Partial differential equations and variational approaches to constant scalar curvature metrics in Kähler geometry

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ABSTRACT

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In this thesis we investigate two approaches to the problem of existence of metrics of constant scalar curvature in a fixed Kähler class. In the first part, we examine the equation for constant scalar curvature under the assumption of toric symmetry, thus reducing the problem to a fourth order nonlinear degenerate elliptic equation for a convex function defined in a polytope in $\mathbb{R}^n$. We obtain partial results on this equation using an associated Monge-Ampère equation to determine the boundary behavior of the solution. In the second part, we consider the asymptotics of certain energy functionals and their relation to stability and the existence of minimizers. We derive explicit formulas for their asymptotic slopes, which allows one to determine whether or not (X, L) is stable, and in some cases rule out the existence of a canonical metric.

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To my parents

Chapter 1

General Introduction

A fundamental theme of modern geometry is to characterize a given geometric structure in terms of a *canonical metric*, a metric with prescribed or optimal curvature properties. A classical example of this phenomenon is the uniformization theorem for Riemann surfaces, which states that a complex structure is characterized by a metric of constant curvature. A guiding problem of modern investigation is the following:

Question 1.0.1. *Given a holomorphic line bundle $L \to X$ such that $c_1(L)$ is a Kähler class, does there exist a Kähler metric $\omega \in c_1(L)$ with constant scalar curvature?*

In the case when $L = K_X^{-\mu}$, a multiple of the canonical bundle, this question reduces to the well-known Kähler-Einstein problem: to ascertain the existence of a Kähler metric ω with

$$Ric(\omega) = \mu\omega. \tag{1.1}$$

The problem of finding Kähler-Einstein metrics has a long history, dating back to Yau's solution of the Calabi conjecture [37] in the case of Kähler manifolds with first Chern class $c_1 = 0$ (equation (1.1) with $\mu = 0$), and the solution by Yau [37] and Aubin [2] for the case $\mu < 0$, which showed the existence and uniqueness of Kähler-Einstein metrics in these cases. The case of Fano manifolds with $\mu > 0$, has been the subject of much investigation; the problem is more subtle as there are known obstructions to the existence of Kähler-Einstein metrics in this case.

The existence of canonical metrics in Kähler geometry in general is now well understood to be linked to certain notions of stability in the sense of geometric invariant theory. Historically, the first result of this type was the theorem of Donaldson-Uhlenbeck-Yau on the equivalence of the existence of Hermitian-Einstein metrics on holomorphic vector bundles with Mumford-Takemoto stability [14; 36]. The problems of Kähler-Einstein metrics and constant scalar curvature metrics may be regarded as analogues of the Hermitian-Einstein problem, where the canonical metric is on the manifold rather than a vector bundle. This leads to two related difficulties: the differential equation is more nonlinear, and the relevant notion of stability is less clear; indeed, identifying the correct notion of stability is a central part of the problem. In addition, having identified a stability condition which guarantees a solution, it is also an interesting problem to give a formulation of the stability condition that is possible to verify in practice, e.g. a numerical condition on an explicitly computable quantity given in terms of algebraic data.

The recent resolution of the Yau-Tian-Donaldson conjecture in the Fano case establishes the equivalence of K-stability (defined in the introduction to part II) and the existence of Kähler-Einstein metrics on Fano manifolds [38; 9; 10; 11; 21; 34; 3]. A more general version of the Yau-Tian-Donaldson conjecture asserts that K-stability is also the necessary and sufficient condition for existence of constant scalar curvature Kähler metrics, providing an answer to Question 1.0.1. This problem may be regarded as somewhat more difficult than the Kähler-Einstein problem, in part because the equation of scalar curvature is of fourth order in the Kähler potential.

In this thesis, we present two approaches to the problem of existence of metrics of constant scalar curvature in a fixed Kähler class. In the first part, we consider the fourth order nonlinear partial differential equation for scalar curvature in the case of toric symmetry. This reduces the problem to a PDE for a convex function defined in a polytope in $\mathbb{R}^n$, known as Abreu's equation. We proceed in an attack on the problem by viewing the equation as a system of equations involving a Monge-Ampère equation and a linearized Monge-Ampère equation. By an analysis of the degenerate Monge-Ampère equation we are able to obtain partial results in establishing the correct geometric boundary behavior of a solution, known as the Guillemin boundary conditions.

In the second part of the thesis, we take a different approach to the problem of canonical metrics, via the minimization of certain convex energy functionals defined on the space of Kähler potentials. The idea in this part is to examine the asymptotics of these functionals along special one-parameter degenerations of the potential along which the functionals are convex. We manage to compute the asymptotic slopes of the functionals along these degenerations, which provides a way of checking whether a polarized variety (X, L) is stable, and ruling out the existence of a canonical metric if there is a degeneration with negative asymptotic slope. This is done by employing methods from algebraic geometry and singular integral analysis. We begin with a study of the asymptotics of the Aubin-Yau functional, which is most directly related to the notion of Chow-Mumford stability and the existence of *balanced metrics* coming from special Kodaira embeddings.

Part I

Abreu's equation for constant scalar curvature metrics on toric varieties

Let ω_0 be a Kähler metric in $c_1(L)$. The space of metrics in $c_1(L)$ may be parametrized by $\omega_\phi = \omega_0 + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\phi$, where ϕ is in the space

$$\mathcal{K} = \{\phi \in C^\infty(X) : \omega_0 + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\phi > 0\}. \tag{1.2}$$

The scalar curvature $S(\omega_\phi)$ is given by the fourth-order expression

$$S(\omega_\phi) = -g^{i\bar{j}}\partial_i\partial_{\bar{j}}\log(\omega_\phi^n). \tag{1.3}$$

Let X be a toric variety of dimension n. Then its image under the moment map is a polytope P in $\mathbf{R}^n$, and a toric Kähler metric on X can be determined by a function $u : \bar{P} \to \mathbb{R}^n$ called the *symplectic potential*, which is the Legendre transform of the Kähler potential in the open torus. Guillemin [22] showed that the symplectic potential of a smooth toric variety satisfies

$$u(x) = \sum_{i=1}^N l_i(x)\log l_i(x) + f(x), \;\; f \in C^\infty(\overline{P}), \, u \text{ convex in } \overline{P} \tag{1.4}$$

where the affine functions $l_i(x)$ defining the faces of P have been appropriately normalized. As shown by Abreu [1], the Kähler metric is an extremal metric if and only if its symplectic potential u satisfies the so-called Abreu equation

$$\sum_{i,j=1}^n \frac{\partial^2 u^{ij}}{\partial x_i \partial x_j} = -A, \tag{1.5}$$

where (u^{ij}) is the inverse of the Hessian (u_{ij}) where A is an affine function. The metric is of constant scalar curvature when A is constant. The Abreu equation is clearly equivalent to the following system of two second-order elliptic equations for the two unknowns (u, φ),

$$\det D^2 u = \varphi^{-1} \tag{1.6}$$

$$U^{ij}\varphi_{ij} = -A \tag{1.7}$$

where (U^{ij}) is the cofactor matrix of the Hessian of u.

The existence of a metric of constant scalar curvature, and hence the solvability of Abreu's equation, has been shown by Donaldson in dimension $n = 2$ to be equivalent to the K-stability of the toric variety X [20]. The same statement is expected to hold in all dimensions, and is known as the Yau-Tian-Donaldson conjecture [16] (see also [32] for a survey). Donaldson also gave interior estimates for Abreu's equation, using in part works of Trudinger-Wang [35] on similar equations arising from the affine Plateau problem. Donaldson's results were subsequently extended by Chen, Li, and Sheng [8], who solved the problem of general prescribed curvatures in dimension two, and also by Chen, Han, Li, and Sheng [7] giving interior estimates for all dimensions.

A major difficulty of solving (1.5) in a polytope is the degeneracy and singular behavior of solutions near the boundary, different parts of which look locally like the intersection of 1 to n half-planes. In the next chapter, we detail an approach to Abreu's equation beginning with study of the Monge-Ampère equation appearing in (1.6). Our main result is Theorem 2.1.1, in which we establish the correct boundary behavior near the edges of a polygon in dimension 2, though the regularity near the corners is still open.

Chapter 2

The Monge-Ampère equation with Guillemin boundary conditions

2.1 Introduction

The aim of this chapter is to study a seemingly new type of boundary value problem for a real Monge-Ampère equation in a convex polytope. More precisely, let $P \subset \mathbf{R}^n$ be a polytope, and let

$$P = \cap_{i=1}^N \{l_i(x) > 0\}, \tag{2.1}$$

be a representation of P as an intersection of half-planes, with $l_i(x)$ an affine function of x for each i. We consider the problem of finding a function $u \in C^0(\overline{P})$ and strictly convex satisfying

$$\det D^2 u(x) = \frac{1}{\varphi(x)} \tag{2.2}$$

$$u(x) - \sum_{i=1}^N l_i(x) \log l_i(x) \in C^\infty(\overline{P}) \tag{2.3}$$

where the given function φ on P is of the form

$$\varphi(x) = h(x) \prod_{i=1}^N l_i(x) \tag{2.4}$$

with $h(x) \in C^\infty(\overline{P})$, $0 < h(x)$. Boundary conditions of the form (2.3) are called Guillemin boundary conditions.

The motivation for this problem comes Abreu's equation (1.5) in toric geometry, which is the equation for a Kähler metric of constant scalar curvature on a toric variety, as described in the introduction to this part. Note that an essential part of the problem is the particular form of the boundary condition, and the fact that the equation takes place on a polytope. For example, a naive version of the problem on a strictly convex domain $D \subset \mathbf{R}^2$ with boundary function $d(x)$ of the form $u - d \log d \in C^3(\overline{D})$, $\det D^2u(x) = O(d^{-1})$, would have no solution, since the boundary asymptotics for u would imply that $\det D^2u(x) \sim d^{-1} \log d^{-1}$ near ∂D. Thus the polytope features of the problem have to be fully taken into account, and they play a major role in our results which we describe next.

We restrict our attention to the case of dimension two. Let n_i be the shortest inward-pointing normal vector in $\mathbb{Z}^n$ to the edge $e_i = \{l_i(x) = 0\}$ of the polytope P, and set $l_i(x) = n_i \cdot x - \lambda_i$ and let T_i be the tangent vector to the face which is the -90 degree rotation of n_i. Any two vectors n_i and n_k form the columns of a 2×2 matrix, the determinant of which is the area of the parallelogram spanned by n_i and n_k. We also denote the N vertices of the polytope P by v_i, $1 \leq i \leq N$, with v_i the intersection of the faces $l_{i-1} = 0$ and $l_i = 0$ (also set $l_0 := l_N$). Then

Theorem **2.1.1.** *Let P be a convex polytope in $\mathbb{R}^2$, and consider the problem (2.2, 2.3) where $h \in C^\infty(\overline{P})$ and $h(x) > 0$.*

(a) *If the equation admits a solution u which is convex in P and satisfies the boundary condition in (2.2, 2.3), then the given function $h(x)$ must satisfy*

$$h(v_i) \quad = \quad \left(\det(n_{i-1} n_i)^2 \prod_{j \neq i-1, i} l_j(v_i) \right)^{-1} \tag{2.5}$$

(b) *Conversely, assume that the given function $h(x)$ satisfies (2.5). Then there exists $\alpha > 0$ depending only on N such that for each choice of values $\{\alpha_i\}_{i=1}^N$, $\alpha_i \in \mathbb{R}$, there is a unique solution $u \in C^\alpha(\overline{P})$ to the equation (2.2), satisfying the following boundary condition*

$$u - \sum_{i=1}^N l_i(x) \log l_i(x) \in C^\infty(\overline{P} \setminus \{v_1, \cdots, v_N\}), \quad \text{and} \quad u(v_i) = \alpha_i, \quad 1 \leq i \leq N. \tag{2.6}$$

At this moment, the regularity of the solution at the corners is still open. We also remark that we show the existence of a unique solution for every choice of values at the vertices. The correct notion of uniqueness for solutions of Abreu's equation is the more restrictive condition that the solution be unique up to the addition of an affine linear function. We conjecture that a solution to Abreu's equation satisfying the Guillemin boundary conditions must also satisfy a kind of *balancing condition* on its values at the vertices of the polytope.

We discuss briefly some of the main steps in the proof of Theorem 2.1.1. A key observation is that, if a solution u exists, then its restriction to the edge e_i is the solution of the following second-order ODE along the edge e_i,

$$\partial^2_{T_i} u = |n_i|^2/\varphi_{n_i} \tag{2.7}$$

Combined with the assigned values of u at the vertices v_i, this equation determines completely the restriction of u to the boundary ∂P of the polytope. Thus, we can obtain u by solving the Monge-Ampère equation (2.2) with this given Dirichlet condition. Because the right hand side of the equation (2.2) blows up near the boundary, and because the domain is a polytope, the solution does not appear to have been written down previously in the literature. However, we show in section §3 that the methods of Cheng-Yau [12] can be suitably extended to produce a generalized solution.

The remaining issue is the regularity. The C^α regularity on $\overline{P}$ is established by constructing suitable barrier functions. The regularity and asymptotic expansion at the edges are modeled on the following problem

$$\det D^2 u(x', x_n) = x_n^{-1} \tag{2.8}$$

near the interior of a face $\{x_n = 0\}$, and

$$\det D^2 u = (x_1 ... x_n)^{-1} \tag{2.9}$$

near a corner. The equation (2.8) is a limit case of the equations studied by Daskalopoulos and Savin in [13] (and in [33] in higher dimensions) of the form

$$\det D^2 v(x,y) = y^\alpha \text{ in } B_1,\ \alpha > -1 \tag{2.10}$$

for which they obtained the behavior of the solution

$$v(x,y) = \frac{1}{2a}x^2 + \frac{a}{(\alpha+2)(\alpha+1)}|y|^{2+\alpha} + O\left((x^2 + |y|^{2+\alpha})^{1+\delta}\right) \tag{2.11}$$

for some $a > 0$, in a neighborhood of the origin. The case of exponent -1 presents a new difficulty from the fact that solutions with quadratic growth on the flat boundary have infinite normal derivative. In our case, we need to combine the techniques of [13] with a careful analysis of the partial Legendre transform of u and of the Monge-Ampère equation.

The chapter is organized as follows: In section 2, we explain the setup and derive the necessary conditions on the right-hand side, as well as the boundary equation, which we solve to give Dirichlet data compatible with the Guillemin boundary conditions. In section 3, we give a Perrón's method argument to solve the Dirichlet problem, ensuring that there exists a solution in the polytope which is Hölder continuous up to the boundary. In section 4, the main part of the paper, we deal with the behavior of the solution near an edge, establishing that under the precise boundary relation, the solution goes like $l_i \log l_i + f$, with f smooth. This completes the proof of Theorem 2.1.1. For the most part we work exclusively in dimension two. This restriction is mainly for simplicity of computation in sections two and three, where the results have clear extensions to higher dimensions, but is essential in section four where we take the partial Legendre transform of the Monge-Ampère equation to yield a quasilinear equation.

