\bar{\partial}\varphi(x) \geq 0$. The sufficiency is proved similarly. □

2.6 Sup-inf convolution

In this section, we recall the sup-convolution technique. In working with viscosity solutions, sup-convolution is the natural regularization technique.

Definition 2.15. *Let $K(r)$ be a non-decreasing function on $[0, \infty)$ and u be a bounded function on Ω. The sup-convolution against kernel K is defined by*

$$u_K(x_0) := \sup\{u(x) - K(|x - x_0|) : y \in \overline{\Omega}\}. \tag{2.38}$$

The inf-convolution is defined by

$$u^K := \inf\{u(x) + K(|x - x_0|) : y \in \overline{\Omega}\}. \tag{2.39}$$

We are specially interested in two kind of kernels. The first kind considers kernels that are modulus of continuity.

Lemma 2.16. *Suppose the kernel K satisfies*

$$K(a + b) \leq K(a) + K(b), \quad \lim_{r \to 0^+} K(r) = 0. \tag{2.40}$$

Then for any bounded function u on Ω, u_K, u^K are uniformly continuous on $\overline{\Omega}$ and

$$\omega_{u^K}(r), \omega_{u_K}(r) \leq K(r). \tag{2.41}$$

In particular, if $K(r) = Cr^\alpha$, then u^K, u_K are α-Hölder continuous.

Proof. We shall only prove the lemma for u_K; the proof for inf-convolution is same.

Fix $x, y \in \overline{\Omega}$, for every $z \in \overline{\Omega}$,

$$\begin{aligned} u_K(x) &\geq u(z) - K(|x-z|) \\ &\geq u(z) - K(|y-z|) - K(|x-y|) \end{aligned} \tag{2.42}$$

Taking supermum over z we conclude

$$u_K(x) - u_K(y) \geq -K(|x-y|). \tag{2.43}$$

Since x, y are chosen arbitrarily, the proof is complete. □

The second kind of kernels are of the form $C|r|^2$. Sup-inf convolution against these kernels are first studied by Jensen (see [CC95] and reference therein).

Lemma 2.17. *Let Ω be bounded domain and $u \in C(\overline{\Omega})$. Let u_ϵ be the sup convolution of u against $\frac{1}{\epsilon}r^2$. Then the following properties holds:*

1. *u^ϵ is Lipschitz continuous with Lipschitz constant smaller than $\frac{2}{\epsilon}\mathrm{diam}(\Omega)^2$.*

2. *u is $\frac{2}{\epsilon}$-semi-convex on Ω.*

3. *u^ϵ decreasing uniformly to u as ϵ tending to zero.*

Corresponding statement holds for inf-convolution.

Proof. The first statement is proved similarly as Lemma 2.16. One only need to notice that

$$|x-y+y-z|^2 \leq |x-y|^2 + |y-z|^2 + 2|x-y||y-z|. \tag{2.44}$$

The semi-convexity follows immediately from the definition. Since u^ϵ is defined as supremum of a family of paraboloid.

To show the last statement, let x^* be the point that realize the supremum in the definition of $u_\epsilon(x)$. Observe that

$$|x - x^*|^2 = \epsilon\, |u(x) - u(x^*)| \le \epsilon \operatorname*{osc}_{\overline{\Omega}} u \tag{2.45}$$

Hence $x^*(\epsilon) \to x$ as $\epsilon \to 0$. Thus for every $x \in \overline{\Omega}$,

$$0 \le u_\epsilon(x) - u(x) \le u(x^*) - u(x) \le \omega(|x^* - x|) \to 0 \tag{2.46}$$

The convergence is uniform as u is a uniform continuous function. The monotonicity is clear. □

Next, we consider the behaviour of continuous plurisubharmonic function under sup-inf convolution.

Lemma 2.18. *Let u be a continuous plurisubharmonic function on a bounded domain Ω. Then u^ϵ, u_ϵ are both plurisubharmonic.*

Proof. Rewrite the definition of sup-convolution as follows

$$u_\epsilon(x) = \sup\{u(x+y) - \frac{1}{\epsilon}\, |y|^2 : y \in \Omega - x_0\} \tag{2.47}$$

Fix $\epsilon > 0$, from the definition and standard diagonal argument, there exists a sequence y_i such that

$$v_i(x) := u(x + y_i) - \frac{1}{\epsilon}\, |y_i|^2 \tag{2.48}$$

converges monotonically and point-wisely to u_ϵ. Clearly v_i's are plurisubharmonic functions.

By Lemma 2.17, u_ϵ is continuous. Thus by Dini's theorem v_i converges uniformly to u_ϵ. The plurisubharmonicity then follows.

The case for inf-convolution is proved similarly. □

Chapter 3

Viscosity Solutions

3.1 Definition and basic properties

3.1.1 Basic definitions

We define the viscosity solutions of the complex Monge-Ampère equation.

Definition 3.1. *Let $f \in C(\Omega)$, we say a continuous plurisubharmonic u satisfies $\mathcal{M}[u] \geq f$ in viscosity sense if the following condition holds: for every plurisubharmonic C^2 function φ on Ω, if φ touches u from above at x, then*

$$\det_{\mathbb{C}}[D^2\varphi](x) \geq f(x). \tag{3.1}$$

In this case, we shall call u a viscosity subsolution. Similarly, we define $M[u] \leq f$ in viscosity sense. u is said to be a viscosity solution to $M[u] = f$ if u satisfies both $M[u] \geq f$ and $M[u] \leq f$ in viscosity sense.

Remark 3.2. *Unlike for general elliptic equations, we only need to test the equation against smooth plurisubharmonic functions, because u is priorly assumed to be plurisubharmonic. By maximum principle, if φ touch u from above at x, then φ is plurisubharmonic near x.*

In the rest of this chapter, unless otherwise mentioned, we assume u is continuous plurisubharmonic and f is continuous.

The following equivalent definition of viscosity solutions is convenient.

Lemma 3.3. *Let u be a continuous plurisubharmonic function on Ω. u satisfies $\mathcal{M}[u] \geq f$ in viscosity sense if and only if the following condition holds: for every plurisubharmonic quadratic polynomial P, if P touches u from above in a neighbourhood of x, then*

$$\det_{\mathbb{C}}(D^2 P) \geq f(x). \tag{3.2}$$

Proof. The necessity follows directly from the definition. To show the sufficiency, let φ touches u from above at x_0. Let P be the second order Taylor expansion of φ at x_0, then, for any $\epsilon > 0$

$$P_\epsilon(x) := P(x) + \epsilon\,|x - x_0| \tag{3.3}$$

touches u from above in a neighbourhood of x_0. By the hypothesis, we conclude that

$$\det_{\mathbb{C}}(D^2 P + 2\epsilon I) \geq f(x_0). \tag{3.4}$$

By the continuity of determinant, we may let ϵ tend to zero and arrive

$$\det_{\mathbb{C}}(D^2 \varphi)(x_0) = \det_{\mathbb{C}}(D^2 P) \geq f(x_0). \tag{3.5}$$

□

Now, we list some basic properties of viscosity solutions of the complex Monge-Ampère equation.

Lemma 3.4. *Let $u, v \in C(\Omega)$ and $f \in C(\overline{\Omega})$.*

1. *If u satisfies $\mathcal{M}[u] \geq f$ in viscosity sense, then $\mathcal{M}[u + \Re(h)] \geq f$ in viscosity sense for all holomorphic function h.*

2. *If $u \in C^2$, then $\mathcal{M}[u] \geq f$ in viscosity sense if any only if $\mathcal{M}[u] \geq f$ in classical sense.*

3. *Suppose $\mathcal{M}[u] \geq f$ in viscosity sense and $w \in C(\Omega)$ is plurisubharmonic. If w touches u from above x and w is C^2 at x, then $M[w](x) \geq f(x)$ in classical sense.*

Corresponding statements are valid for viscosity supersolutions.

Proof. The first statement follows immediately from the definition. The second and third statements follow from an argument that is similar to the one in the proof of Lemma 3.3 □

3.1.2 Properties for Perron method

Now we consider properties of viscosity solutions to the complex Monge-Ampère equation that will be needed to perform the Perron's method.

The first lemma considers the maximum of two subsolutions.

Lemma 3.5. *If $\mathcal{M}[u] \geq f$ and $M[v] \geq f$, then $\mathcal{M}[\max\{u, v\}] \geq f$ in viscosity sense.*

Proof. Let φ be a C^2 function that touches from $\max\{u, v\}$ from above at x_0, then φ touches either u or v from above at x_0. In either case, we have, from the definition of viscosity solution,

$$\det_{\mathbb{C}}(D^2\varphi)(x_0) \geq f(x_0). \tag{3.6}$$

□

Next, we show the convergence property of the solutions.

Lemma 3.6. *Let u_k be a sequence of continuous function on Ω satisfying $\mathcal{M}[u_k] \geq f_k$. Suppose u_k converges uniformly to u on compact subsets of Ω and f_k converges to f on compact subsets of Ω, then $\mathcal{M}[u] \geq f$ in viscosity sense.*

Proof. Let P be a quadratic polynomial that touches u from above at x_0 in $B_r(x_0)$, then, for every $\epsilon > 0$

$$P_\epsilon(x) := P(x) + \epsilon |x - x_0|^2 + c \tag{3.7}$$

touches some u_k from above at $x_k(\epsilon)$ after a proper choice of c. By the uniform convergence,

$$x_k(\epsilon) \to x_0, \qquad \text{as } \epsilon \to 0. \tag{3.8}$$

Since u_k's are subsolutions, we conclude

$$\det_{\mathbb{C}}(D^2P + 2\epsilon I) \geq f_k(x_k). \tag{3.9}$$

Moreover, The proof is completed by letting ϵ go to zero. □

We end this section with a property considers supersolutions.

Lemma 3.7. *Let u be a viscosity subsolution on Ω. If u does not satisfies $\mathcal{M}[u] \geq f$ in viscosity sense, then there exists a continuous plurisubharmonic function $\hat{u}$ such that*

$$\begin{aligned} &\mathcal{M}[\hat{u}] \geq f, \quad \hat{u}|_{\partial\Omega} = u \\ &\hat{u}(x) > u \text{ at some point } x \in \Omega \end{aligned} \tag{3.10}$$

Proof. Since u is not a supersolution, there exists a plurisubharmonic quadratic polynomial P and a point $x_0 \in \Omega$ such that P touches u from below at x_0, but $M(P) > f(x_0) \geq 0$.

Consider

$$\psi := P - \frac{\epsilon}{2}(\left|x - x_0^2\right| - r^2). \tag{3.11}$$

By the continuity of f, there exists r, ϵ small enough such that ψ is plurisubharmonic and

$$\mathcal{M}[\psi] > f(x), \quad \forall z \in B_r(x_0) \subset\subset \Omega, \tag{3.12}$$

Moreover

$$\psi(z_0) > u(z_0), \quad \psi|_{\partial B_r(x_0)} \leq u|_{\partial B_r(x_0)}. \tag{3.13}$$

Now define

$$\hat{u} := \begin{cases} \max\{\psi, u\} & x \in \overline{B_r}(z_0) \\ u & x \in \Omega \setminus B_r(z_0). \end{cases} \tag{3.14}$$

We claim $\hat{u} \in \underline{\mathcal{S}}$.

Clearly, $\hat{u}$ is still a continuous plurisubharmonic and

$$\hat{u}(z_0) = \psi(z_0) = u(z_0) + \epsilon r^2/2 > u(z_0) \tag{3.15}$$

It also follows from the definition that outside ball $B_r(z_0) \subset \Omega$, $\hat{u} = u$, in particular, $\hat{u}|_{\partial\Omega} = u$.

To check that $\hat{u}$ is a viscosity subsolution. Take any quadratic polynomial P such that P touches $\hat{u}$ from above at some $x_0 \in \Omega$.

If $\hat{u}(x_0) = u(x_0)$, then, by the fact that u is a subsolution

$$\mathcal{M}[P] \geq f(x_0). \tag{3.16}$$

If $\hat{u}(x_0) = \psi(x_0) > u(x_0)$, then $x_0 \in B_r(z_0)$ and P touches ψ from above at x_0. In turn

$$M[P] \geq M[\psi] > f(x_0)) \tag{3.17}$$

follows again.

