MATHEMATICAL MODELING OF INSIDER TRADING

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ABSTRACT

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In this thesis, we study insider trading and consider a financial market and an enlarged financial market whose sets of information are respectively represented by the filtrations $\mathbb{F}$ and $\mathbb{G}$. The filtration $\mathbb{G}$ is obtained by initially expanding the filtration $\mathbb{F}$. We also consider that we have a finite trading horizon. First, we show that under certain conditions the enlarged market satisfies no free lunch with vanishing risk (NFLVR) locally and therefore satisfies no arbitrage with respect to admissible simple predictable trading strategies. In addition, we generalize the structure of all the $\mathbb{G}-$ local martingale deflators and find sufficient conditions under which the enlarged market satisfies NFLVR. We apply our results to some recent examples of insider trading that have appeared in newspapers and by doing so, show the limitations of some previous works that have studied the stability of the NFLVR property under an initial expansion.

Second, assuming the enlarged market satisfies NFLVR and markets are incomplete, we define a notion of risk and compare the risk of a market or liquidity trader to the risk of an insider trader. We prove that the risk of an insider is smaller than the risk of a market/liquidity trader under some sufficient conditions that involve their respective trading strategies. We find a relationship between the trading strategies of a market trader and of an insider when the risk neutral measure of the market is used. If an insider trades using the market risk neutral measure and not her own, then her trading strategy should involve not only the stock but also the volatility of the stock.

Finally, assuming that the enlarged market satisfies NFLVR locally, we provide a way for an insider to price her financial claims. We also define a new type of process that we call a quasi-local martingale and prove that the stock price process under local NFLVR is one such process.

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To my parents

Chapter 1

Introduction

Insider trading is a term most individuals and investors have heard but it actually encompasses both legal and illegal activities. The legal version is when corporate insiders buy or sell the stock of the company they work for. Illegal insider trading is the type of misconduct that most people think of when they hear about insider trading. According to the Securities and Exchange Commission (SEC), illegal insider trading refers generally to *buying and selling a security, in breach of a fiduciary duty or other relationship of trust and confidence, while in possession of material, nonpublic information about the security. Insider trading violations may also include "tipping"such information, securities trading by the person "tipped,"and securities trading by those who misappropriate such information.*
According to Wang and Steinberg [63], insider trading was not a regulatory violation forty years ago. In fact, the accepted wisdom on Wall Street was "the only way to make money is to trade on inside information."Nowadays, insider trading has become a global phenomenom with constant accusations of insider trading proliferating in our newspapers (some recent examples include but are not limited to Protess and Goldstein [51], [28], [29], Sorkin [60], Thomas [62], Stewart [59]). Because such behavior undermines public and investors' confidence in the fairness and integrity of the securities markets (see Thomas Newkirk's speech on the U.S. perspective of insider trading [46]), the SEC has the detection and prosecution of insider trading violations as one of its enforcement priorities.
Unfortunately enforcement of the law on insider trading can be difficult for the law itself,

being complex, can be interpreted in mulitple ways by different judges. Moreover, the definition of illegal insider trading may appear blurry and may allow some investors to skirt with the law while remaining in perfect legality. For instance, according to the New York Times (see Sorkin [60]), a novel tactic was used by Mr. William A. Ackman and Mr. J. Michael Pearson in Mr. Ackman's purchase of Allergan (a botox manufacturer) shares ahead of a takeover offer from Valeant Pharmaceuticals (where Mr. Pearson is the CEO). Indeed, Mr. Pearson directly called Mr. Ackman to let him know about his company intent. According to the same New York Times (NYT) article, Mr. Pearson was seeking a parnership in which Mr. Ackman would build a considerable position in Allergan stock and help Valeant press Allergan's board to accept Valeant's deal. Another instance is when the winning streak of Preet Bharara, the U.S. Attorney for the Southern District of New York came to an end (see Stewart's article in the NYT [59]). Indeed Mr. Rengan Rajaratnam, whose brother Raj Rajaratnam was convicted of insider trading, was acquitted of conspiracy to commit securities fraud (Stewart [59]). Allegedly, once Mr. Raj Rajaratnam learned that Intel would be making an important investment in Clearwire (a telecommunication provider), Mr. Rengan Rajaratnam subsequently bought stock of Clearwire for his own account as well as Galleon's (Mr. Raj Rajaratnam's defunct hedge fund) account.

Some of the different SEC's rules addressing illegal insider trading appear somewhat contradictory. For instance in Sorkin's NYT article [60], Mr. Pearson may not have breached a fiduciary duty to Valeant by sharing information about the planned takevoer bid with Mr. Ackman since such an approach is the content of the SEC rule 10b-5. Although, another SEC rule makes it illegal to share information before a tender offer. Complicating the matter, SEC rule 14e-3 says "If any person has taken a substantial step or steps to commence, or has commenced, a tender offer, it will be considered a fraudulent, deceptive or manipulative act for any officer, director, partner or employee or any other person acting on behalf of the offering person or such issuer, to purchase or sell stock."

In such an environment, the fact that the U.S. Attorney office for the Southern District of New York under the supervision of Preet Bharara has racked up a record of eighty-five convictions for one defeat as of July 18, 2014 [59], without even considering the numerous dismissed cases, not only attest to how serious insider trading is in financial markets but also shows how complex and intricated illegal insider trading laws can get. As a consequence, the importance of studying and understanding insider trading has increased over the years. Insider trading can also be defined as the study of market agents having asymmetrical information and investing in the same financial market. One of the first modern attempts to study insider trading within a mathematical framework was by Kyle [43]. Indeed, Kyle, in a financial market with three traders : an informed trader, a market maker, and a noise trader, derived the trading strategy of the informed trader, studied how quickly private information is incoporated into market prices and the effect of noise traders on the volatility of prices. Starting with Kyle's work [43], the interest of researchers in insider trading has grown with a series of papers by Back [6], [7] who gave a continuous time formulation of Kyle's model. Pierre Collin-Dufresne [17] went a step further than Back and allowed the volatility of the noise traders to change stochastically. Such interest has seen a reinforcement from the Mathematical Finance community as surveyed by Wu [64] as of 1999; for a more recent comprehensive bibliography, see the thesis of Aksamit [1] . The insider trading literature deals with two agents: a regular trader and an informed trader or an insider trader who possesses additional information.

In the Mathematical Finance community, the modeling technique used for insider trading is the theory of the expansion of filtrations. The subject of expansion of filtrations began with Itô's seminal paper in 1978 and refers to expanding a filtration $\mathbb{F}$ to get a new filtration $\mathbb{G}$ such that $\mathcal{F}_t \subset \mathcal{G}_t$ for each $t \geq 0$ and martingales remain semimartingales in the larger filtration. It is assumed that the filtrations, $\mathbb{F}$ and $\mathbb{G}$, satisfy the "usual conditions". There are two ways to expand a filtration: an initial expansion and a progressive enlargement.

In the initial expansion framework, the underlying filtration $\mathbb{F}$ is enlarged by the information about some random variable. In the context of insider trading, initial expansion models situations where the insider gets some information at the beginning of the trading interval or at some random time in the trading horizon. The theory of initial expansions was developed in a flurry of papers by Jacod [33], Chaleyat-Maurel & Jeulin [12], Mansuy & Yor [44], Yor [65], Jeulin & Yor [38]. Progressive expansion corresponds to gradually adding a positive random variable to the underlying filtration $\mathbb{F}$. As a consequence, progressive enlargement turns a random time into a stopping time. Developed in a series of papers by

Barlow [8], Jeulin [37], Jeulin & Yor [35], Mansuy & Yor [44], progressive expansion models situations where the extra information the insider trader has comes from a continuous flow of knowledge.

The theory of the expansion of filtrations, as a technique to model insider trading, has already attracted the attention of the mathematical finance community. Indeed, besides insider trading, the techniques of expansion of filtrations can also be used to study the pricing of defaultable claims, to model credit risk and default times, see Coculescu [14], Coculescu et al. [15], and Coculescu and Nikeghbali [16]. In the settings of insider trading, Imkeller [32] used a progressive expansion model with two types of agents, a regular trader and an insider who invests in one risky asset, to model insider trading. He assumed markets are complete and considered continuous time financial models. The insider's additional knowledge is his or her ability to stop at a honest time, which is inaccessible to the regular trader. Examples of random times considered are the last passage times at which a Brownian motion or a one-dimensional diffusion crosses a certain level. Under assumptions on the coefficients of the diffusion describing the price process of the risky asset, the additional drift added by the inside information cannot be removed by an equivalent change of probability measure. As a consequence, Imkeller proved that insiders have free lunches immediately after the honest time, that is right after the knowledge becomes fully available to him or her. Zwierz [66] extended Imkeller's results [32] to any continuous local martingale and general honest times.

Remaining in the context of additional information generated by an honest time, and using Nikeghbali & Yor [48] on the multiplicative decomposition of the Azéma supermartingale, Fontana et al. [27] analyzed the different notions of arbitrage, *No Arbitrage* (NA), *No Arbitrage of the First Kind* (NA1), *No Free Lunch with Vanishing Risk* (NFLVR) and *No Unbounded Increasing Profit* (NUIP), that can be attained by an insider. The notion of Unbounded Increasing Profit has been introduced by Karatzas and Kardaras [40] and represents the strongest possible notion of arbitrage. The notion of Arbitrage of the First Kind is due to Kardaras [41] and has also appeared under the name No Unbounded Profit with Bounded Risk (NUPBR) in Karatzas and Kardaras [40] and Delbaen and Schachermayer [21]. Going further than Imkeller [32] and Zwierz [66], Fontana et al. [27] not only distinguished what types of arbitrage can be realized before, at and after the honest time, but also exhibited the arbitrage strategy.

In the present thesis, we consider continuous time financial models, a finite trading horizon and model insider trading by initially expanding the filtration of the regular trader at a random time. We think this is a better representation of reality than what a progressive expansion model implies. In a progressive enlargement model, the insider knows a random time which in papers by Imkeller [32], Zwierz [66], Fontana et al. [27] is related to the path of the risky asset. Such knowledge by an insider is extremely unlikely. The kinds of additional knowledge we consider are exogenous or endogenous information that might affect the risky asset's price process but do not depend explicitly on its path properties. Additionally, with our approach, we do not need the markets to be complete. Considering three kinds of arbitrage, NA1 (also called NUPBR), NA and NFLVR, our aim is to find conditions under which an insider trader does not have arbitrage and/or a free lunch. In case the insider does not have free lunches, how does her risk neutral measure differ from the regular trader's risk neutral measure? Moreover, we would also like to understand how the risk of an insider compares to the risk of a liquidity or market trader and how an insider who does not have a risk neutral probability measure can price his or her financial instruments. In the case when an informed trader's financial market does not satisfy NFLVR but does satisfy a localized version of NFLVR, we have exhibited a new type of process that we called quasi-local martingales and studied some of its properties.

A work that is related to ours is by Amendinger [3] who wanted to know whether it is possible to show that financial markets are free of arbitrage and complete for the insider under a suitable assumption on the inside information. The suitable assumption found by Amendinger [3] is that the conditional distribution of the extra knowledge with respect to the regular trader's current set of information is equivalent to the distribution of the additional knowledge. Although such an assumption led to all $\mathbb{F}-$ local martingales to be $\mathbb{G}-$ local martingales, it is not practically realistic. Moreover, the author limited himself to complete markets. Grorud & Pontier [30] and Baudoin [9] have used initial expansion to study the possibility of realizing arbitrage opportunities on a finite trading horizon $[0, T]$ but all the arbitrage results are obtained on $[0, T)$. Pikovsky & Karatzas [50] have also used an initial expansion to maximize the logarithmic utility of an informed trader using portfolios that were allowed to anticipate on the future, where the future was represented by the additional knowledge of such an informed investor. The extra information considered by Pikovsky and Karatzas was the terminal value of the prices which were either known exactly or with some uncertainty. Additionally, Aksamit [1] in her PhD thesis studied the NUPBR stability with respect to an initial enlargement of filtration under Jacod's hypothesis. Under conditions less restrictive than Amendinger [3], Aksamit formulates a necessary and sufficient condition such that each parameterized $\mathbb{F}-$ local martingale satisfies NUPBR in $\mathbb{G}$. Unfortunately, the risky asset price process is not a parameterized $\mathbb{F}-$ local martingale. Hence, to the best of our knowledge, we are the first to examine conditions for no arbitrage and free lunches on the whole trading horizon $[0, T]$ under mild but realistic and practical conditions.

This thesis consists of two main parts: The first part focuses on understanding insider trading, on the absence of different types of arbitrage in enlarged filtrations and in applying our results to some of the recent insider trading cases investigated by Preet Bharara, the U.S. Attorney for the Southern District of New York. The second part focusses on comparing the market trader's and insider's risk in a Föllmer-Schweizer sense. We also study how we can define a no arbitrage price in case the insider only satisfies NFLVR locally but not globally. In both parts, we are interested in continuous cases, i.e. cases where local martingales are continuous. It is also important to note that results of the first part can also be applied to progressive expansion when we study arbitrages up to the honest time used to expand the market's filtration.

The different chapters of this thesis are based on the initial expansion of the market filtration. This thesis comprises the following chapters:

Chapter 2: Background and general theory

This chapter recalls results that will be useful for the other chapters of this thesis. We don't provide proofs of these well-known results but set up most of the notation used thereafter.

Chapter 3: Expansion of filtrations and no free lunches

In this chapter, we specify the type of expansion of filtrations we use and our modeling assumptions. Then we study conditions under which the financial market of an insider satisfies NFLVR. We proceed using two approaches: first a direct approach where we exhibit a measure and find conditions under which such a measure is a *bona fide* probability measure. Unfortunately one such conditions is uniform integrability but the case of the three-dimensional Bessel process sheds light into the possibility for an insider to have no arbitrage with respect to a specific type of trading strategies even though his or her market might have arbitrage opportunities if a more general class of trading strategies is allowed. Secondly, we study NA1 and find another set of conditions under which the insider's market satisfies NFLVR. We end the chapter by presenting some examples showing how our results can be applied, and we show, for instance, the limitations of Amendinger's results [3].

Chapter 4: Expansion of filtrations, risk and pricing

In the first section of this chapter, we start off by defining a notion of risk and, assuming the insider's market satisfies NFLVR, we compare the insider's risk to the market/liquidity trader's risk. In the second section, under the assumption that the insider's market doesn't satisfy NFLVR but a localized version of NFLVR, we define what the "optimal" price of a financial claim is for an informed trader. In the same section recalling results from Chapter 3, we exhibit a process that is locally a local martingale with respect to a certain sequence of probability measures. We call those processes quasi-local martingales. Then in the same spirit as in Ruf [56], we extend quasi-local martingale to the whole trading horizon and obtain a Kunita-Watanabe type of decomposition.

Chapter 2

Background and general theory

2.1 Introduction

In this chapter, we introduce well-known results from the general theory of stochastic processes especially as they pertain to the theory of enlargement of filtrations, Kunita-Watanabe and Föllmer-Schweizer decompositions.

We first specifically recall results of the semimartingale theory and the theory of predictable and optional projections before we introduce the theory of enlargement of filtrations. Proofs of all theorems, Corollaries, lemmas and propositions presented in this chapter can be found in Dellacherie [22], Dellacherie and Meyer [23], He, Wang and Yan [31], Karatzas and Shreve [39], Protter [52], and Rogers and Williams [54], [55].

As usual we start with a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ where $\mathbb{F} = (\mathcal{F}_t, t \geq 0)$ is a given filtration satisfying the "usual conditions", and $\mathcal{F} := \mathcal{F}_\infty$.

2.2 Theory of stochastic processes

2.2.1 Adapted stochastic processes

A stochastic process X is a family of random variables such that $(\omega, t) \to X_t(\omega)$ is $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+) -$ measurable, where $\mathcal{B}(\mathbb{R}_+)$ is the Borel $\sigma -$ field on $\mathbb{R}_+$.

Definition 2.2.1. Let X and A be two stochastic processes

- X is continuous if for almost all ω, the map $t \rightarrow X_t(\omega)$ is continuous. X is càdlàg or RCLL(continu à droite, limite à gauche or right continuous with left limits) if for almost all ω the map $t \rightarrow X_t(\omega)$ is càdlàg.

- A is increasing if $A_0 = 0$, A is right continuous and $A_s \leq A_t,\ P-$a.s. for $0 \leq s \leq t$. An increasing process A is integrable if $E_P\left(A_\infty\right) < \infty$. A is called a finite variation process (FV) if almost all of the paths of A are of finite variation on each compact interval of $\mathbb{R}_+$.

We almost always consider that processes are càdlàg unless stated otherwise.

Definition 2.2.2. A stochastic process X is $\mathbb{F}-$ adapted if for any $t \geq 0$, the random variable X_t is $\mathcal{F}_t-$ measurable.

The natural filtration $\mathbb{F}^X$ loosely defined as $\mathbb{F}^X := \sigma\left(X_s; s \leq t\right)$ is the smallest filtration satisfying the "usual conditions" to which the process X is adapted.

2.2.2 Stopping times

A nonnegative random variable $T : \Omega \rightarrow [0, \infty]$ is an $\mathbb{F}-$ stopping time if for any $t \geq 0$, the event $\{T \leq t\}$ is in $\mathcal{F}_t$. We can classify stopping times in three main categories: predictable times, accessible and totally inaccessible times. The most important ones are predictable times and totally inaccessible times.

Definition 2.2.3. A stopping time T is predictable if there exists a sequence of stopping times $\left(T_n\right)_{n\geq 1}$ such that T_n is increasing, $T_n < T$ on $\{T > 0\}$, for all n, and $\lim_{n\rightarrow\infty} T_n = T,$ P-a.s..

The sequence $\left(T_n\right)_{n\geq 1}$ is said to announce T. If X is an adapted continuous process, with $X_0 = 0$, and $T = \inf\{t > 0 : |X_t| \geq c\}$ for some $c > 0$, then T is predictable.

Definition 2.2.4. A stopping time T is accessible if there exists a sequence of predictable stopping times $\left(T_n\right)_{n\geq 1}$ such that

$$P\left(\bigcup_{n=1}^{\infty}\{\omega : T_n(\omega) = T(\omega) < \infty\}\right) \quad = \quad P\left(T < \infty\right)$$

The sequence $(T_n)_{n\geq 1}$ is said to envelop T. Any stopping time that takes on a countable number of values is accessible. Any jump time of a Lévy process is not accessible.

Definition 2.2.5. A stopping time T is totally inaccessible if for every predictable stopping time S,

$$P\{\omega : T(\omega) = S(\omega) < \infty\} \quad = \quad 0.$$

Definition 2.2.6. Let T be a stopping time. The graph of the stopping time T is the subset of $\mathbb{R}_+ \times \Omega$ given by $\{(t, \omega) \,:\, 0 \leq t = T(\omega) < \infty\}$; The graph of T is denoted by $[T]$.

We will need a complement of the concept of a stopping time $\sigma -$ field.

Definition 2.2.7. Let T be a stopping time

- The $\sigma -$ field $\mathcal{F}_{T-}$ of events strictly prior to T is the smallest $\sigma -$ field containing $\mathcal{F}_0$ and all sets of the form $A \cap \{t < T\}$, $t > 0$ and $A \in \mathcal{F}_t$.
- The $\sigma -$ field $\mathcal{F}_T$ of events prior to T is the smallest $\sigma -$ field containing $\mathcal{F}_0$ and all sets of the form $A \cap \{T \leq t\}$, $t > 0$ and $A \in \mathcal{F}_t$.

Of course $\mathcal{F}_{T-} \subset \mathcal{F}_T$ and the stopping time T is $\mathcal{F}_{T-} -$ measurable. Using a monotone class theorem argument, we obtain the following theorem

Theorem 2.2.1. *Let T be a stopping time. Then,*

$$\begin{aligned} \mathcal{F}_{T-} &= \sigma\{H_T\,;\, H \text{ predictable}\} \\ \mathcal{F}_T &= \sigma\{H_T\,;\, H \text{ adapted càdlàg process}\}. \end{aligned}$$

Predictable processes will be defined below.

Theorem 2.2.2. *Let X be a predictable process and T a stopping time. Then $X_T \in \mathcal{F}_{T-}$.*

2.2.3 Predictable and optional $\sigma -$ fields

In this section we work on the space $(\Omega \times \mathbb{R}_+, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+), P \times \lambda)$ where λ is the Lebesgue measure on $\mathbb{R}_+$. The goal is to introduce the notion of predictable and optional $\sigma -$ algebras and processes on $\Omega \times \mathbb{R}_+$.

Definition 2.2.8. The predictable $\sigma-$ algebra $\mathcal{P}(\mathbb{F})$ on $\Omega \times \mathbb{R}_+$ is the smallest $\sigma-$ algebra making all processes in $\mathbb{L}$ (left continuous processes) measurable. A process $H \in \mathcal{P}(\mathbb{F})$ (respectively $\mathbf{b}\mathcal{P}(\mathbb{F})$) is called a predictable process (respectively bounded predictable process).

Definition 2.2.9. The optional $\sigma-$ algebra $\mathcal{O}(\mathbb{F})$ on $\Omega \times \mathbb{R}_+$ is the smallest $\sigma-$ algebra making all càdlàg, adapted processes measurable. We let $\mathcal{O}(\mathbb{F})$ (respectively $\mathbf{b}\mathcal{O}(\mathbb{F})$) denote the processes (respectively bounded processes) that are optional.

Although we will not use them, the following two definitions are often found in the literature.

Definition 2.2.10. The measurable $\sigma-$ algebra $\mathcal{M}(\mathbb{F})$ on $\Omega \times \mathbb{R}_+$ is $\mathcal{M}(\mathbb{F}) := \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$.

Definition 2.2.11. A progressive process X on $\Omega \times \mathbb{R}_+$ is a process such that for each $t \in \mathbb{R}_+$ the mapping $(s, \omega) \to X_s(\omega)$ of $[0,t] \times \Omega$ into $\mathbb{R}$ is measurable with respect of $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$. The progressive $\sigma-$ algebra $\mathcal{A}$ on $\mathbb{R}_+ \times \Omega$ is the smallest $\sigma-$ algebra that makes all progressive processes measurable.

In general, one has the following relationships:

$$\mathcal{P}(\mathbb{F}) \subset \mathcal{O}(\mathbb{F}) \subset \mathcal{A}(\mathbb{F}) \subset \mathcal{M}(\mathbb{F})$$

Proposition 2.2.3. *Let $\mathbb{F}$ be the filtration*

- *The predictable $\sigma-$ algebra $\mathcal{O}$ is the $\sigma-$ algebra generated by the stochastic intervals $]S,T]$ where S and T are two $\mathbb{F}-$ stopping times such that $S \leq T,\ P-\text{a.s.}$.*
- *The optional $\sigma-$ algebra $\mathcal{O}$ is the $\sigma-$ algebra generated by the stochastic intervals $[T, \infty[$ where T is an $\mathbb{F}-$ stopping time.*

If there is no confusion about the filtration with respect to which processes are predictably measurable or optional, we shall often write $\mathcal{P}$ for $\mathcal{P}(\mathbb{F})$ and $\mathcal{O}$ for $\mathcal{O}(\mathbb{F})$. $\mathcal{O} = \mathcal{P}$ if and only if any $\mathbb{F}-$ stopping time is predictable. In general, Protter [52], see Revuz and Yor [53],

$$\mathcal{O} \;=\; \mathcal{P} \vee \sigma\left\{\Delta M\,;\, M \in \text{set of all } \mathbb{F} \text{ martingales}\right\}$$

Martingales will be defined below.

2.2.4 Martingales, local martingales and semimartingales

In this section, we give only the essential results from the theory of continuous time martingales, local martingales and semimartingales.

Definition 2.2.12. A real-valued, adapted process $X = (X_t)_{t\geq 0}$ is called an $\mathbb{F}-$ martingale (respectively supermartingale, submartingale) if

i) $E_P\left\{|X_t|\right\} < \infty,$ for all $t \geq 0$

ii) if $0 \leq s \leq t$, then $E_P\left\{X_t \,|\, \mathcal{F}_s\right\} = X_s,\ P-$a.s. (respectively $E_P\left\{X_t \,|\, \mathcal{F}_s\right\} \leq X_s,\ P-$a.s., respectively $\leq X_s,\ P-$ a.s.)

Lemma 2.2.4. *If X is an $\mathbb{F}-$ martingale then there exists a unique modification Y of X which is càdlàg.*

Since all martingales have right continuous modifications, we will always assume that we are taking the right continuous version. A right continuous martingale can be proved to be actually càdlàg a.s..

Definition 2.2.13. An adapted càdlàg process X is an $\mathbb{F}-$ local martingale if there exists an increasing sequence of stopping times T_n, with $\lim_{n\to\infty} T_n = \infty,\ P-$ a.s. such that $X^{T_n}1_{\{T_n>0\}}$ is an $\mathbb{F}-$ martingale for each n. Such a sequence $(T_n)_{n\geq 1}$ reduces or localizes X and is called the fundamental sequence.

We will need the following definition:

Definition 2.2.14. Let X be a stochastic process. A property π is said to hold locally if it holds up to a stopping time T_n, for each n, where the sequence $(T_n)_{n\geq 1}$ is stricIty increasing $P-$a.s., to a limit T which could be finite $P-$a.s. or take on infinity as a value of a set of positive probability.

Note that a process which is locally a local martingale is also a local martingale. Hence using localization, we define locally bounded, locally square integrable, locally integrable and locally FV processes. Moreover, if X is an $\mathbb{F}-$ local martingale, it is possible to choose the localizing sequence $(T_n)_{n\geq 1}$ such that $X^{T_n}1_{\{T_n>0\}}$ is uniformly integrable.
We denote by

- $\mathcal{M}_{loc}(\mathbb{F}, P)$ the space of all $(\mathbb{F}, P)-$ local martingales;
- $\mathcal{M}^c_{loc}(\mathbb{F}, P)$ the space of all continuous $(\mathbb{F}, P)-$ local martingales;
- $\mathcal{M}^2_{loc}(\mathbb{F}, P)$ the space of all square - integrable $(\mathbb{F}, P)-$ local martingales;
- $\mathcal{M}_{0,loc}(\mathbb{F}, P)$ the space of all $(\mathbb{F}, P)-$ local martingales starting at 0;
- $\mathcal{M}^c_{0,loc}(\mathbb{F}, P)$ the space of all continuous $(\mathbb{F}, P)-$ local martingales starting at 0;
- $\mathcal{M}^2_{0,loc}(\mathbb{F}, P)$ the space of all square - integrable $(\mathbb{F}, P)-$ local martingales starting at 0;

It is often of interest to determine when a local martingale is a actually a martingale. Let $X^*_t = \sup_{s \leq t} |X_s|$ and $X^* = \sup_t |X_t|$

Theorem 2.2.5. *Let X be an $(\mathbb{F}, P)-$ local martingale such that $E_P(X^*_t) < \infty$ for every $t \geq 0$. Then X is a martingale. If $E_P(X^*) < \infty$, then X is a uniformly integrable martingale.*

Theorem 2.2.6. *Let X be an $(\mathbb{F}, P)-$ local martingale. Then X is an $(\mathbb{F}, P)-$ martingale with $E_P\left\{X^2_t\right\} < \infty$, for all $t \geq 0$, if and only if $E_P\left\{[X, X]_t\right\} < \infty$, for all $t \geq 0$. If $E_P\left\{[X, X]_t\right\} < \infty$, then $E_P\left\{X^2_t\right\} = E_P\left\{[X, X]_t\right\}$.*

Using Burkholder-Davis-Gundy inequalities (see Protter [52], chapter IV page 195), it is enough in Theorem 2.2.6 to have $E_P\left\{[X, X]^{\frac{1}{2}}_t\right\} < \infty$, for all $t \geq 0$.
A nonnegative local martingale is a supermartingale. A local martingale that is not a martingale is called a strict local martingale.
It is also important to know when the class of local martingales is stable under stochastic integration.

Theorem 2.2.7. *Let X be an $(\mathbb{F}, P)-$ local martingale and $H \in \mathbb{L}$. Then the stochastic integral $H \cdot X$ is again an $(\mathbb{F}, P)-$ local martingale.*

Theorem 2.2.8. *Let X be a locally square integrable $(\mathbb{F}, P)-$ local martingale and $H \in \mathcal{P}(\mathbb{F})$. Then the stochastic integral $H \cdot X$ exists and is a locallu square integrable local martingale if there exists a sequence of stopping times $(T_n)_{n \geq 1}$ increasing to ∞ $P-$a.s. such that $E_P\left\{\int_0^{T_n} H^2_s d\,[X, X]_s\right\} < \infty$.*

Theorem 2.2.9. *Let X be an $(\mathbb{F}, P)-$ local martingale and let $H \in \mathcal{P}(\mathbb{F})$ be locally bounded,, then the stochastic integral $H \cdot X$ is an $(\mathbb{F}, P)-$ local martingale.*

Theorem 2.2.10. *Let X be a continuous $(\mathbb{F}, P)-$ local martingale and let $H \in \mathcal{P}(\mathbb{F})$ be such that $\int_0^t H_s^2 d\left[X, X\right]_s < \infty,\ P-\text{a.s.}$, for each $t \geq 0$. Then the stochastic integral $H \cdot X$ exists and it is a continuous $(\mathbb{F}, P)-$ local martingale.*

In chapters II and III of Protter [52], semimartingales are defined as good integrators for the theory of stochastic integration.

Definition 2.2.15. An adapted càdlàg process X is a classical semimartingale if there exist processes N, B with $N_0 = B_0 = 0$ such that

$$X_t = X_0 + N_t + B_t$$

where N is a local martingale and B is a FV process.

A classical semimartingale is a semimartingale as defined in chapter II of Protter [52]. A càdlàg local martingale is also a semimartingale.

Theorem 2.2.11. *Let X be a semimartingale. If X has a decomposition $X_t = X_0 + M_t + A_t$ with M a local martingale and A a predictably measurable FV process, $M_0 = A_0 = 0$, then such a decomposition is unique. X is said to be a special semimartingale.*

The following criteria to determine whether a semimartingale is a special semimartingale are useful.

Theorem 2.2.12. *Let X be a semimartingale. X is special if and only if one of the following holds*

- *The process $J_t = \sup_{s \leq t} |\Delta X_s|$ is locally integrable*
- *The process $X_t^* = \sup_{s \leq t} |X_s|$ is locally integrable*

2.2.5 Projection theorems and dual projections

2.2.5.1 Projection theorems

In this section, we introduce the notion of predictable and optional projection of stochastic processes which is related to the notion of filtration shrinkage. We also introduce dual projections which lead to the notion of predictable compensators. Filtration shrinkage involves two filtrations, $\mathbb{F}$ and $\mathbb{G}$, satisfying the "usual conditions" such that $\mathbb{F} \subset \mathbb{G}$.

Theorem 2.2.13 (Stricker's theorem)**.** *Let X be a semimartingale for the $\mathbb{G}$ filtration. If X is adapted to $\mathbb{F}$, then X is a $\mathbb{F}-$ semimartingale.*

What if the subfiltration $\mathbb{F}$ is so small that X is not adapted to it? Then we make X adapted by projecting it onto the subfiltration using the notions defined next.

Definition 2.2.16. Let X be a measurable process which is either bounded or positive or such that for any $\mathbb{G}-$ stopping time S, $X_S 1_{\{S<\infty\}}$ is integrable, then there exists a unique optional process ${}^oX$ such that for any $\mathbb{G}-$ stopping times T one has

$$E_P\left\{X_T 1_{\{T<\infty\}}\right\} = E_P\left\{{}^oX_T 1_{\{T<\infty\}}\right\}.$$

The process ${}^oX$ is called the $\mathbb{F}-$ optional projection of X.

