Three Essays on Dynamic Pricing and Resource Allocation

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ABSTRACT

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This thesis consists of three essays that focus on different aspects of pricing and resource allocation. We use techniques from supply chain and revenue management, scenario-based robust optimization and game theory to study the behavior of firms in different competitive and non-competitive settings. We develop dynamic programming models that account for pricing and resource allocation decisions of firms in such settings.

In Chapter 2, we focus on the resource allocation problem of a service firm, particularly a health-care facility. We formulate a general model that is applicable to various resource allocation problems of a hospital. To this end, we consider a system with multiple customer classes that display different reactions to delays in service. By adopting a dynamic-programming approach, we show that the optimal policy is not simple but exhibits desirable monotonicity properties. Furthermore, we propose a simple threshold heuristic policy that

performs well in our experiments. In Chapter 3, we study a dynamic pricing problem for a monopolist seller that operates in a setting where buyers have market power, and where each potential sale takes the form of a bilateral negotiation. We review the dynamic programming formulation of the negotiation problem, and propose a simple and tractable deterministic "fluid" analogue for this problem. The main emphasis of the chapter is in expanding the formulation to the dynamic setting where both the buyer and seller have limited prior information on their counterparty valuation and their negotiation skill. In Chapter 4, we consider the revenue maximization problem of a seller who operates in a market where there are two types of customers; namely the "investors" and "regular-buyers". In a two-period setting, we model and solve the pricing game between the seller and the investors in the latter period, and based on the solution of this game, we analyze the revenue maximization problem of the seller in the former period. Moreover, we study the effects on the the total system profits when the seller and the investors cooperate through a contracting mechanism rather than competing with each other; and explore the contracting opportunities that lead to higher profits for both agents.

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I consider myself as a very lucky person: I was born into a wonderful family, was endowed with "sufficient" amount of beauty, intelligence and personal traits, but most importantly, have always met wonderful people all my life. In fact, implicating the "valuable" people in my life has always been my motto: People who influenced me, people who inspired me. This note is a tribute to all of them.

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To my mother, To my father

Chapter 1

Introduction

This thesis consists of three essays that utilize various methods of dynamic optimization, revenue management and pricing literature, and game theory to solve problems in the area of pricing and resource allocation. We use techniques from dynamic programming, scenario-based robust optimization and game theory to derive optimal pricing and resource allocation polices.

Allocation of resources among various customer groups or customers arriving at various time points along the sales horizon has been a fundamental problem of production and service firms throughout the history. This problem has elicited interest from various researchers from different fields, and even led to "yield management" to emerge as an independent research field. The revenue loss stemming from unwise pricing practices and inefficient resource allocation schemes can have substantial effects on the profitability of the firms, as well as the social welfare of the society. Hence, many researchers and practitioners have developed sophisticated models of revenue management and resource allocation to address this problem.

Despite the fact that the airline industry was the very first industry which took advantage of dynamic pricing and revenue management techniques, recently many other business sectors are willing to invest in and investigate the potential benefits of revenue management and dynamic pricing. In this thesis also, we start with the application of revenue and supply chain management techniques in a non-conventional area, namely “health care”. To this end, we first focus on the resource allocation problem of a service firm, particularly a health-care facility in Chapter 2. We formulate a general model that is applicable to various resource allocation problems of a hospital. We consider a system with multiple customer classes that display different reactions to the delays in service. By adopting a dynamic-programming approach, we show that the optimal policy is not simple but exhibits desirable monotonicity properties. Furthermore, we propose a simple threshold heuristic policy that performs well in our experiments. Finally, we conclude the chapter by discussing various extensions of the model to extend the applicability of the results across several real-life problems.

Another non-conventional area for the application of pricing and revenue management techniques is the “real-estate”. In Chapter 3, motivated by the revenue maximization problem of a real estate developer, we turn our attention to finding the best dynamic pricing strategy for a monopolist seller that operates in a setting where buyers have market power and each potential sale takes the form of a bilateral negotiation. This problem is again connected to the problem of the previous chapter, especially considering the fact that the revenue management of a monopolist seller that operates in a setting where buyers have market power is essentially a capacity allocation problem. In this setting, buyers arrive sequentially over time and negotiate separately with the seller to purchase one unit of the offered good. The outcome of each such negotiation depends on the valuations of the seller and the buyer for that good, their relative negotiation power, as well as their beliefs for the other party's valuation. We review the dynamic negotiation problem, and propose a simple and tractable deterministic "fluid" analogue for this problem. The main emphasis of the chapter is in expanding the above formulation to the case where both the buyer and seller have limited prior information on their counterparty valuation and their negotiation skill, and mainly analyze the sales process in a dynamic setting. Our first result shows that if both the seller and buyer are bidding so as to minimize their maximum regret over possible counterparty valuation distributions, then it is optimal for them to bid as if the unknown valuation distributions were uniform. Building on this result and the fluid formulation of the dynamic negotiation problem, we characterize the seller's optimal reserve price, i.e., the minimum price that she should be willing to accept for one unit of the good at any given point in time. Finally, we expand on the above ideas to formulate and study the seller's problem in the case where the primitives of the buyer valuation distributions are unknown and non-stationary using ideas from scenario-based robust optimization. Despite the fact that the motivating application is from residential real-estate, the model and proposed approach are generally applicable. This analysis forms and completes Chapter 3.

Finally, motivated by our work in Chapter 3, we consider the problem of a real estate developer from a different angle: In Chapter 3, we were mainly concerned with "naive" buyers who do not strategize over purchase decision or invest in the real estate with the intention of obtaining profit from their investment. However, in practice, despite being a rather non-liquid investment instrument, real estate investment is one of the items investors include in their portfolio: It is common for investors to own multiple pieces of real estate, one of which serves as a primary residence, while the others are used to generate rental income and profits through price appreciation. Hence, in Chapter 4, we consider the revenue maximization problem of a seller who operates in a market where there are two types of customers; namely the “investors” and “regular-buyers”. The regular buyers are similar to the naive buyers of the previous chapter; however the investors purchase the units to resell them later, thus creating a competition against the seller in the latter period of the sales horizon. In a setting that is comprised of two sales periods, we first model and solve the pricing game between the seller and the investors in the latter period, and based on the solution of this problem, we formulate the revenue maximization problem of the seller in the former period. Moreover, we analyze how the total system profits increase when the seller and the investors cooperate through a contracting mechanism rather than competing with each other. Again, the problem takes its roots from the real estate industry, however the results are generally applicable in any duopoly setting with non-flexible capacities.

Chapter 2

Hospital Resource Allocation Problem

2.1 Introduction

In this chapter, we study the problem of dynamically allocating a single resource of fixed capacity to several customer streams. The demand of some customer types can be fully backlogged whereas other demand will be lost if not fulfilled immediately upon arrival. While this kind of a situation arises in many industries, our motivation to study this problem comes from the existence of various patient types in a health-care facility. Some of these patients are of critical condition, or may require immediate attention. Other patients may not require immediate treatment, but the monetary benefits they would bring may be higher than that of the first type of patients. An example is the allocation of operating rooms to emergency and elective surgical operations.

If both cases arrive at the hospital at the same time, the humanitarian (and reputational) concerns favor the admission of the emergency case first. But when the number of emergency patients to arrive during the course of the day is unknown, how a manager should plan the operating room utilization schedule remains to be an unanswered question. In short, in an environment with rising costs and increasing competition, the managers of a health-care facility need to contend with humanitarian versus monetary conflicts, and thus, have to address the tradeoffs arising from this kind of a resource allocation problem. The quality of the provided health care and the monetary aspects of the problem are often interwoven, yielding an even more delicate situation that needs to be handled with utmost care.

In many health-care facilities, the resource allocation problem is considered as being too complicated, and in general, some rule-of-thumb approaches are employed. For instance, in many institutions, a senior floor nurse is entrusted with the process of allocating hospital beds to the patients waiting in the system, the rationale being that this person has the required experience and the knowledge to judge the urgency of the cases. However, this kind of inexact approaches could lead to substantial inefficiencies in terms of social benefits and/or monetary gains to be obtained had an analytical solution methodology been employed (e.g. see Patrick and Puterman (2007) for an example).

Nevertheless, there exist several papers in the literature that analyze the resource allocation problem of a health-care facility using various techniques ranging from queuing to simulation models, optimization, and dynamic programming formulations. For instance Gerchak et al. (1996) focus on the operating rooms' capacity utilization; Green et al. (2006) deal with the effective utilization problem of an MRI center; and starting with Young (1962), a number of authors focus on the allocation of hospital beds to various patient classes. Instead of focusing on the details of some specific resource within an hospital, our work provides an abstract and more general framework for understanding different behavior patterns of various patient classes in terms of their reaction to delays in the service. Our intention is to look for an overarching guideline and intuition that can be useful for any specific resource allocation problem. To achieve this goal, we use a dynamic programming approach which is common in the inventory management literature. We observe that, under certain modeling assumptions, there are similarities between the resource allocation problem of a hospital and the capacity allocation problem of a manufacturing firm that serves multiple types of customers.

Throughout the article, we will focus on the problem of managing the admissions of stochastically arriving patients in an hospital and build our model upon this terminology. However, the reader should note that the derived insights are applicable to various resource management problems of service firms with multiple customer types displaying different reactions to possible delays in service (i.e., lost sales versus backorders).

This chapter is organized as follows. Section 2.2 describes the related literature, which spans revenue management, supply chain and inventory management, as well as several methodologies used in the context of resource allocation in health-care facilities. In section 2.3, we present our model and a dynamic programming formulation. We then prove that the optimal policy has desirable monotonicity properties. While the optimal policy is not simple, we propose a simple threshold-type policy that performs well in our numerical experiments. In section 2.4, we consider several extensions of the model that include features that arise in practice. Finally, in section 2.5, we summarize our work and present avenues for future research.

2.2 Literature Review

Since our work is in the context of health care, we first review the existing literature on the resource allocation problem of a health-care facility. Then, we will review relevant papers on general production-service systems that face demands from multiple customer types.

Health-Care Management Literature. The patient admission problem of a health-care facility has been studied by several researchers, mostly using the tools of simulation or queuing theory. Most of the papers to our knowledge focus on deriving the best "cut-off"-type of policy – which allows for the admission of the 'less serious' patients after a critical number of beds are reserved for the 'critical' patients whose arrival process is stochastic. The classic work of Young (1962) is the first to represent the hospital admissions scheduling as a queuing model. Kolesar (1970) uses Markovian decision models; while Esogbue and Singh (1976) shed more light onto the problem by finding the optimal threshold levels under a linear cost structure. Huang (1995) is able to come up with the number of beds that are required for different days of the week by using a Monte-Carlo simulation model.

Along with queuing models, there are other mathematical modeling approaches to address resource allocation in a health-care facility. A number of decision support systems have been developed with the purpose of helping hospital managers in the bed-allocation decision. The work of McClean and Millard (1995), and that of Mackay (2001) depicting the implementation process of two decision support systems in the South Australian public hospital system are two examples of this line of research. Although the results depend heavily on the distribution of the underlying data or system stability, these methods are proven to be useful in resource allocation management. Regarding other mathematical modeling approaches, Dantzig (1969) is first to develop a scheduling system using a linear programming formulation under deterministic parameters. His work represents the first attempt to use an objective function that explicitly incorporates certain cost elements; in particular, the penalty cost. Later, Harper and Shahani (2002) develop a detailed simulation model in the light of bed occupancies and refusal rate, and Ruth (1981) uses a mixed integer programming formulation to match the demand with the hospital services. For a comprehensive analysis and the summary of the research on this topic, we refer the reader to Milsum et al. (1973) and Smith-Daniels et al. (1988)

Most of these models, however, suffer from strong assumptions that restrict their use. The queuing models assume that the service time (the occupancy time of a bed) and the interarrival times are exponentially distributed, which may not always hold in practice. In fact, Young (1962) tested the use of his queuing model against the results of a simulation model and found that significantly different results were obtained by the two techniques. Similarly, the steady-state Markovian models are criticized regarding their attempt to apply a Markov decision model to a problem that is essentially non-Markovian, which amounts to an oversimplification of the system dynamics. Other models also suffer from oversimplification issues such as the deterministic assumptions (e.g. Dantzig (1969)). In this paper, we adopt a dynamic programming approach which allows modeling the non-stationary stochastic arrival pattern of patients.

Production/Service Systems. Both queuing and dynamic programming models have been widely used in inventory, service and other related areas involving resource allocation problems. Our problem shares some similarities with the problem of selling a single product to multiple customer classes under uncertain arrivals. The early work of Topkis (1968) considers the rationing of inventory to demand from multiple customer classes and shows that a base-stock policy is optimal; our setting differs from this and other inventory models since the amount of capacity available at a health-care facility in a period is fixed and cannot be stored. Duenyas (2000) formulates the problem as a semi-Markov decision process and examines the "due date setting policy" – his work is grounded on the assumption that the manufacturer can sequence the orders in any desired manner, which is clearly not applicable in our context due to the ethical and legal issue of patient rights. Carr and Duenyas (2000) address the admission control and sequencing in a production system with two product classes, and they use a simple two-class M/M/1 queue and come up with optimal switching curves. Maglaras and Zeevi (2005) also use a queueing framework with stationary demand to model two types of demand – these two types are distinguished based on whether the customer receives guaranteed service or "best-effort" service (served only when the server is not too busy), similar to our distinction of customer types.

An interesting work of Carr and Lovejoy (2000) focuses on determining the optimal portfolio of multiple customer segments for a capacitated firm; in this model, however, the demand should remain stationary once the portfolio is calculated.

Gupta and Wang (2006) consider a similar problem of allocating production capacity between two classes of demand, and show the optimality of a policy based on critical numbers. Other important papers involving a single-item, make-to-stock production system with multiple demand classes are papers of Ha (1997a,b) and the work of Sobel and Zhang (2001). Finally, Ding et al. (2006) analyze the tactical problem of allocating inventory to several customer classes when partial backlogging is possible, and they maximize revenue by dynamic pricing through customer discounts where the probability of a denied customer to remain in the system is based on the discount offered to her. This dynamic pricing approach forms the principal difference of their work from ours, as tweaking with prices is not readily acceptable in health-care settings.

In all of the papers mentioned above (except for that of Ding et al. (2006)), the unsatisfied demand is either entirely backlogged or completely lost, but not both. Even though in practice it is quite reasonable that some unsatisfied demand can be backlogged while other demand is lost, there exist only a few papers in the inventory literature that model multiple demand classes of customers based on the stock-out behavior.

Of these, Duran et al. (2008) consider a system with two demand classes, where demand coming from one of these classes is immediately lost if unfulfilled; and the unsatisfied demand of the other class is backlogged for one period. Tang et al. (2007) address a two-class model where higher priority is given to the backorder demand, and the lost-sales demand class is served later. Both papers show the optimality of base-stock, or modified base-stock policies. The paper of Zhou and Zhao (2010) differs from the previous two by involving the assumption that previous backorders can be satisfied in any future period (not necessarily in the immediate next period), and showing that in that setting base-stock policies may no longer be optimal. The main result of this paper is to show that the optimal policy satisfies some monotonicity properties. The main difference of our work from the mentioned papers lies in the different characteristics of inventory and service settings – the manager of an inventory system has the flexibility to decide on the number of units to be ordered in each period and, furthermore, these units can be stored for future use, but in our system only a fixed amount of capacity becomes available in each decision epoch. Hence, our work speaks to one of the basic questions in revenue management: how to make the best use of a limited capacity through allocation.

In summary, we use dynamic programming formulation, as common in the inventory-related literature, to study the dynamic capacity allocation problem in a service context when multiple types of customers are present and the underlying demand pattern may not be stationary. Thus, our model is related to two bodies of literature: the resource allocation problem of the service industries (particularly, the health-care facilities) and the inventory control problem of manufacturing systems with both lost-sales and backorders. In this work, we uncover insights into how the optimal policy behaves, and propose a simple heuristic policy that performs well.

2.3 Basic Model

We consider a system with two patient groups, whom we will refer to as type 1 and type 2 patients. These patient types are distinguished by their behavior upon arrival at the hospital. Type 1 patients wait in the system until they receive service while type 2 patients leave the system (i.e., are "lost") if they cannot be accommodated immediately upon arrival. The modeling of type 1 and type 2 patients is motivated by *elective* surgery patients and *emergency* patients. The elective surgery patients require a surgical operation that is not urgent and they are willing to wait in the system until the necessary resources become available for their treatment (e.g., certain types of plastic surgery patients). By contrast, emergency patients arrive at the hospital in critical condition, and they should either be admitted immediately or be sent to another facility in close proximity.

If a type 2 patient cannot be accommodated immediately, this patient is lost to the system incurring a goodwill penalty of $w_2 > 0$. This penalty cost represents the possibly worsening condition of the patient during his transfer to a nearby facility or damage to the hospital's reputation. Meanwhile, a type 1 patient will wait in the system until she is treated while incurring a goodwill penalty $w_1 > 0$ per unit time she spends awaiting treatment. This cost represents the increasing anxiety of the patient while being enrolled in the waiting list. We denote the expected per-patient revenue from a type 1 patient and a type 2 patient by $r_1 > 0$ and $r_2 > 0$, respectively. Let C denote the capacity of the hospital per day, which is to be allotted to two patient groups. We assume that each patient (type 1 or type 2) consumes exactly one unit of capacity in the time period that she is admitted for treatment. This assumption, which decouples the admission process from the treatment stage by allowing that the system starts fresh with C units of capacity at each new period, may not always hold in practice; but serves as a reasonable approximation for hospital wards for which the length of stay is sufficiently short and does not vary much; or for some specific resources of the hospital (for example, interpret C as the capacity of a certain test required for all newly-admitted patients). In this work, we do not explicitly model the length-of-stay distributions in order to avoid that available capacity depends on the history of patient arrivals; the queuing-based framework would model the length-of-stay feature more appropriately. However, our model and findings can be extended to the case of stochastic and non-identical capacity (section 2.4), which can approximately account for the effect of a patient occupying one unit of resource for several periods.

In each period, we assume that the following sequence of events takes place: At the beginning of each period (for instance, a day) t, the hospital management observes the number of backlogged patients s_t, and based on this observation, decides how many backlogged type 1 patients to admit in the current period (while the rest of the unaquitted type 1 patients remain backlogged). (In section 2.4, we allow the manager to reject some type 1 patients.) The remaining capacity will be protected for type 2 patients who arrive throughout the day. All type 1 arrivals during the course of the day will be placed on the backlogged patients list. We assume that the number of type 1 patient arrivals on any given period t can be approximated by a random variable M_t with a continuous and differentiable cumulative density function (cdf) H_t^1 with the probability density function (pdf) h_t^1, and the distributions are independent but not necessarily identical across periods over the planning horizon. Similarly, type 2 arrivals in each period t is represented by a random variable D_t with cdf H_t^2 and pdf h_t^2, and the distributions of $\{D_1, D_2, \ldots\}$ are independent. We assume that both M_t and D_t have bounded support.

In each period t, let $s_t \geq 0$ denote the number of backlogged type 1 patients at the beginning of the period, and let $x_t \geq 0$ represent the amount of capacity protected for type 2 arrivals, which is the decision made by the hospital management. Then, the expected net revenue to be obtained in period t will be given by:

$$L(s_t, x_t) = r_2 \cdot \mathbf{E}\min\{x_t, D_t\} + r_1 \cdot \mathbf{E}[M_t] - w_1 \cdot (s_t + x_t - C)^+ - w_2 \cdot \mathbf{E}[D_t - x_t]^+ \quad (2.1)$$

And the number of backlogged patients is updated by $s_{t+1} = (s_t - C + x_t)^+ + M_t$ at the end of the period.

Note that in the above expression, the revenue from type 1 patients is collected at the time of their arrival (which is given by the term $r_1 \cdot \mathbf{E}[M_t]$) and the penalty of backlog incurs only after they wait in the system for one period and are still not admitted (which is given by $-w_1 \cdot (s_t + x_t - C)^+$). If the revenue is instead collected at the time of service, we can account for this change by adjusting the value of w_1 to incorporate the time value of revenue. Finally, the term $r_2 \cdot \mathbf{E}\min\{x_t, D_t\}$ represents the revenue collected from type 2 patients arriving during the course of the day, and $-w_2 \cdot \mathbf{E}[D_t - x_t]^+$ is the penalty associated with type 2 patients who cannot be accommodated.

2.3.1 The Structure of the Optimal Policy

The following property of the single-period net revenue is useful in establishing the structural characteristics for the optimal allocation policy. Its proof follows easily from well-known properties of submodularity and concavity.

Lemma 1. $L(s_t, x_t)$ *is submodular and jointly concave in its components.*

We consider a planning horizon of T periods. Let $\alpha \in [0, 1]$ denote the discount factor. The Bellman equation for maximizing total net revenue can be formulated as follows: for $1 \le t \le T$, define

$$f_t(s_t) \;=\; \max_{0 \le x_t \le C} \big[L(s_t, x_t) + \alpha \cdot \mathbf{E}_{M_t}[f_{t+1}((s_t + x_t - C)^+ + M_t)]\big] \;, \quad (2.2)$$

where the terminal condition is given by

$$f_{T+1}(s_{T+1}) \;=\; v \cdot s_{T+1} \;,$$

where $v \le 0$ is some fixed constant.

Let $x_t^*(s_t)$ denote the optimal amount of capacity to reserve for type 2 patients when there are s_t backlogged type 1 patients in the system. If we give priority the to backlogged type 1 patients over newly arriving type 2 patients, then the amount of capacity available for type 2 patients would be $\max\{C - s_t, 0\}$; thus it is easy to see that $C - s_t$ is a lower bound on $x_t^*(s_t)$, i.e.,

$$s_t + x_t^*(s_t) \;\ge\; C \;. \quad (2.3)$$

We are interested in establishing structural properties for the optimal allocation decision $x_t^*(\cdot)$, but it turns out that the optimal decision does not follow a simple form such as a threshold-type policy. However, we establish certain monotonicity properties for $x_t^*(s_t)$. The following result shows that the amount of capacity protected for type 2 arrivals is decreasing in the number of backlogged type 1 patients, while the magnitude of change is at most equivalent to the amount of variation in the number of backlogged patients.

Theorem 1. *For each period $t \in \{1, \ldots, T\}$, $x_t^*(s_t)$ satisfies the following properties:*

(i) $x_t^*(s_t)$ is decreasing in s_t, i.e., $x_t^*(s_t + \epsilon) \le x_t^*(s_t)$ for any $\epsilon > 0$.

(ii) $x_t^*(s_t + \epsilon) \ge x_t^*(s_t) - \epsilon$, for any $\epsilon > 0$.

Proof: See Appendix A.1. □

The proof of Theorem 1 consists of standard arguments and is based on the preservation of joint concavity in f_t functions, from which we find the optimal decision $x_t^*(s_t)$ based on the first-order condition (FOC). Such a partial characterization in Theorem 1 is often the best structural result that can be established in the inventory management and production planning models in the literature. This is consistent with Carr and Duenyas (2000) where the production threshold curve is monotonously decreasing in the number of type 2 orders in the production-queueing context, and also with Green et al. (2006) where the optimal capacity allocation policy belongs to the class of monotone "switching curve" policies in an appointment scheduling context. Such results are useful in motivating heuristic methods as well as in computing the optimal policy function $x_t^*(\cdot)$.

2.3.2 Protect-Constant Policies

Since the optimal policy $x_t^*(\cdot)$ may be difficult to find and to implement, we focus our attention to a simpler policy which we call the *protect-constant* or *protect-θ* policy, where $\theta \in [0, C]$ is a policy parameter. Under this policy, we protect the same amount of capacity θ in each period for the type 2 patients – unless there are more than θ units of capacity available after clearing the backlog, in which case we make all remaining capacity available to type 2 arrivals. Mathematically,

$$x_t(s_t) \quad = \quad \max\{\theta,\ C - s_t\}\ . \tag{2.4}$$

While this class of policies is clearly not optimal, it is easier to define and simpler to implement in practice compared to the optimal policy.

One possible method of selecting the parameter θ in the protect-constant policy is to select the expected number of type 2 patients in a period, i.e., $\theta = E[D_t]$. It is also possible to perform a single-dimensional search to look for the best value of θ within the class of the protect-constant policies. Let θ^* denote the optimal value of θ that maximizes the total net revenue, and we refer to the protect-θ^* policy as the *best protect-constant* policy.

We use numerical experiments to test the performance of the protect-constant policies. We consider a hospital setting where the daily arrival process of elective (type 1) patients is Poisson with rate $\lambda_1 = 8$ while that of the emergency (type 2) patients is Poisson with rate $\lambda_2 = 12$. As a base case, we set

$$r_1 = 5, \quad w_1 = 2, \quad r_2 = 4, \quad w_2 = 4, \quad C = 20 \quad \text{and} \quad T = 40 \ ,$$

but we vary various values in our experiments. Each problem is solved using 500 randomly generated instances assuming a discount factor of $\alpha = 0.99$ and $v = -r_1$ (i.e., if we cannot treat a patient by the end of the horizon, we forfeit the revenue associated with this patient). For each problem case, we have computed the optimal net revenue by computing the optimal policy (OP) using dynamic programming. We have also evaluated the performance of the protect-constant policy with $\theta = E[D_t] = 12$ (EPC) and the best protect-constant policy (BPC). The results are summarized in Tables 2.1-2.5 where t^* denotes the optimal protection level in the best protect constant policy. In each case, we report the *Relative Net Revenue* (RNR), which is the average net-profit of each policy divided by that of the optimal policy, and *Relative Standard Deviation* (RStD) which is the standard deviation of the net-profit divided by the average net-profit. (Thus, for the optimal policy, the RNR should be 100%. The values of RNR and RStD can be negative if the average net-profit is negative.) In Table 2.1, we vary the penalty cost w_2 for not being able to serve a type 2 patient among $\{3, 3.5, 4, 4.5, 5\}$. In Table 2.2, we vary the capacity C among $\{15, 18, 20, 25, 30\}$ and in Table 2.3, we vary the time horizon T among $\{10, 20, 30, 40, 50\}$. Similarly, in Tables 2.4 and 2.5, we vary the Poisson demand parameters for the type 1 and type 2 patient arrivals, respectively.

"Table 2.1 about here"

"Table 2.2 about here"

"Table 2.3 about here"

"Table 2.4 about here"

"Table 2.5 about here"

Table 2.1: Sensitivity Analysis with Respect to the Penalty Coefficient w_2.

	$w_2 = 3$	$w_2 = 3.5$	$w_2 = 4$	$w_2 = 4.5$	$w_2 = 5$
	RNR, PStD	RNR, PStD	RNR, PStD	RNR, PStD	RNR, PStD
OP	100%, 2.04%	100%, 2.14%	100%, 2.23%	100%, 2.32%	100%, 2.48%
EPC	83.69%, 15.32%	83.83%, 15.53%	83.91%, 15.74%	83.99%, 15.96%	84.09%, 16.19%
BPC	99.86%, 2.09% (t^*=8)	99.86%, 2.15% (t^*=8)	99.84%, 2.25% (t^*=8)	99.79%, 2.38% (t^*=8)	99.78%, 2.53% (t^*=9)

Table 2.2: Sensitivity Analysis with Respect to the Capacity C.

	$C = 15$	$C = 18$	$C = 20$	$C = 25$	$C = 30$
	RNR, PStD	RNR, PStD	RNR, PStD	RNR, PStD	RNR, PStD
OP	100%, 4.59%	100%, 2.66%	100%, 2.23%	100%, 3.14%	100%, 3.67%
EPC	−267.75% -15.09%	−7.52% -373.62%	83.91%, 15.74%	99.88%, 3.15%	100%, 3.67%
BPC	99.80%, 4.63% (t^*=1)	99.84%, 2.66% (t^*=5)	99.84%, 2.25% (t^*=8)	99.88%, 3.15% (t^*=12)	100%, 3.67% (t^*=13)

Table 2.3: Sensitivity Analysis with Respect to the Time Horizon T.

	$T = 10$	$T = 20$	$T = 30$	$T = 40$	$T = 50$
	RNR, PStD	RNR, PStD	RNR, PStD	RNR, PStD	RNR, PStD
OP	100%, 4.77%	100%, 3.25%	100%, 2.55%	100%, 2.23%	100%, 2.02%
EPC	94.29%, 7.58%	90.23%, 10.11%	86.95%, 12.86%	83.91%, 15.74%	81.29%, 18.35%
BPC	99.85%, 4.79% (t^*=8)	99.85%, 3.27% (t^*=8)	99.84%, 2.57% (t^*=8)	99.84%, 2.25% (t^*=8)	99.83%, 2.03% (t^*=8)

Table 2.4: Sensitivity Analysis with Respect to the Type-1 Patient Arrival Rate l_1

	$l_1 = 4$	$l_1 = 6$	$l_1 = 8$	$l_1 = 10$	$l_1 = 13$
	RNR, PStD	RNR, PStD	RNR, PStD	RNR, PStD	RNR, PStD
OP	100%, 3.26%	100%, 2.65%	100%, 2.23%	100%, 2.64%	100%, 3.19%
EPC	99.99%, 3.26%	99.51%, 2.67%	83.91%, 15.74%	5.45%, 511.16%	−141.02% -23.49%
BPC	99.99%, 3.26% (t^*=12)	99.87%, 2.66% (t^*=11)	99.84%, 2.25% (t^*=8)	99.84%, 2.63% (t^*=3)	99.78%, 3.25% (t^*=1)

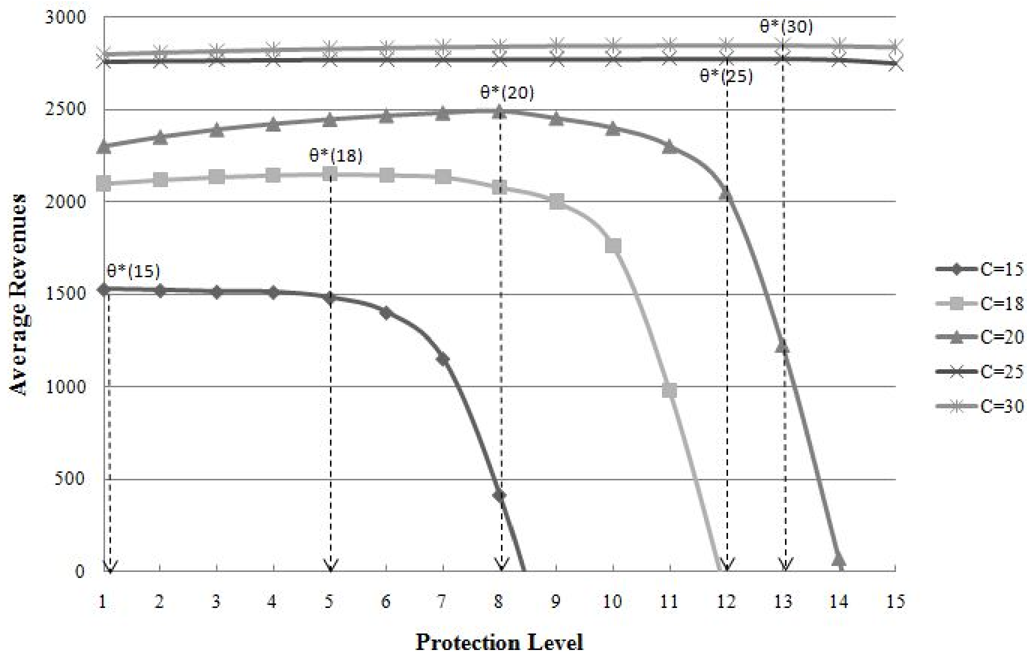
Table 2.5: Sensitivity Analysis with Respect to the Type-2 Patient Arrival Rate l_2.

	$l_2 = 6$	$l_2 = 9$	$l_2 = 12$	$l_2 = 15$
	RNR, PStD	RNR, PStD	RNR, PStD	RNR, PStD
OP	100%, 4.13%	100%, 2.76%	100%, 2.23%	100%, 2.96%
EPC	99.97%, 4.14%	99.81%, 2.78%	83.91%, 15.74%	−47.94% -53.23%
BPC	99.99%, 4.14% (t^*=7)	99.88%, 2.78% (t^*=8)	99.84%, 2.25% (t^*=8)	99.89%, 2.97% (t^*=1)

The numerical results suggest that the simple heuristic of protecting the expected number of type 2 patients does not perform well compared to the optimal policy. Using the optimal policy helps acquiring a significant benefit compared to using the simplest rule-of-thumb approach. The difficulty with the optimal policy, however, is that the dynamic programming computation may be cumbersome and may not be easy to implement. Restricting our attention to the class of protect-constant policies, we note that the best protect-constant policy performs very well under all circumstances. (The RNR values for the best protect-constant policy were 99.78% or higher.) Furthermore, in each case, the net revenue of the protect-θ policy appears to be quasi-concave in θ. Thus, the optimal value θ^* can easily be obtained using, for example, a bisection search method. See Figure 2.1 for an example of how the net revenue depends on the value of θ.

"Figure 2.1 about here"

Figure 2.1: Optimal Protection Levels for Each Capacity Level



2.4 Extensions

In this section, we consider how the analytical results of Section 2.3 is preserved when we introduce additional modeling features.

2.4.1 Time-Varying Stochastic Capacity

In Section 2.3, we have assumed that the capacity in each period is constant at C. However, this assumption may not be realistic in some hospital and service settings since some key resources become available according to a cyclic pattern, or some capacity units are still tied to the patients of previous periods and do not become available immediately. In this section, we consider the case where the capacity available in each period is random.

Let C_t denote a random variable representing the capacity in period t. We assume that the random variables $\{C_t | t = 1, \ldots, T\}$ are independent from each other but not necessarily identically distributed. At the beginning of each period t, the manager observes the realized capacity C_t. Thus, the amount of capacity to be reserved for type 2 patients should depend on C_t as well as s_t, which by $x_t^*(s_t, C_t)$.

For this model, we can show that the properties stated in Theorem 1 continue to hold. (Proof in Appendix A.2.) Moreover, $x_t^*(s_t, C_t)$ satisfies another interesting monotonicity property that, for any $\epsilon \geq 0$,

$$0 \;\leq\; x_t^*(s_t, C_t + \epsilon) - x_t^*(s_t, C_t) \;\leq\; \epsilon \;. \tag{2.5}$$

This result shows that the protection quantity is an increasing function of the capacity, but its sensitivity to the capacity availability is limited.

An interesting generalization of this extension is to allow the available capacity to depend on the number of patients admitted in previous periods. This enables us to involve patients who may require a resource for multiple periods in the model. While it is interesting, we are unable to show the properties of Theorem 1 in this case due to difficulties as the curse of dimensionality, and leave it for future research. However, the results of this section show that the properties stated in Theorem 1 are quite robust, and furthermore, a similar monotonicity property of x_t^* exists with respect to the available capacity C_t, as shown in (2.5).

2.4.2 Rejecting Type 1 Patients

In this section, we consider the possibility of rejecting type 1 patients. We now allow that when a patient of type 1 arrives at the system, the manager may turn her away to ensure high quality of service for those already admitted to the hospital.

We modify the model by introducing another decision at the the end of period t, which determines how many of the new M_t type 1 patients are to be accepted into the system. Note that, at the end of period t, the number of type 1 patients who have arrived in period $t-1$ or earlier and are backlogged equals $(s_t + x_t - C)^+$. Let $a_t \in [0, M_t]$ denote the number of type 1 patients from period t that will join the waiting list. In Section 2.3, we had $a_t = M_t$, but we now modify (2.2) to reflect this decision:

$$\begin{aligned} f_t(s_t) &= \max_{0 \le x_t \le C} \Bigg[L(s_t, x_t) \\ &+ \alpha \cdot \mathbf{E}_{M_t} \left\{ \max_{0 \le a_t \le M_t} -r_1 \cdot (M_t - a_t) + f_{t+1}((s_t + x_t - C)^+ + a_t) \right\} \Bigg] . \end{aligned} \tag{2.6}$$

(Above, the term $-r_1 \cdot (M_t - a_t)$ represents the amount of lost revenue associated with accepting a_t patients only – see the definition of L in (2.1).)

We remark that the value of a_t is chosen at the end of a period after M_t is realized, whereas the x_t decision is made at the beginning of a period.

Based on the concavity properties of f_t, it can be shown that the optimal accept/reject policy for type 1 patients is a threshold policy, i.e., there exists

R_t such that the optimal value of a_t satisfies

$$a_t(z_t) = \begin{cases} 0 & \text{if } z_t > R_t \\ R_t - z_t & \text{if } z_t \leq R_t \text{ and } z_t + M_t \geq R_t \\ M_t & \text{if } z_t + M_t < R_t, \end{cases} \tag{2.7}$$

where $z_t = (s_t + x_t - C)^+$ is the number of outstanding type 1 patients from period $t-1$ or earlier. Furthermore, the monotonicity result of Theorem 1 continues to hold in this case. (Proof in Appendix A.3.)

2.4.3 Multiple Elective Patient Classes

Next, we extend the model of Section 2.3 to the case with multiple classes of type 1 patients. We distinguish these classes with respect to the potential revenue.

Suppose now that there are n classes of type 1 patients, and we use the superscript to identify each of the n type-1 classes, $\{1, \ldots, n\}$. We assume that penalty coefficients satisfy $w_1^1 \geq \cdots \geq w_1^n$, and also the end-of-horizon terminal values per type 1 patient satisfy $v^1 \leq \cdots \leq v^n \leq 0$. These orderings are used to signify that class i patients are more "important" than class j patients, for $i < j$. Let $\mathbf{w}_1 = (w_1^1, \ldots, w_1^n)$ and $\mathbf{v} = (v^1, \ldots, v^n)$. Also, let $\mathbf{r}_1 = (r_1^1, \ldots, r_1^n)$ denote the per-patient revenue vector.

Let $\mathbf{s}_t = (s_t^1, \ldots, s_t^n)$ denote the vector of backlogged type 1 patients at the beginning of a period t, and let x_t denote the amount of capacity protected for type 2 arrivals. Thus, $C - x_t$ units of capacity will be available to serve backlogged type 1 patients. Since it is more costly to delay the service of the patients belonging to a lower-indexed class, a sample path argument can be used to show that the remaining capacity of $C - x_t$ units will be allocated among the type 1 patients in an increasing order of their class indices. Thus, among the type 1 patients backlogged at the beginning of period t, which is denoted by $\mathbf{s}_t$, how many will still be backlogged at the end of the period will be given by the following function:

$$\begin{aligned}\zeta(\mathbf{s}_t, C - x_t) \quad = \quad & \Big(s_t^1 \wedge \left[s_t^1 - C + x_t\right]^+, s_t^2 \wedge \left[s_t^1 + s_t^2 - C + x_t\right]^+, \ldots, \\ & s_t^n \wedge \left[\sum_{i=1}^{n} s_t^i - C + x_t\right]^+\Big) .\end{aligned}$$

Then, the single-period net-profit function of (2.1) is given as

$$L(\mathbf{s}_t, x_t) \quad = \quad r_2 \cdot \mathbf{E} \min\{x_t, D_t\} + \sum_{i=1}^{n} r_1^i \cdot \mathbf{E}[M_t^i] - \sum_{i=1}^{n} w_1^i \cdot z_t^i - w_2 \cdot \mathbf{E}[D_t - x_t]^+ \ ,$$

where $\mathbf{z}_t = (z_t^1, \ldots, z_t^n) = \zeta(\mathbf{s}_t, C - x_t)$, and $\mathbf{M}_t = (M_t^1, \ldots, M_n^t)$ is the vector comprised of independent random variables representing the new type 1 patient arrivals. The dynamic programming formulation in (2.2) can be modified as

$$f_t(\mathbf{s}_t) \quad = \quad \max_{0 \le x_t \le C} \left[L(\mathbf{s}_t, x_t) + \alpha \cdot \mathbf{E}_{M_t}[f_{t+1}(\mathbf{z}_t + \mathbf{M}_t)]\right] \ ,$$

where $f_{T+1}(\mathbf{s}_{T+1}) = \sum_{i=1}^{n} v^i \cdot s_{T+1}^i$.