Thus in both case the desired inequality is verified, hence $\hat{u}$ is a subsolution. □

3.2 Maximum and comparison principle

3.2.1 Alexandroff-Bakelman-Pucci estimate

In this section, we prove a version of Alexandroff-Bakelman-Pucci estimate. This will be a key tool in our study of the solvability of the Dirichlet problem to the complex Monge-Ampère equation.

Lemma 3.8. *Let Ω be a bounded domain containing origin and $w \in C(\overline{\Omega})$ be semi-concave. If*

$$w \geq 0 \ \textit{on}\ \partial\Omega, \quad w(0) = -a < 0 \tag{3.18}$$

then there exists a point $x_0 \in \Omega$ such that w is C^2 at x_0 and

$$D^2w(x_0) > 0, \quad \det(D^2w(x_0)) \geq \left(\frac{a}{3\mathrm{diam}(\Omega)}\right)^n, \quad w(x_0) < 0 \tag{3.19}$$

Proof. Let A be the subset of Ω consists of points on which w can be touched from below by planes (linear functions).

Since u is semi-concave, by 2.10, $T(x) = Dw(x)$ is well-defined Lipschitz map on A.

By the differentiability of Lipschitz map (see [EG92]), T is almost every differentiable and there exists a set Z with zero measure such that

$$DT(x) = D^2w(x), \quad x \in A \setminus Z \tag{3.20}$$

Then, by the area-formula, we conclude that

$$|T(A)| \leq \int_{A\setminus Z} \det(D^2w)\, dx. \tag{3.21}$$

Next, we claim that

$$T(A) \supset B_{a/d}, \quad d = \mathrm{diam}(\Omega). \tag{3.22}$$

Let $l(x)$ be a linear function such that

$$l(x) = v \cdot x + c, \quad v \in B_{a/(3d)}. \tag{3.23}$$

Let x_1 be the point that $l(x)$ touches w from below. We need to show that $x \in \Omega$.

Since $w(x_0) = -a$, we have

$$l(x_0) = v \cdot x_0 + c \leq -a \tag{3.24}$$

Thus

$$c \leq -a + |v|\,|x_0| \leq -\frac{2a}{3}. \tag{3.25}$$

In turn

$$l(x) \leq -\frac{2a}{3} + |v|\,|x_0| \leq -\frac{a}{3} < 0, \quad \forall x \in \partial\Omega. \tag{3.26}$$

Therefore l touches w in the interior and the claim is proved.

Now, claim there exists $x_0 \in A$ that satisfies the desired condition. Clearly, $w(x) < 0$ for all $x \in A$. Suppose that

$$\det(D^2 w(x)) < (\frac{a}{3d})^n, \quad \forall x \in A \cap Z \tag{3.27}$$

Then (3.21) and (3.22) give a contradiction. Therefore the desired point exists in A and the proof is completed. □

3.2.2 Jansen's approximation

We now consider the behaviour viscosity solutions under sup-inf convolution.

Lemma 3.9. *Let $u \in C(\overline{\Omega})$ satisfy $\mathcal{M}[u] \geq f$ in viscosity sense. Then for any compact subdomain Ω', there exists ϵ_0 depending on $\operatorname{dist}(\Omega', \Omega), \operatorname{osc}_\Omega u$ such that, for all $\epsilon < \epsilon_0$, u_ϵ satisfies $\mathcal{M}[u_\epsilon] \geq f_\epsilon$ with*

$$f_\epsilon(x) := \inf \left\{ f(y) \mid y \in B_r(x), r = \left(\epsilon \operatorname*{osc}_\Omega u\right)^{1/2} \right\} \tag{3.28}$$

Similarly, if u satisfies $\mathcal{M}[u] \leq f$ in viscosity sense. Then for any compact subdomain Ω', there exists ϵ_0 depending on $\operatorname{dist}(\Omega', \Omega), \operatorname{osc}_\Omega u$ such that, u^ϵ satisfies $\mathcal{M}[u^\epsilon] \leq f^\epsilon$ with

$$f^\epsilon(x) := \sup \left\{ f(y) \mid y \in B_r(x), r = \left(\epsilon \operatorname*{osc}_\Omega u\right)^{1/2} \right\} \tag{3.29}$$

Proof. Let P be a quadratic polynomial that touches u_ϵ from above at $x_0 \in \Omega'$. Let x_0^* be the point on which the maximum in the definition of u_ϵ is realized.

By Lemma 2.17,

$$|x_0 - x_0^*| \leq r_\epsilon, \quad r_\epsilon = \left(\epsilon \operatorname*{osc}_\Omega u\right)^{1/2} \tag{3.30}$$

Thus, for ϵ sufficiently small $x_0^* \in \Omega$.

Consider the quadratic polynomial Q given by

$$Q(x) := P(x + x_0 - x_0^*) + \frac{1}{\epsilon} |x_0^* - x_0|^2 . \tag{3.31}$$

It is easy to see that Q touches u from above in a neighbourhood of x_0^*. It follows then

$$\det_{\mathbb{C}}(D^2P) = \det_{\mathbb{C}}(D^2Q) \geq f(x_0^*) \geq f_\epsilon(x). \tag{3.32}$$

Thus u_ϵ is a solution. The proof for supersolution case is similar. □

3.2.3 Comparison principle

Proposition 3.10. *Let Ω be a bounded domain in $\mathbb{C}^n$ and $f \in C(\Omega)$. Suppose that u, v are viscosity subsolution and supersolution of the equation $\mathcal{M}[\cdot] = f$ respectively. Then*

$$v \geq u \text{ on } \partial\Omega \ \Rightarrow v \geq u \text{ in } \Omega. \tag{3.33}$$

Proof. Without lose of generality, we may assume $0 \in \Omega$. Let $d = \operatorname{diam}(\Omega)$. Moreover, by replacing v by $v + c$ and u by $u + \mu\,|x|^2, \mu = c/(2d^2)$, we may assume $v > u$ and

$$\mathcal{M}[u] \geq f(x) + \mu. \tag{3.34}$$

The general case will follow from this by taking c to be zero.

Argue by contradiction. Write $w = v - u$. Assume there exists $x_0 \in \Omega$ such that

$$w(x_0) = -a < 0. \tag{3.35}$$

Fix a compact subdomain U of Ω such that $x_0 \in U$ and $w|_U \geq 0$. This choice is possible because w is continuous and $w_{\partial\Omega} > 0$. Regularizing v, u via sup-inf convolution and write $w_\epsilon = v^\epsilon - u_\epsilon$.

Fix any $\epsilon > 0$, denote $E_\epsilon \subset U$ the set consists of points on which $w_\epsilon, v^\epsilon, -u_\epsilon$ are C_2. By Lemma 2.17, $|U \setminus E_\epsilon| = 0$ and w_ϵ is semi-concave. Apply Lemma 3.8, we may choose $x_\epsilon \in E_\epsilon$ such that

$$w(x_\epsilon) < 0, \quad \det{}^{1/n}(D^2u)(x_\epsilon) > 0 \tag{3.36}$$

By Lemma 3.9, for all sufficiently small ϵ

$$\mathcal{M}(v^\epsilon)(x_\epsilon) \leq f^\epsilon(x_\epsilon) \tag{3.37}$$

and

$$\mathcal{M}(u_\epsilon)(x_\epsilon) \geq f_\epsilon(x_\epsilon) + \mu \tag{3.38}$$

Combine (3.36)-(3.38), Minkowski inequality of determinant and Lemma 2.5, we conclude that

$$f^\epsilon(x_\epsilon) \geq f_\epsilon(x_\epsilon) + \mu \tag{3.39}$$

Since μ is independent of ϵ and f^ϵ, f_ϵ converges uniform to f, (3.39) leads to a contradiction when ϵ is taken sufficiently small. □

3.3 Solvability of the Dirichlet problem

The goal of this section is to prove the following theorem.

Theorem 3.11. *Let Ω be a bounded domain in $\mathbb{C}^n$, $f \in C(\Omega)$ and $g \in C(\partial\Omega)$. If there exists a continuous harmonic function h on $\overline{\Omega}$ with $h|_{\partial\Omega} = g$ and a plurisubharmonic function $\underline{u} \in C(\overline{\Omega})$ such that*

$$M[\underline{u}] \geq f, \text{ in } \Omega, \text{ and } \underline{u}|_{\partial\Omega} = g, \tag{3.40}$$

then the Dirichlet Problem

$$\begin{cases} M[u] = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{3.41}$$

has a unique viscosity solution. Moreover the solution u is uniformly continuous and

$$\omega_u(r) \leq 2d\omega(d)r + d^2\omega(r) \tag{3.42}$$

where $\omega = \max\{\omega_{\underline{u}}, \omega_h, \omega_{f^{1/n}}\}$.

3.3.1 The key lemma

Lemma 3.12. *Under assumption of Theorem 3.11. For each subsolution u with $u|_{\partial\Omega} \leq g$, there exists a continuous plurisubharmonic function $\tilde{u}$ such that*

1. *$\tilde{u}$ is a viscosity subsolution.*

2. *$\tilde{u} \geq u$ in $\overline{\Omega}$ and $\tilde{u}|_{\partial\Omega} = g$.*

3. *The modulus of continuity $\omega_{\tilde{u}}$ of $\tilde{u}$ satisfies:*

$$\omega_{\tilde{u}}(r) \leq \omega_u(r) \leq 2d\omega(d)r + d^2\omega(r). \tag{3.43}$$

where $\omega = \max\{\omega_{\underline{u}}, \omega_h, \omega_{f^{1/n}}\}$ and $d = \diamond\Omega$.

Proof. Without lose of generality, we shall assume $0 \in \Omega$. By taking $\sup\{u, \underline{u}\}$, we may assume $u|_{\partial\Omega} = g$ and $u \geq \underline{u}$ in $\partial\Omega$.

Define $\tilde{u}$ as following:

$$\tilde{u}(x) := \sup_{y\in\overline{\Omega}}\{\max[u(x) - \omega(|x-y|) + \frac{\omega(|x-y|)}{2}(|z|^2 - d^2),\ \underline{u}(z)]\}, \tag{3.44}$$

Let z_* for the point where the maximum occurs.

We shall show $\tilde{u}$ satisfies desired properties.

Step 1 We show that $\tilde{u}$ is indeed a subsolution. Let P be quadratic polynomial that touches $\tilde{u}$ from above at $x_0 \in \Omega$; If $\tilde{u}(x_0) = \underline{u}(x_0)$, then

$$\det{}_{\mathbb{C}}(D^2P) \geq f(x_0). \tag{3.45}$$

If $\tilde{u}(x_0) > \underline{u}(x_0)$, then the polynomial

$$Q(x) := P(x + x_0 - x_0^*) + \omega(|x_0 - x_0^*|) - \frac{\tau}{2}(|x|^2 - d^2) \tag{3.46}$$

touches u at x_0^*.

Claim $x_0^* \in \Omega$. Suppose otherwise $x_0^* \in \partial\Omega$,then

$$u(x_0^*) - \omega(|x_0^* - x_0|) + \frac{\omega}{2}(|x_0|^2 - d^2) \leq h(x_0^*) - \omega(|x_0^* - x_0|) \leq h(x_0^*) = \underline{u}(x_0^*), \tag{3.47}$$

which contradicts to the fact that $\tilde{u}(x_0) > \underline{u}(x_0)$.

Since u is a subsolution, we conclude that

$$\det_{\mathbb{C}}(D^2Q) \geq f(x_0^*) \tag{3.48}$$

Apply Minkowski inequality of determinant, we obtain

$$\begin{aligned} \det_{\mathbb{C}}^{1/n}(D^2P) &\geq \det_{\mathbb{C}}^{1/n}(D^2Q) + \omega(|x_0 - x|) \\ &\geq f^{1/n}(x_0^*) + \omega(|x_0 - x|) \geq f^{1/n}(x_0). \end{aligned} \tag{3.49}$$

Therefore, we have shown $\tilde{u}$ is a subsolution.

