A Useful Multivariate Stochastic Integration Result

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Abstract

This paper derives the limiting distribution of the matrix $(1/T)\sum_{t=1}^{T} Z'_{t-1\cdot}\eta_{t\cdot}$ where $Z_{t\cdot} = \sum_{j=1}^{t}\eta_{j\cdot}$ and $\eta_{j\cdot}$ is a suitably restricted mixing process. It does so by the same method as in the scalar case with the aid of Ito's multivariate formula.

Key Words: Nonstationary processes; stationary processes; mixing processes; Ito's formula; functional central limit theorem; stochastic integration.

1 Introduction

Consider the stochastic sequence $\{Z_{n\cdot} : n \in \mathcal{N}\}$, defined on the probability space $(\Omega, \mathcal{A}, \mathcal{P})$, and suppose it is a **nonstationary** process, obeys the initial conditions $Z_{-j\cdot} = 0$, a.c. for $j \geq 0$, and is representable as

$$Z_{n\cdot} = \sum_{j=1}^{n} \eta_{j\cdot}, \tag{1}$$

where η is a suitably restricted process. If we define

$$\zeta_T = \frac{1}{T}\sum_{n=1}^{T} Z'_{n-1\cdot}\eta_{n\cdot}, \tag{2}$$

we are often interested in its limiting distribution. In the scalar case this derivation is rather straightforward since we may write

$$Z_n = Z_{n-1} + \eta_n, \quad Z_n^2 = Z_{n-1}^2 + 2Z_{n-1}\eta_n + \eta_n^2, \tag{3}$$

and thus

$$\sum_{n=1}^{T} Z_{n-1}\eta_n = \frac{1}{2}\sum_{n=1}^{T}\left(Z_n^2 - Z_{n-1}^2 - \eta_n^2\right) = \frac{1}{2}\left(Z_T^2 - \sum_{n=1}^{T}\eta_n^2\right). \quad (4)$$

Hence, under suitable conditions on the η-process so that a functional central limit theorem applies, we may easily conclude that **for scalar** ζ_T,

$$\zeta_T \xrightarrow{\mathrm{d}} \frac{1}{2}\left(\sigma_0^2 B(1)^2 - \sigma_1^2\right). \quad (5)$$

where B indicates the standard Brownian motion (SBM) on $[0,1]$, and σ_0^2, σ_1^2 are suitable parameters.

The simple derivation above, however, may not be employed in the case of multivariate (matrix) ζ_T. This problem was addressed in Phillips (1988), who obtained the limiting distribution with a general argument *de novo.* The purpose of this note is to demonstrate that the same procedure as above, **in conjunction with Ito's formula**, gives the desired result.

2 Formulation and Solution

Consider the stochastic sequence defined in Eq. (1) and let it be desired to obtain the limiting distribution of the matrix ζ_T defined in Eq. (2). We begin with a preliminary result

Proposition 1. Let $\eta = \{\eta_n : n \in \mathcal{N}_+\}$ be a sequence of random variables defined on the probability space $(\Omega, \mathcal{A}, \mathcal{P})$. Let $\mathcal{G}_m = \sigma(\eta_i,\ 1 \le i \le m)$, $\mathcal{G}^{n,k} = \sigma(\eta_i,\ m+k \le i \le n)$, and define the mixing coefficients

$$\alpha_n(k) = \sup_{m \le n-k} \alpha(\mathcal{G}_m,\ \mathcal{G}^{n,k}),$$

for $k \le n-1$, and zero otherwise. Further, define[1]

$$S_n = \sum_{i=1}^{n}\eta_i, \quad \alpha(k) = \sup_{n\in\mathcal{N}_+}\alpha_n(k), \quad (6)$$

$$X_t^n = \frac{S_{[nt]}}{\sqrt{n}}, \quad t \in [0,1]; \quad \text{assume} \quad (7)$$

$$E\eta_n = 0, \quad E\eta_n^2 < \infty, \quad n \in \mathcal{N}; \quad (8)$$

[1] Note that the notation $[a]$ means the integer part of a; thus $[nt]$ means the **largest** integer equal to or less than nt.