2.2 Consequences of the Guillemin boundary conditions

In general, an asymptotic expansion for the solution u near the boundary of a domain will put some constraints on the boundary behavior of $\det D^2u$. In the case of Guillemin boundary conditions on a polytope, these constraints turn out to be quite powerful. This is the contents of Theorem 2.1.1, part (a), which we reformulate as the following separate proposition for convenience:

Lemma 2.2.1. *Let $u \in C^0(\overline{P}) \cap C^\infty(\overline{P} \setminus \{v_1, \cdots, v_N\})$ be a function which satisfies the Guillemin boundary condition (2.3) on $\overline{P} \setminus \{v_1, \cdots, v_N\}$ in the sense that*

$$u(x) - \sum_{i=1}^{N} l_i(x) \log l_i(x) \in C^0(\overline{P}) \cap C^\infty(\overline{P} \setminus \{v_1, \cdots, v_N\}). \tag{2.12}$$

Then

$$\det D^2u = \frac{1}{h(x) \prod_{i=1}^{N} l_i(x)}, \tag{2.13}$$

where $h(x)$ is a function which is in $C^0(\overline{P}) \cap C^\infty(\overline{P} \setminus \{v_1, \cdots, v_N\})$ and satisfies $0 < h(x)$. When the full Guillemin boundary condition (2.3) holds, then $h \in C^\infty(\overline{P})$. Furthermore,

$$h(v_k) = \frac{1}{\det(n_{k-1} n_k)^2 \prod_{j \neq k-1,k} l_j(v_k)}. \tag{2.14}$$

Remark: In the case when the polygon is Delzant, the integral inner normal vectors of two adjacent edges form a basis of $\mathbb{Z}^2$, so $\det(n_{k-1} n_k)^2 = 1$.

Proof. This result and its extension to higher dimension is due to Donaldson in [18]. We perform the calculation globally in two dimensions to obtain the right constant; however, the main point is that the values of h at the vertices do not depend on the potential u.

Writing $u = \sum_{i=1}^N l_i(x) \log l_i(x) + f$, we have

$$D^2 u = \begin{pmatrix} f_{xx} + \sum \frac{(n_i^x)^2}{l_i} & f_{xy} + \sum \frac{n_i^x n_i^y}{l_i} \\ f_{xy} + \sum \frac{n_i^x n_i^y}{l_i} & f_{yy} + \sum \frac{(n_i^y)^2}{l_i} \end{pmatrix}, \tag{2.15}$$

so

$$\begin{aligned} \det D^2 u =& \sum_{i,j} \frac{(n_i^x)^2 (n_j^y)^2}{l_i l_j} - \sum_{i,j} \frac{n_i^x n_i^y n_j^x n_j^y}{l_i l_j} \\ &+ \sum_i \frac{f_{xx}(n_i^y)^2 + f_{yy}(n_i^x)^2 - 2 f_{xy} n_i^x n_i^y}{l_i} + \det D^2 f \\ =& \frac{1}{\prod_k l_k} \Bigg[\sum_{i \neq j} \left((n_i^x)^2 (n_j^y)^2 - n_i^x n_i^y n_j^x n_j^y \right) \prod_{q \neq i,j} l_q \\ &+ \sum_i (f_{xx}(n_i^y)^2 + f_{yy}(n_i^x)^2 - 2 f_{xy} n_i^x n_i^y) \prod_{j \neq i} l_j + \det D^2 f \prod_k l_k \Bigg]. \end{aligned}$$

The term in the brackets is the function $1/h$. When evaluating h at the vertex v_k, both l_{k-1} and l_k are zero, so only the terms from the first sum with $i = k-1$, $j = k$, and $i = k$, $j = k-1$ are nonzero, and therefore

$$\begin{aligned}\frac{1}{h(v_k)} &= \left(\left((n_{k-1}^x)^2(n_k^y)^2 - n_{k-1}^x n_{k-1}^y n_k^x n_k^y\right) + \left((n_k^x)^2(n_{k-1}^y)^2 - n_k^x n_k^y n_{k-1}^x n_{k-1}^y\right)\right) \prod_{q\neq k-1,k} l_q(v_k)\\ &= \left(n_{k-1}^x n_k^y - n_k^x n_{k-1}^y\right)^2 \prod_{q\neq k-1,k} l_q(v_k)\\ &= \det(n_{k-1}n_k)^2 \prod_{q\neq k-1,k} l_q(v_k).\end{aligned}$$

□

Now we determine the restrictions on the Dirichlet boundary data.

Lemma 2.2.2. *Let u be a function which satisfies the Guillemin boundary condition (2.3) near the boundary of the polytope P. Set $\det D^2 u = 1/\varphi$. Then*

$$U^{n_i n_i}\varphi_{n_i}|_{l_i=0} = |n_i|^2 \tag{2.16}$$

where the limit is taken as x approaches any point on the edge away from the vertices.

Proof.

$$\begin{aligned}U^{n_k n_k} &= \begin{pmatrix} n_k^x & n_k^y \end{pmatrix} \begin{pmatrix} f_{yy} + \sum \frac{(n_i^y)^2}{l_i} & -f_{xy} - \sum \frac{n_i^x n_i^y}{l_i} \\ -f_{xy} - \sum \frac{n_i^x n_i^y}{l_i} & f_{xx} + \sum \frac{(n_i^x)^2}{l_i} \end{pmatrix} \begin{pmatrix} n_k^x \\ n_k^y \end{pmatrix}\\ &= (f_{xx}(n_k^y)^2 + f_{yy}(n_k^x)^2 - 2f_{xy}n_k^x n_k^y)\\ &\quad + \left((n_k^x)^2 \sum_i \frac{(n_i^y)^2}{l_i} + (n_k^y)^2 \sum_i \frac{(n_i^x)^2}{l_i} - 2n_k^x n_k^y \sum_i \frac{n_i^x n_i^y}{l_i}\right)\\ &= (f_{xx}(n_k^y)^2 + f_{yy}(n_k^x)^2 - 2f_{xy}n_k^x n_k^y)\\ &\quad + \left((n_k^x)^2 \sum_{i\neq k} \frac{(n_i^y)^2}{l_i} + (n_k^y)^2 \sum_{i\neq k} \frac{(n_i^x)^2}{l_i} - 2n_k^x n_k^y \sum_{i\neq k} \frac{n_i^x n_i^y}{l_i}\right)\end{aligned}$$

since the terms with $i = k$ cancel in the sum in parentheses. Also we have

$$\varphi_{n_k} = h\sum_j (n_k \cdot n_j) \prod_{i\neq j} l_i + D_{n_k} h \prod_j l_j, \tag{2.17}$$

so

$$
\begin{aligned}
U^{n_k n_k}\varphi_{n_k}|_{l_k=0} &= h|n_k|^2 \prod_{i\neq k} l_i \Bigg[(f_{xx}(n_k^y)^2 + f_{yy}(n_k^x)^2 - 2f_{xy}n_k^x n_k^y) \\
&\quad + \left((n_k^x)^2 \sum_{i\neq k} \frac{(n_i^y)^2}{l_i} + (n_k^y)^2 \sum_{i\neq k} \frac{(n_i^x)^2}{l_i} - 2n_k^x n_k^y \sum_{i\neq k} \frac{n_i^x n_i^y}{l_i} \right) \Bigg] \\
&= h|n_k|^2 h^{-1}|_{l_k=0} \\
&= |n_k|^2.
\end{aligned}
$$

□

The boundary equation (2.16) was exploited in [24] in connection with the variational approach to Abreu's equation. In that context, this relation followed from the Euler-Lagrange equation satisfied by a minimizer, but in our context it follows directly from the Guillemin boundary conditions by computation as above.

A very important consequence of the previous lemma is that, up to the values of the solution $u(x)$ at the vertices $v_1, \cdots, v_N$, the Guillemin boundary conditions determine the boundary values of u. Indeed, in dimension two, the cofactor matrix entry U^{nn} is equal to a constant multiple of the second tangential derivative along the edge. We may then interpret this lemma as giving a second-order ODE on each edge for u. We parametrize the i-th edge $e_i = \{l_i(x) = 0\}$, by

$$x = v_i + tT_i.$$

Lemma 2.2.3. *Let $u \in C^2([0,L])$ solve $u_{tt} = \frac{h(t)}{t(L-t)}$ where $h(t)$ is smooth and positive on $[0,L]$. Then*

$$u(t) = h(0)t\log t + h(L)(L-t)\log(L-t) + v(t), \tag{2.18}$$

where v is smooth on $[0,L]$. The function $u(t)$ is determined uniquely by its boundary values $u(0)$ and $u(L)$.

Proof. By Taylor expanding h at 0, we see that $h(0)t\log t$ accounts for the singularity there, and similarly at the other endpoint. What remains on the right-hand side is smooth, and can be integrated twice to obtain v. This proves the desired identity. The second statement is easy, since two solutions of this second order ODE must differ by an affine function of t. □

The following statement now follows readily from the previous two lemmas:

Lemma 2.2.4. *Let $u \in C^0(\overline{P})$ be a strictly convex function on P satisfying the equation (2.2) and the Guillemin boundary condition (2.3). Let $\alpha_i = u(v_i)$ be the values of u at the vertices v_i, and define the function $\hat{u} \in C^0(\partial P)$ as the unique solution on each edge e_i of the equation*

$$\partial_{T_i}^2 \hat{u} = |n_i|^2 \varphi_{n_i}^{-1}, \quad \hat{u}(v_i) = \alpha_i, \quad \hat{u}(v_{i+1}) = \alpha_{i+1}. \tag{2.19}$$

Then the function u is a solution of the Dirichlet problem,

$$\det D^2 u = \frac{1}{\varphi(x)} \text{ on } P, \qquad u_{|\partial P} = \hat{u}. \tag{2.20}$$

2.3 Solution of Dirichlet Problem

We turn now to the proof of Theorem 2.1.1, part (b). In view of Lemma 2.2.4, we shall define the desired solution u as the solution of the Dirichlet problem (2.20), where the Dirichlet data $\hat{u}$ is specified by the values α_i and the function φ.

As a first step, we will first show the existence and uniqueness of generalized solutions to equations of this type, following closely the Perron's method approach of Cheng-Yau. The only new difficulty is that our domain is a polygon, hence not strictly convex. This has consequences for the allowable boundary data and the regularity at the boundary.

Recall the definition of an Alexandroff solution: Let u be a convex function on a domain $\Omega \in \mathbb{R}^n$. For each point $x \in \Omega$, let $B(x) = \{(p_1, ..., p_n) \in \mathbb{R}^n\}$ be the set of hyperplanes $x_{n+1} = \sum p_i x_i + b$ passing through $(x, u(x))$ and lying below the graph of u. To the function u we associate the measure $\mu(u)$, where for any Borel subset E of Ω, $\mu(u)(E) = |B(E)|$. Additionally we define for non-negative $\varphi \in C(\Omega)$ the measure of u with weight φ to be

$$\mu_\varphi(u, E) = \int_E \varphi(x) d\mu(u, x).$$

If $\mu_\varphi(u) = \mu$ where u is a convex function on Ω and μ is a Borel measure, then u is a generalized solution of $\det D^2 u = (1/\varphi)\mu$. In our equation, we take the Borel measure μ to be the ordinary Lebesgue measure. We make repeated use of the following three lemmas, which are now standard:

Lemma 2.3.1. *Let u_i be a sequence of convex functions defined on Ω which converges uniformly on compact sets to a convex function u. Then $\mu(u_i)$ converges to $\mu(u)$ weakly.*

Lemma 2.3.2. *Let u_1 and u_2 be two convex functions defined on a domain Ω with $u_1 = u_2$ on $\partial\Omega$ and $u_1 \geq u_2$ on Ω. Then $\mu(u_2, \Omega) \geq \mu(u_1, \Omega)$.*

Lemma 2.3.3. *Suppose $\varphi(x)$ is a positive and bounded continuous function on Ω. Let u_1, u_2 be convex functions defined on Ω so that*

$$\int_E \phi(x) d\mu(u_1, x) \geq \int_E \phi(x) d\mu(u_2, x) \tag{2.21}$$

for all Borel subsets $E \subset \Omega$. Suppose that $u_1(x) - u_2(x)$ becomes uniformly non-positive as x approaches $\partial\Omega$. Then $u_1 \leq u_2$ on Ω.

First we use a basic proposition taken directly from [12], whose proof we include for convenience.

Proposition 2.3.4. *Let Ω be a polytope in $\mathbb{R}^n$ with vertices $\{v_1, ..., v_N\}$. Suppose $\varphi \in C(\Omega)$, $\varphi \geq 0$, and for all compact sets K in Ω there is a constant $c > 0$ such that $\inf_{x \in K} \varphi(x) \geq c$. Let f be a function which is affine linear on each face of $\partial\Omega$, that is, $f : \partial\Omega \to \mathbb{R}$ such that*

$$f(\sum \lambda_i v_i) = \sum \lambda_i a_i$$

for any $\lambda_i \geq 0$ and $\sum \lambda_i = 1$, $a_1, ..., a_N \in \mathbb{R}$. Then for any Borel measure μ with compact support K contained in Ω and $\mu(\Omega) < \infty$, there exists a unique convex function u on $\bar{\Omega}$ such that $\mu_\varphi(u)$ realizes μ and $u = f$ on $\partial\Omega$.

Proof. First take μ to be a sum of point masses $\mu = \sum_{i=1}^m c_i \delta_{x_i}(x)$, with $c_i > 0$. Let $\mathcal{F}$ denote the family of piecewise linear convex functions w with $w = f$ on $\partial\Omega$ with $\mu_\varphi(w) \leq \mu$ (so the vertices of the polyhedron defined by the graph of w are a subset of the $\{x_i\}$). $\mathcal{F}$ is non-empty since the lower boundary of the convex hull of the data at the vertices is the graph of a piecewise linear function equal to f on the boundary and with mass equal to 0.

Set $\phi(w) = \sum_{i=1}^m w(x_i)$. Then ϕ is bounded below in terms of $\inf d(x_i, \partial\Omega)$, $\inf \varphi(x_i)$, and $\mu(\Omega)$ by the Alexandroff maximum principle. In the topology of uniform convergence, $\mathcal{F}$ is compact, and ϕ is continuous, so ϕ achieves its minimum at some $\bar{w} \in \mathcal{F}$.

Then $\mu_\varphi(\bar{w}) = \mu$: If not, suppose the mass of $\mu_\varphi(\bar{w})$ is strictly less than c_1 at x_1. Then there exists $\varepsilon > 0$ such that the piecewise linear function $\hat{w}$ obtained from $\bar{w}$ by lowering its value at x_1 by

ε, that is, the function whose graph is the convex hull of $(x_1, \bar{w}(x_1) - \varepsilon), \{(x_i, \bar{w}(x_i))\}, \{(v_j, f(v_j))\}$, also has mass less than μ. But $\phi(\hat{w}) < \phi(\bar{w})$, so we get a contradiction.

For a general Borel measure μ with compact support K, we let μ_i be a sequence of sums of point masses converging weakly to μ, and u_i the sequence of piecewise linear functions constructed as above with $\mu_\varphi(u_i) = \mu_i$. The functions u_i are uniformly bounded below as before in terms of $d(K, \partial\Omega)$, $\inf_K \varphi$, $\mu(\Omega)$, so they converge uniformly on compact subsets to u with $\mu_\varphi(u) = \mu$. Since also the u_i have bounded Lipschitz norm in terms of the boundary data and $d(K, \partial\Omega)$, $\inf_K \varphi$, $\mu(\Omega)$, u has bounded Lipschitz norm and $u = f$ on $\partial\Omega$. □

Now we want to solve with more general boundary data. Since we remain in the setting of polygons, which are not strictly convex, we must insist that the boundary data is convex on each face.