Step 2 We now prove the second statement. Same as in the previous step, if $y \in \partial\Omega$, then for any $x \in \overline{\Omega}$

$$u(y) - \omega(|x-y|) + \frac{\omega}{2}(|y|^2 - d^2) \leq h(y) - \omega(|x-y|) < h(y) = \underline{u}(y). \tag{3.50}$$

Hence

$$\tilde{u}|_{\partial\Omega} = \underline{u}|_{\partial\Omega} = g. \tag{3.51}$$

$\tilde{u} \geq u$ in Ω is obvious.

Step 3 To show the last statement, we recall that ω is sub-additivity, i.e., for every $x_1, x_2 \in \overline{\Omega}$,and every $y \in \overline{\Omega}$:

$$\omega(|x_1 - y|) + \omega(|x_2 - y|) \geq \omega(|x_2 - x_1|) \tag{3.52}$$

Therefore

$$\begin{aligned} &\omega(x_1, y)(|x_1|^2 - d^2) - \omega(x_2, y)(|x_2|^2 - d^2) \\ &= \omega(x_1, y)(|x_1|^2 - |x_2|^2) + (\omega(x_1, y) - \omega(x_2, y))(|x_2|^2 - d^2) \\ &\geq -2d\omega(d)\,|x_1 - x_2| - d^2\omega(x_1, x_2) \end{aligned} \tag{3.53}$$

Combine all these, we obtain

$$\tilde{u}(x_1) - \tilde{u}(x_2) \geq -2d\omega(d)\,|x_1 - x_2| - d^2\omega(|x_1 - x_2|) \tag{3.54}$$

Since x_1, x_2 are chosen arbitrarily, the proof is completed □

3.3.2 Proof of the solvability theorem

Proof of Theorem 3.11. The uniqueness follows from the comparison principle (Theorem 3.10).

We first prove the existence and uniqueness. Consider the following collection of plurisubharmonic functions

$$\underline{\mathcal{S}} := \{v \in C(\overline{\Omega}) : \mathcal{M}(v) \geq f(x, v) \text{ in } \Omega,\ v|_{\partial\Omega} \leq g\}, \tag{3.55}$$

and

$$u := \sup\{v : v \in \underline{\mathcal{S}}\}. \tag{3.56}$$

Consider the following sub-family of $\underline{\mathcal{S}}$

$$\tilde{\underline{\mathcal{S}}} := \{\tilde{v} : v \in \underline{\mathcal{S}}\}, \tag{3.57}$$

where $\tilde{v}$ is defined according to Lemma 3.12.

By the Lemma 3.12, $\tilde{\underline{\mathcal{S}}}$ is a equi-continuous subset of $\underline{\mathcal{S}}$ and

$$u = \sup\{\tilde{v} : \tilde{v} \in \tilde{\underline{\mathcal{S}}}\}. \tag{3.58}$$

By Arzelà–Ascoli, u is the uniform limit of a sequence of subsolutions. Hence that $u \in C(\overline{\Omega})$ and it is again a subsolution.

Claim that u is also a supersolution. Argue by contradiction. If u is not a supersolution, then by Lemma 3.7, there exists a subsolution $\hat{u}$ such that $\hat{u} = g$ and $\hat{u} > u$ at some point in Ω. However this contradicts to the maximality of u.

We now estimate the modulus of continuity. Let u be the solution of the equation constructed above. Apply Lemma 3.12 to u, then the resulting $\tilde{u}$ is identical to u. Thus u has the desired modulus of continuity. □

3.4 Relation to pluripotential solution

In this section, we discuss the equivalence between weak solution in the sense of pluripotential theory and in the sense of viscosity.

Proposition 3.13. *Let $f \in C(\Omega)$ be non-negative and $u \in C(\Omega)$. Then $(\sqrt{-1}\partial\bar{\partial}u)^n = f(z)$ in pluripotential-potential sense if and only if that $\mathcal{M}(u) = f(z)$ in viscosity sense.*

Proof. Let u be a pluripotential solution of $\mathcal{M}(u) = f(z)$. First we show that u is a viscosity supersolution.

Let P be a quadratic polynomial that touches u from below at some $x_0 \in \Omega$. We need to verify that $\det_{\mathbb{C}}(D^2P) \leq f(x_0)$. Without loss of generality, we assume that $z_0 = 0$.

Suppose on the contrary that $\det_{\mathbb{C}}(D^2P) \leq f(x_0)$. By continuity of f, there exists some r_0 and ϵ_0 such that

$$\sqrt{-1}\partial\bar{\partial}(P - \frac{\epsilon_0}{2}|x|^2) > 0, \quad \det\left[\sqrt{-1}\partial\bar{\partial}(p - \frac{\epsilon_0}{2}|x|^2\right] > f(x), \ \forall x \in \overline{B_{r_0}}. \tag{3.59}$$

Then

$$\begin{aligned} &P - \frac{\epsilon_0}{2}(|z|^2 - \frac{r_0^2}{2}) < u, \ \text{on } \partial B_{r_0}; \\ &P + \frac{\epsilon_0}{2}\frac{r_0^2}{2} > u(0). \end{aligned} \tag{3.60}$$

But this contradicts to the pluripotential comparison principle. Thus a pluripotential solution u is also a viscosity supersolution.

The proof of u being a subsolution is similar.

Now to prove the converse statement. Let $u \in C(\overline{\Omega})$ satisfy $M_{\mathbb{C}}(\cdot) = f$ in viscosity sense. Solve the Dirichlet problem with data $f, g = u|_{\partial\Omega}$ in pluripotential sense. Denote the unique solution by $\tilde{u}$. By the discuss above $\tilde{u}$ is also a viscosity solution of the Dirichlet problem. The viscosity uniqueness forces that $\tilde{u} = u$. □

Chapter 4

Summary of Current Local Regularity Results

4.1 Review of the current literature

As mentioned in the introduction, there are only three interior regularity results available in the current literature. We first give a detail review of them.

The first one is a gradient estimate of a solution u with a convex level set [Blo09].

Theorem 4.1 (Blocki, 2000). *Let Ω be a bounded convex domain in $\mathbb{C}^n$. If u is a continuous plurisubharmonic function that satisfies*

$$\begin{cases} \mathcal{M}[u] = f(\Omega) \quad \text{in } \Omega \\ u = 0 \text{ on } \partial\Omega \quad f^{1/n} \in C^{1,1}(\Omega) \end{cases} \tag{4.1}$$

then u is Lipschitz in the interior of Ω and for any compact subdomain Ω'

$$|u(x) - u(y)| \leq C\,|x - y|\,, \quad \forall x, y \in \Omega' \tag{4.2}$$

where

$$C = \text{diam}(\Omega)^2 \left(\frac{2\Lambda}{\text{dist}(\Omega', \Omega)} + \|f^{1/n}\|_{C^{0,1}} \left(1 + \frac{\text{diam}(\Omega)}{\text{dist}(\Omega', \Omega)} \right) \right). \tag{4.3}$$

The value of this estimate lies in the fact that it does not the require regularity and curvature information of $\partial\Omega$. However, this estimate is not a special property to the complex Monge-Ampère equation. In fact, one observes from its proof that any concave elliptic operator F of homogeneous 1 will have the same estimate (see next section the proof when $f \equiv 1$).

The insufficiency of the estimate lies in the assumption that u has a convex level set. This is in general not true for solutions of the complex Monge-Ampère equation. In particular, the assumption that u has a convex level set is not invariant under biholomorphic transformations. An interesting question is the following: if the solution u has a convex level set Ω, whether all level sets that contained in Ω are convex. If this is the case, then one can show that u is $C^{1,\alpha}$ in the interior, provided that the complex Monge-Ampère measure is comparable to the Lebesgue measure.

The second result, due to Blocki and Dinew [BD11], states that if a solution D^2u is L^p for a large p, then u is indeed C^2.

Theorem 4.2 (Blocki, Dinew, 2011)**.** *Suppose that u is a continuous plurisubharmonic function that satisfies*

$$\begin{cases} \mathcal{M}[u] = f \in C^{1,1}(B_1) & \text{in } B_1 \\ f^{1/n} \in C^{1,1}(B_1). \end{cases} \tag{4.4}$$

If

$$\begin{aligned} &\theta \le f(x) \le \theta^{-1}, \forall x \in B_1 \\ &\exists p > n(n-1) \text{ such that } \|\Delta u\|_{L^p(B_1)} \le \Lambda, \end{aligned} \tag{4.5}$$

then for every $\alpha > 0$, $u \in C^{2,\alpha}(B_{1/2})$ and

$$\|D^2u\|_{C^{2,\alpha}(B_{1/2})} \le C(\|f^{1/n}\|_{C^{1,1}(B_{1/2})}, \theta, \Lambda, p, \alpha, \|u\|_{L^\infty(B_1)}). \tag{4.6}$$

The key value of the above result lies in the fact that the integrability assumption on Δu is optimal. Block's example discussed in the previous chapter are singular examples with $\Delta u \in L^p, p < n(n-1)$.

The proof follows from a general method given by Trudinger [Tru80]. The special ingredient from the complex Monge-Ampère equation is the Kolodziej's L^∞-estimate (see next section for more details).

The insufficiency of the estimate is that the condition $\Delta u \in L^p$ for $p > n(n-1)$ is difficulty to verify in general.

The last result is given by Dinew, Zhang and Zhang [DZZ]. It states the following:

Theorem 4.3 (Dinew, Zhang, Zhang, 2010)**.** *Suppose that u is a continuous plurisubharmonic function that satisfies*

$$\begin{cases} \mathcal{M}[u] = f & \text{in } B_1 \\ f^{1/n} \in C^\alpha(B_1). \end{cases} \tag{4.7}$$

If

$$\begin{aligned} &\theta \leq f(x) \leq \theta^{-1}, \forall x \in B_1 \\ &\|D^2 u\|_{L^\infty(B_1)} \leq \Lambda, \end{aligned} \tag{4.8}$$

then $u \in C^{2,\alpha}(B_{1/2})$ and

$$\|D^2 u\|_{C^{2,\alpha}(B_{1/2})} \leq C(\|f^{1/n}\|_{C^\alpha(B_{1/2})}, \Lambda, \alpha, \|u\|_{L^\infty(B_1)}). \tag{4.9}$$

The key importance of the above theorem is that D^2u is of exactly same Hölder exponent as f.

4.2 Gradient and Hessian estimates

In this section, we present the proof of Theorem 4.1 and 4.2 for the case the right-hand side $f \equiv 1$. They key in both proof is the concavity property of the determinant.

We first present the proof of Blocki's estimate

Proof of Theorem 4.1 when $f \equiv 1$. By scaling, we may assume

$$B_1 \subset \Omega. \tag{4.10}$$

and 0 is the centre of mass of Ω.

For an arbitrary $y \in B_r, r \leq 1/2$, define

$$T_\alpha(x) := (1-r)x + y. \tag{4.11}$$

It is easy to see that $T(\Omega) \subset \Omega$.

Now, let

$$v(x) := u \circ T(x) + r\left(\frac{|x|^2}{2} - \frac{\operatorname{diam}(\Omega)^2}{2}\right). \tag{4.12}$$

Then

$$\mathcal{M}^{1/n}[v](x) \geq (1-r)\,\mathcal{M}^{1/n}[u](T(x)) + r \geq 1. \tag{4.13}$$

and $v|_{\partial\Omega} \leq 0$.

Therefore, by the comparison principle, we conclude

$$v(0) = u(y) - \operatorname{diam}(\Omega)^2 r \leq u(0). \tag{4.14}$$

This proves the estimate. □

Next we present proof of Blocki and Dinew's estimate.

Proof of Theorem 4.2 when $f \equiv 1$. All constants in the proof will only depends on Λ, n, p. By standard approximation, we may assume $u \in C^4$. All derivatives are performed against the canonical complex coordinates $\{z_1, ..., z_n\}$.