$$\frac{ES_n^2}{n} \to \sigma_0^2 > 0; \tag{9}$$

$$\infty > \sup_{m,n \in \mathcal{N}_+} \frac{E(S_{m+n} - S_m)^2}{n}, \tag{10}$$

and let $\beta \in (2, \infty]$, $\gamma = (2/\beta)$. If the conditions in Eqs. (8), (9), (10) are satisfied and, for a sequence $a = \{a_n : n \in \mathcal{N}_+,\ a_n \in [1, \infty]\}$,

$$\lim_{n \to \infty} \left[\sup_{i \le n} \| \eta_i \|_\beta^2 \left(\sum_{i \ge a_n} \alpha(i)^{1-\gamma} + \frac{a_n^{2-\gamma}}{n^{1-\gamma}} \right) \right] = 0,$$

then the processes X_t^n converge weakly to the standard BM, i.e.

$$X_t^n \xrightarrow{\mathrm{d}} \sqrt{\sigma_0^2} B_t, \quad t \in [0,1].$$

Proof: See Herrndorf (1984b).

The multivariate analog of this result proceeds by replacing the scalar random variable η_n, by the (row) vector $\eta_{n\cdot}$, Eqs. (8), (9) and (10) by

$$E\eta_{n\cdot} = 0, \quad E|\eta_{n\cdot}|^2 < \infty \tag{11}$$

$$E\frac{S_{n\cdot}' S_{n\cdot}}{n} \to \Sigma_0, \tag{12}$$

$$\infty > \sup_{m,n \in \mathcal{N}} \frac{E|S_{m+n\cdot} - S_{m\cdot}|^2}{n}, \tag{13}$$

and the conclusion by

$$X_t^n \xrightarrow{\mathrm{d}} B(t) P_0, \quad t \in [0,1], \tag{14}$$

where B is a standard multivariate (row vector) Brownian motion (SMBM) and $\Sigma_0 = P_0' P_0$, P_0 being the **triangular matrix** of the decomposition of Σ_0. To apply the result above to the case under consideration, suppose $\{\eta_{t\cdot},\ t \in \mathcal{N}\}$, obeys the conditions in Eqs. (11), (12), and (13) as well as the remaining conditions of Proposition 1. Noting that

$$Z_{t\cdot} = Z_{t-1\cdot} + \eta_{t\cdot}, \tag{15}$$

$$Z_{t\cdot}' Z_{t\cdot} = Z_{t-1\cdot}' Z_{t-1\cdot} + \eta_{t\cdot}' \eta_{t\cdot}$$

$$+ Z_{t-1\cdot}' \eta_{t\cdot} + \eta_{t\cdot}' Z_{t-1\cdot},$$

we determine

$$\begin{aligned}
\zeta_T' + \zeta_T &= \frac{1}{T}\sum_{t=1}^{T}\left(\eta_{t\cdot}' Z_{t-1\cdot} + Z_{t-1\cdot}'\eta_{t\cdot}\right) \\
&= \frac{1}{T}\sum_{t=1}^{T}\left(Z_{t\cdot}' Z_{t\cdot} - Z_{t-1\cdot}' Z_{t-1\cdot} - \eta_{t\cdot}'\eta_{t\cdot}\right) \\
&= \frac{1}{T}\left(Z_{T\cdot}' Z_{T\cdot} - \eta_{0\cdot}'\eta_{0\cdot} - \sum_{t=1}^{T}\eta_{t\cdot}'\eta_{t\cdot}\right).
\end{aligned}$$

Since $Z_{T\cdot} = Z_{0\cdot} + \sum_{t=1}^{T}\eta_{t\cdot}$ and in terms of the definition in Eq. (7) we are dealing with

$$\frac{1}{T}Z_{T\cdot}' Z_{T\cdot} = X_{1\cdot}^{'T} X_{1\cdot}^{T} \xrightarrow{\text{d}} P_0' B(1)' B(1) P_0, \tag{16}$$

by Proposition 1 above, and Proposition 28 and Corollary 5 in Dhrymes (1989) pp. 242-243. We may now prove