From the optional projection, it follows that for each stopping time T, we have

$${}^oX_T = E_P\left\{X_T \,|\, \mathcal{F}_T\right\}, \; P-\text{a.s. on } \{T<\infty\}$$

Definition 2.2.17. Let X be a measurable process which is either bounded or positive or such that for any predictable $\mathbb{G}-$ stopping time S, $X_S 1_{\{S<\infty\}}$ is integrable, then there exists a unique predictable process ${}^pX$ such that for any $\mathbb{G}-$ stopping times T one has

$$E_P\left\{X_T 1_{\{T<\infty\}}\right\} = E_P\left\{{}^pX_T 1_{\{T<\infty\}}\right\}.$$

The process ${}^pX$ is called the $\mathbb{F}-$ predictable projection of X.

From the predictable projection, it follows that for each predictable stopping time T, we have

$${}^pX_T = E_P\left\{X_T \,|\, \mathcal{F}_{T-}\right\}, \; P-\text{a.s. on } \{T<\infty\}$$

Proposition 2.2.14. *Let X be a measurable process and Y an optional (respectively predictable) process. If the optional (respectively predictable) projection of X exists, then the optional (respectively predictable) projection of XY exists and iis given by*

$$\begin{aligned} {}^o(XY) &= {}^o(X)\,Y \\ {}^p(XY) &= {}^p(X)\,Y \end{aligned}$$

Proposition 2.2.15. *Let X be a measurable process. If the $\mathbb{F}-$ optional and predictable projections of X exist then ${}^{p}\left({}^{o}X\right) = {}^{p}X$.*

In the special case of martingales and local martingales, the paper of Föllmer and Protter [24] developed a general theory for the projection of martingales and related processes onto smaller filtrations, to which they are not adapted. They also found conditions under which local martingales retain their nature.

Theorem 2.2.16. *Let X be a martingale for the $\mathbb{G}$ filtration. Then the optional projection of X onto $\mathbb{F}$ where $\mathbb{F} \subset \mathbb{G}$ is again a martingale for the $\mathbb{F}$ filtration.*

In the case of local martingales, we have the following theorem.

Theorem 2.2.17. *Let X be a local martingale for the $\mathbb{G}$ filtration and let ${}^{o}X$ denote the optional projection of X onto the subfiltration $\mathbb{F}$. ${}^{o}X$ is a local martingale for $\mathbb{F}$ if there exists a sequence of reducing stopping times $(T_n)_{n\geq 1}$ for X in $\mathbb{G}$ which are also stopping times in $\mathbb{F}$. Conversely, if X is positive and ${}^{o}X$ is a local martingale for $\mathbb{F}$, then a reducing sequence of stopping times for ${}^{o}X$ in $\mathbb{F}$ is also a reducing sequence for X in $\mathbb{G}$.*

Föllmer and Protter [24] studied the inverse $3-$ dimensional Bessel process starting at 1 and its optional projections onto the subfiltrations generated by one and two of the three Brownian components. The inverse $3-$ dimensional Bessel process is in fact a strict local martingale. Föllmer and Protter [24] proved that the optional projection of the inverse Bessel process onto the filtration of one of the Brownian motions is a supermartingale but it is not a local martingale while the optional projection onto the filtration generated by two of the three Brownian motions is a local martingale.

2.2.5.2 Dual projections

In this sectiom we work on either $(\Omega, \mathcal{F}, \mathbb{P}, P)$ or $(\Omega \times \mathbb{R}_+, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+), P \times \lambda)$.
Let A be a (non-adapted) integrable[1] FV process. For any bounded measurable process X, we define the following bounded signed measure on $(\Omega \times \mathbb{R}_+, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+))$

$$\mu_A(X) \quad = \quad E_P\left[\int_0^\infty X(t,\omega)dA(t,\omega)\right].$$

[1] A FV process A is integrable if $E_P\left\{\lim_{t\nearrow\infty}\int_0^t |dA(s,\omega)|\right\} < \infty$

$\mu_A(X)$ is a positive measure if A is an increasing processs.

Theorem 2.2.18 (Doléans)**.** *Suppose that A is an (non-adapted) integrable FV process*

(i) Then A is optional if and only if $\mu_A(X)$ communtes with the optional projection in the sense that

$$E_P\left[\int_0^\infty X\,dA\right] = E_P\left[\int_0^\infty {}^oX\,dA\right]$$

for every bounded measurable process X.

(ii) Moreover, A is predictable if and only if $\mu_A(X)$ communtes with the predictable projection in the sense that

$$E_P\left[\int_0^\infty X\,dA\right] = E_P\left[\int_0^\infty {}^pX\,dA\right]$$

for every bounded measurable process X.

In addition, we have

Theorem 2.2.19. *Let A be a (non-adapted) integrable FV process. Then there exists a unique predictable FV process A^p such that, for every bounded X,*

$$E_P\left[(X\cdot A^p)_\infty\right] = E_P\left[({}^pX\cdot A)_\infty\right].$$

The process A^p is the dual predictable projection or compensator of A. In Theorem 2.2.19, we have required for the FV process to be integrable for the existence and uniqueness of the compensator of that process. It is enough for Theorem 2.2.19 to hold to have A just be of locally integrable total variation. An alternative description of the compensator is given by the next theorem

Theorem 2.2.20. *Let A be a (non-adapted) integrable FV process. Then A^p is the unique predictable process of integrable variation such that ${}^oA - A^p$ is a martingale.*

2.3 Enlargement of filtrations

Expansion of filtrations means we start with a filtration $\mathbb{F}$ satisfying the "usual conditions", then expand it to get a new filtration $\mathbb{G}$ which satisfies the "usual conditions"and such

that $\mathcal{F}_t \subset \mathcal{G}_t$, for each $t \geq 0$. Enlargement of filtrations mirrors filtration shrinkage. The techniques of expansion of filtrations can be used to answer questions in insider trading (Imkeller [32], Zweirz [66], Fontana et al. [27]), modeling default times and pricing defaultable claims (Coculescu [14], Coculescu et al. [15], and Coculescu and Nikeghbali [16]).
A filtration can be expanded in two primary ways:

- **Initial expansion**: initially expanding a filtration corresponds to adding a random variable $L \in \mathcal{F}$ to $\mathcal{F}_0$ or adding L at at an $\mathbb{F}-$ stopping time τ. $\mathbb{G}$ will denote the initial enlargement of the filtration $\mathbb{F}$ with the random variable L, that is the filtration defined by

$$\mathcal{G}_t = \bigcap_{\epsilon>0} \left(\mathcal{F}_{t+\epsilon} \vee \sigma\left(L\right)\right), \;\; t \geq 0$$

- **Progressive expansion**: progressively enlarging a filtration corresponds to slowly adding a random time to $\mathcal{F}_t,\ t \geq 0$. For $\Lambda : \Omega \to [0, \infty]$. a random time, we denote by $\mathbb{G}$ the smallest filtration containing $\mathbb{F}$ that turns Λ into a stopping time.

$$\mathcal{G}_t = \bigcap_{\epsilon>0} \left(\mathcal{F}_{t+\epsilon} \vee \sigma\left(\Lambda \wedge (t+\epsilon)\right)\right), \;\; t \geq 0$$

The questions to address are: under which conditions on Λ or on $\sigma(L)$ do all $\mathbb{F}-$ martingales remain $\mathbb{G}-$ semimartingales? In case we have a positive answer to the first question, what is the corresponding $\mathbb{G}-$ canonical decomposition of a generic $\mathbb{F}-$ semimartingales? To these two questions correspond two hypotheses:

- Hypothesis (H) is satisfied if every $\mathbb{F}-$ martingale is a $\mathbb{G}-$ martingale. If hypothesis (H) is satisfied, it is said that the filtration $\mathbb{F}$ is immersed in the $\mathbb{G}$.
- Hypothesis (H') is satisfied if every $\mathbb{F}-$ martingale is a $\mathbb{G}-$ semimartingale

For the two different types of expansion, we focus on the H' hypothesis.

2.3.1 Initial expansion

Jacod [33] found a condition on the distribution of L such that the hypothesis (H') holds. Assume L is an $(E, \mathcal{E})$ $-$ valued random variable where $\mathbb{E}$ is a standard Borel space and

$\mathcal{E}$ its Borel sets, then there exists a regular conditional distribution $Q_t(\omega, dx)$ which is a version of $E_P\left\{1_{\{L\in dx\}}\,|\,\mathcal{F}_t\right\}$

Theorem 2.3.1 (Jacod's criterion)**.** *Let L be a random variable with values in a standard Borel space $(E, \mathcal{E})$ $-$, and let $Q_t(\omega, dx)$ denote the regular conditional distribution of L given $\mathcal{F}_t$, for each $t \geq 0$. Suppose that for each t there exists a positive $\sigma-$ finite measure η_t on $(E, \mathcal{E})$ $-$ such that $Q_t(\omega, dx) \ll \eta_t(dx)$, $P-$a.s.. Then every $\mathbb{F}$ semimartingale is also a $\mathbb{G}-$ semimartingale.*

The next theorem gives a useful refinement of Jacod's theorem where the family of measures $(\eta_t)_{t\geq 0}$ is replaced by a single measure η.

Theorem 2.3.2. *L is a random variable with values in a standard Borel space $(E, \mathcal{E})$ $-$, and $Q_t(\omega, dx)$ still denotes the regular conditional distribution of L given $\mathcal{F}_t$, for each $t \geq 0$. Then there exists for each t a positive $\sigma-$ finite measure η_t on $(E, \mathcal{E})$ $-$ such that $Q_t(\omega, dx) \ll \eta_t(dx)$, $P-$a.s. if and only if there exists one positive $\sigma-$ finite measure η such that $Q_t(\omega, dx) \ll \eta(dx)$, $P-$a.s., for each $t > 0$. η can be taken to be the distribution of L.*

Under Jacod's criterion, the hypothesis H' holds; Hence we turn our attention to the canonical decomposition of $\mathbb{G}$ semimartingales.

Lemma 2.3.3. *There exists a positive $\mathcal{O}\left(\cap_{s>t}\left(E\otimes\mathcal{F}_s\right)\right)$ $-$ measurable process $(x, \omega, t) \to q_t^x(\omega) := q_t(x, \omega)$, càdlàg in t, such that*

- *(i) For each $x \in E$, q^x is an $\mathbb{F}-$ martingale and if $T^x := \inf\{t : q_t^x = 0\}$, we have that $q^x > 0$ and $q_-^x > 0$ on $[0, T^x[$ and $q^x = 0$ on $[T^x, \infty[$;*
- *(ii) For each t, $q^x(t, \omega)\eta(dx)$ is a version of $Q_t(\omega, dx)$ on $(E, \mathcal{E})$.*

Lemma 2.3.4. *Let $(x, \omega, t) \to Y^x(\omega, t)$ be a nonegative or bounded $\mathcal{P}\left(\cap_{s>t}\left(E\otimes\mathcal{F}_s\right)\right) := \mathcal{E}\otimes\mathcal{P}\left(\mathbb{F}\right)$ $-$ measurable process for each $x \in E$. Then the $\mathbb{F}-$ predictable projection of Y^L is given by the process $t \to \int \eta(dx) q_{t-}^x(\omega) Y^x(t, \omega)$.*

Corollary 2.3.4.1. $T^L = \infty$, $P-$a.s.

To obtain the canonical decomposition of $\mathbb{G}-$ semimartingales, it is enough to obtain the canonical decomposition of $\mathbb{G}-$ local martingales. We start with the case when the $\mathbb{F}-$ local martingale is continuous.

Theorem 2.3.5. *Let M be an $\mathbb{F}-$ continuous local martingale. There exists a $\mathcal{P}\left(\cap_{s>t}\left(E\otimes\mathcal{F}_s\right)\right):=\mathcal{E}\otimes\mathcal{P}\left(\mathbb{F}\right)-$ measurable process $(x,\omega,t)\to k^x(t,\omega)$ such that*

$$\langle q^x, M\rangle \quad = \quad (k^x\, q^x_-)\cdot\langle M, M\rangle$$

Moreover, we have the following properties:

(i) $\int_0^t \left|k_s^L\right| d\left\langle M,M\right\rangle_s < \infty$ *a.s.* $\forall\, t\geq 0$*;*

(ii) The following process is a $\mathbb{G}-$ local martingale

$$\widetilde{M_t} \quad = \quad M_t - \int_0^t k_s^L\, d\left\langle M,M\right\rangle_s.$$

For $\mathbb{F}-$ local martingales that are not necessarily continuous, Jacod defines the following increasing sequence of $\mathbb{F}-$ stopping times.

$$R_n^x \quad := \quad \inf\left\{t\,:\,q_{t-}^x \leq \frac{1}{n}\right\}$$

and we have $\lim_{n\to\infty} R_n^x = T^x$, and

$$\bigcup_{n=1}^{\infty}[0,R_n^x] \quad = \quad \left\{q_-^x > 0\right\}$$

Theorem 2.3.6. *Let M be an $\mathbb{F}-$ local martingale.*

a) For all x not in a set B (where B can depend on M) which is $\eta-$ negligible, and for all n, $[q^x,M]^{R_n^x}$ is a FV process of locally integration total variation. It is therefore possible to define the process $\langle q^x,M\rangle$ on $\bigcup_{n=1}^{\infty}[0,R_n^x]$.

b) There exists an increasing process A and a $\mathcal{P}\left(\cap_{s>t}\left(E\otimes\mathcal{F}_s\right)\right):=\mathcal{E}\otimes\mathcal{P}\left(\mathbb{F}\right)-$ measurable process $(x,\omega,t)\to k^x(t,\omega)$ such for $x\notin B$ such that

$$\langle q^x, M\rangle_t \quad = \quad \int_0^t k_s^x\, q_{s-}^x dA_s,\ P-\text{a.s. on } \bigcup_{n=1}^{\infty}\{t\leq R_n^x\}$$

if M is locally square-integrable, then A can be chosen to equal to $\langle M,M\rangle$.

c) If A and k are as in b), we have

(i) $\int_0^t \left|k_s^L\right| dA_s < \infty$ *a.s.* $\forall\, t > 0$;

(ii) *The following process is a* $\mathbb{G}-$ *local martingale*

$$\widetilde{M_t} \;=\; M_t - \int_0^t k_s^L \, dA_s.$$

Jacod's results do not always necessarily apply. Indeed, let's consider the following example (see Corcuera and Valdivia [18]).

Example 2.3.1. *Let L be the n^{th} jump of a Poisson Process $(N_t)_{t\in[0,T]}$ with intensity μ and let $\mathbb{F}$ be the natural filtration of N. Then*

$$P\left(L > x \,|\, \mathcal{F}_t\right) \;=\; 1_{\{N_x<n, N_t \geq n\}} + 1_{\{N_t<n\}} \int_{(x-t)_+}^{\infty} \frac{\mu e^{-\mu s}\left(\mu s\right)^{n-N_t-1}}{(n-N_t-1)!} ds$$

Therefore, the conditional regular distribution of L given $\mathcal{F}_t$ cannot be dominated by a non-random measure.

Another example of when Jacod's criterion is not applicable can be found in Nikeghbali [49].

2.3.1.1 Progressive enlargement of filtrations

Progressively expanding a filtration corresponds gradually to adding a random time to that filtration in order to create a minimal expanded filtration that turns the random time into a stopping time. Let Λ be the random time. To explain progressive enlargement, we start with Λ being an honest time. We also assume that Λ avoids all $\mathbb{F}-$ stopping times. That is, $P\left(\Lambda = T\right) = 0$ for all $\mathbb{F}-$ stopping times T.

Definition 2.3.1. A random variable is called honest, if for every $t \leq \infty$ there exists an $\mathcal{F}_t-$ measurable random vairable Λ_t such that $\Lambda = \Lambda_t$ on $\{\Lambda \leq t\}$.

Any stopping time is honest.

Theorem 2.3.7. Λ *is an honest time if and only if there exists an optional set* $\Delta \subset [0,\infty] \times \Omega$ *such that* $\Lambda(\omega) = \sup\left\{t \leq \infty \,|\, (t,\omega) \in \Delta\right\}$.

Instead of defining the filtration $\mathbb{G}$ as the smallest filtration satisfying the "usual conditions" and given by

$$\mathcal{G}_t = \bigcap_{u>t} \left(\mathcal{F}_u \vee \sigma\left(\Lambda \wedge u\right)\right), \ t \geq 0$$

We use the fact that Λ is assumed to be an honest time and describe the filtration $\mathbb{G}$ as follows:

Theorem 2.3.8. *Let Λ be an honest time. Define*

$$\mathcal{G}_t = \left\{\Gamma : \Gamma = \left(A \cap \{\Lambda > t\}\right) \cup \left(B \cap \{\Lambda \leq t\}\right), \text{for some } A, B \in \mathcal{F}_t\right\}$$

Then $\mathbb{G} = \left(\mathcal{G}_t\right)_{t\geq 0}$ constitutes a filtration satisfying the "usual conditions".
A process U is $\mathbb{G}-$ predictable if and only if it has the following representation

$$U = H1_{[0,\Lambda]} + K1_{(\Lambda,\infty]}$$

where H and K are $\mathbb{F}-$ predictable processes.

A very important process in the theory of progressive expansion of filtrations is the Azéma supermartingale. The Azéma supermartingale is defined as

$$Z_t = {}^{o}1_{\{\Lambda > t\}}, \ t \geq 0.$$

and associated to Z is the fundamental Λ martingale.

Definition 2.3.2. The martingale M^{Λ} given by $M^{\Lambda} = Z + A^{\Lambda}$ is called the fundamental Λ martingale.

The existence of M^{Λ} and A^{Λ} comes from the Doob-Meyer decomposition of Z. Consequently, the $\mathbb{G}-$ decomposition of an $\mathbb{F}-$ local martingale is given by the following theorem:

Theorem 2.3.9. *Let X be a local martingale for the $\mathbb{F}$ filtration. Then X is a semimartingale for the $\mathbb{G}$ filtration if Λ is an honest time. Moreover, X has a $\mathbb{G}$ decomposition given by*

$$\begin{aligned} X_t &= \left\{X_t - \int_0^{t\wedge\Lambda} \frac{1}{Z_{s-}} d\left\langle X, M^{\Lambda}\right\rangle_s + 1_{\{t\geq\Lambda\}} \int_{\Lambda}^{t} \frac{1}{1-Z_{s-}} d\left\langle X, M^{\Lambda}\right\rangle_s\right\} \\ &+ \left\{\int_0^{t\wedge\Lambda} \frac{1}{Z_{s-}} d\left\langle X, M^{\Lambda}\right\rangle_s - 1_{\{t\geq\Lambda\}} \int_{\Lambda}^{t} \frac{1}{1-Z_{s-}} d\left\langle X, M^{\Lambda}\right\rangle_s\right\} \end{aligned}$$

where $\left\{X_t - \int_0^{t\wedge\Lambda} \frac{1}{Z_{s-}} d\left\langle X, M^{\Lambda}\right\rangle_s + 1_{\{t\geq\Lambda\}} \int_{\Lambda}^{t} \frac{1}{1-Z_{s-}} d\left\langle X, M^{\Lambda}\right\rangle_s\right\}_{t\geq 0}$ is a $\mathbb{G}-$ local martingale.

2.3.1.2 The (H) hypothesis

The (H) hypothesis is widely used in credit default modeling, defaultable claims modeling and default times modeling. If the (H) hypothesis holds, we say that the filtration $\mathbb{F}$ is immersed in the filtration $\mathbb{G}$. Brémaud and Yor [11] proved the following:

Theorem 2.3.10. *The following are equivalent:*

a) The (H) hypothesis holds;

b) For all $t \geq 0$, $\mathcal{G}_t$ and $\mathcal{F}_\infty$ are conditionally independent on $\mathcal{F}_t$.

In the context of progressive enlargement, there exists a class of random times which immerses $\mathbb{F}$ into $\mathbb{G}$ up to the random time. Those random times are called pseudo-stopping times (see Nikeghbali and Yor [47]). Let's recall that the space $\mathcal{H}^1$ is the Banach space of $\mathbb{F}-$ martingales M such that

$$\|M\|_{\mathcal{H}^1} \quad = \quad E_P\left[\sup_{t\geq 0}|M_t|\right] < \infty.$$

Definition 2.3.3. ρ is an $\mathbb{F}-$ pseudo-stopping time if for every $\mathbb{F}-$ martingale M in $\mathcal{H}^1$, we have

$$E_P\left\{M_\rho\right\} = E_P\left\{M_0\right\}. \tag{2.1}$$

It is enough to prove equation (2.1) for bounded martingales. Let's assume the filtration $\mathbb{F}$ is expanded with ρ, then Nikeghbali and Yor [47] proved the following theorem

Theorem 2.3.11. *The following properties are equivalent:*

a) ρ is an $\mathbb{F}-$ pseudo-stopping time;

b) Every $\mathbb{F}-$ local martingale M satisfies

$$\left(M_{t\wedge\rho}\right)_{t\geq 0} \text{ is a } \mathbb{G} \text{ -- local martingale}$$

Hence the (H) hypothesis holds on $[0,\rho]$.

2.4 Kunita-Watanabe decomposition

Let's first start by recalling the definition of orthogonality for two martingales

Definition 2.4.1. The space $\mathbf{M}^2$ of L^2 martingales is all matingales M such that $\sup_t E_P\left(M_t^2\right) < \infty$, and $M_0 = 0,\ P-$a.s..

Definition 2.4.2. Two martingales N, $M \in \mathbf{M}^2$ are said to be strongly orthogonal if their product $L = NM$ is a (uniformly integrable) martingale. N and M are weakly orthogonal if $E_P\left\{M_\infty N_\infty\right\} = 0$.

Strong orthogonality of M and N is equivalent to $[M, N]$ being a uniformly integrable martingale which implies that $\langle M, N\rangle \equiv 0$. Moreover, strong orthogonality implies weak orthogonality.

Definition 2.4.3 (Kunita-Watanabe decomposition)**.** Let M and N be local martingales. The Kunita-Watanabe decomposition of N on M is a decomposition of the form

$$N_t = N_0 + \int_0^t \xi_u dM_u + U_t,\ P-\text{a.s.},\ t \geq 0 \tag{2.2}$$

where $\xi \in \mathcal{L}_{(\text{loc})}(M)$, $U \in \mathcal{M}_{(0,\text{loc})}(\mathbb{F})$ and U is strongly orthogonal to M.
If it exists, the decomposition given in (2.2) is unique in the sense that if

$$N_t = N_0 + \int_0^t \xi_u dM_u + U_t = \widetilde{N}_0 + \int_0^t \widetilde{\xi}_u dM_u + \widetilde{U}_t,\ P-\text{a.s.},\ t \geq 0$$

where (N_0, ξ, U) and $\left(\widetilde{N}_0, \widetilde{\xi}, \widetilde{U}\right)$ satifies the Kunita-Watanabe hypotheses, then

$$\begin{aligned} N_0 &= \widetilde{N}_0 \\ \int_0^t \xi_u dM_u &= \int_0^t \widetilde{\xi}_u dM_u,\ P-\text{a.s.},\ t \geq 0 \\ U_t &= \widetilde{U}_t,\ P-\text{a.s.},\ t \geq 0 \end{aligned}$$

The Kunita-Watanabe decomposition doesn't always exist. Below, we give situations where such a decomposition exists (see Ansel and Stricker [5]).

a) If M and N are locally square integrable local martingales, then the Kunita-Watanabe decomposition exists.

b) If N is square-integrable while M is any local martingale, the decomposition need not exist.

c) If N is any local martingale and M is continuous, then the decomposition exists. Indeed, $N = N^c + N^d$ where N^c and N^d are respectively the continuous part and the purely discontinuous part of N. N^d is strongly orthogonal to all continuous local martingales while N^c is locally square-integrable since it is continuous; Then we are back in the case of N^c to a). Consequently, $N^c = H \cdot M + V$ where V is strongly orthogonal to M and $H \in L_{\text{loc}}(M)$. It is enough to let $U = N^d + V$ to get the decomposition of N as in (2.2).

d) c) cannot be generalized to the case when M is not continuous.

2.5 Föllmer-Schweizer decomposition

For future references, let's define the following:

Definition 2.5.1. X satisfies the structure condition (SC) if X is a special $(\mathbb{F}, P)-$ semi-martingale with canonical decomposition

$$X = X_0 + M + A$$

which satisfies

$$M \in \mathcal{M}^2_{0,loc}(\mathbb{F}, P) \tag{2.3}$$

and

$$A \ll M, \text{ with predictable density } \alpha. \tag{2.4}$$

We also recall the following definition

Definition 2.5.2. A real-valued process Z is called a martingale density for X if Z is a $(\mathbb{F}, P)-$ local martingale with $Z_0 = 1$, $P-$a.s. and such that the product XZ is an $(\mathbb{F}, P)-$ local martingale. If, in addition, Z is strictly positive, Z is called a strict martingale density for X.

Theorem 2.5.1. *Suppose that X admits a strict martingale density Z^* and that either*

$$X \text{ is continuous} \tag{2.5}$$

or

$$X \text{ is a special semimartingale satisfying (2.3)} \tag{2.6}$$

and

$$Z^* \in \mathcal{M}^2_{0,loc}(\mathbb{F}, P). \tag{2.7}$$

Then X satisfies the SC, and

$$\alpha \in L^2_{loc}(M). \tag{2.8}$$

Furthermore, Z^* can be written as

$$Z^* = \varepsilon\left(-\int \alpha dM + L\right), \tag{2.9}$$

where $L \in \mathcal{M}_{0,loc}(\mathbb{F}, P)$ is strongly orthogonal to M. If (2.6) and (2.7), then $L \in \mathcal{M}^2_{0,loc}(\mathbb{F}, P)$; If (2.5) holds, then (2.9) can be simplified to

$$Z^* = \varepsilon\left(-\int \alpha dM\right)\varepsilon(L). \tag{2.10}$$

In the minimal martingale measure theory studied by Schweizer in [58], the minimal martingale measure corresponds to $L = 0$. Let's denote that following strict martingale density by $\widehat{Z}$, then

$$\widehat{Z} = \varepsilon\left(-\int \alpha dM\right) \tag{2.11}$$

Definition 2.5.3 (Föllmer-Schweizer (FS) decomposition)**.** An $\mathcal{F}_T-$ (where T is a fixed time) measurable random variable H is said to admit a generalized Föllmer-Schweizer decomposition if there exist a constant H_0, an $\mathbb{F}-$ predictable $X-$ integrable process ξ^H and an $(\mathbb{F}, P)-$ local martingale L^H strongly orthogonal to M such that H can be written as

$$H = H_0 + \int_0^T \xi^H_s dX_s + L^H_T, \ \ P-\text{a.s.} \tag{2.12}$$

and such that the process $\widehat{Z}\widehat{V}$ is an $(\mathbb{F}, P)-$ martingale, where

$$\widehat{V} := H_0 + \int \xi^H_s dX_s + L^H, \ \ P-\text{a.s.}. \tag{2.13}$$

The decomposition given in (2.12) doesn't always exists. It was introduced in Föllmer and Schweizer [25] and was studied in Schweizer [57], [58], Ansel and Stricker [4] and Monat and Stricker [45].

Chapter 3

Expansion of filtrations and no free lunches

3.1 Introduction

In this chapter, we use Jacod's H' hypothesis to find the decomposition of the discounted stock price in a filtration $\mathbb{G}$ that has been obtained by initially enlarging at a random time the market filtration $\mathbb{F}$. We study conditions for an insider to have no arbitrage and no free lunch with vanishing risk. We also revisit the notion of local no free lunch with vanishing risk and its relation to the notion of no arbitrage with respect to admissible simple predictable integrands.

3.2 General setting, Model and Assumptions

Let $(\Omega, \mathcal{F}, P, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0})$ be a filtered probability space, where the filtration $\mathbb{F}$ satisfies the "usual conditions", P denotes the physical probability measure and $\mathcal{F} := \mathcal{F}_\infty$ is assumed to be separable.

We consider a financial market which is composed of a risky asset S and a bond B, and two traders: a liquidity or market trader and an insider. We also consider that we have a finite trading horizon $[0, T]$, where $T < \infty$. Note that we can easily extend the results from this chapter to the case where we have d, with $d \geq 1$, risky assets. We assume that all $\mathbb{F}-$

local martingales are continuous.

Let L be an $\mathcal{F}-$ random variable that takes values in a Lusin space $(E, \mathbb{E})$ and let $\tau : \Omega \to [0, T]$ be a finite $\mathbb{F}$ - stopping time. We define the filtration $\mathbb{G}$ as the initial expansion of $\mathbb{F}$ with L at the stopping time τ, i.e. the filtration $\mathbb{G}$ is defined by

$$\mathcal{G}_t = \begin{cases} \mathcal{F}_t, & t < \tau \\ \bigcap_{s>t} \left(\mathcal{F}_s \vee \sigma(L)\right), & t \geq \tau \end{cases} \tag{3.1}$$

augmented by the $P-$ null sets of $\mathcal{G} := \mathcal{G}_\infty = \mathcal{F}_\infty$. Hence, the filtration $\mathbb{G}$ also satisfies the "usual conditions". We also suppose that we are in the Jacod paradigm that means

$$Q_t(\omega, dx) \ll \eta(dx) \tag{3.2}$$

where $Q_t(\omega, dx)$ is the regular conditional distribution of L with respect to $\mathcal{F}_t$ and η can be taken to be the distribution of L. Equation (3.2) is the Jacod hypothesis and implies that the H'-hypothesis holds between $\mathbb{F}$ and $\mathbb{G}$ (see [33] and [52]).

W.l.o.g we can also assume that the physical measure P is the risk neutral measure of the market and reduced the modeling steps as follows:

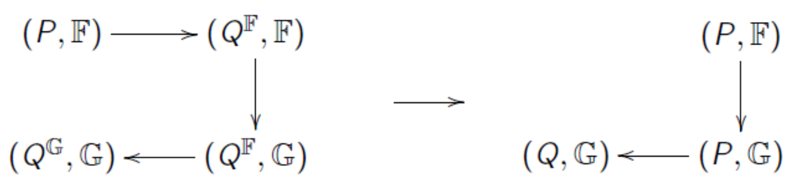

Figure 3.1: Reduced model

Consequently, we can assume that the discounted stock price is an $\mathbb{F}-$ local martingale which implies that it is a $(\mathbb{G}, P)-$ semimartingale.

$$dS_t \quad = \quad dM_t \tag{3.3}$$

where M is a nonnegative $\mathbb{F}-$ continuous local martingale.

We need now to define the notion of admissible trading strategies to model the trading activities of the insider and market trader. We do so following Delbaen and Schachermayer [21].

3.2.1 Admissibility

Let $\mathbb{H}$ denote either the filtration $\mathbb{F}$ or the filtration $\mathbb{G}$; i.e. $\mathbb{H} \in \{\mathbb{F}, \mathbb{G}\}$. We denote by $\mathcal{P}^{\mathbb{H}}(S)$ the set of all $\mathbb{R}-$ valued $\mathbb{H}-$ predictable processes $\varphi = (\varphi)_{t\geq 0}$ which are $S-$ integrable in $\mathbb{H}$.