Let $L[\varphi]$ be the linearized complex Monge-Ampère operator:

$$L[\varphi] := u^{i\bar{j}}\varphi_{\bar{j}i} \tag{4.15}$$

Since $F[u] = 1$, we have for every k

$$L[u_k] = 0, \quad L[\Delta u] \geq 0 \tag{4.16}$$

Fix $q \in (1, \frac{p}{n(n-1)})$, let

$$\alpha = 1 + \frac{p}{qn}, \quad \beta = 2\left(1 + \frac{qn}{p}\right). \tag{4.17}$$

Clearly, $\alpha, \beta \geq 2$; Let

$$\eta(z) := (1 - |z|^2)^\beta, \quad w := \eta \cdot (\Delta u)^\alpha \tag{4.18}$$

By direct calculation,

$$L[w] \geq (\Delta u)^\alpha u^{i\bar{j}} \left(\eta_{i\bar{j}} - \frac{\alpha}{\alpha - 1} \frac{\eta_i \eta_j}{\eta} \right) \tag{4.19}$$

By the choice of η, we have

$$\left| \eta_{i\bar{j}} \right|, \quad \left| \frac{\eta_i \eta_{\bar{j}}}{\eta} \right| \leq C\eta^{1-2/\beta}. \tag{4.20}$$

Thus, we obtain

$$L[w] \geq -Cw^{1-2\beta} (\Delta u)^{2\alpha/\beta} \sum_{ij} \left| u^{i\bar{j}} \right|. \tag{4.21}$$

By standard linear algebra, Δu is bounded in L^p implies $u^{i\bar{j}}$ are bounded in $L^{p/(n-1)}$. It follows then

$$\|(L[w])_-\| \leq C \left(1 + \|w\|_{L^\infty}^{1-2\beta} \right) \tag{4.22}$$

where $f_- := -\min\{f, 0\}$.

Now, solve

$$F[v] = (L[w])_-, \quad v|_{\partial B_1} = 0 \tag{4.23}$$

in viscosity sense. By the arithmetic-geometric inequality, we have

$$L[v] \geq nF^{1/n}[u] \, (L[w])_- \geq -L[w]. \tag{4.24}$$

It implies

$$w \leq (-v), \quad \text{in } B_1. \tag{4.25}$$

On the other hand, by Kolodzieg's estimate, we have

$$-v \leq C\|F[v]\|_q^{1/n} \tag{4.26}$$

Finally, combine (4.22) - (4.26), we have

$$\|w\|_{L^\infty(B_1)} \leq C(1 + \|w\|_{L^\infty(B_1)}^{1-2/\beta}) \tag{4.27}$$

The desired estimate follows. □

4.3 Singular examples

4.3.1 Basic calculations

We consider singular solution to the complex Monge-Ampère equation of the form

$$u(x) = u(|x'|, |(x_{2n-1}, x_{2n})|). \tag{4.28}$$

The following calculation will be needed.

Lemma 4.4. *The following calculation holds:*

1. *Let*

$$u(x) = h(r)g(t), \quad r = |x'| = |(x_1, ..., x_{2n-2})|, t = |(x_{2n-1}, x_{2n})|. \tag{4.29}$$

For all x such that $|x'| \neq 0$, u satisfies

$$\det_{\mathbb{C}}(D^2 u) = \frac{1}{4}\left(\frac{h'}{r}\right)^{n-2} g^{n-2}\left[gh\left(\frac{h'}{r}+h''\right)\left(\frac{g'}{t}+g\right)-(h'g')^2\right] \tag{4.30}$$

2. *Let*

$$u(x) = r + r^{\beta}(1+t^2). \tag{4.31}$$

Then for all x such that $r \neq 0$, u satisfies

$$\det_{\mathbb{C}}(D^2 u) = r^{\beta+1-n}\left(\beta r^{\beta-1}t^2 + \beta r^{\beta-1} + 1\right)\left(\beta^2 r^{\beta-1} + 1\right) \tag{4.32}$$

Proof. Let u be a plurisubharmonic function on $\mathbb{C}^n$ of the form

$$u(x) = h(r)g(t), \quad , r = |x'| = |(x_1, ..., x_{2n-2})|, t = |(x_{2n-1}, x_{2n})| \tag{4.33}$$

We would like to compute its complex Hessian. Fix a point x, we claim: there exists a complex coordinate system such that

$$x = (0, .., 0, r, 0, t). \tag{4.34}$$

The key observation here is that x' is an eigenvector of $\mathfrak{p}(x\otimes x)$. Let $U\in SU(2n)$ such that

$$U\mathfrak{p}(x'\otimes x')U^T = \text{diagonal}. \tag{4.35}$$

U^{-1} gives a complex coordinate transformation that maps $x'/|x'|$ to one of the coordinate axis. Relabelling the coordinate if necessary, we may assume $x/|x| = e_{2n-2}$.

With the same procedure, we find a linear map V on $\mathbb{R}^2$ such that V^{-1} maps $s=(x_{2n-1},x_{2n})$ to $(0,|s|)$.

Then the mapping

$$T = \begin{pmatrix} U^{-1} & 0 \\ 0 & V^{-1} \end{pmatrix} \tag{4.36}$$

gives the required coordinate transformation.

Now, in this new coordinate system, we have

$$D^2u(x) = \begin{pmatrix} g\frac{h'(r)}{r}I_{2n-4} & 0 \\ 0 & M \end{pmatrix} \tag{4.37}$$

where

$$M = \begin{pmatrix} g\frac{h'(r)}{r} & 0 & 0 & 0 \\ 0 & gh''(r) & 0 & h'(r)g'(t) \\ 0 & 0 & \frac{g'(t)}{t}h & 0 \\ 0 & h'(t)g'(t) & 0 & g''(t)h \end{pmatrix} \tag{4.38}$$

By directly calculation, we have

$$\mathfrak{p}(D^2u(x)) = \begin{pmatrix} g\frac{h'(r)}{r}I_{2n-4} & 0 \\ 0 & \mathfrak{p}(M) \end{pmatrix} \tag{4.39}$$

and

$$\mathfrak{p}(M) = \frac{1}{2}\begin{pmatrix} g\frac{h'(r)}{r}+gh''(r) & 0 & h'(t)g'(t) & 0 \\ 0 & g\frac{h'(r)}{r}+gh''(r) & 0 & h'(t)g'(t) \\ h'(t)g'(t) & 0 & \frac{g'(t)}{t}h+g''(t)h & 0 \\ 0 & h'(t)g'(t) & 0 & \frac{g'(t)}{t}h+g''(t)h \end{pmatrix}. \tag{4.40}$$

(4.30) follows immediately.

Now consider

$$u(x) = r + r^{\beta}(1 + t^2) \tag{4.41}$$

Apply the previous calculation with $h(r) = r^{\beta}, g(t) = (1 + t^2)$, we have

$$\mathfrak{p}(D^2 u(x)) = \begin{pmatrix} \left(\beta r^{\beta-2}(1+t^2) + \beta^{-1}\right) I_{2n-4} & 0 \\ 0 & \mathfrak{p}(M) \end{pmatrix} \tag{4.42}$$

and

$$\mathfrak{p}(M) = \frac{1}{2} \begin{pmatrix} \beta^2 r^{\beta-2}(1+t^2) + \frac{1}{r} & 0 & 2r^{\beta-1}t & 0 \\ 0 & \beta^2 r^{\beta-2}(1+t^2) + \frac{1}{r} & 0 & 2r^{\beta-1}t \\ 2r^{\beta-1}t & 0 & 4r^{\beta} & 0 \\ 0 & 2r^{\beta-1}t & 0 & 4r^{\beta} \end{pmatrix}. \tag{4.43}$$

(4.32) follows immediately. □

4.3.2 Partially radial symmetric examples

We now present Blocki's example [Blo97] and its modifications.

Theorem 4.5 (Blocki, 1997)**.** *Let δ_n be a constant only depending on n with $\delta_2 = \infty$; let*

$$\Omega = \{x : |t| \le \delta_n\}. \tag{4.44}$$

There exists a plurisubharmonic function $u \in C^{1,1-\frac{1}{n}(\Omega)}$ but not C^2 such that

$$\mathcal{M}[u] = 1, \quad \text{in } \Omega. \tag{4.45}$$

Proof. Let

$$u = r^{\alpha} g(t). \tag{4.46}$$

By Lemma 4.4 and its proof, we have

$$\det_{\mathbb{C}}(D^2 u) = \frac{\alpha^n}{4} r^{(\alpha-2)(n-2)+2(\alpha-1)} g^{n-2} \left[g\left(g'' + \frac{g'}{t}\right) - (g')^2 \right]. \tag{4.47}$$

and u is psh when

$$\left(\frac{g'}{t} + g''\right) + \alpha^2 r^{-2} g \geq 0, \quad r \neq 0 \tag{4.48}$$

Now, let

$$\alpha = 2 - \frac{2}{n} \tag{4.49}$$

and let g be the solution of the following initial value problem

$$\begin{cases} g\left(g'' + \frac{g'}{t}\right) - (g')^2 = \frac{4g^{n-2}}{\alpha^2} \\ g(0) = 1, g'(0) = 0 \end{cases} \tag{4.50}$$

By the standard ODE theory, there exists δ_n such that $g(t)$ exists for $t < \delta_n$ and $g(t) > 0 \forall t \in [0, \delta_n)$.

Moreover, when $n = 2$

$$g(t) = 1 + t^2 \tag{4.51}$$

is the solution.

Thus, $u(x) = r^\alpha g(t)$ satisfies the required condition. □

Remark 4.6. *By adding trivial variables to the* 2*-D example, i.e., consider*

$$u(z, w, \xi) := |z|\left(1 + |w|^2\right) + |\xi|^2/2, \quad (z, w, \xi) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^{n-2},$$

we obtain singular global solution to $\mathcal{M}[u] = 1$ *in any dimension* $n \geq 2$.

Now we present a more singular solution but with non-constant right-hand side

Theorem 4.7. *Let* $n \geq 3$*, the function*

$$u = r + r^{n-1}(1 + t^2) \tag{4.52}$$

satisfies

$$\mathcal{M}[u] = f \in C^\infty(\mathbb{C}^n), f > 0. \tag{4.53}$$

However $u \in C^{0,1}(\mathbb{C}^n)$ *but not in* $C^{1,\alpha}$ *for any* $\alpha > 0$.

Proof. Follows immediately from Lemma 4.4 □

4.4 Remark on global singular solutions

In the previous section, we have seen that the complex Monge-Ampère equation admits global singular solutions. e.g.,

$$u = |z|\,(1 + |w|^2)$$

solves $\mathcal{M}[u] = 1$ on $\mathbb{C}^2$. The singular set,i.e, the set of points where u is not smooth, is given by

$$\Sigma_u = \{(0, w) : w \in \mathbb{C}\}.$$

Now, consider a holomorphic mapping $F : \mathbb{C}^2 \to \mathbb{C}^2$ (not necessarily injective nor surjective) such that

$$\det(JF) = 1, \quad JF := \begin{pmatrix} F_z^1 & F_w^1 \\ F_z^2 & F_w^2 \end{pmatrix}$$

then $u \circ F$ is again a solution with singular set

$$\Sigma_{u\circ F} = F^{-1}(\Sigma_u).$$

Consider the mapping

$$F(z, w) := (e^z - 1, e^{-z} w)$$

then

$$\Sigma_{u\circ F} = \{(z, w) : e^z = 1, w \in \mathbb{C}\}$$

which consists of infinitely many copies of $\mathbb{C}$.

Given any holomorphic function $f : \mathbb{C} \to \mathbb{C}$, by considering the mapping,

$$F(z, w) := (z, w - f(z)),$$

the graph of f graph can be realized as the singular set of a solution to $\mathcal{M}[u] = 1$ on $\mathbb{C}^2$.

Now, it is interesting to ask whether there are singular set of solutions $\mathcal{M}[u] = 1$ on $\mathbb{C}^2$ with more complicated geometry. For example, can the singular set be of the form

$$\Sigma_u = \{(z, w) : z^2 = w^3\}.$$

More generally, one would like to classify all possible singular sets of global solutions of $\mathcal{M}[u] = 1$ on $\mathbb{C}^n$. Here, we would like to make the following conjecture:

Conjecture: Let u be a solution of $\mathcal{M}[u] = 1$ on $\mathbb{C}^2$, then the singular set Σ_u is an area-minimizing currents.

Chapter 5

Small Perturbation Solutions

5.1 Main statements

We shall consider a family of elliptic operators.