Proposition 2. In the context of Proposition 1,

$$\zeta_T = \frac{1}{T}\sum_{t=1}^{T} Z_{t-1\cdot}'\eta_{t\cdot} \xrightarrow{\text{d}} P_0'\left(\int_0^1 B(s)'\,dB(s)\right)P_0 + \Sigma_1^*, \;\; \Sigma_1^* = \lim_{T\to\infty}\frac{1}{T}\sum_{t<t'}^{T} E\eta_{t\cdot}'\eta_{t'\cdot}.$$

Proof: We first note that, given the initial conditions of the problem, and using the notation of Proposition 1, $Z_{T\cdot} = Z_{0\cdot} + S_{T\cdot} = S_{T\cdot}$,

$$S_{T\cdot} = \sum_{t=1}^{T}\eta_{t\cdot}; \tag{17}$$

thus,

$$\frac{1}{T}ES_{T\cdot}' S_{T\cdot} = \frac{1}{T}\sum_{t=1}^{T}E\eta_{t\cdot}'\eta_{t\cdot} + \frac{1}{T}\sum_{t<t'}^{T}E\eta_{t\cdot}'\eta_{t'\cdot} + \sum_{t>t'}^{T}E\eta_{t\cdot}'\eta_{t'} \rightarrow \Sigma + \Sigma_1^* + \Sigma_1^{*'}. \tag{18}$$

By the conditions of the proposition

$$\frac{1}{T}\sum_{t=1}^{T}\eta_{t\cdot}'\eta_{t\cdot} \xrightarrow{\text{a.c.}} \Sigma. \tag{19}$$

We therefore conclude from Proposition 1 that

$$\zeta_T' + \zeta_T \xrightarrow{\text{d}} P_0' B(1)' B(1) P_0 - \Sigma. \tag{20}$$

We recall that from the Multivariate Ito formula, see Dhrymes (1995), Proposition 5 in Chapter 4, we have for a twice differentiable function h

$$h(X_t) = h(X_0) + \sum_{j=1}^{q} \int_0^t h_i(X_s)\, dX_s^{(i)} + \frac{1}{2} \sum_{i,j=1}^{q} \int_0^t h_{ij}(X_s)\, d[X^{(i)}, X^{(j)}]_s \quad (21)$$

where $X^{(i)}$ indicates the i^{th} component of the vector X and the notation $[X^{(i)}, X^{(j)}]_t$ indicates quadratic (co)variation of the two components, on the interval $[0, t]$. If we take X to correspond to the q-element SMBM, B, and $h(B) = B_i B_j$ where B_i is the i^{th} component of the (row) vector B we find by Eq. (21)

$$B_i(t)B_j(t) = \int_0^t B_i(s)\, dB_j(s) + \int_0^t B_j(s)\, dB_i(s) + \delta_{ij} t,$$

where δ_{ij} is the Kronecker delta; this is so since the components of the SMBM are independent and thus their quadratic (co)variation vanishes. It follows, therefore, that

$$B(t)' B(t) = \int_0^t dB(s)'\, B(s) + \int_0^t B(s)'\, dB(s) + I_q t,$$

and for $t = 1$ we have

$$B(1)' B(1) = \int_0^1 dB(s)'\, B(s) + \int_0^1 B(s)'\, dB(s) + I_q.$$

From the preceding equation we immediately obtain

$$P_0' B(1)' B(1) P_0 - \Sigma = P_0' \left(\int_0^1 dB(s)'\, B(s) + \int_0^1 B(s)'\, dB(s) \right) P_0 + \Sigma_0 - \Sigma. \quad (22)$$

Thus, we conclude

$$\begin{aligned} \zeta_T' + \zeta_T &\xrightarrow{\mathrm{d}} P_0' B(1)' B(1) P_0 - \Sigma \\ &= P_0' \left(\int_0^1 dB(s)'\, B(s) \right) P_0 + \Sigma_1^{*\prime} + P_0' \left(\int_0^1 B(s)'\, dB(s) \right) P_0 + \Sigma_1^*, \\ \zeta_T &= P_0' \left(\int_0^1 B(s)'\, dB(s) \right) P_0 + \Sigma_1^* \end{aligned} \quad (23)$$

q.e.d.