For this model, we can establish results that are analogous to those of Theorem 1. In particular, we can show that the optimal amount of capacity protected for class 2 patients, $x_t^*(\mathbf{s}_t)$ is decreasing in each s_t^i, and the magnitude of this decrease is limited, i.e., $x_t^*(\mathbf{s}_t + \epsilon \cdot \mathbf{e}^i) \ge x_t^*(\mathbf{s}_t) - \epsilon$, for any $\epsilon > 0$, where $\mathbf{e}^i$ is the vector consisting of all zeros except the i'th component being 1. (Proof in Appendix A.4.)

2.5 Conclusion

In this chapter, we have developed a dynamic programming model to solve the resource allocation problem of a health-care facility, and presented the characteristics of the optimal policy and a simple heuristic policy that performs well. Although our model has been developed in a hospital setting, it is readily applicable in other service environments with backlogging and lost sales together. It has been noted earlier that the research on the multiple-demand-inventory *inventory* systems with lost sales and backorders is far from being complete, and our work here has shown that multiple demand classes in *service* systems share similar difficulties. The model we propose and the results we obtain disclose some understanding of the hospital resource management, but there are several questions that this research raises. Future research that incorporate the arbitrary length-of-stay distributions for one or multiple patient types or a limited waiting time for backlogged patients might prove useful.

Chapter 3

An Analysis of Dynamic Bilateral Price Negotiations

3.1 Introduction

Many transactions between a seller and a buyer follow some form of a negotiation. This is typical in business-to-business settings, e.g., in procurement of goods and services, as well as in transactions that involve end consumers for items that are expensive such as cars, furniture, and real-estate. The outcome of each such negotiation depends on the reservation values of the seller and buyer, their respective negotiation skill, and their beliefs about these parameters for their respective counterparties. This process is known as "bilateral price negotiation" and studied extensively in the literature (see Nash (1950), Harsanyi (1956), Myerson (1979), Myerson and Satterthwaite (1983), Myerson (1984), and Chatterjee and Samuelson (1983)). Depending on the market conditions, the seller may enjoy increased market power and as such be able to name her list price in the negotiation, whereas in the other extreme the buyers may essentially submit take-it-or-leave-it bids to the seller. In most settings, actual behavior falls somewhere in between, where the seller and buyer somehow split the difference between the seller's minimum acceptable bid (her reservation price, which may be dynamic) and the buyer's willingness to pay. However, in today's business world, "the shifting balance of power has many stores scrambling for pricing strategies that get beyond the time-worn cycle of markups and discounts" as stated in a recent NY Times article (Clifford (2012)). Thus, there is a power shift towards the consumers, propelled by the Internet and apps, which has rendered the buyers more empowered in haggles, thus demanding much lower prices.

One motivating application for the paper comes from the residential real estate industry, where a developer of a multi-unit project, e.g., a multi-story condominium development, tries to sell various condos to prospective buyers through a sequence of negotiations over time. While for each buyer their respective negotiation could be modeled as a one-off interaction, the seller's behavior should consider the fact that she will engage into a sequence of such negotiations over time. The phenomenon of power shift to the advantage of buyers is also observed in the real estate industry since the explosion of the real-estate bubble in the financial crisis of 2007. Hence, our main focus is the changing trading problem and the new pricing strategies of the sellers in the real-estate setting, even though most of our findings do apply to the general case. [1]

[1] Numerous articles in the press exemplify the phenomenon that "properties once sold at very high monetary terms are now being purchased by the bidders who pay the minimum amount to cover back taxes, interest and fees" Sinclair (2009). Many developers of multi-unit residential projects, such as condos, are advertising in the newspapers, magazines and

In more detail, we study the revenue maximization problem of a firm that has C units of capacity that it wishes to sell over a time horizon of length T to a market of prospective buyers that arrive to the firm according to a Poisson process with rate Λ, each has a willingness-to-pay that is an independent draw from a distribution F_b, and who engages in a bilateral negotiation with the seller for one unit of that good. The salvage value of the seller is private information, and buyers assume that it is drawn form some distribution F_s, and it is constant over time. The reservation price of the seller at time t depends on the salvage value and the state of the sales process, i.e., the time-to-go and remaining capacity. The bilateral negotiation is modeled as a one-off negotiation, where the buyer and seller submit bids and where the unit is warded if the buyer's bid is higher than the seller's bid. When the seller has market power, the transaction price is the seller's posted price (SPP); when the buyer has market power, the transaction price is the buyer's posted price (BPP); in other cases the transaction price splits the difference between the two bids according to a fixed ratio that models the relative negotiation power of the two players. [2]

The ultimate focus of this part is to study this problem primarily in the setting where buyers have market power (BPP), and where the seller and the buyers do not know the distributions F_b, F_s, respectively, and moreover the unknown distribution F_b may be changing over time. This setting is motivated by the real estate application, where there is significant uncertainty about the current and future market conditions, and where the sales horizon is sufficiently long so that the market conditions will change over time; the non-stationarity here

on the internet announcing that all bids are welcome.

[2]A detailed definition of each negotiation mechanism can be found in Bhandari and Secomandi (2009).

is not due to seasonality effects that can be readily incorporated in a setting where the seller would know the evolution of F_b over time, but rather due to changes in underlying market conditions, e.g., such as interest rates, economic conditions, etc., that "modulate" the buyer willingness-to-pay distribution.

Despite the importance and prevalence of negotiation problems in practice, most literature in quantitative dynamic pricing and revenue management has focused on posted price mechanisms (see Gallego and van Ryzin (1994), Varma and Vettas (2001)) and auctions (see Vulcano et al. (2002)). Among the papers that involve revenue management problems in the form of bilateral negotiations, the work Bhandari and Secomandi (2009) is perhaps closest to ours regarding the problem under consideration. However, we consider a setting where the valuation distributions are unknown, and employ an entirely different methodology than Bhandari and Secomandi, who use a stylized MDP to investigate the negotiation processes in a dynamic setting. We also mostly restrict our analysis to a setting where buyers have market power, which is also the main difference of our work from those of Riley and Zeckhauser (1983) and Gallien (2006).

The first modeling and methodological contribution of the paper is in formulating the classical bilateral negotiation problem in an uncertain environment, where buyers and the seller do not have information about F_s, F_b, respectively. There are three natural ways to specify this type of model uncertainty that lead to different formulations and different policy recommendations. The first one is stochastic, wherein the unknown distributions are assumed to be drawn from a given set of possible distributions according to some known probability law, and where the firm's goal is to optimize its expected revenues -potentially risk-adjusted- over all possible market model realizations. Its main shortcoming is that it requires detailed information on the distribution of the model uncertainty, which itself may not be available. The second formulation adopts a worst-case perspective using a max-min criterion for both the seller and buyer, wherein the unknown distributions are assumed to be selected from an appropriate set of possible distributions by an imaginary adversary ("nature") to reduce the seller's revenue or the buyer's surplus, and where the seller's and buyer's objective is to select a biding strategy that optimizes their respective worst-case revenue performance. As a second formulation, both the seller and the buyer adopt a max-min criterion where they aim to optimize their respective worst-case revenues. This criterion may yield overly pessimistic results. To reduce this inherent conservatism, one typically imposes constraints on the decision set of the adversary, that are either ellipsoids (see Ben-Tal and Nemirovski Ben-Tal and Nemirovski (1998), and El-Ghaoui and Lebret El Ghaoui and Lebret (1997)), or polyhedra (see Bertsimas and Sim Bertsimas and Sim (2003), as well as Bertsimas and Thiele Bertsimas and Thiele (2004), Perakis and Sood Perakis and Sood (2003)). Finally, a third approach that reduces the conservatism of max-min formulations while maintaining their appealing low informational requirements is through the use of the competitive ratio or maximum regret criteria, which measure the performance relative to that of a fully-informed decision maker. They have been used extensively in the computer science literature, and have recently been applied in pricing and operations management problems. Specifically, Ball and Queyranne Ball and Queyranne (2009) used a competitive ratio criterion for a single-resource capacity allocation problem, while Bergemann and Schlag Bergemann and Schlag (2008), Eren and Maglaras Eren and Maglaras (2010), and Perakis and Roels Perakis and Roels (2007) adopted the regret criterion to study the monopolist pricing and the newsvendor problems, respectively. Lan et al. Lan et al. (2006)

generalize Ball and Queyranne's analysis and extend it to cover the regret criterion as well. Perakis and Roels (2007) applies similar techniques for network revenue management. Eren and Van Ryzin Eren and van Ryzin (2006) apply these criteria to the problems of product positioning and differentiation. Specifically, Ball and Queyranne Ball and Queyranne (2009), Bergemann and Schlag Bergemann and Schlag (2008), Eren and Maglaras Eren and Maglaras (2010), Perakis and Roels Perakis and Roels (2007), Lan et al. Lan et al. (2006) and Eren and Van Ryzin Eren and van Ryzin (2006) adopt different versions of this idea.

Secondly, focusing on a buyer market, we carry the analysis to the dynamic setting. The key finding is to recognize that in the BPP setting where the seller is simply making accept or reject decisions of the buyer bids can be reduced to a single resource capacity control problem in the form analyzed by Lee and Hersh Lee and Hersh (1998). Specifically, the distribution of buyer bids is analogous to a continuous distribution of fare classes. This observation allows us to completely characterize the structure of the optimal policy. We note in passing that the setting where sellers have market power (SPP) is similarly analogous to the well-studied dynamic pricing problem studied in Gallego and van Ryzin Gallego and van Ryzin (1994).

Next, motivated by the goal of studying the dynamic revenue maximization in settings where the distributional assumptions may be not known and also change over time, we start with a simpler approximated problem where the buyer arrival process is replaced by a deterministic and continuous process. This model can be justified as a limit as the capacity and market potential as captured by Λ grow large, and the sales horizon and distributional assumptions stay unchanged. In the limit model the sales process is continuous, where infinitesimal buyers request infinitesimal quantities of the seller's capacity. This is often referred to as a "fluid" model. The fluid revenue maximization problem admits a static solution, as it could be expected from the mapping of the BPP formulation to the capacity control problem, where the seller accepts all bids above a given threshold.

Finally, the last part of the chapter focuses on the real-life applications where the distributions F_s, F_b are unknown and may vary over time. Motivated by our previous findings regarding the static uncertain problem, we propose a method that a) uses the deterministic fluid model, b) adopts uniform distributions for F_s, F_b, c) considers multiple possible parameter scenarios for the evolution of these distributions, and d) picks a feedback pricing strategy for the seller to optimize its regret relative to the full information problem. This problem can be solved in an open-loop manner to get the best possible strategy for the seller. This, however, can be improved by optimizing over a set of linear feedback bidding rules for the seller, that are motivated by the optimal seller strategy under full information. A set of numerical results show that the regret formulation and the associated uniform distribution assumption lead to good results, i.e., modest revenue loss for the seller, in a variety of settings.

The main contributions of this chapter are as follows: First, Adopting the maximum regret criterion we formulate jointly the buyer and seller bidding problems in the setting where F_s, F_b are unknown to the respective counterparties, and show that the optimal strategies are to bid as if the underlying distributions F_s, F_b were uniform. This formulation and associated result are novel, and important on their own right as they offer a robust analogue of the one-to-one bilateral negotiations problem. Parenthetically, the fact that the uniform distribution appears as the natural assumption under incomplete information is consistent with results derived in the literature. Secondly, we draw attention to the analogy between the dynamic bilateral negotiation problems and the classical revenue management problems; which is a first in the literature. Third, the formulation of the dynamic seller's problem with uncertain F_s, F_b distributions as a problem that assumes that the distributions are uniform, as motivated by the result in the one-to-one setting, is novel and the formulation is itself readily solvable producing a simple and tractable policy that has a good performance.

The remainder of the chapter. In section 3.2, we consider the one-to-one negotiation problems: In section 3.2.1, the classical one-to-one negotiation models are revisited; and in section 3.2.2, we analyze a variant of the classical problem with an added uncertainty element in terms of the valuation distribution functions. In section 3.3, the analysis is carried to a dynamic setting. Section 3.3.1 sheds light on the analogy of the negotiation problems with the revenue management problems. Section 3.3.2 presents the dynamic pricing model that extends the results of the static negotiation problem to a dynamic setting using a fluid model approach. Next, in section 3.4, the results of section 3.2.2 are extended to the dynamic setting again under a regret criterion. In particular, we propose a scenario-based robust optimization approach which is both tractable and takes into account the unfolding uncertainty in the system as time progresses. Numerical illustrations and extensions are presented in Section 3.5. Finally, section 3.6 concludes our findings and presents avenues for further research.

3.2 1-to-1 Bilateral Negotiation Problem

In Section 3.2.1, we present the classical one-to-one negotiation problem that forms the building blocks for the dynamic negotiation problem studied in Section 3.3. Then in Section 3.2.2, we analyze a variant of the classical one-to-one bilateral negotiation problem with an added uncertainty element in terms of the valuation distribution functions which, to best of our knowledge, has not been attempted before.

3.2.1 Classical 1-to-1 Bilateral Negotiation Problem

Although the literature of two-person bargaining games goes back to Nash (1950) and Harsanyi (1956), the first pieces of work to pioneer the analysis of the dynamics of an environment where the buyers have the major market power are those of Myerson (et al.) (Myerson (1979), Myerson and Satterthwaite (1983), Myerson (1984)) and of Chatterjee and Samuelson (1983). However, even in the alluded studies, the negotiation problem is only analyzed within a static context where the game is between a single seller and a single buyer. In this paper, we will extend the previous line of research into a dynamic setting. But first, let us revisit the classical one-to-one negotiation problem of the literature.

The one-to-one bilateral negotiation problem involves the trading interactions between two individuals where one of the individuals (the seller) owns an object that the other (the buyer) wants to buy. Both players are risk neutral. From the seller's perspective the valuation of the buyer for this unit is random variable v_b, distributed according to probability density and distribution functions f_b and F_b with support $[\underline{v}_b,\ \bar{v}_b]$. A symmetric argument holds for the buyer, where he assumes that the seller's valuation for the unit, v_s, is distributed according to cumulative distribution function F_s (with p.d.f. f_s) on the range $[\underline{v}_s,\ \bar{v}_s]$. F_s and F_b are both strictly increasing and differentiable on their supports, and are common knowledge in the sense of Aumann (1976)- that is, each side knows these distributions, knows that they are known by the other side, knows that the latter knowledge is known, and so on and so forth.

The rules of the bargaining game is as follows: At the beginning of the sales interval the seller sets a reservation price $s(v_s)$, then the buyer submits a bid $b(v_b)$, and a successful trade is concluded if $b(v_b)$ exceeds $s(v_s)$. The resulting sales price is $kb(v_b)$ + (1 - k)$s(v_s)$, where $k \in [0, 1]$ is a parameter that determines the bargaining power of the buyers. Specifically, if $k = 0$, the problem reduces to a "seller posted price" (SPP) setting where the entire power to determine the final price lies with the seller: In this case, the trade is concluded at the price $s(v_s)$ as long as $s(v_s) \leq b(v_b)$. At the other extreme, when $k = 1$, the problem simply becomes a "buyer posted price" (BPP) formulation where the sales price is equivalent to the buyer's bid $b(v_b)$, again provided that $s(v_s)$ $\leq b(v_b)$ holds.

Chatterjee and Samuelson (1983) characterize the class of equilibria for the above problem in which player bidding strategies are "well-behaved". In particular, they make the following assumption regarding the buyer and seller bidding functions $s(.)$ and $b(.)$, which is also relevant for our further analyses:

Assumption 1. *In the equilibrium, both $b(.)$ and $s(.)$ are bounded above and below and are strictly increasing and differentiable except possibly at the boundary points.*

Under the above assumption, the following theorem characterizes the equilibrium bidding strategies of the two parties [3]:

Theorem 2. *(Chatterjee & Samuelson 1983) Under Assumption 1, over intervals of seller and buyer values for which the bidding strategies are strictly increasing, the equilibrium bidding strategies of the buyer and the seller must satisfy the linked differential equations respectively:*

$$-kF_s(s^{-1}[b(v_b)])s'(s^{-1}[b(v_b)]) + f_s(s^{-1}[b(v_b)])(v_b - b(v_b)) = 0, \tag{3.1}$$

$$(1-k)(1-F_b(b^{-1}[s(v_s)]))b'(b^{-1}[s(v_s)]) + f_b(b^{-1}[s(v_s)])(v_s - s(v_s)) = 0, \tag{3.2}$$

where $k \in [0,1]$ is the parameter determining the bargaining power of the buyer.

We will not give the details of the proof of the above theorem in this paper. However, we would like to emphasize that this pair of functions is obtained by solving the following two linked "best response problems" of the seller and the buyer simultaneously:

$$\max_{s\in[v_s,\bar{b}]} \int_s^{\bar{b}} (kb + (1-k)s - v_s)g_b(b)db,$$

and

$$\max_{b\in[\underline{s},v_b]} \int_{\underline{s}}^{b} (v_b - kb - (1-k)s)g_s(s)ds,$$

where g_s and g_b are the probability distribution functions (pdf) of the optimal bidding functions $s^*(.)$ and $b^*(.)$ respectively, $\underline{s}$ is the minimum value that the seller's bid can take and $\bar{b}$ is the maximum value that the buyer's bid can assume.

Moreover, observe that the equations (3.1) and (3.2) defining the buyer and the seller bid in the BPP environment (i.e. $k = 1$) take the following simpler

[3] We will use the terms "bidding function" and "bidding strategy" interchangeably throughout the paper.

forms:

$$b^*(v_b) = \{b| - F_s(b) + (v_b - b)f_s(b) = 0\}, \quad \forall v_b \in [\underline{v}_b, \bar{v}_b] \tag{3.3}$$

$$s^*(v_s) = v_s, \quad \forall v_s \in [\underline{v}_s, \bar{v}_s] \tag{3.4}$$

and the same equations produce the following bidding functions in the SPP ($k = 0$) case:

$$b^*(v_b) = v_b, \quad \forall v_b \in [\underline{v}_b, \bar{v}_b] \tag{3.5}$$

$$s^*(v_s) = \{s|1 - F_b(s) + f_b(s)(v_s - s) = 0\}, \quad \forall v_s \in [\underline{v}_s, \bar{v}_s] \tag{3.6}$$

An interesting feature of the seller's optimal bidding function in the BPP setting is its independence from the buyer valuation distribution. The intuition behind this fact is obvious: Since the seller has no influence on determining the final price yet she can reject the offers she considers non-profitable, she is willing to accept any offer above her own valuation to attain a non-negative profit. That makes bidding her own reserve value, v_s, her best response to all bidding functions of the buyers. Thus, given that g_s is essentially identical to f_s in the BPP setting, the bidding function of the buyer assumes the simple form as in (3.3). A symmetrical argument holds in the SPP setting too, justifying the forms of the bidding functions given in equations (3.5) and (3.6).

3.2.2 1-to-1 Bilateral Negotiation Problem in Uncertain Environments

In this subsection, we analyze a variant of the classical one-to-one bilateral negotiation problem with an added uncertainty feature. The classical one-to-one problem of the literature assumes that both individuals have a certain belief regarding the distribution information of their opponent's valuation. However, in real life there are often situations in which the individuals do not have practically any information regarding the valuation of the other party. In particular, we assume that both agents are able to estimate the minimum and the maximum values that their opponent's valuation could assume; however, they do not have any knowledge regarding the distribution of this valuation in its given range.

There are various ways to formulate the uncertainty in this kind of problems, as discussed in Section 3.1 in greater detail. Among these approaches lies the "absolute regret minimization criterion" (ARMC), which basically aims to minimize the maximum revenue gap between the current decision and the best decision over all contingencies. We will use ARMC approach in our analysis. The rationale behind this method is to improve the average quality of decisions under uncertainty; and is introduced by Savage (1951) after observing that another commonly used decision criterion, the maximin criterion, may lead to overly conservative decisions.

Adopting the ARMC approach, the problems that the seller and the buyer need to solve in order to minimize their maximum regret are formulated as follows:

$$\underset{s}{\operatorname{argmin}}\{\max_{b}\max_{s'}[(kb+(1-k)s'-v_s)\cdot 1_{\{b\geq s'\}}-(kb+(1-k)s-v_s)\cdot 1_{\{b\geq s\}}]\} \quad (3.7)$$

and

$$\underset{b}{\operatorname{argmin}}\{\max_{s}\max_{b'}[(v_b-(kb'+(1-k)s))\cdot 1_{\{b'\geq s\}}-(v_b-(kb+(1-k)s))\cdot 1_{\{b\geq s\}}]\} \quad (3.8)$$

In the first of the above problems, the seller tries to select the bid s which minimizes the revenue loss across all bids b of the buyer; where the seller's revenue loss in each instance is the difference between the maximum revenue she could have achieved by bidding her best response s' against a particular bid b of the buyer (i.e. $(kb + (1-k)s' - v_s) \cdot 1_{\{b \geq s'\}}$) and her realized revenue under the selected bid s (i.e. $(kb + (1-k)s - v_s) \cdot 1_{\{b \geq s\}}$). The problem of the buyer is symmetrical.

The equilibrium bidding functions s^*_{ARMC} and b^*_{ARMC} that solve the above problems and are best responses to each other are characterized in the following theorem.

Theorem 3 (**Equivalence of ARMC and the uniform distribution case**)**.** *When each party in the bilateral negotiation game only possesses the support information of the opponent's value distribution and uses ARMC to maximize revenues, the equilibrium bidding functions are given as follows:*

$$s^*_{ARMC}(v_s) = \frac{v_s}{2-k} + \frac{(1-k)\bar{v}_b}{2} + \frac{k(1-k)\underline{v}_s}{2(2-k)}, \forall v_s \in [\underline{v}_s, \bar{v}_s], \tag{3.9}$$

$$b^*_{ARMC}(v_b) = \frac{v_b}{1+k} + \frac{k\underline{v}_s}{2} + \frac{k(1-k)\bar{v}_b}{2(1+k)}, \forall v_b \in [\underline{v}_b, \bar{v}_b]. \tag{3.10}$$

which also happen to be the equilibrium bidding functions of a game where the value distribution of the seller and the buyer are both ***uniform*** *on the given ranges.*

(Proof in Appendix B.1.)

The above result is conceptually significant since it brings a theoretical motivation for assuming uniform distribution for the opponent's valuation when no distribution information is available. In other words, the results of the ARMC analysis support the intuition that the counterparty valuation could be anywhere over its support with equal probabilities when nothing is known regarding its distribution. Therefore, in addition to the fact that nice and neat closed-form solutions can be obtained by assuming that the opponent's value is distributed uniformly in its given range, we have also established a theoretical motivation to use the uniform distribution assumption in this type of games.

Remark 1. *By the above analysis, we have extended the literature on the one-to-one negotiation problem between a single seller and a single buyer where neither the seller nor the buyer know each other's distribution function, but they both know the range of the opponent valuations. It is also possible to analyze a third case where one of the parties is informed about the other's value distribution function, while the other only knows the range of his opponent's valuation. The solution for this case can be found in Appendix B.2.*

3.3 Dynamic Bilateral Negotiation Games

In this section, we turn our attention to the main objective of the chapter and analyze the dynamic problem of a revenue maximizing monopolist who sells a number of homogeneous units over a finite sales horizon to a market of prospective buyers that arrive according to a (possibly non-homogeneous) Poisson process, where each transaction is a one-off bilateral negotiation game.

3.3.1 The Analogy Between the Revenue Management and Bilateral Negotiation Problems

Consider the general revenue maximization problem of a monopolist firm: If the firm operates in a market with imperfect competition, it has the power to determine the sales price. In this setting, the firm's problem is to choose a dynamic pricing strategy which is changing according to the remaining inventory and the expected customer arrivals, with the objective of maximizing expected revenues. If, however, the prices are exogenously determined by competition or through a higher order optimization problem defining the market conditions (as in our case, by the equilibrium in the negotiation game between the buyers and the seller), the firm chooses a dynamic capacity allocation rule that controls when to accept new requests for its products. These two problems, which we will refer to as the "dynamic pricing" and "capacity allocation" problems respectively in the sequel, are the two famous problems of the revenue management and pricing literature and have elicited much attention from various groups of researchers. For instance, the stochastic dynamic pricing game has been extensively analyzed by Gallego and van Ryzin (1994) (which will be referred to as GVR for the rest of the paper) and the capacity rationing problem of a single resource has also been considered by many researchers, involving Lee and Hersh (1998) and Brumelle and McGill (1993) among many others.

In our case, we consider a sequence of independent bilateral negotiation games played between the seller and each one of the buyers arriving over a finite sales horizon. The seller would like to sell C units of a homogeneous good over a sales horizon T where each arriving buyer interacts with the seller for a single unit of the good according to a one-off bilateral negotiation game. Potential customers arrive according to a non-homogeneous Poisson process with instantaneous rate Λ_t. Each customer has a valuation v_b for the product which is an independent draw from a cumulative distribution function F_b. The willingness-to-pay (WtP) value of a buyer with valuation v_b is therefore $b^*(v_b)$, which is characterized by the equation (3.1) regardless of his arrival time t. Next, the seller quotes a minimum reserve price $s_t(v_s)$, and the unit is sold if

$b_t(v_b) = b^*(v_b) > s_t(v_s)$; otherwise the customer leaves without purchasing a unit.

The key observation in the dynamic setting is that the buyers in the system are "naive": they ignore the competition with other buyers in the market, and bid according to the equilibrium bidding function $b^*(\cdot)$ characterized by the equation (3.1). However, the seller will engage into a sequence of such negotiations over time, therefore submits her bid with the objective of maximizing her *overall* revenues. Thus, the seller's bid is no longer determined by the equation (3.2).

First, consider an SPP setting: We know that in this environment the buyer equilibrium bidding function takes the form $b^*_{SPP}(v_b) = v_b$. Hence, given the arrival rate Λ_t, it is possible to define the expected "sales rate" function at the instant t by $N(t) = N(s_t) = \Lambda_t \bar{F}_b(s_t)$ which is a "regular function" in the sense that regular functions are defined in GVR paper provided that s_t lies in a compact set $\mathbf{P}$ (Namely, $N(t)$ is strictly decreasing in s_t, $\lim_{s_t \to \infty} N(s_t) = 0$, and the revenue rate $s_t \Lambda_t \bar{F}_b(s_t)$ is bounded for all $s_t \in \mathbf{P}$ and has a finite maximizer).

We adopt a discrete-time formulation where time has been discretized to small intervals of length δt, indexed by $t = 1, \ldots, T$ such that $\mathbb{P}$(a buyer arrives at $[t - \delta t, t]) = \Lambda_t \delta t + o(\delta t)$ where $o(x)$ implies that $o(x)/x \to 0$ as $x \to 0$. Thus, the random sales amount at instant t, which we will denote by $\xi(t; N)$, is Bernoulli with probability $\mathbb{P}(\xi(t; N) = 1) = N(s_t)\delta t$ and $\mathbb{P}(\xi(t; N) = 0) = 1 - N(s_t)\delta t$ for small δt.

Thus, the seller's revenue maximization problem in an SPP setting could be formulated as follows:

$$\max_{\{s_t, t=1,\ldots T\}} \quad \mathbf{E}_{\xi}[\sum_{t=1}^{T}(s_t - v_s)\xi(t;N)] \tag{3.11}$$

$$\text{subject to} \quad \sum_{t=1}^{T}\xi(t;N) \leq C\, a.s.,\ s_t \in \mathbb{P},\ \forall t. \tag{3.12}$$

In the above formulation, the seller tries to maximize the total profit throughout the sales horizon $[0, T]$. The instantaneous profit at each instant t is given by $(s_t - v_s)\xi(t; N)$, which takes the value $s_t - v_s$ if a buyer arrives at the instant t and trade takes place; and 0 otherwise. The constraint (3.31) ensures that total sales does not exceed the seller's inventory.

A simple observation shows that the above formulation is in fact no different than the formulation of a dynamic pricing problem of a monopolist seller selling a single product in a discrete-time setting; which is readily given in GVR paper. Furthermore, having shown that $N(t)$ is a regular function, it is possible to treat the sales rate $N(t)$ as the control variable and carry the analysis to a demand space. The details of this transformation could be found in GVR and Maglaras and Meissner (2006) papers.

In the other extreme (i.e. the dynamic BPP setting), given the arrival rate Λ_t and the buyer bidding function b^*_{BPP} (having cdf G_b and pdf g_b), it is possible to define the expected "sales rate" function at the instant t as $N(t) = N(s_t) = \Lambda_t \bar{G}_b(s_t)$. Then, the seller's revenue maximization problem in a BPP setting could be formulated as follows:

$$\max_{\{s_t, t=1,\ldots T\}} \quad \mathbf{E}_{\xi, b_t}[\sum_{t=1}^{T}(b_t - v_s)\xi(t;N)] \tag{3.13}$$

$$\text{subject to} \quad \sum_{t=1}^{T}\xi(t;N) \leq C\, a.s.,\ s_t \in \mathbb{P},\ \forall t. \tag{3.14}$$

where $b_t := b^*_{BPP}(v_t)$, and the valuation v_t of the buyer arriving at instant t is randomly drawn from the valuation range of the buyers. However, note that the price s_t now has no direct effect on the revenue at the instant t, except for determining the lower bound of the buyer bids to be admitted. That is, it effectively works as a control that leads to “opening” product classes (buyer bids) that exceed the price level s_t and “closing” classes that bring lower revenue than s_t. Hence, we will proceed by the following approximation to show the connection of the above problem with the capacity allocation problem of the literature: Assume that we can approximate all buyer bids by n finite values; i.e. define $\bar{b}^* \geq b_1 \geq b_2 \geq \ldots \geq b_n \geq \underline{b}^*$ as n finite “fare classes”, where the arrival rate of bid b_i is approximated by $\Lambda_t(\bar{G}_b(b_i) - \bar{G}_b(b_{i-1}))$, $\forall i \in \{2, \ldots, n\}$ and the arrival rate of b_1 is approximated by $\Lambda_t \bar{G}_b(b_1)$ at each instant t. Then, the problem above pours into the following capacity allocation problem of a firm which has discretion as to which product requests to accept at any given time:

$$\max_{\{u(t), t=1,\ldots T\}} \quad \mathbf{E}_\xi[\sum_{t=1}^{T}(b' - v_s)\xi(t; u\Lambda)] \tag{3.15}$$

$$\text{subject to} \quad \sum_{t=1}^{T} e'\xi(t; u\Lambda) \leq C \, a.s., \; u_i(t) \in \{0, 1\}, \; \forall t.$$

where $u_i(t)$'s are the controls that take value 1 when a bid of value b_i is accepted at time t and zero otherwise, $b' = \{b_1, b_2, \ldots, b_n\}$, e' the n-dimensional unit vector, and $\xi(t; u\Lambda)$ denotes the associated sales vector. It is then easily observed that the formulation (3.15) is the discretized version of the capacity control problem of Lee and Hersh (1998).

Thus, if the buyer bids could be approximated by a finite class of fares, the BPP formulation is equivalent to the capacity allocation problem of a seller selling a single resource to multiple demand classes in a perfect competition setting. For details, we refer the reader to Lee and Hersh (1998) and Maglaras and Meissner (2006).

The two above equivalences stem from the fact that while the buyers are naive, the seller will engage into a sequence of negotiation games over the sales horizon, and therefore submits her bid with the objective of maximizing her overall revenues. Hence, in the first game she will pursue a *dynamic pricing strategy* to maximize the revenues to be extracted from the stochastically arriving buyers; whereas in the second game, she will determine the minimum bid to be accepted at each time instant to control *the amount of capacity to be sold*. Therefore, in broad terms, the SPP game reduces to the dynamic pricing and the BPP game to the capacity allocation problems of the literature. We state this result as a proposition.

Proposition 1. *If the buyers in the market are naive, given that the equilibrium bidding functions of the one-to-one game are known, the dynamic SPP game becomes equivalent to the* ***dynamic pricing problem*** *and the dynamic BPP game to the* ***capacity allocation problem*** *of the revenue management literature.*

Although theoretically simple to characterize, the above problems are often very hard to compute numerically. For instance, in the case of the dynamic pricing problem, the authors have adopted a fluid formulation in the GVR paper and developed asymptotically optimal policies. Also, in negotiation problems, there is a continuous stream of buyer bids often with their range and frequency varying in time; therefore it is practically impossible to approximate the buyer bids as finite number of fare classes as we did above. Even if this approximation is valid, the issue of "curse of dimensionality" prevails. Thus, the size of the problem renders the computation of the optimal bids s_t, $\forall t$, almost impossible, which leads us to develop a fluid-formulation equivalent of the dynamic negotiation problems and focus on the analysis in this fluid setting.

Finally, we would like to note that the relationship between the dynamic pricing and capacity allocation problems is extensively studied in Maglaras and Meissner (2006), in which work the authors illustrate how the two problems can be reduced to a common formulation by tracking the aggregate capacity consumption rates. We will pursue a similar objective and help to establish the connection between the two extreme negotiation problems by showing that the structural results of GVR also hold in the BPP setting.

3.3.2 Fluid Formulation of the Dynamic Game

Since analyzing the stochastic dynamic pricing problem of the seller is difficult, we will proceed with a fluid formulation in the hope to obtain possible insights towards the stochastic problem. As commonly known, fluid formulation is a good approximation of the real stochastic problem when number of interactions per unit time is sufficiently large.

To this end, consider the following fluid formulation of the general dynamic negotiations game: Infinitesimal buyers arrive with a (deterministic) rate Λ_t at t, $t \in [0, T]$. The buyer and the seller valuation functions have the same characteristics as defined in the one-to-one setting. Both parties know Λ_t and the distribution function of their opponent. In this setting, the revenue maximization problem of the seller takes the following form:

$$\max_{s_t,\forall t} \quad [\int_0^T r_t(v_s, s_t)dt] \tag{3.16}$$

$$\text{subject to} \quad \int_0^T \Lambda_t[\int_{s_t}^{\bar{b}} g_b(b)\,db]\,dt \leq C \tag{3.17}$$

where $r_t(v_s,\, s_t)$ is the instantaneous net revenue function of the seller at time t when her valuation is v_s and her reservation price s_t; which is given by:

$$r_t(v_s, s_t) = \int_{s_t}^{\bar{b}} \Lambda_t(kb + (1-k)s(v_s) - v_s)\, g_b(b)\, db. \tag{3.18}$$

and g_b is the pdf of the buyer bidding function b characterized in (3.1) and $s(\cdot)$ is given by (3.2). (For the rest of the analysis, the terms $b^*(.)$ and g_b will refer to these functions as well as G_b, which will denote the cdf of b^*, unless stated otherwise.) Observe that we do not need a subscript t in the bidding function $b^*(\cdot)$ of the buyers, since the distribution of b^* is the same at all instances t in the equation (3.18) given that the buyers are naive.

It is usually difficult to find the solution to the above problem if it is modeled as a stochastic control problem in the price space. Therefore, following a similar approach as in GVR, we will analyze the problem by focusing on the optimal *sales rate*, rather than the optimal *pricing policy.* Note that the value of each quantity will follow after the other one is settled.

If the seller sets s_t as the lowest price to be accepted at t, the fraction of buyers that are accepted at that instant is given by $\alpha_t(s_t) = \int_{s_t}^{\bar{b}} g_b(b)\,db = \bar{G}_b(s_t)$, inducing an inverse function:

$$s_t(\alpha_t) = G_b^{-1}(1-\alpha_t).$$

The function $s_t(\alpha_t)$ is well-defined for all $\alpha_t \in [0,1]$ as a result of Assumption 1.

Then, the instantaneous net revenue function of the seller at time t in terms of the fraction of accepted buyers becomes:

$$r_{t,a}(v_s,\alpha_t)=\int_{G_b^{-1}(1-\alpha_t)}^{\bar{b}}\Lambda_t\left(kb+(1-k)(G_b^{-1}(1-\alpha_t))-v_s\right)g_b(b)\,db.$$

Moreover, the seller's revenue maximization problem (3.16)-(3.17) in the price space is equivalent to the following formulation in the demand space:

$$\max_{\alpha_t,\forall t}\quad [\int_0^T r_{t,a}(v_s,\alpha_t)dt] \tag{3.19}$$

$$\text{subject to}\quad \int_0^T \Lambda_t\alpha_t\,dt\le C. \tag{3.20}$$

Provided that $r_{t,a}(v_s,\,\alpha)$ is concave in α, the formulation (3.19)-(3.20) pours into the maximization problem of a concave function over a convex set, and its solution is characterized as in the following Theorem.

Theorem 4. *If $r_{t,a}(v_s,\,\alpha)$ is concave in α, the equilibrium bidding strategy $s_t(.)$, $t\in[0,T]$, of the seller in the dynamic negotiation problem takes the form:*

$$s_t(v_s)=\max\{G_b^{-1}(1-\frac{C}{\int_{t=0}^T\Lambda_t\,dt}),s^*(v_s)\},\forall t\in[0,T], \tag{3.21}$$

where $s^*(v_s)$ is the equilibrium bidding strategy of the seller characterized in (3.2); the equilibrium bidding strategy $b_t(.)$, $t\in[0,T]$, of each infinitesimal buyer arriving at time t is characterized by the equation (3.1), and G_b is its cdf.

Proof. As we have already noted, the buyers are naive: i.e. they neither have the knowledge of the sales rate nor the remaining inventories of the seller. Therefore, they will regard the situation simply as a one-to-one negotiation game and employ the static equilibrium bidding function $b^*(.)$ regardless of their time of arrival.

To see how the seller behaves, note that the problem (3.19)-(3.20) is maximized at the maximizer of $r_{t,a}(., v_s)$, which is $\alpha^* := \bar{G}_b(s^*(v_s))$, as long as it is feasible to admit this fraction at each instant t (i.e. if $\alpha^* \int_0^T \Lambda_t \, dt \leq C$). This case is equivalent to applying the bid $s_t(v_s) = s^*(v_s)$, $\forall t$.

If, on the other hand, $\alpha^* \int_0^T \Lambda_t \, dt > C$, then by the concavity of $r_{t,a}(., v_s)$, it is optimal to admit the constant fraction $\alpha_0 := \frac{C}{\int_{t=0}^T \Lambda_t dt}$ at each t. This second case corresponds to bidding $s_t(v_s) = G_b^{-1}(1 - \frac{C}{\int_{t=0}^T \Lambda_t \, dt})$, $\forall t \in [0, T]$.

Finally, when $\alpha^* > \frac{C}{\int_{t=0}^T \Lambda_t dt}$, the inequality $G_b^{-1}(1 - \frac{C}{\int_{t=0}^T \Lambda_t \, dt}) \geq s^*(v_s)$ holds; and the reverse would be true in the opposite case. So the seller will set her reservation price as $s_t(v_s) = \max\{s^*(v_s), G_b^{-1}(1 - \frac{C}{\int_{t=0}^T \Lambda_t \, dt})\}$, which ends the proof of the theorem. □

The above theorem is in the same spirit as the Proposition 2 of GVR paper and forms the first major result of this section.

Regarding concavity of the instantaneous revenue function of the seller, for instance:

$$g_b'(b) \geq 0, \forall b \in [\underline{b}, \bar{b}] \tag{3.22}$$

is a sufficient condition to ensure that $r_{t,a}(v_s,\, \alpha)$ is concave in α for all $v_s \in [\underline{v}_s, \bar{v}_s]$. This condition simply ensures that the second derivative of the function $r_{t,a}(v_s, \cdot)$ is negative at all α. Observe that both functions F_s, F_b being uniform is one case where Condition (3.22) is satisfied.