Definition 3.2.1. For $a \in \mathbb{R}_+$, an element $\varphi \in \mathcal{P}^{\mathbb{H}}(S)$ is said to be $a-$ admissible if $(\varphi \cdot S)_T = \lim_{t\to T}(\varphi \cdot S)_t$ exists and $(\varphi \cdot S)_t \geq -a$, $P-$ a.s. for all $t \in [0, T]$. If $\mathcal{A}_a^{\mathbb{H}}$ is the set of all $a-$ admissibble $\mathbb{H}-$ trading strategies, we say that φ is an admissible $\mathbb{H}-$ strategy if $\varphi \in \mathcal{A}^{\mathbb{H}} := \bigcup_{a\in\mathbb{R}_+} \mathcal{A}_a^{\mathbb{H}}$.

Of course (admissible) $\mathbb{F}-$ trading strategies are also (admissible) $\mathbb{G}-$ trading strategies. It is important to note that admissibility rules out trading strategies such as the doubling strategy which almost surely generates a profit at the end of the trading horizon with zero probability of losing. Assuming there are no trading constraints, the markets are frictionless, portfolios are self-financing and the spot interest rate is zero, the wealth process generated by a given trading strategy $\varphi \in \mathcal{P}^{\mathbb{H}}(S)$ starting from an initial endowment of $x \in \mathbb{R}_+$ is given by

$$V(x, \varphi) \quad := \quad x + \varphi \cdot S \tag{3.4}$$

3.2.2 Trading strategies

Trading strategies are processes in $\mathcal{P}^{\mathbb{H}}(S)$ and can essentially take two forms

a) Simple predictable integrands

Definition 3.2.2. A trading strategy H is a simple predictable trading strategy if it can be written as

$$H_t \quad = \quad H_0 1_{\{0\}}(t) + \sum_{i=1}^{n} H_i 1_{(\tau_i, \tau_{i+1}]}(t), \quad t \geq 0 \tag{3.5}$$

where $0 = \tau_1 \leq \cdots \leq \tau_{n+1} < \infty$, $P-$a.s. is a finite sequence of stopping times, $H_i \in \mathcal{H}_{\tau_i}$ with $|H_i| < \infty$ a.s., $0 \leq i \leq n$.

The use of simple predictable integrands is motivated by the fact that it is the only type of strategy that corresponds to real-life situations. Simple predictable integrands corresponds to sequences of buy and holds over the trading horizon. We can also require simple predictable strategies to be admissible. It is also possible to define a strategy such that the T_i's are determistic times instead of stopping times.

b) General admissible integrands

Definition 3.2.3. A trading strategy H is a general admissible integrand if it is admissible and

$$H \in \mathcal{P}^{\mathbb{H}}(S) \tag{3.6}$$

General integrands are important for the definition of no free lunch with vanishing risk (defined below) as introduced in Delbaen and Schachermayer [21]. To fit into the context of Delbaen and Schachermayer [21], they have to be admissible.

3.2.3 Notions of arbitrage

There are multiple ways to introduce the different types of arbitrage that exist in the literature. Loosely speaking arbitrage refers to the notion that it should not be allowed in financial markets to find a trading strategy which yields a positive gain with strictly positive probability without taking risk, that is with the probability of losing money being equal to zero. One approach is the one developed in Delbaen and Schchermayer and that approach can be found in [21]. The approach we use can also be found in Fontana et al. [27].

Definition 3.2.4. Let $\mathcal{H}$ and $\mathbb{H}$ represent respectively either the $\sigma-$ field $\mathcal{F}$ or $\mathcal{G}$ and either the filtration $\mathbb{F}$ or $\mathbb{G}$.

a) An element $\varphi \in \mathcal{A}_0^{\mathbb{H}}$ yields an Unbounded Increasing Profit if

$$\begin{aligned} P\left(V(0,\varphi)_s \leq V(0,\varphi)_t, \text{ for all } 0 \leq s \leq t \leq T\right) &= 1 \\ P\left(V(0,\varphi)_T > 0\right) &> 0 \end{aligned}$$

If there exists no such φ, we say that the financial market represented by the quintuplet $\mathcal{M}^{\mathbb{H}} := \left(\Omega, \mathcal{F}, \mathbb{H}, P, S, \mathcal{A}^{\mathbb{H}}\right)$ satisfies the No Unbounded Increasing Profit (NUIP) condition.

b) A non-negative $\mathcal{H}-$ measurable random variable ξ with $P(\xi > 0) > 0$ yields an Arbitrage of the First Kind if for all $x > 0$, there exists an element $\varphi^x \in \mathcal{A}_x^{\mathbb{H}}$ such that $V(x, \varphi^x)_T \geq \xi \; P-$ a.s.. If there exists no such random variable, the financial market $\mathcal{M}^{\mathbb{H}}$ satisfies the No Arbitrage of the First Kind (NA1) condition.

c) An element $\varphi \in \mathcal{A}_0^{\mathbb{H}}$ yields an Arbitrage opportunity if

$$\begin{aligned} P\left(V(0,\varphi)_T \geq 0\right) &= 1 \\ P\left(V(0,\varphi)_T > 0\right) &> 0 \end{aligned}$$

If there is not such a φ, then the financial market satisfies the No Arbitrage (NA) condition.

d) A sequence $(\varphi^n)_{n\in\mathbb{N}} \subset \mathcal{A}^{\mathbb{H}}$ yields a Free Lunch with Vanishing Risk if $\exists\, \epsilon > 0$ and $0 \leq \delta_n \nearrow 1$ s.t.

$$\begin{aligned} P\left(V(0,\varphi^n)_T > -1+\delta_n\right) &= 1 \\ P\left(V(0,\varphi^n)_T > \epsilon\right) &\geq \epsilon \end{aligned}$$

If there is no such sequence, the financial market satisfies the celebrated No Free Lunch with Vanishing Risk (NFLVR) condition.

The notion of NUIP was introduced by Karatzas and Kardaras [40] and is of course the strongest form of arbitrage. It is important to note that NA1 is also known in the literature under the name No Unbounded Profit with Bounded Risk (NUPBR) (see Delbaen and Schachermayer [20], [21] and Karatzas and Kardaras [40]). Nowadays, NFLVR is the gold standard definition for no arbitrage. From the two notions of arbitrage we are interested in, NA and NFLVR, the No Arbitrage condition can be defined for (admissible) simple predictable integrands while NFLVR cannot. We call the No Arbitrage condition restricted to (admissible) simple predictable integrands NAS.

Definition 3.2.5. If we restrict our trading strategies to (admissible) simple predictable strategies as defined in definition 3.2.2 above, then we say that the NAS property holds if NA holds when restricted to simple predictable processes.

From the four notions of arbitrage defined above, only NA1/NUPBR and NFLVR can be defined in probabilistic terms. Before we present those two notions of arbitrage in probabilistic terms, we need to define the following terms:

Definition 3.2.6. Let $\mathcal{H}$ and $\mathbb{H}$ represent respectively either the $\sigma-$ field $\mathcal{F}$ or $\mathcal{G}$ and either the filtration $\mathbb{F}$ or $\mathbb{G}$, then

- A real-valued process V is called a local martingale deflator in $\mathbb{H}$ for S if V is an $\mathbb{H}-$ local martingale with $V_0 = 1,\ P-$a.s. and VS is an $\mathbb{H}-$ local martingale. If in addition V is strictly positive then it is called a strict martingale density for S.
- A probability measure $Q \sim P$ defined on $(\Omega, \mathcal{H})$ is an equivalent local martingale measure if the process S is an $\mathbb{H}-$ local martingale under Q.

In the literature, a local martingale deflator is also called a martingale density (see Schweizer [58] and Choulli and Stricker [13]). The proof of the following theorem can be found in the literature (see Delbaen and Schachermayer [19], Kardaras [41]).

Theorem 3.2.1. *Let $\mathcal{H}$ and $\mathbb{H}$ represent respectively either the $\sigma-$ field $\mathcal{F}$ or $\mathcal{G}$ and either the filtration $\mathbb{F}$ or $\mathbb{G}$, then*

a) NA1 holds in the financial market $\mathcal{M}^{\mathbb{H}}$ if and only if there exists a local martingale deflator in $\mathbb{H}$.

b) NFLVR holds in the financial market $\mathcal{M}^{\mathbb{H}}$ if and only if there exists an equivalent local martingale measure in $\mathbb{H}$.

c) NFLVR is equivalent to NA1/NUPBR and NA.

d) NFLVR holds in the financial market $\mathcal{M}^{\mathbb{H}}$ if and only if there exists an $\mathbb{H}-$ local martingale deflator that is a uniformly integrable $\mathbb{H}-$ martingale.

3.3 General results

The assumption that the physical probability measure P is actually the risk neutral measure of the market is equivalent to assuming that the market $\mathcal{M}^{\mathbb{F}}$ satisfies NFLVR.

Using Jacod's theory, we obtain the following lemma which gives the $(\mathbb{G}, P)-$ semimartingale decomposition of the discounted stock price

Lemma 3.3.1. *There exists a* $\mathcal{P}(\widehat{\mathbb{F}}) := E \bigotimes \mathcal{P}(\mathbb{F})-$ *process* $(x, \omega, t) \to k_t^x(\omega)$ *such that*

$$\langle q^x, S \rangle \quad = \quad (k^x\, q^x) \cdot \langle S, S \rangle \tag{3.7}$$

Moreover, we have the following properties:

(i) $\int_0^t \left| k_s^L \right| \, d \left\langle S, S \right\rangle_s < \infty$ *a.s.* $\forall\ t \geq 0$

(ii) The following process is a $(\mathbb{G}, P)-$ *local martingale*

$$\widetilde{S}_t = S_t - \int_0^t k_s^L \, d \left\langle S, S \right\rangle_s \tag{3.8}$$

and the $(\mathbb{G}, P)$ *decomposition of* S *is*

$$dS_t \quad = \quad \begin{cases} dM_t, & t < \tau \\ d\widetilde{S}_t + k_t^L \, d \left\langle S, S \right\rangle_t, & t \geq \tau \end{cases} \tag{3.9}$$

where $\widetilde{S}$ *is a* $\mathbb{G}$ *- continuous local martingale and* $d \left\langle S, S \right\rangle = d \left\langle \widetilde{S}, \widetilde{S} \right\rangle$*.*

Proof. The above lemma is an application of Theorem 2.3.5 in Chapter 2. ■

The goal of the following section is to find conditions under which insider trading does not necessarily lead to free lunches with vanishing risk or arbitrage opportunities. Applying Theorem 3.2.1, the goal is to find a measure through a Girsanov approach so that the drift in (3.9) is removed. We construct such a measure, denoted by Q, using the process Z which is a solution of the following stochastic differential equation (SDE)

$$dZ_t = -Z_t k_t^L d\widetilde{S}_t; \;\; Z_\tau = 1 \quad (t \geq \tau) \tag{3.10}$$

and we take $dQ = Z_T dP$ as an equivalent measure, with $Q(\Omega) \leq 1$. Therefore, the question is equivalent to finding conditions under which the process Z is a true martingale.

3.3.1 Local NFLVR and NAS

Let's define the following sequence of increasing $\mathbb{G}-$ stopping times

$$T_m = \inf\left\{t > 0 \,:\, \int_0^t \left(k_s^L\right)^2 d\left\langle S, S\right\rangle_s \geq h(m)\right\} \tag{3.11}$$

for some function h.

Theorem 3.3.2 (Local NFLVR)**.** *Suppose*

- $\lim_{m\to\infty} T_m > T$ a.s.
- $\int_0^t \left(k_s^L\right)^2 d\left\langle S, S\right\rangle_s < \infty$ a.s. for each $t \geq 0$.

Then

a) We can find a consistent sequence $\left(Q_m\right)_{m\geq 1}$ of probability measures, where consistency is defined as

$$Q_{m+k}(A) = Q_m(A), \text{ where } A \in \mathcal{G}_{T_m}, \forall\; k \geq 0, \forall\; m \geq 0,$$

such that the market given by $\left(\Omega, \mathcal{F}, \mathbb{G}, Q_m, S, \mathcal{A}^{\mathbb{G}}\right)$ satisfies NFLVR on $[0, T_m]$ for all m.

- *In addition, if the sequence $(Z^{T_m})_{m\geq 0}$ is uniformly integrable*

b) NFLVR holds on $[0, T]$

Proof. W.l.o.g., we can assume that $\tau \equiv 0,\ P-$ a.s.. We are interested in removing the drift in (3.9) by a change of measure. We construct Z as a solution of an SDE of the form

$$dZ_t = Z_t H_t d\widetilde{S}_t\,;\;\; Z_0 = 1 \tag{3.12}$$

then take $dQ = Z_T dP$ as an equivalent measure, with $Q(\Omega) \leq 1$.
To change S into a $(\mathbb{G}, Q)-$ local martingale (assuming for now that Q is a true probability measure) we use Girsanov's theorem to get

$$S_t \;=\; (\widetilde{S}_t - \int_0^t \frac{1}{Z_s} d[Z, \widetilde{S}]_s) + (\int_0^t k_s^L\, d\left\langle S, S\right\rangle_s + \int_0^t \frac{1}{Z_s} d[Z, \widetilde{S}_t]_s) = N_t + C_t \tag{3.13}$$

where N is a $(\mathbb{G}, Q)-$ local martingale. We want $C \equiv 0,\ P-$a.s. and note that by continuity

$$\int_0^t \frac{1}{Z_s} d[Z, \widetilde{S}]_s = \int_0^t \frac{1}{Z_s} Z_s H_s d[\widetilde{S}, \widetilde{S}]_s = \int_0^t H_s d\langle \widetilde{S}, \widetilde{S}\rangle_s = \int_0^t H_s d\langle S, S\rangle_s$$

We need only to take $H_s = -k_s^L$ to get $C \equiv 0,\ P-$ a.s..

In order for what is above to work, we need at a minimum for the stochastic integral $H \cdot \widetilde{S}$ to exist. That is, we need $\int_0^t \left(k_s^L\right)^2 d\langle S, S\rangle_s < \infty$ a.s. for each $t \geq 0$ which holds by assumption.

We construct Q_m by $dQ_m = Z_T^{T_m} dP$ where

$$Z_t^{T_m} = \exp\left(-\int_0^t 1_{[0,T_m]}(s) k_s^L\, d\widetilde{S}_s - \frac{1}{2}\int_0^t 1_{[0,T_m]}(s)(k_s^L)^2\, d\left\langle S, S\right\rangle_s\right) \tag{3.14}$$

By Novikov,

$$E \exp\left(\frac{1}{2}\int_0^{t\wedge T_m} (k_s^L)^2\, d\left\langle S, S\right\rangle_s\right) \leq \exp\left(\frac{1}{2}h(m)\right) < \infty$$

Therefore, Z^{T_m} is a true $\mathbb{G}-$ continuous martingale and the Q_m's defined through Z^{T_m} are *bona fide* probability measures. Therefore, the market given by $\left(\Omega, \mathcal{F}, \mathbb{G}, Q_m, S, \mathcal{A}^{\mathbb{G}}\right)$ satisfies NFLVR on $[0, T_m]\ \forall\, m$.

Since we have not assumed that markets are complete, the sequence of probability measures $(Q_m)_{m\geq 0}$ is not necessarily unique but we choose them in such a way that they form a consistent sequence of probability measures. Since the sequence $(T_m)_{m\geq 0}$ of $\mathbb{G}-$ stopping times is increasing, we have that $\forall\ k \geq 1$,

$$Z_{t\wedge T_{m+k}}^{T_m} = Z_{t\wedge T_m} \Leftrightarrow Z^{T_{m+k}} = Z^{T_m},\ \text{on}\ [|0, T_m|]$$

Hence $\forall\ k \geq 1$, the measures Q_m and Q_{m+k} respectively induced by $Z_{T\wedge T_m}$ and $Z_{T\wedge T_{m+k}}$ are the same on $[|0, T_m|]$.

Let $U^m = Z^{T_m};\ U = Z^{\overline{T}}$ where $\overline{T} = \lim_{m\to\infty} T_m$. Then,

$$\begin{aligned} U_t^m &= 1 + \int_0^t U_s^m 1_{[0,T_m]}(s) k_s^L d\widetilde{S}_s \\ U_t &= 1 + \int_0^t U_s 1_{[0,\overline{T})}(s) k_s^L d\widetilde{S}_s \end{aligned}$$

By stability of SDEs (Chap V, Theorem 15 of [52]), we see that U^m converges to U in *ucp*. Therefore, there exists a subsequence m_k such that $\lim_{m_k\to\infty} (U^{m_k} - U)_t^* = 0$ a.s. for each

$t \geq 0$.

In addition, uniform integrability of $\left(Z^{T_m}\right)_{m\geq 0}$ implies the L_1- convergence of $\left(Z_t^{T_m}\right)_{m\geq 0}$. That is, for t fixed,

$$E(Z_t^{\overline{T}}) = \lim_{m_k\to\infty} E\left(Z_t^{T_{m_k}}\right) = 1 \tag{3.15}$$

By assumption $\overline{T} > T$ a.s., so for each $t \in [0, T]$

$$E(Z_t^{\overline{T}}) = E(Z_t) = 1 \tag{3.16}$$

Consequently, Q defined through Z is a true probabilty measure. We know that S^{T_m} satisfies NFLVR with a corresponding equivalent local martingale probability measure Q_m. Hence, under Q_m, the process S^{T_m} is a $(\mathbb{G}, Q_m)-$ local martingale. Moreover, by the way we choose the Q_m's, for any $n \geq m$, S^{T_m} is also a $(\mathbb{G}, Q_n)-$ local martingale. Since m and n are arbitrary, $S^{\overline{T}}$ is $(\mathbb{G}, Q)-$ local martingale. Consequently NFLVR holds on $[0, T]$. ■

Property a) of Theorem 3.3.2 will be called from now on *Local No Free Lunch with Vanishing Risk*. The second condition of Theorem 3.3.2, $\int_0^t \left(k_s^L\right)^2 d\left\langle S, S\right\rangle_s < \infty$ a.s. for each $t \geq 0$, is exactly the condition that fails in the works of Imkeller [32] and Zweirz [66]; since it represents an integrability condition of the drift that appears in equation (3.9). If on a set of strictly positive probability, that condition doesn't hold, an equivalent local martingale probability measure cannot exist. We would like to show that if NFLVR holds locally then it holds globally. Unfortunately this is not true, the uniform integrability condition of Theorem 3.3.2 cannot be removed and the three-dimensional Bessel process starting at 1 provides a counterexample.

3.3.1.1 The three-dimensional Bessel process

If $(\rho_t)_{t\geq 0}$ is a three-dimensional Bessel process starting at 1 denoted $\mathrm{Bes}^3(1)$, then there is a Brownian motion β such that for all $t \geq 0$

$$\rho_t = 1 + \beta_t + \int_0^t \frac{1}{\rho_s} ds \tag{3.17}$$

Lemma 3.3.3. *The* $\mathrm{Bes}^3(1)$ *satisfies Local NFLVR.*

Proof. We know that the $\text{Bes}^3(1)$ never reaches 0 even at time 0 since the process is assumed to start at 1 (see [53]). Therefore, $\int_0^t \left(1/\rho_s^2\right) ds < \infty,\ P-\text{a.s.}$ holds for all $t > 0$, since ρ is a.s. continuous and bounded away from zero for each path. If we suppose that we started with the Brownian motion β and expanded the filtration of β in such a way that $k^L = 1/\rho$, then

$$\int_0^t \left(k_s^L\right)^2 ds = \int_0^t \left(1/\rho_s^2\right) ds \tag{3.18}$$

$$< \infty,\ P-\text{a.s.} \tag{3.19}$$

Let's define the following sequence of stopping times

$$T_m = \inf\left\{t > 0 \,:\, \int_0^t \left(1/\rho_s^2\right) ds \geq m\right\}$$

then $T_m \nearrow \infty$, therefore by Theorem 3.3.2, the $\text{Bes}^3(1)$ satisfies Local NFLVR on $[0, T]$. ■

The methods used in Delbaen and Schachermayer [20] prove that the $\text{Bes}^3(1)$ process satisfies the NA property with respect to simple integrands, hence with respect to admissible simple integrands but Delbaen and Schachermayer [21] proved that the $\text{Bes}^3(1)$ does not satisfy NFLVR on $[0, T]$ for any fixed T. Therefore, the $\text{Bes}^3(1)$ provides an example of a process that satisfies Local NFLVR but not NFLVR. Nevertheless, there is an important fact that came to light while studying the $\text{Bes}^3(1)$, despite the fact that NFLVR fails on any compact interval with a fixed ending time, it still satisfies NAS, shedding light on a possible connection between Local NFLVR and NAS.

3.3.2 NAS

Theorem 3.3.4 (Local NFLVR and NAS). *Suppose*

- *Local NFLVR holds*

then an insider does not have arbitrage opportunities on $[0, T]$ *with respect to admissible simple predictable integrands.*

Proof. Let's consider the sequence of stopping times $(T_n)_{n\geq 1}$, defined in (3.11), increasing to $\overline{T}$ such that NFLVR holds on $[0, T_n]$ for each n. We want to prove that local NFLVR implies

NA for simple *admissible* integrands. We focus on NA for simple admissible integrands because if NFLVR holds on $[0, T_n]$ for each n, then NA holds on $[0, T_n]$ for each n for general admissible integrands, hence also for simple admissible integrands.

Suppose that NA fails to hold for simple admissible predictable processes. That means there exists a trading strategy H^* that gives arbitrage; then it does not give arbitrage on $[0, T_n]$ for each n, otherwise we would immediately have a contradiction. H^* can be written as follows:

$$H^* = \sum_{k=1}^{n} f_k 1_{(\tau_k, \tau_{k+1}]}, \text{ where } f_k \in \mathcal{G}_{\tau_k} \tag{3.20}$$

and we have $0 \leq \tau_1 \leq \cdots \leq \tau_{n+1} \leq T$, $P -$ a.s.. Let $\xi = \tau_{n+1} = \sup_k \tau_k$.

Let's first suppose that $\xi < \overline{T}$ a.s.

We have that $T_n \nearrow \overline{T}$ and $\xi < \overline{T}$ a.s., therefore given ω, there exists $T_n(\omega)$ such that $\xi(\omega) < T_n(\omega)$.

Let

$$\Lambda_n = \{\omega : \xi(\omega) < T_n(\omega)\} \tag{3.21}$$

then, $\bigcup_{n \geq 1} \Lambda_n = \left\{\xi < \overline{T}\right\}$. Hence, $P\left(\bigcup_{n \geq 1} \Lambda_n\right) = P\left(\xi < \overline{T}\right) = 1$.

We have assumed that H^* yields arbitrage; That means

$$(H^* \cdot S)_0 = 0, \text{ a.s.}$$

$$(H^* \cdot S)_{\overline{T}} \geq 0 \tag{3.22}$$

$$P\left((H^* \cdot S)_{\overline{T}} > 0\right) > 0 \tag{3.23}$$

Let $\Lambda = \left\{(H^* \cdot S)_{\overline{T}} > 0\right\}$, then by equation (3.23), we know that $P(\Lambda) > 0$. Since $P\left(\bigcup_{n \geq 1} \Lambda_n\right) = 1$, we have $\Lambda = \bigcup_{n \geq 1} (\Lambda_n \cap \Lambda)$. Consequently, there exists at least one Λ_n with $P(\Lambda_n \cap \Lambda) > 0$. Let's call one of such Λ_n, Λ_{n^*}. Therefore $P(\Lambda_{n^*} \cap \Lambda) > 0$.

On the set $\Lambda_{n^*} \cap \Lambda$, $\xi < T_{n^*}$ we have that

$$(H^* \cdot S)_{\overline{T}} = (H^* \cdot S)_{T_{n^*}} \tag{3.24}$$

Equation (3.24) implies $P\left((H^* \cdot S)_{T_{n^*}} > 0\right) > 0$ which violates local NFLVR.

Note that if there is arbitrage on $\left[0, \overline{T}\right]$, the preceding such arbitrage happens either right before $\overline{T}$ or at $\overline{T}$.

Now let's suppose that $\xi = \overline{T}$ a.s. (It is enough to have ξ being equal to $\overline{T}$ on a set of positive probability)

Based on the proof above, it is enough to consider the following type of admissible trading strategies

$$H = f1_{(\gamma,\overline{T}]}, \text{ where } f \in \mathcal{G}_\gamma \tag{3.25}$$

γ is a $\mathbb{G}-$ stopping time. Moreover, without loss of generality, it is enough to consider $f = 1$ and $f = -1$. Let's first suppose that $f = 1$ and assume that NAS fails to hold; That means there exists an admissible trading strategy $H_t^* = 1_{(\gamma,\overline{T}]}(t)$ that gives arbitrage. The above statement is equivalent to

$$\begin{cases} P\left(S_{\overline{T}} - S_\gamma \geq 0\right) = 1 \\ P\left(S_{\overline{T}} - S_\gamma > 0\right) > 0 \end{cases} \tag{3.26}$$

<u>Claim</u>: If the discounted stock price is continuous, then

$$\left\{S_{\overline{T}} - S_\gamma > 0\right\} \subseteq \bigcup_{n\geq 1} \{S_{T_n} - S_\gamma > 0\} \tag{3.27}$$

Let's instead prove that $\bigcap_{n\geq 1} \{S_{T_n} - S_\gamma \leq 0\} \subseteq \left\{S_{\overline{T}} - S_\gamma \leq 0\right\}$.

If $S_{T_n} - S_\gamma \leq 0$ for each n then, since S is continuous, $\lim_{n\to\infty} S_{T_n} = S_{\overline{T}}$. Hence, by continuity $S_{\overline{T}} - S_\gamma \leq 0$

$$\Rightarrow \ 0 < P\left(S_{\overline{T}} - S_\gamma > 0\right) \leq P\left(\bigcup_{n\geq 1} \{S_{T_n} - S_\gamma > 0\}\right) \tag{3.28}$$

Equation (3.28) implies that there exists at least one $n \geq 1$, called n^*, such that $P\left(S_{T_{n^*}} - S_\gamma > 0\right) > 0$ which violates local NFLVR.

Now let's assume that $f = -1$. The same argument as the one for the case $f = 1$ can be applied just by changing signs.

Hence if the strategy $H_t^* = f1_{(\gamma,\overline{T}]}(t)$, where $f \in \mathcal{G}_\gamma$ gives arbitrage, local NFLVR is violated. We just proved that if there is an arbitrage right before $\overline{T}$ or at $\overline{T}$, such arbitrage cannot be exploited using admissible simple predictable trading strategies. In other words, to be able to exploit that arbitrage, the trading strategy should not be constant on $\left(\gamma, \overline{T}\right]$. ■

To achieve our goals, we have, so far, focused on the second point of Theorem 3.2.1, but Theorem 3.2.1 part d) also showed that NFLVR is equivalent to the existence of a uniformly integrable $\mathbb{G}-$ martingale deflator. The existence of a $\mathbb{G}-$ local martingale deflator for the discounted price process shows that it satisfies NA1/NUPBR.

3.4 NA1 and NFLVR

Theorem 3.4.1. *Suppose*

- $\left[q^L, S\right] = k^L q^L \cdot [S, S]$

then, the process $1/q^L$ *is a* $\mathbb{G}-$ *local martingale deflator for* S *and therefore the enlarged market* $(\Omega, \mathcal{F}, \mathbb{G}, S, \mathcal{A}^{\mathbb{G}})$ *satisfies the NA1/NUPBR condition.*

Proof. The process $1/q^L$ is well-defined since $T^L = \inf\left\{t : q_t^L = 0\right\} = \infty$ a.s. (see Jacod [33]).
$(q^x, x \in E)$ is a parameterized $\mathbb{F}-$ local martingale, where a process $(X^x, x \in E)$ is called a parameterized $\mathbb{F}-$ local martingale if for each $x \in E$ the process X^x is $\mathbb{F}-$ local martingale (see Aksamit [1]). Therefore by Theorem 7.1 of Aksamit [1]

$$\overline{q}^L \quad := \quad q^L - \frac{1}{q^L} \cdot \left[q^L, q^L\right] \tag{3.29}$$

is a $(\mathbb{G}, P)-$ local martingale. Therefore, by Itô's formula, the process $1/q^L$ satisfies the following SDE

$$\begin{aligned}
\frac{1}{q^L} &= 1 - \frac{1}{\left(q^L\right)^2} \cdot q^L + \frac{1}{2}\frac{2}{\left(q^L\right)^3} \cdot \left[q^L, q^L\right] \\
&= 1 - \frac{1}{\left(q^L\right)^2} \cdot \left(\overline{q}^L + \frac{1}{q^L} \cdot \left[q^L, q^L\right]\right) + \frac{1}{\left(q^L\right)^3} d\left[q^L, q^L\right] && (3.30) \\
&= 1 - \frac{1}{\left(q^L\right)^2} \cdot \overline{q}^L && (3.31)
\end{aligned}$$

where equation (3.30) follows from (3.29). (3.31) shows that $1/q^L$ is a strictly positive $\mathbb{G}-$ local martingale with $1/q_0^L = 1,\ P-$a.s.. Using integration by parts, we obtain

$$d\left(\frac{S}{q^L}\right) \quad = \quad \frac{1}{q^L} dS + S\, d\frac{1}{q^L} + d\left[S, \frac{1}{q^L}\right]$$

$$\begin{aligned}
&= \frac{1}{q^L}\left(d\widetilde{S} + k^L d\left\langle S,S\right\rangle\right) - \frac{S}{\left(q^L\right)^2}d\overline{q}^L - \frac{1}{\left(q^L\right)^2}d\left[S,\overline{q}^L\right] && (3.32)\\
&= \frac{1}{q^L}\left(d\widetilde{S} + k^L d\left\langle S,S\right\rangle\right) - \frac{S}{\left(q^L\right)^2}d\overline{q}^L - \frac{1}{\left(q^L\right)^2}d\left[S,\, q^L - \frac{1}{q^L}\cdot\left[q^L,q^L\right]\right] && (3.33)\\
&= \frac{1}{q^L}\left(d\widetilde{S} + k^L d\left\langle S,S\right\rangle\right) - \frac{S}{\left(q^L\right)^2}d\overline{q}^L - \frac{1}{\left(q^L\right)^2}d\left[S,\, q^L\right]\\
&= \frac{1}{q^L}d\widetilde{S} + \frac{k^L}{q^L}d\left\langle S,S\right\rangle - \frac{S}{\left(q^L\right)^2}d\overline{q}^L - \frac{1}{\left(q^L\right)^2}d\left[S,\, q^L\right]\\
&= \frac{1}{q^L}d\widetilde{S} + \frac{k^L}{q^L}d\left\langle S,S\right\rangle - \frac{S}{\left(q^L\right)^2}d\overline{q}^L - \frac{1}{\left(q^L\right)^2}k^L q^L d\left[S,S\right] && (3.34)\\
&= \frac{1}{q^L}d\widetilde{S} + \frac{k^L}{q^L}d\left\langle S,S\right\rangle - \frac{S}{\left(q^L\right)^2}d\overline{q}^L - \frac{k^L}{q^L}d\left[S,S\right]\\
&= \frac{1}{q^L}d\widetilde{S} + \frac{k^L}{q^L}d\left[S,S\right] - \frac{S}{\left(q^L\right)^2}d\overline{q}^L - \frac{k^L}{q^L}d\left[S,S\right] && (3.35)\\
&= \frac{1}{q^L}d\widetilde{S} - \frac{S}{\left(q^L\right)^2}d\overline{q}^L
\end{aligned}$$

Equations (3.32) and (3.33) follow from (3.9) and (3.29). (3.34) and (3.35) follow by assumption and from the continuity of S. Since $\widetilde{S}$ and $\overline{q}^L$ are $\mathbb{G}-$ local martingales, so is S/q^L. Consequently, $1/q^L$ is a local martingale deflator in $\mathbb{G}$. ■

The assumption of Theorem 3.4.1 is an extension of (3.7). Instead of equation (3.7) being valid for each $x \in E$, we want it to be valid even when the processes involved, q^x and k^x, are composed with L. That condition is a sufficient condition for the financial market $\mathcal{M}^{\mathbb{G}}$ to satisfy No Arbitrage of the First Kind. Aksamit [1] studied the stability of NA1 under initial enlargement for parameterized $\mathbb{F}-$ local martingales; Hence, her results are not applicable to any discounted price process S that is not a parameterized $\mathbb{F}-$ local martingale. Amendinger [3] studied the stability of NFLVR under initial enlargement of filtrations under the following restrictive condition

$$Q_t(\omega, dx) \sim \eta(dx) \tag{3.36}$$

Under Amendinger's condition, the process $1/q^L$ was also studied and Amemdinger proved that it can be used to define an equivalent local martingale measure. Under our condition, the process $1/q^L$ cannot necessarily be used to construct an equivalent local martingale measure but it gives us at least a $\mathbb{G}-$ local martingale deflator under a less restrictive condition than (3.36).