Proposition 2.3.5. *Let Ω be a polygon in $\mathbb{R}^2$. Let $f : \partial\Omega \to \mathbb{R}$ be convex on each edge and continuous, and φ, μ as in Proposition 2.3.4. Then there is a unique continuous convex function u on $\bar{\Omega}$ such that $\mu_\varphi(u)$ realizes μ and $u = f$ on $\partial\Omega$.*

Proof. We approximate the solution of this problem with the solutions of Proposition 2.3.4 by taking a sequence of sets of vertices where $A_1 = \{v_1, ..., v_N\}$, the set of vertices of Ω, and A_n containing all the points in the nth dyadic subdivision of each of the edges. The same proof shows that for each set of vertices A_n, the Dirichlet problem can be solved in Ω for a continuous convex function u_n with boundary data equal to $f(x_i)$ at each point x_i of A_n and linear on the edges in between, since we can still form the non-empty family of piecewise linear convex functions in $\overline{\Omega}$ matching the boundary data with mass less than a sum of point masses.

The u_n are uniformly bounded below and decreasing, and thus converge to a continuous solution u with $u = f$ on $\partial\Omega$. Note that the convergence is not necessarily Lipschitz in the corners since the boundary data need not be Lipschitz there. □

Now we must allow the measure μ to have support up to the boundary.

Theorem **2.3.6.** *Let $P = \cap_{i=1}^N \{l_i > 0\}$ be a polygon in $\mathbb{R}^2$. Let $f : \partial P \to \mathbb{R}$ be continuous and convex, with second tangential derivatives $f_{tt} < C/d$, where d is the distance to the nearest vertex. Let φ be a smooth function such that there exist positive constants a, A where $a \prod l_i \le \varphi \le A \prod l_i$,* *and $\mu = F d\lambda$, a bounded continuous function times Lebesgue measure. Then there exists a unique continuous convex function u on $\overline{P}$ such that $\mu_\varphi(u)$ realizes μ. Moreover, for all $0 < \alpha < 1/N$, $u \in C^\alpha(\overline{P})$, with norm bounded in terms of C, $\mu(P)$, a, and A.*

Proof. Let $\{h_i\}$ be an increasing sequence of cutoff functions with compact support in P such that $0 \leq h_i \leq 1$ and for any compact subset K of P, for i sufficiently large, h_i is identically equal to 1 on K. From the previous proposition, we have a sequence of functions u_i such that $u_i = f$ on ∂P and $\mu_\varphi(u_i)$ realizes $h_i\mu$. By Lemma 2.3.3, the u_i are a decreasing sequence of functions. To establish the existence of a limit with the stated boundary regularity, we must find a lower barrier. This step is more difficult for a polygon than in the case of a uniformly convex domain because of the lack of a C^2 convex defining function.

Lemma 2.3.7. *For $0 < \alpha < 1/N$, the function $\phi(x) = \left(\prod_{1\leq i\leq N} l_i(x)\right)^\alpha$ is strictly concave in P.*

Assuming the lemma, we let $\tilde{f}$ be any smooth extension of the boundary values to the interior bounded by the values of f and with Hessian bounded by $C/\prod l_i$ and let $v(x) = \tilde{f} + A(-\phi(x))$. Then $v = f$ on ∂P with $\det D^2 v \sim l_i^{2\alpha-2}$ near $l_i = 0$ and $\det D^2 v \sim (l_i l_j)^{2\alpha-2}$ near the corner $l_i = l_j = 0$, so for A sufficiently large, $\det D^2 v > 1/\phi$, and so $v \leq u_i$ for each i. Therefore u_i converges uniformly on compact subsets to a continuous convex function u on $\overline{P}$ such that $u = f$ on ∂P and $\mu_\varphi(u)$ realizes μ. Since v is Hölder continuous at the boundary and is a lower barrier for u, we obtain that $u \in C^\alpha(\bar{P})$. □

Proof of Lemma 2.3.7. Note that for a single corner, one can easily see by direct calculation of the Hessian that the function $((y + \lambda x)y)^\alpha$ is concave for $0 < \alpha \leq 1/2$ and strictly concave for $0 < \alpha < 1/2$ in the region $\{y + \lambda x > 0\} \cap \{y > 0\}$. For the barrier in the whole polygon, we show that the function $\phi(x)$ is strictly concave on any line segment contained in P. When restricted to a line parametrized by t, we have

$$\phi(x) = \left(\prod_{1\leq i\leq N} (a_i + b_i t)\right)^\alpha,$$

and therefore

$$\begin{aligned}
\phi_{tt} &= \alpha\phi\left(\alpha\left(\sum_{i=1}^{N}\frac{b_i}{a_i+b_it}\right)^2-\sum_{i=1}^{N}\left(\frac{b_i}{a_i+b_it}\right)^2\right)\\
&= \alpha\phi\left(\alpha\left(\sum_{i,j=1}^{N}\left(\frac{b_i}{a_i+b_it}\frac{b_j}{a_j+b_jt}\right)\right)-\sum_{i=1}^{N}\left(\frac{b_i}{a_i+b_it}\right)^2\right)\\
&\le \alpha\phi\left(\frac{\alpha}{2}\sum_{i,j=1}^{N}\left(\left(\frac{b_i}{a_i+b_it}\right)^2+\left(\frac{b_j}{a_j+b_jt}\right)^2\right)-\sum_{i=1}^{N}\left(\frac{b_i}{a_i+b_it}\right)^2\right)\\
&= \alpha\phi\,(N\alpha-1)\sum_{i=1}^{N}\left(\frac{b_i}{a_i+b_it}\right)^2,
\end{aligned}$$

which is negative if $0<\alpha<1/N$.

□

Hence for every choice of values at the vertices $\{\alpha_k\}$, there exists a unique continuous convex solution u to the Dirichlet problem (2.20), which is Hölder continuous of exponent α for any $\alpha<1/N$ at the boundary. Restricting this solution to any uniformly convex subdomain that does not touch the boundary, we have a solution of a Monge-Ampère equation with uniformly bounded right-hand side, so by the results of Cafarelli [5], the solution is in fact smooth in the interior.

2.4 Behavior Near the Interior of an Edge

Now we investigate the behavior of the solution u at the boundary near an edge and away from the vertices. We take our edge to be a segment of $\{y=0\}$ containing an interval around $(0,0)$. Our goal is to show that in a small half-disc $B_r^+(0)$, $u=y\log y+f$, where $f\in C^\infty(\overline{B_r^+(0)})$. The main technique is the partial Legendre transform as in [13], which is useful in dimension two, where the transformed function satisfies a quasilinear equation. (Another way to understand why the dimension two case is simpler, without reference to the partial Legendre transform, is as follows: In dimension two, the second tangential derivative u_{xx} is a solution to linear equation with 0 right-hand side, so it is possible to obtain a positive lower bound for u_{xx}.)

As a model, consider the degenerate Monge-Ampère equation

$$\det D^2u=\frac{1}{y}\ \text{in}\ \mathbb{R}^2_{y>0},\quad u|_{y=0}=\frac{1}{2}x^2. \tag{2.22}$$

We perform a partial Legendre transform in the x-variable as follows. Let

$$(p, q) = (u_x, y) \tag{2.23}$$

and define

$$u^*(p, q) = xu_x(x, y) - u(x, y). \tag{2.24}$$

This change of variables yields

$$\frac{\partial(p,q)}{\partial(x,y)} = \begin{pmatrix} u_{xx} & 0 \\ u_{xy} & 1 \end{pmatrix}, \qquad \frac{\partial(x,y)}{\partial(p,q)} = \begin{pmatrix} 1/u_{xx} & 0 \\ -u_{xy}/u_{xx} & 1 \end{pmatrix}, \tag{2.25}$$

so we compute $u^*_p = x$, $u^*_{pp} = 1/u_{xx}$, $u^*_q = -u_y$, $u^*_{qq} = -\det D^2u/u_{xx}$. Identifying q with y, the transformed function satisfies the equation

$$(\det D^2u)u^*_{pp} + u^*_{yy} = 0, \tag{2.26}$$

which reduces in the model case to the linear degenerate equation

$$\frac{1}{y}u^*_{pp} + u^*_{yy} = 0 \tag{2.27}$$

with boundary data

$$u^*(p, 0) = \left(\frac{1}{2}x^2\right)^* = \frac{1}{2}p^2$$

along the flat boundary. Additionally, note that the partial Legendre transform of a solution of the Monge-Ampère equation is necessarily convex in the tangential, or p-direction, and concave in the y-direction. A model solution of this equation is $u^*(p, y) = \frac{1}{2}p^2 - y\log y$.

Now let us consider the problem

$$\begin{cases} Lu = \frac{1}{y}u_{pp} + u_{yy} = 0 & \text{in } B_1^+(0) \\ u = g & \text{on } \partial B_1^+(0) \end{cases} \tag{2.28}$$

in a half-ball $B_1^+(0) = B_1(0) \cap \{y > 0\}$ with arbitrary boundary data. We begin by establishing the existence of solutions to this equation by approximating by solutions to uniformly elliptic equations. We use a Bernstein technique to control the derivatives; the effect of the degeneracy is that we can only control the derivatives in the direction parallel to the edge.

Proposition 2.4.1. *For any* $g \in C^0(\partial B_1^+(0)) \cap C^4(\{y=0\} \cap \overline{B_1^+})$*, there is a unique strong solution* u *of (2.28) in the sense that* $u \in C^2(B_1^+(0)) \cap C^\alpha(\overline{B_1^+(0)})$ *for all* $\alpha < 1$ *and* $u = g$ *on* $\partial B_1^+(0)$*. Furthermore,*

$$\max_{\overline{B_1^+}} u \le \max_{\partial B_1^+} g, \qquad \min_{\overline{B_1^+}} u \ge \min_{\partial B_1^+} g, \tag{2.29}$$

and

$$\max_{\overline{B_{1/2}^+}} |u_p| \le C(\|g\|_{C^2}), \qquad \max_{\overline{B_{1/2}^+}} |u_{pp}| \le C(\|g\|_{C^4}). \tag{2.30}$$

Proof. For $\epsilon > 0$ sufficiently small, let $\eta_\epsilon(y) \in C^\infty(-\infty,\infty)$ such that

$$\eta_\epsilon(y) = 1/y \text{ for } y > 2\epsilon; \quad \eta_\epsilon(y) = 1/\epsilon \text{ for } y \le \epsilon.$$

By the standard theory of uniformly elliptic equations, the equation

$$L_\epsilon u := \eta_\epsilon u_{pp} + u_{yy} = 0 \text{ in } B_1^+, u = g \text{ on } \partial B_1^+, \tag{2.31}$$

has a unique solution $u^\epsilon \in C^2(B_1^+) \cap C^\alpha(\overline{B_1^+})$. By the maximum principle, u^ϵ satisfies (2.29). We will find uniform estimates for u^ϵ and take $\epsilon \to 0$ to obtain the desired solution.

In this setting, it is important to establish that the solution is continuous up to the boundary. While it is clear that any limit of the u^ε will satisfy (2.29), without any better control than the L^∞ norm there is nothing to prevent the graph of the limit from becoming vertical on $\{y=0\}$, which is to say that possibly $\lim_{y\to 0} u(p,y) \ne g(p)$. To see that we will have continuity to the prescribed boundary values we construct barriers as follows. For each point $(p_0, 0)$ with $-1 < p_0 < 1$, let $P_{p_0}^+(p)$ and $P_{p_0}^-(p)$ be the tangent parabolas to g at p_0 opening up and down, respectively. Set

$$\begin{aligned} v_{p_0,\delta}^+(p,y) &:= P_{p_0}^+(p) - By\log y + Cy + \delta, \\ v_{p_0,\delta}^-(p,y) &:= P_{p_0}^-(p) + B'y\log y - C'y - \delta, \end{aligned}$$

with B, B', C, C' positive constants to be chosen below, and $\delta > 0$ small. We compute

$$L_\epsilon v_{p_0,\delta}^+(p,y) = \eta_\epsilon A - B/y < 0$$

for $B > A$, where $A > 0$, the quadratic coefficient in $P^+_{p_0}$ such that the parabola lies above g on $\{y = 0\}$, depends on $\|g\|_{C^2}$, and C is chosen large enough, depending on $\|g\|_{C^0}$, so that $v^+_{p_0,\delta}$ lies above g on the half-circle. The function $v^+_{p_0,\delta}$ is thus a supersolution that lies above the solution u^ϵ for each p_0 and each δ. Similar considerations for $v^-_{p_0,\delta}$ give that

$$v^-_{p_0,\delta}(p, y) \leq u^\epsilon(p, y) \leq v^+_{p_0,\delta}(p, y)$$

for all $\delta > 0$, therefore

$$g(p_0, 0) + B'y \log y - C'y \leq u^\epsilon(p_0, y) \leq g(p_0, 0) - By \log y + Cy,$$

and

$$|u^\epsilon(p_0, y) - g(p_0)| \leq |Dy \log y|$$

for D independent of ϵ.

We may now take ϵ to 0 to obtain a solution u of (2.28) in $C^2(B^+_1(0)) \cap C^\alpha(\overline{B^+_1(0)})$, which is unique since u satisfies (2.29) since solutions of our equation can have no interior maxima or minima.

Now we use the same argument as in [13] to obtain a bound on u_p. We show

$$L(Cu^2 + \varphi^2 u_p^2) \geq 0 \tag{2.32}$$

for a solution u and a cutoff function φ where $\varphi = 1$ in $B^+_{1/2}$, $\varphi = 0$ in $B^+_1 \setminus B^+_{3/4}$, and $\varphi_y = 0$ for all $y \leq 1/4$. We compute

$$L(u^2) = 2(u_p^2/y + u_y^2)$$

and

$$L(Cu^2 + \varphi^2 u_p^2) = 2C(u_p^2/y + u_y^2) + 2\varphi^2(u_{pp}^2/y + u_{py}^2) + L(\varphi^2)u_p^2 \\ + 8(\varphi_p u_y)(\varphi u_{pp})/y + 8(\varphi_y u_p)(\varphi u_{py}). \tag{2.33}$$

Since also we may assume

$$L(\varphi^2) \geq -C_1/y, \quad |\varphi_y u_p| \leq C_1 |u_p|/y^{1/2},$$

so

$$8(\varphi_y u_p)(\varphi u_{py}) \geq -C_1^2 u_p^2/y - \varphi^2 u_{py}^2,$$

and similarly for the other mixed term, we see that the right-hand side of (2.33) can be made non-negative in B_1^+ for sufficiently large C. Hence

$$\sup_{B_{1/2}^+} |u_p| \leq C^{1/2} \sup_{B_1^+} |u| + \sup_{\{y=0\}} |g_p|^2.$$

Further, since $Lu_p = 0$, the same argument implies that $|u_{pp}| \leq C$ in $B_{1/2}^+$, using the regularity of g on the flat boundary.

□

Remark: We shall have need for similar C^0 bounds for u (depending additionally on b, c, f) for the more general equation

$$u_{pp} + yu_{yy} + b(p,y)u_p + c(p,y)u = f(p,y) \tag{2.34}$$

where b, c, f are bounded. This follows from the same arguments as above. Note that the sign of c does not matter here, since barriers of the form $v = Cy \log y$ are bounded, go to 0 as y goes to 0, and have $yv_{yy} = C$.