3.3.3 The Informed Buyers in BPP Setting

Until this far, we have assumed the buyers in the market are “naive buyers” who do not have the knowledge of (or, simply ignore) the seller’s inventory level and the competition in the market. However, in many real life situations, this may not be the case, and the buyers could observe the inventory level of the seller and the number of other buyers in the market. In this subsection, we will briefly analyze the case where the buyers have the knowledge of the current inventory level x_t of the seller and the information regarding the future market size; and base their decisions on this knowledge.

We will restrict our attention to the BPP setting for this subsection, to see the possibility whether the competition in the market can nullify the buyers' advantage of being in a buyer's market.

To this end, consider a multi-period setting with discrete sales periods indexed forward as $t \in \{1, 2, \ldots, T\}$. We will employ backward induction to analyze and solve the problem.

At $t = T$, when the seller and all the buyers know x_T (the number of units in seller's hand), F_b, F_s and Λ_T, there are two possible cases:

(i) $\Lambda_T < x_T$: In this case, the competition among buyers is not of significance, since there is ample capacity to serve everyone in the market. So the buyers will again bid as if they participate in a one-to-one game, i.e. according to $b^*(\cdot)$ given by (3.3), and the seller will bid her best response against b^*, which is $s^*(v_s) = v_s$.

(ii) $\Lambda_T > x_T$: In this case there is not sufficient capacity to serve the entire market. Hence, the seller's revenue maximization problem takes the form:

$$\begin{aligned} &\max_{\alpha_T} \Lambda_T \int_{G_{b_T}^{-1}(1-\alpha_T)}^{\bar{b}_T} (b - v_s)\, g_{b_T}(b)\, db \\ \text{subject to} \quad &\Lambda_T \alpha_T \leq x_T \end{aligned} \tag{3.23}$$

where b_T is the bidding function of the buyers at $t = T$, $G_{b_T}(.)$ its cdf, and

$g_{b_T}(.)$ is the pdf. The above problem is clearly maximized at the seller bid $s_T(v_s) = \max\{v_s, G_{b_T}^{-1}(1 - \frac{x_T}{\Lambda_T})\}$; i.e. the seller is willing to accept all bids above the maximum of her own valuation and market clearing price.

On the other hand, the revenue maximization problem of each buyer now becomes:

$$\max_b \int_{\underline{s}_T}^{b} (v_b - b) g_{s_T}(s) ds \tag{3.24}$$

where g_{s_T} is the pdf of the bidding function $s_T(\cdot)$ of the seller.

When these two problems are solved simultaneously, the equilibrium bidding functions at $t = T$ are found to take the following forms:

Theorem 5. *At $t = T$, when $\Lambda_T > x_T$, the equilibrium bidding functions of the buyer and the seller take the forms:*

$$s_T(v_s) = max\{(F_b^{-1}(1 - \frac{x_T}{\Lambda_T}), v_s\}, \tag{3.25}$$

and

$$b_T(v_b) = \begin{cases} v_b, & \text{if } v_b < F_b^{-1}(1 - \frac{x_T}{\Lambda_T}) \\ F_b^{-1}(1 - \frac{x_T}{\Lambda_T}), & \text{if } b^*(v_b) \leq F_b^{-1}(1 - \frac{x_T}{\Lambda_T}) \leq v_b \\ b^*(v_b), & \text{if } F_b^{-1}(1 - \frac{x_T}{\Lambda_T}) < b^*(v_b) \end{cases}$$

where $b^*(\cdot)$ is the equilibrium bidding function of the one-to-one game (i.e. $b^*_{BPP}(\cdot)$ given by (3.3)).

The proof of the above Theorem simply follows by showing that the pair of bidding functions $s_T(\cdot)$ ad $b_T(\cdot)$ satisfy the equations (3.23) and (3.24); and are best responses to each other. However, we note that in this game the equilibrium bidding functions no longer comply to the conditions stated in Assumption 1.

Remark 2. *When competition among buyers comes into play, the game in the last period becomes similar to one where all winning buyers pay the first losing bid (=last winning bid), i.e. a* ***Secondary Price Auctioning Mechanism*** *(except for buyers with very high valuations, for whom bidding $b^*(v_b)(>$ first losing bid) is a better option in order to minimize the risk of "no trade" in the contingency of high valuations of the seller). Note that this result is also consistent with the findings of Vulcano et al. (2002).*

Similarly, for $t = 1, 2, \ldots, T-1$, the problem can be solved by backwards induction provided that the seller's revenue function for $t+1, \ldots, T$ is known. However, characterizing the closed form solutions of the bidding functions grow to be gradually more complicated as we move to earlier time periods. For instance, consider $t = T-1$. Again, if $\Lambda_{T-1} + \Lambda_T < x_{T-1}$, all buyers act as if in a one-to-one negotiation game against the seller. However, if $x_{T-1} < \Lambda_{T-1} + \Lambda_T$, the buyers and the seller should solve simultaneous revenue maximization problems as follows: Let $V_T(x_T)$ denote the seller's maximum revenue function when she has x_T units of inventory left at $t = T$. This function should be easy to characterize since we already know how the agents will act at $t = T$ for each level of inventory x_T given Λ_T. Hence, the seller's revenue maximization problem at $t = T-1$ takes the following form:

$$\begin{aligned} & \max_{\alpha_{T-1}} \Lambda_{T-1} \int_{G^{-1}_{b_{T-1}}(1-\alpha_{T-1})}^{\bar{b}_{T-1}} (b - v_s)\, g_{b_{T-1}}(b)\, db \\ & +V_T(x_{T-1} - \Lambda_{T-1}\alpha_{T-1}) \\ \text{subject to} \quad & \Lambda_{T-1}\alpha_{T-1} \leq x_{T-1} \end{aligned}$$

where $b_{T-1}(.)$ is the equilibrium bidding function of the buyers at $t = T-1$ and $G_{b_{T-1}}$ its cdf, where the function $b_{T-1}(.)$ is characterized by the following revenue maximization problem of buyers:

$$\max_b \int_{\underline{s}_{T-1}}^{b} (v_b - b) g_{s_{T-1}}(s) ds$$

where $s_{T-1}(.)$ is the equilibrium bidding function of the seller at $t = T - 1$. As before, s_{T-1} and b_{T-1} should be best responses to each other.

Note that the above results hold when the buyers can *only* observe the current inventory level and know the future market size information. If, on the other hand, they can also observe the past sales data, then they can infer further information regarding the valuation of the seller aside from its distribution information; in which case the above analysis fails to hold any longer.

3.3.4 Dynamic Negotiation Games under Uncertainty

In this part, we study a variant of dynamic negotiation problems where the primitives of the buyer valuation distribution are unknown. This type of multi-stage stochastic optimization problems has elicited much interest from various research communities and there are several established methodologies to expound them involving dynamic programming, stochastic programming and robust optimization. However, the problem usually remains hard to solve analytically. Therefore, in practice, one would typically solve the recursions numerically or resort to some approximations such as approximate dynamic programming or simulation. In a similar manner, we will introduce a class of policies that is motivated by the structure identified in the deterministic version of the problem as well as the solution of the stochastic one-to-one problem and confirm that these policies achieve "good" performance in the dynamic stochastic problem.

The problem setting is as follows: At each instant t, $t \in [0,T]$, independent dynamic negotiation games take place between the seller and the entire population of infinitesimal buyers whose valuation distribution function is revealed only at the instant t. The players know each other's distribution range for all $t \in [0,T]$ (and suppose that, for convenience, this range does not change across time). In this situation, the ARMC approach is again a viable choice for all parties. However, at this point we need to make the following assumption to ignore the "learning effect" for the seller (otherwise, the seller's problem becomes trivial as she employs the optimal pricing policy as soon as she finds out the value distribution function of buyers either by pure observation, or by inferring from the instantaneous sales amount).

Assumption 2. *The seller can neither observe the buyer value distribution function, F_b, nor the sales amount until the end of the sales horizon.*

Although the above assumption might seem unrealistic, it is in fact equivalent to assuming that the buyers' valuation distribution is continuously changing over time. Hence, observing the past sales will not help the seller in predicting the future sales. Moreover, with no information regarding the future based on current observations, the seller employs a stationary bidding policy starting at the instant t, i.e. $s_t = s$, $\forall t$.

With these observations, we are ready to state and prove the following Theorem, which emphasizes the analogy of the dynamic stochastic problem with the stochastic one-to-one problem.

Theorem 6. *The dynamic stochastic problem with unknown valuation functions but known ranges reduces to the dynamic deterministic problem of section 3.3.2, with F_s and F_b being uniform distribution functions on their given ranges* *at each t.*

(Proof in Appendix B.3.)

3.3.5 A Comparison of Seller Revenues in the Dynamic SPP vs. BPP Settings

By the analysis in Section 3.3.2, we know that the seller's optimal bidding strategy is given by:

$$s_t(v_s) = \max\{v_s, G_b^{-1}(1 - \frac{C}{\int_{t=0}^{T} \Lambda_t \, dt})\}$$

in the BPP market; and by:

$$s_t(v_s) = \max\{s^*_{SPP}(v_s), F_b^{-1}(1 - \frac{C}{\int_{t=0}^{T} \Lambda_t \, dt})\}$$

in the SPP market. This result makes the analogy between the two problems more explicit. However, since BPP is a mechanism that favors the buyers; the two optimal strategies, despite having essentially the same static structure, may produce very different revenues in SPP and BPP environments. This result will be more explicit in the numerical examples section. Still, in this part, we would like to briefly analyze the relationship between the seller revenues in the two above settings under the assumption of uncertainty; i.e. in the case where all distributions are assumed to be uniform in their respective ranges.

In this case, letting $\alpha_0 := 1 - \frac{C}{\int_{t=0}^{T} \Lambda_t \, dt}$, the seller's bidding function becomes:

$$s_t(v_s) = \max\{v_s, \frac{v_s}{2} + \frac{\alpha_0(\bar{v}_b - \underline{v}_b) + \underline{v}_b}{2}\}$$

in the BPP market, and:

$$s_t(v_s) = \max\{\frac{v_s + \bar{v}_b}{2}, \alpha_0(\bar{v}_b - \underline{v}_b) + \underline{v}_b\}$$

in the SPP market (which are found by the equations (3.3), (3.4); and (3.5), (3.6) respectively). Thus, the seller revenues take values:

$$\int_{t=0}^{T} \Lambda_t \Big[\int_{\max\{2v_s - \underline{v}_s, \alpha_0(\bar{v}_b - \underline{v}_b) + \underline{v}_b\}}^{\bar{v}_b} (\frac{\underline{v}_s}{2} + \frac{v_b}{2} - v_s)\, f_b(v_b)\, dv_b \Big]\, dt$$

in the BPP market, and:

$$\int_{t=0}^{T} \Lambda_t \Big[\int_{\max\{\frac{v_s + \bar{v}_b}{2}, \alpha_0(\bar{v}_b - \underline{v}_b) + \underline{v}_b\}}^{\bar{v}_b} (\max\{\frac{v_s + \bar{v}_b}{2}, \alpha_0(\bar{v}_b - \underline{v}_b) + \underline{v}_b\} - v_s)\, f_b(v_b)\, dv_b \Big]\, dt$$

in the SPP market, respectively.

Now consider the following cases:

(i) $2v_s - \underline{v}_s \le \alpha_0(\bar{v}_b - \underline{v}_b) + \underline{v}_b$, $\frac{v_s + \bar{v}_b}{2} \le \alpha_0(\bar{v}_b - \underline{v}_b) + \underline{v}_b$:

This situation might correspond to a case where the clearance value is relatively high (i.e. C is low), v_s relatively low, or $\bar{v}_b$ low. In this case, the revenue figures become:

$$rev(BPP) = \int_{t=0}^{T} \Lambda_t \Big[\int_{\alpha_0(\bar{v}_b - \underline{v}_b) + \underline{v}_b}^{\bar{v}_b} (\frac{\underline{v}_s}{2} + \frac{v_b}{2} - v_s)\, f_b(v_b)\, dv_b \Big]\, dt$$

and

$$rev(SPP) = \int_{t=0}^{T} \Lambda_t \Big[\int_{\alpha_0(\bar{v}_b - \underline{v}_b) + \underline{v}_b}^{\bar{v}_b} (\alpha_0(\bar{v}_b - \underline{v}_b) + \underline{v}_b - v_s)\, f_b(v_b)\, dv_b \Big]\, dt$$

Since $\frac{\underline{v}_s}{2} + \frac{v_b}{2} \le \frac{v_s}{2} + \frac{\bar{v}_b}{2} \le \alpha_0(\bar{v}_b - \underline{v}_b) + \underline{v}_b$ and the sales volumes are the same, $rev(BPP) < rev(SPP)$ in this case.

(ii) $2v_s - \underline{v}_s > \alpha_0(\bar{v}_b - \underline{v}_b) + \underline{v}_b$, $\frac{v_s + \bar{v}_b}{2} \le \alpha_0(\bar{v}_b - \underline{v}_b) + \underline{v}_b$:

In this case, possibly v_s is relatively large. The revenue figures become:

$$rev(BPP) = \int_{t=0}^{T} \Lambda_t \Big[\int_{2v_s - \underline{v}_s}^{\bar{v}_b} (\frac{\underline{v}_s}{2} + \frac{v_b}{2} - v_s)\, f_b(v_b)\, dv_b \Big]\, dt$$

and

$$rev(SPP) = \int_{t=0}^{T} \Lambda_t \Big[\int_{\alpha_0(\bar{v}_b - \underline{v}_b) + \underline{v}_s}^{\bar{v}_b} (\alpha_0(\bar{v}_b - \underline{v}_b) + \underline{v}_b - v_s)\, f_b(v_b)\, dv_b \Big]\, dt$$

Since $\frac{\underline{v}_s}{2} + \frac{v_b}{2} \leq \frac{v_s}{2} + \frac{\bar{v}_b}{2} \leq \alpha_0(\bar{v}_b - \underline{v}_b) + \underline{v}_b$ again, and the sales volume under BPP setting is lower, we again claim that $rev(BPP) < rev(SPP)$.

(iii) $2v_s - \underline{v}_s > \alpha_0(\bar{v}_b - \underline{v}_b) + \underline{v}_b$, $\frac{v_s+\bar{v}_b}{2} > \alpha_0(\bar{v}_b - \underline{v}_b) + \underline{v}_b$:

This situation may correspond to a case where v_s or $\bar{v}_b$ relatively high, or C is high. Now the revenues take values:

$$rev(BPP) = \int_{t=0}^{T} \Lambda_t \left[\int_{2v_s - \underline{v}_s}^{\bar{v}_b} (\frac{\underline{v}_s}{2} + \frac{v_b}{2} - v_s)\, f_b(v_b)\, dv_b \right] dt$$

and

$$rev(SPP) = \int_{t=0}^{T} \Lambda_t \left[\int_{\frac{v_s+\bar{v}_b}{2}}^{\bar{v}_b} (\frac{v_s + \bar{v}_b}{2} - v_s)\, f_b(v_b)\, dv_b \right] dt$$

respectively. Interestingly, for seller valuation ranges with high $\underline{v}_s$, it is possible to observe situations in which the BPP revenues can reach the level of, or even exceed the SPP revenues.

(iv) $2v_s - \underline{v}_s \leq \alpha_0(\bar{v}_b - \underline{v}_b) + \underline{v}_b$, $\frac{v_s+\bar{v}_b}{2} > \alpha_0(\bar{v}_b - \underline{v}_b) + \underline{v}_b$:

In this case, $\underline{v}_s$ or $\bar{v}_b$ is likely to be large. While the revenues become:

$$rev(BPP) = \int_{t=0}^{T} \Lambda_t \left[\int_{\alpha_0(\bar{v}_b - \underline{v}_b) + \underline{v}_b}^{\bar{v}_b} (\frac{\underline{v}_s}{2} + \frac{v_b}{2} - v_s)\, f_b(v_b)\, dv_b \right] dt$$

and

$$rev(SPP) = \int_{t=0}^{T} \Lambda_t \left[\int_{\frac{v_s+\bar{v}_b}{2}}^{\bar{v}_b} (\frac{v_s + \bar{v}_b}{2} - v_s)\, f_b(v_b)\, dv_b \right] dt$$

they are difficult to compare, since the sales volume under the BPP setting is higher. Therefore, it might again be possible to observe cases where the two revenue figures approach each other, especially if v_s is relatively low.

In conclusion, despite the intuition that BPP favors the buyers more, for some realizations of the parameter values (i.e. C high, v_s low, etc.), the two revenue figures might be similar.

3.4 Applications in Non-Stationary Environments

In this section, we consider dynamic stochastic problems where the support of valuation distributions are unknown and non-stationary. This type of multi-stage stochastic optimization problems has elicited much interest from various research communities and there are several established methodologies to expound them involving dynamic programming, stochastic programming and robust optimization. However, the problem usually remains hard to solve analytically. Therefore, in practice, one would typically solve the recursions numerically or resort to some approximations such as approximate dynamic programming or simulation. In a similar manner, we will introduce a class of policies that is motivated by the structure identified in the deterministic version of the problem as well as the solution of the stochastic one-to-one problem and confirm that these policies achieve "good" performance in the dynamic stochastic problem.

Before proceeding with the analysis, first we would like to shed light on the relevance of the results of Lan et al. (2006) and Lobel and Perakis (2010) to our problem, where both papers analyze the capacity rationing problem of a seller operating under limited demand information. Both papers employ a robust formulation approach and resort to "absolute regret minimization criterion" (ARMC) approach among others. The resulting optimal policies are in the form of a nested booking policy. However, as pointed out earlier, despite the analogy between the dynamic BPP problem of this paper and the classical capacity rationing problem, restricting the buyer bids to a fixed set of discrete fares and characterizing the worst-case scenario by a specific sequence of buyer arrivals (as is the case in these papers) would only be analyzing a special case of the general stochastic BPP problem. Therefore, we will proceed with the more general form of the problem where we allow for a continuous range of buyer bids changing dynamically over time, and assume no specific sequence or volume of buyer arrivals.

The problem with added time-varying nature of the valuations could seem to be far-fetched to the reader; however, it is commonly observed in some business settings, particularly in the real-estate sector. For instance, the following example describes such a setting:

Example: Consider a condo-developer who has $C = 375$ units to sell over $T = 15$ bi-monthly intervals. Assume that the market conditions remain stationary within an interval, but there is an observable transition in the buyer valuation distribution at the end of each period. In particular, the buyers' valuation in period t, $t \in \{1, 2, \ldots, 15\}$ is uniform in the range $[\mu(t) - \$300K, \mu(t) + \$300K]$ for an unknown and non-stationary parameter $\mu(t)$ given by the equation:

$$\mu(t) = \mu(t-1) + \delta(t), \text{ for } t = 2, 3, \ldots T; \quad \mu(1) = \$600K$$

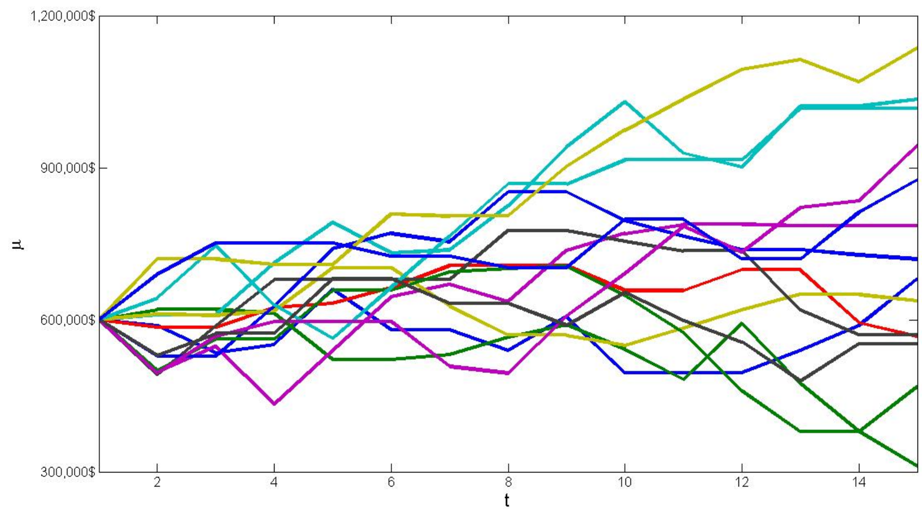
where $\delta(t)$ is the noise factor with the following distribution:

$$\delta(t) = \begin{cases} \mathbf{U}[-\$120K, 0], & \text{w.p } 0.4 \\ \$0, & \text{w.p } 0.2 \\ \mathbf{U}[0, \$120K] & \text{w.p } 0.4 \end{cases} \forall t$$

Some possible μ paths are given in Figure 3.1.

As exemplified above, the problem setting we will consider is as follows: At each discrete period t, $t \in \{1, 2, ...T\}$, independent dynamic negotiation games take place between the seller and an entire population of infinitesimal buyers

Figure 3.1: Examples of Parameter Paths



whose valuation distribution function and the valuation range are revealed at the beginning of period t. We will focus on a BPP setting, in the light of the previous discussion on the motivation problem.

Suppose the distribution function for buyer valuations in period t is denoted by $F_{\mu(t)}$ and is characterized by the single parameter $\mu(t)$, which is unknown and varying in time. The seller regards $F_{\mu(t)}$ as uniform distribution over an unknown range (the reasoning for this assumption comes from the solution of the dynamic problem with ARMC approach, i.e. Theorem 6). To parameterize the uniform distribution with a single variable, we will assume that the length of the distribution support, $l(t)$, is known and given at all times t; but the middle point of the range, $\mu(t)$, remains unknown. In particular:

$$\mu(t) = \hat{\mu}(t) + \delta(t),\ \forall t \in \{1, 2, \ldots, T\}, \tag{3.26}$$

where $\hat{\mu}(t)$ is the forecasted value of the parameter and $\delta(t)$ is the unknown noise factor. Hence, given $\mu(t)$, the seller regards the pdf of the buyer valuations as:

$$f_{\mu(t)}(v_b) := \begin{cases} \frac{1}{l(t)} & \text{if } \mu(t) - \frac{l(t)}{2} \leq v_b \leq \mu(t) + \frac{l(t)}{2} \\ 0 & \text{otherwise} \end{cases} \tag{3.27}$$

We make no distributional assumptions regarding δ, except that it lies in a basic compact algebraic set $\mathbf{\Delta}$. This approach gives us more freedom than a classical stochastic optimization model which often requires uncertainty factors to be independent across time periods.

As the buyers also regard the seller's valuation as uniformly distributed in its given range, they bid according to the uniform bidding function $b(v_b) = \frac{v_b}{2} + \frac{v_s}{2}$ (which follows from the equilibrium bidding function of the one-to-one BPP game).

For simplicity, we will pursue the analysis on the example problem stated above.

If we resize the problem by dividing all monetary values by \$300K, and carry the analysis to a fluid setting, we obtain the following robust optimization problem:

$$\max_{\{s_t, t=1,...15\}} \quad [\sum_{t=1}^{15} \frac{100}{4}((1.5 + \sum_{i=1}^{t} \delta(t))^2 - 4s_t^2)] \tag{3.28}$$

$$\text{subject to} \quad 100 \sum_{t=1}^{15}((1.5 + \sum_{i=1}^{t} \delta(t)) - 2s_t) \leq a_t \, \forall t, \tag{3.29}$$

$$\sum_{t=1}^{15} a_t \leq 375 \quad a.s. \, \forall t \tag{3.30}$$

$$-0.2 \leq \delta(t) \leq 0.2, \, \forall t. \tag{3.31}$$

which takes the following form, if we would like to make the objective function independent of noise factors:

$$\max_{\{s_t, t=1,\ldots 15\}} \quad z \tag{3.32}$$

$$\text{subject to} \quad z \leq [\sum_{t=1}^{15} \frac{100}{4}((1.5 + \sum_{i=1}^{t} \delta(i))^2 - 4s_t^2)], \tag{3.33}$$

$$100((1.5 + \sum_{i=1}^{t} \delta(i)) - 2s_t) \leq a_t \, \forall t, \tag{3.34}$$

$$\sum_{t=1}^{15} a_t \leq 375 \quad a.s. \, \forall t \tag{3.35}$$

$$-0.2 \leq \delta(t) \leq 0.2, \quad a_t \geq 0, \, \forall t. \tag{3.36}$$

which is an uncertain quadratically-constrained (QC) problem. This class of problems is analyzed in great detail by many researchers, including A. Ben-Tal and Roos (2002), who build an SDP which approximates the NP-hard robust counterpart, and Goldfarb and Iyengar (2003) who reformulate the uncertain QC problem as an SOCP problem and solve the latter. Although the solution methodologies proposed in these papers require much less computational effort than solving the original problem, the main problem with the above formulation is that it essentially leads to an open-loop solution (i.e. a pricing policy s_t that does not make use of the past disturbances), therefore yielding conservative results for practical use.

However, the hope is that, rather than open-loop policies that do not take into account the system dynamics, some simple but tractable functional forms might be sufficient for good performances, if not for optimality. "Affine policies" of past disturbances, i.e. pricing policies of the form: $s_t = m_t + \sum_{i=1}^{t} B_{t,i}\delta(i)$, could be one such policy. This approach is not new in the literature. It has been originally advocated in the context of stochastic programming (see Garstka and J.-B.Wets (1974), and references therein), where such policies are known as decision rules. More recently, the idea has received renewed interest in robust optimization (A. Ben-Tal and Nemirovski (2004)), and has been extended to various contexts for solving specific types of optimization problems, which vary from linear and quadratic programs (A. Ben-Tal and Nemirovski (2005), Kerrigan and Maciejowski (2004)) to conic and semidefinite (A. Ben-Tal and Nemirovski (2005), Bertsimas and Brown (2007)).

The following formulation of D. Bertsimas and Parrilo (2010) which is motivated by the inventory replenishment problems in uncertain environments seems to be closest to our case. The authors consider a one-dimensional, discrete, linear, time-varying dynamical system:

$$x_{k+1} = \alpha_k x_k + \beta_k u_k + \gamma_k w_k$$

where $\alpha_k, \beta_k, \gamma_k$ are known scalars, the initial state x_1 is specified, w_k's are bounded ($w_k \in W_k$) random disturbances, and the goal is to find the controls u_k's that obey the constraints $u_k \in L_k, U_k$, for known and fixed values L_k, U_k, and minimizing the following Bellman equation:

$$J_k(x_k) = \min_{L_k \leq u_k \leq U_k} [c_k u_k + \max_{w_k \in W_k} [h_k(x_k + u_k + w_k) + J_{k+1}(x_k + u_k + w_k)]]$$

where the function $h_k : \mathbf{R} \to \mathbf{R} \cup \{\infty\}$ is convex and coercive; c_k's fixed and known.

Note that our problem can almost be expressed in the same way as the above system. In particular, denoting x_t as the current inventory level of the seller, s_t as the controls and $\delta(t)$ as the noise factors, the dynamic system equation becomes:

$$x_{t+1} = x_t - (\hat{\mu}(t) + 0.5l(t) + \delta(t) - 2s_t)$$

and the Bellman equation takes the form:

$$J_k(x_k) = \min_{L_k \le s_k \le U_k} [\max_{\delta_k \in \Delta_k} [\frac{\Lambda_k}{l(k)}((s_k)^2 - (s_k^*(x_k))^2) + J_{k+1}(x_k + u_k + w_k)]]$$

where $L_k = \min\{v_s, 0.5\Lambda_k(\hat{\mu}(k) + 0.5l(k) + \delta(k) + \underline{v}_s - \frac{x_k}{l(t)})\}$, $U_k = 0.5(\hat{\mu}(k) + 0.5l(k) + \delta(k) + \underline{v}_s)$ and $s_k^*(x_k) = \left[(\sum_{t=k}^{T} \Lambda_t \frac{\mu(t)+0.5l(t)}{l(t)}) - x_k\right]/(2\sum_{t=k}^{T} \frac{\Lambda_t}{l(t)}) + 0.5\underline{v}_s$. Clearly, in our setting, L_k, U_k are not known in advance and are revealed during the course of time, and $c_k = 0$, $\forall k$. But the main difference between the two settings arises from the fact that in our problem, $h_k(.)$ function is a function of "future" noise factors as clear in the definition of $s_k^*(x_k)$. Moreover, since the seller cannot sell more than her capacity, there are linear constraints that couple controls s_t across different time steps.

D. Bertsimas and Parrilo (2010) were able to show that the optimal controls in the above problem are affine functions of the past disturbances. However, they also show that the optimality of affine policies is easily violated even when the problem assumptions are relaxed slightly. For instance, they show that optimality no longer holds when there exist linear constraints coupling the controls u_t across different time-steps, which is exactly the case in our problem due to the capacity constraints. Hence, in our problem, affine policies are no longer optimal, but hopefully yield results that are close to optimal.

Moreover, we have an additional structural support in the favor of using affine policies. To see this, let us take a step back and consider the deterministic problem. The form of the optimal pricing policy in the deterministic problem is given in the following Proposition.

Proposition 2. *If μ path is known at $t = 1$ (i.e. δ is known), the optimal bidding function of the seller having valuation v_s in a dynamic negotiation* *game under BPP setting is given by:*

$$\hat{s}_t(v_s) = max\{b^*(v_b^0), v_s\},\ \forall t \in \{1, 2, \ldots, T\}$$

where v_b^0 *satisfies:* $\sum_{t=1}^{T} \Lambda_t \int_{v_b^0}^{\mu(t)+0.5l(t)} f_{\mu(t)}(v_b)dv_b = C$.

Hence, the optimal clairvoyant policy is a stationary policy where the optimal bid is given by:

$$s_t^*(v_s) = \max\{v_s, \frac{\left[\sum_{t=1}^{T} \Lambda_t \frac{\mu(t)+0.5l(t)}{l(t)}\right] - C}{2\sum_{t=1}^{T} \frac{\Lambda_t}{l(t)}} + \frac{v_s}{2}\},\ \forall t,$$

when the buyer valuations are uniform. This policy can also be written in feedback form:

$$s_t^*(v_s) = \max\{v_s, \frac{\sum_{t'=t}^{T} \Lambda_{t'}(\frac{\mu(t')+0.5l(t')}{l(t')}) - x(t)}{2\sum_{t'=t}^{T} \frac{\Lambda_{t'}}{l(t')}} + \frac{v_s}{2}\},\ \forall t.$$

where $x(t)$ is the inventory amount at the beginning of period t. That is, given the μ path, the optimal policy assumes the following feedback form (provided that $s_t^* \geq v_s$):

$$s_t^* = A_t + B_t x(t) + \sum_{i \geq t} C_{i,t}\mu(i) \tag{3.37}$$

for some known coefficients A_t, B_t and $C_{i,t}$, $\forall t$.

Thus, inspired by the optimal policy of the deterministic problem, a candidate closed-loop policy for the stochastic problem could be defined as:

$$s_t = A_t + B_t x(t) + C_{t,t}\mu(t) + \sum_{i>t} C_{i,t}(\hat{\mu}(i) + \mathbf{E}[\delta(i)]) \tag{3.38}$$

for some appropriate constants A_t, B_t and $C_{i,t}$, $\forall t$, $\forall i > t$.

However, optimizing over the coefficients A_t, B_t and $C_{i,t}$ violates the convex nature of the maximum regret minimization problem (3.32)-(3.36), since $x(t)$

is dependent on s_i, $i = 1, 2, \ldots, t-1$, $\forall t$. Fortunately, noting that both $x(t)$ and $\mu(t)$ are functions of the past noise factors $\delta(i), i \leq t$; it is possible to recover the form (3.38) by defining the optimal pricing policy s as an affine function of the past uncertainties:

Proposition 3. *Defining:*

$$s_t = m_t + \sum_{j=1}^{t} B_{t,j}\delta(j) \tag{3.39}$$

the formulation (3.38) can be recovered.

(Proof in Appendix B.4.) The proof simply follows from observing that if the seller bid is defined as in (3.39), the current capacity $x(t)$ can be expressed as an affine function of previous noise factors $\delta(j)$, $j \leq t$, in a uniform distribution setting. Thus, the equivalence of the two formulations follows.

However, the opposite of this claim is not true, i.e. it is not possible to recover equations (3.39) by (3.38). That is because the degree of freedom is larger for the set of equations (3.39) (i.e. given the values of m_t and $B_{t,i}$, $\forall t, i \leq t$, there are more than one solution for A_t, B_t, $C_{i,t}$, $\forall t, i > t$). To see this, consider the simplest case, T=2. We know that for any given δ path, the feedback heuristic pricing policy can be expressed as:

$$\begin{aligned} s_{1,f} &= A_1 + B_1 x_1 + C_{11}\delta(1) \\ s_{2,f} &= A_2 + B_2(x_1 - (\hat{\mu}(1) + 0.5l(1) + \delta(1) - 2s_{1,f})) + C_{22}\delta(2) \end{aligned}$$

The closed-loop heuristic policy, on the other hand, is expressed as:

$$\begin{aligned} s_{1,c} &= m_1 + K_{11}\delta(1) \\ s_{2,c} &= m_2 + K_{21}\delta(1) + K_{22}\delta(2) \end{aligned}$$

If $s_{i,c} = s_{i,c}$, $i = 1, 2$ holds, for any given values of A_1, B_1, C_{11} and C_{22}, it can be shown that:

$$\begin{aligned} m_1 &= A_1 + B_1 x_1 \\ m_2 &= A_2 + B_2 x_1 - B_2(\hat{\mu}(1) + 0.5 l(1)) + 2B_2 A_1 + 2B_2 B_1 x_1 \\ K_{11} &= C_{11} \\ K_{21} &= -B_2 + 2B_2 C_{11} \\ K_{22} &= C_{22} \end{aligned}$$

But since the degree of freedom is larger for the second set of equations, it is not possible to trace back A_1, A_2 and B_1 given the values of m_1, m_2, K_{11}, K_{21} and K_{22} (i.e. there are more than 1 possible solution for these variables). The case for $T > 2$ is similar.

Hence, supported by previous research and the structural form of optimal policy in the deterministic problem, we confine our search to affine pricing policies. On the example case, the formulation becomes as follows:

$$\begin{aligned} \max_{\{m_t, B_{t,i}, t=1,\ldots 15, i=1,\ldots,t\}} \quad & z \\ \text{subject to} \quad & z \leq \frac{100}{4}[\sum_{t=1}^{15}((1.5 + \sum_{i=1}^{t}\delta(i))^2 - 4s_t^2)] \\ & 100((1.5 + \sum_{i=1}^{t}\delta(i)) - 2s_t) \leq a_t \,\forall t, \\ & \sum_{t=1}^{15} a_t \leq 375 \quad a.s. \,\forall t \\ & s_t = m_t + \sum_{i=1}^{t} B_{t,i}\delta(t) \,\forall t \\ & -0.2 \leq \delta(t) \leq 0.2, \quad a_t \geq 0, \,\forall t. \end{aligned}$$

Solving the above problem could still be computationally challenging. Therefore, rather than accounting for the whole uncertainty set, we will sample an appropriate number N (which is to be found by trial-and-error) of scenarios and model the pricing problem of the seller under this approximation with the objective of "minimizing the worst case regret". This approach ensures tractability and is supported by previous works. For instance, Perakis and Roels (2007) argue that rather than spanning the entire uncertainty set in the formulation, accounting for the twenty-fifth and seventy-fifth percentiles produces much better regret-minimizing policies which perform substantially better on average without deteriorating much in terms of the worst-case regret performance. Therefore, even with a moderate number N, we hope to appropriately represent the uncertainty set while avoiding computational complexity. Observe that if $N \to \infty$, the problem becomes equivalent to finding the heuristic that would minimize the maximum regret in the entire uncertainty set.

To this end, we define the following quantities:

$\Pi(s, \delta)$:= net revenues to be obtained by the pricing policy s under the realized noise vector δ;

$\Pi^*(\delta)$:= maximum revenues to be obtained under the realized noise vector δ.

Clearly, for the example problem (3.28)-(3.31):

$$\Pi(s, \delta) = \sum_{t=1}^{15} 0.25((1.5 + \sum_{i=1}^{t} \delta(i))^2 - 4(s_t)^2)$$

$$\Pi^*(\delta) = \sum_{t=1}^{15} 0.25((1.5 + \sum_{i=1}^{t} \delta(i))^2 - 4(s^*(\delta)_t)^2)$$

where $s^*(\delta)_t = \max\{v_s, \frac{\left[\sum_{t=1}^{15} 100 \times (1.5 + \sum_{i=1}^{t} \delta(i))\right] - 375}{2 \sum_{t=1}^{15} 100}\}$, $\forall t$,.

And the final form of the problem to be solved is the following:

$$\begin{aligned}
\min_{\{m_t, B_{t,i}, t=1,\ldots 15, i=1,\ldots,t\}} \quad & z \\
\text{subject to} \quad & z \geq [\Pi^*(\delta^j) - \Pi(s^j, \delta^j)], \quad \forall j = 1, 2, \ldots, N, \\
& 100((1.5 + \sum_{i=1}^{t} \delta^j(i)) - 2s_t^j) \leq a_t^j, \quad \forall t, \forall j = 1, 2, \ldots, N, \\
& \sum_{t=1}^{15} a_t^j \leq 375 \quad a.s. \forall j = 1, 2, \ldots, N, \\
& s_t^j = m_t + \sum_{i=1}^{t} B_{t,i} \delta^j(t), \quad \forall t, \forall j = 1, 2, \ldots, N, \\
& a_t^j \geq 0, \quad \forall t, \forall j = 1, 2, \ldots, N.
\end{aligned}$$

In the following section, we compare the performances of the affine closed-loop policy and some more basic heuristic methodologies.

3.5 Numerical Results

3.5.1 Comparison of BPP and SPP settings

We will start the numerical analysis by comparing the seller revenues in the BPP and SPP settings in order to illustrate how the BPP mechanism favors the buyers to the disadvantage of the seller. To this end, we tabulate the maximum revenue obtained by the seller in the BPP environment as a percentage of the revenues collected in the SPP setting, under various buyer distribution functions F_b and various ranges for the seller distribution F_s [4]. We first tabulate the results as we vary the load factor (i.e. $\frac{C\Lambda^*}{T}$ ratio where Λ^* is the

[4] To be able to obtain closed-form buyer bidding functions, we assumed F_s is uniform in all cases.

revenue-maximizing sales rate for SPP environment), and then for different seller values (i.e. where v_s is represented as the corresponding fractile value of the valuation range $[\underline{v}_s,\ \bar{v}_s]$). Recall that at each instant t, trade occurs if the bid of the arriving buyer $b^*(v_b)$ is greater than or equal to the bid of the seller $s^*(v_s)$. Also recall that, given F_s and F_b, the equilibrium bidding functions $s^*(.)$ and $b^*(.)$ are found by solving the pair of ODE's (3.1) and (3.2) simultaneously; where k=0 in the SPP and $k = 1$ in the BPP environment. Moreover, the resulting bidding functions $b^*(.)$ and $s^*(.)$ are given in equation pairs (3.3) and (3.4) in a BPP and by the pair (3.5)-(3.6) in an SPP setting.

In the following scenarios, Λ=1 and T=50, while C is varied to give the desired ratio.

We pick 200 sample paths under each distribution, where the parameters defining the distribution are varied randomly. However, we ensure that there is trade with probability one at each period t at least under the SPP setting (i.e. the upper bound of buyer bids always exceeds the maximum seller bid) in each of the sample paths generated. We then note the median, 0.1^{th} and 0.9^{th} fractiles of the (BPP/SPP) revenue ratios under all scenarios as in the following two tables.