Corollary 3.4.1.1. *Under the same assumption as in Theorem 3.4.1, NFLVR holds on* $[0,T]$ *if and only if* $E_P\left[1/q_\infty^L\right]=1$.

Proof. Being a positive $\mathbb{G}-$ local martingale, the process $1/q^L$ is therefore a $\mathbb{G}-$ supermartingale. Consequently, by Fatou's lemma, it is a uniformly integrable $\mathbb{G}-$ martingale if and only if $E_P\left[1/q_\infty^L\right]=E_P\left[1/q_0^L\right]=1$. ∎

Theorem 3.4.1 shows that there exists at least one local martingale deflator for the discounted price process S in $\mathbb{G}$. A very natural question is therefore to find the general structure of all $\mathbb{G}-$ local martingale deflators.

Lemma 3.4.2. *Under the same assumption as in Theorem 3.4.1, if* $D=(D_t)_{t\geq 0}$ *is a local martingale deflator for* S *in* $\mathbb{G}$*, then*

$$D \equiv \frac{1}{q^L}\varepsilon\left(\widetilde{X}\right) \tag{3.37}$$

where $\widetilde{X}$ *is a continuous* $\mathbb{G}-$ *semimartingale.*

Proof. Under the assumption of Theorem 3.4.1, the set of all local martingale deflators in $\mathbb{G}$ is not empty since $1/q^L$ is in it. Let D be a local martingale deflator in $\mathbb{G}$, then it can be written as (see Schweizer [58] and (3.9))

$$D = \varepsilon\left(-\int k_t^L d\widetilde{S}_t+\widetilde{R}\right) \tag{3.38}$$

where $\widetilde{R}$ is a $\mathbb{G}-$ local martingale and $\left[\widetilde{S},\widetilde{R}\right]\equiv 0,\ P-$a.s.. By (3.29), we know that $\overline{q}^L$ is a $\mathbb{G}-$ local martingale, therefore, we can find a $\mathbb{G}-$ local martingale, $\overline{R}$, starting at 0 and strongly orthogonal to $\widetilde{S}$ such that

$$d\overline{q}^L = k^L q^L d\widetilde{S}+d\overline{R} \tag{3.39}$$

with $\left[\widetilde{S},\overline{R}\right]\equiv 0,\ P-$a.s.. Then by (3.31), we have

$$\begin{aligned} d\frac{1}{q^L} &= -\frac{1}{\left(q^L\right)^2}d\overline{q}^L \\ &= -\frac{1}{\left(q^L\right)^2}\left(k^L q^L d\widetilde{S}+d\overline{R}\right) \\ &= \frac{1}{q^L}\left(-k^L d\widetilde{S}\right)-\frac{1}{\left(q^L\right)^2}d\overline{R} \end{aligned}$$

$\overline{R}$ is a $\mathbb{G}-$ local martingale, hence, we can find a $\mathbb{G}-$ local martingale, U, starting at 0 and strongly orthogonal to $\widetilde{R}$ such that

$$d\overline{R} = -q^L d\widetilde{R} + dU \tag{3.40}$$

with $\left[\widetilde{R}, U\right] \equiv 0,\ P-$ a.s.. Consequently,

$$\begin{aligned} d\frac{1}{q^L} &= \frac{1}{q^L}\left(-k^L d\widetilde{S}\right) - \frac{1}{\left(q^L\right)^2} d\overline{R} \\ &= \frac{1}{q^L}\left(-k^L d\widetilde{S}\right) - \frac{1}{\left(q^L\right)^2}\left\{-q^L d\widetilde{R} + dU\right\} \\ &= \frac{1}{q^L}\left(-k^L d\widetilde{S} + d\widetilde{R}\right) - \frac{1}{\left(q^L\right)^2} dU \\ &= \frac{1}{q^L}\left(-k^L d\widetilde{S} + d\widetilde{R} - \frac{1}{q^L} dU\right) \end{aligned}$$

We have that $\left[\widetilde{S}, \widetilde{R}\right] \equiv 0,\ P-$ a.s. and $\left[\widetilde{R}, U\right] \equiv 0,\ P-$ a.s. but it does not necessarily mean that $\left[\widetilde{S}, U\right] \equiv 0,\ P-$ a.s.. Consequently, using a Gram-Schmidt approach to make $\widetilde{S}$ and a process related to U strongly orthogonal, we let

$$\begin{aligned} A &= \widetilde{S} \\ B &= U - \frac{d\left[U, \widetilde{S}\right]}{d\left[\widetilde{S}, \widetilde{S}\right]} \cdot \widetilde{S} \end{aligned} \tag{3.41}$$

From (3.41), we have

$$\begin{aligned} \left[B, \widetilde{S}\right] &= \left[U - \frac{d\left[U, \widetilde{S}\right]}{d\left[\widetilde{S}, \widetilde{S}\right]} \cdot \widetilde{S}, \widetilde{S}\right] \\ &= \left[U, \widetilde{S}\right] - \left[\frac{d\left[U, \widetilde{S}\right]}{d\left[\widetilde{S}, \widetilde{S}\right]} \cdot \widetilde{S}, \widetilde{S}\right] \\ &= \left[U, \widetilde{S}\right] - \frac{d\left[U, \widetilde{S}\right]}{d\left[\widetilde{S}, \widetilde{S}\right]} \cdot \left[\widetilde{S}, \widetilde{S}\right] \\ &= \left[U, \widetilde{S}\right] - \left[U, \widetilde{S}\right] = 0,\ P-\text{a.s.} \end{aligned} \tag{3.42}$$

similarly,

$$\left[B, \widetilde{R}\right] = \left[U - \frac{d\left[U, \widetilde{S}\right]}{d\left[\widetilde{S}, \widetilde{S}\right]} \cdot \widetilde{S}, \widetilde{R}\right]$$

$$
\begin{aligned}
&= \left[U, \widetilde{R}\right] - \left[\frac{d\left[U, \widetilde{S}\right]}{d\left[\widetilde{S}, \widetilde{S}\right]} \cdot \widetilde{S}, \widetilde{R}\right] \\
&= \left[U, \widetilde{R}\right] - \frac{d\left[U, \widetilde{S}\right]}{d\left[\widetilde{S}, \widetilde{S}\right]} \cdot \left[\widetilde{S}, \widetilde{R}\right] = 0, \ P-\text{a.s.} \qquad (3.43)
\end{aligned}
$$

From (3.41), we also have that $U = B + \frac{d[U,\widetilde{S}]}{d[\widetilde{S},\widetilde{S}]} \cdot \widetilde{S}$. Let $J := \frac{d[U,\widetilde{S}]}{d[\widetilde{S},\widetilde{S}]}$, then

$$
\begin{aligned}
d\frac{1}{q^L} &= \frac{1}{q^L}\left(-k^L d\widetilde{S} + d\widetilde{R} - \frac{1}{q^L} dU\right) \\
d\frac{1}{q^L} &= \frac{1}{q^L}\left(-k^L d\widetilde{S} + d\widetilde{R} - \frac{1}{q^L} dB - \frac{J}{q^L} d\widetilde{S}\right) \\
&= \frac{1}{q^L}\left(-\left[k^L + \frac{J}{q^L}\right] d\widetilde{S} + d\widetilde{R} - \frac{1}{q^L} dB\right) \qquad (3.44)
\end{aligned}
$$

From (3.44), we obtain

$$
\begin{aligned}
\frac{1}{q^L} &= \varepsilon\left(-\int \left[k^L + \frac{J}{q^L}\right] d\widetilde{S} + \widetilde{R} - \int \frac{1}{q^L} dB\right) \\
&= \varepsilon\left(-\int \left[k^L + \frac{J}{q^L}\right] d\widetilde{S} + \widetilde{R} + \int k^L \frac{J}{q^L} d\left[\widetilde{S}, \widetilde{S}\right] - \int \frac{1}{q^L} dB - \int k^L \frac{J}{q^L} d\left[\widetilde{S}, \widetilde{S}\right]\right)
\end{aligned}
$$

Let $X := -\int \left[k^L + \frac{J}{q^L}\right] d\widetilde{S} + \widetilde{R} + \int k^L \frac{J}{q^L} d\left[\widetilde{S}, \widetilde{S}\right]$ and $Y := -\int \frac{1}{q^L} dB + \int k^L \frac{J}{q^L} d\left[\widetilde{S}, \widetilde{S}\right]$, then

$$
\begin{aligned}
[X, Y] &= \left[-\int \left[k^L + \frac{J}{q^L}\right] d\widetilde{S} + \widetilde{R} + \int k^L \frac{J}{q^L} d\left[\widetilde{S}, \widetilde{S}\right], -\int \frac{1}{q^L} dB - \int k^L \frac{J}{q^L} d\left[\widetilde{S}, \widetilde{S}\right]\right] \\
&= \int \frac{1}{q^L}\left[k^L + \frac{J}{q^L}\right] d\left[\widetilde{S}, B\right] - \int \frac{1}{q^L} d\left[\widetilde{R}, B\right] = 0, \ P-\text{a.s.}
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\frac{1}{q^L} &= \varepsilon\left(X + Y\right) \\
&= \varepsilon\left(X\right)\varepsilon\left(Y\right) \\
&= \varepsilon\left(-\int \left[k^L + \frac{J}{q^L}\right] d\widetilde{S} + \widetilde{R} + \int k^L \frac{J}{q^L} d\left[\widetilde{S}, \widetilde{S}\right]\right) \varepsilon\left(-\int \frac{1}{q^L} dB - \int k^L \frac{J}{q^L} d\left[\widetilde{S}, \widetilde{S}\right]\right) \\
&= \varepsilon\left(-\int k^L d\widetilde{S} + \widetilde{R} - \int \frac{J}{q^L} d\widetilde{S} + \int k^L \frac{J}{q^L} d\left[\widetilde{S}, \widetilde{S}\right]\right) \\
&\times \ \varepsilon\left(-\int \frac{1}{q^L} dB - \int k^L \frac{J}{q^L} d\left[\widetilde{S}, \widetilde{S}\right]\right) \\
&= \varepsilon\left(-\int k^L d\widetilde{S} + \widetilde{R}\right) \varepsilon\left(-\int \frac{J}{q^L} d\widetilde{S}\right) \varepsilon\left(-\int \frac{1}{q^L} dB - \int k^L \frac{J}{q^L} d\left[\widetilde{S}, \widetilde{S}\right]\right) \qquad (3.45)
\end{aligned}
$$

$$
\begin{aligned}
&= D\,\varepsilon\left(-\int \frac{J}{q^L} d\widetilde{S}\right)\varepsilon\left(-\int \frac{1}{q^L} dB - \int k^L \frac{J}{q^L} d\left[\widetilde{S},\widetilde{S}\right]\right) \\
&= D\,\varepsilon\left(-\int \frac{J}{q^L} d\widetilde{S} - \int \frac{1}{q^L} dB - \int k^L \frac{J}{q^L} d\left[\widetilde{S},\widetilde{S}\right]\right) \qquad (3.46)
\end{aligned}
$$

$$
D = \frac{1}{q^L}\,\varepsilon\left(\int \frac{1}{q^L} dU + \int k^L \frac{J}{q^L} d\left[\widetilde{S},\widetilde{S}\right] + \int \left(\frac{1}{q^L}\right)^2 d\,[U,U]\right) \qquad (3.47)
$$

(3.45) and (3.46) follow from Theorem 38 in Chapter II of [52]. (3.47) follows from the Corollary to Theorem 38 and from the fact that $[U,U] = [B,B] + J^2 \cdot [\widetilde{S},\widetilde{S}]$ since $U = B + J \cdot \widetilde{S}$. Therefore, the Lemma follows if we set $\widetilde{X} := \int \frac{1}{q^L} dU + \int k^L \frac{J}{q^L} d\left[\widetilde{S},\widetilde{S}\right] + \int \left(\frac{1}{q^L}\right)^2 d\,[U,U]$. ■

3.5 Applications

In this section, One of our examples concerns an insider trading case that recently appeared on the news. We then generalize that case and introduce a stochastic volatility model with inside information.

3.5.1 Jacod's finite expansion

Let's consider an insider who possesses knowledge of the results of the development of a medicine. The insider's knowledge is equivalent to knowing one of the following sets:

$$
\begin{aligned}
B_i &= \{\text{The medicine works more than } i\,\text{x}\,10\% \text{ of the time}\},\ i \in \{1,\cdots,9\} \\
B_{10} &= \{\text{The medicine is better than alternatives}\} \\
B_{11} &= \{\text{Side effects are minimal}\}
\end{aligned}
$$

In Ben Protess and Matthew Goldstein's article [28] that appeared in the New York Times, it is mentioned that Mr. Martoma, a former trader of SAC Advisors, is accused of obtaining secret information from a doctor about clinical trials for an Alzheimer's drug. In his indictment, Mr. Martoma is accused of having learned that the trials produced negative results. Although the different sets described above are used as an illustrating example, they can be used to model the extra information Mr. Martoma supposedly obtained. Mr. Martoma has now been convicted of insider trading (See Alexandra Stevenson and Matthew Goldstein's article in the New York Times [61]).

The sets $(B_i)_{i=1}^{11}$ do not form a partition of the sample space but they can be made into one by the following procedure:

$$
\begin{aligned}
A_1 &= B_1 \\
A_2 &= B_2 \cap B_1^c \\
A_i &= B_i \cap \left(\cup_{j=1}^{i-1} B_j\right)^c, \quad i \in \{3, \cdots, 11\} \\
A_{12} &= \Omega \cap \left(\cup_{k=1}^{11} A_k\right)^c
\end{aligned}
$$

More generally, let $\mathcal{A} = (A_1, A_2, \cdots, A_k)$ be a sequence of events such that $A_i \cap A_j = \varnothing,\ i \neq j$ and $\cup_{i=1}^{k} A_i = \Omega$. $\mathbb{G}$ is the filtration generated by $\mathbb{F}$ and $\mathcal{A}$, and the information represented by L can be modeled as $L = \sum_{i=1}^{k} a_i 1_{A_i}$. For instance, in the case above, $L = \sum_{i=1}^{12} a_i 1_{A_i}$ with $a_{12} = 0$, since $\bigcup_{i=1}^{11} A_i = \bigcup_{i=1}^{11} B_i \subset \Omega$. By Corollary 3 (p. 371 of [52]), we know that the H' hypothesis is satisfied, hence every $\mathbb{F}$ semimartingale is a $\mathbb{G}$ semimartingale. The distribution of L is given by $\eta(dx) = \sum_{i=1}^{k} P(A_i)\varepsilon_{a_i}(dx)$ where ε_{a_i} denotes the point mass at a_i. The regular conditional distribution of L given $\mathcal{F}_t$ has density with respect to η given by

$$
q_t^x = \sum_{i=1}^{k} \frac{P(L = a_i \,|\, \mathcal{F}_t)}{P(A_i)} 1_{\{x=a_i\}} = \sum_{i=1}^{k} \frac{N_t^i}{P(A_i)} 1_{\{x=a_i\}}
$$

where $N_t^i := P(L = a_i \,|\, \mathcal{F}_t)$ is a bounded $(\mathbb{F}, P)-$ martingale. To find the $(\mathbb{G}, P)-$ semimartingale decomposition of the discounted price process, we have to find an $\mathbb{F}-$ predictable process k^x, $x \in E := \{a_i\}_{i=1}^{k}$ such that $\langle q^x, S\rangle = (k^x\, q^x) \cdot \langle S, S\rangle$.

$$
\begin{aligned}
\langle q^x, S\rangle_t &= \sum_{i=1}^{k} \frac{1}{P(A_i)} \left\langle N^i, S\right\rangle_t 1_{\{x=a_i\}} \\
&= \sum_{i=1}^{k} \frac{J_t^i}{P(A_i)} \langle S, S\rangle_t\, 1_{\{x=a_i\}}
\end{aligned}
$$

where Kunita-Watanabe implies that for all i, there exists an $\mathbb{F}-$ predictable process, J^i, such that $\forall\ i,\ d\left\langle N^i, S\right\rangle_t = J_t^i\, d\langle S, S\rangle_t$. Hence, we choose k^x so that, for all t, it satisfies the following equation:

$$
k_t^x\, q_t^x = \sum_{i=1}^{k} \frac{J_t^i}{P(A_i)} 1_{\{x=a_i\}} \Rightarrow k_t^x = \frac{\sum_{i=1}^{k} \frac{J_t^i}{P(A_i)} 1_{\{x=a_i\}}}{q_t^x} = \frac{\sum_{i=1}^{k} \frac{J_t^i}{P(A_i)} 1_{\{x=a_i\}}}{\sum_{i=1}^{k} \frac{N_t^i}{P(A_i)} 1_{\{x=a_i\}}}
$$

After we composed the k^x process obtained above with L, we have the following

$$k_t^L = \frac{\sum_{i=1}^k \frac{J_t^i}{P(A_i)} 1_{\{L=a_i\}}}{\sum_{i=1}^k \frac{N_t^i}{P(A_i)} 1_{\{L=a_i\}}} = \frac{\sum_{i=1}^k \frac{J_t^i}{P(A_i)} 1_{\{L=a_i\}}}{q_t^L} \tag{3.48}$$

Proposition 3.5.1. *The insider's financial market* $\mathcal{M}^{\mathbb{G}} := \left(\Omega, \mathcal{F}, \mathbb{F} \vee \sigma(\mathcal{A}), P, S, \mathcal{A}^{\mathbb{G}}\right)$ *satisfies Local NFLVR, and therefore the NAS condition.*

Before we prove the proposition just above, we notice that if Local NFLVR holds for the insider's enlarged financial market described just above, then Imkeller's results are not applicable, because there is a possibility that an equivalent local martingale measure exits.

Proof. We first need to check that $\int_0^t \left(k_s^L\right)^2 d\langle M, M\rangle_s < \infty$ a.s. for each $t \geq 0$
Using the Martingale Representation theorem and since we have not assume that markets are complete, we can write $N^i = J^i \cdot S + X$ where $[S, X] = \langle S, X\rangle = 0$. Therefore,

$$\langle N^i, N^i\rangle = \int_0^{\cdot} (J_s^i)^2 d\langle S, S\rangle_s + \langle X, X\rangle \tag{3.49}$$

$$\geq \int_0^{\cdot} (J_s^i)^2 d\langle S, S\rangle_s \tag{3.50}$$

$N_t^i = P(L = a_i \,|\, \mathcal{F}_t) = E(1_{\{L=a_i\}} \,|\, \mathcal{F}_t)$ is bounded for all t which implies that $\langle N^i, N^i\rangle < \infty$ a.s.. Consequently,

$$\int_0^t (J_s^i)^2 d\langle S, S\rangle_s < \infty \;\; a.s. \;\; \forall\, t \geq 0, \;\; \forall\, i \in \{1, \cdots, k\} \tag{3.51}$$

and,

$$\begin{aligned}
\left|\int_0^t \left(k_s^L q_s^L\right)^2 d\langle S, S\rangle_s\right| &= \left|\int_0^t \left(\sum_{i=1}^k \frac{J_t^i}{P(A_i)} 1_{\{L=a_i\}}\right)^2 d\langle S, S\rangle_s\right| \\
&= \left|\sum_{i=1}^k \int_0^t \left(\frac{J_t^i}{P(A_i)}\right)^2 1_{\{L=a_i\}} d\langle S, S\rangle_s\right| \\
&\leq \sum_{i=1}^k \int_0^t \left(\frac{J_t^i}{P(A_i)}\right)^2 d\langle S, S\rangle_s < \infty, \quad \forall\, t \geq 0 \\
&= \sum_{i=1}^k \frac{1}{P(A_i)^2} \int_0^t (J_t^i)^2 d\langle S, S\rangle_s < \infty, \quad \forall\, t \geq 0
\end{aligned} \tag{3.52}$$

where the last inequality follows from (3.51). Therefore,

$$\infty > \int_0^t \left(k_s^L q_s^L\right)^2 d\langle S, S\rangle_s$$

$$\geq \quad C^2 \int_0^t \left(k_s^L\right)^2 d\left\langle S, S\right\rangle_s \tag{3.53}$$

(3.53) follows from the fact that q^L has continuous paths and $T^L = \inf\left\{t : q_t^L = 0\right\} = \infty$ a.s.; then pathwise we have $q_t(\omega, L(\omega)) \geq C(\omega) > 0$ for some C.
Let $T_{1/m}^L = \inf\left\{t > 0 : q_t^L \leq 1/m\right\}$ be a sequence of increasing $\mathbb{G}-$ stopping times, then

$$\lim_{m\to\infty} T_{1/m}^L \quad = \quad \infty \text{ a.s.} \tag{3.54}$$

It can be easily seen that $\lim_{t\to\infty} q_t^L = 1$ a.s. since $\lim_{t\to\infty} q_t^x = 1$ a.s. where the null set depends on x and there is only a finite number of them. Moreover, $q_0^L = 1$ and $T^L = \infty$ a.s..
Consequently, $\lim_{m\to\infty} T_{1/m}^L = \infty$ a.s.
Let $\zeta_m = \inf\left\{t > 0 : \left|\sum_{i=1}^k \frac{1}{P(A_i)^2} \int_0^t (J_t^i)^2 d\left\langle S, S\right\rangle_s\right| \geq m\right\}$ be a sequence of increasing $\mathbb{F}$ (hence $\mathbb{G}$) stopping times, then

$$\text{The stopping times } \zeta_m \text{ are a.s. infinite from some } n \text{ on.} \tag{3.55}$$

For each $i \in \{1, \cdots, k\}$, (3.55) comes from the fact that

$$\lim_{t\to\infty} \int_0^t (J_s^i)^2 d\left\langle S, S\right\rangle_s \leq \lim_{t\to\infty} \left\langle N^i, N^i\right\rangle_t = \left\langle N^i, N^i\right\rangle_\infty$$

$N^i \in H^2$ since it is bounded. Hence, by Chapter IV Proposition (1.23) of [53], $\left\langle N^i, N^i\right\rangle$ is integrable. Therefore,

$$\lim_{t\to\infty} \int_0^t (J_s^i)^2 d\left\langle S, S\right\rangle_s < \infty.$$

Let's now define $T_m = T_{1/m}^L \wedge \zeta_m \nearrow \infty$ as $m \nearrow \infty$. Hence,

$$\lim_{m\to\infty} T_m > T \text{ a.s.} \tag{3.56}$$

Hence, by Theorem 3.3.2, the insider's financial market satisfies the Local NFLVR on $[0, T]$ and by applying Theorem 3.3.4, it also satisfies the NAS condition. ■

We now investigate the possibility that the enlarged financial market satisfies the No Arbitrage of the First Kind condition.

Proposition 3.5.2. *The insider's financial market* $\mathcal{M}^{\mathbb{G}} = \left(\Omega, \mathcal{F}, \mathbb{F} \vee \sigma\left(\mathcal{A}\right), P, S, \mathcal{A}^{\mathbb{G}}\right)$ *satisfies the NA1 condition.*

Proof. By Theorem 3.4.1, it is enough to check that $\left[q^L, S\right] = k^L q^L \cdot [S, S]$. From equation (3.48), we already know that

$$k_t^L q_t^L \;=\; \sum_{i=1}^{k} \frac{J_t^i}{P(A_i)} 1_{\{L=a_i\}} \tag{3.57}$$

Hence, from (3.57), it follows that

$$k_t^L q_t^L \cdot [S, S] \;=\; \sum_{i=1}^{k} \frac{J_t^i}{P(A_i)} 1_{\{L=a_i\}} \cdot [S, S] \tag{3.58}$$

Therefore, we only have to compute $\left[q^L, S\right]$

$$\begin{aligned}
\left[q^L, S\right]_t &= \left[\sum_{i=1}^{k} \frac{N^i}{P(A_i)} 1_{\{L=a_i\}}, S\right]_t \\
&= \sum_{i=1}^{k} \frac{1_{\{L=a_i\}}}{P(A_i)} \cdot \left[N^i, S\right]_t \\
&= \sum_{i=1}^{k} \frac{1_{\{L=a_i\}}}{P(A_i)} \cdot \left\langle N^i, S\right\rangle_t \\
&= \sum_{i=1}^{k} \frac{1_{\{L=a_i\}}}{P(A_i)} J_t^i \cdot \langle S, S\rangle_t \\
&= \sum_{i=1}^{k} \frac{1_{\{L=a_i\}}}{P(A_i)} J_t^i \cdot [S, S]_t
\end{aligned} \tag{3.59}$$

The result then follows from equations (3.58) and (3.59). ■

From the statements below equation (3.54), we have that $q_\infty^L = \lim_{t\to\infty} q_t^L = 1$, hence the following holds

Theorem 3.5.3. *The enlarged financial market $\mathcal{M}^{\mathbb{G}}$ satisfies NFLVR on $[0, T]$.*

Proof. By Corollary 3.4.1.1, it is enough to have $E_P\left(1/q_\infty^L\right) = 1$. The result immediately follows since we have already noticed that $q_\infty^L = 1$. ■

The enlarged market $\mathcal{M}^{\mathbb{G}}$ satisfies NFLVR, hence a condition stronger than NAS holds. That is $\mathcal{M}^{\mathbb{G}}$ also satisfies NA with respect to general admissible integrands. Although in practice only NAS, not NA, is applicable because real-life trading strategies are sequences of buy and hold which are modeled with simple predictable integrands.

3.5.2 Jacod's countable expansion

We now consider the case of Jacod's countable expansion as orginally studied by Jacod, see Protter [52]. The case of the Jacod's finite expansion is most likely enough to model most insider trading cases where the extra knowledge is equivalent to knowing one of a finite number of outcomes, like the results of some experiment, the board of directors of a company deciding to acquire or not another company, etc.. Hence, the extra knowledge can be represented by sets as seen with the example of $(B_i)_{1\leq i\leq 11}$ but the case of countable expansion presents in itself some interesting mathematical challenges.

Let $\mathcal{B} = (A_1, A_2, \cdots)$ be a sequence of events such that $A_i \cap A_j = \varnothing,\ i \neq j$ and $\cup_{i=1}^{\infty} A_i = \Omega$. $\mathbb{G}$ is still the filtration generated by $\mathbb{F}$ and $L = \sum_{i=1}^{\infty} a_i 1_{A_i}$. The distribution of L is given by $\eta(dx) = \sum_{i=1}^{\infty} P(A_i)\varepsilon_{a_i}(dx)$ where ε_{a_i} denotes the point mass at a_i. The regular conditional distribution of L given $\mathcal{F}_t$ has density with respect to η given by

$$q_t^x = \sum_{i=1}^{\infty} \frac{P(L = a_i \,|\, \mathcal{F}_t)}{P(A_i)} 1_{\{x=a_i\}}$$

with $N_t^i := P(L = a_i \,|\, \mathcal{F}_t)$, and the processes J^i exist by Kunita-Watanabe. Following a procedure similar to the finite expansion case, we obtain

$$\begin{aligned} k_t^L q_t^L &= \sum_{i=1}^{\infty} \frac{J_t^i}{P(A_i)} 1_{\{L=a_i\}} \\ \left(k_t^L q_t^L\right)^2 &= \sum_{i=1}^{\infty} \left(\frac{J_t^i}{P(A_i)}\right)^2 1_{\{L=a_i\}} \end{aligned} \tag{3.60}$$

Proposition 3.5.4. *The insider's financial market* $\mathcal{M}^{\mathbb{G}} = \left(\Omega, \mathcal{F}, \mathbb{F} \vee \sigma(\mathcal{B}), P, S, \mathcal{A}^{\mathbb{G}}\right)$ *satisfies Local NFLVR, and therefore the NAS condition.*

Proof. As in case of the Jacod's finite expansion, to prove that $\int_0^t \left(k_s^L\right)^2 d\langle S, S\rangle_s < \infty$ a.s. $\forall\, t \geq 0$, it is enough to prove that $\int_0^t \left(k_s^L q_s^L\right)^2 d\langle S, S\rangle_s < \infty$ a.s. $\forall\, t \geq 0$.

Let

$$r = \sum_{i=2}^{\infty} P(A_i)^6 + P(A_1) - \sum_{i=2}^{\infty} P(A_i)^5 P(A_1) < \infty$$

and let's define a probability measure R equivalent to P as follows:

$$dR = \frac{1}{r}\left(\sum_{i=2}^{\infty} P(A_i)^5 1_{\{L=a_i\}} + (1 - \sum_{i=2}^{\infty} P(A_i)^5) 1_{\{L=a_1\}}\right) dP \tag{3.61}$$

$$:= \quad X\,dP$$

By Cauchy-Schwarz and using the fact that S is a $(\mathbb{F}, P)-$ continuous local martingale, we have the following inequalities:

$$\begin{aligned}
E_R\left(\int_0^t \left(k_s^L q_s^L\right)^2 d\left\langle S,S\right\rangle_s\right) &= E_R\left(\int_0^t \left(k_s^L q_s^L\right)^2 d\left[S,S\right]_s\right) \\
&= E_R\left(\int_0^t \sum_{i=1}^{\infty}\left(\frac{J_s^i}{P(A_i)}\right)^2 1_{\{L=a_i\}} d\left[M,M\right]_s\right) \\
&= E_R\left(\sum_{i=1}^{\infty}\frac{1_{\{L=a_i\}}}{P(A_i)^2}\int_0^t \left(J_s^i\right)^2 d\left[S,S\right]_s\right) \\
&= \sum_{i=1}^{\infty}\frac{1}{P(A_i)^2} E_R\left(1_{\{L=a_i\}}\int_0^t \left(J_s^i\right)^2 d\left[S,S\right]_s\right) \\
&\leq \sum_{i=1}^{\infty}\frac{1}{P(A_i)^2}\left\{E_R\left(1_{\{L=a_i\}}\right)^2\right\}^{1/2}\left\{E_R\left(\int_0^t \left(J_s^i\right)^2 d\left[S,S\right]_s\right)^2\right\}^{1/2} \\
&\leq \sum_{i=1}^{\infty}\frac{R(A_i)^{1/2}}{P(A_i)^2}\left\{E_R\left(\int_0^t \left(J_s^i\right)^2 d\left[S,S\right]_s\right)^2\right\}^{1/2} \\
&\leq \sum_{i=1}^{\infty}\frac{R(A_i)^{1/2}}{P(A_i)^2}\left\{E_R\left([N^i,N^i]_t^2\right)\right\}^{1/2}
\end{aligned}$$

where the last inequality follows from equation (3.50).