Definition 5.1. *Given constants $\delta, \theta, K > 0$, the family $\mathcal{F}_{\delta,\theta,K}$ consists of functions $F : \mathrm{Sym}(2n) \to \mathbb{R}$ that satisfy the following conditions:*

H1 *For every $M \in \mathrm{Sym}(2n)$*

$$F(M+P) \geq F(M), \quad \forall P \geq 0. \tag{5.1}$$

H2 $F(0) = 0$.

H3 *For every M with $\|M\| \leq \delta$*

$$\theta^{-1}\|P\| \geq F(M+P) - F(M) \geq \theta\|P\|, \forall P \geq 0, \|P\| \leq \delta. \tag{5.2}$$

H4 *F is twice differentiable in the set $\{M \mid \|M\| \leq \delta\}$ and*

$$\left|D^2F(M)\right| \leq K. \tag{5.3}$$

We will prove the following more general theorem.

Theorem 5.2 (Savin, 2007)**.** *For every $\delta > 0$, if $F \in \mathcal{F}_{\delta,\theta,K}$ and*

$$f(0) = 0, \quad \int_0^1 \frac{-\omega_f(r)\log r}{r}\,dr < \infty, \tag{5.4}$$

then there exist constants μ, ρ, C only depending on $n, \delta, \theta, K, \omega_f$ such that, if

$$\begin{aligned} &F[u] := F(D^2u) = f, \ \text{in } B_1 \\ &\|u\|_{L^\infty(B_1)} \le \mu, \end{aligned} \tag{5.5}$$

then there exists a quadratic polynomial P such that

$$\begin{aligned} &F(D^2P) = 0, \quad \|D^2P\| \le \delta \\ &\|u - P\|_{L^\infty(B_r)} \sim o(r^2), \quad \text{as } r \to 0. \end{aligned} \tag{5.6}$$

Remark 5.3. *The condition on the modulus of continuity of f seems a bit unnatural. The optimal condition that one would like to obtain is that f is Dini continuous, i.e.,*

$$\int_0^1 \frac{\omega_f(r)}{r}\,dr < \infty.$$

However, we cannot reach this generality at this moment.

In the case that $f \equiv 0$, Theorem 5.2 was first introduced by Savin in [Sav07] where Theorem 5.2 has been proved for more general elliptic equations. Though not stated explicitly, the proof in [Sav07] essentially covers the case that f has proper modulus of continuity. However, the result and method in [Sav07] seem not well-known in the study of the complex Monge-Ampère equations. For the purpose of completeness, we shall present a detailed proof of Theorem 5.2.

5.2 Setup and the main ideas

Fix $\delta > 0$, in the rest of this chapter, we shall refer constants that only depends on $n, \delta, \theta, K, \beta, \omega_f$ as universal constants. Denote

$$\eta(r) := -\omega_f(r)\log r \tag{5.7}$$

The proof of Theorem 5.2 is divided into two parts.

In the first part, we want to establish a modified version of Harnack inequality (see Prop.5.9) for f has small oscillation around 0 . As the standard Harnack inequality, it will provide us necessary compactness in performing a blow-up. The underline idea of the proof is essentially same to the classical proof of Krylov-Safonov Harnack inequality (see [CC95]). The key observation here is that in the classical proof, one only need the equation to be evaluated on test functions that touches v. In the case v is very close to zero, we can choose test functions with very small Hessian. For this part, we follows exactly same proof as given in [Sav07].

In the second part, we obtain estimate of D^2u through a blow-up argument. The formal idea is the following: Let $v = \epsilon h$, by a formal Taylor expansion

$$F(D^2u) - 1 = \epsilon \Delta h + O(\epsilon^2). \tag{5.8}$$

Thus h satisfies $\Delta h = 1$ in turn h and v are regular. However, the formal argument is unsatisfactory because the $O(\epsilon^2)$ depends on the seize of D^2w which is not under control. This issue is resolved through the compactness. In this part, we modify slightly the argument in [Sav07] to cover the case f is non-constant.

Before moving on, we list some immediate consequences of the structure conditions of $\mathcal{F}$.

Lemma 5.4. *If $F : \mathrm{Sym}(2n) \to \mathbb{R}$ belongs to $\mathcal{F}_{\delta,\theta,K}$, then*

1. *For every $M \in \mathrm{Sym}(2n)$ such that $\|M\| \leq \delta$*

$$\theta\|M^+\| - (2n-1)\theta^{-1}\|M^-\| \leq F(M) \leq (2n-1)\theta^{-1}\|M^+\| - \theta\|M^-\|, \tag{5.9}$$

2. *There exists a universal constant c_0 such that: for all $a \leq c_0\delta$, If $F(M) \leq 1+a$ and $M \geq -aI$, then*

$$M \leq C_0 aI. \tag{5.10}$$

Proof. Statement 1 follows immediately from **H3** of the Definition 5.1. To prove the second statement, Let e be the eigenvector that corresponds to the largest eigenvalue of M and take C such that $Me = (Ca)e$. Since $M \geq -aI$,

$$M \geq Cae \otimes e - aI. \tag{5.11}$$

First, claim that $Ca \leq \delta$. Suppose otherwise, then

$$M \geq \delta e \otimes e - aI. \tag{5.12}$$

and by the first statement, we have

$$\begin{aligned} c_0\delta \geq a \geq F(M) - 1 &\geq F(\delta e \times e - aI) \\ &\geq \theta\delta - \theta^{-1}(n-1)a \geq \theta\delta - \theta^{-1}(n-1)c_0\delta. \end{aligned} \tag{5.13}$$

This leads to a contradiction when c_0 is small.

Next, as we have shown that $Ca \leq \delta$, we can apply Statement 1 to conclude that

$$a \geq F(M) - 1 \geq \theta Ca - \theta^{-1}(n-1)a \tag{5.14}$$

which implies that

$$C \leq \theta^{-1} + \theta^{-2}(n-1). \tag{5.15}$$

This completes the proof. □

5.3 Measure estimate and localization

In this section, we establish two important tools in the proof.

First, we introduce the concept of contact sets.

Definition 5.5. *For each closed subset $E \subset \overline{B}_1$ and positive number a, we define the contact set*

$$A_a(E) := \{x \mid \exists y \in E \text{ such that } P_{a,y} \text{ touches } u \text{ from below at } x.\} \tag{5.16}$$

where

$$P_{a,y} := -\frac{a}{2}\left|x-y\right|^2 + c, \quad c \in \mathbb{R}. \tag{5.17}$$

We set $A_a := A_a(\overline{B}_1)$.

The following properties of contact sets will be useful

Lemma 5.6. *Let* E *be a closed subset of* $\overline{B}_1$ *and* $a > 0$*, then the following statement holds*

1. $A_a(E)$ *is closed hence measurable.*

2. *If* $E \subset F$*, then* $A_a(E) \subset A_a(F)$*.*

3. *If* $a < b$*, then* $A_a(\overline{B}_1) \subset A_b(\overline{B}_1)$*.*

Proof. Statement 1 and 2 are straight forward to check. To show 3, let $x \in A_a(E)$ and $y_a \in \overline{B}_1$ be its corresponding point. It is easy to check that the polynomial

$$P(x) := -\frac{b}{2}\left|x-y_b\right|^2 + c_b \tag{5.18}$$

with

$$y_b = \left(1 - \frac{a}{b}\right)x + \frac{a}{b}y_a \tag{5.19}$$

will touch u at x after a proper choice of c_b. □

5.3.1 Measure estimate

We now state and prove an ABP-type measure estimate.

Lemma 5.7. *There exists a constant* c_1 *such that: for every* $\nu \in (0,1)$*, every* $\delta' \leq c_0\delta$ *and every* $E \subset B_1$*, if* $F \in \mathcal{F}_{\delta,\theta,K}$*,*

$$F[u] \leq 1 + \nu\delta' \tag{5.20}$$

and

$$A_a(E) \subset B_1 \tag{5.21}$$

then for all $a \in (\nu\delta', c_0\delta 0)$,

$$|A_a(E)| \geq c_1 |E| . \tag{5.22}$$

Proof. It suffices to prove the lemma for u being semi-concave in B_1. The general case follows from jansen's approximation.

By Lemma 2.10, the map

$$T(x) := x + \frac{1}{a}\nabla w(x) \tag{5.23}$$

is a well-defined Lipschitz map on A. By the assumption that $A \subset \Omega$, we conclude that

$$T(A) = E \tag{5.24}$$

Since u is semi-concave, we know by Theorem 2.9 that there exists a set Z such that $|B_1 \setminus Z| = 0$ and w is C_2 on all $z \in Z$.

Fix $x \in A \cap Z$, by the definition of contact set and 5 of Lemma5.4, we have

$$- aI \leq D^2 u(x) \leq CI. \tag{5.25}$$

Thus, by the area formula, we conclude from (5.24), (5.25) that

$$|E| \leq \int_{A\setminus Z} \det(D^2 w)\, dx \leq C^n |A| . \tag{5.26}$$

This proves the desired estimate. □

5.3.2 Localization

The ABP measure estimate suggests that the good sets (contact sets) are not small in measure. To obtain point-wise estimate, we need a finer information on their distribution.

Lemma 5.8. *For any* $\nu \in (0, 1)$ *and* $\delta' \leq c_0\delta$, *if* $F \in \mathcal{F}_{\delta,\theta,K}$ *and*

$$F[u] \leq 1 + \nu\delta' \tag{5.27}$$

then there exist constants c, C *such that, for all* $a \in (\nu\delta', C_2^{-1}c_0\delta)$*, if*

$$\overline{B}_r(x_0) \subset B_1, \quad \overline{B}_r(x_0) \cap A_a \neq \emptyset, \tag{5.28}$$

then

$$\frac{\left|A_{Ca} \cap B_{r/8}(x_0)\right|}{|B_r(x_0)|} \geq c \tag{5.29}$$

Proof. Without loss of generality assume

$$x_1 \in B_r(x_0) \cap A_a \neq \emptyset \tag{5.30}$$

Otherwise we replace r by $r + \epsilon$ and the result follows by letting $\epsilon \to 0$.

Denote by $y_1 \in \overline{B}_1$ the vertex of the tangent paraboloid

$$P_{y_1}(x) := w(x_1) + \frac{a}{2}\left|x_1 - y_1\right|^2 - \frac{a}{2}\left|x - y_1\right|^2 \tag{5.31}$$

that touches w from below at x_1.

We devide the proof into three steps.

Step 1 Claim: there exists a point $z \in \overline{B}_{r/16}(x_0)$ such that

$$u(z) - P_{y_1}(z) \leq Car^2. \tag{5.32}$$

Let φ be the radially symmetric continuous function

$$\varphi : \overline{B}_1 \to \mathbb{R}, \quad \varphi(x) := \begin{cases} \alpha^{-1}\left(|x|^{-\alpha} - 1\right), & \frac{1}{16} \leq |x| \leq 1 \\ \alpha^{-1}\left(16^{\alpha} - 1\right), & |x| \leq \frac{1}{16} \end{cases} \tag{5.33}$$

where α is a large constant.

Construct a function ψ by adding a rescaling of the above function to the tangent paraboloid $P_{y_1}(x)$, i.e.,

$$\psi : \overline{B}_1 \to \mathbb{R}, \quad \psi(x) := P_{y_1}(x) + ar^2\varphi\left(\frac{x - x_0}{r}\right). \tag{5.34}$$

We claim that ψ satisfies $F[\psi] \geq 1 + \nu\delta'$ in the region

$$r/16 < |x - x_0| < r. \tag{5.35}$$

Indeed, using Lemna 5.4 we find

$$F[\psi] - F[0] \geq \lambda a \left[(\alpha+1)t^{-\alpha-2} - 1\right] - (n-1)\Lambda a(1 + t^{-\alpha-2}) \geq a \tag{5.36}$$

where $t = |x - x_0| / r$.

Now let z be the point where

$$\min_{x \in \overline{B}_r(x_0)} (u - \psi) \tag{5.37}$$

is realized.

If $x \in \partial\overline{B}_r(x_0)$ then

$$u(x) \geq P_{y_1} = \psi(x). \tag{5.38}$$

However, the value in (5.35) is negative since $x_1 \in B_r(x_0)$ and

$$u(x_1) - \psi(x_1) = P_{y_1}(x_1) - \psi(x_1) < 0. \tag{5.39}$$

Therefore, from the above consideration, we deduce that the minimum cannot be realized in the region given by (5.32). In conclusion

$$z \in \overline{B}_{r/16} \tag{5.40}$$

and

$$u(z) < \psi(z) \leq P_{y_1}(z) + Car^2 \tag{5.41}$$

which proves the claim.