“Table 3.1 about here”

Table 3.1: BPP vs. SPP under various C values

	$F_b \sim \mathbf{U}[\underline{v}_b, \bar{v}_b]$ $\underline{v}_b \sim [1,3], \bar{v}_b = \underline{v}_b + 2$	$F_b \sim \mathbf{N}(\mu_b, \sigma_b)$ $\mu_b \sim [1,3]\sigma_b \sim [0.5, 1]$	$F_b \sim \mathbf{exp}(\mu_b)$ $\mu_b \sim [0.2, 0.8]$
	10%, 50%, 90%	10%, 50%, 90%	10%, 50%, 90%
$C = 5$	4.76%, 51.32%, 53.55%	44.63%, 56.72%, 64.17%	20.45%, 32.82%, 43.00%
$C = 10$	9.52%, 51.32%, 53.55%	43.85%, 54.65%, 61.76%	20.45%, 37.33%, 60.26%
$C = 20$	19.04%, 52.18%, 53.39%	47.10%, 56.64%, 68.99%	32.86%, 65.16%, 93.40%
$C = 30$	28.56%, 53.42%, 63.18%	49.38%, 67.32%, 85.47%	47.90%, 81.50%, 103.02%
$C = 45$	41.32%, 59.13%, 82.16%	65.38%, 82.59%, 96.61%	55.58%, 86.29%, 106.04%

“Table 3.2 about here”

Table 3.2: BPP vs. SPP under various v_s values

	$F_b \sim \mathbf{U}[\underline{v}_b, \bar{v}_b]$ $\underline{v}_b \sim [1,3], \bar{v}_b = \underline{v}_b + 2$	$F_b \sim \mathbf{N}(\mu_b, \sigma_b)$ $\mu_b \sim [1,3]\sigma_b \sim [0.5,1]$	$F_b \sim \mathbf{exp}(\mu_b)$ $\mu_b \sim [0.2, 0.8]$
	10%, 50%, 90%	10%, 50%, 90%	10%, 50%, 90%
$v_s = 0\%$	19.04%, 52.18%, 53.39%	47.10%, 56.64%, 68.99%	32.86%, 65.16%, 93.40%
$v_s = 20\%$	16.49%, 46.15%, 46.88%	41.42%, 50.97%, 60.42%	27.90%, 56.92%, 79.35%
$v_s = 40\%$	13.49%, 38.44%, 38.92%	32.41%, 42.14%, 47.93%	22.20%, 46.19%, 59.74%
$v_s = 60\%$	9.89%, 28.74%, 29.87%	17.79%, 30.03%, 34.57%	15.63%, 32.87%, 39.53%
$v_s = 80\%$	5.49%, 16.44%, 16.91%	4.96%, 15.50%, 19.16%	8.22%, 16.74%, 19.58%

As clear from the Tables, net revenues of the seller in a BPP setting are generally less than the revenues in the list price (SPP) environment as expected. The discrepancy between the two figures tends to be larger when the capacity is too low, the valuation of the seller is too high, or the uncertainty in the system increases; which is also in alignment with the results of the analysis in section 3.3.5. Again consistent with the given section, for high values of the load factor, the profits under BPP setting may approach or surpass the profits under the SPP setting. This situation can be explained as follows: Recall that in the SPP setting, the sales price is determined by the seller bid only. Thus, the seller faces a dilemma by setting a lower bid and attaining a higher volume of trade with a smaller premium per trade; or setting a higher bid but risking spoilage. In the BPP setting, on the other hand, she sets her own reserve value without taking into account the premium she will earn, since solely the buyer bids determine the final price. Also, recall that the buyer bids in the BPP setting are given by $b^*_{BPP}(v_b) = 0.5\underline{v}_s + 0.5v_b$, which approach the buyer bids in the SPP setting, namely $b^*_{SPP}(v_b) = v_b$, for the ranges of seller valuation for which $\underline{v}_s$ is high relative to average buyer valuation. Therefore, when BPP/SPP ratio is over 1, it is likely that this is due to the combination of the two factors: The seller sets a high price to capture a higher premium from the buyers whose valuation exceeds her price, therefore attains a low sales volume under the SPP setting; while she is able to attain a higher sales volume under the BPP setting together with an increasing set of prices rather than all buyers paying a single list price as in the SPP setting.

3.5.2 The Effect of the Negotiation Parameter

In this subsection we analyze the effect of the “buyer's negotiation power” (which is reflected in the parameter k) on the seller revenues. To this end, we consider a dynamic setting where the buyers and the seller both have uniform valuation distributions on the ranges $[\underline{v}_b, \bar{v}_b] = [1, 3]$ and $[\underline{v}_s, \bar{v}_s] = [0.5, 1.5]$ respectively. Assume that the buyers arrive according to a Poisson distribution with rate $\Lambda = 1$ per period, for a sales horizon of $T = 50$ periods. Recall that the buyer and the seller bidding functions in the dynamic problem for a given value of the parameter k take the forms:

$$b^*(v_b) = \frac{v_b}{1+k} + \frac{k\underline{v}_s}{2} + \frac{k(1-k)\bar{v}_b}{2(1+k)}, \; \forall v_b \in [\underline{v}_b, \bar{v}_b] \tag{3.40}$$

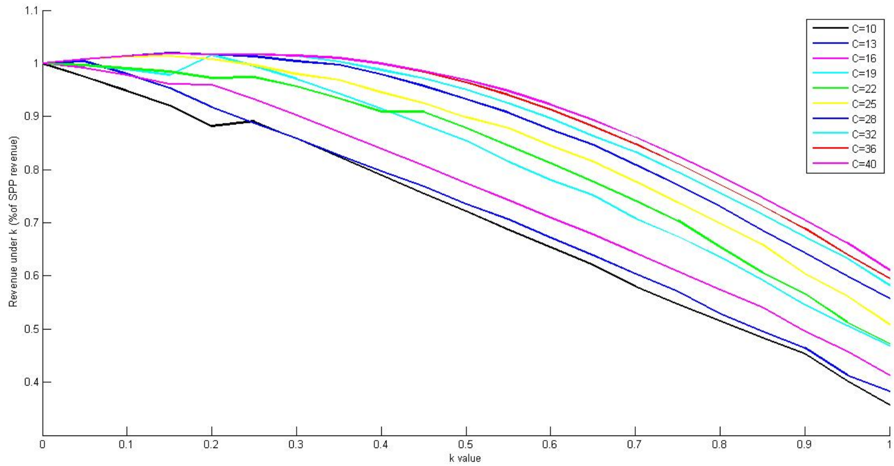
$$s_t^*(v_s) = \max\{v_s, G_b^{-1}(1 - \frac{x(t)}{\int_{\tau=t}^{T} \Lambda_\tau d\tau})\}, \; \forall t, \forall v_s \in [\underline{v}_s, \bar{v}_s] \tag{3.41}$$

respectively, where $x(t)$ is the remaining inventory at t, and $G_b(.)$ is the cdf of the buyer bidding function $b(.)$.

We vary the value of k from 0 (which corresponds to an SPP setting) to 1 (which corresponds to a BPP setting) and use 500 random instances. The ratio of average seller revenues to the average revenues under the SPP setting at various levels of seller capacity (starting from $C = 10$ to $C = 40$ in increments of 3) is given in the Figure 3.2.

An interesting observation in this experiment is that the seller might actually

Figure 3.2: Seller revenues (as a percentage of revenue at $k = 0$) for various k and C values



benefit from a slight shift in negotiation power for higher load factor values, $\frac{C\Lambda}{T}$. This is mainly due to the fact that the buyer bids might first increase and then decrease in k for lower-valued buyers (For instance, take a buyer with $v_b = 1.2$. His bid will be equivalent to $b(v_b) = v_b = 1.2$ for $k = 0$; $b(v_b) = \frac{v_b}{1.2} + \frac{0.2\times 0.5}{2} + \frac{0.2\times 0.8\times 3}{2\times 1.2} = 1.25$ for $k = 0.2$, and $b(v_b) = \frac{v_b}{2} + \frac{0.5}{2} = 0.85$ for $k = 1$). As the load factor increases, it is more prevalent to accept lower-valued buyers, who now bid highest at moderate values of k rather than at $k = 0$ or at $k = 1$. This observation is also consistent with the results of the previous subsection which suggests that the seller can also do as good or better under BPP setting than under the SPP setting if she has abundant capacity.

3.5.3 An Analysis about the Effect of Uniform Distribution Assumption

Next, we would like to investigate the seller's loss when she does not have the real distribution information and assumes that the buyers' valuations are distributed uniformly in their range, which is a natural conclusion of the ARMC approach. Our experiments contrast the revenues obtained by the seller in the "no distribution information" setting to the revenues in the "full-information" setting. To this end, consider the revenue maximization problem of a seller who wants to sell a number of units over T=15 time periods, where the market size is Poisson with rate Λ=100 per period.

We restricted the buyer valuations to belong in the range $[\underline{v}_b,\ \bar{v}_b]$ for each distribution in each period t, $t \in \{1, 2, \ldots, 15\}$. For the Normal and Gumbel distributions, we extracted the mean as the midpoint of the range and selected the standard deviation σ by assuming that the range is equal to $\pm 3\sigma$. For the exponential distribution we assumed that the valuation of a typical consumer is given by $\underline{v}_b + w$ where w is exponentially distributed in $[0,\ \bar{v}_b - \underline{v}_b]$ and its rate parameter μ is selected so that the probability that w lies in that range is 99.5% (this is consistent with the $\pm 3\sigma$ assumption of the Normal distribution). In each test case, we assumed that the buyers bid believing that the seller's valuation distribution is uniform in the range $[\underline{v}_s,\ \bar{v}_s] =$ [\$750K, \$2000K]; which induces a buyer bidding function $b^*(v_b) = \min\{v_b, 0.5v_b + 0.5\underline{v}_s\}$.

The sets of results summarized in Tables 3.3-3.4 illustrate the performance of the policy under uniform distribution assumption in a variety of settings as we varied the range of the WtP distribution, the inventory of the seller, and also the seller valuation. In Table 3.3, the seller's valuation is assumed to be fixed at v_s = \$1000K, and the inventory of the seller and the buyer valuation range are varied to test different cases. In Table 3.4, the inventory of the seller is fixed at $C = 500$ where the valuation range of the buyers and the seller's valuation are varied. We display the revenues of the no-information case as a percentage of the revenues of the full-information case, which is the maximum revenues to be achieved.

"Table 3.3 about here"

Table 3.3: The Ratio of Seller Revenues under Uniform Distribution Assumption to the Revenues in Full Information Setting

	$[\underline{v}_b, \bar{v}_b]$ = [\$500K, \$1500K] C = 250, 500, 750	$[\underline{v}_b, \bar{v}_b]$ = [\$1000K, \$2000K] C = 250, 500, 750	$[\underline{v}_b, \bar{v}_b]$ = [\$1000K, \$3000K] C = 250, 500, 750
Exp.	96.49%, 100%, 100%	44.83%, 71.19%, 90.65%	47.13%, 63.95%, 80.34%
Normal	92.65%, 100%, 100%	86.85%, 97.00%, 100%	92.14%, 98.35%, 100%
Gumbel	85.15%, 100%, 100%	32.56%, 60.41%, 82.50%	37.84%, 52.64%, 73.59%

"Table 3.4 about here"

Table 3.4: The Ratio of Seller Revenues under Uniform Distribution Assumption to the Revenues in Full Information Setting

	$[\underline{v}_b, \bar{v}_b]$ = [\$500K, \$1500K] v_s =\$0.75M, \$1M, \$1.5M	$[\underline{v}_b, \bar{v}_b]$ = [\$1000K, \$2000K] v_s =\$0.75M, \$1M, \$1.5M	$[\underline{v}_b, \bar{v}_b]$ = [\$1000K, \$3000K] v_s =\$0.75M, \$1M, \$1.5M
Exp.	80.46%, 100%, 100%	80.47%, 80.46%, 100%	68.34%, 68.52%, 98.40%
Normal	97.87%, 100%, 100%	98.35%, 97.00%, 100%	98.86%, 98.35%, 99.28%
Gumbel	98.24%, 100%, 100%	70.49%, 60.41%, 100%	61.21%, 52.64%, 98.19%

As the figures in the Table 3.3 and 3.4 suggest, the uniform distribution assumption performs especially well when the underlying distribution is normal. It may perform poorly for the exponential and Gumbel distribution settings under very low capacity and moderate seller values; which mainly stems from the fact that if the underlying distribution is too skewed, the uniform distribution assumption yields a significant miscalculation in the value of the optimal bid in the critical settings as the ones described. If the capacity is sufficiently large, the initial mishap could be remedied quickly as the bid of the uniform distribution assumption converges to the real optimal bid value in the earlier periods of the sales horizon, hence resulting in low revenue loss. If the seller valuation is too large, again the two revenue figures are equivalent or very close to each other, which is because in this case the optimal bid is the seller valuation itself regardless of the underlying distribution.

3.5.4 Stochastic Dynamic BPP Problem

Example 1 (continued): Recall the problem of a seller who has C=375 units to sell over T=15 time periods, where the market size is Poisson with rate Λ=100 per period. The buyer valuation distribution is uniform on the range $[\mu(t) - \$300K, \mu(t) + \$300K]$, with the parameter μ changing according to the formula:

$$\mu(t) = \mu(t-1) + \delta(t)$$

with $\mu(1)$=\$600K, where $\delta(t)$ is the random drift given by:

$$\delta(t) = \begin{cases} -d, & \text{w.p } 0.4 \\ 0, & \text{w.p } 0.2 \quad \forall t \\ d, & \text{w.p } 0.4 \end{cases}$$

where $d \sim \mathbf{U}[0, \$120K]$. Suppose the buyers bid according to the bidding function $b^*(v_b) = 0.5v_b$. In the base case, assume we do not account for the salvage value of the seller, hence $v_s = 0$.

We solve the scenario-based optimization problem for various seller valuation (v_s), capacity (C), and noise-size ($d := |\delta|$) values with N=150 scenarios; and compare the results with the simple "expected value (EV)" heuristic, where all stochastic variables in the problem are assumed to take their expected values, and with the solution of the uncertain QC-formulation given in (3.32)-(3.36), which we call "uncertain QC (UQC)" solution. For all policies, we compute the revenues obtained by applying the proposed bid values on a random sample of 1000 scenarios, and state the average of these revenues as a percentage of the absolute upper bound of revenues, i.e. the revenues produced by the "clairvoyant" policy. We also investigate the effect of reformulating and resolving the problem at the beginning of each period according to current capacity and buyer valuation values. All problems are solved via the CVX package developed by Grant and Boyd (2008) for MATLAB using a version 7.5.0 and on a computer that has 4 GB of RAM. The results are given in the Tables 3.5, 3.6, 3.7 and 3.8[5].

"Table 3.5 about here"

Table 3.5: Changing v_s

	$v_s = 0$	$v_s = \$120K$	$v_s = \$180K$	$v_s = \$240K$
closed-loop	93.28%	91.21%	89.32%	86.24%
EV heuristic	70.81%	60.44%	57.72%	60.18%
UQC solution	43.12%	43.17%	44.17%	47.13%
closed-loop (res.)	96.36%	95.35%	93.98%	91.10%
EV (resolved)	90.17%	88.84%	87.62%	85.73%

"Table 3.6 about here"

Table 3.6: Changing C

	$C = 185$	$C = 375$	$C = 560$	$C = 750$
closed-loop	90.80%	93.28%	94.94%	95.74%
EV heuristic	66.64%	70.81%	74.47%	78.51%
UQC solution	82.17%	43.12%	29.48%	22.80%
closed-loop (res.)	93.58%	96.36%	97.72%	97.97%
EV (resolved)	82.80%	90.17%	93.16%	94.85%

"Table 3.7 about here"

[5]Note that since the problem is resolved in each period, the z^* value which indicates the in-sample performance is not applicable for the iterative closed-loop policy and the expected value heuristics.

Table 3.7: Changing d

	d = \$60K	d = \$120K	d = \$240K	d = \$480K
closed-loop	96.63%	93.28%	86.50%	77.91%
EV heuristic	72.19%	70.81%	66.76%	59.93%
UQC solution	15.12%	43.12%	50.71%	50.68%
closed-loop (res.)	98.86%	96.36%	89.51%	80.93%
EV (resolved)	92.36%	90.17%	85.64%	78.92%

"Table 3.8 about here"

Table 3.8: Changing N

$N = 50$		$N = 150$		$N = 200$		$N = 250$	
z^*	avg. rev	z^*	avg. rev	z^*	avg. rev.	z^*	avg. rev
\$84K (5.9%)	92.78%	\$100K (6.2%)	93.28%	\$122K (7.6%)	92.85%	\$124K (7.7%)	93.07%

Here are a few remarks to note:

1. The closed-loop policy always outperforms the open-loop formulation of the uncertain QC problem, and the simple expected value heuristic. Moreover, as clear from the figures, there exist significant gains in resolving the problem at the beginning of each period with the current data and more accurate future forecast figures.

2. The expected value heuristic also performs well if it is resolved at each period. This phenomenon can be explained by the fact that feedback-type policies perform well if tracked in a smart manner; and is also in accordance with the findings of the literature, e.g. see Besbes and Maglaras (2009) for a similar argument again regarding the real-estate sector.

3. The gap between the simple expected value heuristic (resolved) and the closed-loop heuristic (resolved) tends to become larger in the capacity-constrained settings. This result is quite intuitive, since the scenario-

based approach can account for various states of the world and prevent the inventory from depleting much earlier than the end of the period; whereas the myopic approach could lead to shortages in most of the cases.

4. Finally, to see the effect of using a certain number of random scenarios in the optimization model, we also compute the worst-case regret (z^*) within the scenarios used in the optimization and state its ratio to the average optimal revenues of the scenarios used in the model. First, we observe that the worst case regret value is relatively low with respect to average revenues in almost all cases. This is also an indication that the proposed closed-loop formulation does not produce too conservative results. Second, as seen from the results in the Table 3.8, both the in-sample (i.e. maximum regret) and out-of-sample (i.e. the revenue gap in other scenarios) performances of the closed-loop policy do not change much by increasing the number of random scenarios used in the optimization problem after a critical number is reached. Moreover, this critical number of scenarios is expected to be as low as $N = 150$ for a problem of the above size.

3.6 Conclusion

In this chapter, we discussed the dynamic negotiation problems, and in particular the transactions in a buyer's market. We started with the classical one-to-one negotiation problem and discussed how it is extended to account for uncertainty in valuation distributions. Next, we extended our analysis to capture various situations in the dynamic environment: Starting with formulating and solving the deterministic fluid problem, we were able to observe the structural properties of the optimal pricing policy; which is the stationary nature of this policy. We were then able to extend the analysis to the problems with uncertain parameters, and offer tractable and effective solution methodologies for real life applications with uncertain and time-varying parameters. Moreover, we established the theoretical motivation to use uniform distribution in the situations where the distribution information of the valuation of the opponent party is not readily available; and emphasized the connection between the dynamic negotiation problems and the revenue management problems.

Our results also offer various avenues of future research: First, there could be several forms of negotiation problems to be analyzed in a dynamic setting from the game theoretical perspective we presented. Of these, the games that involve strategic buyers who could choose their time of purchase is of utmost interest. Also, the closed-loop formulation we developed or the structural results we presented regarding the nature of the optimal pricing policies for the certain set of problems discussed in this work might be inspiring and insightful in the formulation and solution of various other scenario-based robust optimization problems.

Chapter 4

Pricing Problem of a Monopolist in the Presence of Investors

4.1 Introduction and Literature Review

In this chapter, we analyze the revenue maximization problem of a seller who operates in a market where two type of customers exist: the "investors" and "regular buyers".

The problem is motivated by the revenue maximization problem of a real-estate developer who wants to sell a number of units in a market where investors are prevalent. In this setting, the regular buyers are defined to be the customers who purchase the units with self-inhabitance purposes whereas investors are the buyers who purchase the units with the intention of re-selling them again within the sales period of the developer at a higher price, possibly cannibalizing some of the developer's demand. Although in practice, investors in the real estate market may make purchase or sell decisions upon various financial factors including their expected rate of return, risk management factors, or portfolio variance criteria among many others; we do not incorporate financial details in our formulation. Instead, we focus on the game-theoretical perspective of the interaction between the developer and the investors. Thus, our model is applicable to various business settings that include a "pricing game in a duopoly".

The price and quantity duopoly games have extensively been studied in the literature, and various modeling approaches are applicable to these games. For an extensive review of oligopoly pricing game models, we refer the reader to Puu and Sushko (2002). Bertrand, Cournot and Stackelberg games are perhaps among the oldest and the most well-studied models. Various versions of these models are studied in the literature. For instance, Tasnadi (1999) expands a Bertrand-Edgeworth duopoly into a two-stage game in which during the first stage the firms can select their "rationing rule" that determines how the market demand is shared between two parties with different prices. He shows that under certain conditions the efficient rationing rule is an equilibrium action of the first stage. Birge et al. (1998) address joint capacity and pricing decisions for two substitutable products where the demands are uniformly distributed. They later consider the case where each product is managed by a product manager trying to maximize individual product profits rather than overall firm profits, and analyze how optimal price and capacity decisions are affected. There are fundamental differences between the setting of their paper and of ours: For instance, the capacity level is a strategic decision in their paper whereas it is a problem parameter in our work; and they restrict their attention to Bertrand and Stackelberg games. Still, if the substitutable products of this paper are thought of as the same type of goods that are sold by different players, the results are similar. Herk (1993) also considers a two-stage model of duopolistic capacity choice and subsequent price competition, and shows that consumer switching costs can deter some consumers from seeking service at a low-price firm that lacks sufficient capacity to serve the entire market. Moreover, if consumers are approximately risk neutral with respect to service reliability, then capacity-constrained duopoly competition has a unique, subgame-perfect equilibrium in which firms choose Cournot capacities and prices. Deneckere and Peck (1995) consider a two-stage game in which firms simultaneously select prices and capacities; and the customers select a firm based on the maximum utility attained, which depends on the firm's price-service pair. Similarly, Bansal and Maglaras (2009) study the dynamic pricing problem of a monopolist firm in presence of strategic customers that differ in their valuations and risk preferences, and they show through an asymptotic analysis that the 'two-price point' strategy is near-optimal. The main difference between this line of research and our work is that the selling parties in our paper have fixed capacities once the second stage of the sales period starts, and these capacity levels are observed by the customers in the market. Thus, in our framework, it is not possible to assume that the customers base their decisions on maximizing a utility function of expected service rates. Moreover, the nature of the problem does not align with this idea, since the real-estate transactions involve expensive items that do not have the same expected service rate concept as in the daily transactions. We define the pricing game between the developer and the investors in a way that potential buyers choose from which vendor to buy according to the price differential between them and their own valuation for the unit.

Having characterized the pricing game between two competing sellers, we later show that both parties are better off in equilibrium by "cooperating" rather than "competing". This is a well-known result of the supply chain and pricing literature: Federgruen and Bernstein (Bernstein et al. (2002), Bernstein and Federgruen (2003), Bernstein and Federgruen (2005)) prove the advantages of centralized decision making process where a single party determines the quantity replenishment values with the objective of maximizing the overall profits of the entire supply chain, over the decentralized decision making, where each agent gives replenishment orders with the objective of maximizing his/her own profit only. Similarly, Cachon (2003) presents an extensive analysis of the game between a supplier and a retailer, and shows how different forms of contracts could be used to lead the retailer to act towards maximizing the system profits. Contracting schemes have also been used widely in the revenue management literature to establish some form of a cooperation. For instance, Gallego and Kou (2008) develop the concept of the "callable products", which is based on the idea of selling unused capacity to low-valuation buyers and re-buying this capacity later in case high-valued demand arises. Similarly, Gallego and Sahin (2006) show the benefits of options over spot-selling and forward-selling mechanisms in retail markets.

Our contributions are twofold: First, we formulate and analyze a new pricing game between two parties where both players have fixed and inflexible capacities. Next, we compare the optimal revenues obtained in this setting with the revenues of the situation in which, rather than playing a pricing game, the agents cooperate and establish their prices accordingly. Hence, it is possible to design various contracting schemes to be able to capture the additional benefits obtained by cooperation, one of which we state and analyze numerically.

The organization of this chapter is as follows: In section 4.2, we formulate and analyze the problem where each agent makes his/her own price decision in a two-period model. In this setting, investors purchase a number of units from the developer in the first period, and resell them in the latter. We start by characterizing the equilibrium of the game between the developer and the investors in the latter period; and work our way back to the former period, where we characterize the revenue maximizing strategy of the developer. In section 4.3, we analyze the "centralized system" where the pricing decisions are made by a single party (the developer) and compute the revenues of this system. In the next section 4.4, we analyze the additional profit opportunities by centralizing the pricing decisions. Furthermore, based on our analysis of the previous part, in 4.5 we suggest an easily applicable contracting scheme to be offered to the investors that leads to an increase in the revenues of both parties, i.e. a win-win situation. Finally, we conclude by stating further avenues of research in 4.6.

4.2 The Decentralized Model

In this part, we consider the problem in a two-period setting, where there exists no cooperation between the agents. We will refer to the two periods of the problem as $t = 0$ and $t = 1$. At $t = 0$, there are both regular buyers and investors in the market. At $t = 1$, all buyers in the market are regular buyers. In both periods, the willingness-to-pay (WtP) values of the regular buyers are uniformly distributed [1]. The WtP values of regular buyers at $t = 0$ are known to lie within the interval $[\underline{v}_0, \bar{v}_0] = [0, \bar{v}_0]$. The exact range of WtP distribution at $t = 1$, which is $[\underline{v}_1, \bar{v}_1] = [0, \bar{v}_1]$, is unknown at $t = 0$ and revealed before the pricing decisions of $t = 1$ are made and sales take place. However, we assume that $\bar{v}_1 = \bar{v}_0 + \gamma$ for some random variable γ with a cumulative distribution function $G(\cdot)$ (and a continuous pdf g) on the known and bounded range $[-a, a]$. Thus, it is possible to define the unknown support of WtP distribution of buyers at $t = 1$ by a single parameter, which is γ. Finally, the market size of regular buyers at $t = 0$ is given as Λ_0, whereas the market size at $t = 1$, Λ_1, is again a random variable whose probability distribution function is $f(\cdot)$ at $t = 0$ and exact value is only revealed at $t = 1$.

We will adopt a fluid formulation framework and assume that all buyers are infinitesimal. Hence, the sales amount at a certain price level takes its expected value.

The sequence of events in this two-period problem is as follows: At $t = 0$, the developer (who will also be referred to as the "seller" interchangeably throughout the chapter) has C units to sell, and announces a sales price p_0^s. First, the regular buyers whose WtP is higher than this price, i.e. a total number of $D_0^r := \Lambda_0 \frac{\bar{v}_0 - p_0^s}{\bar{v}_0}$ regular buyers, purchase the units. Then the investors, who base their purchase decision on the expectation of their net revenue at $t = 1$, put a deposit payment (dp_0^s per unit where d is a predetermined constant, $d \in (0, 1)$) for a total number of $D_0^i \leq C - \Lambda_0 \frac{\bar{v}_0 - p_0^s}{\bar{v}_0}$ units that maximizes their expected revenues.

We assume that after these transactions occur but before the next sales period

[1] The uniform distribution argument is based on Chapter 3 where the negotiating parties assume that their opponents' valuation is distributed uniformly within its range if no other distribution information is available.

starts, the information regarding the WtP range of regular buyers at $t = 1$ (i.e. the exact value of γ) is revealed. Let us call the time at which this information is revealed as $t = 0.5$ to avoid confusion. We assume that this is the time for the investors to decide what portion of the units they put a down payment for, D_0^i, to claim. We assume that all investors act like a single body and collectively claim q_i ($q_i \leq D_0^i$) of these units. Then, they pay an additional amount of $q_i \times (1-d)p_0^s$ to the seller at $t = 0.5$ and forfeit $dp_0^s(D_0^i - q_i)$. While they make this decision, the exact value of Λ_1 is still unknown.

At $t = 1$, the value of Λ_1 is revealed, and the investors and the developer choose their respective prices p_1^i and p_1^s simultaneously. Suppose the developer has q_s units left after the investors claimed q_i units. Finally the market demand each of them observes is calculated as follows: Assume w.l.o.g. that $p_1^i \leq p_1^s$. Then, the minimum WtP value of customers who purchase from the higher-priced vendor (the developer in this case) is given by:

$$x(p_1^s, p_1^i) = \min\{\frac{\bar{v}_1 + p_1^s}{2} + (p_1^s - p_1^i), \bar{v}_1\} \tag{4.1}$$

Given $x(p_1^s, p_1^i)$, all buyers who have valuations between $[p_1^i, x(p_1^s, p_1^i)]$ will purchase the units from the lower-price agent (i.e. the investors in this case), whereas the customers whose valuations lie in the range $[x(p_1^s, p_1^i), \bar{v}_1]$ buy from the agent with the higher price. In other words, the demand the higher-priced agent observes takes the value $\Lambda_1 \frac{\bar{v}_1 - x(p_1^s, p_1^i)}{\bar{v}_1}$, while the demand the lower-priced agent observes is equivalent to $\Lambda_1 \frac{x(p_1^s, p_1^i) - p_1^i}{\bar{v}_1}$. Note that as $p_1^i \to p_1^s$, $x \to \frac{\bar{v}_1 + p_1^s}{2}$, i.e. when the two prices are equivalent, the market demand is split between the two agents equally; whereas if $p_1^i << p_1^s$, $x = \bar{v}_1$ i.e. if the gap between p_1^s and p_1^i is considerably large, all buyers in the market prefer the lower-price agent. The case with $p_1^s \leq p_1^i$ is symmetrical and omitted for the sake of brevity.

The higher-valued customers being usually less price-sensitive is a common assumption in the literature, and the above model aims to capture this effect. In other words, provided that there does not exist a considerable price difference between substitutable alternatives, some high-valued buyers are content with paying higher prices to be able to acquire the units without spending much time or effort in the process of searching for cheaper alternatives, whereas the lower-valued buyers may wait longer in the market to buy at a lower price later (for an exemplification of this phenomenon, we refer the reader to Su (2007)). Also note that by using this model, we were able to differentiate between the buyers who purchase from the lower-priced vendor and those who purchase from the higher-priced one in a smooth manner. Using a model where the demand sharply shifts to the lower-priced vendor if his price drops even slightly below the price of his competitor, as in Tasnadi (1999), might also be admissible; however it will not carry the smooth nature of Bertrand duopolies where the demand is linear, increases in the competitor's price and decreases in one's own price.

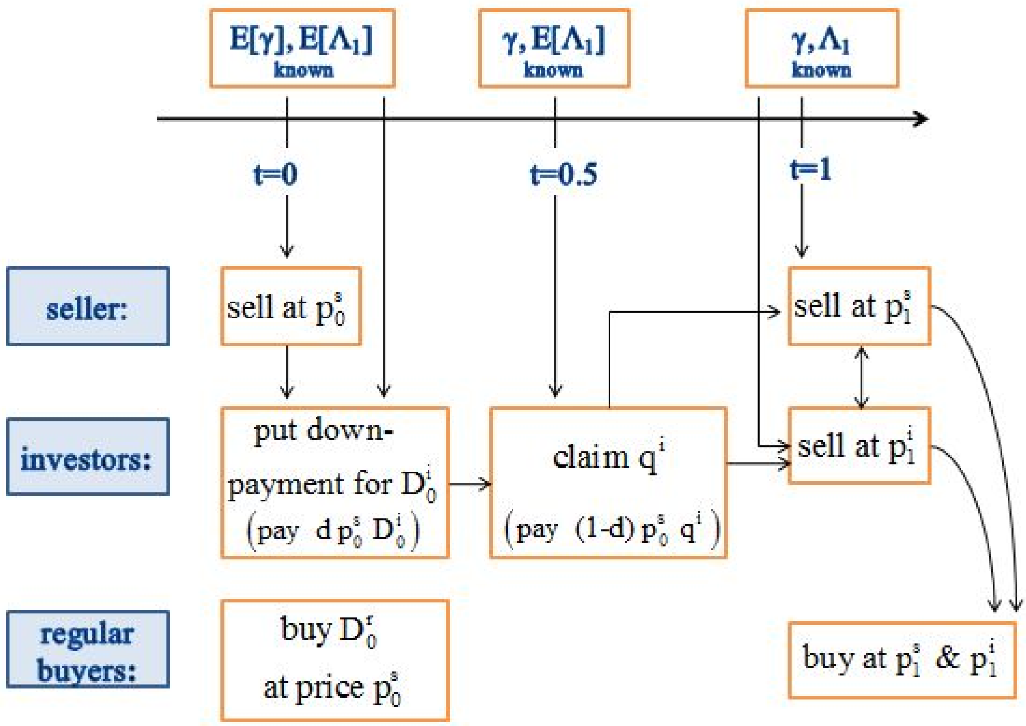
Figure 4.1 shows the sequence of events.

"Figure 4.1 about here"

4.2.1 The Pricing Problem at $t = 1$

We start with the problem at $t = 1$. Define π_1^s and π_1^i as the profit function of the seller and the investor respectively at $t = 1$. Recall that q_s and q_i are the inventory values of the developer and the investors respectively at $t = 1$. We denote the maximum profit to be obtained by the seller as $\Pi_1^s(\cdot)$ and the maximum profit to be obtained by the investors as $\Pi_1^i(\cdot)$. The revenue maximization problems of the agents at $t = 1$ can be stated in their general

Figure 4.1: Sequence of Events



forms as follows: [2]

$$
\begin{aligned}
\Pi_1^s(\gamma, \Lambda_1, q_s, q_i) &= \max_{p_1^s} \pi_1^s(p_1^s|\gamma, \Lambda_1, p_1^{i*}, q_s, q_i) \\
&:= \max_{p_1^s} p_1^s \min \big\{q_s, \\
&\quad \Lambda_1 [\frac{\bar{v}_1 - x(p_1^s, p_1^{i*})}{\bar{v}_1} \mathbf{1}_{\{p_1^{i*} \le p_1^s\}} + \frac{x(p_1^{i*}, p_1^s) - p_1^s}{\bar{v}_1} \mathbf{1}_{\{p_1^{i*} \ge p_1^s\}}] \big\}
\end{aligned} \tag{4.2}
$$

[2] Although γ is not expressed explicitly in the equations, observe that the revenue functions are also functions of γ since it is involved in the definition of $\bar{v}_1$.

$$\begin{aligned}\Pi_1^i(\gamma,\Lambda_1,q_s,q_i) &= \max_{p_1^i} \pi_1^i(p_1^i|\gamma,\Lambda_1,p_1^{s*},q_s,q_i) \\ &:= \max_{p_1^i} p_1^i \min\big\{q_i, \\ &\quad \Lambda_1[\frac{\bar{v}_1 - x(p_1^i,p_1^{s*})}{\bar{v}_1}\mathbf{1}_{\{p_1^{s*}\le p_1^i\}} + \frac{x(p_1^{s*},p_1^i)-p_1^i}{\bar{v}_1}\mathbf{1}_{\{p_1^{s*}\ge p_1^i\}}]\big\}\end{aligned} \tag{4.3}$$

where p_1^{s*} and p_1^{i*} are the maximizers of Π_1^s and Π_1^i respectively. That is, given the value of Λ_1 and their respective inventories, both parties try to select the best-response price that would bring them the greatest profit under the optimal price selection of the other party. We do not assume a Stackelberg game where the one of the players is the leader and the other the follower; but rather assume that both agents choose prices simultaneously. Hence the above problems should be solved together and the optimal prices should be best-responses to each other.

To be able to solve the two problems more easily, we first characterize some properties of the equilibrium as in the following Theorem.

Theorem 7. *If $q_s < q_i$, then there is at least an equilibrium in which the prices are such that $p_1^{s*} \ge p_1^{i*}$; and vice versa.*

We will prove Theorem 7 by characterizing the equilibrium prices and show that the claim of the Theorem holds in all cases. To this end, we start by assuming that Theorem 7 holds. Then, w.l.o.g, for $q_s < q_i$, the two revenue maximization problems simplify to:

$$\begin{aligned}\Pi_1^s(\gamma,\Lambda_1,q_s,q_i) &= \max_{p_1^s\ge p_1^{i*}} \pi_1^s(p_1^s|\gamma,\Lambda_1,p_1^{i*},q_s,q_i) \\ &= \max_{p_1^s\ge p_1^{i*}} p_1^s \min\big\{q_s, \Lambda_1[\frac{\bar{v}_1 - x(p_1^s,p_1^{i*})}{\bar{v}_1}]\big\}\end{aligned}$$

$$\Pi_1^i(\gamma, \Lambda_1, q_s, q_i) = \max_{p_1^i \leq p_1^{s*}} \pi_1^i(p_1^i | \gamma, \Lambda_1, p_1^{s*}, q_s, q_i)$$
$$= \max_{p_1^i \leq p_1^{s*}} p_1^i \min\left\{q_i, \Lambda_1[\frac{x(p_1^{s*}, p_1^i) - p_1^i}{\bar{v}_1}]\right\}$$

Next, to simplify the two maximization problems even further, observe that the following Proposition is true (proof in Appendix C.1).

Proposition 4. *If prices p_1^{s*} and $p_1^{i*}(\leq p_1^{s*})$ constitute an equilibrium of the pricing game between the developer and the investors, then* $\frac{\bar{v}_1+p_1^{s*}}{2}+(p_1^{s*}-p_1^{i*}) \leq \bar{v}_1$.

This proposition allows us to restate $x(p_1^s, p_1^i)$ as $x(p_1^s, p_1^i) = \frac{\bar{v}_1+p_1^s}{2} + (p_1^s - p_1^i)$. Then, the above problems simplify to:

$$\Pi_1^s(\gamma, \Lambda_1, q_s, q_i) = \max_{p_1^s \geq p_1^{i*}} p_1^s \min\left\{q_s, \Lambda_1[\frac{\bar{v}_1 - \frac{\bar{v}_1+p_1^s}{2} - (p_1^s - p_1^{i*})}{\bar{v}_1}]\right\}$$
$$\Pi_1^i(\gamma, \Lambda_1, q_s, q_i) = \max_{p_1^i \leq p_1^{s*}} p_1^i \min\left\{q_i, \Lambda_1[\frac{\frac{\bar{v}_1+p_1^{s*}}{2} + (p_1^{s*} - p_1^i) - p_1^i}{\bar{v}_1}]\right\}$$

First, we consider the unconstrained problems, i.e. without taking into account the inventory values q_s, q_i. These two problems can be expressed as maximizing the functions $\phi_1(.|p_1^{i*})$ and $\phi_2(.|p_1^{s*})$ respectively, where:

$$\phi_1(p_1^s|p_1^{i*}) = p_1^s \Lambda_1[\frac{\bar{v}_1 - \frac{\bar{v}_1+p_1^s}{2} - (p_1^s - p_1^{i*})}{\bar{v}_1}] \tag{4.4}$$

and

$$\phi_2(p_1^i|p_1^{s*}) = p_1^i \Lambda_1[\frac{\frac{\bar{v}_1+p_1^{s*}}{2} + (p_1^{s*} - p_1^i) - p_1^i}{\bar{v}_1}] \tag{4.5}$$

Observe that both functions $\phi_1(.|p_1^{i*})$ and $\phi_2(.|p_1^{s*})$ are concave and are maximized at $p_s^* = \frac{2p_1^{i*}+\bar{v}_1}{6}$ and $p_i^* = \frac{\bar{v}_1+3p_1^{s*}}{8}$ respectively. Thus, the revenue maximization problems $\Pi_1^s(\gamma, \Lambda_1, q_s, q_i)$ and $\Pi_1^i(\gamma, \Lambda_1, q_s, q_i)$ are maximization problems of concave functions over a convex set, and the optimal prices for these two problems take the following values respectively:

$$p_1^{s*} = \begin{cases} \max\{\frac{2p_1^{i*}+\bar{v}_1}{6}, p_1^{i*}\} & \text{if } \frac{\Lambda_1}{\bar{v}_1}(\frac{\bar{v}_1}{4} + \frac{p_1^{i*}}{2}) \leq q_s \\ \max\{\frac{2}{3}(\frac{\bar{v}_1}{2} + p_1^{i*} - \frac{\bar{v}_1 q_s}{\Lambda_1}), p_1^{i*}\}, & \text{if } \frac{\Lambda_1}{\bar{v}_1}(\frac{\bar{v}_1}{4} + \frac{p_1^{i*}}{2}) > q_s \end{cases} \tag{4.6}$$

and

$$p_1^{i*} = \begin{cases} \min\{\frac{3p_1^{s*}+\bar{v}_1}{8}, p_1^{s*}\} & \text{if } \frac{\Lambda_1}{\bar{v}_1}(\frac{\bar{v}_1}{4} + \frac{3p_1^{s*}}{4}) \leq q_i \\ \min\{\frac{1}{2}(\frac{\bar{v}_1}{2} + \frac{3p_1^{s*}}{2} - \frac{\bar{v}_1 q_i}{\Lambda_1}), p_1^{s*}\}, & \text{if } \frac{\Lambda_1}{\bar{v}_1}(\frac{\bar{v}_1}{4} + \frac{3p_1^{s*}}{4}) > q_i \end{cases} \tag{4.7}$$

The equivalence (4.6) is explained as follows: If the total amount of sales at price p_s^*, which is computed as $\frac{\Lambda_1}{\bar{v}_1}(\frac{\bar{v}_1}{4} + \frac{p_1^{i*}}{2})$, is lower than the seller's inventory q_s, then there are two cases: If p_s^* is greater than the investors' price p_1^{i*}, it is the optimal price for the constrained problem. Otherwise, investors' price p_1^{i*} is the best price to set by concavity of the revenue function $\Pi_1^s(\gamma, \Lambda_1, q_s, q_i)$. If, on the other hand, p_s^* leads to a sales figure that is higher than the seller inventories, then we compute the price level at which the seller's inventory is cleared, which is given by $\frac{2}{3}(\frac{\bar{v}_1}{2} + p_1^{i*} - \frac{\bar{v}_1 q_s}{\Lambda_1})$. Provided that this price level exceeds the investor price p_1^{i*}, it is the optimal price again by concavity of $\Pi_1^s(\gamma, \Lambda_1, q_s, q_i)$. Otherwise, she simply sells at p_1^{i*}. The explanation for the optimal prices of the investors is symmetrical, so we skip this part.