We cannot perform a proper Taylor expansion at a point on the flat boundary since we expect that $|u_y^\epsilon(p,0)|$ will go to infinity as ϵ goes to zero. Nevertheless it is still true that

$$\begin{aligned}
u(p,y) &= u(p,0) + \int_0^y u_y(p,s)ds \\
&= u(p,0) + \int_0^y \left(C_\delta + \int_\delta^s u_{yy}(p,t)dt \right) ds \\
&= u(p,0) + \int_0^y \left(C_\delta + \int_\delta^s -\frac{u_{pp}(p,t)}{t} dt \right) ds,
\end{aligned}$$

and since u_{pp} solves the same equation as u, we have that $u_{pp}(p,t) = u_{pp}(p,0) + O(t \log t)$ (this requires $g \in C^4$ on the flat boundary). Hence

$$\begin{aligned}
u(p,y) &= u(p,0) + \int_0^y \left(C_\delta + \int_\delta^s -\frac{u_{pp}(p,0) + O(t\log t)}{t} dt \right) ds \\
&= u(p,0) + \int_0^y (-u_{pp}(p,0)(\log s - \log\delta + O(s\log s) + C)\, ds \\
&= u(p,0) + -u_{pp}(p,0) y \log y + u_{pp}(p,0)(1 - \log\delta) y + Cy + O(y^2 \log y) \\
&= u(p,0) - u_{pp}(p,0) y \log y + O(y).
\end{aligned}$$

Now for each n, $\partial_p^n u$ solves the same equation, so as long as the boundary function g possesses $n+2$ continuous derivatives along the flat boundary, the same estimates and the same type of expansion will hold for $\partial_p^n u$. In particular,

$$u_{pp} = u_{pp}(p,0) - u_{pppp}(p,0)y\log y + O(y), \tag{2.35}$$

and now we can use the equation $u_{yy} = -u_{pp}/y$ to expand to the next order in y:

$$u(p,y) = u(p,0) - u_{pp}(p,0)y\log y + u_1(p)y + \frac{1}{2}u_{pppp}(p,0)y^2\log y + O(y^2), \tag{2.36}$$

and so on.

Now we return to using u to denote the solution of the Monge-Ampère equation, and u^* for its partial Legendre transform. If u satisfies the Dirichlet problem for the MA-equation in the polytope, then its partial Legendre transform u^* satisfies

$$\frac{u^*_{pp}}{\varphi(x,y)} + u^*_{yy} = 0, \qquad u^*_{pp}(p,0) = \frac{1}{u_{xx}(x,0)} = h(x,0) \tag{2.37}$$

and $u^*_{pp} > 0$ (here we have absorbed the factor $\prod_{j\neq i} l_j(x,0)$ into h). Care is needed in understanding this equation: the function $\varphi(x,y)$ depends on p through the Legendre transform, in that at the point (p,y), $x = u^*_p(p,y)$, so the equation has a non-linear dependence on u^*_p. If we can show that the solution to this equation has an expansion in terms of y^n and $y^n \log y$ like the solution to the model equation, then we can use the boundary condition to determine the coefficient functions. We state the existence of such an expansion as a lemma:

Lemma 2.4.2. *If u^* solves (2.37), then in some small half-ball around 0, for each $k \in \mathbb{N}$, u^* has an expansion along the boundary as*

$$u^*(p,y) = u^*(p,0) + \sum_{i=1}^{k} \frac{1}{i!}\hat{u}^*_i(p)y^i \log y + \sum_{i=1}^{k-1} \frac{1}{i!}u^*_i(p)y^i + O(y^k) \tag{2.38}$$

where $\hat{u}^*_i$ and u^*_i are smooth functions.

Now we can prove the main theorem of this section, which completes the proof of the regularity stated in Theorem 2.1.1, part (b).

Theorem **2.4.3.** *Let u be the solution of the Dirichlet problem (2.20), with right-hand side given by (2.4) and boundary data given by (2.19), and suppose $l_1(x,y) = y$. Then for any point q on the* *edge* $\{y = 0\} \cap \partial P$*, there is a small half-ball around* q *such that* $u(x, y) = y \log y + f(x, y)$*, with* $f \in C^\infty(\overline{B}_r^+(q))$*.*

Proof. Since u satisfies $u_{xx} = 1/h(x, 0)$ on $\{y = 0\}$, its partial Legendre transform u^* satisfies the equation (2.37) with the boundary condition. Assuming Lemma 2.4.2, u^* has an expansion as in (2.38). Now we use (2.37) to write

$$\begin{aligned} u^*(p, y) &= -\int^y \int^{y'} \frac{u^*_{pp}(p, y'')}{\varphi(x, y'')} dy'' dy' \\ &= -\int^y \int^{y'} \frac{u^*_{pp}(p, y'')/h(x, y'')}{y''} dy'' dy' \\ &= -\int^y \int^{y'} \frac{(u_0^{*\prime\prime}(p) + \hat{u}_1^{*\prime\prime}(p) y'' \log y'' + ...)(1/h(x, 0) + a_1(x) y'' + ...)}{y''} dy'' dy', \end{aligned}$$

where we have also used a polynomial expansion for $1/h(x, y)$. We compute the $y^i \log y$ terms of the expansion explicitly: The $y \log y$ term can only come from two integrations of $1/y$, which only occurs in the very first term in the expansion, so

$$\hat{u}_1^*(p) = -u_0^{*\prime\prime}(p)/h(x, 0) \equiv -1 \tag{2.39}$$

by the boundary condition. Similarly we can compute the higher coefficients:

$$\hat{u}_2^*(p) = -\hat{u}_1^{*\prime\prime}(p)/h(x, 0) \equiv 0, \tag{2.40}$$

$$\hat{u}_3^*(p) = -\hat{u}_2^{*\prime\prime}(p)/h(x, 0) - \hat{u}_1^{*\prime\prime}(p) a_1(x) \equiv 0, \tag{2.41}$$

and in the same way, all the higher coefficients on $y^n \log y$ terms are identically 0. Thus the solution of (2.37) with the particular boundary data, is of the form

$$u^*(p, y) = u^*(p, 0) - y \log y + \sum_{i=1}^{N} \frac{1}{i!} u_i^*(p) y^i + o(y^N). \tag{2.42}$$

We obtain the theorem by taking the partial Legendre transform back to u. Note that this theorem did not use any of the prescribed data of the function h at the vertices. □

It remains only to show that the partial Legendre transform u^* possesses such an expansion, which will be established by a perturbation argument, contained in the following two propositions.

Assume for simplicity that $a(0, 0) = 1$. Let $Q_r = \{y \leq r - x^2\} \cap \{y \geq 0\}$.

Proposition 2.4.4. *Let u solve*

$$Lu = u_{pp} + ya(p,y)u_{yy} + bu_p + cu = f, \qquad u|_{\partial Q_1} = g, \tag{2.43}$$

where $a, b, c, f \in C(\overline{Q_1})$, $g \in C^4(\{y = 0\})$. Fix $0 < \alpha < 1$. Then there exists ε_0, r_0 small depending on α such that if

$$|1 - a| + |b| + |c| + |f| < \varepsilon_0 \text{ in } Q_1 \tag{2.44}$$

then for some $P(p,y) = P_0(p) + P_1(p)y \log y + P_2(p)y$ with $|P_0| + |P_1| + |P_2| \le C$ depending only on g, we have

$$|u - P| < r_0^{1+\alpha} \text{ in } Q_{r_0}. \tag{2.45}$$

Proof. Let w be the solution of the model equation

$$w_{pp} + yw_{yy} = 0, \qquad w|_{\partial Q_{3/4}} = u. \tag{2.46}$$

Then in $Q_{1/2}$,

$$|L(u - w)| = |y(1 - a)w_{yy} - bw_p - cw + f| \le C\varepsilon_0, \tag{2.47}$$

from which we see that $u - w$ solves the equation with small right-hand side and zero boundary data. Comparing this function to a barrier $v = C\varepsilon_0 y \log y$, we obtain

$$\max_{Q_{1/2}} |u - w| \le C'\varepsilon_0. \tag{2.48}$$

We set

$$P_1(p,y) := w_0(p) + w_1(p)y \log y + w_2(p)y, \tag{2.49}$$

the first terms in the expansion for w. We have $|w - P_1| = |w - (w(p,0) + w_1(p)y \log y + w_2(p)y))| \le Cy^{1+\alpha}$ for all $\alpha < 1$, and therefore

$$|u - P_1| \le C\varepsilon_0 + Cr_0^{1+\alpha} \le C'r_0^{1+\alpha'} \tag{2.50}$$

in Q_{r_0}, for $\varepsilon_0 \le cr_0^{1+\alpha}$. □

If we additionally assume a weighted Hölder condition, we can iterate this comparison to obtain

Proposition 2.4.5. *Let u be as in Proposition 2.4.4. Then for fixed $0 < \alpha < 1$, there exists $\varepsilon > 0$ such that if for all λ sufficiently small,*

$$\max\{\operatorname{osc}_{Q_\lambda} a, \operatorname{osc}_{Q_\lambda} b, \operatorname{osc}_{Q_\lambda} c, \operatorname{osc}_{Q_\lambda} f\} < \varepsilon\lambda^\alpha, \tag{2.51}$$

then $u(p,y) = u(p,0) + u_1(p)y \log y + u_2(p)y + o(y^{1+\alpha})$.

Proof. Fix $r = r_0$ as in Proposition 2.4.4, with $w_{pp} + yw_{yy} = 0$ on Q_1, equal to u on $\partial Q_{3/4}$, $P_1 = w_0(p) + w_1(p)y\log y + w_2(p)y$ the first terms in the expansion of w as before. Set

$$\tilde{u}(p,y) = \frac{(u-P_1)(r^{1/2}p, ry)}{r^{1+\alpha}}, \tag{2.52}$$

$$\tilde{L}v = v_{pp} + ya(r^{1/2}p, ry)v_{yy} + r^{1/2}bv_p + rcv, \tag{2.53}$$

$$\tilde{f}(p,y) = \tilde{L}\tilde{u}(p,y), \tag{2.54}$$

Then

$$\begin{aligned}
|\tilde{f}| =& \Bigg|\frac{f(r^{1/2}p,ry)}{r^\alpha} - \Big[r^{-\alpha}w_0''(r^{1/2}p) + w_1''(r^{1/2}p)r^{1-\alpha}y\log(ry) + w_2''(r^{1/2}p)r^{1-\alpha}y \\
&+ r^{-\alpha}ya(r^{1/2}p,ry)(\frac{w_1}{y}) \\
&+ r^{-1/2-\alpha}b(r^{1/2}p,ry)\Big(r^{1/2}w_0'(r^{1/2}p) + r^{3/2}w_1'(r^{1/2}p)y\log(ry) + r^{3/2}w_2'(r^{1/2}p)y\Big) \\
&+ r^{-\alpha}c(r^{1/2}p,ry)\left(w_0(r^{1/2}p) + rw_1(r^{1/2}p)y\log(ry) + rw_2(r^{1/2}p)y\right)\Big]\Bigg| \\
\le& \left|\frac{(1-a(r^{1/2}p,ry))w_1}{r^\alpha}\right| \\
&+ \left|\frac{f(r^{1/2}p,ry)}{r^\alpha}\right| + \left|\frac{w_0''(r^{1/2}p) + w_1(r^{1/2}p) + b(r^{1/2}p,ry)w_0'(r^{1/2}p) + c(r^{1/2}p,ry)w_0(r^{1/2}p)}{r^\alpha}\right| \\
&+ Cr^\beta \\
\le& C\varepsilon + C\varepsilon + \left|\frac{b(r^{1/2}p,ry)w_0'(r^{1/2}p) + c(r^{1/2}p,ry)w_0(r^{1/2}p)}{r^\alpha}\right| + Cr^\beta \\
\le& C'\varepsilon + C''\varepsilon + Cr^\beta \\
\le& C'''\varepsilon.
\end{aligned}$$

In the second line, we have combined all terms that go as a positive power of r in Cr^β, and in the third line we use that $w_0''(r^{1/2}p) + w_1(r^{1/2}p) = 0$.

Thus by taking $\varepsilon = \varepsilon_0/C$, where ε_0 is as in Proposition 2.4.4 and C is a constant depending only on the size of $|a|, |b|, |c|, |f|, |g|$, we can apply Proposition 2.4.4 again and compare $\tilde{u}$ to $\tilde{w}$, the solution of $w_{pp} + yw_{yy} = 0$ in Q_r matching $\tilde{u}$ on the boundary. Again, $\tilde{w} = \tilde{w}_0(p) + \tilde{w}_1(p)y\log y + \tilde{w}_2(p)y + o(y^{1+\alpha})$ (in fact $\tilde{w}_0 = \tilde{w}_1 = 0$ since $\tilde{w} = 0$ on the flat boundary). We can thus iterate the comparison, since $\tilde{f}$ also satisfies $\operatorname{osc}_{Q_r} \tilde{f} < Cr^\alpha$. For example,

$$\left| \frac{u(r^{1/2}p, ry) - P_1(r^{1/2}p, ry)}{r^{1+\alpha}} - (\tilde{w}_1(p) y \log y + \tilde{w}_2(p) y) \right|_{L^\infty(Q_r)} < Cr^{1+\alpha} \tag{2.55}$$

so

$$\left| u(p,y) - P_1(p,y) - \left(r^\alpha \tilde{w}_1(p/r^{1/2}) y \log(y/r) + r^\alpha \tilde{w}_2(p/r^{1/2}) \right) \right|_{L^\infty(Q_{r^2})} < Cr^{2(1+\alpha)} \tag{2.56}$$

We obtain for each k a function $P_k = a_k(p) + b_k(p) y \log y + c_k(p) y$ with coefficient functions bounded by the $Cr^{k\alpha}$, where C depends on the L^∞ norms of the original coefficients. Therefore the constant C_k in the size of the right-hand side $|\tilde{f}^{(k)}| \leq C_k \varepsilon$ remains bounded as k goes to infinity. The sum of the P_k is thus bounded by a convergent geometric series, and so

$$\left| u - \sum_{k=1}^{n} P_k \right|_{L^\infty(Q_{r^n})} \leq Cr^{n(1+\alpha)}, \tag{2.57}$$

and we get the conclusion of the lemma by taking a limit of the $\sum_{k=1}^n P_k$. □

Remark: Note that the smallness condition on the functions b, c, and f may be always be satisfied close to a point on the flat boundary by first performing a rescaling as above.

We now show that the hypotheses of Proposition 2.4.5 are satisfied by the partial Legendre transform equation (2.37) for u^*, as well as by the equations satisfied by its derivatives $\partial_p^k u^*$. First, we must verify that the coefficient function $\varphi(x,y) = ya(p,y)$ satisfies the weighted Hölder condition (2.51). If we define the function $\hat{a}(x,y)$ by $\varphi(x,y) = y\hat{a}(x,y)$, then as a function of (p,y), we have

$$a(p,y) = \hat{a}(u_p^*(p,y), y) \tag{2.58}$$

and

$$a(r^{1/2}p, ry) = \hat{a}(u_p^*(r^{1/2}p, ry), ry). \tag{2.59}$$

Since by assumption $\hat{a}$ is smooth, we must show that u_p^* satisfies a similar weighted Hölder condition.

Lemma 2.4.6. *Suppose u satisfies*

$$\det D^2 u = \frac{1}{y\hat{a}(x,y)}, \tag{2.60}$$

Then there exist $C, \alpha > 0$ such that its partial Legendre transform u^* in (p, y) satisfies

$$\left| u_p^*(r^{1/2}p, ry) - u_p^*(0,0) \right| \leq Cr^\alpha \tag{2.61}$$

for all $r > 0$ sufficiently small.

Proof. We may assume that $u_p^*(0,0) = u^*(0,0) = 0$. We will exploit the fact that since u^* arises as the partial Legendre transform of a solution of a Monge-Ampère equation, it satisfies the equation (2.37) and is strictly convex in the p-direction.