Let

$$R_n^k = \inf\left\{t \,:\, \sum_{i=1}^{k}[N^i,N^i]_t \geq n\right\}$$

then $\Upsilon_n = R_n^n \nearrow \infty$ as $n \to \infty$ since $\sum_{i=1}^k [N^i,N^i]_t < \infty$ a.s. for each $t \geq 0$ and $k \geq 0$.

Consequently,

$$\begin{aligned}
E_R\left(\int_0^{t\wedge\Upsilon_n} \left(k_s^L q_s^L\right)^2 d\left\langle S,S\right\rangle_s\right) &\leq \sum_{i=1}^{\infty}\frac{R(A_i)^{1/2}}{P(A_i)^2}\left\{E_R\left(\int_0^{t\wedge\Upsilon_n} \left(J_s^i\right)^2 d\left[S,S\right]_s\right)^2\right\}^{1/2} \\
&\leq \sum_{i=1}^{\infty}\frac{R(A_i)^{1/2}}{P(A_i)^2}\left\{E_R\left([N^i,N^i]_{t\wedge\Upsilon_n}^2\right)\right\}^{1/2} \\
&\leq n\sum_{i=1}^{\infty}\frac{R(A_i)^{1/2}}{P(A_i)^2} \\
&= n\left\{\frac{R(A_1)^{1/2}}{P(A_1)^2} + \sum_{i=2}^{\infty}\frac{R(A_i)^{1/2}}{P(A_i)^2}\right\}
\end{aligned}$$

We have $R(A_i) = E_P\left(1_{A_i}X\right)$ where $X = \frac{dR}{dP}$. Hence,

$$R(A_i) = \begin{cases} \frac{P(A_1) - \sum_{j=2}^{\infty} P(A_i)^5 P(A_1)}{r} & i = 1 \\ \frac{P(A_i)^6}{r} & i \neq 1 \end{cases}$$

So,

$$\begin{aligned} \sum_{i=2}^{\infty} \frac{R(A_i)^{1/2}}{P(A_i)^2} &= \frac{1}{r}\sum_{i=2}^{\infty} \frac{P(A_i)^{6/2}}{P(A_i)^2} \\ &= \frac{1}{r}\sum_{i=2}^{\infty} P(A_i) = \frac{1 - P(A_1)}{r} < \infty \end{aligned}$$

Therefore, $\int_0^{t\wedge\Upsilon_n}\left(k_s^L q_s^L\right)^2 d\left\langle S,S\right\rangle_s < \infty$ R-a.s., $\forall t \geq 0$. That means, $\int_0^{t\wedge\Upsilon_n}\left(k_s^L q_s^L\right)^2 d\left\langle S,S\right\rangle_s < \infty$ P-a.s., $\forall t \geq 0$. Since, $\Upsilon_n \nearrow \infty$, then $\int_0^{t}\left(k_s^L q_s^L\right)^2 d\left\langle S,S\right\rangle_s < \infty$ P-a.s., $\forall t \geq 0$. By a reasoning similar to the finite case, we obtain that $\int_0^{t}\left(k_s^L\right)^2 d\left\langle S,S\right\rangle_s < \infty$ P-a.s., $\forall t \geq 0$.
Let $T_{1/m}^L = \inf\left\{t > 0 : q_t^L \leq 1/m\right\}$ be an increasing sequence of $\mathbb{G}-$ stopping times. Since, $\lim_{m\to\infty} T_{1/m}^x = T^x$ where $T^x = \inf\left\{t > 0 : q_t^x = 0\right\}$, and $T^L = \infty$ a.s., we have $\lim_{m\to\infty} T_{1/m}^L = \infty$ a.s.
Let $\zeta_m = \inf\left\{t > 0 : \left|\int_0^t \sum_{i=1}^{\infty}\left(\frac{J_s^i}{P(A_i)}\right)^2 1_{\{L=a_i\}} d\left\langle S,S\right\rangle_s\right| \geq m\right\}$ be a sequence of increasing $\mathbb{F}$ (hence $\mathbb{G}$) stopping times. Notice that

$$\int_0^t \left(k_s^L q_s^L\right)^2 d\left\langle S,S\right\rangle_s = \int_0^t \sum_{i=1}^{\infty}\left(\frac{J_s^i}{P(A_i)}\right)^2 1_{\{L=a_i\}} d\left\langle S,S\right\rangle_s$$

We proved above that $\int_0^{t}\left(k_s^L q_s^L\right)^2 d\left\langle S,S\right\rangle_s < \infty$ P-a.s., $\forall t \geq 0$. Hence, $\lim_{m\to\infty}\zeta_m = \infty$.
Let's now define $T_m = T_{1/m}^L \wedge \zeta_m \nearrow \infty$ as $m \nearrow \infty$. Hence,

$$\lim_{m\to\infty} T_m > T \text{ a.s.}$$

The result of the proposition follows. ■

The following two results are similar to the case of Jacod's finite expansion and the proofs henceforth will not be presented since they are essentially the same as above

Proposition 3.5.5. *The insider's financial market* $\mathcal{M}^{\mathbb{G}} = \left(\Omega, \mathcal{F}, \mathbb{F}\vee\sigma(\mathcal{B}), P, S, \mathcal{A}^{\mathbb{G}}\right)$ *satisfies the NA1 condition.*

Theorem 3.5.6. *The enlarged financial market* $\mathcal{M}^{\mathbb{G}} = \left(\Omega, \mathcal{F}, \mathbb{F}\vee\sigma(\mathcal{B}), P, S, \mathcal{A}^{\mathbb{G}}\right)$ *satisfies NFLVR on* $[0,T]$.

3.5.3 Stochastic volatility model with additional information

In the cases of the Jacod finite and countable expansions, the insider does not know the future value of the stock but her extra knowledge is intrinsic to the stock and has a discrete distribution. Such a knowledge is a realistic modeling of the different information levels of asymmetric market agents because stock prices are determined by supply and demand. Going a step further than a discrete modeling of the inside information, we consider the case when the insider's extra knowledge has a continuous distribution by using a Gaussian expansion of the regular trader's filtration.

Suppose B^1 and B^2 are standard Brownian motions. Let the regular trader's filtration $\mathbb{F}$, satisfying the "usual conditions", be heuristically described as

$$\mathcal{F}_t = \sigma\left(B_u^1, B_u^2; u \leq t\right)$$

Let's suppose the discounted stock price follows a stochastic volatility model described below

$$\begin{aligned} dS_t &= \gamma(S_t, B_t^2)dB_t^1 \\ d\left\langle B^1, B^2 \right\rangle_t &= \rho dt \\ L &= \int_0^\infty g(s)dB_s^2 \end{aligned}$$

In the model described just above, the insider has information that can affect the volatility of the discounted price process. This type of extra information fits into our setting since it is not related to the path property of the traded discounted stock price. This approach is in contrast to the type of extra knowledge that can be found in the literature. For instance some of the extra information usually considered are the last time the discounted stock price is equal to a certain value or the last time the discounted stock price is equal to its running maximum or minimum. Let's first assume that

$$a = \inf\left\{t : \int_t^\infty g^2(s)ds = 0\right\} = \infty \tag{3.62}$$

$$0 < \gamma_t, \ \forall\, t \geq 0 \tag{3.63}$$

For example, if $g(t) = \lambda e^{-\lambda t}$, $t \geq 0$, $\lambda > 0$, then equation (3.62) is satisfied. The extra information is represented by a Gaussian random variable L. The Gaussian expansion of the

Brownian filtration was first studied by Chaleyat-Maurel and Jeulin [12] and was revisited in Protter [52].

To find the $(\mathbb{G}, P)-$ decomposition of S, we need to find for each $x \in \mathbb{R}$, q^x and k^x. We have the following:

$$\begin{aligned} P\left(L < x \,|\, \mathcal{F}_t\right) &= P\left(\int_0^t g(s)dB_s^2 + \int_t^\infty g(s)dB_s^2 < x \,|\, \mathcal{F}_t\right) \\ &= P\left(\int_t^\infty g(s)dB_s^2 < x - \int_0^t g(s)dB_s^2 \,|\, \mathcal{F}_t\right) \\ &= P\left(\int_t^\infty g(s)dB_s^2 < x - \int_0^t g(s)dB_s^2 \,|\, \sigma\left(B_u^1; u \le t\right)\right) \end{aligned}$$

By assumption, $d\left\langle B^1, B^2\right\rangle_t = \rho dt$ which implies that $B^2 = \rho\, B^1 + \sqrt{1-\rho^2}\, Z$ where Z is independent of B^1 and a standard BM with respect to $\sigma\left(B_u^1, Z_u; u \le t\right) = \sigma\left(B_u^1, B_u^2; u \le t\right)$. Therefore, $\int_t^\infty g(s)dB_s^2$ is independent of $\sigma\left(B_u^1; u \le t\right)$ and

$$\begin{aligned} P\left(L < x \,|\, \mathcal{F}_t\right) &= P\left(\int_t^\infty g(s)dB_s^2 < x - \int_0^t g(s)dB_s^2 \,|\, \sigma\left(B_u^1; u \le t\right)\right) \\ &= \Phi^v\left(x - \int_0^t g(s)dB_s^2\right) \end{aligned}$$

where Φ^v is the cumulative distribution function of a centered Gaussian random variable with variance equal to $v_t = \int_t^\infty g^2(s)ds$. Φ^v is well-defined since $a = \infty$. Consequently,

$$Q_t(\omega, dx) = \phi^v\left(x - \int_0^t g(s)dB_s^2\right) dx \tag{3.64}$$

where

$$q_t^x = \frac{1}{\sqrt{2\pi v_t}} \exp\left(-\frac{1}{2v_t}\left(x - \int_0^t g(s)dB_s^2\right)^2\right) \tag{3.65}$$

Let $X = \int g(s)dB_s^2$, then using Itô's formula, the SDE satisfied by q^x for all x is computed as follows

$$\begin{aligned} dq_t^x &= -\frac{1}{2v_t}\frac{\partial v_t}{\partial t} q_t^x\, dt - \frac{(x - X_t)^2}{2v_t^2}\frac{\partial v_t}{\partial t} q_t^x\, dt \\ &+ \frac{(x - X_t)}{v_t} q_t^x\, dX_t + \frac{1}{2}\left[-\frac{q_t^x}{v_t} - \frac{(x - X_t)^2}{v_t^2} q_t^x\right] d\left[X, X\right]_t \end{aligned}$$

We have $d\left[X, X\right]_t = g^2(t)\, dt$ and $\frac{\partial v_t}{\partial t} = -g^2(t)$. Hence,

$$dq_t^x = \frac{g^2(t)}{2v_t} q_t^x\, dt + g^2(t)\frac{(x - X_t)^2}{2v_t^2} q_t^x\, dt$$

$$
\begin{aligned}
&+ \quad \frac{(x-X_t)}{v_t} q_t^x g(t)\, dB_t^2 - \frac{g^2(t)}{2v_t} q_t^x\, dt - g^2(t) \frac{(x-X_t)^2}{2v_t^2} q_t^x\, dt \\
&= \quad \frac{(x-X_t)}{v_t} q_t^x g(t)\, dB_t^2
\end{aligned}
$$

Therefore, for all t, we have

$$
\begin{aligned}
d\langle q^x, S\rangle_t &= \rho\, \gamma_t\, q_t^x \frac{g(t)\left(x - \int_0^t g(s) dB_s^2\right)}{v_t} dt \\
&= \rho\, q_t^x \frac{g(t)\left(x - \int_0^t g(s) dB_s^2\right)}{\gamma_t v_t} d\langle S, S\rangle_t \\
&= \rho\, q_t^x \frac{g(t)\left(x - \int_0^t g(s) dB_s^2\right)}{\gamma_t v_t} d\,[S, S,]_t
\end{aligned}
$$

which implies that for all t,

$$
\begin{aligned}
k_t^x &= \frac{\rho\, g(t)\left(x - \int_0^t g(s) dB_s^2\right)}{v_t\, \gamma_t} \\
k_t^L &= \frac{\rho\, g(t)\left(L - \int_0^t g(s) dB_s^2\right)}{\gamma_t\, \int_t^\infty g^2(s) ds} \\
&= \frac{\rho\, g(t)\left(\int_0^\infty g(s) dB_s^2 - \int_0^t g(s) dB_s^2\right)}{\gamma_t\, \int_t^\infty g^2(s) ds} \\
&= \frac{\rho\, g(t) \int_t^\infty g(s) dB_s^2}{\gamma_t\, \int_t^\infty g^2(s) ds} \qquad (3.66)
\end{aligned}
$$

By (3.63) and (3.64), the predictable processes k^x and k^L are well-defined, and $\widetilde{S} = S - \int k_s^L d\,[S,S]_s$ is a $(\mathbb{G}, P)-$ local martingale. Then, the $(\mathbb{G}, P)$ decomposition of S is given by

$$
dS_t = d\widetilde{S}_t + \rho\, g(t) \frac{\int_t^\infty g(s) dB_s^2}{\gamma_t\, \int_t^\infty g^2(s) ds} d\,[S,S]_t \qquad (3.67)
$$

The next theorem is an application of Amendinger's results.

Theorem 3.5.7. *If $a = \infty$, the enlarged market $\mathcal{M}^{\mathbb{G}}$ satisfies NFLVR on $[0,T]$.*

Proof. From equation (3.62), since $a = \infty$, $Q_t(\omega, dx)$ is equivalent to dx, then Amendinger [3] and Amendinger et al. [2] proved that $\frac{1}{q^L}$ is a $(\mathbb{G}, P)-$ martingale. Let's define a measure Q on $(\Omega, \mathcal{G}_T)$ as follows:

$$
Q(A) \quad = \quad \int_A \frac{1}{q^L} dP, \quad A \in \mathcal{G}_T \qquad (3.68)
$$

Then, S is a $(\mathbb{G}, Q)-$ local martingale if and only if S/q^L is a $(\mathbb{G}, P)-$ local martingale. We have that

$$q^L = \varepsilon\left(\int_0^{\cdot} \frac{L - X_t}{v_t} g(t)\, dB_t^2\right) \Rightarrow \frac{1}{q^L} = \varepsilon\left(-\int_0^{\cdot} \frac{L - X_t}{v_t} g(t)\, dB_t^2 + \int_0^{\cdot} \left(\frac{L - X_t}{v_t} g(t)\right)^2 dt\right)$$

which implies

$$d\frac{1}{q_t^L} = \frac{1}{q_t^L}\left(-\frac{L - X_t}{v_t} g(t)\, dB_t^2 + \left(\frac{L - X_t}{v_t} g(t)\right)^2 dt\right)$$

Consequently,

$$\begin{aligned} d\left(\frac{S}{q^L}\right)_t &= \frac{1}{q_t^L} dS_t + S_t d\left(\frac{1}{q^L}\right)_t + d\left[S, \frac{1}{q^L}\right]_t \\ &= \frac{1}{q_t^L}\left[d\widetilde{S}_t + \rho g(t)\frac{L - X_t}{\gamma_t\, v_t} d\,[S,S]_t\right] + S_t d\left(\frac{1}{q^L}\right)_t - \frac{1}{q_t^L}\frac{L - X_t}{v_t} g(t)\gamma_t \rho\, dt \\ &= \frac{1}{q_t^L}\left[d\widetilde{S}_t + \rho g(t)\frac{L - X_t}{\gamma_t\, v_t} d\,[S,S]_t\right] + S_t d\left(\frac{1}{q^L}\right)_t - \frac{1}{q_t^L}\frac{L - X_t}{\gamma_t\, v_t} g(t)\rho\, d\,[S,S]_t \\ &= \frac{1}{q_t^L} d\widetilde{S}_t + S_t d\left(\frac{1}{q^L}\right)_t \end{aligned}$$

and S/q^L is a $(\mathbb{G}, P)-$ local martingale. So Q is a probability measure that turns S into a $(\mathbb{G}, Q)-$ local martingale. The probability measure Q is a risk neutral measure for the insider. Therefore, the enlarged market given by $\mathcal{M}^{\mathbb{G}}$ satisfies NFLVR on $[0, T]$. ■

In general, when the discounted stock price is an $(\mathbb{F}, P)-$ local martingale, then equation (3.9) gives the $(\mathbb{G}, P)-$ decomposition of S. If for each x, $Q_t(\omega, dx) \sim \eta(dx)$ then $Z = 1/q^L$ is a $(\mathbb{G}, P)-$ martingale and a candidate risk neutral measure for the insider is given by equation (3.68). The Amendinger condition [3] that for each x, $Q_t(\omega, dx) \sim dx$ fails whenever $a < \infty$ since $Q_t(\omega, dx)$ no longer has the same support as dx; and therefore his results do not help us here. This situation happens quite easily, for instance if one considers functions g with compact support.

Let's now assume that

$$a \;=\; \inf\left\{t \,:\, \int_t^{\infty} g^2(s)ds = 0\right\} < \infty \tag{3.69}$$

Despite the fact that $a < \infty$, equations (3.64), (3.65), (3.66) and (3.67) are still valid but only on the interval $[0, a)$.

Proposition 3.5.8. *The insider's financial market* $\mathcal{M}^{\mathbb{G}} = \left(\Omega, \mathcal{F}, \mathbb{G}, S, \mathcal{A}^{\mathbb{G}}\right)$ *satisfies Local NFLVR and therefore NAS on* $[0,T]$ *if* $[0,T] \subset [0,a)$.

Proof. Let's first prove that $\int_0^t \left(k_s^L\right)^2 d\left\langle S,S\right\rangle_s < \infty$ a.s. $\forall\ t \in [0,a)$ where k^L is given by (3.66). To do so, let's compute the following:

$$\begin{aligned}
E\left(\int_0^t \left(k_s^L\right)^2 d\left\langle S,S\right\rangle_s\right) &= \rho^2 E\left(\int_0^t \left(\frac{g(s)\int_s^\infty g(u)dB_u^2}{\gamma_s \int_s^\infty g^2(u)du}\right)^2 d\left\langle S,S\right\rangle_s\right) \\
&= \rho^2 E\left(\int_0^t \left(\frac{g(s)\int_s^\infty g(u)dB_u^2}{\gamma_s \int_s^\infty g^2(u)du}\right)^2 \gamma_s^s ds\right) \\
&= \rho^2 \int_0^t \frac{g^2(s)}{\left(\int_s^\infty g^2(u)du\right)^2} E\left(\int_s^\infty g(u)dB_u^2\right)^2 ds \\
&= \rho^2 \int_0^t \frac{g^2(s)}{\left(\int_s^\infty g^2(u)du\right)^2} E\left(\int_s^\infty g^2(u)du\right)^2 ds \\
&= \rho^2 \int_0^t \frac{g^2(s)}{\int_s^\infty g^2(u)du} ds
\end{aligned}$$

Note that

$$g^2(s) \leq \int_s^\infty g^2(u)du \Rightarrow \frac{g^2(s)}{\int_s^\infty g^2(u)du} \leq 1$$

which implies

$$\int_0^t \frac{g^2(s)}{\int_s^\infty g^2(u)du} ds < \infty \quad \forall\ t \in [0,a) \tag{3.70}$$

then $\int_0^t \left(k_s^L\right)^2 d\left\langle S,S\right\rangle_s < \infty$ a.s. $\forall\ t \in [0,a)$.

Second let's prove that if $[0,T] \subset [0,a)$, then $\lim_{m\to\infty} T_m > T$ a.s..

Let $u > 0$ and define the following $\mathbb{G}-$ stopping times

$$\theta_m = \inf\left\{0 < t \leq (T+u) \wedge a \ : \ \int_0^t \left(k_s^L\right)^2 d\left\langle S,S\right\rangle_s \geq m\right\}$$

Then on $\left\{\int_0^{(T+u)\wedge a} \left(k_s^L\right)^2 d\left\langle S,S\right\rangle_s < \infty\right\}$, the stopping times θ_m are a.s. $(T+u) \wedge a$ from some n on. If (3.70) holds, then the θ_m are a.s. $(T+u) \wedge a$ from some n on. Let

$$T_m = \inf\left\{t \wedge a \ : \ \int_0^t \left(k_s^L\right)^2 d\left\langle S,S\right\rangle_s \geq m\right\},$$

then $\{\theta_m\} \subseteq \{T_m\}$. Therefore, if $[0,T] \subset [0,a)$, $\lim_{m\to\infty} T_m > T$ a.s.. Consequently, Local NFLVR and therefore NAS hold on $[0,T]$. ■

Proposition 3.5.9. *The insider's financial market* $\mathcal{M}^{\mathbb{G}} = \left(\Omega, \mathcal{F}, \mathbb{G}, S, \mathcal{A}^{\mathbb{G}}\right)$ *satisfies the NA1 condition on* $[0,T]$ *if* $[0,T] \subset [0,a)$.

Proof. It is enough to prove that $\left[q^L, S\right] = k^L q^L \cdot [S,S]$ on $[0,a)$. For $t \in [0,a)$, the process q^L is well-defined and we have

$$\begin{aligned}
d\left[q^L, S\right]_t &= d\left[1 + \int \frac{L - X_t}{v_t} q_t^L g(t) dB_t^2, S_0 + \int \gamma_t dB_t^1\right]_t \\
&= \frac{L - X_t}{v_t} q_t^L g(t) \gamma_t d\left[B_t^2, B_t^1\right]_t \\
&= \rho \frac{L - X_t}{v_t} g(t) q_t^L \gamma_t dt \\
&= \rho \frac{L - X_t}{\gamma_t v_t} g(t) q_t^L \gamma_t^2 dt \\
&= k_t^L q_t^L \gamma_t^2 dt \\
&= k_t^L q_t^L d\left[S, S\right]_t
\end{aligned}$$

Hence the result follows by Theorem 3.4.1. ■

Theorem 3.5.10. *The insider's financial market* $\mathcal{M}^{\mathbb{G}} = \left(\Omega, \mathcal{F}, \mathbb{G}, S, \mathcal{A}^{\mathbb{G}}\right)$ *does not satisfy NFLVR on* $[0,T]$ *if* $[0,T] \subset [0,a)$ *or* $[0,a) \subset [0,T]$.

Proof. For all $t \in [0,a)$, we have

$$q_t^L = \frac{1}{\sqrt{2\pi v_t}} \exp\left(-\frac{1}{2v_t}\left(\int_t^\infty g(s) dB_s^2\right)^2\right)$$

As $t \nearrow a$, $v_t \downarrow 0$; Since $\left(\int_t^\infty g(s) dB_s^2\right)_{t \geq 0}$ is a Gaussian process, we have

$$\begin{aligned}
q_a^L &:= \lim_{t \to a} q_t^L \\
&= \lim_{v_t \downarrow 0} q_t^L = 0
\end{aligned}$$

Hence by Corollary 3.4.1.1, NFLVR does not holds on $[0,T]$ ■

An insider whose extra information is represented by $L = \int_0^\infty g(s) dB_s^2$ such that $a < \infty$ will have NAS and NA1 but not NFLVR. Therefore, such an insider does not have a pricing measure but she can still use the resulting sequence of probability measures $(Q_m)_{m \geq 0}$ to price her financial instruments. How to use the sequence of probability measures $(Q_m)_{m \geq 0}$ in pricing will be one of the goals of the next chapter.

3.6 Conclusion

In the present chapter, we have studied conditions under which the extra information obtained by an insider gives rise to arbitrage and free lunches. We have shown that under simple conditions, the insider's enlarged financial market satisfies NAS which is enough since admissible simple predictable integrands are the only type of trading strategies that can be used in practice. Therefore, although the theory of derivatives pricing requires a risk neutral measure which is only ensured by the presence of NFLVR (see Delbaen and Schachermayer [19]), the enlarged market is consistent with the no arbitrage condition that is required for a well-functioning financial market. Moreover, even though the enlarged market does not satisfy NFLVR, the insider can still price her financial contracts using the sequence of probability measures that are exhibited under local NFLVR.

One important assumption of the present chapter is the continuity of all $\mathbb{F}-$ local martingales. A possible extension of the current work is to consider discounted price processes which have jumps. We do not believe the consideration of such processes will be a big departure from our work because the presence of jumps will just require extra conditions to obtain Local NFLVR, then NAS and NFLVR.

Chapter 4

Expansion of filtrations, risk and pricing

4.1 Introduction

In this chapter, we are first interested in comparing the insider's risk to the market/liquidity trader's risk. The insider's filtration $\mathbb{G}$ has been obtained by initially enlarging the market filtration $\mathbb{F}$. Under the assumption that markets are incomplete, both traders can only partially hedge their risk exposure. Second, still assuming that the financial markets of both the liquidity/market trader and the insider are incomplete, we are interested in pricing insider's financial claims. If the insider's market has No Free Lunch with Vanishing Risk (NFLVR), then he or she can use his or her risk neutral measure to price claims, but what if the insider's market only has Local NFLVR? From Chapter 3, we show that if Local NFLVR holds, then there exists a consistent sequence of probability measures such the financial market given by $\left(\Omega, \mathcal{F}, \mathbb{G}, Q_m, S, \mathcal{A}^{\mathbb{G}}\right)$ satisfies NFLVR on $[0, T_m]$ for all m, where $\mathcal{A}^{\mathbb{G}}$ is the set of $\mathbb{G}-$ admissible trading strategies and $(T_m)_{m\geq 1}$ a carefuly chosen sequence of $\mathbb{G}-$ stopping times. In that case, the consistent sequence of probability measures $(Q_m)_{m\geq 1}$ can be used to price the insider's financial instruments, yielding one price, independent of m.

4.2 General settings

Let $(\Omega, \mathcal{F}, P, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0})$ be a filtered probability space, where the filtration $\mathbb{F}$ satisfies the "usual conditions", P denotes the physical probability measure and $\mathcal{F} := \mathcal{F}_\infty$ is assumed to be separable. The insider's filtration $\mathbb{G}$ is defined as the initial expansion of $\mathbb{F}$ with L at the stopping time τ, where τ is an $\mathbb{F}-$ stopping time and L represents the insider's extra information, augmented by the $P-$ null sets of $\mathcal{G} := \mathcal{G}_\infty = \mathcal{F}_\infty$. The filtration $\mathbb{G}$ also satisfies the "usual conditions".

We consider a financial market which is composed of a risky asset S and a bond B, and two traders: a liquidity trader also known as a "market trader," and an insider. Additionally, we assume there are no trading constraints, the markets are frictionless and the spot interest rate is zero. We also consider that we have a finite trading horizon $[0, T]$, where $T < \infty$. We assume that all $\mathbb{F}-$ local martingales are continuous.

Consider a contingent claim expiring at time T and given by the following random variable

$$H \quad \in \quad \mathcal{L}^{2+\Delta}(\Omega, \mathcal{F}_T, P) \tag{4.1}$$

for some $\Delta > 0$.

To sell the claim $H(\omega)$ at time t, where $t \in [T_m(\omega), T_{m+1}(\omega)]$ for a given ω and some m, we use Q_m in the case where the insider has a pricing measure Q, we use Q. In the case where the insider's market only satisfies Local NFLVR, we show that $E_{Q_m}(H)$ is optimal in some sense.

4.3 Föllmer-Schweizer (FS) theory

4.3.1 FS setting and goal

Let's assume that the discounted stock price is an $\mathcal{H}^2$ semimartingale (see Protter [52] for the definition of the space of $\mathcal{H}^2$ semimartingales) with the following decomposition:

$$S \quad = \quad S_0 + M + A \tag{4.2}$$

where M is $P-$ square-integrable martingale and A a predictable finite variation process.

To hedge against the financial claim H, a portfolio strategy involving the discounted stock S

and a riskless bond B is used and should yield H at time T. Note that with the assumption that the spot interest rate is zero, $B \equiv 1$. Let ξ_t and η_t respectively represent the amounts of stock and bond held at time t. We assume that the process $\xi = (\xi_t)_{0\le t\le T}$ is predictable while $\eta = (\eta_t)_{0\le t\le T}$ is allowed to be adapted. The value of the resulting portfolio at time t and the cost accumulated up to time t are respectively given by the following processes

$$V_t = \xi_t S_t + \eta_t, \ \ 0 \le t \le T \tag{4.3}$$

$$C_t = V_t - \int_0^t \xi_s dS_s, \ \ 0 \le t \le T. \tag{4.4}$$

In the Föllmer-Schweizer setting [25], admissible trading strategies are strategies (ξ, η) such that the processes V and C are square-integrable, have right-continuous paths and satisfy

$$V_T = H, \ P-\text{a.s.} \tag{4.5}$$

An admissible trading strategy should also satisfy the following integrability condition

$$E_P\left[\int_0^T \xi_t^2 d\left\langle S,S\right\rangle_t + \left(\int_0^T |\xi_t|\, d\,|A|_t\right)^2\right] < \infty \tag{4.6}$$

First let's assume that S is actually a $P-$ square-integrable martingale. Therefore by Kunita-Watanabe, we have the following:

$$H = H_0 + \int_0^T \xi_t^H dS_t + L_T^H, \ P-\text{a.s..} \tag{4.7}$$

where $H_0 \in \mathcal{F}_0$ and L^H is a $P-$ square integrable martingale strongly orthogonal to S. Therefore, in an incomplete market, a typical claim carries an intrinsic risk (represented by L^H) and the goal of FS was to find a dynamic portfolio strategy which reduces the actual risk (represented by the the accumalated cost) to the intrinsic component. In that context, Föllmer and Sondermann [26] introduced the following risk-minimization criterion: We look for an admissible trading strategy which minimizes at each time t, the remaining accumulated cost

$$E_P\left[(C_T - C_t)^2 \mid \mathcal{F}_t\right] \tag{4.8}$$

over all admissible continuations of that strategy from time t on (see Föllmer and Sondermann [26] for the definition of an admissible continuation of an admissible trading strategy). Föllmer and Sondermann [26] proved that the accumulated cost process is mean-self financing. That means the cost process C associated to a risk-minimizing strategy is a martingale.

Theorem 4.3.1 (Föllmer and Sondermann [26]). *If the Kunita-Watanabe decomposition given in* (4.7) *holds, the risk-minimizing strategy is given by*

$$\xi \quad := \quad \xi^H\,, \;\; \eta := V - \xi \cdot S \tag{4.9}$$

where

$$V_t \quad := \quad H_0 + \int_0^t \xi_s^H dS_s + L_t^H\,, \;\; 0 \leq t \leq T. \tag{4.10}$$

The process V can also be computed using the càdlàg version of the martingale

$$V_t \quad = \quad E_P\left[H \mid \mathcal{F}_t\right], \;\; 0 \leq t \leq T. \tag{4.11}$$

Hence, if S is a $P-$ martingale, the risk minimizing trading strategy is found using the Kunita-Watanabe decomposition of the claim. The problem is that the discounted price process is not a $P-$ martingale but a $P-$ semimartingale; Therefore, the Kunita-Watanabe decomposition given in (4.7) does not hold. One way to approach the problem is to use a Girsanov approach to construct a probability measure $Q \sim P$ under which the discounted price process is a $Q-$ square-integrable martingale. The issue that arises from this approach is the fact that in incomplete markets, such an equivalent martingale measure might not be unique. FS [25] proved that there exists a minimal equivalent martingale measure $\widehat{P} \sim P$ such that the optimal trading strategy can be computed in terms of $\widehat{P}$.