Now we proceed to determining the optimal prices by solving the revenue maximization problems of the two agents simultaneously. Let C_1 be the total inventory of the seller after the regular sales of $t = 0$ is cleared (i.e. $C_1 = q_s + q_i = C - \Lambda_0 \frac{\bar{v}_0 - p_0^s}{\bar{v}_0}$). The equilibrium prices of the pricing game between the seller and the investors at $t = 1$ are characterized in the following Proposition:

Proposition 5. *If $q_i \geq 0.5C_1$, the equilibrium prices of the two agents that maximize the revenue functions (4.2) and (4.3) respectively are characterized as follows:*

$$
\begin{aligned}
&(p_1^{s*}, p_1^{i*}) \\
&= \begin{cases}
(\frac{10\bar{v}_1}{42}, \frac{9\bar{v}_1}{42}), & \textit{if } \Lambda_1(\frac{15}{42}) \leq q_s \textit{ and } \Lambda_1(\frac{18}{42}) \leq q_i \\
(\frac{5\bar{v}_1}{9} - \frac{8q_s\bar{v}_1}{9\Lambda_1}, \frac{\bar{v}_1}{3} - \frac{q_s\bar{v}_1}{3\Lambda_1}), & \textit{if } \Lambda_1(\frac{15}{42}) > q_s \textit{ and } \Lambda_1(\frac{2}{3} - \frac{2q_s}{3\Lambda_1}) \leq q_i \\
(\frac{\bar{v}_1}{3} - \frac{2q_i\bar{v}_1}{9\Lambda_1}, \frac{\bar{v}_1}{2} - \frac{2q_i\bar{v}_1}{3\Lambda_1}), & \textit{if } \Lambda_1(\frac{1}{2} - \frac{q_i}{3\Lambda_1}) \leq q_s \textit{ and } \Lambda_1(\frac{18}{42}) > q_i \\
(\bar{v}_1 - \frac{2(2q_s+q_i)\bar{v}_1}{3\Lambda_1}, \bar{v}_1 - \frac{(q_i+q_s)\bar{v}_1}{\Lambda_1}), & \textit{otherwise}
\end{cases}
\end{aligned}
\tag{4.8}
$$

Similarly, if $q_i < 0.5C_1$, the equilibrium prices are characterized as the following:

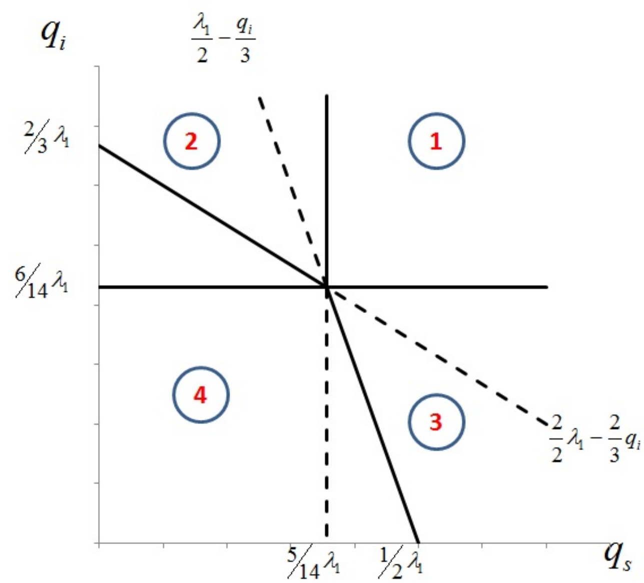
$$
\begin{aligned}
&(p_1^{s*}, p_1^{i*}) \\
&= \begin{cases}
(\frac{9\bar{v}_1}{42}, \frac{10\bar{v}_1}{42}), & \textit{if } \Lambda_1(\frac{15}{42}) \leq q_i \textit{ and } \Lambda_1(\frac{18}{42}) \leq q_s \\
(\frac{\bar{v}_1}{3} - \frac{q_i\bar{v}_1}{3\Lambda_1}, \frac{5\bar{v}_1}{9} - \frac{8q_i\bar{v}_1}{9\Lambda_1}), & \textit{if } \Lambda_1(\frac{15}{42}) > q_i \textit{ and } \Lambda_1(\frac{2}{3} - \frac{2q_i}{3\Lambda_1}) \leq q_s \\
(\frac{\bar{v}_1}{2} - \frac{2q_s\bar{v}_1}{3\Lambda_1}, \frac{\bar{v}_1}{3} - \frac{2q_s\bar{v}_1}{9\Lambda_1}), & \textit{if } \Lambda_1(\frac{1}{2} - \frac{q_s}{3\Lambda_1}) \leq q_i \textit{ and } \Lambda_1(\frac{18}{42}) > q_s \\
(\bar{v}_1 - \frac{(q_i+q_s)\bar{v}_1}{\Lambda_1}, \bar{v}_1 - \frac{2(2q_i+q_s)\bar{v}_1}{3\Lambda_1}), & \textit{otherwise}
\end{cases}
\end{aligned}
\tag{4.9}
$$

(Proof in Appendix C.2.)

Consider the equivalence (4.8): The first set of prices (p_1^{s*}, p_1^{i*}) in the equivalence corresponds to the case where the maximizers of unconstrained revenue maximization problems of both agents ($\phi_1(.|p_1^{i*})$ and $\phi_2(.|p_1^{s*})$) lead to sales figures below their respective inventories. Hence, these prices are also optimal for the constrained problems. The second set of prices in (4.8) corresponds to the case where the seller runs out of capacity if she sets the unconstrained revenue maximizing price; so both prices are adjusted accordingly. The third case is symmetrical to the second case, except that now the investors run out of capacity if the agents both set their unconstrained revenue maximizing prices. Finally, the fourth set of prices are observed when the market size is very large and both agents run out of capacity under the revenue maximizing prices of unconstrained case. The explanation for the equivalence (4.9) is symmetrical, so we omit it.

Figure 4.2 will help visualize various cases of the problem for $q_i^* \geq 0.5C_1$ (the regions are numbered as in the same sequence as they appear in the equation (4.8)).

Figure 4.2: Regions of Equivalence (4.8)



Also observe that in (4.8), the case where $\Lambda_1 \frac{15}{42} > q_s$ and $\Lambda_1(\frac{2}{3} - \frac{2q_s}{3\Lambda_1}) \leq q_i$ is only possible if $q_s < \frac{5}{6}q_i$; and $\Lambda_1(\frac{1}{2} - \frac{q_i}{3\Lambda_1}) \leq q_s$ and $\Lambda_1 \frac{18}{42} > q_i$ is only attainable if $\frac{5}{6}q_i < q_s < q_i$. A symmetrical statement holds for (4.9). Hence, for a fixed pair of (q_s, q_i) values, the second and third cases stated in each of the equivalences (4.8) and (4.9) cannot occur at the same time. Thus, for each given (q_s, q_i) pair, we observe at most three cases stated in the equivalence (4.8) (or in (4.9)) throughout the entire range of Λ_1 values.

Moreover, it is important to note that the inequality $p_1^{i*} \leq p_1^{s*}$ holds in all cases under the equivalence (4.8) as claimed in Theorem 7; and similarly $p_1^{s*} \leq p_1^{i*}$ holds in all cases under the equivalence (4.9). Moreover, $x(p_1^{s*}, p_1^{i*}) \leq \bar{v}_1$ as stated before. This proves Theorem 7.

4.2.2 The Quantity-Claiming Problem at $t = 0.5$

Now let us go back one step and analyze the q_i-selection problem of the investors at $t = 0.5$ after the value of γ is revealed. Recall that $D_0^i \leq C_1$ is the total amount the investors have put a down payment for, hence they could claim $q_i \leq D_0^i$ and leave the rest $C_1 - q_i$ to the seller to sell at $t = 1$. Let $\pi_{0.5}^i(\gamma, C_1 - q_i, q_i, p_0^s)$ denote the expected profit function of the investors for a given γ value by claiming q_i units, when the seller's remaining number of units after the regular sales of $t = 0$ is cleared is C_1 and the seller's price at $t = 0$ was p_0^s. Then, after the value of γ is revealed, the q_i-selection problem of the investors at $t = 0.5$ takes the following form:

$$\begin{aligned}
\Pi_{0.5}^i(\gamma, D_0^i, C_1, p_0^s) &= \max_{q_i \leq D_0^i} \pi_{0.5}^i(\gamma, C_1 - q_i, q_i, p_0^s) \\
&:= \max_{q_i \leq D_0^i} \int_{\Lambda_1} \Pi_1^i(\gamma, \Lambda_1, C_1 - q_i, q_i) f(\Lambda_1) d\Lambda_1 - (1-d) p_0^s q_i
\end{aligned} \tag{4.10}$$

Recall that for every $(q_s,\ q_i)$ choice, we have three cases regarding the values the optimal prices could take; and therefore three different forms of the revenue function $\Pi_1^i(\cdot)$. Thus, for instance for $q_i < \frac{5}{6} q_s$ (i.e. $q_i < \frac{5}{11} C_1$), the expected revenue function of investors at $t = 0.5$ takes the following form, where each integrand corresponds to different ranges of Λ_1 values that lead to different optimal price pairs defined in (4.9):

$$\begin{aligned}
\pi^i_{0.5}(\gamma, q_s, q_i, p^s_0) = &\int_{\Lambda_1=\Lambda_{min}}^{\max\{\frac{14q_i}{5},\Lambda_{min}\}} \frac{\Lambda_1 15}{42}\frac{10\bar{v}_1}{42} f(\Lambda_1)d\Lambda_1 \\
+ &\int_{\max\{\frac{14q_i}{5},\Lambda_{min}\}}^{\min\{\Lambda_{max},\max\{\Lambda_{min},(\frac{3}{2}q_s+q_i)\}\}} (\frac{5\bar{v}_1}{9} - \frac{8\bar{v}_1 q_i}{9\Lambda_1})q_i f(\Lambda_1)d\Lambda_1 \\
+ &\int_{\min\{\Lambda_{max},\max\{\Lambda_{min},(\frac{3}{2}q_s+q_i)\}\}}^{\Lambda_{max}} (\bar{v}_1 - \frac{2\bar{v}_1(2q_i+q_s)}{3\Lambda_1})q_i f(\Lambda_1)d\Lambda_1 \\
-&(1-d)p^s_0 q_i
\end{aligned} \tag{4.11}$$

And for $\frac{5}{6}q_s < q_i < q_s$, the revenue function takes the following values:

$$\begin{aligned}
\pi^i_{0.5}(\gamma, q_s, q_i, p^s_0) = &\int_{\Lambda_1=\Lambda_{min}}^{\max\{\frac{14q_s}{6},\Lambda_{min}\}} \frac{\Lambda_1 15}{42}\frac{10\bar{v}_1}{42} f(\Lambda_1)d\Lambda_1 \\
+ &\int_{\max\{\frac{14q_s}{6},\Lambda_{min}\}}^{\min\{\Lambda_{max},\max\{\Lambda_{min},(2q_i+\frac{2}{3}q_s)\}\}} \frac{3\Lambda_1}{2\bar{v}_1}(\frac{\bar{v}_1}{3} - \frac{2q_s\bar{v}_1}{9\Lambda_1})^2 f(\Lambda_1)d\Lambda_1 \\
+ &\int_{\min\{\Lambda_{max},\max\{\Lambda_{min},(2q_i+\frac{2}{3}q_s)\}\}}^{\Lambda_{max}} (\bar{v}_1 - \frac{2\bar{v}_1(2q_i+q_s)}{3\Lambda_1})q_i f(\Lambda_1)d\Lambda_1 \\
-&(1-d)p^s_0 q_i
\end{aligned} \tag{4.12}$$

The above forms of $\pi^i_{0.5}(\cdot)$ follow from the equivalence (4.9); the first one involving the ranges of Λ_1 values under which the optimal prices take the forms $(p^{s*}_1, p^{i*}_1) = (\frac{9\bar{v}_1}{42}, \frac{10\bar{v}_1}{42})$, $(p^{s*}_1, p^{i*}_1) = (\frac{\bar{v}_1}{3} - \frac{q_i\bar{v}_1}{3\Lambda_1}, \frac{5\bar{v}_1}{9} - \frac{8q_i\bar{v}_1}{9\Lambda_1})$ and $(p^{s*}_1, p^{i*}_1) = (\bar{v}_1 - \frac{(q_i+q_s)\bar{v}_1}{\Lambda_1}, \bar{v}_1 - \frac{2(2q_i+q_s)\bar{v}_1}{3\Lambda_1})$ respectively as Λ_1 increases; and the second corresponding to the situation where optimal prices take the values $(p^{s*}_1, p^{i*}_1) =$

$(\frac{9\bar{v}_1}{42}, \frac{10\bar{v}_1}{42})$, $(p_1^{s*}, p_1^{i*}) = (\frac{\bar{v}_1}{2} - \frac{2q_s\bar{v}_1}{3\Lambda_1}, \frac{\bar{v}_1}{3} - \frac{2q_s\bar{v}_1}{9\Lambda_1})$ and $(p_1^{s*}, p_1^{i*}) = (\bar{v}_1 - \frac{(q_i+q_s)\bar{v}_1}{\Lambda_1}, \bar{v}_1 - \frac{2(2q_i+q_s)\bar{v}_1}{3\Lambda_1})$ respectively based on the value Λ_1 assumes. Performing a symmetrical analysis for the case where $q_i > 0.5C_1$, the function $\pi_{0.5}^i$ takes the following form for each range of q_i [3]:

$$\pi_{0.5}^i(\gamma, C_1 - q_i, q_i, p_0^s)$$
$$= \begin{cases} \int_{\Lambda_1=\Lambda_{min}}^{\frac{14q_i}{5}} \frac{\Lambda_1 15}{42}\frac{10\bar{v}_1}{42} f(\Lambda_1)d\Lambda_1 + \int_{\frac{14q_i}{5}}^{(\frac{3}{2}(C_1-q_i)+q_i)} (\frac{5\bar{v}_1}{9} - \frac{8\bar{v}_1 q_i}{9\Lambda_1})q_i f(\Lambda_1)d\Lambda_1 \\ \quad + \int_{(\frac{3}{2}(C_1-q_i)+q_i)}^{\Lambda_{max}} (\bar{v}_1 - \frac{2\bar{v}_1(2q_i+(C_1-q_i))}{3\Lambda_1})q_i f(\Lambda_1)d\Lambda_1 - (1-d)p_0^s q_i, \\ \quad \textbf{if} \quad q_i \le \frac{5}{11}C_1; \\ \int_{\Lambda_1=\Lambda_{min}}^{\frac{14(C_1-q_i)}{6}} \frac{\Lambda_1 15}{42}\frac{10\bar{v}_1}{42} f(\Lambda_1)d\Lambda_1 \\ \quad + \int_{\frac{14(C_1-q_i)}{6}}^{(2q_i+\frac{2}{3}(C_1-q_i))} \frac{3\Lambda_1}{2\bar{v}_1}(\frac{\bar{v}_1}{3} - \frac{2\bar{v}_1(C_1-q_i)}{9\Lambda_1})^2 f(\Lambda_1)d\Lambda_1 \\ \quad + \int_{(2q_i+\frac{2}{3}(C_1-q_i))}^{\Lambda_{max}} (\bar{v}_1 - \frac{2\bar{v}_1(2q_i+(C_1-q_i))}{3\Lambda_1})q_i f(\Lambda_1)d\Lambda_1 - (1-d)p_0^s q_i, \\ \quad \textbf{if} \ \frac{5}{11}C_1 < q_i \le \frac{1}{2}C_1; \\ \int_{\Lambda_1=\Lambda_{min}}^{\frac{14q_i}{6}} \frac{\Lambda_1 18}{42}\frac{9\bar{v}_1}{42} f(\Lambda_1)d\Lambda_1 + \int_{\frac{14q_i}{6}}^{(\frac{2}{3}q_i+2(C_1-q_i))} q_i(\frac{\bar{v}_1}{2} - \frac{2\bar{v}_1 q_i}{3\Lambda_1})f(\Lambda_1)d\Lambda_1 \\ \quad + \int_{(\frac{2}{3}q_i+2(C_1-q_i))}^{\Lambda_{max}} (\bar{v}_1 - \frac{\bar{v}_1(q_i+(C_1-q_i))}{\Lambda_1})q_i f(\Lambda_1)d\Lambda_1 - (1-d)p_0^s q_i, \\ \quad \textbf{if} \ \frac{1}{2}C_1 < q_i \le \frac{6}{11}C_1; \\ \int_{\Lambda_1=\Lambda_{min}}^{\frac{14(C_1-q_i)}{5}} \frac{\Lambda_1 18}{42}\frac{9\bar{v}_1}{42} f(\Lambda_1)d\Lambda_1 + \int_{\frac{14(C_1-q_i)}{5}}^{(\frac{3}{2}q_i+C_1-q_i)} 2\frac{\Lambda_1}{\bar{v}_1}(\frac{\bar{v}_1}{3} - \frac{\bar{v}_1(C_1-q_i)}{3\Lambda_1})^2 f(\Lambda_1)d\Lambda_1 \\ \quad + \int_{(\frac{3}{2}q_i+C_1-q_i)}^{\Lambda_{max}} (\bar{v}_1 - \frac{\bar{v}_1(C_1)}{\Lambda_1})q_i f(\Lambda_1)d\Lambda_1 - (1-d)p_0^s q_i, \\ \quad \textbf{if} \ \frac{6}{11}C_1 < q_i \end{cases}$$

The maximizer value q_i^* of $\Pi_{0.5}^i(\cdot)$ should be evaluated numerically. Let this value be denoted by $q_i^*(\gamma, D_0^i, C_1, p_0^s)$. Then the following Proposition holds.

Proposition 6.
- *$q_i^*(\gamma, D_0^i, C_1, p_0^s)$ is decreasing in p_0^s;*
- *$q_i^*(\gamma, D_0^i, C_1, p_0^s)$ is increasing in D_0^i and C_1.*

[3] Here, for notational convenience, we implicitly make the assumption that Λ_1 values allow the realization of all three cases. To be technically correct, the boundaries of Λ_1 ranges in the equivalence (4.13) should be defined as in the equivalences (4.11) and (4.12)

The first claim is straightforward to see as the function $\pi^i_{0.5}(\gamma, q_s, q_i, p^s_0)$ is a linear decreasing function of p^s_0 in all cases. Again it is easy to observe that $q^*_i(\gamma, D^i_0, C_1, p^s_0)$ increases as D^i_0 increases, since this value does not appear in the $\pi^i_{0.5}(\gamma, q_s, q_i, p^s_0)$ functions and is only constraining the feasible region. The claim about C_1 is proved by showing the supermodularity of all $\pi^i_{0.5}(\gamma, C_1 - q_i, q_i, p^s_0)$ functions in (C_1, q_i) (which simply follows from observing that $\frac{\partial^2 \pi^i_{0.5}(\gamma, C_1 - q_i, q_i, p^s_0)}{\partial C_1 \partial q_i} \leq 0$ in all cases).

4.2.3 The Quantity Selection Problem of Investors at $t = 0$

We could now go back one more step to the investors' problem of "selecting the optimal number of units D^i_0 to put a down payment for" at $t = 0$. Defining $\pi^i_0(D^i_0, p^s_0)$ as the expected net profit of investors when they put a down payment for D^i_0 units given that the seller's price is p^s_0; the investors' problem at $t = 0$ will be formulated as:

$$\Pi^i_0(p^s_0) = \max_{D^i_0 \leq C_1(p^s_0)} \pi^i_0(D^i_0, p^s_0) := \max_{D^i_0 \leq C_1(p^s_0)} \mathbf{E}_\gamma[\Pi^i_{0.5}(\gamma, D^i_0, C_1, p^s_0)] - dp^s_0 D^i_0 \tag{4.13}$$

Assume that solving the above problem leads to $D^{i*}_0 := D^{i*}_0(p^s_0)$ as the optimal quantity the investors put a deposit for at $t = 0$ when the seller sets the sales price p^s_0. We cannot say anything about the uniqueness of this optimal value without further analysis. However, for the rest of the chapter we assume that it is unique.

4.2.4 The Price Setting Problem of the Seller at $t = 0$

Finally, the seller's price-setting problem at $t = 0$ takes the form:

$$\Pi_0^s(C, \Lambda_0) = \max_{p_0^s} \Big[p_0^s \frac{\Lambda_0(\bar{v}_0 - p_0^s)}{\bar{v}_0 - \underline{v}_0} + dp_0^s D_0^{i*} + \mathbf{E}_\gamma \big[(1-d) p_0^s q_i^*(\gamma, D_0^{i*}, C_1(p_0^s), p_0^s)$$
$$+ \mathbf{E}_{\Lambda_1} [\Pi_1^s(\gamma, \Lambda_1, C_1(p_0^s) - q_i^*(\gamma, D_0^{i*}, C_1(p_0^s), p_0^s), q_i^*(\gamma, D_0^{i*}, C_1(p_0^s), p_0^s))] \big] \Big].$$

4.3 The Centralized Model

In the model of the previous section, when all demand uncertainty is resolved at $t = 1$, both parties try to maximize their own profits by playing a game against each other. However, by the well-known result of supply chain and game theory literature, the total system profit in a centralized system (where the decisions are made by a single agent) is always larger than or equal to the sum of the profits in a decentralized system (where all agents make their own decisions) (e.g. for an example, please refer to Bernstein and Federgruen (2003)). In this section, we will analyze this phenomenon in greater detail.

Recall the two revenue maximization problems of the seller and the investors at $t = 1$ when the values of γ and Λ_1 are revealed, and the investors have claimed the quantity q_i^* that maximizes the revenue function at $t = 0.5$ (i.e. the maximizer of the equation (4.10)): First, assume optimal $q_i^* > 0.5C$. Recall that the decentralized problems are then stated as:

$$\Pi_1^s(\gamma, \Lambda_1, q_s, q_i) = \max_{p_1^s \geq p_1^{i*}} p_1^s \min \big\{ q_s, \Lambda_1 [\frac{\bar{v}_1 - x(p_1^s, p_1^{i*})}{\bar{v}_1}] \big\}$$

$$\Pi_1^i(\gamma, \Lambda_1, q_s, q_i) = \max_{p_1^i \leq p_1^{s*}} p_1^i \min \big\{ q_i, \Lambda_1 [\frac{x(p_1^{s*}, p_1^i) - p_1^i}{\bar{v}_1}] \big\}$$

On the other hand, if the pricing decisions are made by a single agent, the centralized problem will be formulated as follows:

$$
\begin{aligned}
\Pi_1(\gamma, \Lambda_1, q_s, q_i) \quad = \quad & \max_{p_1^i, p_1^s} \big[p_1^s \min\big\{q_s, \Lambda_1[\frac{\bar{v}_1 - x(p_1^s, p_1^i)}{\bar{v}_1}]\big\} \\
& + p_1^i \min\big\{q_i, \Lambda_1[\frac{x(p_1^s, p_1^i) - p_1^i}{r}]\big\}\big]\mathbf{1}_{\{p_1^s \geq p_1^i\}} \\
& + \big[p_1^s \min\big\{q_s, \Lambda_1[\frac{x(p_1^i, p_1^s) - p_1^s}{\bar{v}_1}]\big\} \\
& + p_1^i \min\big\{q_i, \Lambda_1[\frac{\bar{v}_1 - x(p_1^i, p_1^s)}{\bar{v}_1}]\big\}\big]\mathbf{1}_{\{p_1^s < p_1^i\}} \qquad (4.14)
\end{aligned}
$$

It could be shown that the centralized system revenues are higher if the price of the agent with higher inventory is less than or equal to the price of the other agent, as was the case in the decentralized setting, which we state as a Proposition:

Proposition 7. *The optimal prices in the formulation (4.14) are such that if* $q_i > q_s$, $p_1^s \geq p_1^i$ *and if* $q_s > q_i$, $p_1^i \geq p_1^s$.

(Proof in Appendix C.3.)

By this claim, for $q_i^* > 0.5C_1$ the above problem will be equivalent to:

$$
\begin{aligned}
\Pi_1(\gamma, \Lambda_1, q_s, q_i) \quad = \quad & \max_{p_1^i \leq p_1^s} p_1^s \min\big\{q_s, \Lambda_1[\frac{\bar{v}_1 - x(p_1^s, p_1^i)}{\bar{v}_1}]\big\} \\
& + p_1^i \min\big\{q_i, \Lambda_1[\frac{x(p_1^s, p_1^i) - p_1^i}{\bar{v}_1}]\big\}
\end{aligned}
$$

which can also be formulated as:

$$
\begin{aligned}
\Pi_1(\gamma, \Lambda_1, q_s, q_i) \quad = \quad & \max_{p_1^s \geq p_1^i} p_1^s \Lambda_1[\frac{\bar{v}_1 - x(p_1^s, p_1^i)}{\bar{v}_1}] + p_1^i \Lambda_1[\frac{x(p_1^s, p_1^i) - p_1^i}{\bar{v}_1}] \\
& \text{subject to} \\
& \Lambda_1[\frac{\bar{v}_1 - x(p_1^s, p_1^i)}{\bar{v}_1}] \leq q_s \\
& \Lambda_1[\frac{x(p_1^s, p_1^i) - p_1^i}{\bar{v}_1}] \leq q_i \qquad (4.15)
\end{aligned}
$$

The above formulation is concave in both p_1^s and p_1^i. Call the prices that are the maximizers of the above problem as the centralized prices $p_1^{i*}(c)$ and $p_1^{s*}(c)$. Via the characterization of KKT conditions, these prices can be shown to take the following optimal values under different cases:

$$(p_1^{s*}(c), p_1^{i*}(c)) = \begin{cases} (\frac{13}{23}\bar{v}_1, \frac{11}{23}\bar{v}_1), & \text{if } q_s \geq \Lambda_1\frac{3}{23},\ q_i \geq \Lambda_1\frac{9}{23} \\ (\frac{2}{3}\bar{v}_1 - \frac{7q_s\bar{v}_1}{9\Lambda_1}, \frac{\bar{v}_1}{2} - \frac{q_s\bar{v}_1}{6\Lambda_1}), & \text{if } q_s < \Lambda_1\frac{3}{23},\ q_i \geq \Lambda_1\frac{1}{2} - \frac{5}{6}q_s \\ (\frac{\bar{v}_1}{2} + \frac{q_i\bar{v}_1}{6\Lambda_1}, \frac{5}{8}\bar{v}_1 - \frac{3q_i\bar{v}_1}{8\Lambda_1}), & \text{if } q_s \geq \Lambda_1\frac{3}{8} - \frac{5}{8}q_i,\ q_i < \Lambda_1\frac{9}{23} \\ (\bar{v}_1 - \frac{2(2q_s+q_i)\bar{v}_1}{3\Lambda_1}, \bar{v}_1 - \frac{(q_i+q_s)\bar{v}_1}{\Lambda_1}), & \text{otherwise} \end{cases} \tag{4.16}$$

And the revenue in each case is stated as:

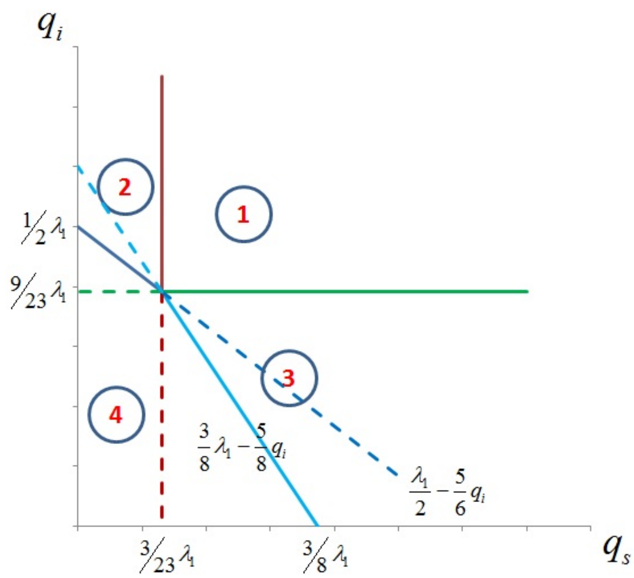
$$\Pi_1(\gamma, \Lambda_1, q_s, q_i) = \begin{cases} \frac{6\Lambda_1}{23}\bar{v}_1, \text{ if } q_s \geq \frac{3\Lambda_1}{23},\ q_i \geq \frac{9\Lambda_1}{23} \\ \frac{1}{6}\bar{v}_1 q_s - \frac{23}{36}\frac{(q_s)^2\bar{v}_1}{\Lambda_1} + \frac{\Lambda_1}{4}\bar{v}_1, \text{ if } q_s < \frac{3\Lambda_1}{23},\ q_i \geq \frac{\Lambda_1}{2} - \frac{5}{6}q_s \\ \frac{3}{8}\bar{v}_1 q_i - \frac{23}{48}\frac{\bar{v}_1(q_i)^2}{\Lambda_1} + \frac{3\Lambda_1}{16}\bar{v}_1, \text{ if } q_s \geq \frac{3\Lambda_1}{8} - \frac{5}{8}q_i,\ q_i < \frac{9\Lambda_1}{23} \\ q_s(\bar{v}_1 - \frac{2(2q_s+q_i)\bar{v}_1}{3\Lambda_1}) + q_i(\bar{v}_1 - \frac{(q_i+q_s)\bar{v}_1}{\Lambda_1}), \text{ otherwise} \end{cases} \tag{4.17}$$

Again, to help visualize various cases of the problem for $q_i^* \geq 0.5C_1$, please refer to the Figure 4.3.

4.4 The Price of Anarchy

In this section, we will compare the profits of the decentralized system versus the centralized system under various cases with respect to the relationship between the market size and the inventories of the two agents.

Figure 4.3: Regions of Equivalence (4.16)



- $\Lambda_1(\frac{15}{42}) \leq q_s$ and $\Lambda_1(\frac{18}{42}) \leq q_i$:

  Recall that in this case, the optimal prices in the decentralized system are $p_1^{s*} = \frac{10\bar{v}_1}{42}$ and $p_1^{i*} = \frac{9\bar{v}_1}{42}$, leading to a total revenue of:

$$\begin{aligned} rev(decen) &= \Pi_1^s(\gamma, \Lambda_1, q_s, q_i) + \Pi_1^i(\gamma, \Lambda_1, q_s, q_i) \\ &= \frac{10\Lambda_1}{42}\frac{15\bar{v}_1}{42} + \frac{9\Lambda_1}{42}\frac{18\bar{v}_1}{42} \\ &= \Lambda_1\frac{26}{147}(\bar{v}_1) \end{aligned}$$

  We know that the optimal prices of the formulation given in (4.15) are $p_1^{i*}(c) = \frac{11}{23}\bar{v}_1$ and $p_1^{s*}(c) = \frac{13}{23}\bar{v}_1$ provided $\Lambda_1\frac{3}{23} \leq q_s$ and $\Lambda_1\frac{9}{23} \leq q_i$, which is indeed the case for this range of q_i and q_s values. Hence, total revenues under this set of prices amount to $rev(cen) = \Pi_1(\gamma, \Lambda_1, q_s, q_i) = \Lambda_1\frac{6}{23}\bar{v}_1$ as known from (4.17). Hence, the revenue difference between the centralized and decentralized systems is $\Lambda_1\frac{284}{23\cdot147}\bar{v}_1$, which is 47.5% of the revenues in the decentralized system.

- $\frac{15\Lambda_1}{42} > q_s$ and $\frac{2}{3}\Lambda_1 - \frac{2q_s}{3} \leq q_i$:

  Recall that in this case, the decentralized system optimal prices are $p_1^{s*} = \frac{5\bar{v}_1}{9} - \frac{8q_s\bar{v}_1}{9\Lambda_1}$ and $p_1^{i*} = \frac{\bar{v}_1}{3} - \frac{q_s\bar{v}_1}{3\Lambda_1}$, leading to a total revenue of:

$$\begin{aligned} rev(decen) &= \Pi_1^s(\gamma, \Lambda_1, q_s, q_i) + \Pi_1^i(\gamma, \Lambda_1, q_s, q_i) \\ &= (\frac{5\bar{v}_1}{9} - \frac{8q_s\bar{v}_1}{9\Lambda_1})q_s + \frac{\Lambda_1}{\bar{v}_1}(\frac{\bar{v}_1}{3} - \frac{q_s\bar{v}_1}{3\Lambda_1})(\frac{2\bar{v}_1}{3} - \frac{2q_s\bar{v}_1}{3\Lambda_1}) \\ &= \frac{1}{9}\bar{v}_1 q_s - \frac{6}{9}\frac{(q_s)^2\bar{v}_1}{\Lambda_1} + \Lambda_1\frac{2}{9}\bar{v}_1 \end{aligned}$$

  In this case, based on the values of q_i and q_s, two subcases can be observed:

  - $\frac{15\Lambda_1}{42} > q_s \geq \Lambda_1\frac{3}{23}$: The maximizer prices of the unconstrained revenue function, namely $p_1^{i*}(c) = \frac{11}{23}\bar{v}_1$ and $p_1^{s*}(c) = \frac{13}{23}\bar{v}_1$, produce a sales quantity $\Lambda_1\frac{3}{23}$ for the seller; hence these prices are again optimal, leading to a total system revenue of $rev(cen) = \Lambda_1\frac{6}{23}\bar{v}_1$. It could be shown that:

$$\begin{aligned} rev(cen) - rev(decen) &= \Lambda_1\frac{6}{23}\bar{v}_1 - (\frac{1}{9}\bar{v}_1 q_s - \frac{6}{9}\frac{(q_s)^2\bar{v}_1}{\Lambda_1} + \Lambda_1\frac{2}{9}\bar{v}_1) \\ &= \Lambda_1\frac{8}{23\cdot 9}\bar{v}_1 - \frac{1}{9}\bar{v}_1 q_s + \frac{6}{9}\frac{(q_s)^2\bar{v}_1}{\Lambda_1} \\ &\geq 0 \end{aligned}$$

    for $\frac{15\Lambda_1}{42} > q_s \geq \Lambda_1\frac{3}{23}$. Furthermore, the ratio of additional revenue to be captured by centralized price decisions to the total revenue of the decentralized system is:

$$\frac{rev(cen) - rev(decen)}{rev(decen)} = \frac{\Lambda_1\frac{8}{23\cdot 9}\bar{v}_1 - \frac{1}{9}\bar{v}_1 q_s + \frac{6}{9}\frac{(q_s)^2\bar{v}_1}{\Lambda_1}}{\frac{1}{9}\bar{v}_1 q_s - \frac{6}{9}\frac{(q_s)^2\bar{v}_1}{\Lambda_1} + \Lambda_1\frac{2}{9}\bar{v}_1}$$

which can be shown to increase in q_s; hence attains its minimum value at $q_s = \Lambda_1 \frac{3}{23}$ which is 15.8%; and its maximum value at $q_s = \Lambda_1 \frac{15}{42}$, which is 47.5%.

– $q_s < \Lambda_1 \frac{3}{23}$: In this case, as clear from (4.16), the formulation (4.15) is maximized at $p_1^{i*}(c) = \frac{\bar{v}_1}{2} - \frac{q_s \bar{v}_1}{6\Lambda_1}$, leading to $p_1^{s*}(c) = \frac{2\bar{v}_1}{3} - \frac{7q_s \bar{v}_1}{9\Lambda_1}$ and a total revenue of $rev(cen) = \frac{1}{6}\bar{v}_1 q_s - \frac{23}{36}\frac{(q_s)^2\bar{v}_1}{\Lambda_1} + \Lambda_1 \frac{1}{4}\bar{v}_1$. The difference between the two revenue figures is therefore:

$$rev(cen) - rev(decen) = \frac{1}{18}\bar{v}_1 q_s + \frac{1}{36}\frac{(q_s)^2\bar{v}_1}{\Lambda_1} + \Lambda_1 \frac{1}{36}\bar{v}_1$$

which is clearly positive, and increasing in $\bar{v}_1$ and q_s. Furthermore, the ratio of additional revenue to be captured by centralized price decisions to the total revenue of the decentralized system is:

$$\frac{rev(cen) - rev(decen)}{rev(decen)} = \frac{\frac{1}{18}\bar{v}_1 q_s + \frac{1}{36}\frac{(q_s)^2\bar{v}_1}{\Lambda_1} + \Lambda_1 \frac{1}{36}\bar{v}_1}{\frac{1}{9}\bar{v}_1 q_s - \frac{6}{9}\frac{(q_s)^2\bar{v}_1}{\Lambda_1} + \Lambda_1 \frac{2}{9}\bar{v}_1}$$

which can be shown to increase in q_s, and attains a value between 12.5% and 47.5%.