There are constants c, C such that

$$\frac{1}{2}p^2 - cy \log y < u^* < \frac{1}{2}p^2 - Cy \log y, \tag{2.62}$$

or

$$|u^*(p, y_0) - \frac{1}{2}p^2| < Cy_0 \log y_0. \tag{2.63}$$

in each slice of fixed y_0 small. We estimate the difference $|u_p^*(p, y_0) - p|$ for $|p| < 1/2$. By subtracting a linear function in p, it suffices to bound $|u_p^*(0, y_0)|$. Let $\varepsilon = Cy_0 \log y_0$. For the upper bound, we note that since u^* is convex in p, we have

$$\frac{1}{2}p^2 + \varepsilon \geq u^*(p, y_0) \geq u^*(0, y_0) + u_p^*(0, y_0)p \geq -\varepsilon + u_p^*(0, y_0)p. \tag{2.64}$$

The maximum of $u_p^*(0, y_0)$ is realized when the line $-\varepsilon + u_p^*(p, y_0)p$ is tangent to the graph of $\frac{1}{2}p^2 + \varepsilon$, which occurs at (p_ε, y_0) with slope p_ε, hence

$$\frac{1}{2}p_\varepsilon^2 + 2\varepsilon \geq p_\varepsilon^2, \tag{2.65}$$

so

$$u_p^*(0, y_0) \leq C\sqrt{\varepsilon}. \tag{2.66}$$

The lower bound is obtained in the same way looking in the negative p-direction. Hence

$$|u_p^*(p, y_0) - p| < C\sqrt{y_0 \log y_0}. \tag{2.67}$$

It follows that for any $\alpha < 1/2$, $|u_p^*(r^{1/2}p_0, ry_0)| < Cr^\alpha$. □

Proof of Lemma 2.4.2. It follows from Lemma 2.4.6 that the partial Legendre transform u^* satisfies the hypotheses of Proposition 2.4.5, and therefore

$$u^*(p,y) = u^*(p,0) + \hat{u}_1^*(p)y\log y + u_2^*(p)y + o(y^{1+\alpha}). \tag{2.68}$$

Now we differentiate and examine the equations satisfied by the derivatives $\partial_p^k u^*$:

$$(u_p^*)_{pp} + y\hat{a}(x,y)(u_p^*)_{yy} + \hat{a}_x(x,y)yu_{yy}^*(u_p^*)_p = 0, \tag{2.69}$$

and

$$(u_{pp}^*)_{pp}+y\hat{a}(x,y)(u_{pp}^*)_{yy}+\hat{a}_x(x,y)yu_{yy}^*(u_{pp}^*)_p+\left(\hat{a}_x(x,y)yu_{yyp}^* + \hat{a}_{xx}(x,y)yu_{yy}^*u_{pp}^*\right)(u_{pp}^*) = 0. \tag{2.70}$$

We can then obtain that u^* has an expansion to all orders as follows: Since yu_{yy}^* is bounded and $\hat{a}(x,y)$ is smooth, $\operatorname{osc}_{Q_\lambda} \hat{a}_x(x,y)yu_{yy}^* \leq c\lambda^\alpha$, so, possibly after a rescaling, the equation satisfied by u_p^*, (2.69), satisfies the hypotheses of Proposition 2.4.5, and so u_p^* also admits such an expansion to order $o(y^{1+\alpha})$. It follows that yu_{yyp}^* is also bounded, and so is u_{pp}^*, so we may also apply the Proposition 2.4.5 to the equation (2.70), and thus

$$u_{pp}^*(p,y) = u_{pp}^*(p,0) + \hat{u}_1^{*\prime\prime}(p)y\log y + u_1^{*\prime\prime}(p)y + o(y^{1+\alpha}). \tag{2.71}$$

Since $yau_{yy}^* = -u_{pp}^*$, we obtain the next two terms in the expansion for u^*:

$$u^*(p,y) = u^*(p,0) + \hat{u}_1^*(p)y\log y + u_1^*(p)y + \hat{u}_2^*(p)y^2\log y + u_2^*(p)y^2 + o(y^{2+\alpha}). \tag{2.72}$$

We can bootstrap to higher orders inductively: at each step, the equation satisfied by $\partial_p^k u^*$ satisfies an equation of the form (2.43) with coefficients in of the form $\partial_x^j \hat{a} * C(p,y)$, where $C(p,y)$ is bounded by the expansion for $\partial_p^{k-1}u^*$, so we can expand $\partial_p^k u^*$ to order $o(y^{1+\alpha})$. To obtain further terms in the expansion of u^*, one can now repeat essentially the argument of Propositions 2.4.4 and 2.4.5, comparing u^* with the solution of

$$L_k w = w_{pp} + yA_k w_{yy} = 0 \tag{2.73}$$

where A_k is the k-th order Taylor approximation to $\hat{a}(x,y)$. □

Part II

Asymptotics of energy functionals

In this part, we pursue the variational approach to the problem of canonical metrics in Kähler geometry. Mabuchi first realized that the constant scalar curvature equation $R(\omega) = \bar{R}$ for a metric in a fixed Kähler class could be written as the Euler-Lagrange equation of a functional, now called the Mabuchi K-energy. The K-energy may be defined in terms of its variation:

$$\delta K = -\frac{1}{V} \int_x \delta\phi (R - \bar{R}) \omega_\phi^n \tag{2.74}$$

where $\bar{R}$ is the average scalar curvature, a topological invariant. The Mabuchi K-energy may be written in an explicit form $K_{\omega_0}(\phi)$ by choosing a reference metric ω_0:

$$K_{\omega_0}(\phi) = \frac{1}{V} \left[\int_X \left(\log(\frac{\omega_\phi^n}{\omega_0^n}) \omega_\phi^n - \phi \sum_{j=0}^{n-1} Ric(\omega_0) \omega_\phi^j \omega_0^{n-1-j} + \frac{\bar{R}}{n+1} \phi \sum_{j=0}^{n} \omega_\phi^j \omega_0^{n-j} \right) \right]. \tag{2.75}$$

Donaldson showed that the K-energy is convex in an appropriate sense in the space of Kähler potentials [17], so that if the class $c_1(L)$ admits a constant scalar curvature metric ω_ϕ, then ϕ is a minimizer of the K-energy. A major problem is to determine when the K-energy is bounded below. We need only consider certain special one-parameter degenerations of the Kähler class along which the K-energy restricts to become a convex function. Along these paths, the K-energy an asymptotic slope that is related to algebraic stability invariants, and determines the properness of the energy. In particular, the existence of a degeneration along which the asymptotic slope is negative is an obstruction to the existence of a minimizer for the functional.

Definition: The pair (X, L) is *K-semistable* if for all test configurations, the Donaldson-Futaki invariant ≥ 0. (X, L) is *K-polystable* if for all normal test configurations (see [25]), the Donaldson-Futaki invariant is ≥ 0, with equality only if the test configuration is the product test configuration.

We do not define the Donaldson-Futaki invariant or the general notion of test configuration here, but mention only that conjecturally these conditions are equivalent to the corresponding statements about the asymptotic slopes of the K-energy along the Bergman geodesics we consider below. The precise relation between the asymptotic slope of the K-energy and the Donaldson-Futaki invariant in the Fano case is discussed in [3], in which he shows that the Donaldson-Futaki invariant is equal to the asymptotic slope plus a non-negative error term q, which is zero in the case of a normal test configuration. The Yau-Tian-Donaldson conjecture states that there exists a metric of constant scalar curvature in $c_1(L)$ if and only if (X, L) is K-polystable. We note also that asymptotics of energy functionals are also of considerable interest in the study of partition functions over Bergman metrics [23].

A related functional, which appears as a term in the K-energy, and is important in its own right, is the Aubin-Yau functional. The *Aubin-Yau functional* $F^0_{\omega_0}(\phi)$ is given by

$$F^0_{\omega_0}(\phi) = \frac{1}{n+1}\frac{1}{V}\int_X \phi \sum_{i=0}^{n} \omega_0^i \wedge \omega_\phi^{n-i}. \tag{2.76}$$

The significance of the Aubin-Yau functional in Kähler geometry is discussed extensively in [32]. Minimizers of $F^0_{\omega_0}(\phi)$ in the space of Bergman metrics are called *balanced metrics*. Zhang [39] proved that the existence of a balanced metric is equivalent to Chow-Mumford stability (see also [30]). Donaldson showed that the existence of a cscK metric implies the existence of a balanced metric, hence the Chow-Mumford stability and the existence of a minimizer for $F^0_{\omega_0}(\phi)$ in the space $\mathcal{K}_k$. In this case, the asymptotic slope of the Aubin-Yau functional is necessarily positive. The asymptotic slope is thus a numerical stability invariant called the *Chow weight*.

We develop a relatively simple and completely explicit framework for determining the asymptotics of the Aubin-Yau functional and the K-energy along Bergman geodesics, which are finite-dimensional approximations to the geodesics in the infinite-dimesnional space of all Kähler potentials, and are equivalent to test configurations. The Bergman geodesics are given by $\mathbb{C}^*$-actions on the Kodaira embedding generated by a bases of sections of L^k, and approximate the infinite-dimensional geodesics as $k \to \infty$ [32]. In this context, we may reduce the determination of the asymptotics of these functionals to the analysis of some algebraic singular integrals. We begin by computing the asymptotic slope of the Aubin-Yau functional. Later we will apply the same techniques to determine the asymptotic slope of the K-energy. From the formula for the asymptotic slope, it is possible to check whether the relevant stability condition is satisfied.

Chapter 3

Asymptotics of the Aubin-Yau functional

3.1 Introduction

Let (X, ω_0) be a Kähler manifold of complex dimension n with reference Kähler metric ω_0, $\omega_\phi = \omega_0 + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\phi$, $Ric(\omega_0) = -\frac{\sqrt{-1}}{2\pi}\log\omega_0^n$ the Ricci form of ω_0, and $V = \int_X \omega_0^n$. The functionals described below are defined on the space of Kähler potentials

$$\mathcal{K} = \{\phi \in C^\infty(X), \omega_0 + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\phi > 0\}. \tag{3.1}$$

Definition: The *Aubin-Yau functional* $F^0_{\omega_0}(\phi)$ is given by

$$F^0_{\omega_0}(\phi) = \frac{1}{n+1}\frac{1}{V}\int_X \phi \sum_{i=0}^{n} \omega_0^i \wedge \omega_\phi^{n-i}. \tag{3.2}$$

As mentioned in the introduction to this part of the thesis, the minimizer of $F^0_{\omega_0}(\phi)$ in the space of Bergman metrics is called a *balanced metric*, and the existence of a balanced metric has been shown to be equivalent to the GIT notion of Chow-Mumford stability.

Briefly, let us say how the Aubin-Yau functional is related to other functionals in the literature. It is related to the J-functional

$$J_{\omega_0}(\phi) = \frac{\sqrt{-1}}{2\pi V}\int_X \sum_{i=0}^{n-1} \frac{(i+1)}{(n+1)} \partial\phi \wedge \bar{\partial}\phi \wedge \omega_\phi^{n-i-1} \wedge \omega_0^i \tag{3.3}$$

by

$$F^0_{\omega_0}(\phi) = \frac{1}{V}\int_X \phi\omega_0^n - J_{\omega_0}(\phi). \tag{3.4}$$

In the special case $[\omega_0] = [K_X^{-1}]$, $F^0_{\omega_0}$ is related to the functional $F_{\omega_0}(\phi)$ by

$$F_{\omega_0}(\phi) = -F^0_{\omega_0}(\phi) - \log\left(\frac{1}{V}\int_X e^{h_{\omega_0}-\phi}\omega_0^n\right), \qquad Ric(\omega_0) - \omega_0 = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}h_{\omega_0}. \tag{3.5}$$

Minimizers of $F_{\omega_0}(\phi)$ are Kähler-Einstein metrics. Its asymptotics are discussed to establish the necessity of K-stability for existence of a KE metric and the issue of uniqueness of KE metrics in [4; 3]. Note the surprising sign on the $F^0_{\omega_0}$ term, given that both $F^0_{\omega_0}$ and F_{ω_0} are convex along the Bergman geodesics we will define below.

Here is the setup for the degenerations we will consider: Let $L \to X$ be a very ample line bundle, with $S = \{S_0, ..., S_N\}$ a basis of sections of $H^0(X, L)$ furnishing a Kodaira embedding

$$X \ni z \mapsto \iota_S(z) = [S_0(z), ..., S_N(z)] \in \mathbb{P}^N. \tag{3.6}$$

Then the line bundle L is the pullback of the restriction to $\iota(X)$ of the hyperplane bundle $\mathcal{O}_{\mathbb{P}^N}(1)$. We consider the action of one-parameter subgroups $\sigma_t \in SL(N+1, \mathbb{C})$ acting diagonally as

$$\sigma_t \cdot S = (t^{a_0}S_0, ..., t^{a_N}S_N), \qquad a_0 + ... + a_N = 0. \tag{3.7}$$

Under this action, X acquires a corresponding family of Kähler metrics

$$\omega_t = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\|\sigma_t \cdot S\|^2, \qquad \|\sigma_t \cdot S\|^2 = \sum_{j=0}^N |t|^{2a_j}|S_j|^2 \tag{3.8}$$

which are the restrictions to $\sigma_t \cdot \iota(X)$ of the Fubini-Study metric on $\mathbb{P}^N$. Written in terms of potentials, we have $\omega_t = \omega_0 + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\phi$, where our reference metric is $\omega_0 = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\|S\|^2$, and

$$\phi_t = \log\frac{\|\sigma_t \cdot S\|^2}{\|S\|^2} = \log\frac{\sum_{j=0}^N |t|^{2a_j}|S_j|^2}{\sum_{j=0}^N |S_j|^2}. \tag{3.9}$$

The finite dimensional space of such potentials as the basis of sections varies is called the *Bergman space* $\mathcal{K}_1$. We may also consider larger Bergman spaces $\mathcal{K}_k$ as we consider powers of the line bundle L^k with larger bases of sections. Note that if ϕ is a potential in $\mathcal{K}_1$, then $k\phi$ is a potential in $\mathcal{K}_k$, and furthermore,

$$F^0_{k\omega_0}(k\phi) = kF^0_{\omega_0}(\phi), \tag{3.10}$$

so for our purposes it suffices to look at a single line bundle L.

We may assume that $a_0 \geq ... \geq a_N$, and we call the sections with weight equal to a_N *sections of lowest weight*. The path $t \mapsto \phi$ defined above in the space of Kähler potentials is called a *Bergman geodesic*.

It is known that along such a one-parameter subgroup, $F^0_{\omega_0}(\phi)$ is convex in $u = \log(1/|t|)$. We aim to describe the asymptotic behavior of $F^0_{\omega_0}$ as $u \to \infty$, or equivalently, as $|t| \to 0$, and in particular, to determine its asymptotic slope. We employ analysis to establish that the singular behavior of the functional is $O(\log|t|)$, and use some algebra to determine the precise coefficient. The following theorem is in [27]:

THEOREM **3.1.1.** *Let X be a Riemann surface. Then*

$$F^0_{\omega_0}(\phi_t) = \{-2a_N - \frac{1}{V} \sum_{zeroes\ of\ S_N} \sum_{\alpha=1}^{M} p^2_\alpha(m_\alpha - m_{\alpha+1})\} \log \frac{1}{|t|} + O(1) \tag{3.11}$$

as $t \to 0$, where p_α, m_α refer to the data of the Newton polygon.

Since the expression in braces above is positive, the formula gives another proof of the Chow-Mumford stability of curves. Another proof of Theorem 3.1.1 is given in [31], along with the slope of the Mabuchi functional. Our approach is inspired by [31], as well as earlier works on asymptotics of oscillatory integrals in [28; 29].