4.3.2 FS results

Definition 4.3.1. [Minimal martingale measure] An equivalent martingale measure $\widehat{P}$ will be called minimal if

$$\widehat{P} = P, \;\; \text{on } \mathcal{F}_0, \tag{4.12}$$

and if any $P-$ square-integrable martingale which is orthognal to M under P remains a martingale under $\widehat{P}$:

$$L \in \mathcal{M}^2 \text{ and } \langle L, M\rangle = 0 \Longrightarrow L \text{ is a martingale under } \widehat{P}. \tag{4.13}$$

By assumption, the discounted price satisfies no arbitrage in the sense of NFLVR under P. Hence, by Delbaen and Schachermayer [19]

$$A \ll \langle S, S \rangle \tag{4.14}$$

Equation (4.14) is equivalent to the existence of a $\mathcal{P}(S)-$ process, $(\alpha_t)_{0 \leq t \leq T}$, such that

$$A_t = \int_0^t \alpha_s d \langle S, S \rangle_s \, , \; 0 \leq t \leq T.$$

Using the $\mathbb{F}-$ predictable process α, we can give a mathmatical expression of the equivalent minimal martingale measure and study its existence and uniqueness.

Theorem 4.3.2 (Föllmer and Schweizer [25])**.** *The minimal martingale measure* $\widehat{P}$

a) is uniquely determined.

b) exists if and only if

$$\widehat{G}_t = \exp\left(-\int_0^t \alpha_s dM_s - \frac{1}{2}\int_0^t \alpha_s^2 d \langle S, S \rangle_s\right), \; 0 \leq t \leq T \tag{4.15}$$

is a $P-$ *square-integrable martingale. Hence,* $\frac{d\widehat{P}}{dP} = \widehat{G}_T$.

c) preserves orthogonality: Any $L \in \mathcal{M}^2$ *with* $\langle L, M \rangle = 0$ *under* P *satisfies* $\langle L, X \rangle = 0$ *under* $\widehat{P}$.

The minimal martingale measure is minimal in the sense that it minimizes the relative entropy $H(.|P)$ among all martingale measures Q with fixed expectation $E_Q\left[\int_0^T \alpha_s^2 d \langle S, S \rangle_s\right]$; where the relative entropy is given by the following functional

$$H(Q|P) = \begin{cases} \int \left(\log \frac{dQ}{dP}\right) dP & \text{if } Q \ll P \\ \infty & \text{otherwise} \end{cases}$$

Consequently S is a $P-$ semimartingale, but a $\widehat{P}-$ square-integrable martingale. Hence, the problem of computing the optimal strategy when S is a $P-$ semimartingale reverts back to the Föllmer and Sondermann case but the different processes of interest (such as the accumulated cost process C, the portfolio value V) are now computed under the equivalent minimal martingale measure.

Theorem 4.3.3 (Föllmer and Schweizer [25]). *The optimal strategy* $\left(\xi^H, \eta^H\right)$ *is uniquely determined and given by*

$$\xi^H = \frac{d\langle V,S\rangle}{d\langle S,S\rangle}, \ \eta^H := V - \xi^H \cdot S \tag{4.16}$$

where V *denotes the càdlàg version of the following martingale*

$$V_t = E_{\widehat{P}}\left[H \mid \mathcal{F}_t\right], \ 0 \le t \le T. \tag{4.17}$$

It is important to note that the risk minimization criterion in the context of FS is to look for an admissible trading strategy that minimizes

$$E_{\widehat{P}}\left[\left(C_T - C_t\right)^2 \mid \mathcal{F}_t\right] \tag{4.18}$$

over all admissible continuations of that strategy from time t on.
In FS, the remaining actual risk of a trader is measured by either $E_P\left[\left(C_T - C_t\right)^2 \mid \mathcal{F}_t\right]$ or $E_{\widehat{P}}\left[\left(C_T - C_t\right)^2 \mid \mathcal{F}_t\right]$ depending on whether the discounted stock price is either a $P-$ martingale or a $\widehat{P}-$ martingale and is minimized in such a way that it ends up being equal to the intrinsic risk that cannot be traded away by the trader. We define the risk of a financial claim as the intrinsic risk a liquidity/market trader or an insider takes when he or she trades in the asset H. Hence, what we define as risk is equivalent to the FS minimum risk at time zero if both traders use their optimal admissible trading strategies in their respective filtration.

4.4 Comparison of the market trader and insider's risk in an incomplete market

Throughout this section, we will assume that the insider's market satisfies NFLVR, and that $H \in L^{2+\Delta}(dP)$ for some $\Delta > 0$ (see (4.27))

4.4.1 Kunita-Watanabe decompositions of H

The possible Kunita-Watanabe decompositions of H are:

Kunita - Watanabe decompositions of H	Filtration and Probability measure
$H = \alpha + \int_0^T \xi_t^{\mathbb{F}} dS_t + U_T$	$(\mathbb{F}, P)$
$H = \alpha + \int_0^T \widetilde{\xi}_t^{\mathbb{G}} d\widetilde{S}_t + V_T$	$(\mathbb{G}, P)$
$H = \beta + \int_0^T \xi_t^{\mathbb{G}} dS_t + K_T$	$(\mathbb{G}, Q)$

Table 4.1: Kunita - Watanabe decompositions of H

where S is a local martingale respectively in $(\mathbb{F}, P)$ and $(\mathbb{G}, Q)$ and $\widetilde{S}$ a local martingale in $(\mathbb{G}, P)$; U and V are respectively $(\mathbb{F}, P)$ and $(\mathbb{G}, P)$ local martingales starting at 0 and strongly orthogonal to S and $\widetilde{S}$ under P, while K is a $(\mathbb{G}, Q)-$ local martingale starting at 0 and strongly orthogonal to S under Q.

Given the assumption made on the financial claim, the following lemma holds for every decomposition in Table 4.1 but we state it using only the $(\mathbb{F}, P)-$ decomposition of H.

Lemma 4.4.1. *Suppose the sequence of $\mathbb{F}$ stopping times that reduces both $\xi^{\mathbb{F}} \cdot S$ and U denoted by $(\Theta_n)_{n\geq 1}$ is such that*

$$\lim_{n\to\infty} E_P\left(U_{T\wedge\Theta_n}^2\right) \quad \leq \quad E_P\left(U_T^2\right) \tag{4.19}$$

then

$$E_P(H^2) \quad = \quad \alpha^2 + E_P\left(\int_0^T \left(\xi_t^{\mathbb{F}}\right)^2 d\left[S, S,\right]_t\right) + E_P\left(U_T^2\right) \tag{4.20}$$

A similar relationship as the one given by equation (4.20) is also satisfied by the decompositions in $(\mathbb{G}, P)$ and $(\mathbb{G}, Q)$ under hypotheses similar to the one in equation (4.19).

Proof. Let

$$H_t \quad := \quad E_P\left(H \,|\, \mathcal{F}_t\right), \; 0 \leq t \leq T, \tag{4.21}$$

then $(H_t)_{0\leq t\leq T}$ is a uniformly integrable $(\mathbb{F}, P)-$ martingale. Projecting the $(\mathbb{G}, P)-$ decomposition of H onto $\mathcal{F}_t$, we get

$$H_t \quad = \quad \alpha + \int_0^t \xi_s^{\mathbb{F}} dS_s + U_t, \; 0 \leq t \leq T \tag{4.22}$$

Since $\left(\int_0^t \xi_s^{\mathbb{F}} dS_s\right)_t$ and U are local martingales, we can find a sequence of $\mathbb{F}-$ stopping times reducing both local martingales. Such a sequence can be constructed by taking the minimum of the reducing sequences of both local martingales. $(\Theta_n)_{n\geq 1}$ is such a reducing sequence. Then, for each n

$$H_{t\wedge\Theta_n} = \alpha + \int_0^{t\wedge\Theta_n} \xi_s^{\mathbb{F}} dS_s + U_{t\wedge\Theta_n}, \; 0 \leq t \leq T. \tag{4.23}$$

Hence,

$$\begin{aligned} E_P\left(H_{t\wedge\Theta_n}^2\right) &= \alpha^2 + E_P\left(\int_0^{t\wedge\Theta_n} \xi_s^{\mathbb{F}} dS_s\right)^2 + E_P\left(U_{t\wedge\Theta_n}\right)^2 \\ &+ 2\alpha E_P\left(\int_0^{t\wedge\Theta_n} \xi_s^{\mathbb{F}} dS_s\right) + 2\alpha E_P\left(U_{t\wedge\Theta_n}\right) + 2E_P\left(U_{t\wedge\Theta_n}\int_0^{t\wedge\Theta_n} \xi_s^{\mathbb{F}} dS_s\right) \\ &= \alpha^2 + E_P\left(\int_0^{t\wedge\Theta_n} \xi_s^{\mathbb{F}} dS_s\right)^2 + E_P\left(U_{t\wedge\Theta_n}\right)^2 + 2E_P\left(\int_0^{t\wedge\Theta_n} \xi_s^{\mathbb{F}} d\left[U,S\right]_s\right) \qquad (4.24) \\ &= \alpha^2 + E_P\left(\int_0^{t\wedge\Theta_n} \left(\xi_s^{\mathbb{F}}\right)^2 d\left[S,S\right]_s\right) + E_P\left(U_{t\wedge\Theta_n}\right)^2 \qquad (4.25) \end{aligned}$$

where equation (4.24) follows from the fact that for each n, we can choose the sequence $(\Theta_n)_{n\geq 1}$ such that $(U_{t\wedge\Theta_n})_{0\leq t\leq T}$ and $\left(\int_0^{t\wedge\Theta_n} \xi_s^{\mathbb{F}} dS_s\right)_{0\leq t\leq T}$ are $\mathcal{H}^2$ martingales in $(\mathbb{F}, P)$. Equation (4.25) follows from the orthogonality of S and U. Hence, taking limits we obtain

$$\begin{aligned} \lim_{n\to\infty} E_P\left(H_{t\wedge\Theta_n}^2\right) &= \alpha^2 + \lim_{n\to\infty} E_P\left(\int_0^{t\wedge\Theta_n} \left(\xi_s^{\mathbb{F}}\right)^2 d\left[S,S\right]_s\right) \\ &+ \lim_{n\to\infty} E_P\left(U_{t\wedge\Theta_n}\right)^2 \qquad (4.26) \end{aligned}$$

First let's focus on $\lim_{n\to\infty} E_P\left(H_{t\wedge\Theta_n}^2\right)$.

$\left(H_t^2\right)_{0\leq t\leq T}$ is a uniformly integrable $(\mathbb{F}, P)$ − submartingale by the Theorem of de la Vallée-Poussin. Indeed,

$$\begin{aligned} \sup_t E_P\left(H_t^{2+\Delta}\right) &= \sup_t E_P\left(\left\{E_P\left(H \mid \mathcal{F}_t\right)\right\}^{2+\Delta}\right) \\ &\leq \sup_t E_P\left(E_P\left(H^{2+\Delta} \mid \mathcal{F}_t\right)\right) \qquad (4.27) \\ &= \sup_t E_P\left(H^{2+\Delta}\right) \\ &= E_P\left(H^{2+\Delta}\right) < \infty, \text{ (by assumption on } H). \end{aligned}$$

where (4.27) follows from Jensen's inequality. Similarly, fix t and let $\left(X_n := H_{t\wedge\Theta_n}^2\right)_{n\geq 1}$. Then $(X_n)_{n\geq 1}$ is a discrete time uniformly integrable submartingale with respect to the filtration $(\mathcal{F}_{t\wedge\Theta_n})_{n\geq 1}$. We have

$$E_P\left(X_{n+1} \mid \mathcal{F}_{t\wedge\Theta_n}\right) = E_P\left(H_{t\wedge\Theta_{n+1}}^2 \mid \mathcal{F}_{t\wedge\Theta_n}\right)$$

$$\begin{aligned} &= E_P\left(\left[E_P\left\{H \mid \mathcal{F}_{t\wedge\Theta_{n+1}}\right\}\right]^2 \mid \mathcal{F}_{t\wedge\Theta_n}\right) \\ &\leq E_P\left(E_P\left\{H^2 \mid \mathcal{F}_{t\wedge\Theta_{n+1}}\right\} \mid \mathcal{F}_{t\wedge\Theta_n}\right) \qquad (4.28) \\ &= E_P\left\{H^2 \mid \mathcal{F}_{t\wedge\Theta_n}\right\} \\ &= X_n,\ P-\text{a.s.} \qquad (4.29) \end{aligned}$$

and

$$\begin{aligned} \sup_n E_P\left(X_n^{2+\Delta}\right) &= \sup_n E_P\left(H_{t\wedge\Theta_n}^{2+\Delta}\right) \\ &= \sup_n E_P\left(\left[E_P\left\{H \mid \mathcal{F}_{t\wedge\Theta_n}\right\}\right]^{2+\Delta}\right) \\ &\leq \sup_n E_P\left(E_P\left\{H^{2+\Delta} \mid \mathcal{F}_{t\wedge\Theta_n}\right\}\right) \qquad (4.30) \\ &= \sup_n E_P\left\{H^{2+\Delta}\right\} \\ &= E_P\left\{H^{2+\Delta}\right\} < \infty \end{aligned}$$

where (4.28) and (4.30) follow from Jensen's inequality. Moreover, $\left\{E_P\left(H_{t\wedge\Theta_n}^2\right)\right\}_{n\geq1}$ is a nondecreasing sequence of real numbers, hence its limit exists. Since $\left(H_{t\wedge\Theta_n}^2\right)_{n\geq1}$ is a uniformly integrable discrete time submartingale, then for each t, we have

$$\begin{aligned} H_{t\wedge\Theta_n}^2 &\underset{n\to\infty}{\rightarrow} H_t^2,\ P-\text{a.s. and uniform integrability of } \left(H_{t\wedge\Theta_n}^2\right)_{n\geq1} \\ \Rightarrow E_P\left(H_{t\wedge\Theta_n}^2\right) &\underset{n\to\infty}{\rightarrow} E_P\left(H_t^2\right). \qquad (4.31) \end{aligned}$$

Second, let's look at $\lim_{n\to\infty} E_P\left(\int_0^{t\wedge\Theta_n}\left(\xi_s^{\mathbb{F}}\right)^2 d\,[S,S]_s\right)$. By the Monotone convergence theorem,

$$\lim_{n\to\infty} E_P\left(\int_0^{t\wedge\Theta_n}\left(\xi_s^{\mathbb{F}}\right)^2 d\,[S,S]_s\right) = E_P\left(\int_0^{t}\left(\xi_s^{\mathbb{F}}\right)^2 d\,[S,S]_s\right),\ 0\leq t\leq T \qquad (4.32)$$

Consequently, $\lim_{n\to\infty} E_P\left(U_{t\wedge\Theta_n}^2\right)$ exists as well; Indeed from equation (4.26), we have that it is the difference of two limits that exist.

Finally, let's focus on $\lim_{n\to\infty} E_P\left(U_{t\wedge\Theta_n}\right)^2$. By Fatou's lemma

$$\begin{aligned} E_P\left(\liminf_{n\to\infty} U_{T\wedge\Theta_n}^2\right) &\leq \liminf_{n\to\infty} E_P\left(U_{T\wedge\Theta_n}^2\right) \\ &= \lim_{n\to\infty} E_P\left(U_{T\wedge\Theta_n}^2\right). \qquad (4.33) \end{aligned}$$

Equation (4.33) and the hypothesis given by equation (4.19) imply

$$E_P\left(\liminf_{n\to\infty} U_{T\wedge\Theta_n}^2\right) = \lim_{n\to\infty} E_P\left(U_{T\wedge\Theta_n}^2\right) \qquad (4.34)$$

To finish the proof of the lemma, it is enough by equations (4.26), (4.31) to prove that $\lim_{n\to\infty} U_{t\wedge\Theta_n} = U_t, P-$a.s. which would imply that $\lim_{n\to\infty} U^2_{t\wedge\Theta_n} = U^2_t, P-$a.s..
For each fix t, we have

$$\begin{aligned} E_P\left\{U_{t\wedge\Theta_{n+1}} \,|\, \mathcal{F}_{t\wedge\Theta_n}\right\} &= E_P\left\{E_P\left\{U_T \,|\, \mathcal{F}_{t\wedge\Theta_{n+1}}\right\} \,|\, \mathcal{F}_{t\wedge\Theta_n}\right\} \\ &= E_P\left\{U_T \,|\, \mathcal{F}_{t\wedge\Theta_n}\right\} \\ &= U_{t\wedge\Theta_n} \end{aligned} \tag{4.35}$$

Moreover,

$$\begin{aligned} U_{t\wedge\Theta_n} &= H_{t\wedge\Theta_n} - \alpha - \int_0^{t\wedge\Theta_n} \xi^{\mathbb{F}}_s dS_s \\ \left(U_{t\wedge\Theta_n}\right)^2 &= \left(H_{t\wedge\Theta_n} - \alpha - \int_0^{t\wedge\Theta_n} \xi^{\mathbb{F}}_s dS_s\right)^2 \\ E_P\left(U_{t\wedge\Theta_n}\right)^2 &= E_P\left(H_{t\wedge\Theta_n} - \alpha - \int_0^{t\wedge\Theta_n} \xi^{\mathbb{F}}_s dS_s\right)^2 \\ &\leq 3\alpha^2 + 3E_P\left(H_{t\wedge\Theta_n}\right)^2 + 3E_P\left(\int_0^{t\wedge\Theta_n} \xi^{\mathbb{F}}_s dS_s\right)^2 \qquad (4.36) \\ &= 3\alpha^2 + 3E_P\left(H_{t\wedge\Theta_n}\right)^2 + 3E_P\int_0^{t\wedge\Theta_n} \left(\xi^{\mathbb{F}}_s\right)^2 d\left[S,S\right]_s \\ \sup_n E_P\left(U_{t\wedge\Theta_n}\right)^2 &\leq 3\alpha^2 + 3\sup_n E_P\left(H_{t\wedge\Theta_n}\right)^2 + 3\sup_n E_P\int_0^{t\wedge\Theta_n} \left(\xi^{\mathbb{F}}_s\right)^2 d\left[S,S\right]_s \\ &= 3\alpha^2 + 3E_P\left(H_t\right)^2 + 3E_P\int_0^{t} \left(\xi^{\mathbb{F}}_s\right)^2 d\left[S,S\right]_s < \infty \qquad (4.37) \end{aligned}$$

where in (4.36) we use the fact that $(a+b+c)^2 \leq 3a^2 + 3b^2 + 3c^2$. Equations (4.35) and (4.37) imply that $\left(U_{t\wedge\Theta_n}\right)_n$ is a uniformly integrable discrete time martingale. Hence, for each t, $\lim_{n\to\infty} U_{t\wedge\Theta_n} = U_t, P-$a.s..
Consequently, equation (4.34) is equivalent to

$$E_P\left(U^2_T\right) = \lim_{n\to\infty} E_P\left(U^2_{T\wedge\Theta_n}\right). \tag{4.38}$$

The result of the lemma follows by plugging equations (4.31), (4.32) and (4.38) into (4.26) at time T and using the fact that $H \in \mathcal{F}_T$. ■

The hypotheses given in Lemma 4.4.1 are not very restrictive. Indeed, in the proof of Lemma 4.4.1, we prove that $\left(U_{t\wedge\Theta_n}\right)_n$ is a uniformly integrable discrete time martingale.

Therefore, Doob's maximal inequality implies

$$E_P\left(\sup_{0\le n<\infty} U^2_{T\wedge\Theta_n}\right) \;\le\; 4E_P\left(U_T^2\right). \tag{4.39}$$

Hence, from Fatou's lemma and the uniform integrability of $(U_{t\wedge\Theta_n})_n$, we have

$$E_P\left(U_T^2\right) \;\le\; E_P\left(\sup_{0\le n<\infty} U^2_{T\wedge\Theta_n}\right) \le 4E_P\left(U_T^2\right). \tag{4.40}$$

For the rest of this chapter, we assume that (4.19) holds for the $(\mathbb{F}, P)$ − representation of H. We assume similar hypotheses for the $(\mathbb{G}, P)$ and $(\mathbb{G}, Q)$ representations of H as well. Although we change filtrations and probability when going from one representation to the other, the Kunita-Watanabe representations found in Table 4.1 are not completely independent from each other. We have the following lemma

Lemma 4.4.2. *Let $Z_T := \frac{dQ}{dP}$ be the Radon-Nikodym derivative and let define the process Z as the càdlàg version of the following martingale $Z_t := E_P\left\{Z_T \,|\, \mathcal{G}_t\right\},\ 0 \le t \le T$. Then using the processes V, K and $\widetilde{S}$ given in Table 4.1,*

$$\begin{aligned} V &= ZK + Kk^L Z\cdot\widetilde{S},\ P-\text{a.s.},\\ K &= \frac{1}{Z}V - Vk^L\frac{1}{Z}\cdot S,\ Q-\text{a.s.}. \end{aligned}$$

Proof. We know that ZK is a $(\mathbb{G}, P)$ − local martingale since K is a $(\mathbb{G}, Q)$ − local martingale. The integration by parts formula gives us the following:

$$\begin{aligned} d(ZK) &= ZdK + KdZ + d[Z,K]\\ &= ZdK - Kk^L Zd\widetilde{S} \end{aligned} \tag{4.41}$$

Equation (4.41) follows from the facts that $dZ = -Zk^L d\widetilde{S}$ and

$$[S,K]\equiv 0 \;\;\Rightarrow\;\; \left[\widetilde{S},K\right]\equiv 0 \tag{4.42}$$

$$\Rightarrow\;\; [Z,K]\equiv 0 \tag{4.43}$$

Consequently,

$$d\left[ZK,\widetilde{S}\right] \;=\; -Kk^L S\, d[S,S] \tag{4.44}$$

From (4.44), we see that ZK and $\widetilde{S}$ are not orthogonal to each other. Therefore, our goal here is to use a Gramm-Schmidt approach to construct out of the $(\mathbb{G}, P)-$ local martingale, ZK, a $(\mathbb{G}, P)-$ local martingale that is orthogonal to $\widetilde{S}$; Then we could use the uniqueness of the Kunita-Watanabe decomposition to obtain a relationship between K and V. Let define the following processes

$$\begin{aligned} A &= \widetilde{S} \\ B &= ZK - \frac{d\left[ZK, \widetilde{S}\right]}{d\left[\widetilde{S}, \widetilde{S}\right]} \cdot \widetilde{S} \\ &= ZK - \frac{d\left[ZK, \widetilde{S}\right]}{d\left[S, S\right]} \cdot \widetilde{S} \\ &= ZK + Kk^L Z \cdot \widetilde{S} \end{aligned}$$

We claim that

$$V \equiv ZK + Kk^L Z \cdot \widetilde{S}, \;\; P-\text{a.s.} \tag{4.45}$$

To prove (4.45), it is enough to prove that B is a $(\mathbb{G}, P)-$ local martingale and is orthogonal to $\widetilde{S}$.

B is a $(\mathbb{G}, P)-$ local martingale since it is the sum of ZK, which is a $(\mathbb{G}, P)-$ local martingale and the stochastic integral of the $\mathbb{G}$ predictable process $Kk^L Z$ with respect to $\widetilde{S}$. To prove B is strongly orthogonal to $\widetilde{S}$, we need to compute the quadratic covariation of B and $\widetilde{S}$.

$$\begin{aligned} [B, A] = \left[B, \widetilde{S}\right] &= \left[ZK + Kk^L Z \cdot \widetilde{S}, \widetilde{S}\right] \\ &= \left[ZK, \widetilde{S}\right] + \left[Kk^L Z \cdot \widetilde{S}, \widetilde{S}\right] \\ &= -Kk^L S \, d\left[S, S\right] + Kk^L S \, d\left[S, S\right] \qquad (4.46) \\ &= 0 \end{aligned}$$

where (4.46) follows from (4.44). Consequently, equation (4.45) follows by the uniqueness of the Kunita-Watanabe decomposition.

Similarly, $1/Z$ is a $(\mathbb{G}, Q)-$ local martingale since by Itô's formula

$$d\frac{1}{Z} = -\frac{1}{Z^2} dZ + \frac{1}{2} \frac{2}{Z^3} d\left[Z, Z\right]$$

$$
\begin{aligned}
&= -\frac{1}{Z^2}\left(-Zk^L d\widetilde{S}\right) + \frac{1}{Z^3}Z^2\left(k^L\right)^2 d\left[S,S\right] \\
&= \frac{1}{Z}k^L d\widetilde{S} + \frac{1}{Z}\left(k^L\right)^2 d\left[S,S\right] \\
&= \frac{1}{Z}k^L\left(d\widetilde{S} + k^L d\left[S,S\right]\right) \\
&= \frac{1}{Z}k^L dS
\end{aligned}
$$

Hence $1/Z$ is a $(\mathbb{G}, Q)-$ local martingale since S is. Using an approach similar to the one above, we can also prove that

$$
\begin{aligned}
K &\equiv \frac{1}{Z}V - \frac{d\left[\frac{1}{Z}V, S\right]}{d\left[S,S\right]}\cdot S, \;\; Q-\text{a.s.} \\
&= \frac{1}{Z}V - Vk^L\frac{1}{Z}\cdot S, \;\; Q-\text{a.s..} \qquad (4.47)
\end{aligned}
$$

Hence, the results of the Lemma follow. ■

Additionally from Chapter 3, we have for each t the following

$$
d\widetilde{S}_t = dS_t - k_t^L d\left[S,S\right]_t \qquad (4.48)
$$

Consequently, in $(\mathbb{G}, P)$

$$
H = \alpha + \int_0^T \widetilde{\xi}_t^{\mathbb{G}} dS_t - \int_0^T \widetilde{\xi}_t^{\mathbb{G}} k_t^L d\left[S,S\right]_t + V_T, \;\; P-\text{a.s.} \qquad (4.49)
$$

The insider holds a trading strategy with both the stock and the volatility of the stock. In a different modeling situation, Kyle [43], Back [6] and Collin-Dufresne [17] with very different hypotheses have all observed that an optimal trading strategy for the insider (where the insider's optimal trading strategy is found while maximizing the return of the insider given his current set of knowledge) should involve the volatility in some way. In our work, this is also the case, although we show exactly how the insider's behavior changes, in the sense that a use of the volatility index should be part of his or her trading strategy. In Kyle [43], Back [6] and Collin-Dufresne [17], the trading strategy involves only the stock itself but the amount of stock the insider should trade at each time t is a function of the volatility, while our results show that it is natural also to trade the volatility of the stock.

Since one of our goals is to compare the market/liquidity trader's risk with the insider's, we need to make precise the notion of risk for the insider and market trader.

4.4.2 Insider and market trader's risk

As already mentioned above, we define the risk as the remainder of the financial claim H that the market trader or insider cannot trade away. Consequently, the market trader's risk is represented by $E_P(U_T^2)$, while the insider's risk is represented by $E_Q(K_T^2)$. Our goal therefore is to compare $E_P(U_T^2)$, as the risk of the market trader, versus $E_Q(K_T^2)$, a measure of the insider's risk. Two difficulties arise in that comparison: the processes U and K are defined in different filtrations but also with respect to different probability measures. Whence, to achieve our goal, we will first fix the probability measure P and compare $E_P(U_T^2)$ versus $E_P(V_T^2)$, then fix the filtration $\mathbb{G}$ and compare $E_P(V_T^2)$ versus $E_Q(K_T^2)$.

4.4.2.1 $E_P(V_T^2)$ and $E_P(U_T^2)$

To be able to compare $E_P(V_T^2)$ versus $E_P(U_T^2)$, we need to find the relationship between $\xi^{\mathbb{F}}$ and $\widetilde{\xi}^{\mathbb{G}}$.

Lemma 4.4.3. *Let's suppose there exists a sequence of reducing stopping times $(\gamma_n)_{n\geq 1}$ for V in $\mathbb{G}$ which are also $\mathbb{F}-$ stopping times, then*

$$\xi^{\mathbb{F}} -^{\circ\mathbb{F}} (\widetilde{\xi}^{\mathbb{G}}) \quad = \quad \frac{d\left[{}^{\circ\mathbb{F}}(V), S\right]}{d\,[S,S]}. \tag{4.50}$$

Proof. We have

$$\alpha + \int_0^T \xi_t^{\mathbb{F}} dS_t + U_T = \alpha + \int_0^T \widetilde{\xi}_t^{\mathbb{G}} dS_t - \int_0^T \widetilde{\xi}_t^{\mathbb{G}} k_t^L d\,[S,S]_t + V_T \tag{4.51}$$

and if we project both sides of the inequalities onto $\mathcal{F}_t$, by Brémaud and Yor [11], we obtain the following

$$\int_0^t \xi_u^{\mathbb{F}} dS_u + U_t = \int_0^t {}^{\circ\mathbb{F}}(\widetilde{\xi}^{\mathbb{G}})_u\, dS_u - \int_0^t {}^{\circ\mathbb{F}}(\widetilde{\xi}^{\mathbb{G}} k^L)_u\, d\,[S,S]_u +^{\circ\mathbb{F}} (V)_t. \tag{4.52}$$

Since there exists a sequence of reducing stopping times $(\gamma_n)_{n\geq 1}$ for V in $\mathbb{G}$ which are also $\mathbb{F}-$ stopping times, the optional projection of V is an $(\mathbb{F}, P)-$ local martingale by Theorem 3.7 in [24]. On the left side of (4.52) we have a local martingale and on the right side, we have a semimartingale. Therefore, since S is continuous,

$${}^{\circ\mathbb{F}}(\widetilde{\xi}^{\mathbb{G}} k^L) \cdot [S,S] \equiv 0 \tag{4.53}$$

That implies

$$^{o\mathbb{F}}(\widetilde{\xi}^{\mathbb{G}}k^L) = 0,\ \overline{P} - \text{a.e.}. \tag{4.54}$$

where $d\overline{P}(d\omega, dt) = d\left[S,S\right]_t(\omega)P(d\omega)$. Hence, from equation (4.52), we are left with

$$\begin{aligned}
\int_0^t \xi_u^{\mathbb{F}} dS_u + U_t &= \int_0^t {}^{o\mathbb{F}}(\widetilde{\xi}^{\mathbb{G}})_u\, dS_u + {}^{o\mathbb{F}}(V)_t \\
\int_0^t \xi_u^{\mathbb{F}} dS_u - \int_0^t {}^{o\mathbb{F}}(\widetilde{\xi}^{\mathbb{G}})_u\, dS_u &= {}^{o\mathbb{F}}(V)_t - U_t \\
\left(\xi^{\mathbb{F}} - {}^{o\mathbb{F}}(\widetilde{\xi}^{\mathbb{G}})\right)\cdot S &= {}^{o\mathbb{F}}(V) - U
\end{aligned}$$

which implies

$$\left[\left(\xi^{\mathbb{F}} - {}^{o\mathbb{F}}(\widetilde{\xi}^{\mathbb{G}})\right)\cdot S, S\right] = \left[{}^{o\mathbb{F}}(V) - U, S\right] \tag{4.55}$$

$$\left(\xi^{\mathbb{F}} - {}^{o\mathbb{F}}(\widetilde{\xi}^{\mathbb{G}})\right)\cdot [S,S] = \left[{}^{o\mathbb{F}}(V), S\right] - [U,S] \tag{4.56}$$

$$\left(\xi^{\mathbb{F}} - {}^{o\mathbb{F}}(\widetilde{\xi}^{\mathbb{G}})\right)\cdot [S,S] = \left[{}^{o\mathbb{F}}(V), S\right] \tag{4.57}$$

where equation (4.57) follows from the fact that S is strongly orthogonal to U (see Table 4.1). Consequently,

$$\xi^{\mathbb{F}} - {}^{o\mathbb{F}}(\widetilde{\xi}^{\mathbb{G}}) = \frac{d\left[{}^{o\mathbb{F}}(V), S\right]}{d\,[S,S]}.$$

Hence, the Lemma follows. ■

In the proof of Lemma 4.4.3, we have that ${}^{o\mathbb{F}}(\widetilde{\xi}^{\mathbb{G}}k^L) = 0$ on the support of $d\,[S,S]$. That means that the insider's trading stategy that involves the volatility of the stock is not seen by the market/liquidity trader in a financial market where only the market/liquidity trader and insider trade. This is in contrast to the findings of Collin-Dufresne [17] that the insider trades more aggressively when measured price impact is low, and therefore that more information gets into prices.