- $\Lambda_1(\frac{1}{2} - \frac{q_i}{3\Lambda_1}) \leq q_s$ and $\Lambda_1(\frac{18}{42}) > q_i$:

Recall that in this case, the decentralized system prices are $p_1^{s*} = \frac{\bar{v}_1}{3} - \frac{2q_i\bar{v}_1}{9\Lambda_1}$ and $p_1^{i*} = \frac{\bar{v}_1}{2} - \frac{2q_i\bar{v}_1}{3\Lambda_1}$ leading to a total revenue of:

$$\begin{aligned}
rev(decen) &= \Pi_1^s(\gamma, \Lambda_1, q_s, q_i) + \Pi_1^i(\gamma, \Lambda_1, q_s, q_i) \\
&= \Lambda_1 \frac{3}{2}(\frac{1}{3} - \frac{2q_i}{9\Lambda_1})^2 + (\frac{\bar{v}_1}{2} - \frac{2q_i\bar{v}_1}{3\Lambda_1})q_i \\
&= \frac{5}{18}\bar{v}_1 q_i - \frac{16}{27}\frac{\bar{v}_1(q_i)^2}{\Lambda_1} + \Lambda_1 \frac{1}{6}\bar{v}_1
\end{aligned}$$

Regarding the centralized system, again there are two subcases:

- $\frac{18}{42}\Lambda_1 > q_i \geq \frac{9}{23}\Lambda_1$: The maximizer prices of the unconstrained revenue function, namely $p_1^{i*}(c) = \frac{11}{23}\bar{v}_1$ and $p_1^{s*}(c) = \frac{13}{23}\bar{v}_1$, produce a sales quantity $\frac{9}{23}\Lambda_1$ for the investors; hence these prices are again optimal, leading to a total system revenue of $rev(cen) = \Lambda_1\frac{6}{23}\bar{v}_1$. Then the ratio of additional revenue to be captured by centralized price decisions to the total revenue of the decentralized system is:

$$\frac{rev(cen) - rev(decen)}{rev(decen)} = \frac{\Lambda_1\frac{13}{23\cdot 6}\bar{v}_1 - \frac{5}{18}\bar{v}_1 q_i + \frac{16}{27}\frac{(q_i)^2\bar{v}_1}{\Lambda_1}}{\frac{5}{18}\bar{v}_1 q_i - \frac{16}{27}\frac{(q_i)^2\bar{v}_1}{\Lambda_1} + \Lambda_1\frac{1}{6}\bar{v}_1}$$

  which can be shown to increase in q_i; hence attains its minimum value at $q_i = \Lambda_1\frac{9}{23}$ which is 41.2%; and its maximum value at $q_i = \Lambda_1\frac{18}{42}$, which is 47.5%.

- $q_i < \frac{9}{23}\Lambda_1$: In this case, as clear from (4.16), the formulation (4.15) is maximized at $p_1^{s*}(c) = \frac{\bar{v}_1}{2} + \frac{q_i\bar{v}_1}{6\Lambda_1}$ and $p_1^{i*}(c) = \frac{5\bar{v}_1}{8} - \frac{3q_i\bar{v}_1}{8\Lambda_1}$; and leads to a total system revenue of $rev(cen) = \frac{3}{8}\bar{v}_1 q_i - \frac{23}{48}\frac{\bar{v}_1(q_i)^2}{\Lambda_1} + \Lambda_1\frac{3}{16}\bar{v}_1$. Then the ratio of additional revenue to be captured by centralized price decisions to the total revenue of the decentralized system is:

$$\frac{rev(cen) - rev(decen)}{rev(decen)} = \frac{\frac{7}{72}\bar{v}_1 q_i + \frac{49}{27\cdot 16}\frac{\bar{v}_1(q_i)^2}{\Lambda_1} + \Lambda_1\frac{1}{48}\bar{v}_1}{\frac{5}{18}\bar{v}_1 q_i - \frac{16}{27}\frac{\bar{v}_1(q_i)^2}{\Lambda_1} + \Lambda_1\frac{1}{6}\bar{v}_1}$$

  which can be shown to increase in q_i; hence attains its minimum value at $q_i = 0$ which is 12.5%; and its maximum value at $q_i = \Lambda_1\frac{9}{23}$, which is 41.2%.

- $\Lambda_1\frac{15}{42} > q_s$ and $\Lambda_1(\frac{2}{3}) - \frac{2}{3}q_s > q_i$ OR $\Lambda_1(\frac{18}{42}) > q_i$ and $\Lambda_1(\frac{1}{2}) - \frac{1}{3}q_i > q_s$:

  Recall that in this case, $p_1^{s*} = \bar{v}_1 - \frac{2(2q_s+q_i)\bar{v}_1}{3\Lambda_1}$ and $p_1^{i*} = \bar{v}_1 - \frac{(q_i+q_s)\bar{v}_1}{\Lambda_1}$

leading to a total revenue of:

$$
\begin{aligned}
rev(decen) &= \Pi_1^s(\gamma, \Lambda_1, q_s, q_i) + \Pi_1^i(\gamma, \Lambda_1, q_s, q_i) \\
&= q_s(\bar{v}_1 - \frac{2(2q_s + q_i)\bar{v}_1}{3\Lambda_1}) + (\bar{v}_1 - \frac{(q_i + q_s)\bar{v}_1}{\Lambda_1})q_i \\
&= \bar{v}_1 q_i + \bar{v}_1 q_s - \frac{5}{3}\frac{\bar{v}_1 q_s q_i}{\Lambda_1} - \frac{4}{3}\frac{\bar{v}_1 (q_s)^2}{\Lambda_1} - \frac{\bar{v}_1 (q_i)^2}{\Lambda_1}
\end{aligned}
$$

As before, there are different cases to consider in the centralized system:

- $q_i \geq \frac{9}{23}\Lambda_1$ and $q_s \geq \Lambda_1 \frac{3}{23}$: In this case, the maximizer prices of the unconstrained revenue function, namely $p_1^{i*}(c) = \frac{11}{23}\bar{v}_1$ and $p_1^{s*}(c) = \frac{13}{23}\bar{v}_1$ are optimal for the centralized system, and the system revenue is equivalent to $rev(cen) = \Lambda_1 \frac{6}{23}\bar{v}_1$. It could be shown that the ratio $\frac{rev(cen) - rev(decen)}{rev(decen)}$ could be as high as 47.5% in this range of (q_i, q_s) values.
- $q_s < \Lambda_1 \frac{3}{23}$ and $q_i \geq \Lambda_1 \frac{9}{23}$: In this case, the formulation (4.15) is maximized at $p_1^{i*}(c) = \frac{\bar{v}_1}{2} - \frac{q_s \bar{v}_1}{6\Lambda_1}$ and $p_1^{s*}(c) = \frac{2\bar{v}_1}{3} - \frac{7q_s\bar{v}_1}{9\Lambda_1}$, leading to $rev(cen) = \frac{1}{6}\bar{v}_1 q_s - \frac{23}{36}\frac{(q_s)^2\bar{v}_1}{\Lambda_1} + \Lambda_1 \frac{1}{4}\bar{v}_1$. It could be shown that $rev(cen) > rev(decen)$ for this range of (q_i, q_s) values.
- $q_s \geq \Lambda_1 \frac{3}{23}$ and $q_i < \Lambda_1 \frac{9}{23}$: In this case, the formulation (4.15) is maximized at $p_1^{s*}(c) = \frac{\bar{v}_1}{2} + \frac{q_i \bar{v}_1}{6\Lambda_1}$ and $p_1^{i*}(c) = \frac{5\bar{v}_1}{8} - \frac{3q_i\bar{v}_1}{8\Lambda_1}$; leading to $rev(cen) = \frac{3}{8}\bar{v}_1 q_i - \frac{23}{48}\frac{\bar{v}_1 (q_i)^2}{\Lambda_1} + \Lambda_1 \frac{3}{16}\bar{v}_1$. It could be shown that $rev(cen) > rev(decen)$ for this range of (q_i, q_s) values.
- $q_s < \Lambda_1 \frac{3}{23}$ and $q_i < \Lambda_1 \frac{9}{23}$: In this case the optimal central prices are equivalent to the optimal prices of the decentralized system. Hence, the revenues in both cases are equivalent.

The case where $q_i^* \leq 0.5C_1$ is symmetrical, so we skip the analysis.

In conclusion, the additional profit opportunities arising from centralized decision making lies between 0% to 47.5% depending on the realization of Λ_1.

4.5 Numerical Analysis: Evaluating a Candidate Contracting Scheme

As clear from the analysis in the previous section, it is possible to increase the revenues of both agents considerably by centralizing the pricing decisions. At this point, various contracting schemes can be proposed to ensure cooperation between the two agents. We will proceed with the following scheme, which combines the spirit of "callable products" of the revenue management and the "revenue sharing contracts" of the supply chain literature.

In this scheme, the seller offers the so-called "callable units" at $t = 0$ at a price p_0^c ($\leq p_0^s$) together with the regular units which are sold at p_0^s. However, the inferior callable products come with a call option for the seller: If the extra profit attained by centralizing the pricing decisions surpasses a certain percentage or a certain amount (say, K_1), the seller exercises her call option at $t = 1$ by paying a percentage K_0 on the top of their purchase price (or, again a fixed option price per unit) to the investors, and gains the right to recall and reprice the units of the investors. Thus, at $t = 1$, rather than having to compete with the investors, the seller can price the units according to the centralized system optimal (We still assume that the investor units and the seller units should be priced differently. The reasoning for this practice could be the fact that during the time period the investors owned the units, they made structural changes inside these units that lead them to be perceived as different than seller-owned units).

Provided that the expected profit of the investors under the contracting scheme is higher than their expected profits in the previous system, the risk-neutral investors would agree to this scheme. Or, it is possible to act cautiously, and design a pricing scheme that leaves the investors with an equivalent or higher revenue at each realization of the unknown random variables.

Let $\mathbf{A}$ denote the subset of the state space of Λ_1 values that lead to a recall and let $\mathbf{A}'$ denote the contingencies that do not lead to a recall at $t = 1$. That is:

$$\Lambda_1 \in \mathbf{A} \Rightarrow \Pi_1(\gamma, \Lambda_1, q_s, q_i) \geq (1 + K_1)\big[\Pi_1^s(\gamma, \Lambda_1, q_s, q_i) + \Pi_1^i(\gamma, \Lambda_1, q_s, q_i)\big]$$
$$\Lambda_1 \in \mathbf{A}' \Rightarrow \Pi_1(\gamma, \Lambda_1, q_s, q_i) < (1 + K_1)\big[\Pi_1^s(\gamma, \Lambda_1, q_s, q_i) + \Pi_1^i(\gamma, \Lambda_1, q_s, q_i)\big]$$

where $\Pi_1(\cdot)$ is the centralized system expected profits and $\Pi_1^s(\cdot)$ and $\Pi_1^i(\cdot)$ are the expected profits of the seller and the investors in the decentralized system respectively.

Also, let $D_0^{i,new}(K_0, K_1, p_0^c, p_0^{s,new})$ denote the callable units purchased by the investors under the scheme $(K_0, K_1, p_0^c, p_0^{s,new})$ where $p_0^{s,new}$ is the new price for regular units at $t = 0$. Moreover, let $q_{i,c}(\gamma, D_0^{i,new}, C_1, p_0^c)$ denote the units to be claimed at $t = 0.5$. Then, the pricing scheme $(K_0, K_1, p_0^c, p_0^{s,new})$ produces the following expected profit functions for the seller [4]:

[4] We denote the seller profit functions under the contracting scheme $(K_0, K_1, p_0^c, p_0^{s,new})$ by adding a superscript "new".

$$
\begin{aligned}
\Pi_1^{s,new}(\gamma, \Lambda_1, q_s, q_{i,c}) &= \begin{cases} \Pi_1(\gamma, \Lambda_1, q_s, q_{i,c}) - (1+K_0)dp_0^c \times q_{i,c}, & \text{if } \Lambda_1 \in \mathbf{A} \\ \Pi_1^s(\gamma, \Lambda_1, q_s, q_{i,c}), & \text{if } \Lambda_1 \in \mathbf{A}' \end{cases} \\
\Pi_0^{s,new}(C, \Lambda_0, K_0, p_0^c) &= \max_{p_0^{s,new}} \Bigg[p_0^{s,new} \frac{\Lambda_0(\bar{v}_0 - p_0^{s,new})}{\bar{v}_0} + dp_0^c D_0^{i,new} \\
&\quad + \mathbf{E}_{\gamma,\Lambda_1}[\Pi_1^{s,new}(\gamma, \Lambda_1, C_1(p_0^{s,new}) - q_{i,c}, q_{i,c}))] \Bigg]
\end{aligned}
$$

and the following profit functions for the investors:

$$
\begin{aligned}
\Pi_1^{i,new}(\gamma, \Lambda_1, q_s, q_{i,c}) &= \begin{cases} (1+K_0)dp_0^c \times q_{i,c}, & \text{if } \Lambda_1 \in \mathbf{A} \\ \Pi_1^i(\gamma, \Lambda_1, q_s, q_{i,c}), & \text{if } \Lambda_1 \in \mathbf{A}' \end{cases} \\
\Pi_{0.5}^{i,new}(\gamma, D_0^{i,new}, C_1, p_0^c) &= \max_{q_{i,c} \le D_0^{i,new}} \int_{\Lambda_1} \Pi_1^{i,new}(\gamma, \Lambda_1, C_1 - q_{i,c}, q_{i,c}) f(\Lambda_1) d\Lambda_1 \\
&\quad -(1-d)p_0^c q_{i,c} \\
\Pi_0^{i,new}(p_0^c, p_0^{s,new}) &= \max_{D_0^{i,new}} \left[\mathbf{E}_\gamma[\Pi_{0.5}^{i,new}(\gamma, D_0^{i,new}, C_1, p_0^c)] - dp_0^c D_0^{i,new} \right]
\end{aligned}
$$

Clearly, for the seller to offer and the investors to accept the contracting scheme, the following inequalities must hold:

$$
\begin{aligned}
\Pi_0^{s,new}(C, \Lambda_0, K_0, K_1, p_0^c) &\ge \Pi_0^s(C, \Lambda_0) \\
\Pi_0^{i,new}(p_0^c, p_0^{s,new}) &\ge \Pi_0^i(p_0^{s,new})
\end{aligned}
$$

The only problem with the above situation could be the fact that offering callable units might also be appealing to the regular buyers who initially had no intention of making profit from buying and reselling the units. But in practice, regular buyers are prone to incurring additional costs during the process of reselling the units. This phenomenon stems from several reasons. For instance, unlike investors, regular individuals who would like to sell their houses do not have efficient channels to sell the units; hence this arbitrage opportunity does not come as naturally in their part as it does for the investors. Thus, by setting the fixed premium K_0 such that their total revenue per recalled unit minus their transaction costs is negative, it is possible to render the process of investing in the callable units non-profitable for regular buyers.

In practice, it is difficult to compute the K_0, K_1 and p_0^c values optimally, and the nature of the contract changes based on how the developer and the investor agree to split the profits. Therefore, rather than attempting to find the optimal values based on some assumptions on sharing the profit, we will evaluate the performances of various contracting schemes with for a number of (K_0, K_1, p_0^c) triplets.

Numerical Example: Consider the problem of a developer who wants to sell $C = 100$ units. The market size at $t = 0$ is Λ_0=50, while it can take values Λ_1=50, 100 or 150 with equal probabilities at $t = 1$. Assume that the willingness to pay values of the regular buyers at $t = 0$ is uniformly distributed in the range $[0, \$200, 000]$, and the upper bound of valuations vary uniformly between \$100,000 and \$300,000 at $t = 1$ (i.e., $\gamma \sim \mathbf{U}[-\$100, 000, \$100, 000]$).

Under the above setting, the optimal price of the seller at $t = 0$ turns out to be $p_0^{s*} = \$72, 000$. The investors then make a deposit for only $D_0^i(p_0^s) = 15$ units. The expected seller and investor profits then take the values $\Pi_0^s = \$6.648M$ and $\Pi_0^i = \$330K$.

On the other hand, various contracting schemes $(K_0, K_1, p_0^c, p_0^{s,new})$ produce revenue figures as stated in Table 4.1 and Table 4.2.

Table 4.1: Performances of Various Contracting Schemes (changing K_1 and the ratio $p_0^{s,new}/p_0^c$)

	$\Pi_0^{s,new}$ (% of Π_0^s)	$\Pi_0^{i,new}$ (% of Π_0^i)	D_0^i
$K_0 = 5\%, K_1 = 10\%, p_0^c = \$88K, p_0^{s,new} = 1.2 \times p_0^c$	\$7.290M (110%)	\$457K (138%)	39
$K_0 = 5\%, K_1 = 10\%, p_0^c = \$88K, p_0^{s,new} = 1.1 \times p_0^c$	\$7.156M (108%)	\$362K (110%)	72
$K_0 = 5\%, K_1 = 10\%, p_0^c = \$88K, p_0^{s,new} = 1.05 \times p_0^c$	\$7.110M (107%)	\$390K (118%)	73
$K_0 = 5\%, K_1 = 8\%, p_0^c = \$88K, p_0^{s,new} = 1.2 \times p_0^c$	\$7.290M (110%)	\$457K (138%)	39
$K_0 = 5\%, K_1 = 8\%, p_0^c = \$88K, p_0^{s,new} = 1.1 \times p_0^c$	\$7.156M (108%)	\$362K (110%)	72
$K_0 = 5\%, K_1 = 8\%, p_0^c = \$88K, p_0^{s,new} = 1.05 \times p_0^c$	\$7.110M (107%)	\$390K (118%)	73
$K_0 = 5\%, K_1 = 20\%, p_0^c = \$96K, p_0^{s,new} = 1.2 \times p_0^c$	\$7.396M (111%)	\$340K (103%)	42
$K_0 = 5\%, K_1 = 20\%, p_0^c = \$80K, p_0^{s,new} = 1.1 \times p_0^c$	\$7.154M (108%)	\$370K (112%)	61
$K_0 = 5\%, K_1 = 20\%, p_0^c = \$80K, p_0^{s,new} = 1.05 \times p_0^c$	\$7.070M (106%)	\$384K (116%)	60

Table 4.2: Performances of Various Contracting Schemes (changing K_0 and the ratio $p_0^{s,new}/p_0^c$)

	$\Pi_0^{s,new}$ (% of Π_0^s)	$\Pi_0^{i,new}$ (% of Π_0^i)	D_0^i
$K_0 = 5\%, K_1 = 10\%, p_0^c = \$88K, p_0^{s,new} = 1.2 \times p_0^c$	\$7.29M (110%)	\$457K (138%)	39
$K_0 = 5\%, K_1 = 10\%, p_0^c = \$88K, p_0^{s,new} = 1.1 \times p_0^c$	\$7.15M (108%)	\$362K (110%)	72
$K_0 = 5\%, K_1 = 10\%, p_0^c = \$88K, p_0^{s,new} = 1.05 \times p_0^c$	\$7.11M (107%)	\$390K (118%)	73
$K_0 = 2\%, K_1 = 10\%, p_0^c = \$88K, p_0^{s,new} = 1.2 \times p_0^c$	\$7.29M (110%)	\$452K (137%)	39
$K_0 = 2\%, K_1 = 10\%, p_0^c = \$88K, p_0^{s,new} = 1.1 \times p_0^c$	\$7.17M (108%)	\$350K (106%)	73
$K_0 = 2\%, K_1 = 10\%, p_0^c = \$88K, p_0^{s,new} = 1.05 \times p_0^c$	\$7.12M (107%)	\$379K (115%)	72
$K_0 = 10\%, K_1 = 10\%, p_0^c = \$88K, p_0^{s,new} = 1.2 \times p_0^c$	\$7.28M (110%)	\$467K (141%)	39
$K_0 = 10\%, K_1 = 10\%, p_0^c = \$88K, p_0^{s,new} = 1.1 \times p_0^c$	\$7.14M (107%)	\$381K (115%)	73
$K_0 = 10\%, K_1 = 10\%, p_0^c = \$88K, p_0^{s,new} = 1.05 \times p_0^c$	\$7.09M (107%)	\$410K (124%)	72

Clearly, the contracting scheme brings higher profits for both the seller and the investors. (Although the investors seem to benefit more based on the percentage increase in their profits, this may in fact not be the case: In the above example, we do not take into account the production costs of the developer, therefore the percentage increase is on gross profits, which might be much higher when net profits are considered.)

Another observation is the fact that investors put a down payment for a greater number of units in the existence of contracting options. However, their profit margins are still lower compared to the developer; which enhances the assumption that it is not profitable for regular buyers to invest in this kind of a business. (For instance, for an additional transaction cost of as low as \$30K per unit on the top of investor costs, it is not profitable for the regular buyers to invest in the regular units.)

Also observe that the price of the contracting option and the price of the regular units both increase in the presence of the contracting scheme. That is mainly because, due to the existence of higher profit opportunities at $t = 1$ by a more efficient management, the developer prefers to keep a greater number of units to be sold in the later period. This phenomenon may also help increase the market value of the units.

Finally, the effect of varying K_0 or K_1 is not huge on the profits of the two parties at least within the above set of values.

As a side note we would like to emphasize that the above contracting schemes are only some selections among the set of feasible contracts. For instance, for the base case, it is possible to select another contracting scheme with $(K_0, K_1, p_0^c, p_0^{s,new}) = (5\%, 10\%, \$84K, \$92K)$ which brings the expected revenues of $\Pi_1^s = \$7.017M$ to the seller and $\Pi_1^i = \$481K$ to the investors. That is, by modulating the contracting price, it is possible to shift the additional profits to either side, which might itself be a negotiation tool among the developer and the investors in the design of the contract.

Keeping the contracting scheme variables fixed at $K_0 = 5\%$, $K_1 = 10\%$, and $p_0^{s,new} = 1.1 \times p_0^c$, the selection of the best contracting price and the effect of the problem parameters on the profits of both systems (with or without contract) are analyzed in Table 4.3.

The results in the above table suggest the following insights: Contracting schemes are especially helpful in highly uncertain environments (e.g. $\gamma \sim \mathbf{U}[-200K, 200K]$); and in the cases where buyer valuations are lower (e.g.

Table 4.3: Different Problem Settings

	$\Pi_0^s, \Pi_0^i, p_0^{s*}$	$\Pi_0^{s,new}(\% of \Pi_0^s), \Pi_0^{i,new}(\% of \Pi_0^i), p_0^{c*}$
$\gamma \sim \mathbf{U}[-50K, 50K]$, $\bar{v}_0$=200K, Λ_0=50	6.65M, 330K, 72K	7.16M (108%) 362K (110%), 88K
$\gamma \sim \mathbf{U}[-50K, 50K]$, $\bar{v}_0$=200K, Λ_0=50	6.82M, 301K, 72K	7.15M (105%), 344K (114%), 88K
$\gamma \sim \mathbf{U}[-200K, 200K]$, $\bar{v}_0$=200K, Λ_0=50	6.41M, 288K, 88K	7.21M (112%), 444K (154%), 112K
$\gamma \sim \mathbf{U}[-100K, 100K]$, $\bar{v}_0$=150K, Λ_0=50	4.83M, 203K, 60K	5.53M (114%), 281K (124%), 76K
$\gamma \sim \mathbf{U}[-100K, 100K]$, $\bar{v}_0$=250K, Λ_0=50	8.44M, 421K, 88K	8.91M (106%), 491K (117%), 108K
$\gamma \sim \mathbf{U}[-100K, 100K]$, $\bar{v}_0$=200K, Λ_0=75	7.91M, 246K, 88K	8.12M (103%), 409K (166%), 92K
$\gamma \sim \mathbf{U}[-100K, 100K]$, $\bar{v}_0$=200K, Λ_0=100	8.99M, 197K, 100K	9.03M (100.9%), 284K (144%), 104K

$\bar{v}_0$=150K). Moreover, if the seller encounters a larger market upfront (e.g. $\Lambda_0 = 100$), the usefulness of contract options is again diminished.

4.6 Conclusion

In this chapter, we characterized the pricing and resource allocation decision of a monopolist seller in the presence of demand-cannibalizing investors. The setting is defined as a two-stage game: In the first stage, the problem of the seller is to decide on the sales price of her units, which affects both the current profits and the revenue to be obtained in the later period as a result of the number of investors purchasing from the current price. In the latter stage, the problem pours into a pricing game between two competing sellers with fixed capacities. We first characterized the solution of the pricing game in the second stage, and based on the solution of this game, we formulated and analyzed the solution of the revenue maximization game of the seller in the previous period. Moreover, we analyzed the situation in a setting where pricing decision in the latter period is made by a central party (the seller), and showed that both parties are better off in equilibrium by "cooperating" rather than "competing", which is a result consistent with the previous results in supply chain and revenue management literature. We also proposed a candidate contracting scheme that helps coordinate the system and reach system optimal profits.

The analysis of this chapter promises a wide avenue for future research: For instance, it is possible to suggest various other forms of coordinating contracts, and their performances should be evaluated in different markets. As another research avenue, the game could be carried to a multi-period setting: In that case, the timing of sales and purchase decisions appears as an additional decision variable for the investors; and the seller needs to decide how many units to offer at each instant in addition to setting the unit prices.

Bibliography

A. Ben-Tal, A. N., C. Roos. 2002. Robust solutions of uncertain quadratic and conic-quadratic problems. *SIAM Journal on Optimization* **13**(2) 535–560.

A. Ben-Tal, E. G., A. Goryashko, A. Nemirovski. 2004. Adjustable robust solutions of uncertain linear programs. *Mathematical Programming* **99**(2) 351376.

A. Ben-Tal, S. B., A. Nemirovski, 2005. Control of uncertainty-affected discrete time linear systems via convex programming.

Ball, M. O., M. Queyranne. 2009. Toward robust revenue management: Competitive analysis of online booking. *Operations Research* **57**(4) 950–963.

Bansal, M., C. Maglaras. 2009. Dynamic pricing when customers strategically time their purchase: asymptotic optimality of a two price policy. *Journal of Revenue and Pricing Management* **8** 42–66.

Ben-Tal, A., A. Nemirovski. 1998. Robust convex optimization **23**(4) 769–805.

Bergemann, D., K. Schlag. 2008. Pricing without priors. *Journal of the European Economic Association* **6**(2-3) 560–569.

Bernstein, F., F. Chen, A. Federgruen, 2002. Coordinating supply chains with simple pricing schemes: The role of vendor managed inventories.

Bernstein, F., A. Federgruen. 2003. Pricing and replenishment strategies in a distribution system with competing retailers. *Operations Research* **51**(3) 409–426.

Bernstein, F., A. Federgruen. 2005. Decentralized supply chains with competing retailers under demand uncertainty. *Management Science* **51**(1) 18–29.

Bertsimas, D., D. B. Brown. 2007. Constrained stochastic lqc: A tractable approach. *IEEE Transactions on Automatic Control* **52**(10) 18261841.

Bertsimas, D., M. Sim. 2003. The price of robustness **52** 35–53.

Bertsimas, D., A. Thiele. 2004. A robust optimization approach to supply chain manage- ment. *Proceedings of the 14th IPCO* 86–100.

Besbes, O., C. Maglaras, 2009. Dynamic pricing with financial milestones: Feedback-form policies. Working Paper.

Bhandari, A., N. Secomandi, 2009. Comparison of negotiation mechanisms in revenue management. Working Paper.

Birge, J. R., J. Drogosz, I. Duenyas. 1998. Setting single-period optimal capacity levels and prices for substitutable products. *The International Journal of Flexible Manufacturing Systems* **10** 407430.

Brumelle, S., J. McGill. 1993. Airline seat allocation with multiple nested fare classes **41**(1) 127–137.

Cachon, G. P., 2003. Supply chain coordination with contracts.

Carr, S., I. Duenyas. 2000. Optimal admission control and sequencing in a make-to-stock/ make-to-order production system. *Operations Research* **48**(5) 709–720.

Carr, S., W. Lovejoy. 2000. The inverse newsvendor problem: Choosing an optimal portfolio for capacitated resources. *Management Science* **46**(7) 912–927.

Chatterjee, K., W. Samuelson. 1983. Bargaining under incomplete information. *Operations Research* **31**(5) 835–861.

Clifford, S., 2012. The ny times article in the adress http://www.nytimes.com/2012/03/28/business/retailers-rush-to-adjust-to-price-smart-shoppers.html.

D. Bertsimas, D. A. I., P. A. Parrilo. 2010. Optimality of affine policies in multi-stage robust optimization. *Mathematics of Operations Research* **35** 363–394.

Dantzig, G. B. 1969. A hospital admission problem. *Technical Report. Operations Research House, Stanford University, Stanford, California* .

Deneckere, R., J. Peck. 1995. Competition over price and service rate when demand is stochastic: a strategic analysis. *Journal of Economics* **26**(1) 148–162.

Ding, Q., P. Kouvelis., J. Milner. 2006. Dynamic pricing through customer discounts for optimizing multi-class customers demand fulfillment. *Operations Research* **54**(1) 169–183.

Duenyas, I. 2000. Single facility due date setting with multiple customer classes. *Operations Research* **48**(5) 709–720.

Duran, S., T. Liu, D. Simchi-Levi, J. Swann. 2008. Policies utilizing tactical inventory for service-differentiated customers. *Operations Research Letters* **36**(2) 259–264.

El Ghaoui, L., H. Lebret. 1997. Robust solutions to least-squares problems with uncertain data. **18**(4) 1035–1064.

Eren, S., C. Maglaras. 2010. Monopoly pricing with limited demand information. *Journal of revenue and pricing management* **9**(1-2) 23–48.

Eren, S., G. van Ryzin. 2006. Product line positioning without marked information. *Working Paper, Columbia University, New York, NY 10027* .

Esogbue, O., A. Singh. 1976. A stochastic model for an optimal priority bed distribution problem in a hospital ward. *Operations Research* **24**(5) 884–898.

Gallego, G., S. Kou. 2008. Revenue management of callable products. *Management Science* **54**(3) 550–564.

Gallego, G., O. Sahin, 2006. Inter-temporal valuations, product design and revenue management.

Gallego, G., G. van Ryzin. 1994. Optimal dynamic pricing of inventories with stochastic demand over finite horizons. *Management Science* **40** 999–1020.

Gallien, J. 2006. Dynamic mechanism design for online commerce. *Operations Research* **54** 291–310.

Garstka, S. J., R. J.-B.Wets. 1974. On decision rules in stochastic programming. *Mathematical Programming* **7**(1) 117–143.

Gerchak, Y., D. Gupta, M. Henig. 1996. Reservation planning for elective surgery under uncertain demand for emergency surgery. *Management Science* **42**(3) 321–334.

Goldfarb, D., G. Iyengar. 2003. Robust convex quadratically constrained programs. *Mathematical Programming* **97**(3) 495–515.

Grant, M., S. Boyd. 2008. *CVX: MATLAB Software for Disciplined Convex programming.*

Green, L. V., S. Savin, B. Wang. 2006. Managing patient service in a diagnostic medical facility. *Operations Research* **54**(1) 11–25.

Gupta, D., L. Wang, 2006. Manufacturing capacity revenue management. Forthcoming in *Operations Research*.

Ha, A. Y. 1997a. Inventory rationing in a make-to-stock production system with several demand classes and lost sales. *Management Science* **43**(8) 1093–1103.

Ha, A. Y. 1997b. Optimal dynamic scheduling policy for a make-to-stock production system. *Operations Research* **45**(1) 42–53.

Harper, R., A. Shahani. 2002. Modeling for the planning and management of bed capacities in hospitals. *Journal of the Operational Research Society* **53**(1) 11–18.

Harsanyi, J. C. 1956. Approaches to the bargaining problem before and after the theory of games. *Econometrica* **24**(2) 144–157.

Herk, L. 1993. Consumer choice and cournot behavior in capacityconstrained duopoly competition. *Journal of Economics* **24**(3) 399–417.

Huang, X. 1995. A planning model for requirement of emergency beds. *IMA Journal of Mathematics Applied in Medicine & Biology* **12**(3-4) 345–353.

Kerrigan, E., J. Maciejowski. 2004. Properties of a new parameterization for the control of constrained systems with disturbances. *Proceedings of the 2004 American Control Conference* **5**(5) 46694674.

Kolesar, P. 1970. A markovian model for hospital admission scheduling. *Management Science* **16**(6) 384–396.

Lan, Gao, M. Ball, I. Karaesmen. 2006. Revenue management with limited demand information. *Working Paper, University of Maryland, College Park, MD* .

Lee, T. C., M. Hersh. 1998. A model for dynamicairline seat inventory control with multiple seat bookings. *Transportation Science* **27**(3) 252–265.

Lobel, G. Perakis, 2010. Network dynamic pricing through scenario based optimization. Working Paper.

Mackay, M. 2001. Practical experience with bed occupancy management and planning systems: an australian view. *Health Care Management Science* **4**(1) 47–56.

Maglaras, C., J. Meissner. 2006. Dynamic pricing strategies for multi-product revenue management problems **8**(2) 136–148.

Maglaras, C., A. Zeevi. 2005. Pricing and design of differentiated services: Approximate analysis and structural insights. *Operations Research* **53**(2) 242–262.

McClean, S., P. Millard. 1995. A decision support system for bed-occupancy management and planning hospitals. *IMA Journal of Mathematics Applied in Medicine & Biology* **12**(3-4) 249–257.

Milsum, J., E. Turban, I. Vertinsky. 1973. Hospital admission systems: Their evaluation and management. *Management Science* **19**(6) 646–666.

Myerson, R. B. 1979. Incentive compatibility and the bargaining problem. *Econometrica* **47**(1) 61–73.

Myerson, R. B. 1984. Two person bargaining problems with incomplete information. *Econometrica* **2**(2) 461–487.

Myerson, R. B., M. A. Satterthwaite. 1983. Efficient mechanisms for bilateral trading. *Journal of Economic Theory* **29** 265–281.

Nash, J. C. 1950. The bargaining problem. *Econometrica* **18** 151–162.

Patrick, J., M. L. Puterman. 2007. Improving resource utilization for diagnostic services through flexible inpatient scheduling: A method for improving resource utilization. *Journal of Operations Research Society* **58**(1) 235–245.

Perakis, G., G. Roels. 2007. Robust controls for network revenue management. *Working Paper, UCLA, Los Angeles, CA 90095* .

Perakis, G., A. Sood. 2003. Competitive multi-period pricing with fixed inventories. *Working Paper, MIT, Cambridge, MA,* .

Porteus, E. L. 2002. *Foundations of Stochastic Inventory Theory*. Stanford University Press.

Puu, T., I. Sushko. 2002. *Oligopoly Dynamics*. Springer.

Riley, J., R. Zeckhauser. 1983. Optimal selling strategies: When to haggle, when to hold firm. *Quarterly Journal of Economics* **98**(2) 267–289.

Ruth, R. J. 1981. Mixed integer programming model for regional planning of a hospital inpatient service. *Management Science* **27**(5) 521–533.

Sinclair, J., 2009. The internet blog in the adress http://jsmineset.com/.

Smith-Daniels, V. L., S. B. Schweikhart, D. E. Smith-Daniels. 1988. Capacity management in health care services: Review and future research directions. *Decision Science* **19**(4) 889–919.

Sobel, M. J., R. Zhang. 2001. Inventory policies for systems with stochastic and deterministic demand. *Operations Research* **49**(1) 157–162.

Su, X. 2007. Inter-temporal pricing with strategic customer behavior. *Management Science* **53**(3) 726–741.

Tang, Y., D. Xu, W. Zhou. 2007. Inventory rationing in a capacitated system with backorders and lost sales. *Proceedings of the 2007 IEEE IEEM* 1579–1583.

Tasnadi, A. 1999. A two-stage bertrand-edgeworth game. *Economics Letters* **65** 353–358.

Topkis, D. M. 1968. Optimal ordering and rationing policies in a nonstationary dynamic inventory model with n demand classes. *Management Science* **15**(3) 160–176.

Varma, G. D., N. Vettas. 2001. Optimal dynamic pricing with inventories. *Economic Letters* **72** 335–340.

Vulcano, G., G. van Ryzin, C. Maglaras. 2002. Optimal dynamic auctions for revenue management. *Management Science* **45**(11) 1388–1407.

Young, J. P., 1962. A queuing theory approach to the control of hospital inpatient census. Unpublished Doctoral Dissertation, The Johns Hopkins University, IE dept.

Zhou, Y., X. Zhao. 2010. A two-demand-class inventory system with lost sales and backorders. *Operations Research Letters* **38**(4) 261–266.

Appendix A

Appendix to Chapter 2

A.1 Proof of Theorem 1

First, define the following function based on (2.2):

$$G_t(s_t, x_t) \quad = \quad L(s_t, x_t) + \alpha \cdot \mathbf{E}_{M_t}[f_{t+1}((s_t + x_t - C)^+ + M_t)] \ .$$

Using an inductive argument, it can be shown that $G_t(s_t, x_t)$ is jointly concave in its arguments for all t, and $f_t(s_t)$ is also concave and decreasing. The details of this argument is standard and can be found in many of the papers mentioned in the literature; in particular, we refer the reader to the proof of Lemma 4.3 and Theorem 4.2 in Porteus (2002).

Note the optimal decision $x_t^*(s_t)$ is obtained from maximizing a concave function $G_t(s_t, x_t)$ with respect to x_t. From (2.3), it suffices to restrict the feasible set of x_t to $[(C - s_t)^+, C]$. Then, from the definition of G_t above and the definition of $L(s_t, x_t)$ in (2.1),

$$\frac{\partial G_t(s_t, x_t)}{\partial x_t} = (r_2 + w_2)(1 - H_t^2(x_t)) - w_1 + \alpha \frac{\partial \mathbf{E}_{M_t}[f_{t+1}((s_t + x_t - C) + M_t)]}{\partial x_t} . \tag{A-1}$$

(i) Consider the right-hand-side of (A-1). Since f_{t+1} is concave, the partial derivative in the last term is decreasing in s_t. It follows that $\partial G_t(s_t, x_t)/\partial x_t$ is decreasing in s_t. Note also that the upper-bound of the feasible region $C - s_t$ is decreasing in s_t. Therefore, we conclude that $x_t^*(s_t)$ is decreasing in s_t.

(ii) From (2.3) and its proof, it can be shown that $G_t(s_t, x_t)$ is increasing in x_t if $x_t \leq (C - s_t)^+$. Thus, the optimal solution $x_t^*(s_t)$ must satisfy the first-order condition that the partial derivative $\partial G_t(s_t, x_t)/\partial x_t$ at $x_t = x_t^*(s_t)$ is zero.

Thus,

$$\begin{aligned} 0 &= \frac{\partial G_t(s_t, x_t)}{\partial x_t}\Big|_{x_t = x_t^*(s_t)} \\ &= (r_2 + w_2)(1 - H_t^2(x_t^*(s_t))) - w_1 + \\ &\quad \alpha \frac{\partial \mathbf{E}_{M_t}[f_{t+1}((s_t + x_t - C) + M_t)]}{\partial x_t}\Big|_{x_t = x_t^*(s_t)} . \end{aligned}$$

Now, observe that

$$\begin{aligned} &\frac{\partial \mathbf{E}_{M_t}[f_{t+1}((s_t + x_t - C) + M_t)]}{\partial x_t}\Big|_{x_t = x_t^*(s_t)} \\ &= \frac{\partial \mathbf{E}_{M_t}[f_{t+1}((s_t + \epsilon + x_t - C) + M_t)]}{\partial x_t}\Big|_{x_t = x_t^*(s_t) - \epsilon} , \end{aligned}$$

hence:

$$
\begin{aligned}
&\frac{\partial G_t(s_t+\epsilon, x_t)}{\partial x_t}\Big|_{x_t=x_t^*(s_t)-\epsilon} \\
&= (r_2+w_2)(1-H_t^2(x_t^*(s_t)-\epsilon)) - w_1 \\
&+\alpha\frac{\partial \mathbf{E}_{M_t}[f_{t+1}((s_t+\epsilon+x_t-C)+M_t)]}{\partial x_t}\Big|_{x_t=x_t^*(s_t)-\epsilon} \\
&\geq 0\ .
\end{aligned}
$$

This result shows that $G_t(s_t+\epsilon, x_t)$ is increasing with respect to x_t when $x_t = x_t^*(s_t)-\epsilon$. Thus, we conclude that $x_t^*(s_t+\epsilon) \geq x_t^*(s_t)-\epsilon$.