We derive an analogous formula for the Aubin-Yau functional in complex dimension $n = 2$ using a similar asymptotic calculation of singular integrals with certain modifications in order to deal with the new complications in higher dimensions. We expect that the approach is valid in all dimensions with analogous formulas for the slope, but in this thesis we stick to dimension 2 for concreteness and ease of notation.

Here are the results of the chapter: First, we observe that for all the integrals that do not involve the highest power of ω_ϕ, the entire contribution to the slope is from the lowest weight:

THEOREM **3.1.2.** *Assume X has dimension $n = 2$. For $1 \leq i \leq n$,*

$$\frac{1}{V} \int_X \phi_t \omega_0^i \wedge \omega_\phi^{n-i} = -2a_N \log \frac{1}{|t|} + O(1) \tag{3.12}$$

as $t \to 0$.

The main result of this chapter is the following formula for the slope:

THEOREM **3.1.3.** *Assume X has dimension $n = 2$, $L \to X$ is very ample, and that the zero sets of a basis of sections $\{S_0, \ldots, S_N\}$ intersect with normal crossings. Then along a Bergman geodesic (3.9) we have*

$$F^0_{\omega_0}(\phi_t) = \mu \log \frac{1}{|t|} + O(1) \tag{3.13}$$

as $|t| \to 0$ where the asymptotic slope μ is given by

$$\mu = (-2a_N) - \frac{1}{(n+1)V} \sum_{Sing(\tilde{D}) \cap Z(\tilde{S_N})} \sum_{\substack{faces\ F_c \\ of\ \mathfrak{N}}} 16 d_c \sum_{\{i,j,k,l\}^*} D_4(i,j,k,l) \int_0^\infty \int_0^\infty \frac{x^{2(p_i+p_j+p_k+p_l)-1} y^{2(r_i+r_j+r_k+r_l)-1}}{(\sum_\alpha x^{2p_\alpha} y^{2r_\alpha})^4} dx dy \tag{3.14}$$

where $\mathfrak{N}$ is the Newton polytope of the data at a singular point with normal crossings, the exponents p_i, r_i, etc. refer to the data of the Newton diagram, and the final sum is over sets of four indices $\{i, j, k, l\}$ corresponding to an unordered selection of four points of the Newton diagram, all lying on the face F_c (not necessarily vertices of F_c), at least three of which are distinct, and not all collinear. The sum over α is over all points of the Newton diagram lying on the face F_i. The term d_c is defined by describing the equation of the face F_i in (p, r, q)-space as

$$F_c = \{a_c p + b_c r + q = d_c\} \cap \mathfrak{N}, \tag{3.15}$$

and is positive. The positive, symmetric function denoted $D_4(i, j, k, l)$, actually a function of $(p_i, r_i, p_j, r_j, p_k, r_k, p_l, r_l)$, is defined in (3.33) below. The integrals in the formula are all convergent.

Remark: We note that as written, the asymptotic slope μ is a difference of positive terms: a positive trivial contribution $-2a_N$ from the lowest weight, minus the positive nontrivial contribution.

As a corollary, note also that the formula shows that the slope is linear and homogeneous in the weights a_i or q_i, at least for fixed geometries of the Newton polytope.

The outline of the chapter is as follows: in section two, we describe the proof of Theorem 3.1.3. We begin by isolating the contribution from the lowest weight. We then calculate the lowest order terms that appear in the volume forms, making use of some algebraic identities that give us the determinant-like quantities $D_4(i,j,k,l)$. At this point, we introduce and describe the important features of the Newton diagram associated to a one-parameter subgroup, and carry out the computation of the singular part of the integrals. In the final section, we carry through the slope calculation for some simple examples.

3.2 Details of the Slope Calculation

It is convenient to utilize the notation from [31] and isolate the lowest power of $|t|$ as follows:

$$\phi = \log \frac{|\sigma S|^2}{|S|^2} - 2a_N \log \frac{1}{|t|}, \quad |\sigma S|^2 = \sum_{j=0}^{N} |t|^{2q_j} |S_j|^2, \quad |S|^2 = \sum_{j=0}^{N} |S_j|^2, \tag{3.16}$$

where the exponents

$$q_j = a_j - a_N \geq 0 \tag{3.17}$$

are the *non-negative weights*. Note that at least one of the non-negative weights is equal to 0. By the assumption that the basis of sections furnishes a smooth Kodaira embedding, there is no point on X where all of the sections vanish. This implies that $\log |S|^2$ is bounded on X, and therefore

$$\int_X \log |S|^2 \omega_0^i \omega_\phi^{n-i} \leq CV = O(1), \tag{3.18}$$

so we may drop this term from ϕ for the calculation of the asymptotic slope. It is then trivial to compute the contribution to the slope from the section of lowest weight, since

$$\begin{aligned} \frac{1}{V} \int_X -2a_N \log \frac{1}{|t|} \omega_0^i \omega_\phi^{n-i} &= -2a_N \log \frac{1}{|t|} \frac{1}{V} \int_X \omega_0^i \omega_\phi^{n-i} \\ &= -2a_N \log \frac{1}{|t|} \end{aligned} \tag{3.19}$$

for each $0 \leq i \leq n$. These $n+1$ terms account for the overall contribution of $-2a_N$ (which is non-negative since $a_N \leq 0$) to the asymptotic slope.

We set out to determine the nontrivial contribution to the slope, that is, to compute the singular part of

$$A_i(t) = \int_X \log |\sigma S|^2 \omega_0^i \omega_\phi^{n-i}. \tag{3.20}$$

Here is the basic idea: The singular part of the global integral $A_i(t)$ may be calculated by integrating only over neighborhoods of isolated points, namely the transverse intersection points of the zero divisor of the section(s) of lowest weight with itself and the zero divisors of the other sections.

Proof. Observe first that the integrand of $A_i(t)$ is bounded away from the union of the zero sets of the sections of lowest weight. Let s_N be a section of lowest weight. Suppose that in a neighborhood of a smooth point $p = 0$ of $\{S_N = 0\}$, we may take complex coordinates z, w (possibly after a resolution) in which $\{S_N = 0\} = \{z = 0\}$, and each of the other sections in this trivialization may be written in the form $s_i = z^{p_i} u_i(z, w)$, where u_i is a unit. Then it will follow from the calculations below that the volume forms ω_0^2, $\omega_0 \wedge \omega_\phi$, and ω_ϕ^2 only contain terms of strictly higher order in $|z|$ and $|w|$, and thus there is no contribution to the $\log |t|$ term in $F^0_{\omega_0}$ by Lemma 3.2.7. □

In general, we recall Hironaka's result on resolution of singularities: There exists a resolution $\mu : \tilde{X} \to X$ such that $\mu^* D + Exc(\mu) = \tilde{D}$ has simple normal crossing support. On $\tilde{X}$, the nontrivial contributions to the slope come from a finite set of points of intersections with the other divisors with $\mu^* D_N$.

Assume that we have a set of coordinates in a neighborhood of a point, taken to be the origin, at which the sections vanish with normal crossings. This means that we our sections S_j are written in these coordinates as

$$S_j = z^{p_j} w^{r_j} u_j(z, w), \tag{3.21}$$

where the $u_j(z, w)$ are holomorphic functions that do not vanish at the origin. For a more detailed account of Hironaka's theorem and its use in the analysis of integrals see [29].

3.2.1 Description of the volume form

We must first compute ω_ϕ, $\omega_0 \wedge \omega_\phi$, and ω_ϕ^2. We find

$$\omega_\phi = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log |\sigma S|^2 \tag{3.22}$$

$$= \frac{\sqrt{-1}}{2\pi |\sigma S|^4} \sum_{i,j} |t|^{2q_i + 2q_j} \left(|S_j|^2 \partial S_i \wedge \bar{\partial} \bar{S}_i - S_i \bar{S}_j \partial S_j \wedge \bar{\partial} \bar{S}_i \right) \tag{3.23}$$

$$= \frac{\sqrt{-1}}{2\pi |\sigma S|^4} \sum_{i,j} |t|^{2(q_i + q_j)} |u_i(0)|^2 |u_j(0)|^2 |z|^{2(p_i + p_j - 1)} |w|^{2(r_i + r_j - 1)}$$
$$\big((p_i^2 - p_i p_j) |w|^2 dz \wedge d\bar{z} + (p_i r_i - p_j r_i) \bar{z} w dz \wedge d\bar{w} + (p_i r_i - p_i r_j) z \bar{w} dw \wedge d\bar{z} + (r_i^2 - r_j r_i) |z|^2 dw \wedge d\bar{w} \big)$$
$$+ O(...). \tag{3.24}$$

Here by $O(...)$ we mean all higher order terms in $|z|$ and $|w|$. Taking the wedge product yields

$$\begin{aligned}\omega_\phi^2 = \frac{1}{4\pi^2|\sigma S|^8} \sum_{i,j,k,l} |t|^{2(q_i+q_j+q_k+q_l)}|z|^{2(p_i+p_j+p_k+p_l-1)}|w|^{2(r_i+r_j+r_k+r_l-1)}|u_i|^2|u_j|^2|u_k|^2|u_l|^2 \\ 2\left[(p_i^2 - p_ip_j)(r_k^2 - r_kr_l) - (p_ir_i - p_jr_i)(p_kr_k - p_kr_l)\right]\sqrt{-1}dz\wedge d\bar{z}\wedge\sqrt{-1}dw\wedge d\bar{w} \\ + O(...), \qquad (3.25)\end{aligned}$$

where we have used the symmetry in the indices $(i,j) \leftrightarrow (k,l)$ to obtain twice the quantity in brackets. The quantity in the brackets we will denote by the symbol $(ijkl)$, and it may be simplified as

$$\begin{aligned}(ijkl) &= (p_i^2 - p_ip_j)(r_k^2 - r_kr_l) - (p_ir_i - p_jr_i)(p_kr_k - p_kr_l) \\ &= (p_ir_k - p_kr_i)(p_i - p_j)(r_k - r_l). \qquad (3.26)\end{aligned}$$

The other volume forms are the same except for the factors of $|t|$ that only appear in ω_ϕ:

$$\begin{aligned}\omega_0\wedge\omega_\phi =& \frac{1}{4\pi^2|S|^4|\sigma S|^4} \sum_{i,j,k,l} |t|^{2(q_i+q_j)}|z|^{2(p_i+p_j+p_k+p_l-1)}|w|^{2(r_i+r_j+r_k+r_l-1)}|u_i|^2|u_j|^2|u_k|^2|u_l|^2 \\ & [(ijkl)+(klij)]\sqrt{-1}dz\wedge d\bar{z}\wedge\sqrt{-1}dw\wedge d\bar{w} \\ & + O(...). \qquad (3.27)\end{aligned}$$

We will see later that it is sufficient to consider only these lowest order terms. From now on, we will also assume that $u_j(0) = 1$ for $j = 0, ..., N$, which we may do since the asymptotic slope is independent of the basis.

The appearance of the determinant-like quantity $(ijkl)$ is a novel feature in dimension $n > 1$. In particular, it rules out terms where the same index is taken in each of the four sums. Let us make some simple observations about the symbol $(ijkl)$. First, $(ijkl) = 0$ if $i = j$, $k = l$, or $i = k$. If i and j are distinct indices, the only possibly non-zero symbols involving only i and j are $(ijji)$ and $(jiij)$. But

$$\begin{aligned}(ijji) &= (p_ir_j - p_jr_i)(p_i - p_j)(r_j - r_i) \\ &= -(p_jr_i - p_ir_j)(p_j - p_i)(r_i - r_j) \\ &= -(jiij), \qquad (3.28)\end{aligned}$$

so $(ijji)+(jiij)=0$, and therefore there are no terms in the lowest order part of the volume form with only two distinct indices taken from the sum.

Now consider the case of three distinct indices:

Lemma 3.2.1. *The nonzero symbols $(ijkl)$ consisting of a set of three indices with one repeated have sum*

$$(ijki)+(ikji)+(jiik)+(kiij)+(jiki)+(kiji) = (p_jr_i - p_kr_i - p_ir_j + p_kr_j + p_ir_k - p_jr_k)^2 \tag{3.29}$$

$$= ([ij]-[ik]+[jk])^2, \tag{3.30}$$

where $[ij] = p_ir_j - p_jr_i$. The sum represents the square of the area of any parallelogram with three vertices $\{(p_i,r_i),(p_j,r_j),(p_k,r_k)\}$, and is nonnegative and symmetric in the indices i,j,k.

And the remaining case of four distinct indices:

Lemma 3.2.2. *The summation of symbols $(ijkl)$ over all permutations of 4 distinct indices yields*

$$\sum_{\sigma\in S_4}(\sigma(i)\sigma(j)\sigma(k)\sigma(l)) = ([ij]-[ik]+[jk])^2+([jk]-[jl]+[kl])^2+([kl]-[ki]+[li])^2+([li]-[lj]+[ij])^2 \tag{3.31}$$

These two algebraic identities may be verified by a computer algebra system.

As a corollary, it is clear from these formulas that the lowest order terms in the volume form are non-negative, and are equal to 0 if the four indices correspond to collinear points. We set

$$D_3(i,j,k) = ([ij]-[ik]+[jk])^2 \tag{3.32}$$

and

$$D_4(i,j,k,l) = \begin{cases} D_3(i,j,k)+D_3(j,k,l)+D_3(k,l,i)+D_3(l,i,j) & \text{if all indices are distinct} \\ \frac{1}{2}\left(D_3(i,j,k)+D_3(j,k,l)+D_3(k,l,i)+D_3(l,i,j)\right) & \text{if any two indices are the same.} \end{cases} \tag{3.33}$$

The factor of 1/2 in the case of a repeated index compensates for the overcounting by transposing the slots of the repeated index, and so $D_4(i,j,k,k) = D_3(i,j,k)$.

We may thus rewrite the lowest-order part of ω_ϕ^2 as a sum of positive terms as

$$\omega_\phi^2 = \frac{1}{|\sigma S|^8} \sum_{\{i,j,k,l\}^*} \left[2D_4(i,j,k,l)|t|^{2(q_i+q_j+q_k+q_l)}|z|^{2(p_i+p_j+p_k+p_l)-2}|w|^{2(r_i+r_j+r_k+r_l)-2} \frac{\sqrt{-1}}{2\pi} dz \wedge d\bar{z} \frac{\sqrt{-1}}{2\pi} dw \wedge d\bar{w} \right] + O(...) \tag{3.34}$$

and similarly for ω_0^2 and $\omega_0 \wedge \omega_\phi$.

3.2.2 Newton diagram

The analysis of the singular integrals appearing in the Aubin-Yau functional is well facilitated by appealing to the geometry of the Newton polytope.

Definition: We call the set of points $\{(p_j, r_j, q_j) \in \mathbb{R}_+^3\}_{j=0}^N$ the *Newton diagram* of the data. The *Newton polytope* $\mathfrak{N}$ is the region given by

$$\mathfrak{N} = ConvexHull \left(\bigcup_{j=0}^{N} \{(p_j, r_j, q_j) + \mathbb{R}_+^3\} \right), \tag{3.35}$$

that is, the unbounded convex polytope which is the convex hull of the union of the positive orthant $\mathbb{R}_+^3 = \{(p, r, q) \in \mathbb{R}^3 | p, r, q \geq 0\}$ translated to each point in the Newton diagram.

The vertices of $\mathfrak{N}$ are a subset of the data $\{p_v, r_v, q_v\}_{v=1}^L$.