Step 2 Claim: for every $y \in \overline{B}_{r/64}(z)$, the polynomial

$$P(x; y) := P_{y_1}(x) - \frac{C'a}{2} |x - y|^2 + c_y \tag{5.42}$$

touches v from below in B_1.

The opening of the above paraboloid is $(C' + 1)\, a$ and the vertex is

$$T(y) := \frac{C'}{C+1} y + \frac{1}{C'+1} y_0 \in B_1 \tag{5.43}$$

From (5.32) we find

$$c_y \le Car^2 + C'\frac{a}{2}\left(\frac{r}{64}\right)^2. \tag{5.44}$$

If $|x-z| \ge r/16$ and C' is large

$$\begin{aligned} &P_{y_1}(x) - C'\frac{a}{2}|x-y|^2 + c_y \\ &\le P_{y_1}(x) - C'\frac{a}{2}\left(\frac{r}{32}\right)^2 + Car^2 + C'\frac{a}{2}\left(\frac{r}{64}\right)^2 < u(x). \end{aligned} \tag{5.45}$$

Thus, the contact points are inside $B_{r/16} \subset B_{r/8}(x_0) \subset B_1$. This proves the claim.

Step 3 Complete the proof. Let

$$E = T(B_{r/64}(z_0)) \tag{5.46}$$

By Step 2, we know

$$A_{C_2 a}(E) \subset B_1, \quad C_2 = C' + 1. \tag{5.47}$$

Therefore, we can apply Lemma 5.7 to conclude

$$|A_{C_2 a}| \ge |A_{C_2 a} E| \ge |E| = \left(\frac{C'}{C'+1}\right)^n \left|B_{r/64}\right|. \tag{5.48}$$

The desired estimate follows. □

5.4 Oscillation decay and compactness

Recall that for uniformly elliptic equation, the oscillation of a solution over B_r decays as r tends to zero. This shows that the solutions are uniformly Hölder and hence the family of solutions are compact.

The goal of this section is to show that oscillation of v over B_r decays for r not too small.

Proposition 5.9. *Suppose that $F \in \mathcal{F}_{\delta,\theta,K}$ and*

$$\|u\|_{L^\infty(B_1)} \le \delta' \le c_0\delta, \quad u(0) = 0. \tag{5.49}$$

There exists a small constant ν such that if

$$1-\nu\delta' \le F[u] \le 1+\nu\delta', \text{ in } B_1 \tag{5.50}$$

then

$$\|u\|_{L^\infty(B_{1/2})} \le (1-\nu)\delta'. \tag{5.51}$$

As an immediate corollary, we have

Corollary 5.10. *If $F \in \mathcal{F}_{\delta,\theta,K}$ and for some k,*

$$\|u\|_{L^\infty(B_1)} \le \delta' \le 2^{-2k} c_0 \delta, \quad u(0)=0 \tag{5.52}$$

and

$$1-\nu\delta' \le F[w] \le 1+\nu\delta', \tag{5.53}$$

then for all $\rho \ge 2^{-(k+1)}$

$$\|u\|_{L^\infty(B_\rho)} \le 2\rho^\beta \delta', \tag{5.54}$$

where β is a small constant.

5.4.1 Oscillation decay in measure

We first show that the measure of contact set of w with large opening is very large compare to B_1. This is done via a measure covering argument.

Lemma 5.11. *Let $\nu \in (0,1)$, $\delta' \le c_0\delta$ and $a \in (\nu\delta', c_0\delta)$, if $F \in \mathcal{F}_{\delta,\theta,K}$,*

$$F[u] \le 1+\nu\delta' \tag{5.55}$$

and

$$A_a \cap B_{1/2} \ne \emptyset, \tag{5.56}$$

then for all k such that $C_2^k a \le c_0\delta$,

$$\left|B_{1/2} \setminus A_{C^k a}\right| \le (1-c_3)^k \left|B_{1/2}\right|. \tag{5.57}$$

Proof. Let $D_k := A_{C^k a} \cap B_{1/2}$. The proof is divided into two steps.

Step 1 Claim: for all $x \in B_{1/2}$

$$\left|B_{r/3}(x) \cap D_{k+1}\right| \geq c\left|B_{1/3} \cap B_r(x)\right| \tag{5.58}$$

where

$$r := \operatorname{dist}(x_0, D_k). \tag{5.59}$$

To show this, let

$$x' := x - \frac{r}{6}\frac{x}{|x|}. \tag{5.60}$$

Notice that

$$B_{r/6}(x') \subset B_{r/2}(x) \cap B_{1/2}. \tag{5.61}$$

From Lemma 5.8 and

$$\operatorname{dist}(x', D_k) \leq r + \frac{r}{6} = \frac{7}{6}r, \tag{5.62}$$

we conclude that

$$\left|B_{r/6}(x') \cap D_{k+1}\right| \geq c'\left|B_{1/2} \cap B_r(x)\right| \tag{5.63}$$

which proves the claim.

Step 2 Now we perform a Vitali covering argument. For each $x \in B_{1/2}$, we consider the ball of centre x and radius

$$r_x = \operatorname{dist}(x, D_k). \tag{5.64}$$

By Vitali's covering lemma (see [EG92]), we can choose a sequence of balls $B_{r_i}(x_i)$ that covers $B_{1/2} \setminus D_k$ and $B_{r_i/3}$ are disjoint.

We have

$$\begin{aligned}\left|B_{1/2} \setminus D_k\right| &\leq \sum_i \left|B_{r_i}(x_i) \cap B_{1/2}\right| \\ &\leq c^{-1} \sum \left|B_{r_i/2} \cap (D_{k+1} \setminus D_k)\right| \leq c^{-1}\left|D_{k+1} \setminus D_k\right|.\end{aligned} \tag{5.65}$$

In conclusion,

$$\left|B_{1/2} \setminus D_{k+1}\right| \leq \left|B_{1/3} \setminus D_k\right| - \left|D_{k+1} \setminus D_k\right| \leq (1 - c_3)\left|B_{1/2} \setminus D_k\right| \tag{5.66}$$

and the lemma is proved. □

5.4.2 Proof of the oscillation decay property

Now, we give the proof of Proposition 5.9.

Proof of Proposition 5.9. We proceed the proof in three steps.

Step 1 Let

$$E_+ = \{u \le \delta'/4\} \cap B_{1/3}, \quad E_- = \{u \ge -\delta'/4\} \cap B_{1/3} \tag{5.67}$$

Claim:

$$|E_+|, |E_-| \ge c. \tag{5.68}$$

To estimate E_+, we slide from below the paraboloids

$$-16\delta' |x-y|^2 + c_y, \quad |y| \le 1/3. \tag{5.69}$$

Since $w(0) = 0$, and $w \ge -\delta'$, we see that the contact points belong to E_+. Then Lemma 5.7 implies the desired estimate. Flip the above picture, we obtain the estimate of E_-.

Step 2 We prove the lower bound of u by contradiction. Suppose that there exists a point $x_0 \in B_{1/3}$ such that

$$u(x_0) \ge -\delta' + \nu\delta'. \tag{5.70}$$

Let $a = 72\nu\delta'$. By a proper choice of c_{x_0}, the paraboloid

$$-\frac{a}{2}|x-x_0|^2 + c_{x_0} \tag{5.71}$$

touches u from below in $B_{1/2}$. Therefore

$$A_a \cap B_{1/2} \ne \emptyset. \tag{5.72}$$

By Lemma 5.11 and the fact that

$$u(x) = P_{C^k a, y}(x) \le C^k \nu\delta' \quad \forall x \in A_{C^k a} \cap B_{1/2}, \tag{5.73}$$

we conclude that

$$\left|\{u \ge -\delta' + C^k\nu\delta\} \cap B_{1/2}\right| \le (1-c_3)^k \left|B_{1/2}\right| \tag{5.74}$$

Now, we first choose k such that

$$(1-c_3)^k \left|B_{1/2}\right| \leq c/2 \tag{5.75}$$

then choose ν such that

$$C^k \nu \leq 1/2. \tag{5.76}$$

Then (5.74) and

$$|E_-| = \left|\{u \geq -\delta/4\} \cap B_{1/2}\right| \geq c \tag{5.77}$$

contradict the fact that $(x, w(x))$ is a graph on $B_{1/2}$.

Step 3 To complete the proof, we are left to estimate $u|_{B_{1/2}}$ from above. This follows simply via applying Step 2 to $-u$ and $-F(-M)$. □

5.4.3 Compactness

In this section, we prove the Corollary 5.10 and discuss its implication in compactness.

Proof of Corollary 5.10. The proof contains two steps.

Step 2 Claim: if

$$\|u\|_{L^\infty(B_r)} \leq \delta' \leq r^2 c_0 \delta, \quad w(0) = 0 \tag{5.78}$$

then

$$\|u\|_{L^\infty(B_{r/2})} \leq (1-\nu)\delta'. \tag{5.79}$$

This follows from a simple scaling. Fix $r < 1$, let

$$u_r(x) := r^{-2} u(rx), \quad x \in B_1. \tag{5.80}$$

Then w_r satisfies

$$\|u_r\|_{L^\infty(B_1)} \leq r^{-2}\delta' \tag{5.81}$$

and

$$|F[u_r] - 1| \leq \nu\delta' \leq r^{-2}\nu\delta'. \tag{5.82}$$

Thus, we can apply Proposition 5.9 to conclude that

$$\|u_r\|_{L^\infty(B_{1/2})} \leq (1-\nu)r^{-2}\delta' \tag{5.83}$$

The claim then follows.

Step 2 By standard interpolation argument, it suffices to prove the corollary for dyadic balls, that is,

$$\|u\|_{L^\infty(B_{2^{-j}})} \leq (1-\nu)^j\delta', \quad j \leq k+1. \tag{5.84}$$

This follows directly from an inductive application of Step 1. □

The following lemma explains the relation between compactness of solutions and Proposition 5.9.

We introduce the following terminology.

Definition 5.12. *A continuous function $w \in B_1$ is said to have γ-Hölder modulus of continuity outside a ρ neighbourhood with normal C if*

$$\|w - w(x)\|_{L^\infty(B_r)} \leq Cr^\alpha, \quad \forall r \geq \rho. \tag{5.85}$$

Lemma 5.13. *Let w_k be a sequence of continuous function on B_1. Assume w_k has β-Hölder continuity outside ρ_k neighbourhood normal C. If C, β are independent of k and $\rho_k \to 0$, then w_k converges uniformly on compact subsets to a continuous function w.*

Proof. Let $\tilde{w}_k$ the inf-convolution of w_k against kernel $|x-y|^\beta$

By Lemma 2.17, $\tilde{w}_k$ is γ-Hölder continuous with normal C. By ArzelàAscoli, we conclude that $\tilde{w}_k$ converges uniformly on compact sets to w.

On the other hand,

$$|\tilde{w}(x) - w(x)| = \left|w(x^*) - w(x) + C\,|x-x^*|^\beta\right| \tag{5.86}$$

where x^* is the point to realize the minimum in the definition of inf-convolution. By the β-Hölder continuity outside ρ_k, we conclude that

$$|\tilde{w}(x) - w(x)| \leq 2C\rho_k^\beta, \quad \forall x \in B_1. \tag{5.87}$$

Therefore, w_k also converges uniformly to w. This proves the lemma. □

5.5 Proof of the main statement

We now complete the proof of Theorem 5.2. We denote by $P(N, x)$ the quadratic polynomial

$$P(N, x) = \frac{1}{2}x^T N x + l(x) \tag{5.88}$$

where $l(x)$ is a linear function.

The key is the following proposition.