Under an additional assumption, the quadratic covariation of ${}^{o\mathbb{F}}(V)$ and S can be computed in terms of V, the extra-information and the volatility of the stock.

Lemma 4.4.4. *Let's suppose there exists a sequence of reducing stopping times $(\kappa_n)_{n\geq 1}$ for $\widetilde{S}$ in $\mathbb{G}$ which are also $\mathbb{F}-$ stopping times and under the same assumption as in Lemma 4.4.3, we have*

$$\left[{}^{o\mathbb{F}}(V), S\right]_t = E_P\left(V_t \int_0^t k_u^L d\,[S,S]_u \mid \mathcal{F}_t\right) \tag{4.58}$$

Proof. Under our assumptions, Theorem 3.7 in [24] implies that $Y :=^{\circ\mathbb{F}} (V)$ and S are $(\mathbb{F}, P)-$ local martingales. That means that $YS - [Y, S]$ is an $(\mathbb{F}, P)-$ local martingale. Let $0 \leq s < t$ and $(\tau_n)_{n\geq 1}$ be the reducing sequence of $YS - [Y, S]$ then $(\tau_n \wedge \gamma_n \wedge \kappa_n)_{n\geq 1}$ also reduces $YS - [Y, S]$. Then by stopping, we have

$$\begin{aligned}
Y_s S_s &= E_P\left(Y_s S_s \,|\, \mathcal{F}_s\right) \\
&= E_P\left(E_P\left(V_s \,|\, \mathcal{F}_s\right) S_s \,|\, \mathcal{F}_s\right) \\
&= E_P\left(E_P\left(S_s V_s \,|\, \mathcal{F}_s\right) \,|\, \mathcal{F}_s\right) \\
&= E_P\left(S_s V_s \,|\, \mathcal{F}_s\right) \\
&= E_P\left(V_s \widetilde{S}_s + V_s \int_0^s k_t^L d\left[S, S\right]_t \,|\, \mathcal{F}_s\right) \qquad (4.59) \\
&= E_P\left(V_s \widetilde{S}_s + V_s A_s \,|\, \mathcal{F}_s\right) \\
&= E_P\left(E_P\left(V_t \widetilde{S}_t \,|\, \mathcal{G}_s\right) + V_s A_s \,|\, \mathcal{F}_s\right) \qquad (4.60) \\
&= E_P\left(E_P\left(V_t \widetilde{S}_t \,|\, \mathcal{G}_s\right) \,|\, \mathcal{F}_s\right) + E_P\left(V_s A_s \,|\, \mathcal{F}_s\right) \\
&= E_P\left(V_t \widetilde{S}_t \,|\, \mathcal{F}_s\right) + E_P\left(V_s A_s \,|\, \mathcal{F}_s\right) \\
&= E_P\left(E_P\left(V_t \widetilde{S}_t \,|\, \mathcal{F}_t\right) \,|\, \mathcal{F}_s\right) + E_P\left(V_s A_s \,|\, \mathcal{F}_s\right) \\
&= E_P\left(E_P\left(V_t S_t - V_t A_t \,|\, \mathcal{F}_t\right) \,|\, \mathcal{F}_s\right) + E_P\left(V_s A_s \,|\, \mathcal{F}_s\right) \\
&= E_P\left(E_P\left(V_t S_t \,|\, \mathcal{F}_t\right) \,|\, \mathcal{F}_s\right) - E_P\left(E_P\left(V_t A_t \,|\, \mathcal{F}_t\right) \,|\, \mathcal{F}_s\right) + E_P\left(V_s A_s \,|\, \mathcal{F}_s\right) \\
&= E_P\left(S_t E_P\left(V_t \,|\, \mathcal{F}_t\right) \,|\, \mathcal{F}_s\right) - E_P\left(V_t A_t \,|\, \mathcal{F}_s\right) + E_P\left(V_s A_s \,|\, \mathcal{F}_s\right) \\
&= E_P\left(S_t Y_t \,|\, \mathcal{F}_s\right) - E_P\left(V_t A_t \,|\, \mathcal{F}_s\right) + E_P\left(V_s A_s \,|\, \mathcal{F}_s\right) \qquad (4.61)
\end{aligned}$$

where $A_s := \int_0^s k_t^L d\left[S, S\right]_t$. Equation (4.59) is due to (4.48); (4.60) comes from the fact the $V\widetilde{S}$ is a $(\mathbb{G}, P)-$ local martingale since $\left[V, \widetilde{S}\right] \equiv 0$ and the sequence of $\mathbb{F}-$ stopping times $(\tau_n \wedge \gamma_n \wedge \kappa_n)_{n\geq 1}$ that are also $\mathbb{G}-$ stopping times reduces it. Additionally, $YS - [Y, S]$ is an $(\mathbb{F}, P)-$ local martingale, hence by stopping, using $(\tau_n \wedge \gamma_n \wedge \kappa_n)_{n\geq 1}$, we obtain

$$\begin{aligned}
Y_s S_s - \left[Y, S\right]_s &= E_p\left(Y_t S_t - \left[Y, S\right]_t \,|\, \mathcal{F}_s\right) \\
Y_s S_s &= E_p\left(Y_t S_t - \left[Y, S\right]_t \,|\, \mathcal{F}_s\right) + \left[Y, S\right]_s \\
&= E_p\left(Y_t S_t \,|\, \mathcal{F}_s\right) - E_p\left(\left[Y, S\right]_t \,|\, \mathcal{F}_s\right) + \left[Y, S\right]_s \\
&= E_p\left(Y_t S_t \,|\, \mathcal{F}_s\right) - E_p\left(\left[Y, S\right]_t \,|\, \mathcal{F}_s\right) + E_p\left(\left[Y, S\right]_s \,|\, \mathcal{F}_s\right) \qquad (4.62)
\end{aligned}$$

From (4.61) and (4.62), we have

$$\begin{cases} Y_s S_s - [Y,S]_s & = & E_p\left(Y_t S_t - [Y,S]_t \mid \mathcal{F}_s\right) \\ Y_s S_s - E_p\left(V_s A_s \mid \mathcal{F}_s\right) & = & E_p\left(Y_t S_t - E_p\left(V_t A_t \mid \mathcal{F}_t\right) \mid \mathcal{F}_s\right) \end{cases}$$

Therefore by Corollary 2 on page 72 of [52], uniqueness of $[Y,S]$ implies that

$$\begin{aligned} [Y,S]_t &= \left[{}^{\circ\mathbb{F}}(V), S\right]_t \\ &= E_P\left(V_t A_t \mid \mathcal{F}_t\right) \\ &= E_P\left(V_t \int_0^t k_u^L d\,[S,S]_u \mid \mathcal{F}_t\right). \end{aligned} \tag{4.63}$$

Hence, the Lemma. ■

We will need the following lemma.

Lemma 4.4.5. *Let f be a convex function and X an $\mathbb{F}-$ adapted process such that ${}^{\circ\mathbb{F}}X$ and ${}^{\circ\mathbb{F}}f(X)$ exist, then*

$$f\left({}^{\circ\mathbb{F}}X_t\right) \le {}^{\circ\mathbb{F}}f\left(X_t\right), \text{ a.s. } \forall\, t \ge 0 \tag{4.64}$$

with the inequality being strict when the function f is strictly convex.

Proof. The proof follows from Jensen's inequality. ■

Let's now return to the problem of computing $E_P(V_T^2)$ and $E_P(U_T^2)$.

Theorem 4.4.6. *Under the same assumption as in Lemma 4.4.3 and if*

$$E_P\left[{}^{\circ\mathbb{F}}(V), \xi^{\mathbb{F}} \cdot S\right]_T \quad < \quad 0 \tag{4.65}$$

then

$$E_P(U_T)^2 \quad > \quad E_P(V_T)^2.$$

Hence for $E_P(U_T)^2$ to be strictly greater than $E_P(V_T)^2$, it is necessary that in expectation, what the market sees of the insider's intrinsic risk in $(\mathbb{G}, P)$ is negatively related to the market trader's trading strategy at the expiration date of the financial claim, in the sense of equation (4.65).

Proof. By Lemma 4.4.1, we have

$$\begin{cases} E_P(H^2) &= \alpha^2 + E_P\left(\int_0^T \left(\xi_t^{\mathbb{F}}\right)^2 d\,[S,S]_t\right) + E_P(U_T)^2 \\ E_P(H^2) &= \alpha^2 + E_P\left(\int_0^T \left(\widetilde{\xi}_t^{\mathbb{G}}\right)^2 d\,[S,S]_t\right) + E_P(V_T)^2 \end{cases}$$

Hence, if $J_t = \frac{d\left[{}^{o\mathbb{F}}(V),S\right]_t}{d[S,S]_t} \neq 0$ for each $t \in [0,T]$, we have

$$\begin{aligned}
0 &= E_P\left(\int_0^T \left[\left(\xi_t^{\mathbb{F}}\right)^2 - \left(\widetilde{\xi}_t^{\mathbb{G}}\right)^2\right] d\,[S,S]_t\right) + E_P(U_T)^2 - E_P(V_T)^2 \\
E_P(V_T)^2 - E_P(U_T)^2 &= E_P\left(\int_0^T \left[\left(\xi_t^{\mathbb{F}}\right)^2 - \left(\widetilde{\xi}_t^{\mathbb{G}}\right)^2\right] d\,[S,S]_t\right) \\
E_P(U_T)^2 - E_P(V_T)^2 &= E_P\left(\int_0^T \left[\left(\widetilde{\xi}_t^{\mathbb{G}}\right)^2 - \left(\xi_t^{\mathbb{F}}\right)^2\right] d\,[S,S]_t\right) \\
&= E_P\left(\int_0^T \left[{}^{o\mathbb{F}}\left(\widetilde{\xi}_t^{\mathbb{G}}\right)^2 - \left(\xi_t^{\mathbb{F}}\right)^2\right] d\,[S,S]_t\right) \\
&\geq E_P\left(\int_0^T \left[\left({}^{o\mathbb{F}}\widetilde{\xi}_t^{\mathbb{G}}\right)^2 - \left(\xi_t^{\mathbb{F}}\right)^2\right] d\,[S,S]_t\right) \qquad (4.66) \\
&= E_P\left(\int_0^T \left[\left(\xi_t^{\mathbb{F}} - J_t\right)^2 - \left(\xi_t^{\mathbb{F}}\right)^2\right] d\,[S,S]_t\right) \\
&= E_P\left(\int_0^T \left[-2\xi_t^{\mathbb{F}} J_t + J_t^2\right] d\,[S,S]_t\right) \\
&\geq -2\,E_P \int_0^T \xi_t^{\mathbb{F}} J_t d\,[S,S]_t \\
&= -2\,E_P \int_0^T \xi_t^{\mathbb{F}} d\,[J\cdot S,S]_t \\
&= -2\,E_P \int_0^T \xi_t^{\mathbb{F}} d\left[{}^{o\mathbb{F}}(V),S\right]_t \\
&= -2\,E_P \int_0^T d\left[{}^{o\mathbb{F}}(V),\xi^{\mathbb{F}}\cdot S\right]_t \\
&= -2\,E_P \left[{}^{o\mathbb{F}}(V),\xi^{\mathbb{F}}\cdot S\right]_T \qquad (4.67)
\end{aligned}$$

where (4.66) comes from Lemma 4.4.5. The result of the theorem follows from equation (4.65). ∎

4.4.2.2 $E_P(V_T^2)$ and $E_Q(K_T^2)$

Just as above, to be able to compare $E_P(V_T^2)$ versus $E_Q(K_T^2)$, we need to find the relationship between $\widetilde{\xi}^{\mathbb{G}}$ and $\xi^{\mathbb{G}}$.

Lemma 4.4.7. *On the support of* $d\,[S,S]$,

$$Z_t\xi_t^{\mathbb{G}} - \widetilde{\xi}_t^{\mathbb{G}} \;=\; E_P\left\{k_t^L H \,|\, \mathcal{G}_t\right\},\ \forall\, t$$

where Z *is defined as in Lemma 4.4.2.*

Proof. From the $(\mathbb{G},P)$ and $(\mathbb{G},Q)$ representations of H, we have

$$\begin{cases} E_P\{H\,|\,\mathcal{G}_t\} = \alpha + \int_0^t \widetilde{\xi}_u^{\mathbb{G}} d\widetilde{S}_u + V_t, & (\mathbb{G},P) \\ E_Q\{H\,|\,\mathcal{G}_t\} = \beta + \int_0^t \xi_u^{\mathbb{G}} dS_u + K_t, & (\mathbb{G},Q) \end{cases}$$

Since $\left(E_Q\{H\,|\,\mathcal{G}_t\}\right)_{t\geq 0}$ is a $(\mathbb{G},Q)-$ martingale, it follows that $\left(Z_tE_Q\{H\,|\,\mathcal{G}_t\}\right)_{t\geq 0}$ is a $(\mathbb{G},P)-$ martingale and

$$\left(Z_tR_t := Z_tE_Q\{H\,|\,\mathcal{G}_t\}\right) = \left(E_P\{H\,|\,\mathcal{G}_t\} := \overline{R}_t\right) \tag{4.68}$$

Using integration by parts, we obtain the following

$$\begin{aligned} d(ZR)_t &= Z_tdR_t + R_tdZ_t + d\,[Z,R]_t \\ &= Z_t\xi_t^{\mathbb{G}}dS_t + Z_tdK_t + R_tdZ_t - k_t^L\xi_t^{\mathbb{G}}Z_td\,[S,S]_t \qquad (4.69) \\ &= Z_t\xi_t^{\mathbb{G}}\left(dS_t - k_t^Ld\,[S,S]_t\right) + Z_tdK_t + R_tdZ_t \\ &= Z_t\xi_t^{\mathbb{G}}d\widetilde{S}_t + Z_tdK_t + R_tdZ_t \\ &= Z_t\xi_t^{\mathbb{G}}d\widetilde{S}_t + Z_tdK_t - R_tk_t^LZ_td\widetilde{S}_t \qquad (4.70) \end{aligned}$$

Equation (4.70) comes from the fact that Z solves the following SDE: $dZ_t = -k_t^LZ_td\widetilde{S}_t$; while (4.69) is due to the fact that

$$[S,K] = 0 \Rightarrow \left[\widetilde{S},K\right] = 0 \tag{4.71}$$

by the continuity of S and $\widetilde{S}$. That implies $d\,[Z,R]_t = -k_t^L\xi_t^{\mathbb{G}}Z_td\,[S,S]_t$. Using (4.68), we therefore have

$$\begin{aligned} Z_t\xi_t^{\mathbb{G}}d\widetilde{S}_t + Z_tdK_t - R_tk_t^LZ_td\widetilde{S}_t &= \widetilde{\xi}_t^{\mathbb{G}}d\widetilde{S}_t + dV_t \\ \left(Z_t\xi_t^{\mathbb{G}} - \widetilde{\xi}_t^{\mathbb{G}}\right)d\widetilde{S}_t &= dV_t - Z_tdK_t + R_tk_t^LZ_td\widetilde{S}_t \qquad (4.72) \end{aligned}$$

From (4.71), (4.72) and using the fact that $\left[\widetilde{S},V\right] = 0$, we obtain for all t

$$\left(Z_t\xi_t^{\mathbb{G}} - \widetilde{\xi}_t^{\mathbb{G}}\right)d\,[S,S]_t \;=\; R_tk_t^LZ_td\,[S,S]_t \tag{4.73}$$

Hence on the support of $d\,[S,S]$,

$$
\begin{aligned}
Z_t \xi_t^{\mathbb{G}} - \widetilde{\xi}_t^{\mathbb{G}} &= Z_t R_t k_t^L, \ \forall\, t \\
&= \overline{R}_t k_t^L, \ \forall\, t \\
&= E_P \left\{ k_t^L H \,|\, \mathcal{G}_t \right\}, \ \forall\, t
\end{aligned} \tag{4.74}
$$

The result of the Lemma follows. ■

Going back to computing $E_P(V_T^2)$ and $E_Q(K_T^2)$, we have the following theorem:

Theorem 4.4.8. *Assume the finite variation process* $\left(\int_0^t \xi_s^{\mathbb{G}} d\,[S,S]_s\right)_{t\geq 0}$ *satisfies enough integrability conditions for* $\left(\int_0^t \left[\int_0^u \xi_s^{\mathbb{G}} d\,[S,S]_s\right] dZ_u\right)_{t\geq 0}$ *to remain a martingale. Additionally, if*

$$
E_P\left\{ZR\left(\widetilde{\xi}^{\mathbb{G}} k^L\right)\cdot [S,S]\right\}_T - E_P\left\{Z\xi^{\mathbb{G}}\left(\widetilde{\xi}^{\mathbb{G}} - \xi^{\mathbb{G}}\right)\cdot [S,S]\right\}_T > \mathrm{Var}_Q(H) - \mathrm{Var}_P(H)
$$

then,

$$
E_P(V_T^2) \quad > \quad E_Q(K_T^2).
$$

Proof. By Lemma 4.4.1, we have

$$
\begin{cases}
E_P(H^2) = \alpha^2 + E_P\left(\int_0^T \left(\widetilde{\xi}_t^{\mathbb{G}}\right)^2 d\,[S,S]_t\right) + E_P(V_T)^2 \\
E_Q(H^2) = \beta^2 + E_Q\left(\int_0^T \left(\xi_t^{\mathbb{G}}\right)^2 d\,[S,S]_t\right) + E_Q(K_T)^2
\end{cases}
$$

Hence,

$$
\begin{aligned}
E_P(H^2) - E_Q(H^2) &= \alpha^2 - \beta^2 + E_P\left(\int_0^T \left(\widetilde{\xi}_t^{\mathbb{G}}\right)^2 d\,[S,S]_t\right) - E_Q\left(\int_0^T \left(\xi_t^{\mathbb{G}}\right)^2 d\,[S,S]_t\right) \\
&+ \ E_P(V_T)^2 - E_Q(K_T)^2 \\
E_Q(K_T)^2 - E_P(V_T)^2 &= \alpha^2 - \beta^2 + E_P\left(\int_0^T \left(\widetilde{\xi}_t^{\mathbb{G}}\right)^2 d\,[S,S]_t\right) - E_Q\left(\int_0^T \left(\xi_t^{\mathbb{G}}\right)^2 d\,[S,S]_t\right) \\
&+ \ E_Q(H^2) - E_P(H^2) \\
&= \alpha^2 - \beta^2 + E_P\left(\int_0^T \left(\widetilde{\xi}_t^{\mathbb{G}}\right)^2 d\,[S,S]_t\right) - E_P\left(Z_T\int_0^T \left(\xi_t^{\mathbb{G}}\right)^2 d\,[S,S]_t\right) \\
&+ \ E_Q(H^2) - E_P(H^2) \qquad (4.75) \\
&= \alpha^2 - \beta^2 + E_P\left(\int_0^T \left(\widetilde{\xi}_t^{\mathbb{G}}\right)^2 d\,[S,S]_t\right) - E_P\left(\int_0^T Z_t\left(\xi_t^{\mathbb{G}}\right)^2 d\,[S,S]_t\right)
\end{aligned}
$$

$$
\begin{aligned}
& + \quad E_Q(H^2) - E_P(H^2) \\
& = \quad \alpha^2 - \beta^2 + E_P\left(\int_0^T \left[\left(\widetilde{\xi}_t^{\mathbb{G}}\right)^2 - Z_t\left(\xi_t^{\mathbb{G}}\right)^2\right] d\left[S,S\right]_t\right) + E_Q(H^2) \\
& - \quad E_P(H^2) \\
E_P(V_T)^2 - E_Q(K_T)^2 & = \quad \beta^2 - \alpha^2 + E_P\left(\int_0^T \left[Z_t\left(\xi_t^{\mathbb{G}}\right)^2 - \left(\widetilde{\xi}_t^{\mathbb{G}}\right)^2\right] d\left[S,S\right]_t\right) + E_P(H^2) \\
& - \quad E_Q(H^2) \\
& = \quad (E_Q(H))^2 - (E_P(H))^2 + E_P(H^2) - E_Q(H^2) \\
& + \quad E_P\left(\int_0^T \left[Z_t\left(\xi_t^{\mathbb{G}}\right)^2 - \left(\widetilde{\xi}_t^{\mathbb{G}}\right)^2\right] d\left[S,S\right]_t\right) \\
& = \quad \mathrm{Var}_P(H) - \mathrm{Var}_Q(H) \\
& + \quad E_P\left(\int_0^T \left[Z_t\left(\xi_t^{\mathbb{G}}\right)^2 - \left(\widetilde{\xi}_t^{\mathbb{G}}\right)^2\right] d\left[S,S\right]_t\right) \qquad (4.76)
\end{aligned}
$$

Equation (4.75) follows from the integration by parts formula. Indeed,

$$
\begin{aligned}
d\left(Z_t \int_0^t \xi_s^{\mathbb{G}} d\left[S,S\right]_s\right) & = Z_t \xi_t^{\mathbb{G}} d\left[S,S\right]_t + \left(\int_0^t \xi_s^{\mathbb{G}} d\left[S,S\right]_s\right) dZ_t \\
& + d\left[\int_0^{\cdot} \xi_s^{\mathbb{G}} d\left[S,S\right]_s, Z\right]_t \\
& = Z_t \xi_t^{\mathbb{G}} d\left[S,S\right]_t + \left(\int_0^t \xi_s^{\mathbb{G}} d\left[S,S\right]_s\right) dZ_t
\end{aligned}
$$

Where the last equality holds because $\int \xi_s^{\mathbb{G}} d\left[S,S\right]_s$ is a finite variation process and Z is continuous (see Theorem 28 page 75 of [52]). We have $Z_0 \int_0^0 \xi_s^{\mathbb{G}} d\left[S,S\right]_s = 1 \times 0 = 0$ and since by assumption $\left(\int_0^t \left[\int_0^u \xi_s^{\mathbb{G}} d\left[S,S\right]_s\right] dZ_u\right)_{t \geq 0}$ is still a martingale, then

$$
\begin{aligned}
E_P\left(Z_T \int_0^T \xi_s^{\mathbb{G}} d\left[S,S\right]_s\right) & = E_P\left(\int_0^T Z_t \xi_t^{\mathbb{G}} d\left[S,S\right]_t\right) + E_P\left(\int_0^T \left[\int_0^t \xi_s^{\mathbb{G}} d\left[S,S\right]_s\right] dZ_t\right) \\
& = E_P\left(\int_0^T Z_t \xi_t^{\mathbb{G}} d\left[S,S\right]_t\right)
\end{aligned}
$$

From (4.74), we have

$$
\begin{aligned}
Z\xi^{\mathbb{G}} - \widetilde{\xi}^{\mathbb{G}} & = \overline{R} k^L \\
Z\xi^{\mathbb{G}} \widetilde{\xi}^{\mathbb{G}} - \left(\widetilde{\xi}^{\mathbb{G}}\right)^2 & = \widetilde{\xi}^{\mathbb{G}} \overline{R} k^L \\
\left(\widetilde{\xi}^{\mathbb{G}}\right)^2 & = Z\xi^{\mathbb{G}} \widetilde{\xi}^{\mathbb{G}} - \widetilde{\xi}^{\mathbb{G}} \overline{R} k^L \qquad (4.77)
\end{aligned}
$$

where $\overline{R}$ is defined in (4.68). Plugging (4.77) into (4.76), we get

$$
E_P(V_T)^2 - E_Q(K_T)^2 = \mathrm{Var}_P(H) - \mathrm{Var}_Q(H) + E_P\left(\int_0^T \left[Z_t\left(\xi_t^{\mathbb{G}}\right)^2 - \left(\widetilde{\xi}_t^{\mathbb{G}}\right)^2\right] d\left[S,S\right]_t\right)
$$

$$
\begin{aligned}
&= \mathrm{Var}_P(H) - \mathrm{Var}_Q(H) \\
&+ E_P\left(\int_0^T \left[Z_t\left(\xi_t^{\mathbb{G}}\right)^2 - Z_t\xi_t^{\mathbb{G}}\widetilde{\xi}_t^{\mathbb{G}} + \widetilde{\xi}_t^{\mathbb{G}}\overline{R}_t k_t^L\right] d\,[S,S]_t\right) \\
&= \mathrm{Var}_P(H) - \mathrm{Var}_Q(H) + E_P\left(\int_0^T \left[Z_t\xi_t^{\mathbb{G}}\left(\xi_t^{\mathbb{G}} - \widetilde{\xi}_t^{\mathbb{G}}\right) + \widetilde{\xi}_t^{\mathbb{G}} k_t^L \overline{R}_t\right] d\,[S,S]_t\right) \\
&= \mathrm{Var}_P(H) - \mathrm{Var}_Q(H) + E_P\left(\int_0^T \left[\widetilde{\xi}_t^{\mathbb{G}} k_t^L \overline{R}_t - Z_t\xi_t^{\mathbb{G}}\left(\widetilde{\xi}_t^{\mathbb{G}} - \xi_t^{\mathbb{G}}\right)\right] d\,[S,S]_t\right) \\
&= E_P\left(\int_0^T \left\{d\left[\widetilde{\xi}^{\mathbb{G}} k^L \overline{R}\cdot S, S\right]_t - d\left[\left(\widetilde{\xi}^{\mathbb{G}} - \xi^{\mathbb{G}}\right)\cdot S, Z\xi^{\mathbb{G}}\cdot S\right]_t\right\}\right) \\
&+ \mathrm{Var}_P(H) - \mathrm{Var}_Q(H) \\
&= E_P\left[\widetilde{\xi}^{\mathbb{G}} k^L \overline{R}\cdot S, S\right]_T - E_P\left[\left(\widetilde{\xi}^{\mathbb{G}} - \xi^{\mathbb{G}}\right)\cdot S, Z\xi^{\mathbb{G}}\cdot S\right]_T \\
&+ \mathrm{Var}_P(H) - \mathrm{Var}_Q(H)
\end{aligned}
$$

Therefore, if $E_P\left\{ZR\left(\widetilde{\xi}^{\mathbb{G}} k^L\right)\cdot[S,S]\right\}_T - E_P\left\{Z\xi^{\mathbb{G}}\left(\widetilde{\xi}^{\mathbb{G}} - \xi^{\mathbb{G}}\right)\cdot[S,S]\right\}_T > \mathrm{Var}_Q(H) - \mathrm{Var}_P(H)$, then $E_P(V_T)^2 > E_Q(K_T)^2$. ■

It is important to note that the trading strategy in the volatility of the stock, $\widetilde{\xi}^{\mathbb{G}} k^L$ (equation (4.49)), and the difference in payoffs provided by the insider's trading strategies in $(\mathbb{G}, P)$ and $(\mathbb{G}, Q)$, appear in the conditions of Theorem 4.4.8.

Our goal is to compare the intrinsic risk of the insider to the intrinsic risk of the market/liquidity trader.

Theorem 4.4.9. *Let's suppose*

- *there exists a sequence of reducing stopping times for V in $\mathbb{G}$ which are also $\mathbb{F}-$stopping times*
- $E_P\left[{}^{\circ\mathbb{F}}(V), \xi^{\mathbb{F}}\cdot S\right]_T < 0$
- *the finite variation process $\left(\int_0^t \xi_s^{\mathbb{G}} d\,[S,S]_s\right)_{t\geq 0}$ satisfies enough integrability conditions for $\left(\int_0^t \left[\int_0^u \xi_s^{\mathbb{G}} d\,[S,S]_s\right] dZ_u\right)_{t\geq 0}$ to remain a martingale*
- $E_P\left\{ZR\left(\widetilde{\xi}^{\mathbb{G}} k^L\right)\cdot[S,S]\right\}_T - E_P\left\{Z\xi^{\mathbb{G}}\left(\widetilde{\xi}^{\mathbb{G}} - \xi^{\mathbb{G}}\right)\cdot[S,S]\right\}_T > \mathrm{Var}_Q(H) - \mathrm{Var}_P(H)$

where $\overline{R} = E_P\left(H\,|\,\mathcal{G}_\cdot\right)$, then

$$E_Q(K_T^2) < E_P(V_T^2) < E_P(U_T^2). \tag{4.78}$$

Proof. Theorem 4.4.9 is a straightforward application of Theorem 4.4.6 and of Theorem 4.4.8. ■

Theorem 4.4.9 goes a bit against the intuition that even though the insider still undertakes risk, his or her risk has to be smaller than the market/liquidity trader's risk given the asymmetry of information that exist in the market. Indeed, we have found that the insider's intrinsic risk is not always smaller than the market/liquidity trader's intrinsic risk. Under certain conditions, the market/liquidity trader undertakes less risk than the insider, with risk interpreted in the Föllmer-Schweizer sense.

The second objective of this chapter is to answer the following question: how would an insider whose market only satisfies Local NFLVR price his or her financial claims? Note that if the insider's market satisfies NFLVR, then he or she can easily price financial instruments using his or her risk neutral measure denoted here by Q. Using the representation in Table 4.1, the market/liquidity trader price for H is given by $E_P(H) = \alpha$ while the insider price for the same claim is given by $E_Q(H) = \beta$. Therefore, if the insider is interested in selling or buying H, depending on whether $\alpha < \beta$ or $\alpha > \beta$, the insider knows whether the market price is a good or a bad deal. Surpringly, when the insider market only satisfies Local NFLVR, similar statements can also be made.

4.5 Local NFLVR and Pricing

Under Local NFLVR (See Chapter 3), the discounted stock price is no longer a $(\mathbb{G}, Q)-$ local martingale but there exists a sequence of $\mathbb{G}-$ stopping times $(T_n)_n$ and a consistent sequence of probability measures $(Q_n)_n$ such that S^{T_n} is a $(\mathbb{G}, Q_n)-$ local martingale. Therefore, by Kunita-Watanabe, we have the following representation for each n

$$E_{Q_n}\left(H \,|\, \mathcal{G}_{T\wedge T_n}\right) = \beta_n + \int_0^{T\wedge T_n} \xi_t^{n\mathbb{G}} dS_t^{T_n} + K_T^n, \quad Q_n - \text{a.s.} \tag{4.79}$$

which implies

$$\begin{aligned} E_{Q_n}\left(E_{Q_n}\left(H \,|\, \mathcal{G}_{T\wedge T_n}\right) |\, \mathcal{G}_t\right) &= E_{Q_n}\left(H \,|\, \mathcal{G}_{t\wedge T_n}\right) \\ &= \beta_n + \int_0^{t\wedge T_n} \xi_u^{n\mathbb{G}} dS_u^{T_n} + K_t^n, \quad Q_n - \text{a.s.} \end{aligned} \tag{4.80}$$

where K^n is a $(\mathbb{G}, Q_n)-$ local martingale which is orthogonal to S^{T_n} for each n and

$$\xi^{n\mathbb{G}} = \frac{d\left[E_{Q_n}\left(H \,|\, \mathcal{G}_{\cdot\wedge T_n}\right), S^{T_n}\right]}{d\left[S^{T_n}, S^{T_n}\right]}, \quad Q_n - \text{a.s.} \tag{4.81}$$

Theorem 4.5.1. *We have*

$$\beta_n = \beta, \;\forall\, n.$$

Before we prove Theorem 4.5.1, let's first prove the following two lemmas.