Fix t with $0 < |t| < 1$. We examine the sum $|\sigma S|^2 = \sum_{i=0}^N |t|^{2q_i}|z|^{2p_i}|w|^{2r_i}$ that appears in the denominator and as a factor of $\log |\sigma S|^2$ in the integrals under consideration. It is convenient to describe the regions in the z, w variables where each particular term is *dominant*; for fixed z, w, t, we say that the term $|t|^{2q_i}|z|^{2p_i}|w|^{2r_i}$ dominates the other terms in the sum if

$$|t|^{2q_i}|z|^{2p_i}|w|^{2r_i} \geq |t|^{2q_j}|z|^{2p_j}|w|^{2r_j} \text{ for all } j \neq i. \tag{3.36}$$

It is simple to describe the regions where the various terms dominate in terms of variables α, β where we set $|z| = |t|^\alpha$, $|w| = |t|^\beta$, where α, β are real and non-negative. The term $|t|^{2q_i}|z|^{2p_i}|w|^{2r_i}$ dominates when

$$|t|^{2(q_i+p_i\alpha+r_i\beta)} \geq |t|^{2(q_j+p_j\alpha+r_j\beta)} \tag{3.37}$$

for all other points (p_j, r_j, q_j) in the diagram, or

$$q_j + p_j\alpha + r_j\beta \geq q_i + p_i\alpha + r_i\beta, \qquad j \neq i. \tag{3.38}$$

Together with the constraints $\alpha \geq 0$, $\beta \geq 0$, these inequalities describe the region M_i where $|t|^{2q_i}|z|^{2p_i}|w|^{2r_i}$ dominates as the intersection of a set of half-planes. M_i describes the set of planes passing through the point lying below all the other points of the diagram, so it has empty interior unless the point (p_i, r_i, q_i) is a vertex of the Newton polytope.

Lemma 3.2.3. *The positive quadrant $\alpha, \beta \geq 0$ is partitioned into polygonal regions M_v, $v = 1, ..., L$, on which the term $|t|^{2q_v}|z|^{2p_v}|w|^{2r_v}$ dominates. For each vertex with $(p_v, r_v) \neq (0,0)$, the region M_v is the convex hull of the points $(m^z_{v,i}, m^w_{v,i})$, where i ranges over the number of faces of $\mathfrak{N}$ incident to the vertex, with normal vector $(m^z_{v,i}, m^w_{v,i}, 1)$.*

Note that the set of points $(m^z_{v,i}, m^w_{v,i})$ forms a natural dual of the Newton polyhedron. Every Newton polytope for test configurations of this form also contains vertical faces $x = 0$ and $y = 0$, which do not play a role in the analysis.

In the interior of the region where $|t|^{2q_v}|z|^{2p_v}|w|^{2r_v}$ dominates, we may Taylor expand to obtain

$$\log|\sigma S|^2 = \log(|t|^{2q_v}|z|^{2p_v}|w|^{2r_v}) + O(\frac{\sum_{j\neq v}|t|^{2q_j}|S_j|^2}{|t|^{2q_v}|z|^{2p_v}|w|^{2r_v}}), \tag{3.39}$$

$$\frac{1}{|\sigma S|^2} = \frac{1}{|t|^{2q_v}|z|^{2p_v}|w|^{2r_v}} + O(\frac{\sum_{j\neq v}|t|^{2q_j}|S_j|^2}{(|t|^{2q_v}|z|^{2p_v}|w|^{2r_v})^2}). \tag{3.40}$$

At the boundary of the region M_v, the term $|t|^{2q_v}|z|^{2p_v}|w|^{2r_v}$ shares the same order as other terms.

3.2.3 Singular integral analysis

We aim to calculate the divergent term in the Aubin-Yau functional as $|t| \to 0$, which is of the form $\log(1/|t|)$. Let U be a small polydisk around the origin, which we may take to have radius 1. Subtracting off contributions of order $O(1)$, we calculate the contribution from the term $\int_U \phi\omega_\phi^2$ as follows:

$$\int_U \phi\omega_\phi^2 = \int_U \frac{\log|\sigma S|^2}{4\pi^2|\sigma S|^8} \sum_{\{i,j,k,l\}} |t|^{2(q_i+q_j+q_k+q_l)}|z|^{2(p_i+p_j+p_k+p_l-1)}|w|^{2(r_i+r_j+r_k+r_l-1)}$$

$$2D_4(i,j,k,l)\sqrt{-1}dz\wedge d\bar{z}\wedge\sqrt{-1}dw\wedge d\bar{w} + O(1) \tag{3.41}$$

$$= 8\int_0^1\int_0^1 \sum_{\{i,j,k,l\}} D_4(i,j,k,l)\frac{|t|^{2(q_i+q_j+q_k+q_l)}x^{2(p_i+p_j+p_k+p_l)-1}y^{2(r_i+r_j+r_k+r_l)-1}}{|\sigma S|^8}\log|\sigma S|^2 dxdy + O(1), \tag{3.42}$$

where we integrate out the angular variables. In the last line, we let $x = |z|, y = |w|$. We must be careful with all factors of 2 and π in this calculation, since we must compare these localized integrals to the global contribution from the lowest weight. The factor of 4 comes from the change to polar coordinates in both z and w:

$$\frac{\sqrt{-1}}{2\pi} dz \wedge d\bar{z} = \frac{\sqrt{-1}}{2\pi}(du + \sqrt{-1}dv) \wedge (du - \sqrt{-1}dv) \tag{3.43}$$

$$= \frac{2}{2\pi} du \wedge dv \tag{3.44}$$

$$= \frac{2}{2\pi} x dx \wedge d\theta \tag{3.45}$$

Integrating each of the θ variables cancels a factor of 2π in denominator.

We set

$$A(t) = 8 \int_0^1 \int_0^1 \sum_{\{i,j,k,l\}} D_4(i,j,k,l) \frac{|t|^{2(q_i+q_j+q_k+q_l)} x^{2(p_i+p_j+p_k+p_l)-1} y^{2(r_i+r_j+r_k+r_l)-1}}{|\sigma S|^8} \log |\sigma S|^2 dxdy$$

and set to determine the asymptotic behavior of $A(t)$. We begin with two very simple lemmas.

Lemma 3.2.4. *Let $A, B, \{p_i\}_{i=0}^N, \{r_i\}_{i=0}^N$ be non-negative integers such that $A + B \geq \min_i\{p_i + r_i\}$ and $A + B \leq \max_i\{p_i + r_i\} - 3$. Then the integral*

$$I = \int_0^\infty \int_0^\infty \frac{x^A y^B}{\sum_{i=0}^n x^{p_i} y^{r_i}} dxdy < +\infty. \tag{3.46}$$

Proof. We set $r^2 = x^2 + y^2$ and let $I = I_1 + I_2$ where the domain of integration of I_1 is $r < 1$ and that of I_2 is $r > 1$. We have

$$I_1 \leq \int_0^1 \int_0^1 \left| \frac{x^A y^B}{(\sum_i x^{2p_i} y^{2r_i})^4} \right| dxdy \tag{3.47}$$

$$\leq c \int_0^1 \frac{r^{A+B+1}}{r^{\min_i\{p_i+r_i\}}} dr \tag{3.48}$$

$$\leq c \int_0^1 r dr \tag{3.49}$$

$$< \infty. \tag{3.50}$$

Similarly,

$$I_2 \leq c \int_1^\infty \frac{r^{A+B+1}}{r^{\max_i\{p_i+r_i\}}} dr \tag{3.51}$$

$$\leq c \int_1^\infty r^{-2} \tag{3.52}$$

$$< \infty. \tag{3.53}$$

□

Note that this argument also shows

Lemma 3.2.5.

$$I = \int_0^\infty \int_0^\infty \frac{x^A y^B}{\sum_i x^{p_i} y^{r_i}} \log P(x,y) dxdy < +\infty, \tag{3.54}$$

where $P(x,y)$ is a positive polynomial.

Now we return to the setting of our Bergman geodesic and the integral $A(t)$. First we prove that we may take the upper limits of integration to $+\infty$ without changing the singular part of the integral. This will facilitate the scaling we wish to make later on.

Lemma 3.2.6. *Suppose $\{i,j,k,l\}$ are a set of indices so that*

$$2(p_i + p_j + p_k + p_l + r_i + r_j + r_k + r_l) - 1 \le 4 \max_{\alpha: q_\alpha = 0} \{2(p_\alpha + r_\alpha)\} - 2. \tag{3.55}$$

Then

$$\int_0^1 \int_0^1 \frac{|t|^{2(q_i+q_j+q_k+q_l)} x^{2(p_i+p_j+p_k+p_l)-1} y^{2(r_i+r_j+r_k+r_l)-1}}{|\sigma S|^8} \log |\sigma S|^2 dxdy =$$
$$\int_0^\infty \int_0^\infty \frac{|t|^{2(q_i+q_j+q_k+q_l)} x^{2(p_i+p_j+p_k+p_l)-1} y^{2(r_i+r_j+r_k+r_l)-1}}{|\sigma S|^8} \log |\sigma S|^2 dxdy + O(1)$$

Proof. We have

$$\int_1^\infty \int_1^\infty \frac{|t|^{2(q_i+q_j+q_k+q_l)} x^{2(p_i+p_j+p_k+p_l)-1} y^{2(r_i+r_j+r_k+r_l)-1}}{|\sigma S|^8} \log |\sigma S|^2 dxdy \le \tag{3.56}$$
$$\int_1^\infty \int_1^\infty \frac{|t|^{2(q_i+q_j+q_k+q_l)} x^{2(p_i+p_j+p_k+p_l)-1} y^{2(r_i+r_j+r_k+r_l)-1}}{(\sum_{\alpha:q_\alpha=0} x^{2p_\alpha} y^{2r_\alpha})^4} \log |\sigma S|^2 dxdy. \tag{3.57}$$

By taking $|t|$ sufficiently small, we may write the term $\log |\sigma S|^2$ as

$$\log |\sigma S|^2 \le \log(2 \sum_{\alpha:q_\alpha=0} x^{2p_\alpha} y^{2r_\alpha}) \tag{3.58}$$
$$= \log(\sum_{\alpha:q_\alpha=0} x^{2p_\alpha} y^{2r_\alpha}) + \log 2 \tag{3.59}$$

By (3.55), the integral

$$\int_1^\infty \int_1^\infty \frac{x^{2(p_i+p_j+p_k+p_l)-1} y^{2(r_i+r_j+r_k+r_l)-1}}{(\sum_{\alpha:q_\alpha=0} x^{2p_\alpha} y^{2r_\alpha})^4} \log |\sigma S|^2 = O(1)$$

by Lemma 3.2.5, and the same holds for the same integral multiplied by $|t|^{2(q_i+q_j+q_k+q_l)}$, a non-negative power of $|t|$. □

Let us remark that in the case when the inequality (3.55) is not satisfied, then

$$\int_0^1\int_0^1 \frac{|t|^{2(q_i+q_j+q_k+q_l)}x^{2(p_i+p_j+p_k+p_l)-1}y^{2(r_i+r_j+r_k+r_l)-1}}{|\sigma S|^8}\log|\sigma S|^2 dxdy \leq$$
$$\int_0^0\int_0^1 \frac{|t|^{2(q_i+q_j+q_k+q_l)}x^{2(p_i+p_j+p_k+p_l)-1}y^{2(r_i+r_j+r_k+r_l)-1}}{(\sum_{\alpha:q_\alpha=0}x^{2p_\alpha}y^{2r_\alpha})^4}\log|\sigma S|^2 dxdy$$
$$= O(1), \tag{3.60}$$

again by Lemma 3.2.5, since necessarily $2(p_i+p_j+p_k+p_l+r_i+r_j+r_k+r_l)-1 \geq 4\min_{\alpha:q_\alpha=0}\{2(p_\alpha+r_\alpha)\}+1$. Therefore such terms cannot contribute to the slope.

We are now ready to prove the main lemma in which we calculate the nontrivial contribution to the slope:

Lemma 3.2.7. *Let $\{i,j,k,l\}$ be a set of indices such that $D_4(i,j,k,l) \neq 0$ and such that (3.55) holds. Then*

$$I(t) = \int_0^\infty\int_0^\infty \frac{|t|^{2(q_i+q_j+q_k+q_l)}x^{2(p_i+p_j+p_k+p_l)-1}y^{2(r_i+r_j+r_k+r_l)-1}}{|\sigma S|^8}\log|\sigma S|^2 dxdy$$
$$= 2d_F\log|t|\int_0^\infty\int_0^\infty \frac{x^{2(p_i+p_j+p_k+p_l)-1}y^{2(r_i+r_j+r_k+r_l)-1}}{\sum_\alpha x^{2p_\alpha}y^{2r_\alpha}}dxdy + O(1) \tag{3.61}$$

where the sum is over indices α where the points $(p_\alpha, r_\alpha, q_\alpha)$ lie on the face F of $\mathfrak{N}$, where F is a subset of the plane given by the equation $m_F^x x + m_F^y y + z = d_F$.

Proof. We compute the integral by rescaling. Let $x \to |t|^{m^x}x$, $y \to |t|^{m^y}y$, where $m^x, m^y \geq 0$. The integral becomes

$$I(t) = \int_0^\infty\int_0^\infty \frac{|t|^{2(q_i+q_j+q_k+q_l)}x^{2(p_i+p_j+p_k+p_l)-1}y^{2(r_i+r_j+r_k+r_l)-1}}{|\sigma S|^8}\log|\sigma S|^2 dxdy$$
$$= \int_0^\infty\int_0^\infty \frac{|t|^{2((q_i+q_j+q_k+q_l)+m^x(p_i+p_j+p_k+p_l)+m^y(r_i+r_j+r_k+r_l))}x^{2(p_i+p_j+p_k+p_l)-1}y^{2(r_i+r_j+r_k+r_l)-1}}{(\sum_{a=0}^N |t|^{2(q_a+m^xp_a+m^yr_a)}x^{2p_a}y^{2r_a})^4}$$
$$\log(\sum_{u=0}^N |t|^{2(q_u+m^xp_u+m^yr_u)}x^{2p_u}y^{2r_u})dxdy. \tag{3.62}$$

Setting $\gamma = q_v + m^xp_v + m^yr_v = \min_u\{q_u + m^xp_u + m^yr_u\}$, we may factor this term out of the denominator. The integrand thus acquires an overall factor of

$$|t|^{2((q_i+q_j+q_k+q_l)+m^x(p_i+p_j+p_k+p_l)+m^y(r_i+r_j+r_k+r_l)-4\gamma)}.$$

Now the minimum γ is realized at $q_v+m^x p_v+m^y r_v$ if (p_v, r_v, q_v) is a vertex of $\mathfrak{N}$ and $(m^x, m^y) \in M_v$. We observe

$$[(q_i - q_v) + m^x(p_i - p_v) + m^y(r_i - r_v)] + \cdots + [(q_l - q_v) + m^x(p_l - p_v) + m^y(r_l - r_v)] \geq 0. \quad (3.63)$$

If this exponent is strictly greater than 0, then by Lemma 3.2.5, the integral is O(1) as $t \to 0$. Equality is obtained only if each term in brackets is 0. In this case, the point (m^x, m^y) must be contained in $M_i \cap M_j \cap M_k \cap M_l$, and since at least three of i, j, k, l must be distinct, (m^x, m^y) must lie at a corner of M_v, or in other words, each of the points $(p_i, r_i, q_i), ..., (p_l, r_l, q_l)$ lies on a common face F of $\mathfrak{N}$ with normal vector $(m^x, m^y, 1)$, and $\gamma = d_F$. This shows that only sets of indices corresponding to four points lying on a single face of the Newton polytope contribute to the asymptotic slope, and moreover, that all higher-order terms in $\omega_0^i \wedge \omega_\phi^{n-i}$ do not contribute to the slope.