□

A.2 Proof for Section 2.4.1

We modify the Bellman equation introduced in (2.2) to include the capacity of the current period t:

$$
f_t(s_t) \;=\; \mathbf{E}_{C_t}\left[\max_{0\leq x_t\leq C_t}\left[L(s_t,x_t,C_t)+\alpha\cdot\mathbf{E}_{M_t}[f_{t+1}((s_t+x_t-C_t)^+ + M_t)]\right]\right]\ ,
$$

where the single-period net-revenue function (2.1) is now modified to

$$
\begin{aligned}
L(s_t,x_t,C_t) \;&=\; r_2\mathbf{E}\min\{x_t,D_t\}+r_1\mathbf{E}[M_t]-w_1(s_t+x_t-C_t)^+ \\
&\quad -w_2\mathbf{E}[D_t-x_t]^+\ .
\end{aligned}
$$

Similar to the proof of Theorem 1, we define the following function:

$$
G_t(s_t,x_t,C_t) \;=\; L(s_t,x_t,C_t)+\alpha\cdot\mathbf{E}_{M_t,C_t}[f_{t+1}((s_t+x_t-C_t)^+ + M_t]
$$

An inductive argument similar to the one used Theorem 1 can be used to show that each f_t is concave in s_t, and G_t is jointly concave in (s_t, x_t) for any fixed C_t, for all t. Thus, given the capacity value C_t and the state variable s_t, there exists an optimal value of x_t, denoted by $x_t^*(s_t, C_t)$, that maximizes the function $G_t(s_t, x_t, C_t)$. As before, from (2.3), it suffices to restrict the feasible set of x_t to $[(C_t - s_t)^+, \infty)$. Then, the above definition of G_t implies

$$\frac{\partial G_t(x_t, s_t, C_t)}{\partial x_t} = (r_2+w_2)(1-H_t^2(x_t))-w_1+\alpha\frac{\partial \mathbf{E}_{M_t}[f_{t+1}((s_t + x_t - C_t) + M_t)]}{\partial x_t}\,. \tag{A-2}$$

(i) Consider the right-hand-side of (A-2). Since f_{t+1} is a concave function, the partial derivative in the last term is decreasing in s_t. It follows that $\partial G_t(x_t, s_t, C_t)/\partial x_t$ is decreasing in s_t. Note also that the lower-bound of the feasible region $(C_t - s_t)^+$ is decreasing in s_t. Therefore, we conclude that $x_t^*(s_t, C_t)$ is decreasing in s_t for fixed C_t.

(ii) As in the proof of Theorem 1, $x_t^*(s_t, C_t)$ satisfies the first-order condition that the partial derivative $\partial G_t(x_t, s_t, C_t)/\partial x_t$ at $x_t = x_t^*(s_t, C_t)$ is zero. Thus,

$$\begin{aligned} 0 &= \frac{\partial G_t(x_t, s_t, C_t)}{\partial x_t}\Big|_{x_t=x_t^*(s_t,C_t)} \\ &= (r_2 + w_2)(1 - H_t^2(x_t^*(s_t, C_t))) - w_1 \\ &\quad +\alpha\frac{\partial \mathbf{E}_{M_t}[f_{t+1}((s_t + x_t - C_t) + M_t)]}{\partial x_t}\Big|_{x_t=x_t^*(s_t,C_t)} \end{aligned} \tag{A-3}$$

Observe that

$$\begin{aligned} &\frac{\partial \mathbf{E}_{M_t}[f_{t+1}((s_t + x_t - C_t) + M_t)]}{\partial x_t}\Big|_{x_t=x_t^*(s_t,C_t)} \\ = &\frac{\partial \mathbf{E}_{M_t}[f_{t+1}((s_t + \epsilon + x_t - C_t) + M_t)]}{\partial x_t}\Big|_{x_t=x_t^*(s_t,C_t)-\epsilon}\,. \end{aligned}$$

Thus, by comparing the partial derivative of $G_t(x_t, s_t + \epsilon, C_t)$ with respect to

x_t at $x_t^*(s_t, C_t) - \epsilon$ to (A-3), we obtain that

$$\begin{aligned} &\frac{\partial G_t(x_t, s_t+\epsilon, C_t)}{\partial x_t}\Big|_{x_t = x_t^*(s_t,C_t)-\epsilon} \\ &= (r_2+w_2)(1-H_t^2(x_t^*(s_t,C_t)-\epsilon)) - w_1 \\ &\quad +\alpha \frac{\partial \mathbf{E}_{M_t}[f_{t+1}((s_t+\epsilon+x_t-C_t)+M_t)]}{\partial x_t}\Big|_{x_t=x_t^*(s_t,C_t)-\epsilon} \end{aligned}$$

is nonnegative since H_t^2 is a CDF, an increasing function. This result shows that $G_t(x_t, s_t+\epsilon, C_t)$ is increasing with respect to x_t when $x_t = x_t^*(s_t, C_t) - \epsilon$, which implies that $x_t^*(s_t+\epsilon, C_t) \geq x_t^*(s_t, C_t) - \epsilon$.

(iii) Next, we prove that $0 \leq x_t^*(s_t, C_t+\epsilon) - x_t^*(s_t, C_t) \leq \epsilon$.

Since $x_t^*(s_t, C_t)$ satisfies the first-order condition,

$$\begin{aligned} 0 &= \frac{\partial G_t(x_t, s_t, C_t)}{\partial x_t}\Big|_{x_t=x_t^*(s_t,C_t)} \\ &= (r_2+w_2)(1-H_t^2(x_t^*(s_t,C_t))) - w_1 \\ &\quad + \alpha \frac{\partial \mathbf{E}_{M_t}[f_{t+1}(s_t+x_t-C_t+M_t)]}{\partial x_t}\Big|_{x_t=x_t^*(s_t,C_t)} . \end{aligned}$$

As before, observe that:

$$\begin{aligned} &\frac{\partial \mathbf{E}_{M_t}[f_{t+1}((s_t+x_t-C_t)+M_t)]}{\partial x_t}\Big|_{x_t=x_t^*(s_t,C_t)} \\ &= \frac{\partial \mathbf{E}_{M_t}[f_{t+1}((s_t+x_t-C_t-\epsilon)+M_t)]}{\partial x_t}\Big|_{x_t=x_t^*(s_t,C_t)+\epsilon} . \end{aligned}$$

Moreover, $1 - H_t^2(x_t^*(s_t, C_t)) \geq 1 - H_t^2(x_t^*(s_t, C_t) + \epsilon)$. Thus,

$$\begin{aligned} 0 &\geq (r_2+w_2)(1-H_t^2(x_t^*(s_t,C_t)+\epsilon)) - w_1 \\ &\quad +\alpha \frac{\partial \mathbf{E}_{M_t}[f_{t+1}((s_t+x_t-C_t-\epsilon)+M_t)]}{\partial x_t}\Big|_{x_t=x_t^*(s_t,C_t)+\epsilon} \\ &= \frac{\partial G_t(x_t, s_t, C_t+\epsilon)}{\partial x_t}\Big|_{x_t=x_t^*(s_t,C_t)+\epsilon} . \end{aligned}$$

Hence, $G_t(x_t, s_t, C_t+\epsilon)$ is decreasing with respect to x_t when $x_t = x_t^*(s_t, C_t)+\epsilon$, which implies that $x_t^*(s_t, C_t+\epsilon) \le x_t^*(s_t, C_t)+\epsilon$. Moreover,

$$\begin{aligned} 0 &\le (r_2+w_2)(1-H_t^2(x_t^*(s_t,C_t))) - w_1 \\ &\quad +\alpha \frac{\partial \mathbf{E}_{M_t}[f_{t+1}((s_t+x_t-C_t-\epsilon)^+ + M_t)]}{\partial x_t}\Big|_{x_t=x_t^*(s_t,C_t)} \\ &= \frac{\partial G_t(x_t,s_t,C_t+\epsilon)}{\partial x_t}\Big|_{x_t=x_t^*(s_t,C_t)} , \end{aligned}$$

where the inequality follows from the fact that f_{t+1} is a concave function, which implies

$$\begin{aligned} &\frac{\partial \mathbf{E}_{M_t}[f_{t+1}((s_t+x_t-C_t-\epsilon)^+ + M_t)]}{\partial x_t}\Big|_{x_t=x_t^*(s_t,C_t)} \\ \ge\ &\frac{\partial \mathbf{E}_{M_t}[f_{t+1}((s_t+x_t-C_t) + M_t)]}{\partial x_t}\Big|_{x_t=x_t^*(s_t,C_t)} . \end{aligned}$$

Thus, $G_t(x_t, s_t, C_t+\epsilon)$ is increasing with respect to x_t when $x_t = x_t^*(s_t, C_t)$, which completes our proof.

□

A.3 Proof for Section 2.4.2

As in the proof of Theorem 1, define the following function:

$$\begin{aligned} G_t(s_t,x_t) &= L(s_t,x_t) \\ &+ \alpha \mathbf{E}_{M_t}\left[\max_{0\le a_t\le M_t} -r_1(M_t-a_t) + f_{t+1}((s_t+x_t-C)^+ + a_t)\right] . \end{aligned} \tag{A-4}$$

As before, an inductive argument establishes that each G_t is jointly concave, and that each f_t is concave and decreasing.

We first address the structure of the optimal policy given in (2.7). Define

$$\phi_t(\omega) = r_1\omega + f_{t+1}(\omega) .$$

Note that ϕ_t is a concave function. Let the maximizer of ϕ_t be denoted by R_t. Observe that the optimization problem in the definition of (A-4) is related to maximizing ϕ_t; in particular, it is equivalent to

$$\max_{z_t \le u_t \le z_t + M_t} \phi_t(u_t) \qquad \text{where} \quad z_t = (s_t + x_t - C)^+ \ .$$

By the concavity of ϕ_t and the convexity of the interval $[z_t, z_t + M_t]$, it can be verified that the optimal solution to this maximization problem is to set $u_t = R_t$ if R_t is feasible; otherwise, we choose u_t such that $z_t + u_t$ is as close to the target R_t as possible. This is the optimal policy given in (2.7).

Furthermore, the properties that $x_t^*(s_t)$ is decreasing in s_t and that $x_t^*(s_t+\epsilon) \ge x_t^*(s_t) - \epsilon$, for any $\epsilon > 0$, follows directly from (2.7).

□

A.4 Proof for Section 2.4.3

It is convenient to define a mapping between the state vector $\mathbf{s}_t$ and $\overline{\mathbf{s}}_t = (\overline{s}_t^1, \ldots, \overline{s}_t^n)$, where we define $\overline{s}_t^i = s_t^1 + s_t^2 + \cdots + s_t^i$. Thus, $\overline{s}_t^i$ refers to the sum of all backlogged patients in classes 1 through i. Then, $\zeta(\mathbf{s}_t, C - x_t) = \overline{\zeta}(\overline{\mathbf{s}}_t, C - x_t)$, where

$$\overline{\zeta}(\overline{\mathbf{s}}_t, C - x_t) \;\; = \;\; \left(\left[\overline{s}_t^1 - C + x_t\right]^+, \left[\overline{s}_t^2 - C + x_t\right]^+, \ldots, \left[\overline{s}_t^n - C + x_t\right]^+ \right) \ .$$

Let $\overline{w}_1^i = w_1^i - w_1^{i+1}$ where we let $w_1^{n+1} = 0$ for convenience. Define

$$\begin{aligned} \overline{L}(\overline{\mathbf{s}}_t, x_t) \;\; = \;\; & r_2 \cdot \mathbf{E} \min\{x_t, D_t\} + \sum_{i=1}^n r_1^i \cdot \mathbf{E}[M_t^i] - \sum_{i=1}^n \overline{w}_1^i \cdot \left[\overline{s}_t^i - C + x_t\right]^+ \\ & - w_2 \cdot \mathbf{E}[D_t - x_t]^+ \ . \end{aligned}$$

Then, the dynamic programming formulation can be written as

$$\begin{aligned}\overline{f}_t(\bar{\mathbf{s}}_t) &= \max_{0\le x_t\le C} \overline{G}_t(\bar{\mathbf{s}}_t, x_t)\ , \qquad\qquad \text{where} \\ \overline{G}_t(\bar{\mathbf{s}}_t, x_t) &= \overline{L}(\bar{\mathbf{s}}_t, x_t) + \alpha\cdot \mathbf{E}_{\mathbf{M}_t}[\overline{f}_{t+1}(\overline{\zeta}(\bar{\mathbf{s}}_t, C-x_t)+\mathbf{M}_t)]\ .\end{aligned}$$

For the terminal value, we let $\overline{f}_{T+1}(\bar{\mathbf{s}}_{T+1}) = \sum_{i=1}^n v^i\cdot(\bar{s}^i_{T+1} - \bar{s}^{i-1}_{T+1}) = \sum_{i=1}^n (v^i - v^{i+1})\cdot \bar{s}^i_{T+1}$, where $v^{n+1}=0$. Recall that $v^i - v^{i+1} \le 0$ for each i by assumption.

Note that $\overline{L}(\bar{\mathbf{s}}_t, x_t)$ is a jointly concave function. Then, it can be shown using induction that each $\overline{G}_t(\bar{\mathbf{s}}_t, x_t)$ is jointly concave and each $\overline{f}_t(\bar{\mathbf{s}}_t)$ is also concave and decreasing.

Now, let $\bar{x}^*_t(\bar{\mathbf{s}}_t)$ denote the value of x_t maximizing $\overline{G}_t(\bar{\mathbf{s}}_t, x_t)$ with respect to x_t. As before, it suffices to restrict the feasible set of x_t to $[(C-\bar{s}^n_t)^+, C]$. Then, from the definition of $\overline{G}_t$ and the definition of $\overline{L}$, we can obtain the partial derivative of $\overline{G}_t(\bar{\mathbf{s}}_t, x_t)$ with resect to x_t everywhere except finite points. For $x_t \in ((C-\bar{s}^i_t)^+, (C-\bar{s}^{i-1}_t)^+)$:

$$\begin{aligned}\frac{\partial \overline{G}_t(\bar{\mathbf{s}}_t, x_t)}{\partial x_t} &= (r_2+w_2)(1-H^2_t(x_t)) - \sum_{j=1}^n \overline{w}^j_1\cdot \mathbf{1}_{[x_t > C-\bar{s}^j_t]} \\ &\quad + \alpha\frac{\partial \mathbf{E}_{\mathbf{M}_t}[\overline{f}_{t+1}(\overline{\zeta}(\bar{\mathbf{s}}_t, C-x_t)+\mathbf{M}_t)]}{\partial x_t}\ . \end{aligned} \tag{A-5}$$

where $\mathbf{1}$ is the indicator function. (We remark that the second term in the right-hand-side of (A-5) is simply $\overline{w}^i_1 + \cdots + \overline{w}^n_1 = w^i_1$ since x_t is in the interval $((C-\bar{s}^i_t)^+, (C-\bar{s}^{i-1}_t)^+)$. We write this way so that it becomes more clear how this expression depends on $\bar{\mathbf{s}}_t$.)

(i) Based on (A-5), it can be shown as before that $\partial\overline{G}_t(\bar{\mathbf{s}}_t, x_t)/\partial x_t$ is decreasing in $\bar{\mathbf{s}}$. (To be technically correct, we should consider *one-sided* partial derivatives, but we use the term "derivative" for expositional simplicity.) To sketch the argument for this assertion, observe that the first term in (A-5) is independent of $\bar{\mathbf{s}}$, the second term is also decreasing in $\bar{\mathbf{s}}$, and finally, the third term is also decreasing in $\bar{\mathbf{s}}$ since we have

$$\frac{\partial \mathbf{E}_{\mathbf{M}_t}[\overline{f}_{t+1}(\overline{\zeta}(\bar{\mathbf{s}}_t, C - x_t) + \mathbf{M}_t)]}{\partial x_t} = \frac{\partial \mathbf{E}_{\mathbf{M}_t}[\overline{f}_{t+1}((\bar{\mathbf{s}}_t - C + x_t)^+ + \mathbf{M}_t)]}{\partial x_t}$$
$$= \sum_{j=1}^{n} \frac{\partial \mathbf{E}_{\mathbf{M}_t}[\overline{f}_{t+1}((\bar{\mathbf{s}}_t - C + x_t)^+ + \mathbf{M}_t)]}{\partial \bar{s}_t^j} \cdot \mathbf{1}_{[x_t > C - \bar{s}_t^j]} \quad \text{(A-6)}$$

where each partial derivative in the right-most expression is non-positive and decreasing in $\bar{s}_t^j$ since $\overline{f}_{t+1}$ is concave and decreasing.

Thus, we conclude that $\overline{x}^*(\bar{\mathbf{s}}_t)$ is decreasing in $\bar{\mathbf{s}}_t$. In particular, for any $\epsilon > 0$, we have $\overline{x}^*(\bar{\mathbf{s}}_t) \geq \overline{x}^*\left(\bar{\mathbf{s}}_t + \epsilon \cdot (\mathbf{e}^i + \mathbf{e}^{i+1} + \cdots + \mathbf{e}^n)\right)$. Thus, we obtain $x_t^*(\mathbf{s}_t) \geq x_t^*(\mathbf{s}_t + \epsilon \cdot \mathbf{e}^i)$ for any $\epsilon > 0$.

(ii) From the definition of $\overline{\zeta}$, it is easy to show that for any $\bar{\mathbf{s}}_t$, x_t and $\epsilon > 0$,

$$\overline{\zeta}(\bar{\mathbf{s}}_t + \epsilon \cdot (\mathbf{e}^i + \cdots + \mathbf{e}^n), C - (x_t - \epsilon)) \;\; \leq \;\; \overline{\zeta}(\bar{\mathbf{s}}_t, C - x_t) \qquad \text{for any } i \; .$$

Then, since the expression in (A-6) is decreasing in $\bar{\mathbf{s}}_t$, it follows that

$$\frac{\partial \mathbf{E}_{\mathbf{M}_t}[\overline{f}_{t+1}(\overline{\zeta}(\bar{\mathbf{s}}_t + \epsilon \cdot (\mathbf{e}^i + \cdots + \mathbf{e}^n), C - x_t) + \mathbf{M}_t)]}{\partial x_t}\Big|_{x_t - \epsilon}$$
$$\geq \;\; \frac{\partial \mathbf{E}_{\mathbf{M}_t}[\overline{f}_{t+1}(\overline{\zeta}(\bar{\mathbf{s}}_t, C - x_t) + \mathbf{M}_t)]}{\partial x_t}\Big|_{x_t} \; .$$

Also, since H_t^2 is a CDF, it follows that

$$(1 - H_t^2(x_t - \epsilon)) \;\; \geq \;\; (1 - H_t^2(x_t)) \; .$$

Finally,

$$\sum_{j=1}^{i-1} \overline{w}_1^j \cdot \mathbf{1}_{[x_t - \epsilon > C - \bar{s}_t^j]} \; + \; \sum_{j=i}^{n} \overline{w}_1^j \cdot \mathbf{1}_{[x_t - \epsilon > C - \bar{s}_t^j - \epsilon]} \;\; \leq \;\; \sum_{j=1}^{n} \overline{w}_1^j \cdot \mathbf{1}_{[x_t > C - \bar{s}_t^j]}$$

Therefore, from (A-5), we obtain that, for any $\bar{\mathbf{s}}_t$ and x_t,

$$\frac{\partial \overline{G}_t(\bar{\mathbf{s}}_t + \epsilon \cdot (\mathbf{e}^i + \cdots + \mathbf{e}^n), x_t)}{\partial x_t}\Big|_{x_t - \epsilon} \quad \geq \quad \frac{\partial \overline{G}_t(\bar{\mathbf{s}}_t, x_t)}{\partial x_t}\Big|_{x_t} .$$

Applying the above inequality to $x_t = \overline{x}^*(\bar{\mathbf{s}}_t)$, we obtain

$$\overline{x}^* \left(\bar{\mathbf{s}}_t + \epsilon \cdot (\mathbf{e}^i + \mathbf{e}^{i+1} + \cdots + \mathbf{e}^n)\right) \geq \overline{x}^*(\bar{\mathbf{s}}_t) - \epsilon.$$

Thus, $x_t^*(\mathbf{s}_t + \epsilon \cdot \mathbf{e}^i) \geq x_t^*(\mathbf{s}_t) - \epsilon$.

□

Appendix B

Appendix to Chapter 3

B.1 Proof of Theorem 3

First, note that any optimal strategy should satisfy $b(v_b) \leq v_b$ and $s(v_s) \geq v_s$ to be feasible; hence, this will be our implicit assumption throughout the analysis. Also assume that the optimal strategies are nondecreasing in the valuations of the bidders, which will also be shown to hold.

In the minimax absolute regret minimization problem (3.7) of the seller, the innermost maximization takes the following values depending on the relationship among b, s and v_s:

$$\max_{s'}[(kb + (1-k)s' - v_s)1_{\{b \geq s'\}} - (kb + (1-k)s - v_s)1_{\{b \geq s\}}]$$
$$= \begin{cases} 0 & \text{if } b < v_s \\ (b - v_s) & \text{if } v_s \leq b \leq s \\ (b - v_s) - (kb + (1-k)s - v_s) & \text{if } b > s \end{cases}$$

That is, if the buyer bid is less than the seller's valuation, the optimal bid returns zero net profits for the seller and so does any bid s the seller selects above her own valuation. If, however, the buyer bid is above the seller valuation, the seller achieves her optimal profit by selecting the same bid as the buyer; which is the situation in the second and third cases in the above equivalence. Observe that in the second case, the seller loses the chance of trade by bidding s since this bid is too large; whereas in the third case, she loses additional revenue she could have obtained if she had increased her bid to the point b.

Hence, the mathematical quantity to be minimized by selecting the appropriate s takes the form:

$$\begin{aligned}
&\max_{b} \max_{s'} [(kb + (1-k)s' - v_s) \cdot 1_{\{b \geq s'\}} - (kb + (1-k)s - v_s) \cdot 1_{\{b \geq s\}}] \\
&= \begin{cases} 0 & \text{if } b < v_s \\ (s - v_s) & \text{if } v_s \leq b \leq s \\ (1-k)(\bar{b} - s) & \text{if } b > s \end{cases} \\
&= \max\{(s - v_s), (1-k)(\bar{b} - s)\} \qquad \text{(B-1)}
\end{aligned}$$

where $\bar{b}$ is the unknown maximum value of the buyer's bid b. Thus, the problem of the seller is reduced to selecting the bid to minimize the maximum of two regret values arising in either of the two situations: In situation 1, the seller overbids and loses the chance to obtain positive return; whereas in situation 2, she bids too low and loses the chance to obtain higher profits.

Since the first of the quantities inside the maximization in (B-1) is increasing and the second is decreasing in s, the minimizer is attained at the intersection point, i.e:

$$\begin{aligned} & s^*_{ARMC}(v_s) = \underset{s}{\operatorname{argmin}} \max\{(s - v_s), (1-k)(\bar{b} - s)\} \\ \Rightarrow \quad & s^*_{ARMC}(v_s) - v_s = (1-k)(\bar{b} - s^*_{ARMC}(v_s)) \\ \Rightarrow \quad & s^*_{ARMC}(v_s) = \frac{v_s}{2-k} + \frac{(1-k)}{2-k}\bar{b} \end{aligned}$$

Via a symmetrical analysis for the buyers, we obtain $b^*_{ARMC}(v_b) = \frac{v_b}{1+k} + \frac{k}{1+k}\underline{s}$. Finally, since s^* and b^* should be best responses to each other and are increasing in v_s and v_b respectively, we find that the functions $s^*_{ARMC}(v_s) = \frac{v_s}{2-k} + \frac{(1-k)\bar{v}_b}{2} + \frac{k(1-k)\underline{v}_s}{2(2-k)}$ and $b^*_{ARMC}(v_b) = \frac{v_b}{1+k} + \frac{k\underline{v}_s}{2} + \frac{k(1-k)\bar{v}_b}{2(1+k)}$ are the solutions of the above equations and they satisfy all previously made assumptions. Furthermore, when the equations (3.1) and (3.2) are solved simultaneously for a game where both valuations are distributed **uniformly** on the given ranges, it is seen that the resulting equilibrium bidding functions are identical to the functions given by (3.9) and (3.10).

□

B.2 Analysis of the One-to-one Negotiation Problem between an Informed and Uninformed Agent

Consider the one-to-one negotiation problem between a single seller and a single buyer. We know the solution to this problem when (i) both the seller and the buyer know each other's distribution function, (ii) neither the seller nor the buyer know each other's distribution function, but they both know the range of the opponent valuations and employ ARMC approach to decide their bid. In this note, we will analyze a third option, i.e. (iii) the case where the seller knows the buyer distribution function, F_b, while the buyer only knows the range the seller's valuation can come from, $[\underline{v}_s, \bar{v}_s]$ (or, vice-versa).

During the analysis, we will implicitly assume that the equilibrium bidding functions $s(.)$ and $b(.)$ are increasing in the seller and the buyer valuations respectively. At the end of the analysis, we will indeed show that this claim is true, provided that the following assumption holds:

Assumption: F_b is a distribution function with decreasing hazard rate (DFR).

The revenue maximization problem of the seller takes the form:

$$\begin{aligned}\Pi_s(s, v_s) &= \max_{s\in[v_s,\bar{b}]} \int_s^{\bar{b}} (kb + (1-k)s - v_s) g_b(b) db, \\ &= \max_{s\in[v_s,b(\bar{v}_b)]} \int_{b^{-1}(s)}^{\bar{v}_b} (kb(v_b) + (1-k)s - v_s) f_b(v_b) dv_b,\end{aligned}$$

which is maximized at the value s that satisfies the following equation:

$$(1-k)(1-F_b(b^{-1}[s]))b'(b^{-1}[s]) - f_b(b^{-1}[s])(s - v_s) = 0 \qquad \text{(B-2)}$$

At this point, the seller does not know the function $b(.)$, or its derivative $b'(.)$. Thus, we turn our attention to the buyer's problem, which takes the form:

$$\begin{aligned}&\operatorname*{argmin}_b \big\{ \max_s \max_{b'} [(v_b - (kb' + (1-k)s)) \cdot 1_{\{b' \geq s\}} \\ &-(v_b - (kb + (1-k)s)) \cdot 1_{\{b \geq s\}}]\big\} \\ &= \operatorname*{argmin}_b \big\{ \max\{(v_b - b), k(b - \underline{s})\}\big\} \qquad \text{(B-3)}\end{aligned}$$

given that the buyer employs ARMC approach, by the analysis in the proof of Theorem 3.

Observe that, if the buyer is able to characterize the value of the lowest seller bid, $\underline{s} = s(\underline{v}_s)$, the solution of the equation (B-3) leads to the following bidding function:

$$b(v_b) = \frac{v_b}{k+1} + \frac{k\underline{s}}{k+1} \tag{B-4}$$

Hence, the seller's problem is equivalent to finding the s value that satisfies:

$$(1-k)(1-F_b(s(k+1)-k\underline{s}))\frac{1}{k+1} - f_b(s(k+1)-k\underline{s})(s-v_s) = 0$$

by inserting the appropriate values of b and b' into the equation (B-2).

Finally, the value of $\underline{s}$ is found from the equation:

$$(1-k)(1-F_b(\underline{s}))\frac{1}{k+1} - f_b(\underline{s})(\underline{s}-\underline{v}_s) = 0$$

which is then used to characterize the final form of the function $b(.)$.

Moreover, both $b(.)$ and $s(.)$ become nondecreasing in the respective valuations provided that F_b is a function with decreasing failure rate (which is a sufficient condition, but not a necessary one for $s(.)$ to be an increasing function). A symmetrical problem can be solved for the case where the buyer knows the seller distribution function, F_s, while the seller only knows the range buyer's valuation can come from, i.e. $[\underline{v}_b, \bar{v}_b]$.

□

B.3 Proof of Theorem 6

As before, our implicit assumptions are that the optimal strategies satisfy $b(v_b) \leq v_b$ and $s(v_s) \geq v_s$; and that the optimal strategies are nondecreasing in the valuations of the bidders.

Since buyers are naive (i.e. they play a one-to-one game with the seller regardless of the seller's inventory or other buyers in the market), their problem takes the form:

$$\begin{aligned}&\operatorname*{argmin}_{b}\{\max_{s}\max_{b'}[(v_b-(kb'+(1-k)s))\cdot 1_{\{b'\geq s\}}\\&-(v_b-(kb+(1-k)s))\cdot 1_{\{b\geq s\}}]\}\\&=\max\{(v_b-b),k(b-\underline{s})\}\end{aligned}$$

as before. As they assume that the seller is also playing a one-to-one game with them, they simply compute their optimal bidding strategy by solving the two ARMC problems simultaneously, therefore reaching at the same equilibrium bidding function as in the one-to-one game, i.e. b^*_{ARMC}.

On the other hand, the seller's problem is now different: Given that the bidding function of the buyers is b^*_{ARMC}, how should she select the bid $s_t = s$, $\forall t$, that would minimize her maximum regret for all distribution functions F_b of buyers?

$$\begin{aligned}&\operatorname*{argmin}_{s}\{\max_{F_b}\max_{s'}\left[\int_{t=0}^{T}\Lambda_t[\int_{b^{-1}(s')}^{\bar{v}_b}(kb(v_b)+(1-k)s'-v_s)f_b(v_b)dv_b]dt\right.\\-&\left.\int_{t=0}^{\min\{T,T'\}}\Lambda_t[\int_{b^{-1}(s)}^{\bar{v}_b}(kb(v_b)+(1-k)s-v_s)f_b(v_b)dv_b]dt\right]\}\end{aligned}$$

where s' is such that $\int_0^T \Lambda_t \int_{b^{-1}(s')}^{\bar{v}_b} f_b(v_b)dv_b dt = C$ (provided that $\int_0^T \Lambda_t f_b(\bar{v}_b)dt \leq C$), and T' is such that $\int_0^{T'} \Lambda_t \int_{b^{-1}(s)}^{\bar{v}_b}[f_b(v_b)dv_b]dt = C$, if $s < s'$.

Hence, regarding the inner maximization problem, we have two cases:

Case (i): $s < s'$:

This case indicates the situation that the seller underbids and fails to capture a higher profit. This loss is at its maximum when all buyers have the highest valuation, i.e. $f_b(\bar{v}_b) = 1$. Thus, the two inner maximization problems take the form:

$$\max_{F_b}\max_{s'}\Big\{\int_{t=0}^{T}\Lambda_t[\int_{b^{-1}(s')}^{\bar{v}_b}(kb(v_b)+(1-k)s'-v_s)f_b(v_b)dv_b]dt$$
$$-\int_{t=0}^{\min\{T,T'\}}\Lambda_t[\int_{b^{-1}(s)}^{\bar{v}_b}(kb(v_b)+(1-k)s-v_s)f_b(v_b)dv_b]dt\Big\}$$
$$=\int_{t=0}^{T}\Lambda_t[(kb(\bar{v}_b)+(1-k)(b(\bar{v}_b))-v_s)]dt$$
$$-\int_{t=0}^{T}\Lambda_t[(kb(\bar{v}_b)+(1-k)s-v_s)]dt$$
$$=((1-k)(b(\bar{v}_b)-s))\min\{C,\int_{t=0}^{T}\Lambda_t\}dt$$

Case (ii): $s > s'$:

This case indicates the fact that the seller overbids and fails to sell a proportion of her inventories. This loss is at its maximum when all buyers bid just slightly below the seller's bid s, i.e. $f_b(b^{-1}(s-\epsilon)) = 1$ for small $\epsilon > 0$. Thus, the two inner maximization problems take the form:

$$\max_{F_b}\max_{s'}\Big\{\int_{t=0}^{T}\Lambda_t[\int_{b^{-1}(s')}^{\bar{v}_b}(kb(v_b)+(1-k)s'-v_s)f_b(v_b)dv_b]dt$$
$$-\int_{t=0}^{T}\Lambda_t[\int_{b^{-1}(s)}^{\bar{v}_b}(kb(v_b)+(1-k)s-v_s)f_b(v_b)dv_b]dt\Big\}$$
$$=\int_{t=0}^{T}\Lambda_t[(k(s-\epsilon)+(1-k)(s-\epsilon)-v_s)]dt-0$$
$$=(s-v_s)\min\{C,\int_{t=0}^{T}\Lambda_t\}$$

Thus, combining the two cases, the seller should bid to minimize the two maximum regrets, i.e. $s = \operatorname{argmin}\max\{(s-v_s)\min\{C,\int_{t=0}^{T}\Lambda_t\},(1-k)(\bar{b}-s)\min\{C,\int_{t=0}^{T}\Lambda_t\}\}$. But note that these two regret terms are simply the same terms as in the one-to-one game, only multiplied by a coefficient $\min\{C, \int_{t=0}^{T} \Lambda_t\}$. Thus, we arrive at the same result as in the one-to-one game; i.e. the seller bids as if F_b is uniform on its given range; which also validates the buyers' bidding game.

□

B.4 Proof of Proposition 3

Recall that $\mu(t) = \hat{\mu}(t) + \delta(t)$ and that, whenever the affine form (3.39) of s_t is applied,

$$\begin{aligned} x(t) &= x(t-1) - \frac{\Lambda_{t-1}}{l(t-1)}[\mu(t-1) + 0.5l(t-1) - b^{*-1}(s_{t-1})] \\ &= x(t-1) - \frac{\Lambda_{t-1}}{l(t-1)}[\mu(t-1) + 0.5l(t-1) - 2(s_{t-1} - \underline{v}_s)] \\ &= C - \sum_{t'=1}^{t-1} \frac{\Lambda_{t'}}{l(t')}[\mu(t') + 0.5l(t') - 2(s_{t'} - \underline{v}_s)] \\ &= C - K(t) + \sum_{t'=1}^{t-1} \frac{\Lambda_{t'}}{l(t')}[-\delta(t') + \sum_{k=1}^{t'} B_{t',k}\delta(k)] \end{aligned}$$

where $K(t) := \sum_{t'=1}^{t-1} \frac{\Lambda_{t'}}{l(t')}[\hat{\mu}(t') + 0.5l(t') - 2m_{t'} + 2\underline{v}_s]$ is a constant. Hence, the inventory level $x(t)$ is an affine function of the past uncertainties. Replacing it in the equation (3.38):

$$
\begin{aligned}
s_t &= A_t + B_t x(t) + C_{t,t}\mu(t) + \sum_{j>t} C_{j,t}(\hat{\mu}(j) + \mathbf{E}[\delta(j)]) \\
&= A_t + B_t\big(C - K(t) + \sum_{t'=1}^{t-1} \frac{\Lambda_{t'}}{l(t')}[-\delta(t') + \sum_{k=1}^{t'} B_{t',k}\delta(k)]\big) + C_{t,t}\mu(t) \\
&\quad + \sum_{j>t} C_{j,t}(\hat{\mu}(j) + \mathbf{E}[\delta(j)]) \\
&= m_t + \sum_{k=1}^{t} B_{t,k}\delta(k)
\end{aligned}
$$

where

$$
\begin{aligned}
m_t &= A_t + B_t(C - K(t)) + \sum_{j\geq t} C_{j,t}\hat{\mu}(j) + \sum_{j>t} \mathbf{E}[\delta(j)], \\
B_{t,k} &= B_t\big[\sum_{t'=k}^{t-1} \frac{\Lambda_{t'}}{l(t')} B_{t',k} + \frac{\Lambda_k}{l(k)}(-1 + B_{k,k})\big], \ \text{for } k < t, \\
B_{t,t} &= C_{t,t}.
\end{aligned}
$$

Thus, for appropriate coefficients m_t and $B_{t,k}$, $\forall k \leq t$, the two formulations are equivalent.

□

Appendix C

Appendix to Chapter 4

C.1 Proof of Proposition 4

Proof. First, observe that the higher price agent (in this case the seller) will always choose her best response price p_1^s for any given p_1^{i*} such that $\bar{v}_1 > x(p_1^s, p_1^{i*})$ in order to be able to sell (since, otherwise, because of the price differential, the market demand is entirely captured by the investors).

Next, consider the lower price agent (the investors, here). For a given p_1^s, if the investors choose p_1^i such that $\frac{\bar{v}_1+p_1^s}{2} + (p_1^s - p_1^i) \leq \bar{v}_1$, then the maximizer of the function $p_1^i \Lambda_1[\frac{(\frac{\bar{v}_1+3p_1^{s*}}{2} - p_1^i) - p_1^i}{\bar{v}_1}]$ is given by $\frac{\bar{v}_1+3p_1^s}{8}$, and the revenue at this price is $\frac{\Lambda_1}{\bar{v}_1}\frac{(\bar{v}_1+3p_1^s)^2}{32}$. If, on the other hand, the investors choose the maximum price that leads to $x(p_1^s, p_1^i) = \bar{v}_1$, i.e. the price $\frac{3p_1^s-\bar{v}_1}{2}$, then the revenue they obtain is $\frac{\Lambda_1}{\bar{v}_1}\frac{(3\bar{v}_1-3p_1^s)(3p_1^s-\bar{v}_1)}{4}$. They will not choose any lower price than $\frac{3p_1^s-\bar{v}_1}{2}$ since they already capture the whole market demand at this price level. But observe that:

$$\begin{aligned}
& \frac{\Lambda_1}{\bar{v}_1}\frac{(\bar{v}_1+3p_1^s)^2}{32}-\frac{\Lambda_1}{\bar{v}_1}\frac{(3\bar{v}_1-3p_1^s)(3p_1^s-\bar{v}_1)}{4} \\
= & \frac{\Lambda_1}{\bar{v}_1}\frac{25\bar{v}_1^2-90\bar{v}_1p_1^s+81(p_1^s)^2}{32} \\
= & \frac{\Lambda_1}{\bar{v}_1}\frac{(5\bar{v}_1^2-9p_1^s)^2}{32} \\
\geq & \; 0
\end{aligned}$$

i.e. the revenue obtained by choosing the price level that leads to nonpositive sales for the seller is always lower than the revenue obtained otherwise. Thus, the investor is better off by increasing his price and allowing some fraction of the market demand to be captured by the seller. Note that we did not take into account the capacity restriction (i.e. the fact that the sales cannot exceed q_i) here; but since decreasing the price p_1^i increases the probability of running out of capacity even further and therefore decreases the revenues, the result does not change. □

C.2 Proof of Proposition 5

We will only consider the price pairs in the case of $q_i \geq 0.5C_1$ since the results in the case $q_i < 0.5C_1$ are symmetrical.

First, note that the prices stated in the Proposition 5 are simply found by replacing p_1^{i*} in the characterization of optimal seller prices (i.e. the equivalence (4.6)) by their values in the equivalence (4.7), and by replacing p_1^{s*} in the equation (4.7) by their correspondent values given in equation (4.6). The first set of prices (p_1^{s*}, p_1^{i*}) in the equivalence (4.8) corresponds to the case where the maximizer of unconstrained revenue maximization problems of both agents

($\phi_1(.)$ and $\phi_2(.)$) lead to sales figures that are less than the inventories of both agents. That is, solving $p_1^{i*} = \frac{3p_1^{s*}+\bar{v}_1}{8}$ and $p_1^{s*} = \frac{2p_1^{i*}+\bar{v}_1}{6}$ together, we found that $p_1^{i*} = \frac{9\bar{v}_1}{42}$ and $p_1^{s*} = \frac{10\bar{v}_1}{42}$; and the unconstrained sales figures acquire the values $\frac{18\Lambda_1}{42}$ and $\frac{15\Lambda_1}{42}$ for the investors and the seller respectively. The price pair $(p_1^{s*}, p_1^{i*}) = (\frac{10\bar{v}_1}{42}, \frac{9\bar{v}_1}{42})$ is therefore optimal for the ranges of Λ_1 values such that $q_s \geq \frac{15\Lambda_1}{42}$ and $q_i \geq \frac{18\Lambda_1}{42}$. The second set of prices corresponds to the case where the seller runs out of capacity if she sets the unconstrained revenue maximizing price; so both prices are adjusted accordingly (i.e. at optimum, the seller's sales is exactly equivalent to her inventories; and the investors choose their optimal best response price that maximizes their unconstrained revenue maximization problem. That is, $p_1^{s*} = \frac{2}{3}(\frac{\bar{v}_1}{2} + p_1^{i*} - \frac{\bar{v}_1 q_s}{\Lambda_1})$ and $p_1^{i*} = \frac{3p_1^{s*}+\bar{v}_1}{8}$). The third case is symmetrical to the second case, except that now the prices are adjusted in a way that the sales of investors' is exactly equivalent to their inventory. Finally, the fourth set of prices are observed when the market size is very large and both agents run out of capacity under the revenue maximizing prices of unconstrained case. Therefore, the prices (p_1^{s*}, p_1^{i*}) are found by solving:

$$\begin{aligned} \frac{\Lambda_1}{\bar{v}_1}(\bar{v}_1 - x(p_1^{s*}, p_1^{i*})) &= q_s \\ \frac{\Lambda_1}{\bar{v}_1}(x(p_1^{s*}, p_1^{i*}) - p_1^{i*}) &= q_i \end{aligned}$$

simultaneously.