Proposition 5.14. *Under assumption of Theorem 5.2, there exist small universal constants σ, C_1, ρ_1 such that, for every $r < \rho_1$, if*

$$\|u - P(N, x)\|_{L^\infty(B_r)} \leq r^2\eta(r) \tag{5.89}$$

and

$$F(N) = 0, \quad \|N\| \leq \frac{\delta}{2}, \tag{5.90}$$

then there exists N' such that

$$\|u - P(N'.x)\|_{L^\infty(B_{\sigma r})} \leq (\sigma r)^2\eta(\sigma r) \tag{5.91}$$

and

$$F(N') = 0, \quad \|N' - N\| \leq C\eta(r). \tag{5.92}$$

Proof. The proof is proceed via contradiction. Let σ, C be fixed constants that will be specified later in the proof. Suppose the statement if false, then there exist sequences

$$\begin{aligned} &r_k \to 0, \quad F_k \in \mathcal{F}_{\delta,\theta,K}, \quad f_k \in C^0(B_1) \\ &u_k \in C^0(B_1), \quad N_k \in \mathrm{Sym}(2n) \end{aligned} \tag{5.93}$$

such that for all $k \geq 0$

$$\begin{aligned} & f_k(0) = 1, \quad \|N_k\| \leq \delta/2 \\ & F_k(D^2 u_k) = f_k \text{ in } B_1, \quad \int_0^1 r^{-1}\eta(r)\, dr \leq A \\ & \|u_k - P(N_k, x)\|_{L^\infty(B_{r_k})} \leq r_k^2 \eta(r_k), \end{aligned} \tag{5.94}$$

but there exists no $N_k' \in \mathrm{Sym}(2n)$ that satisfies

$$\begin{aligned} & \|u_k - P(N_k', x)\|_{L^\infty(B_{\sigma r_k}} \leq (\sigma r_k)^2 \eta(\sigma r) \\ & F(N_k') = 0, \quad \|N_k' - N_k\| \leq C\eta(r). \end{aligned} \tag{5.95}$$

We shall arrive a contradiction in three steps.

Step 1: Let $w_k : B_1 \to [0,1]$ be defined by

$$u_k(r_k x) = P(N_k, r_k x) + r^2 \eta(r) w(r_k x), \quad x \in B_1 \tag{5.96}$$

Claim, w_k converges uniformly to w.

Define

$$\tilde{F}_k(M) := \frac{1}{\eta(r_k)} \left[F_k(N_k + \eta(r_k)M) - F(N^k) \right] \tag{5.97}$$

The function w_k satisfies

$$\tilde{F}_k(D^2 w_k) = \frac{1}{\eta(r_k)} f_k, \text{ in } B_1. \tag{5.98}$$

and $\tilde{F}_k \in \mathcal{F}_{\tilde{\delta}_k, \theta, K}$ with

$$\tilde{\delta}_k = \delta \eta^{-1}(r_k) \tag{5.99}$$

For each $x_0 \in B_{1/2}$,

$$v_k = w_k - w_k(x_0) \tag{5.100}$$

satisfies

$$\left| \tilde{F}_k(D^2 v_k) \right| \leq C(K, \delta)\eta(r_k) + \frac{1}{-\log r} \tag{5.101}$$

Now let $\delta' = 2$, for any $l \in \mathbb{N}$, we can take k sufficiently large, we can insure

$$\begin{aligned} & \delta' \leq 2 \leq \frac{c_0 \delta}{2^{2l}\eta(r_k)} \quad \|v_k\|_{B_{1/2}(x_0)} \leq \delta' \\ & \left| \tilde{F}_k(D^2 v_k) \right| \leq \nu \delta'. \end{aligned} \tag{5.102}$$

Therefore, by Corollary 5.10, we conclude that

$$\|v_k\| \leq 4r^\gamma, \quad \forall r \geq 2^{-(l+1)} \geq \frac{1}{2\sqrt{2}}\sqrt{c_0\delta\eta(r_k)} \tag{5.103}$$

It follows that w_k has a Hölder modulus of continuity outside a $r_k', r_k' \to 0$.

By Lemma 5.13, we conclude that w_k converges uniformly to a continuous function w in $B_{1/2}$. This completes the Step 1.

Step 2 Claim: w satisfies

$$\operatorname{tr}\left(ND^2 w\right) = 0, \quad \text{in } B_{1/2} \tag{5.104}$$

where N is a limit of N_k modulo subsequences.

It suffices to show w is a supersolution, the other case follows similarly. Assume by contradiction that we can touch w from below at x_* by a smooth function φ and

$$\operatorname{tr}\left(ND^2\varphi\right)(x_*) > \epsilon > 0. \tag{5.105}$$

Then

$$\varphi - \epsilon' \left|x - x_*\right|^2 + c \tag{5.106}$$

touches w_k from below at $x_k, x_k \to x_*$. We have

$$\begin{aligned} 0 &\geq \frac{1}{\eta(r_k)}\left[F_k(N_k + \eta(r_k)M) - F(N^k)\right] \\ &\geq \operatorname{tr}(N_k D^2\varphi) - n\epsilon' - C(\varphi, K)\eta(r_k) \\ &\geq \epsilon/2, \quad \text{as } k \to 0. \end{aligned} \tag{5.107}$$

This is a contradiction and the claim is proved.

Step 3 Reach the contradiction. Follows from standard theory of linear equation, there exists $\tilde{N} \in \operatorname{Sym}(2n)$ and universal constants C such that for every $\sigma < 1/2$

$$\begin{aligned} &\operatorname{tr}\left(N \cdot \tilde{N}\right) = 0, \quad \|\tilde{N}\| \leq C \\ &\|w - P(\tilde{N}, x)\|_{L^\infty(B_\sigma} \leq C\sigma^3. \end{aligned} \tag{5.108}$$

Let σ be chosen so that

$$C\sigma < \frac{1}{3}\eta(\sigma) \tag{5.109}$$

Let

$$\tilde{N}_k := \tilde{N} + s_k I \tag{5.110}$$

where s_k is choose so that

$$F(N_k + r_k^\beta \tilde{N}_k) = 0. \tag{5.111}$$

Such s_k always exists for large k, because for s_k such that $s_k\eta(r_k) \leq \delta/2$

$$\frac{1}{\eta(r_k)} F_k(N_k + \eta(r_k)(\tilde{N} + sI)) = Cs_k + \operatorname{tr} N_k\tilde{N} + Cs\delta/2, \quad 0 < \frac{s}{s_k} < 1 \tag{5.112}$$

whose sign will only depends on s_k when r

Moreover, (5.112) implies that for k large,

$$|s_k| \leq \eta^\beta/3. \tag{5.113}$$

By the uniform continuity, we have for k large

$$|w_k - w|_{L^\infty(B_\sigma)} \leq \frac{1}{3}\sigma^2\eta(\sigma) \tag{5.114}$$

Combine (5.108), (5.113) and (5.114), we can take k large enough so that

$$\|w_k - P(\tilde{N}_k, x)\|_{L^\infty(B_\sigma)} \leq \sigma^2\eta(\sigma). \tag{5.115}$$

Therefore, the polynomial $P(N_k', x)$ with

$$N_k' = N_k + \eta(r_k)\tilde{N}_k \tag{5.116}$$

satisfies

$$\begin{aligned} &\|u_k - P(N_k', x)\|_{L^\infty(B_{\sigma r})} \leq (\sigma r)^2\eta(\sigma r) \\ &F(N_k') = 0, \quad \|N_k' - N_k\| \leq C\eta(r_k). \end{aligned} \tag{5.117}$$

However, this contradicts to our starting hypothesis (5.95). □

Proof of Theorem 5.2. Since

$$\int_0^1 \frac{\eta(r)}{r}\, dr < \infty \tag{5.118}$$

we can choose $\rho \leq \rho_1$ so that

$$C\sum_{k=1}^{\infty}\eta(\sigma^k\rho) \leq \delta/2. \tag{5.119}$$

Then choose

$$\mu := \frac{1}{2}\rho^2\eta(\rho). \tag{5.120}$$

Define

$$P_0 := 0. \tag{5.121}$$

The triple (u, P_0, B_ρ) verifies the hypothesis of Proposition 5.14. Starting with (u, P_0), inductively apply Proposition 5.14, we obtain a sequence of quadratic polynomials

$$P_k = P(M_k, x) = \frac{1}{2}x^T M_k x + b_k \cdot x + c_k \tag{5.122}$$

that satisfies:

$$\|u - P_k\|_{L^\infty(B_{\sigma^k\rho})} \leq (\sigma^k\rho)^2\eta(\sigma^k\rho), \quad F(M_k) = 0, \tag{5.123}$$

It follows then

$$(\sigma^k r_0)^2\|M_k - M_{k+1}\|, \left(\sigma^k r_0\right)\|b_k - b_{k+1}\|, |c_k - c_{k+1}| \leq (\sigma^k\rho)^2\eta(\sigma^k\rho) \tag{5.124}$$

From the above estimate, we conclude that P_k's form a Cauchy sequence and converge uniformly to

$$P = P(M, x) = \frac{1}{2}x^T M x + b \cdot x + c \tag{5.125}$$

in B_{r_0}.

Clearly $F(M) = 0$. Consider

$$\|u - P\|_{L^\infty(B_r)}. \tag{5.126}$$

Fix $r < r_0$, there exists k such that

$$\sigma^{k+1}\rho < r \leq \sigma^k\rho \tag{5.127}$$

By (5.123) and (5.124), we have,

$$\begin{aligned} \|u - P\|_{L^\infty(B_r)} &\le \|u - P_k\|_{L^\infty(B_{\sigma^k \rho})} + \|P_k - P\|_{L^\infty(B_{\sigma^k \rho})} \\ &\le C_1 (\sigma^k \rho)^2 \eta(\sigma^k \rho) \le \frac{C_1 \eta(\sigma^k \rho)}{\sigma^2} r^2 \end{aligned} \tag{5.128}$$

Observe that the term

$$\frac{C_1 \eta(\sigma^k \rho)}{\sigma^2} \to 0, \qquad \text{as } r \to 0. \tag{5.129}$$

Therefore, we conclude u is C^2 at 0. Clearly $D^2 P = M$ satisfies $F(D^2 P) = 0$.

□

Chapter 6

Proof of the Main Theorems

6.1 Proof of Theorem 1.4 and its corollary

Theorem 1.4 will follow from Theorem 5.2 and the following lemma.

Lemma 6.1. *Let $F : \mathrm{Sym}(2n) \to \mathbb{R}$ be defined by*

$$F_\lambda(M) := \begin{cases} \det^{1/2}(\lambda I + \mathfrak{p}(M)) - 1, & I + \mathfrak{p}(M) \geq 0 \\ -1 & \textit{otherwise.} \end{cases} \tag{6.1}$$

There exists constant θ, K only depending on the dimension n such that, for every $\delta < \min\{\lambda, 1/2\}$, $F_\lambda \in \mathcal{F}_{\delta,\theta,K}$.

Proof. This follows directly from the properties of the determinant. □

Proof of Theorem 1.4. Apply Theorem 5.2 to F_1, then the desired conclusion follows.

□

Now, we proof the corollaries.

6.1.1 Proof of Corollary 1.5

Proof of Corollary 1.5. We shall show that

$$D^2u(x) - I = 0 \quad \forall x \in \mathbb{C}^n. \tag{6.2}$$

Fix $x \in \mathbb{C}^n$, by a translation of coordinates, we may assume $x = 0$. For every ϵ, let μ_ϵ be the constant produced by Theorem 1.4.

By the comparison principle and the growth condition, we conclude that

$$\|u - \tfrac{|x|^2}{2}\|_{L^\infty(B_R)} = o(R^2).$$

Therefore, we can take R_ϵ large so that

$$w_\epsilon(x) := \frac{1}{R_\epsilon^2} u(R_\epsilon x) - \frac{1}{2}|x|^2 \tag{6.3}$$

satisfies

$$\|w_\epsilon\|_{L^\infty(B_1)} = \frac{1}{R_\epsilon^2}\|u - \frac{1}{2}|x|^2\|_{L^\infty(B_{R_\epsilon})} \leq \mu_\epsilon. \tag{6.4}$$

Then by apply Theorem 1.4 to (w, F_1, δ), we conclude that

$$\|D^2u(0) - I\| = \|D^2w(0)\| \leq \epsilon. \tag{6.5}$$

The conclusion follows by taking ϵ to zero. □

6.1.2 Proof of Corollary 1.6

Proof of Corollary 1.6. Clearly, the α-Hölder modulus of continuity satisfies

$$\int_0^1 -\frac{Cr^\alpha \log r}{r} < \infty. \tag{6.6}$$

Given $x_0 \in B_{1/3}$ and $\epsilon > 0$, we will show

$$\|D^2u_k(x_0) - D^2\varphi(x_0)\| \leq \epsilon' := \min\{\epsilon, \frac{1}{2}\|D^2\varphi\|_{L^\infty(B_{1/3})} \min_{x\in\overline{B}_{1/3}} \mathrm{dist}(D^2\varphi(x), \partial\mathcal{P})\} \tag{6.7}$$

for all k sufficiently large. $C^{2,\alpha}$-convergence then follows from [DZZ].