Remark 1. It should be noted that while the conditions under which Proposition 1 is proved are rather complex, they allow for heterogeneity so long as the moments are governed by the last condition of the proposition. A similar result may be proved under somewhat more restrictive but simpler conditions. To this end, we have

Proposition 1a. Let ξ be a strictly stationary sequence and suppose

$$E|\xi_1|^{2+\delta} < \infty, \quad \text{for arbitrary } \delta > 0;$$

then ξ obeys a standard CLT.

If, in addition, the sequence is α-mixing and

$$\sum_{n=1}^{\infty} \alpha_n^{\delta/2+\delta} < \infty,$$

then ξ obeys a FCLT.

Proof: For the first part see Ibragimov and Linnik (1971); for the second part see Oodaira and Yoshihara (1972).

For ϕ-mixing (uniform mixing) sequences we have

Proposition 1b. Let ξ be a strictly stationary (uniform mixing) sequence and suppose in addition it obeys a **Lindeberg** condition, i.e.

$$\frac{1}{\sigma_n^2} \sum_{j=1}^{n} \int_{A_{jn}} |\xi_j(\omega)|^2 \, d\mathcal{P} = \frac{n}{\sigma_n^2} \int_{A_{1n}} |\xi_1|^2 d\mathcal{P} \rightarrow 0,$$

where $A_{jn} = \{\omega : |\xi_j(\omega)| \geq \epsilon \sigma_n\}$, for every $\epsilon > 0$. Then ξ obeys a FCLT.

Proof: See Peligrad (1985). Note, for example, that if the first (left) version of the condition above holds then ξ would obey a FCLT, even if it were **only covariance stationary** instead of **strictly** stationary.

A somewhat weaker form is given in Billingsley (1968), viz.

Proposition 1c. Suppose the sequence ξ satisfies

$$\sum_{n=1}^{\infty} \phi_n^{1/2} < \infty;$$

then the series

$$\sigma^2 = E\xi_1 + 2 \sum_{k=1}^{\infty} E\xi_1 \xi_{1+k}$$

converges absolutely and ξ obeys a FCLT.

Proof: See Billingsley (1968), pp. 174-177.

For ρ-mixing sequences, the strongest results pertain to **covariance stationary** sequences. We first give a result that pertains to the behavior of σ_n^2 and the spectral density of the process.

Proposition 1d. Let ξ be covariance stationary and ρ-mixing;

i. if $\sigma_n^2 \to \infty$ then $\sigma_n^2 = nh(n)$, where h is a slowly varying (positive) function on R_+;

ii. if $\sum_{j=1}^{\infty} \rho(2^j) < \infty$, ξ has a **continuous** spectral density, say f, and if $f(0) \neq 0$, then

$$\sigma_n^2 = 2\pi f(0)n[1 + o(1)].$$

Proof: See Ibragimov (1975), and Ibragimov and Rozanov (1978).

Convergence results are given in the two propositions below.

Proposition 1e. Let ξ be a ρ-mixing covariance stationary sequence:

i. if

$$\sigma_n^2 \to \infty \quad \text{and} \quad \sum_{j=1}^{\infty} \rho(2^j) < \infty$$

then ξ obeys a standard CLT;

ii. if, in addition,

$$\sum_{j=1}^{\infty} [\rho(2^j)]^{1/2} < \infty$$

then ξ obeys a FCLT.

Proof: For the proof of part i, see Ibragimov (1975); for part ii see Peligrad (1982).

Proposition 1f. Let ξ be strictly stationary, ρ-mixing and suppose

$$E|\xi_1|^r < \infty, \quad \text{for} \quad r > 2.$$

The following are true:

i. ξ obeys a FCLT;

ii. for every $k \in [1, r]$

$$E\left|\frac{S_n}{\sigma_n}\right|^k \rightarrow m_k$$

where m_k is the k^{th} absolute moment of the standard normal distribution.

Proof: See Ibragimov (1975) for part i; for part ii see Peligrad (1985).

For more extensive discussion of FCLT the reader may consult Dhrymes (1995), Chapter 5.

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