Observe that the inequality $p_1^{i*} \leq p_1^{s*}$ holds in all cases as claimed in Theorem 7. Moreover, $x(p_1^{s*}, p_1^{i*}) \leq \bar{v}_1$ as claimed.

So we only need to show that neither of the agents wants to deviate from his/her equilibrium price. Since p_1^{s*} is the seller's best response price to p_1^{i*} and so is p_1^{i*} to p_1^{s*} given $p_1^{s*} \geq p_1^{i*}$; we need to check if there exists some other price $p_1^s < p_1^{i*}$ where the seller is better off than at p_1^{s*}, or some price $p_1^i > p_1^{s*}$

where the investors have higher revenue than that at p_1^{i*}. We will analyze all cases in detail:

- $\frac{15\Lambda_1}{42} \leq q_s$ and $\frac{18\Lambda_1}{42} \leq q_i$:

  Recall that in this case, $p_1^{s*} = \frac{10\bar{v}_1}{42}$, $p_1^{i*} = \frac{9\bar{v}_1}{42}$ and $x(p_1^{s*}, p_1^{i*}) = \frac{27\bar{v}_1}{42}$. Thus, the seller revenue is $\Pi_1^s(p_1^{s*} = \frac{10\bar{v}_1}{42}) = \frac{10\bar{v}_1}{42}\frac{15\Lambda_1}{42}$ and the investor revenue $\Pi_1^i(p_1^{i*} = \frac{9\bar{v}_1}{42}) = \frac{9\bar{v}_1}{42}\frac{18\Lambda_1}{42}$.

  First, consider the seller: The revenue maximization problem to find the revenue maximizing price $p_1^s < p_1^{i*}$ is given by:

  $$\max_{p_1^s < \frac{9\bar{v}_1}{42}} p_1^s \min\left\{q_s, \Lambda_1\left[\frac{\left(\frac{\bar{v}_1 + 3\frac{9\bar{v}_1}{42}}{2} - p_1^s\right) - p_1^s}{\bar{v}_1}\right]\right\}$$

  which is maximized at $p_1^s = \frac{23\bar{v}_1}{112}$ (provided $q_s > \frac{69\Lambda_1}{168}$) and the maximum revenue obtained by setting $p_1^s \leq p_1^{i*}$ therefore is $\frac{23\bar{v}_1}{112}\frac{69\Lambda_1}{168}$. But clearly $\frac{23\bar{v}_1}{112}\frac{69\Lambda_1}{168} < \frac{10\bar{v}_1}{42}\frac{15\Lambda_1}{42}$. Hence, the seller is better off by setting the price p_1^{s*}.

  Similarly, the revenue maximization problem the investors need to solve to find the revenue maximizer price $p_1^i > p_1^{s*}$ is given by:

  $$\max_{p_1^i > \frac{10\bar{v}_1}{42}} p_1^i \min\left\{q_i, \Lambda_1\left[\frac{\bar{v}_1 - \frac{\bar{v}_1 + 3p_1^i}{2} + \frac{10\bar{v}_1}{42}}{\bar{v}_1}\right]\right\}$$

  which is maximized at $p_1^i = \frac{31\bar{v}_1}{126}$ and the revenue obtained at this price level is $\frac{31\bar{v}_1}{126}\frac{31\Lambda_1}{84}$. But clearly $\frac{31\bar{v}_1}{126}\frac{31\Lambda_1}{84} < \frac{9\bar{v}_1}{42}\frac{18\Lambda_1}{42}$. Hence, the investors are also better off by setting the price p_1^{i*}.

- $\frac{15\Lambda_1}{42} > q_s$ and $\frac{2}{3}\Lambda_1 - \frac{2q_s}{3} \leq q_i$:

  Recall that in this case, $p_1^{s*} = \frac{5\bar{v}_1}{9} - \frac{8q_s\bar{v}_1}{9\Lambda_1}$, $p_1^{i*} = \frac{\bar{v}_1}{3} - \frac{q_s\bar{v}_1}{3\Lambda_1}$ and $x(p_1^{s*}, p_1^{i*}) = \bar{v}_1 - \frac{q_s\bar{v}_1}{\Lambda_1}$. Thus, the seller revenue is $\Pi_1^s(p_1^{s*} = \frac{5\bar{v}_1}{9} - \frac{8q_s\bar{v}_1}{9\Lambda_1}) = (\frac{5\bar{v}_1}{9} - \frac{8q_s\bar{v}_1}{9\Lambda_1})q_s$

and the investor revenue assumes the value $\Pi_1^i(p_1^{i*} = \frac{\bar{v}_1}{3} - \frac{q_s \bar{v}_1}{3\Lambda_1}) = (\frac{2\Lambda_1}{3} - \frac{2q_s}{3})(\frac{\bar{v}_1}{3} - \frac{q_s \bar{v}_1}{3\Lambda_1})$.

First, consider the seller: She is already selling all her inventory at the price $p_1^{s*} > p_1^{i*}$, so decreasing her price below p_1^{i*} could only lower her revenues. Thus, given that p_1^{s*} is her best response price to p_1^{i*} among all prices that exceed p_1^{i*}, she does not deviate.

Next, the revenue maximization problem of the investors to find the revenue maximizer price $p_1^i > p_1^{s*}$ is given by:

$$\max_{p_1^i > \frac{5\bar{v}_1}{9} - \frac{8q_s\bar{v}_1}{9\Lambda_1}} p_1^i \min\left\{q_i, \Lambda_1[\frac{\bar{v}_1 - \frac{\bar{v}_1 + 3p_1^i}{2} + \frac{5\bar{v}_1}{9} - \frac{8q_s\bar{v}_1}{9\Lambda_1}}{\bar{v}_1}]\right\}$$

which is maximized at $p_1^i = \frac{1}{3}(\frac{19\bar{v}_1}{18} - \frac{8q_s\bar{v}_1}{9\Lambda_1})$ and the revenue obtained at this price level is $\Lambda_1(\frac{19}{54} - \frac{8q_s}{27\Lambda_1})(\frac{19\bar{v}_1}{36} - \frac{8q_s\bar{v}_1}{18\Lambda_1})$. But note that:

$$\begin{aligned} & (\frac{\bar{v}_1}{3} - \frac{q_s\bar{v}_1}{3\Lambda_1})(\frac{2\Lambda_1}{3} - \frac{2q_s}{3}) - (\frac{19\bar{v}_1}{54} - \frac{8q_s\bar{v}_1}{27\Lambda_1})(\frac{19\Lambda_1}{36} - \frac{8q_s}{18}) \\ = \; & \frac{\Lambda_1}{\bar{v}_1}[2(\frac{\bar{v}_1}{3} - \frac{q_s\bar{v}_1}{3\Lambda_1})^2 - \frac{1}{6}(\frac{19\bar{v}_1}{18} - \frac{8q_s\bar{v}_1}{9\Lambda_1})^2] \\ \geq \; & 0 \end{aligned}$$

since:

$$\begin{aligned} & [2\sqrt{3}(\frac{\bar{v}_1}{3} - \frac{q_s\bar{v}_1}{3\Lambda_1}) - (\frac{19\bar{v}_1}{18} - \frac{8q_s\bar{v}_1}{9\Lambda_1})] \\ = \; & 0.099\bar{v}_1 - 0.266 q_s \frac{\bar{v}_1}{\Lambda_1} \geq 0 \end{aligned}$$

which follows from $\frac{15\Lambda_1}{42} > q_s$. Hence, the investors are also better off by staying below the price of the seller.

- $(\frac{\Lambda_1}{2} - \frac{q_i}{3}) \leq q_s$ and $\frac{18\Lambda_1}{42} > q_i$:

  First, observe that this case is only possible when $q_i > q_s \geq \frac{5q_i}{6}$; i.e. $\frac{1}{2}C_1 \leq q_i \leq \frac{6}{11}C_1$.

Recall that in this case, $p_1^{s*} = \frac{\bar{v}_1}{3} - \frac{2q_i\bar{v}_1}{9\Lambda_1}$, $p_1^{i*} = \frac{\bar{v}_1}{2} - \frac{2q_i\bar{v}_1}{3\Lambda_1}$ and $x(p_1^{s*}, p_1^{i*}) = \frac{\bar{v}_1}{2} + \frac{q_i\bar{v}_1}{3\Lambda_1}$. Thus, the seller revenue is $\Pi_1^s(p_1^{s*} = \frac{\bar{v}_1}{3} - \frac{2q_i\bar{v}_1}{9\Lambda_1}) = \frac{\Lambda_1}{\bar{v}_1}\frac{3}{2}(\frac{\bar{v}_1}{3} - \frac{2q_i\bar{v}_1}{9\Lambda_1})^2$ and the investor revenue $\Pi_1^i(p_1^{i*} = \frac{\bar{v}_1}{2} - \frac{2q_i\bar{v}_1}{3\Lambda_1}) = (\frac{\bar{v}_1}{2} - \frac{2q_i\bar{v}_1}{3\Lambda_1})q_i$.

First, consider the seller: The revenue maximization problem she needs to solve to find the revenue maximizing price $p_1^s < p_1^{i*}$ is given by:

$$\max_{p_1^s \leq \frac{\bar{v}_1}{2} - \frac{2q_i\bar{v}_1}{3\Lambda_1}} p_1^s \min\left\{q_s, \Lambda_1\left[\frac{\left(\frac{\bar{v}_1 + 3(\frac{\bar{v}_1}{2} - \frac{2q_i\bar{v}_1}{3\Lambda_1})}{2} - p_1^s\right) - p_1^s}{\bar{v}_1}\right]\right\}$$

which is maximized at $p_1^s = \frac{5\bar{v}_1}{16} - \frac{q_i\bar{v}_1}{4\Lambda_1}$ provided $q_s > \Lambda_1(\frac{5}{8} - \frac{q_i}{2\Lambda_1})$ and at $\min\{p_1^{i*}, \frac{5\bar{v}_1}{8} - \frac{\bar{v}_1(q_i+q_s)}{2\Lambda_1}\}$ otherwise. There are a few cases to consider:

(i) First, suppose $q_s > \frac{5\Lambda_1}{8} - \frac{q_i}{2}$ holds. Then, the revenue obtained at the price $p_1^s = \frac{5\bar{v}_1}{16} - \frac{q_i\bar{v}_1}{4\Lambda_1}$ is given by $\frac{\Lambda_1}{\bar{v}_1}2(\frac{5\bar{v}_1}{16} - \frac{q_i\bar{v}_1}{4\Lambda_1})^2$. Thus, the seller is better off by setting the lower price if:

$$\begin{aligned} & \frac{\Lambda_1}{\bar{v}_1}2\left(\frac{5\bar{v}_1}{16} - \frac{q_i\bar{v}_1}{4\Lambda_1}\right)^2 - \frac{\Lambda_1}{\bar{v}_1}\frac{3}{2}\left(\frac{\bar{v}_1}{3} - \frac{2q_i\bar{v}_1}{9\Lambda_1}\right)^2 \geq 0 \\ \iff & q_i \leq 0.414\Lambda_1 \end{aligned}$$

But note that $q_s < q_i$ and $q_s > \frac{5\Lambda_1}{8} - \frac{q_i}{2}$ together lead to $q_i > 0.416\Lambda_1$, which then leads to a contradiction along with the inequality $q_i \leq 0.414\Lambda_1$. Hence, this case cannot hold.

(ii) Next, consider the case where $q_s < \frac{5\Lambda_1}{8} - \frac{q_i}{2}$ and the revenue maximizing price is $\frac{5\bar{v}_1}{8} - \frac{\bar{v}_1(q_i+q_s)}{2\Lambda_1}$. Then, $\frac{5\bar{v}_1}{8} - \frac{r(q_i+q_s)}{2\Lambda_1} \leq p_1^{i*} = \frac{\bar{v}_1}{2} - \frac{2q_ir}{3\Lambda_1}$. Along with the definition of the case range (i.e. $\frac{\Lambda_1}{2} - \frac{q_i}{3} \leq q_s$ and $\frac{18\Lambda_1}{42} > q_i$), and the assumption that $q_s < q_i$, it is possible to show that $q_s \in [0.375\Lambda_1, 0.416\Lambda_1]$. Then, the difference in revenue figures by setting the price $\frac{5\bar{v}_1}{8} - \frac{\bar{v}_1(q_i+q_s)}{2\Lambda_1}$ and the price p_1^{s*} is given by:

$$\Delta\left(\frac{5\bar{v}_1}{8} - \frac{\bar{v}_1(q_i+q_s)}{2\Lambda_1}, \frac{\bar{v}_1}{3} - \frac{2q_i\bar{v}_1}{9\Lambda_1}\right) = q_s\left(\frac{5\bar{v}_1}{8} - \frac{\bar{v}_1(q_i+q_s)}{2\Lambda_1}\right) - \frac{\Lambda_1}{\bar{v}_1}\frac{3}{2}\left(\frac{\bar{v}_1}{3} - \frac{2q_i\bar{v}_1}{9\Lambda_1}\right)^2$$

It could be shown that the revenue difference $\Delta(\frac{5\bar{v}_1}{8} - \frac{\bar{v}_1(q_i+q_s)}{2\Lambda_1}, \frac{\bar{v}_1}{3} - \frac{2q_i\bar{v}_1}{9\Lambda_1})$ is nonpositive for all q_s values in the range $q_s \in [0.375\Lambda_1, 0.416\Lambda_1]$.

(To check this claim, note that Δ has partial derivative $\frac{\partial\Delta}{\partial q_s} = \frac{5\bar{v}_1}{8} - \frac{\bar{v}_1(q_i+q_s)}{2\Lambda_1} - \frac{3\Lambda_1}{2\bar{v}_1}2(\frac{\bar{v}_1}{3} - \frac{2(C_1-q_s)\bar{v}_1}{9\Lambda_1})\frac{2\bar{v}_1}{9\Lambda_1} = 0.403\bar{v}_1 - \frac{\bar{v}_1}{\Lambda_1}(0.351(q_i+q_s)+0.222q_s)$ (which follows by noting that $q_i + q_s = C_1$, a fixed value). Moreover, $\frac{\partial\Delta}{\partial q_s} > 0$ for all $q_s \in [0.375\Lambda_1, 0.416\Lambda_1]$. Thus, checking the revenue difference at the maximum value q_s could attain, i.e. $q_s = 0.416\Lambda_1$ and all associated $q_i \geq q_s$ values that satisfy the initial assumptions, it could be verified that the revenue difference $\Delta(\frac{5\bar{v}_1}{8} - \frac{\bar{v}_1(q_i+q_s)}{2\Lambda_1}, \frac{\bar{v}_1}{3} - \frac{2q_i\bar{v}_1}{9\Lambda_1})$ is nonpositive.)

Thus, the seller cannot do better by decreasing her price below p_1^{i*}.

Next, consider the investors: The revenue maximization problem of the investors to find the revenue maximizer price $p_1^i > p_1^{s*}$ is given by:

$$\max_{p_1^i > \frac{\bar{v}_1}{3} - \frac{2q_i\bar{v}_1}{9\Lambda_1}} p_1^i \min\left\{q_i, \Lambda_1[\frac{\bar{v}_1 - \frac{\bar{v}_1+3p_1^i}{2} + \frac{\bar{v}_1}{3} - \frac{2q_i\bar{v}_1}{9\Lambda_1}}{\bar{v}_1}]\right\}$$

which is maximized at $p_1^i = \frac{5\bar{v}_1}{18} - \frac{2q_i\bar{v}_1}{27\Lambda_1}$ provided that $q_i \geq 0.375\Lambda_1$. But note that, the revenue difference:

$$\Delta(\frac{5\bar{v}_1}{18} - \frac{2q_i\bar{v}_1}{27\Lambda_1}, \frac{\bar{v}_1}{2} - \frac{2q_i\bar{v}_1}{3\Lambda_1}) = \frac{1.5\Lambda_1}{\bar{v}_1}(\frac{5\bar{v}_1}{18} - \frac{2q_i\bar{v}_1}{27\Lambda_1})^2 - (\frac{\bar{v}_1}{2} - \frac{2q_i\bar{v}_1}{3\Lambda_1})q_i$$

is nonpositive for $q_i \in [0.375\Lambda_1, 0.428\Lambda_1]$. If, on the other hand, $q_i < 0.375\Lambda_1$, then the investor revenues are maximized at the price where investor sales is exactly equivalent to the inventory. But this is already the case at p_1^{i*}. Hence, the investors are also better off by staying at the price level p_1^{i*}.

- $\Lambda_1\frac{15}{42} > q_s$ and $\Lambda_1\frac{18}{42} > q_i$:

Recall that in this case, $p_1^{s*} = \bar{v}_1 - \frac{2(2q_s+q_i)\bar{v}_1}{3\Lambda_1}$, $p_1^{i*} = \bar{v}_1 - \frac{(q_i+q_s)\bar{v}_1}{\Lambda_1}$ and $x(p_1^{s*}, p_1^{i*}) = \bar{v}_1 - \frac{q_s\bar{v}_1}{\Lambda_1}$. Thus, the seller revenue is $\Pi_1^s(p_1^{s*}) = q_s(\bar{v}_1 - \frac{2(2q_s+q_i)\bar{v}_1}{3\Lambda_1})$ and the investor revenue $\Pi_1^i(p_1^{i*}) = (\bar{v}_1 - \frac{(q_i+q_s)\bar{v}_1}{\Lambda_1})q_i$.

First, consider the seller: She is already selling all her inventory at the price $p_1^{s*} > p_1^{i*}$, so decreasing her price below p_1^{i*} could only lower her revenues. Thus, given that p_1^{s*} is her best response price to p_1^{i*} among all prices that exceed p_1^{i*}, she does not deviate.

Next, the revenue maximization problem of the investors to find the revenue maximizer price $p_1^i > p_1^{s*}$ is given by:

$$\max_{p_1^i > \bar{v}_1 - \frac{2(2q_s+q_i)\bar{v}_1}{3\Lambda_1}} p_1^i \min\left\{q_i, \Lambda_1\left[\frac{\bar{v}_1 - \frac{\bar{v}_1+3p_1^i}{2} + \bar{v}_1 - \frac{2(2q_s+q_i)\bar{v}_1}{3\Lambda_1}}{\bar{v}_1}\right]\right\}$$

which is maximized at $p_1^i = \frac{\bar{v}_1}{2} - \frac{2(2q_s+q_i)\bar{v}_1}{9\Lambda_1}$, provided $\frac{\bar{v}_1}{2} - \frac{2(2q_s+q_i)\bar{v}_1}{9\Lambda_1} \geq p_1^{s*}$. Otherwise, by concavity of the revenue function, the investors' best response price in the range $p_1^i \geq p_1^{s*}$ is p_1^{s*} itself, which proves that p_1^{i*} is indeed the optimal price to start with. Thus, assume $\frac{\bar{v}_1}{2} - \frac{2(2q_s+q_i)\bar{v}_1}{9\Lambda_1} \geq p_1^{s*}$, i.e. $2q_s + q_i > \frac{9}{8}\Lambda_1$, which (combined with the fact that $q_s < \frac{15}{42}\Lambda_1$) leads to $q_i \geq \frac{23}{56}\Lambda_1$. The investor revenues at the price $\frac{\bar{v}_1}{2} - \frac{2(2q_s+q_i)\bar{v}_1}{9\Lambda_1}$ is $\frac{3\Lambda_1}{2\bar{v}_1}(\frac{\bar{v}_1}{2} - \frac{2(2q_s+q_i)\bar{v}_1}{9\Lambda_1})^2$. But note that, the revenue difference:

$$\begin{aligned}&\Delta\left(\frac{\bar{v}_1}{2} - \frac{2(2q_s+q_i)\bar{v}_1}{9\Lambda_1}, \bar{v}_1 - \frac{(q_i+q_s)\bar{v}_1}{\Lambda_1}\right)\\ &= \frac{3\Lambda_1}{2\bar{v}_1}\left(\frac{\bar{v}_1}{2} - \frac{2(2q_s+q_i)\bar{v}_1}{9\Lambda_1}\right)^2 - \left(\bar{v}_1 - \frac{(q_i+q_s)\bar{v}_1}{\Lambda_1}\right)q_i\end{aligned}$$

is nonpositive for $q_i \in [\frac{23}{56}\Lambda_1, \frac{18}{42}\Lambda_1]$, $q_s < q_i$ and $2q_s + q_i > \frac{9}{8}\Lambda_1$. Hence, the investors are again better off by staying below the price of the seller.

A similar result could be established in the cases where $\Lambda_1\frac{15}{42} > q_s$ and $\Lambda_1(\frac{2}{3} - \frac{2q_s}{3\Lambda_1}) > q_i$; and $\Lambda_1(\frac{1}{2} - \frac{q_i}{3\Lambda_1}) > q_s$ and $\Lambda_1(\frac{18}{42}) > q_i$ along the same lines as in this case, and we omit the details.

□

C.3 Proof of Proposition 7

Consider the centralized pricing problem:

$$\begin{aligned}\Pi_1(\gamma,\Lambda_1,q_s,q_i) &= \max_{p_1^i,p_1^s}\Big[p_1^s\min\Big\{q_s,\Lambda_1[\frac{\bar{v}_1-x(p_1^s,p_1^i)}{\bar{v}_1}]\Big\} \\ &\quad +p_1^i\min\Big\{q_i,\Lambda_1[\frac{x(p_1^s,p_1^i)-p_1^i}{\bar{v}_1}]\Big\}\Big]\mathbf{1}_{\{p_1^s\geq p_1^i\}} \\ &\quad +\Big[p_1^s\min\Big\{q_s,\Lambda_1[\frac{x(p_1^i,p_1^s)-p_1^s}{\bar{v}_1}]\Big\} \\ &\quad +p_1^i\min\Big\{q_i,\Lambda_1[\frac{\bar{v}_1-x(p_1^i,p_1^s)}{\bar{v}_1}]\Big\}\Big]\mathbf{1}_{\{p_1^s<p_1^i\}}\end{aligned}$$

Assume $q_i > q_s$. Suppose that the claim of the Proposition 7 holds. Then, the solution of the above problem is equivalent to the solution of the "Problem 1" defined as:

$$\begin{aligned}\Pi_1^1(\gamma,\Lambda_1,q_s,q_i) &= \max_{p_1^i,p_1^s} p_1^s\min\Big\{q_s,\Lambda_1[\frac{\bar{v}_1-x(p_1^s,p_1^i)}{\bar{v}_1}]\Big\} \\ &\quad +p_1^i\min\Big\{q_i,\Lambda_1[\frac{x(p_1^s,p_1^i)-p_1^i}{\bar{v}_1}]\Big\}\end{aligned}$$

which can be reformulated as:

$$\begin{aligned}\Pi_1^1(\gamma,\Lambda_1,q_s,q_i) &= \max_{p_1^s\geq p_1^i} p_1^s\Lambda_1[\frac{\bar{v}_1-x(p_1^s,p_1^i)}{\bar{v}_1}]+p_1^i\Lambda_1[\frac{x(p_1^s,p_1^i)-p_1^i}{\bar{v}_1}] \\ &\quad \text{subject to} \\ &\quad \Lambda_1[\frac{\bar{v}_1-x(p_1^s,p_1^i)}{\bar{v}_1}]\leq q_s \\ &\quad \Lambda_1[\frac{x(p_1^s,p_1^i)-p_1^i}{\bar{v}_1}]\leq q_i\end{aligned}$$

Observe that the above formulation is concave in both p_1^s and p_1^i; and its solution is as follows:

- If $q_s \geq \Lambda_1 \frac{3}{23}$ and $q_i \geq \Lambda_1 \frac{9}{23}$, the optimal prices are $(p_1^{s*}, p_1^{i*}) = (\frac{13}{23}\bar{v}_1, \frac{11}{23}\bar{v}_1)$ and the total system revenue is $\Pi_1^1 = \Lambda_1 \frac{6}{23}\bar{v}_1$.

- If $q_s < \Lambda_1 \frac{3}{23}$ and $q_i \geq \frac{\Lambda_1}{2} - \frac{5}{6}q_s$, the optimal prices are $(p_1^{s*}, p_1^{i*}) = (\frac{2}{3}\bar{v}_1 - \frac{7q_s\bar{v}_1}{9\Lambda_1}, \frac{\bar{v}_1}{2} - \frac{q_s\bar{v}_1}{6\Lambda_1})$ and the total system revenue is $\Pi_1^1 = \frac{1}{6}\bar{v}_1 q_s - \frac{23}{36}\frac{(q_s)^2\bar{v}_1}{\Lambda_1} + \Lambda_1 \frac{1}{4}\bar{v}_1$.

- If $q_s \geq \Lambda_1 \frac{3}{8} - \frac{5}{8}q_i$ and $q_i < \Lambda_1 \frac{9}{23}$, the optimal prices are $(p_1^{s*}, p_1^{i*}) = (\frac{\bar{v}_1}{2} + \frac{q_i\bar{v}_1}{6\Lambda_1}, \frac{5}{8}\bar{v}_1 - \frac{3q_i\bar{v}_1}{8\Lambda_1})$ and the total system revenue is $\Pi_1^1 = \frac{3}{8}\bar{v}_1 q_i - \frac{23}{48}\frac{\bar{v}_1(q_i)^2}{\Lambda_1} + \Lambda_1 \frac{3}{16}\bar{v}_1$.

- If $q_s < \Lambda_1 \frac{3}{23}$ and $q_i < \frac{\Lambda_1}{2} - \frac{5}{6}q_s$; or if $q_s < \Lambda_1 \frac{3}{8} - \frac{5}{8}q_i$ and $q_i < \Lambda_1 \frac{9}{23}$; then the optimal prices are $(p_1^{s*}, p_1^{i*}) = (\bar{v}_1 - \frac{2(2q_s+q_i)\bar{v}_1}{3\Lambda_1}, \bar{v}_1 - \frac{(q_i+q_s)\bar{v}_1}{\Lambda_1})$ and the total system revenue is $\Pi_1^1 = q_s(\bar{v}_1 - \frac{2(2q_s+q_i)\bar{v}_1}{3\Lambda_1}) + q_i(\bar{v}_1 - \frac{(q_i+q_s)\bar{v}_1}{\Lambda_1})$.

Next, assume that the claim of the Proposition does not hold; i.e. $p_1^{s*} < p_1^{i*}$. Then, the solution of the original problem is equivalent to the solution of the "Problem 2" below:

$$
\begin{aligned}
\Pi_1^2(\gamma, \Lambda_1, q_s, q_i) &= \max_{p_1^i, p_1^s} p_1^s \min\left\{q_s, \Lambda_1[\frac{x(p_1^i, p_1^s) - p_1^s}{\bar{v}_1}]\right\} \\
&\quad + p_1^i \min\left\{q_i, \Lambda_1[\frac{\bar{v}_1 - x(p_1^i, p_1^s)}{\bar{v}_1}]\right\}
\end{aligned}
$$

which is equivalent to:

$$
\begin{aligned}
\Pi_1^2(\gamma, \Lambda_1, q_s, q_i) &= \max_{p_1^s \geq p_1^i} p_1^s \Lambda_1[\frac{x(p_1^i, p_1^s) - p_1^s}{\bar{v}_1}] + p_1^i \Lambda_1[\frac{\bar{v}_1 - x(p_1^i, p_1^s)}{\bar{v}_1}] \\
&\quad \text{subject to} \\
&\quad \Lambda_1[\frac{\bar{v}_1 - x(p_1^i, p_1^s)}{\bar{v}_1}] \leq q_i \\
&\quad \Lambda_1[\frac{x(p_1^i, p_1^s) - p_1^s}{\bar{v}_1}] \leq q_s
\end{aligned}
$$

There are again a few cases to consider:

- If $q_i \geq \Lambda_1 \frac{3}{23}$ and $q_s \geq \Lambda_1 \frac{9}{23}$, the optimal prices are $(p_1^{s*}, p_1^{i*}) = (\frac{11}{23}\bar{v}_1, \frac{13}{23}\bar{v}_1)$ and the total system revenue is $\Pi_1^2 = \Lambda_1 \frac{6}{23}\bar{v}_1$.

- If $q_i < \Lambda_1 \frac{3}{23}$ and $q_s \geq \frac{\Lambda_1}{2} - \frac{5}{6}q_i$, the optimal prices are $(p_1^{s*}, p_1^{i*}) = (\frac{\bar{v}_1}{2} - \frac{q_i \bar{v}_1}{6\Lambda_1}, \frac{2}{3}\bar{v}_1 - \frac{7q_s\bar{v}_1}{9\Lambda_1})$ and the total system revenue is $\Pi_1^2 = \frac{1}{6}\bar{v}_1 q_i - \frac{23}{36}\frac{(q_i)^2\bar{v}_1}{\Lambda_1} + \Lambda_1\frac{1}{4}\bar{v}_1$. (However, this case is never attained for $q_i > q_s$.)

- If $q_i \geq \Lambda_1 \frac{3}{8} - \frac{5}{8}q_s$ and $q_s < \Lambda_1 \frac{9}{23}$, the optimal prices are $(p_1^{s*}, p_1^{i*}) = (\frac{5}{8}\bar{v}_1 - \frac{3q_s\bar{v}_1}{8\Lambda_1}, \frac{\bar{v}_1}{2} + \frac{q_s\bar{v}_1}{6\Lambda_1})$ and the total system revenue is $\Pi_1^2 = \frac{3}{8}\bar{v}_1 q_s - \frac{23}{48}\frac{\bar{v}_1(q_s)^2}{\Lambda_1} + \Lambda_1\frac{3}{16}\bar{v}_1$.

- If $q_i < \Lambda_1 \frac{3}{23}\bar{v}_1$ and $q_s < \frac{\Lambda_1}{2} - \frac{5}{6}q_i$; or if $q_i < \Lambda_1\frac{3}{8} - \frac{5}{8}q_s$ and $q_s < \Lambda_1\frac{9}{23}$; then the optimal prices are $(p_1^{s*}, p_1^{i*}) = (\bar{v}_1 - \frac{(q_i+q_s)\bar{v}_1}{\Lambda_1}, \bar{v}_1 - \frac{2(2q_i+q_s)\bar{v}_1}{3\Lambda_1})$ and the total system revenue is $\Pi_1^2 = q_i(\bar{v}_1 - \frac{2(2q_i+q_s)\bar{v}_1}{3\Lambda_1}) + q_s(\bar{v}_1 - \frac{(q_i+q_s)\bar{v}_1}{\Lambda_1})$.

Observe that some of the cases in the characterization of Π_1^2 can never be attained for $q_i > q_s$. Combining the two analyses, the comparison of optimal values of Problem 1 and Problem 2 are as following:

- If both $q_i > \Lambda_1\frac{9}{23}$ and $q_s > \Lambda_1\frac{9}{23}$, the optimal solutions of the two problems are equivalent; i.e. $\Pi_1^1 = \Pi_1^2 = \Lambda_1\frac{6}{23}\bar{v}_1$.

- If $q_i > \Lambda_1\frac{9}{23}$ but $\Lambda_1\frac{3}{23} < q_s < \Lambda_1\frac{9}{23}$, then $\Pi_1^1 = \Lambda_1\frac{6}{23}\bar{v}_1$ but $\Pi_1^2 = \frac{3}{8}\bar{v}_1 q_s - \frac{23}{48}\frac{\bar{v}_1(q_s)^2}{\Lambda_1} + \Lambda_1\frac{3}{16}\bar{v}_1$. It could be shown that $\Pi_1^1 \geq \Pi_1^2$ for this range of (q_i, q_s) values by noting that the function Π_1^2 increases in q_s for $q_s \in (\Lambda_1\frac{3}{23}, \Lambda_1\frac{9}{23})$ and attains it maximum value which is equivalent to Π_1^1 at $q_s = \Lambda_1\frac{9}{23}$.

- If $\Lambda_1\frac{3}{23} < q_s < q_i < \Lambda_1\frac{9}{23}$, then $\Pi_1^1 = \frac{3}{8}\bar{v}_1 q_i - \frac{23}{48}\frac{\bar{v}_1(q_i)^2}{\Lambda_1} + \Lambda_1\frac{3}{16}\bar{v}_1$. There are two cases for the second problem: If $q_i \geq \Lambda_1\frac{3}{8} - \frac{5}{8}q_s$, then $\Pi_1^2 = \frac{3}{8}\bar{v}_1 q_s - \frac{23}{48}\frac{\bar{v}_1(q_s)^2}{\Lambda_1} + \Lambda_1\frac{3}{16}\bar{v}_1$. Otherwise, $\Pi_1^2 = q_i(\bar{v}_1 - \frac{2(2q_i+q_s)\bar{v}_1}{3\Lambda_1}) + q_s(\bar{v}_1 - \frac{(q_i+q_s)\bar{v}_1}{\Lambda_1})$.

By noting that $q_i(\bar{v}_1 - \frac{2(2q_i+q_s)\bar{v}_1}{3\Lambda_1}) + q_s(\bar{v}_1 - \frac{(q_i+q_s)\bar{v}_1}{\Lambda_1}) \leq \frac{3}{8}\bar{v}_1 q_s - \frac{23}{48}\frac{\bar{v}_1(q_s)^2}{\Lambda_1} + \Lambda_1\frac{3}{16}\bar{v}_1$ for $q_i < \Lambda_1\frac{3}{8} - \frac{5}{8}q_s$ and that the function $\zeta(x) = \frac{3}{8}\bar{v}_1 x - \frac{23}{48}\frac{\bar{v}_1 x^2}{\Lambda_1} + \Lambda_1\frac{3}{16}\bar{v}_1$ is increasing in the range $x \in (\Lambda_1\frac{3}{23}, \Lambda_1\frac{9}{23})$, it could be shown that $\Pi_1^1 \geq \Pi_1^2$ in both cases.

- If $q_s < \Lambda_1\frac{3}{23} < q_i < \Lambda_1\frac{9}{23}$, there are three cases:
  - If $q_i < \Lambda_1\frac{3}{8} - \frac{5}{8}q_s$, then $\Pi_1^1 = q_s(\bar{v}_1 - \frac{2(2q_s+q_i)\bar{v}_1}{3\Lambda_1}) + q_i(\bar{v}_1 - \frac{(q_i+q_s)\bar{v}_1}{\Lambda_1})$ and $\Pi_1^2 = q_i(\bar{v}_1 - \frac{2(2q_i+q_s)\bar{v}_1}{3\Lambda_1}) + q_s(\bar{v}_1 - \frac{(q_i+q_s)\bar{v}_1}{\Lambda_1})$. It could be shown that $\Pi_1^1 \geq \Pi_1^2$ for this range of (q_i, q_s) values, since $(\bar{v}_1 - \frac{2(2q_i+q_s)\bar{v}_1}{3\Lambda_1}) < (\bar{v}_1 - \frac{(q_i+q_s)\bar{v}_1}{\Lambda_1}) < (\bar{v}_1 - \frac{2(2q_s+q_i)\bar{v}_1}{3\Lambda_1})$ and $q_s < q_i$.
  - If $\Lambda_1\frac{3}{8} - \frac{5}{8}q_s \leq q_i < \Lambda_1\frac{1}{2} - \frac{5}{8}q_s$, then $\Pi_1^1 = q_s(\bar{v}_1 - \frac{2(2q_s+q_i)\bar{v}_1}{3\Lambda_1}) + q_i(\bar{v}_1 - \frac{(q_i+q_s)\bar{v}_1}{\Lambda_1})$ and $\Pi_1^2 = \frac{3}{8}\bar{v}_1 q_s - \frac{23}{48}\frac{\bar{v}_1(q_s)^2}{\Lambda_1} + \Lambda_1\frac{3}{16}\bar{v}_1$. It could be shown that $\Pi_1^1 \geq \Pi_1^2$ for this range of (q_i, q_s) values.
  - If $q_i \geq \Lambda_1\frac{1}{2} - \frac{5}{8}q_s$, then $\Pi_1^1 = \frac{1}{6}\bar{v}_1 q_s - \frac{23}{36}\frac{\bar{v}_1(q_s)^2}{\Lambda_1} + \Lambda_1\frac{1}{4}\bar{v}_1$ and $\Pi_1^2 = \frac{3}{8}\bar{v}_1 q_s - \frac{23}{48}\frac{\bar{v}_1(q_s)^2}{\Lambda_1} + \Lambda_1\frac{3}{16}\bar{v}_1$. It could be shown that $\Pi_1^1 \geq \Pi_1^2$ for $q_s < \Lambda_1\frac{3}{23}$.
- If $q_s < \Lambda_1\frac{3}{23} < \Lambda_1\frac{9}{23} < q_i$, there are two cases:
  - If $\Lambda_1\frac{3}{8} - \frac{5}{8}q_s \leq q_i < \Lambda_1\frac{1}{2} - \frac{5}{8}q_s$, then $\Pi_1^1 = q_s(\bar{v}_1 - \frac{2(2q_s+q_i)\bar{v}_1}{3\Lambda_1}) + q_i(\bar{v}_1 - \frac{(q_i+q_s)\bar{v}_1}{\Lambda_1})$ and $\Pi_1^2 = \frac{3}{8}\bar{v}_1 q_s - \frac{23}{48}\frac{\bar{v}_1(q_s)^2}{\Lambda_1} + \Lambda_1\frac{3}{16}\bar{v}_1$. It could be shown that $\Pi_1^1 \geq \Pi_1^2$ for this range of (q_i, q_s) values.
  - If $q_i \geq \Lambda_1\frac{1}{2} - \frac{5}{8}q_s$, then $\Pi_1^1 = \frac{1}{6}\bar{v}_1 q_s - \frac{23}{36}\frac{\bar{v}_1(q_s)^2}{\Lambda_1} + \Lambda_1\frac{1}{4}\bar{v}_1$ and $\Pi_1^2 = \frac{3}{8}\bar{v}_1 q_s - \frac{23}{48}\frac{\bar{v}_1(q_s)^2}{\Lambda_1} + \Lambda_1\frac{3}{16}\bar{v}_1$. It could be shown that $\Pi_1^1 \geq \Pi_1^2$ for $q_s < \Lambda_1\frac{3}{23}$.
- If $q_s < q_i < \Lambda_1\frac{3}{23}$, then then $\Pi_1^1 = q_s(\bar{v}_1 - \frac{2(2q_s+q_i)\bar{v}_1}{3\Lambda_1}) + q_i(\bar{v}_1 - \frac{(q_i+q_s)\bar{v}_1}{\Lambda_1})$ and $\Pi_1^2 = q_i(\bar{v}_1 - \frac{2(2q_i+q_s)\bar{v}_1}{3\Lambda_1}) + q_s(\bar{v}_1 - \frac{(q_i+q_s)\bar{v}_1}{\Lambda_1})$. It is easy to see that $\Pi_1^1 \geq \Pi_1^2$

for this range of (q_i, q_s) values since $(\bar{v}_1 - \frac{2(2q_i+q_s)\bar{v}_1}{3\Lambda_1}) < (\bar{v}_1 - \frac{(q_i+q_s)\bar{v}_1}{\Lambda_1}) < (\bar{v}_1 - \frac{2(2q_s+q_i)\bar{v}_1}{3\Lambda_1})$.

Hence, in all cases the optimal value attained in the first problem is greater than or equal to the optimal value of the second problem. Therefore, the formulation (4.14) could be reduced to the problem defined as Problem 1; which proves the claim of the Proposition.

□