Let h be a poly-harmonic quadratic polynomial such that

$$D^2\varphi(x_0) = \mathfrak{p}(D^2\varphi(x_0)) + D^2h. \tag{6.8}$$

By replacing u_k, φ by $u_k - h, \varphi - h$, we may assume

$$\mathfrak{p}(D^2\varphi(x_0)) = D^2\varphi(x_0) \tag{6.9}$$

It follows that we can take a linear transformation L such that

$$L(0) = x_0, \quad D^2\varphi \circ L(0) = I, \quad [L, J] = 0. \tag{6.10}$$

Let $w_k : B_1 \to \mathbb{R}$ be defined by

$$w_k(x) := \lambda_k r^{-2}\left(u \circ L(rx) - \frac{|x|^2}{2}\right) \tag{6.11}$$

where r is a small constant that will be specified later and

$$\lambda_k = f_k^{-1/n}(x_0). \tag{6.12}$$

w_k satisfies the equation

$$F_{\lambda_k}(D^2 w) = \lambda_k^{-1/n} f_k \circ L - 1 \in C^\infty \tag{6.13}$$

Since $f_k \circ L$ converges uniformly to $f \circ L$, for sufficiently large k

$$\lambda_k \in (\frac{3}{4}, \frac{4}{3}) \tag{6.14}$$

Therefore, for all k sufficiently large, $F_{\lambda_k} \in \mathcal{F}_{\epsilon'/2,\theta,K}$ where θ, K only depends on φ.

Now, we can apply Theorem 5.2 to obtain a small constant μ such that if

$$\|w_k\|_{L^\infty(B_1)} \leq \mu \tag{6.15}$$

then

$$\|D^2 w_k(0)\| \leq \epsilon'/2. \tag{6.16}$$

On the other hand, by the definition of w_k, we have

$$\|w_k\|_{B_1} \leq \frac{1}{\lambda_k r^2}\|u_k - \varphi\|_{L^\infty(B_r(x_0))} + \|D^2\varphi\|_{C^\alpha(B_{1/3})} r^\alpha. \tag{6.17}$$

Now, we first take

$$r^\alpha < \frac{\mu}{2\|D^2\varphi\|_{C^\alpha(B_{1/3})}} \tag{6.18}$$

then take k large so that

$$\frac{1}{\lambda_k r^2}\|u_k - \varphi\|_{L^\infty(B_r(x_0))} < \mu/2. \tag{6.19}$$

In this way, we conclude that w_k satisfies (6.15) for all k large. (6.7) then follows immediately. $\square$

6.2 Proof of Theorem 1.7

Theorem 1.7 follows immediately from Theorem 5.2 and the following proposition.

Lemma 6.2. *Let $f \in C^0(B_1)$ and $p > n(n-1)$. If $u \in W^{2,p}$ is a viscosity solution of*

$$\mathcal{M}[u] = f, \text{ in } B_1, \tag{6.20}$$

then for every $\mu > 0$, exists ϵ_μ depending on $\mu, \|u\|_{L^\infty}, n$ such that, If $|f-1|_{B_1} \leq \epsilon_\mu$, then there exists a universal constant r_0, C_0 and a quadratic polynomial P_0 such that

$$\begin{aligned} &\|u - P_0\|_{B_{r_0}} \leq \mu r_0^2 \\ &\mathcal{M}[P_0] = 1, \|D^2 P_0\| \leq C_0. \end{aligned} \tag{6.21}$$

Proof. Suppose the statement is false, that is, there exists a number $\mu > 0$ and sequences

$$\epsilon_k \to 0, u_k \in W^{2,p}, f_k \in C^0(B_1) \tag{6.22}$$

such that

$$\|u_k\|_{W^{2,p}} \leq \Lambda, \|u_k\|_{L^\infty} \leq A, \|f_k - 1\|_{L^\infty(B_1)} \leq \epsilon_k, \tag{6.23}$$

but no required P_k exists.

Since $\|\Delta u_k\|_{L^p} \leq \Lambda$, modulo subsequences, u_k converges uniformly to some $u \in C^0(B_1)$. By standard harmonic analysis

$$u \in W^{2,p}, \quad \|u\|_{W^{2,p}(B_1)} \leq \Lambda. \tag{6.24}$$

Since

$$M[u_k] = f_k, \quad f_k \to 1 \text{ uniformly }, \tag{6.25}$$

we conclude that u satisfies

$$\mathcal{M}[u] = 1 \text{ in } B_1 \tag{6.26}$$

By Thm.4.2, there exists a polynomial P_0 such that $\mathcal{M}[P_0] = 1$, $\|D^2 P_0\| \leq C_0$ and

$$\|u - P_0\|_{L^\infty(B_r)} \leq C r^3, \forall r < 1/2 \tag{6.27}$$

where C_0, C are universal constants.

Fix r_0 small so that

$$Cr_0 = \mu_0/2, \tag{6.28}$$

then fix k large so that

$$\|u_k - u\|_{L^\infty(B_{r_0})} \le \mu_0 r_0^2/2. \tag{6.29}$$

We then find that

$$\|u_k - P_0\|_{L^\infty(B_{r_0})} \le \mu_0 r_0^2. \tag{6.30}$$

This leads to a contradiction. □

Proof of Theorem 1.7. Fix $p > n(n-1)$, let C_1 be a universal constant such that if u satisfies

$$\begin{cases} \mathcal{M}[u] = 1 & \text{in } B_1 \\ \|\Delta u\|_{L^p(B_1)} \le \Lambda \end{cases} \tag{6.31}$$

then

$$\|D^2 u\|_{L^\infty(B_{1/2})} \le C_1 \tag{6.32}$$

By Theorem 4.2, such a constants exists.

Let μ be the constant produced by Theorem 1.4 with respect to

$$\delta = \frac{1}{2C^{n-1}}. \tag{6.33}$$

Take r_1 according to ϵ_μ such that

$$\tilde{f}(x) := f(r_1 x) \quad \text{satisfies } \left|\tilde{f}(x) - 1\right|_{B_1} \le \epsilon_\mu. \tag{6.34}$$

Let

$$\tilde{u}(x) := \frac{1}{r_1^2} u(r_1 x), \quad x \in B_1, \tag{6.35}$$

then $\tilde{u}$ satisfies

$$\mathcal{M}[\tilde{u}] = \tilde{f} \text{ in } B_1. \tag{6.36}$$

By Lemma (6.2), we conclude that there exists universal P_0, r_0 such that

$$\|\tilde{u} - P_0\|_{L^\infty(B_{r_0})} \leq \mu_0 r_0^2, \quad \mathcal{M}[P_0] = 1 \tag{6.37}$$

Let

$$\hat{u} := \frac{1}{r_0^2}\tilde{u}(r_0 x), \quad \hat{P}_0 := \frac{1}{r_0^2} P_0(r_0 x) \tag{6.38}$$

Then

$$\mathcal{M}[\hat{P}_0] = 1 \text{ and } \left|\hat{u} - \hat{P}_0\right|_{B_1} \leq \mu_0 \tag{6.39}$$

Moreover

$$\mathcal{M}[\hat{u}](x) = M[\tilde{u}](r_0 x) = \tilde{f}(r_0 x) := \hat{f}(x), \quad x \in B_1. \tag{6.40}$$

Now, by an argument that is similar to the one given in the proof of Corollary 1.6, we conclude that

$$\|D^2 u - D^2 P\|_{L^\infty(B_{r_0 r_1/2})} = \|D^2 \hat{u} - D^2 \hat{P}\|_{L^\infty(B_{1/2})} \leq \delta. \tag{6.41}$$

Higher regularity of u follows from [DZZ]. □

Bibliography

[Aub76] Aubin. Equations du type Monge-Ampère sur les variétés kähleriennes compacts. *Bull. Sc. Math.*, 120:119–121, 1976.

[BD11] Z. Blocki and S. Dinew. A local regularity of the complex Monge-Ampère equation. *Math. Ann.*, 351(2):411–416, 2011.

[Blo97] Z. Blocki. Onthe regularity of the complex MongeAmpère operator. in:Complex Geometric Analysis in Pohang. *Contemp. Math.*, 222(1):181–189, 1997.

[Blo09] Z. Blocki. A gradient estimate in the calabi-yau theorem. *Math. Ann.*, 344(2):317–327, 2009.

[BT76] E Bedford and B.A. Taylor. The dirichlet problem for a complex Monge-Ampère equation. *journal*, 82(1):102–104, 1976.

[BT82] E Bedford and B.A.. Taylor. A new capacity for plurisubharmonic functions. *Acta Math.*, 149:1–40, 1982.

[CC95] L.A. Caffarelli and X. Cabré. *Fully nonlinear elliptic equations.*, volume 43 of *Amer. math. Soc. Colloq. Publ.* American Mathematical Society, 1995.

[Ceg04] U. Cegrell. The general definition of the complex Monge-Ampère operator. *Ann. Inst. Fourier (Grenoble)*, 54:159–179, 2004.

[CIL92] M.G. Crandall, H. Ishii, and P. Lions. User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc.*, 27(1):1–67, 1992.

[CKNS85] L. Caffarelli, J.J. Kohn, L. Nirenberg, and J. Spruck. The Dirichlet problem for nonlinear second-order elliptic equations. II. Complex Monge-Ampère, and uniformly elliptic, equations. *Comm. Pure Appl. Math.*, 38(2):209–252, 1985.

[Dem98] J.-P. Demailly. Regularization of closed positive currents and intersection theory. *Acta Math*, 180:69–117, 1998.

[DZZ] S. Dinew, X. Zhang, and X.-w. Zhang. The $C^{2,\alpha}$-estimate of complex Monge-Ampère equation. *Indiana Univ. Math. J.*

[EG92] L.C. Evans and R.F. Gariepy. *Measure theory and fine properties of functions.* Studies in Advanced Mathematics. CRC Press, 1992.

[EGZ11] P. Eyssidieux, V Guedj, and A. Zeriahi. Viscosity solutions to degenerate complex Monge-Ampère equations. *Comm. Pure Appl. Math.*, 64(8):1059–1094, 2011.

[GL10] B. Guan and Q. Li. Complex Monge-Ampére equations and totally real submanifolds. *Adv. Math.*, 225(3):1185–1223, 2010.

[Gua98] B. Guan. The Dirichlet problem for Monge-Ampère equation and regularity of the pluri-complex green function. *Comm. Anal. Geom.*, 6(4):687–703, 1998.

[Kol98] S. Kolodziej. The complex Monge-Ampère equation. *Acta Math*, 180:69–117, 1998.

[Kol05] S. Kolodziej. *the complex Monge-Ampère equation and the pluripotential theory.*, volume 178 of *Mem. Amer. Math. Soc.* American Mathematical Society, 2005.

[Lem83] L. Lempert. Solving the degenerate Monge-Ampère equation with one concentrated singularity. *Math. Ann.*, 263:515–532, 1983.

[PS09] D.H. Phong and J. Sturm. On pointwise gradient estimates for the complex Monge-Ampère equation. *preprint*, 2009.

[Sav07] O. Savin. Small perturbation solutions for elliptic equations. *Comm. Partial Differential Equations*, 32(4-6):557–578, 2007.

[Tru80] N.S. Trudinger. Local estimates for subsolutions and supersolutions of general second order elliptic quasilinear equations. *Invent. Math.*, 61:67–79, 1980.

[Yau78] S.-T. Yau. On the ricci curvature of a compact khler manifold and the complex Monge-Ampère equation. I. *Comm. Pure Appl. Math.*, 31(3):339–411, 1978.