For a given term with all four indices corresponding to points on a face F of the Newton diagram, after scaling by the coordinates of the normal vector of the face, we have

$$I(t) = \int_0^\infty \int_0^\infty \frac{x^{2(p_i+p_j+p_k+p_l)-1} y^{2(r_i+r_j+r_k+r_l)-1}}{(\sum_\alpha x^{2p_\alpha} y^{2r_\alpha} + |t|P(x, y, |t|))^4} (\log |t|^{2\gamma} + \log(\sum_\alpha x^{2p_\alpha} y^{2r_\alpha} + |t|P(x, y, |t|))) dxdy \quad (3.64)$$

$$= 2\gamma \log |t| \int_0^\infty \int_0^\infty \frac{x^{2(p_i+p_j+p_k+p_l)-1} y^{2(r_i+r_j+r_k+r_l)-1}}{(\sum_\alpha x^{2p_\alpha} y^{2r_\alpha})^4} dxdy + O(1) \quad (3.65)$$

where the sum is over all the indices α of points on the face F, and $P(x, y, |t|)$ is a polynomial. The last line is obtained by Taylor expanding the denominator and $\log(\sum_\alpha x^{2p_\alpha} y^{2r_\alpha} + |t|P(x, y, |t|))$ term. We may use Lemma 3.2.5 since if all the points $(p_i, r_i, q_i), ..., (p_l, r_l, q_l)$ lie on the face F and $D_4(i, j, k, l) \neq 0$, then

$$2(p_i + p_j + p_k + p_l + r_i + r_j + r_k + r_l) - 1 \leq 4 \max_\alpha \{2(p_\alpha + r_\alpha)\} - 2 \quad (3.66)$$

and

$$2(p_i + p_j + p_k + p_l + r_i + r_j + r_k + r_l) - 1 \geq 4 \min_\alpha \{2(p_\alpha + r_\alpha)\} + 1. \quad (3.67)$$

□

Now let us show there is no non-trivial contribution to the slope from the terms in the Aubin-Yau functional involving ω^2 and $\omega \wedge \omega_\phi$; that is, their only contribution is from the highest weight.

For the integral $\int_X \log|\sigma S|^2 \omega_0^n$ this is easy to see, since ω_0^n is bounded independent of t and $\log|\sigma S|^2 \leq \log|S_N|^2 + c$ is integrable on X, therefore $\int_X \log|\sigma S|^2 \omega_0^n \leq C = O(1)$.

It remains to show $\int_X \log|\sigma S|^2 \omega_0 \wedge \omega_\phi$ is bounded as $|t| \to 0$. This can be seen by computing as before:

$$\int_U \log|\sigma S|^2 \omega \wedge \omega_\phi = \int_U \frac{\log|\sigma S|^2}{4\pi^2 |S|^4 |\sigma S|^4} \sum_{i,j,k,l} |t|^{2(q_i+q_j)} |z|^{2(p_i+p_j+p_k+p_l-1)} |w|^{2(r_i+r_j+r_k+r_l-1)}$$

$$((ijkl) + (klij))\sqrt{-1} dz \wedge d\bar{z} \wedge \sqrt{-1} dw \wedge d\bar{w} + O(1) \tag{3.68}$$

$$= 4 \int_0^\infty \int_0^\infty \sum_{i,j,k,l} ((ijkl) + (klij)) \log|\sigma S|^2$$

$$\frac{|t|^{2(q_i+q_j)} x^{2(p_i+p_j+p_k+p_l)-1} y^{2(r_i+r_j+r_k+r_l)-1}}{|S|^4 |\sigma S|^4} dx dy + O(1) \tag{3.69}$$

Under the scaling $x \to |t|^{m^x} x$, $y \to |t|^{m^y} y$, we may pull out a factor of

$$|t|^{2((q_i+q_j)+m^x(p_i+p_j+p_k+p_l)+m^y(r_i+r_j+r_k+r_l)-2(q_\alpha+m^x p_\alpha+m^y r_\alpha))},$$

where $q_\alpha + m^x p_\alpha + m^y r_\alpha = \min_i\{q_i + m^x p_i + m^y r_i\}$. We find that the exponent

$$[(q_i - q_\alpha) + m^x(p_i - p_\alpha) + m^y(r_i - r_\alpha)] + [(q_j - q_\alpha) + m^x(p_j - p_\alpha) + m^y(r_j - r_\alpha)]$$

$$+[m^x p_k + m^y r_k] + [m^x p_l + m^y r_l] \geq 0 \tag{3.70}$$

with equality only if each term in brackets is 0. But this can only happen when $(p_k, r_k) = (p_l, r_l) = (0,0)$, in which case $(ijkl) = (klij) = 0$. It follows that the lowest order integral is a convergent integral multiplied by a positive power of $|t|$, which is $O(1)$ as $|t| \to 0$.

Thus we have proven Theorem 3.1.2, and when combined with Lemma 3.2.7, we obtain Theorem 3.1.3.

3.3 Examples

The formula in (3.1.3) is most easily applicable in the case of toric surfaces. A polarized toric surface (X, L) is associated to a polygon P in the first quadrant of $\mathbb{R}^2$ with integral vertices including the point $(0,0)$. The lattice points (p_i, r_i) of $\overline{P}$ are in 1-1 correspondence with a basis of sections $S_i = x^{p_i} y^{r_i}$ of L in the coordinates of an open, dense subset of X. As a consequence, there is a natural set of coordinates suitable for integration on X so that the basis of sections are monomials, and there is only one point in the outermost sum in (3.14). A nice reference for the relevant background on toric varieties is [26].

We must be careful with the computation of our normalized volume. The volume of X, $V = \int_X \omega_0^2$ is related to the Euclidean volume of the polygon P by

$$Vol_{Euc}(P) = \int_X \frac{\omega_0^2}{2} = \frac{V}{2}. \tag{3.71}$$

3.3.1 Projective space $\mathbb{P}^2$

The bundle $\mathcal{O}(1)$ over $\mathbb{P}^2$ is represented by a triangle with vertices $(0,0),(1,0),(0,1)$. The area of the triangle is $1/2$, so $V = \int_X \omega_0^2 = 1$. We may specify a test-configuration or Bergman geodesic by assigning a non-negative weight over each point. The lowest weight contribution is twice the average of the non-negative weights. The setup is symmetric with respect to the points $(1,0)$ and $(0,1)$, so there are not many essentially different configurations. Here are the possibilities:

1. The weight at $(0,0)$ is 0. In this case, the Newton polytope is trivial, consisting of the entire positive orthant, and there is no non-trivial contribution to the slope. The slope is positive and comes entirely from the lowest weight.

2. The weight at $(0,0)$ is greater than zero, which we may take to be 1 by the linear homogeneity of the slope in the weights. At least one of the remaining weights must be zero. We take our Newton diagram to be $\{(0,0,1),(0,1,q),(1,0,0)\}$. There are two possibilities for q:

   (a) $q > 1$: In this case, the Newton polytope has only one non-trivial face, given by the equation $x + z = 1$, and only the points $(0,0,1)$ and $(1,0,0)$ lie on it. There is no non-trivial contribution to the slope. The slope is equal to $\mu = 2(1+q)/3$.

   (b) $0 \le q \le 1$: Again the Newton polytope consists of just the face $F : x + (1-q)y + z = 1 = d_F$. Now all three points lie on the face. We have $D_4(1,2,3,3) = 1$. The slope is given by

   $$\mu = \frac{2(1+q)}{3} - \frac{1}{3} 16 \cdot 1 \cdot 1 \cdot (I_1 + I_2 + I_3) \tag{3.72}$$

   where the three integrals come from the three choices of repeated index. We may compute

$$I_1 = \int_0^\infty \int_0^\infty \frac{xy}{(1+x^2+y^2)^4} dxdy = \frac{1}{24}, \tag{3.73}$$

$$I_2 = \int_0^\infty \int_0^\infty \frac{x^3y}{(1+x^2+y^2)^4} dxdy = \frac{1}{24}, \tag{3.74}$$

$$I_3 = \int_0^\infty \int_0^\infty \frac{xy^3}{(1+x^2+y^2)^4} dxdy = \frac{1}{24}, \tag{3.75}$$

so the slope comes to $\mu = 2(1+q)/3 - 2/3 = q/3$, which is 0 if $q = 0$. In particular, $\{(0,0,1),(0,1,0),(1,0,0)\}$ is the configuration with the smallest slope, and it is non-negative.

3.3.2 First Hirzebruch surface

We may represent this toric surface as the convex polygon with vertices $\{(0,0),(2,0),(1,1),(0,1)\}$ with volume 3/2, and a Bergman geodesic is specified by a choice of weights for a Newton diagram $\{P_0 : (0,0,q_{00}), P_1 : (1,0,q_{10}), P_2 : (2,0,q_{20}), P_3 : (0,1,q_{01}), P_4 : (1,1,q_{11})\}$. Now there are many more possibilities for the shape of the Newton polytope. Note that if, for example, $q_{00} = 0$, $q_{10} = 0$, or $q_{01} = 0$, then the Newton polytope will be of the same types as for $\mathbb{P}^2$.

Let us give two examples of test configurations. If we seek slopes that are as small as possible, we want the Newton diagrams to consist entirely of points on the boundary of the Newton polytope. In particular, if $q_{20} = 0$, we must have

$$q_{10} \le \frac{1}{2} q_{00}.$$

The maximal number of faces each containing at least three points of the Newton polytope is three, and these may occur in two shapes: $F_1 = \{P_0, P_1, P_4\}, F_2 = \{P_0, P_3, P_4\}, F_3 = \{P_1, P_2, P_4\}$ or $\tilde{F}_1 = \{P_0, P_1, P_3\}, \tilde{F}_2 = \{P_1, P_3, P_4\}, \tilde{F}_3 = \{P_1, P_2, P_4\}$. The first case occurs if

$$q_{10} + q_{01} - q_{00} - q_{11} > 0, \tag{3.76}$$

and the second case if the inequality is reversed.

Let us take as an example the first case. By solving the inequalities (3.38), we obtain the equations of the faces of the polytope:

$$F_1 = \{(q_{00} - q_{10})p + (q_{10} - q_{11})r + q = q_{00}\} \tag{3.77}$$

$$F_2 = \{(q_{01} - q_{11})p + (q_{00} - q_{11})r + q = q_{00}\} \tag{3.78}$$

$$F_3 = \{q_{10}p + (q_{10} - q_{11})r + q = 2q_{10}\} \tag{3.79}$$

The contribution from each face requires the evaluation of three integrals of the form

$$I_{ijkl} = \int_0^\infty \int_0^\infty \frac{x^{2(p_i+p_j+p_k+p_l)-1}y^{2(r_i+r_j+r_k+r_l)-1}}{(\sum_{\alpha\in F} x^{2p_\alpha}y^{2r_\alpha})^4} dxdy.$$

For example, on face F_3,

$$I_{1241} = \int_0^\infty \int_0^\infty \frac{x^9 y}{(x^2 + x^2y^2 + x^4)^4} dxdy = \frac{1}{24}, \tag{3.80}$$

and in fact all of the integrals are the same as the integrals appearing in the $\mathbb{P}^2$ calculation, and are all equal to 1/24. Also, all the relevant factors $D_4(i,j,k,l)$ are equal to 1. For the total slope we have

$$\mu = \frac{2(q_{00} + q_{10} + q_{01} + q_{11} + q_{20})}{5} - \frac{1}{3}\frac{1}{2\cdot 3/2}16\left(\frac{q_{00}}{8} + \frac{q_{00}}{8} + \frac{2q_{10}}{8}\right) \tag{3.81}$$

$$= \frac{-2(q_{00} + q_{10}) + 18(q_{01} + q_{11})}{45} \tag{3.82}$$

Combining the inequality (3.76) with $q_{10} < q_{00}/2$, we have $q_{01} > q_{00}/2 + q_{11}$, so

$$\mu > \frac{-5q_{00}/2 + 18(q_{01} + q_{11})}{45} > \frac{13q_{00}/2 + 36q_{11}}{45}. \tag{3.83}$$

For the final example, suppose all of the points of the Newton diagram lie on a single face and $q_{20} = 0$. Setting $q_{00} = 1$, we have that the equation of the face must be $F : 1/2x + cy + z = 1$, where $0 \le c \le 1/2$. The sum

$$m = \sum_{\{i,j,k,l\}^*} D_4(i,j,k,l) \int_0^\infty \int_0^\infty \frac{x^{2(p_i+p_j+p_k+p_l)-1}y^{2(r_i+r_j+r_k+r_l)-1}}{\sum_\alpha x^{2p_\alpha}y^{2r_\alpha}} dxdy$$

in (3.14) for the non-trivial part of the slope contains 32 terms: 5 choices for sets of 4 distinct indices, and 9 choices for sets of 3 indices with one repeated (the set $\{0,1,2\}$ excluded for being collinear), each of which gives three terms by the choice of the repeated index. The integrals in m

can be computed by Mathematica, for example:

$$I_{0013} = \int_0^\infty \int_0^\infty \frac{xydxdy}{(1+x^2+y^2+x^2y^2+x^4)^4} = \frac{7(-9+2\sqrt{3}\pi)}{648} \tag{3.84}$$

$$I_{0113} = \int_0^\infty \int_0^\infty \frac{x^3ydxdy}{(1+x^2+y^2+x^2y^2+x^4)^4} = \frac{6-\sqrt{3}\pi}{108} \tag{3.85}$$

$$I_{0133} = \int_0^\infty \int_0^\infty \frac{xy^3dxdy}{(1+x^2+y^2+x^2y^2+x^4)^4} = \frac{9-\sqrt{3}\pi}{324} \tag{3.86}$$

$$I_{0014} = \int_0^\infty \int_0^\infty \frac{x^3ydxdy}{(1+x^2+y^2+x^2y^2+x^4)^4} = \frac{6-\sqrt{3}\pi}{108} \tag{3.87}$$

$$I_{0114} = \int_0^\infty \int_0^\infty \frac{x^5ydxdy}{(1+x^2+y^2+x^2y^2+x^4)^4} = \frac{-9+2\sqrt{3}\pi}{648} \tag{3.88}$$

$$I_{0144} = \int_0^\infty \int_0^\infty \frac{x^5y^3dxdy}{(1+x^2+y^2+x^2y^2+x^4)^4} = \frac{45-8\sqrt{3}\pi}{648} \tag{3.89}$$

$$I_{0124} = \int_0^\infty \int_0^\infty \frac{x^7ydxdy}{(1+x^2+y^2+x^2y^2+x^4)^4} = \frac{-9+2\sqrt{3}\pi}{648} \tag{3.90}$$

$$\vdots$$

and so on. Incredibly, the overall sum m of these integrals weighted by the numbers $D_4(i,j,k,l)$ is rational: $m = 3/8$. The total asymptotic slope is

$$\mu = \frac{2(q_{00}+q_{10}+q_{01}+q_{11}+q_{20})}{5} - \frac{1}{3}\frac{1}{2\cdot 3/2}16q_{00}\frac{3}{8} \tag{3.91}$$

$$= \frac{2(q_{10}+q_{01}+q_{11}+q_{20})}{5} - \frac{4q_{00}}{15}. \tag{3.92}$$

Setting $q_{00} = 1$, μ attains its smallest value when $q_{10} = q_{01} = 1/2$, $q_{20} = q_{11} = 0$, in which case $\mu = 2/15$.

Part III

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