Lemma 4.5.2. *For all $n \geq 1$, we have*

$$\xi^{n\mathbb{G}} = \left\{\xi^{(n+1)\mathbb{G}}\right\}^{T_n}, \quad Q_n - \text{a.s..} \tag{4.82}$$

Proof. Fix n. It is known $\left(S^{T_{n+1}}\right)^2 - \left[S^{T_{n+1}}, S^{T_{n+1}}\right]$ is a local $(\mathbb{G}, Q_{n+1})-$ local martingale by Corollary 2 on page 72 of [52], then $Z^{n,n+1}\left(S^{T_{n+1}}\right)^2 - Z^{n,n+1}\left[S^{T_{n+1}}, S^{T_{n+1}}\right]$ is a $(\mathbb{G}, Q_n)-$ local martingale where

$$Z_t^{n,n+1} = \frac{dQ_{n+1}|\mathcal{G}_t}{dQ_n|\mathcal{G}_t} = \exp\left(-\int_{T_n}^{t\wedge T_{n+1}} k_s^L\, d\widetilde{S}_s - \frac{1}{2}\int_{T_n}^{t\wedge T_{n+1}} (k_s^L)^2\, d\left[S, S\right]_s\right). \tag{4.83}$$

Therefore, $\left\{Z^{n,n+1}\left(S^{T_{n+1}}\right)^2 - Z^{n,n+1}\left[S^{T_{n+1}}, S^{T_{n+1}}\right]\right\}^{T_n}$ is still a $(\mathbb{G}, Q_n)-$ local martingale. Since $\left\{Z^{n,n+1}\right\}^{T_n} \equiv 1, \quad Q_n - \text{a.s..}$

We obtain the following set of equalities

$$\begin{aligned}
\left\{Z^{n,n+1}\left(S^{T_{n+1}}\right)^2 - Z^{n,n+1}\left[S^{T_{n+1}}, S^{T_{n+1}}\right]\right\}^{T_n} &= \left(S^{T_n}\right)^2 - \left\{\left[S^{T_{n+1}}, S^{T_{n+1}}\right]\right\}^{T_n} \\
&= \left(S^{T_n}\right)^2 - \left\{\left[S, S\right]^{T_{n+1}}\right\}^{T_n} \qquad (4.84) \\
&= \left(S^{T_n}\right)^2 - \left[S, S\right]^{T_n} \\
&= \left(S^{T_n}\right)^2 - \left[S^{T_n}, S^{T_n}\right] \qquad (4.85)
\end{aligned}$$

where (4.84) and (4.85) follow from Theorem 22 Chapter II of [52]. Consequently, continuity, Corollary 2 on page 72 of [52] and (4.85) imply

$$\left[S^{T_{n+1}}, S^{T_{n+1}}\right]^{T_n} = \left[S^{T_n}, S^{T_n}\right], \quad Q_n - \text{a.s.} \tag{4.86}$$

Similarly, $E_{Q_{n+1}}\left(H \,|\, \mathcal{G}_{\cdot\wedge T_{n+1}}\right) S^{T_{n+1}} - \left[E_{Q_{n+1}}\left(H \,|\, \mathcal{G}_{\cdot\wedge T_{n+1}}\right), S^{T_{n+1}}\right]$ is a local $(\mathbb{G}, Q_{n+1})-$ local martingale by Corollary 2 on page 72 of [52], then

$$Z^{n,n+1} E_{Q_{n+1}}\left(H \,|\, \mathcal{G}_{\cdot\wedge T_{n+1}}\right) S^{T_{n+1}} - Z^{n,n+1}\left[E_{Q_{n+1}}\left(H \,|\, \mathcal{G}_{\cdot\wedge T_{n+1}}\right), S^{T_{n+1}}\right]$$

is a $(\mathbb{G}, Q_n)$ − local martingale. Therefore,

$$\left\{Z^{n,n+1}E_{Q_{n+1}}\left(H\,|\,\mathcal{G}_{\cdot\wedge T_{n+1}}\right)S^{T_{n+1}} - Z^{n,n+1}\left[E_{Q_{n+1}}\left(H\,|\,\mathcal{G}_{\cdot\wedge T_{n+1}}\right), S^{T_{n+1}}\right]\right\}^{T_n}$$

is still a $(\mathbb{G}, Q_n)$− local martingale. Since $\left\{Z^{n,n+1}\right\}^{T_n} \equiv 1, \;\; Q_n -$ a.s., we obtain

$$\begin{aligned}
&\left\{Z^{n,n+1}E_{Q_{n+1}}\left(H\,|\,\mathcal{G}_{\cdot\wedge T_{n+1}}\right)S^{T_{n+1}} - Z^{n,n+1}\left[E_{Q_{n+1}}\left(H\,|\,\mathcal{G}_{\cdot\wedge T_{n+1}}\right), S^{T_{n+1}}\right]\right\}^{T_n} \\
&\qquad = E_{Q_{n+1}}\left(H\,|\,\mathcal{G}_{\cdot\wedge T_{n+1}\wedge T_n}\right)S^{T_n} - \left[E_{Q_{n+1}}\left(H\,|\,\mathcal{G}_{\cdot\wedge T_{n+1}\wedge T_n}\right), S^{T_n}\right] \\
&\qquad = E_{Q_n}\left(H\,|\,\mathcal{G}_{\cdot\wedge T_n}\right)S^{T_n} - \left[E_{Q_n}\left(H\,|\,\mathcal{G}_{\cdot\wedge T_n}\right), S^{T_n}\right] \qquad (4.87)
\end{aligned}$$

(4.87) comes from the fact we can choose a consistent sequence of probability measures $(Q_n)_{\{n\geq 0\}}$ such that Q_n and Q_{n+1} agree on $\mathcal{G}_{T_n}$, therefore also on $\mathcal{G}_{t\wedge T_n}$ since $\mathcal{G}_{t\wedge T_n} = \mathcal{G}_{T_n} \cap \mathcal{G}_t$. Hence, continuity, Corollary 2 on page 72 of [52] and (4.87) imply

$$\left[E_{Q_{n+1}}\left(H\,|\,\mathcal{G}_{\cdot\wedge T_{n+1}}\right), S^{T_{n+1}}\right]^{T_n} = \left[E_{Q_n}\left(H\,|\,\mathcal{G}_{\cdot\wedge T_n}\right), S^{T_n}\right], \;\; Q_n - \text{a.s.} \qquad (4.88)$$

(4.86) and (4.88) imply that

$$\xi^{n\mathbb{G}} \;\; = \;\; \left\{\xi^{(n+1)\mathbb{G}}\right\}^{T_n}, \;\; Q_n - \text{a.s..} \qquad (4.89)$$

Hence, Lemma 4.5.2 holds. ■

Lemma 4.5.3. *For all n, we have*

$$\left\{K^{(n+1)}\right\}^{T_n} \;\; = \;\; K^n, \;\; Q_n - \text{a.s.} \qquad (4.90)$$

Proof. We know that $\left[S^{T_n}, K^n\right] \equiv 0, \;\; Q_n -$ a.s. and $S^{T_n}K^n$ is a $(\mathbb{G}, Q_n)$− local martingale. Similarly, $\left[S^{T_{n+1}}, K^{(n+1)}\right] \equiv 0, \;\; Q_{n+1} -$ a.s. and $S^{T_{n+1}}K^{(n+1)}$ is a $(\mathbb{G}, Q_{n+1})$− local martingale. Therefore, $Z^{n,n+1}S^{T_{n+1}}K^{(n+1)}$ is a $(\mathbb{G}, Q_n)$− local martingale, hence so is

$$\left\{Z^{n,n+1}S^{T_{n+1}}K^{(n+1)}\right\}^{T_n} \;\; = \;\; S^{T_n}\left\{K^{(n+1)}\right\}^{T_n}.$$

That means $\left[S^{T_n}, \left(K^{(n+1)}\right)^{T_n}\right] \equiv 0, \;\; Q_n -$ a.s.. Consequently, $\left(E_{Q_n}\left(H\,|\,\mathcal{G}_{t\wedge T_n}\right)\right)_{0\leq t\leq T}$ can also be written as

$$E_{Q_n}\left(H\,|\,\mathcal{G}_{t\wedge T_n}\right) \;\; = \;\; \beta_n + \int_0^{t\wedge T_n} \widetilde{\xi}_u^{n\mathbb{G}} dS_u^{T_n} + \left(K^{(n+1)}\right)_t^{T_n}, \; Q_n - \text{a.s..} \qquad (4.91)$$

Uniqueness of the Kunita-Watanabe decomposition implies that

$$\left\{K^{(n+1)}\right\}^{T_n} = K^n, \;\; Q_n - \text{a.s..} \qquad (4.92)$$

Hence, Lemma 4.5.3 holds. ■

Proof of Theorem 4.5.1. To prove Theorem 4.5.1, it is enough to prove that $\beta_{n+1} = \beta_n$ for each n. Fix n. By Lemma 4.5.2 and Lemma 4.5.3, we have $\xi^{n\mathbb{G}} = \left\{\xi^{(n+1)\mathbb{G}}\right\}^{T_n}, \quad Q_n - \text{a.s.}$ and $\left\{K^{(n+1)}\right\}^{T_n} = K^n, \quad Q_n - \text{a.s.}$.

We have

$$E_{Q_n}\left(H \,|\, \mathcal{G}_{t\wedge T_n}\right) = \beta_n + \int_0^{t\wedge T_n} \xi_u^{n\mathbb{G}} dS_u^{T_n} + K_t^n, \quad Q_n - \text{a.s.} \tag{4.93}$$

$$E_{Q_{n+1}}\left(H \,|\, \mathcal{G}_{t\wedge T_{n+1}}\right) = \beta_{n+1} + \int_0^{t\wedge T_{n+1}} \xi_u^{(n+1)\mathbb{G}} dS_u^{T_{n+1}} + K_t^{(n+1)}, \quad Q_{n+1} - \text{a.s.} \tag{4.94}$$

Multipling equation (4.94) by $Z^{n,n+1}$, we get

$$\begin{aligned} Z^{n,n+1} E_{Q_{n+1}}\left(H \,|\, \mathcal{G}_{t\wedge T_{n+1}}\right) &= Z^{n,n+1}\beta_{n+1} + Z^{n,n+1}\int_0^{t\wedge T_{n+1}} \xi_u^{(n+1)\mathbb{G}} dS_u^{T_{n+1}} \\ &+ Z^{n,n+1} K_t^{(n+1)}, \quad Q_{n+1} - \text{a.s.} \end{aligned} \tag{4.95}$$

stopping equation (4.95) at T_n, we obtain

$$\begin{aligned} \left\{Z^{n,n+1} E_{Q_{n+1}}\left(H \,|\, \mathcal{G}_{t\wedge T_{n+1}}\right)\right\}^{T_n} &= \left\{Z^{n,n+1}\beta_{n+1}\right\}^{T_n} + \left\{Z^{n,n+1}\int_0^{t\wedge T_{n+1}} \xi_u^{(n+1)\mathbb{G}} dS_u^{T_{n+1}}\right\}^{T_n} \\ &+ \left\{Z^{n,n+1} K_t^{(n+1)}\right\}^{T_n}, \; Q_n - \text{a.s.} \end{aligned} \tag{4.96}$$

and since $\left\{Z^{n,n+1}\right\}^{T_n} \equiv 1, \quad Q_n - \text{a.s.}$, it implies that

$$\begin{aligned} E_{Q_{n+1}}\left(H \,|\, \mathcal{G}_{t\wedge T_{n+1}\wedge T_n}\right) &= \beta_{n+1} + \int_0^{t\wedge T_{n+1}\wedge T_n} \xi_u^{(n+1)\mathbb{G}} dS_u^{T_{n+1}} + \left\{K_t^{(n+1)}\right\}^{T_n}, \; Q_n - \text{a.s.} \\ E_{Q_n}\left(H \,|\, \mathcal{G}_{t\wedge T_n}\right) &= \beta_{n+1} + \int_0^{t\wedge T_n} \xi_u^{n\mathbb{G}} dS_u^{T_n} + K_t^n, \; Q_n - \text{a.s.} \end{aligned} \tag{4.97}$$

where (4.97) follows from (4.83) and (4.90). Uniqueness of the Kunita-Watanabe decomposition implies that $\beta_n = \beta_{n+1}$. ■

From the proof of Lemma 4.5.3, we have exhibited processes which are locally local martingales. The next subsection makes such processes more precise.

4.5.1 Quasi-local martingales

Let $\mathcal{H}$ and $\mathbb{H}$ represent respectively either the $\sigma-$ field $\mathcal{F}$ or $\mathcal{G}$ and either the filtration $\mathbb{F}$ or $\mathbb{G}$.

Definition 4.5.1. An $\mathbb{H}-$ adapted process X is a quasi-local martingale if there exist a sequence of $\mathbb{H}-$ stopping times increasing a.s. to a limit τ which could be finite a.s. or take on infinity and is an $\mathbb{H}-$ stopping time, and a sequence of probability measures $(Q_n)_{n\geq 1}$ such that $Q_{n+k}|_{[0,T_n]} = Q_n$, for all $k \geq 1$, and X is an $(\mathbb{H}, Q_n) -$ local martingale on $[0, T_n]$, for all n.

The process X is well-defined on $[0, \tau)$.

Remark. We have already encountered a couple of quasi-local martingales. For instance,

a) If Local NFLVR holds then the discounted price process is a quasi-local martingale on $[0, \overline{T})$ (where $\overline{T}$ was defined in Chapter 3 as the limit of the sequence of $\mathbb{G}-$ stopping times $(T_n)_n$ defined in Chapter 3 equation (3.11)).

b) From Lemma 4.5.3, we have that $\left\{K^{(n+1)}\right\}^{T_n} = K^n,\ Q_n -$ a.s. which means that if we define the following process

$$K \;=\; K^n, \text{ on } [0, T \wedge T_n]$$

then K is a quasi-local martingale on $[0, T)$.

c) The process defined by $\{E_{Q_n}(H \mid \mathcal{G}_{t\wedge T_n})\}_t$ is also a quasi-local martingale on $[0, T)$.

Since for each n, X is a Q_n- local martingale on $[0, T_n]$, we can refine the definition of a quasi-local martingale using martingales instead of local martingales. There exists $R_{n,k} \nearrow T_n$ such that $X^{R_{n,k}}$ is a Q_n- martingale $[0, R_{n,k}]$. Moreover, for every ϵ, there exists δ such that for every n and k,

$$Q_n\left(R_{n,k} < T_n - \frac{\epsilon}{2^k}\right) < \frac{\delta}{2^k}$$

Consequently, we can find a subsequence $k(n) \nearrow \infty$ such that $R_{n,k(n)} \nearrow \tau$.

Definition 4.5.2. An $\mathbb{H}-$ adapted process X is a quasi-local martingale if there exist a sequence of $\mathbb{H}-$ stopping times increasing a.s. to a limit τ which could be finite a.s. or take on infinity or be an $\mathbb{H}-$ stopping time, and a sequence of probability measures $(Q_n)_{n\geq 1}$ such that $Q_{n+m}|_{[0,R_{n,k(n)}]} = Q_n$, for all $m \geq 1$, and X is an $(\mathbb{H}, Q_n) -$ martingale on $\left[0, R_{n,k(n)}\right]$, for all n.

As mentioned above, quasi-local martingales are well-defined on $[0,\tau)$. Under certain conditions, we can extend its definition to $[0,\tau]$ in the spirit of Ruf [56]. We start by an extension of the Martingale Convergence Theorem to the case of quasi-local martingales.

Let $k \geq 1$ and let $(Y_n)_{0\leq n\leq k}$ be a Q_k submartingale with respect to a discrete time filtration $\mathbb{V} = (\mathcal{V}_n)_{0\leq n\leq k}$. Following the standard notation, see for example [34], we let U_k be the number of upcrossings of an interval $[a,b]$ $(a,b\in\mathbb{R})$ before time k.

Lemma 4.5.4. *Let $k \geq 1$ and let $(Y_n)_{0\leq n\leq k}$ be a $Q_k -$ submartingale where the sequence of measures $(Q_k)_{k\geq 1}$ satisfies the following: $Q_{k+m} = Q_k$ on $\mathcal{V}_k$ for all $m \geq 1$. If*

$$\sup_k E_{Q_k}\left\{(Y_k)^+\right\} \quad < \quad \infty, \tag{4.98}$$

then $Y_\infty := \lim_{k\to\infty} Y_k$ exists.

Proof. Fix $k \geq 1$. If U_k is the number of upcrossings of an interval $[a,b]$ $(a,b\in\mathbb{R})$ before time k, then by Doob's Upcrossing Inequality, we obtain

$$\begin{aligned} E_{Q_k}(U_k) &\leq \frac{1}{b-a}E_{Q_k}\left\{(Y_k-a)^+\right\} \\ &\leq \frac{1}{b-a}E_{Q_k}\left\{(Y_k)^+\right\} + \frac{|a|}{b-a} \end{aligned}$$

We define the number of upcrossings of the process Y as follows

$$U^*(a,b;Y) \quad = \quad \lim_{k\to\infty} E_{Q_k}\{U_k\} \tag{4.99}$$

Note that $U_k \leq U_{k+1}$ $P-$ a.s. which implies that $E_{Q_{k+1}}(U_k) \leq E_{Q_{k+1}}(U_{k+1})$. In addition, since $(Q_k)_{k\geq 1}$ is a consistent sequence of probability measures and U_k is $\mathcal{V}_k -$ measurable, then $E_{Q_k}(U_k) = E_{Q_{k+1}}(U_k)$. Hence, $E_{Q_k}(U_k) \leq E_{Q_{k+1}}(U_{k+1})$. By assumption we also have $\sup_k E_{Q_k}\left\{(Y_k)^+\right\} < \infty$.

Consequently $\{E_{Q_k}(U_k)\}_{k\geq 1}$ is a nondecreasing sequence of nonnegative numbers; Hence, $U^*(a,b;Y)$ exists and $U^*(a,b;Y) < \infty,$ $P-$ a.s.. Then the process $(Y_n)_{n\geq 0}$ upcrosses the interval $[a,b]$ only finitely often $P-$ a.s.. The rest of the proof follows as the proof of the Martingale Convergence Theorem (see for example Jacod and Protter [34]). ■

We can apply the results just above to some of the processes listed in Remark 4.5.1.

Proposition 4.5.5. *[Extension of quasi-local martingales]. If*

$$\sup_n E_{Q_n}(H^+) < \infty. \tag{4.100}$$

Then $H^* := \lim_{n\to\infty} E_{Q_n}\{H \,|\, \mathcal{G}_{T\wedge T_n}\}$ *exists* $P-$a.s..

In addition, if

$$\sup_n E_{Q_n}(H^2) < \infty, \tag{4.101}$$

$$\sup_n E_{Q_n}\left[\int_0^{T\wedge T_n} \left(\xi_t^{n\mathbb{G}}\right)^2 d\,[S,S]_t\right] < \infty. \tag{4.102}$$

Then $K^* := \lim_{n\to\infty} K_{T\wedge T_n}$ *exists* $P-$a.s..

Proof. Let $Y_n = E_{Q_n}\{H \,|\, \mathcal{G}_{T\wedge T_n}\}$. Fix $k \geq 1$, then $(Y_n)_{0\leq n\leq k}$ is a Q_k- martingale with respect to the filtration given by $\mathcal{V}_n := \mathcal{G}_{T\wedge T_n}$. Indeed, let $n < n+1 \leq k$, then

$$\begin{aligned} E_{Q_k}\{X_{n+1} \,|\, \mathcal{V}_n\} &= E_{Q_k}\{X_{n+1} \,|\, \mathcal{G}_{T\wedge T_n}\} \\ &= E_{Q_{n+1}}\{X_{n+1} \,|\, \mathcal{G}_{T\wedge T_n}\} \qquad (4.103) \\ &= E_{Q_{n+1}}\left\{E_{Q_{n+1}}\left\{H \,|\, \mathcal{G}_{T\wedge T_{n+1}}\right\} \,|\, \mathcal{G}_{T\wedge T_n}\right\} \\ &= E_{Q_n}\{H \,|\, \mathcal{G}_{T\wedge T_n}\} \end{aligned}$$

Equation (4.103) follows from the fact that $Q_k = Q_{n+1}$ on $\mathcal{G}_{T\wedge T_{n+1}}$. If

$$\begin{aligned} \sup_k E_{Q_k}\left[\{E_{Q_k}(H \,|\, \mathcal{G}_{T\wedge T_k})\}^+\right] &\leq \sup_k E_{Q_k}\left[E_{Q_k}\left(H^+ \,|\, \mathcal{G}_{T\wedge T_k}\right)\right] \qquad (4.104) \\ &= \sup_k E_{Q_k}\left[H^+\right] \\ &< \infty \qquad (4.105) \end{aligned}$$

where (4.104) follows from Jensen's inequality and (4.105) by assumption. Then H^* exists $P-$a.s. by Lemma 4.5.4.

Similarly, let $\widetilde{Y}_n = K_T^n = K_{T\wedge T_n}$. Fix $k \geq 1$, then $\left(\widetilde{Y}_n\right)_{0\leq n\leq k}$ is a Q_k- martingale with respect to the filtration given by $\mathcal{V}_n := \mathcal{G}_{T\wedge T_n}$. Indeed, let $n < n+1 \leq k$ and since Q_k and Q_n agree on $\mathcal{G}_{T\wedge T_n}$, we have the following

$$\begin{aligned} E_{Q_k}\left(K_{T\wedge T_{n+1}} \,|\, \mathcal{G}_{T\wedge T_n}\right) &= E_{Q_n}\left(K_{T\wedge T_{n+1}} \,|\, \mathcal{G}_{T\wedge T_n}\right) \\ &= K_{T\wedge T_n} \qquad (4.106) \end{aligned}$$

where (4.106) follows from the fact that $\left\{K^{(n+1)}\right\}^{T_n} = K^n,\ Q_n - \text{a.s.}$. Moreover,

$$E_{Q_n}\left\{K^2_{T\wedge T_n}\right\} \leq 3\beta + 3E_{Q_n}\left[\int_0^{T\wedge T_n}\left(\xi_t^{n\mathbb{G}}\right)^2 d\left[S,S\right]_t\right] + 3E_{Q_n}\left(H^2\right)$$

where we use the fact that $(a+b+c)^2 \leq 3a^2+3b^2+3c^2$. Hence by equations (4.101) and (4.102)

$$\sup_n E_{Q_n}\left\{K^2_{T\wedge T_n}\right\} < \infty$$

Consequently, it follows from Lemma 4.5.4 that K^* exists $P - \text{a.s.}$. ■

Note that from the representation in (4.79), we have that for each n, $E_{Q_n}\left[\int_0^{T_n}\left(\xi_t^{n\mathbb{G}}\right)^2 d\left[S,S\right]_t\right] < \infty$ but not $\sup_n E_{Q_n}\left[\int_0^{T_n}\left(\xi_t^{n\mathbb{G}}\right)^2 d\left[S,S\right]_t\right] < \infty$. The sufficient conditions to obtain the existence of K^* can probably be weakened but we haven't found yet how to do so.

4.5.2 Process-wise extension of the Kunita-Watanabe decomposition

From the extension of quasi-local martingales to $[0,T]$ or $[0,\overline{T}]$, two very natural questions arise. Taking the limit as n goes to ∞ in equation (4.79), what does the resulting equation mean? Moreover, H is the financial claim while H^* represents the limit of the different projections of H onto the subfiltrations $\mathcal{G}_{T\wedge T_n}$'s. Is H equal to H^*? Under Local NFLVR, the answer to the second question is no and it can be intuitively understood by the fact under Local NFLVR, there is no risk neutral Q since there need not exist a limit of the sequence of consistent probability measures $(Q_n)_n$.

Lemma 4.5.6. *Suppose*

$$\sup_n E_{Q_n}(H^+) < \infty,$$

then, there exists a random variable $0 < \Psi \in \mathcal{G}_{T+} = \mathcal{G}_T$ *such that*

$$H^* = \Psi H \tag{4.107}$$

Proof. By definition of H^* we have

$$\begin{aligned} H^* &= \lim_{n\to\infty} E_{Q_n}\left(H \,|\, \mathcal{G}_{T\wedge T_n}\right) \\ &= \lim_{n\to\infty} E_P\left(Z^n H \,|\, \mathcal{G}_{T\wedge T_n}\right) \end{aligned} \tag{4.108}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} Z^n E_P\left(H \,|\, \mathcal{G}_{T\wedge T_n}\right) && (4.109)\\
&= \left\{\lim_{n\to\infty} Z^n\right\}\left\{\lim_{n\to\infty} E_P\left(H \,|\, \mathcal{G}_{T\wedge T_n}\right)\right\} \\
&= \left\{\lim_{n\to\infty} Z^n\right\}\left\{E_P\left(H \,|\, \mathcal{G}_{T}\right)\right\} && (4.110)\\
&= \left\{\lim_{n\to\infty} Z^n\right\} H && (4.111)\\
&= \Psi H
\end{aligned}
$$

where in (4.108), $Z^n = \frac{dQ_n}{dP}$. Note that $Z^n = Z_T^{T_n}$ where Z is defined in Chapter 3, equation (3.14). Equation (4.109) follows from the fact that $Z^n = Z_{T\wedge T_n} \in \mathcal{G}_{T\wedge T_n}$. Since $\{E_P\left(H \,|\, \mathcal{G}_{T\wedge T_n}\right)\}_{n\geq 1}$ is a discrete time closed martingale, therefore (4.110) follows from the Martingale Convergence Theorem (See Jacod and Protter [34]). By assumption, $H \in \mathcal{F}_T$ and since $\mathcal{F}_T \subset \mathcal{G}_T$, we have $H \in \mathcal{G}_T$, hence (4.111). By assumption, H^* exists. Hence since the left side exists and is finite $P-$a.s. and on the right side we have $H < \infty, P-$a.s., then $\Psi < \infty, P-$a.s.. For each n, $Z^n > 0$, hence so is $\Psi := \lim_{n\to\infty} Z^n$. ∎

Note that the existence of Ψ can also be derived from the extended Doob's upcrossing inequality (Lemma 4.5.4). Now, we focus on obtaining a representation such as the one given in (4.79) at the fixed time T. Lemma 4.5.6 shows that an insider whose market satisfies Local NFLVR sells or buys the claim H but can only hedge sequentially; that means if the insider hedges $E_{Q_n}\left(H \,|\, \mathcal{G}_{T\wedge T_n}\right)$ for each n, and as $n \to \infty$, then he or she doesn't recover H but rather, H^*.

Theorem 4.5.7. *Suppose* (4.100), (4.101) *and* (4.102) *hold, then*

$$
\begin{aligned}
H^* &= \beta + \int_0^T \xi_t dS_t + K^*, \ P-\text{a.s.} && (4.112)\\
\text{where } \xi &= \xi^{n\mathbb{G}}, \ \text{on } [0, T\wedge T_n]
\end{aligned}
$$

Equation (4.112) should be understood as one process being equal to another at a fixed time T.

Proof. If (4.100), (4.101) and (4.102) hold, then H^* and K^* exist $P-$a.s. by Proposition 4.5.5. Moreover, from equation (4.80), we have

$$
\infty > \int_0^{t\wedge T_n} \left(\xi_s^{n\mathbb{G}}\right)^2 d\left[S,S\right]_s^{T_n}, \ Q_n - \text{a.s.} \tag{4.113}
$$

$$
\begin{aligned}
&= \int_0^{t\wedge T_n} \left(\xi_s^{n\mathbb{G}}\right)^2 d\,[S,S]_s^{T_n}\,,\ P-\text{a.s.} \\
&= \int_0^{t\wedge T_n} (\xi_s)^2\, d\,[S,S]_s \qquad (4.114)
\end{aligned}
$$

Then, the Monotone Convergence Theorem implies

$$
\int_0^t (\xi_s)^2\, d\,[S,S]_s \quad < \quad \infty,\ P-\text{a.s.}. \qquad (4.115)
$$

Hence, for all $t \in [0,\overline{T})$, $\int_0^t \xi_t dS_t < \infty,\ P-\text{a.s.}$. Since $T < \overline{T},\ P-\text{a.s.}$, then $\int_0^T \xi_t dS_t < \infty,\ P-\text{a.s.}$. Therefore

$$
\begin{aligned}
E_{Q_n}\left(H\,|\,\mathcal{G}_{T\wedge T_n}\right) &= \beta_n + \int_0^{T\wedge T_n} \xi_t^{n\mathbb{G}} dS_t^{T_n} + K_T^n,\quad Q_n-\text{a.s.} \\
E_{Q_n}\left(H\,|\,\mathcal{G}_{T\wedge T_n}\right) &= \beta_n + \int_0^{T\wedge T_n} \xi_t^{n\mathbb{G}} dS_t^{T_n} + K_T^n,\quad P-\text{a.s.} \qquad (4.116)
\end{aligned}
$$

and the result of the theorem follows by taking limits of the left and right sides of (4.116). ■

From Lemma 4.5.6, we have found a relationship between the financial claim H and H^*. We would like now to compare $E_{Q_n}\left(H\,|\,\mathcal{G}_{T\wedge T_n}\right)$ and $E_{Q_n}\left(H^*\,|\,\mathcal{G}_{T\wedge T_n}\right)$ for each n.

Lemma 4.5.8. *Let's assume H^* exists and suppose $H^* \in L^1(dQ_n),\ n \geq 1$ and $H \geq 0$, then*

$$
E_{Q_n}\left(H^*\,|\,\mathcal{G}_{T\wedge T_n}\right) \quad \leq \quad E_{Q_n}\left(H\,|\,\mathcal{G}_{T\wedge T_n}\right),\ \ P-\text{a.s.}, \qquad (4.117)
$$

for each n.

Proof. Since H^* exists and $H^* \in L^1(dQ_n),\ n \geq 1$, we have that $0 \leq E_{Q_n}\left(H^*\,|\,\mathcal{G}_{T\wedge T}\right)$ and is well-defined P almost surely. Let fix $n = n_0$ and

$$
\begin{aligned}
E_{Q_{n_0}}\left(H^*\,|\,\mathcal{G}_{T\wedge T_{n_0}}\right) &= E_{Q_{n_0}}\left(\lim_{k\to\infty} E_{Q_k}\left(H\,|\,\mathcal{G}_{T\wedge T_k}\right)\,|\,\mathcal{G}_{T\wedge T_{n_0}}\right) \\
&\leq \liminf_{k\to\infty} E_{Q_{n_0}}\left(E_{Q_k}\left(H\,|\,\mathcal{G}_{T\wedge T_k}\right)\,|\,\mathcal{G}_{T\wedge T_{n_0}}\right) \qquad (4.118) \\
&= \liminf_{k\to\infty} E_{Q_k}\left(E_{Q_k}\left(H\,|\,\mathcal{G}_{T\wedge T_k}\right)\,|\,\mathcal{G}_{T\wedge T_{n_0}}\right) \qquad (4.119) \\
&= \liminf_{k\to\infty} E_{Q_k}\left(H\,|\,\mathcal{G}_{T\wedge T_{n_0}}\right) \\
&= \liminf_{k\to\infty} E_{Q_{n_0}}\left(H\,|\,\mathcal{G}_{T\wedge T_{n_0}}\right) \qquad (4.120) \\
&= E_{Q_{n_0}}\left(H\,|\,\mathcal{G}_{T\wedge T_{n_0}}\right),\ \ Q_{n_0}-\text{a.s.}
\end{aligned}
$$

$$= \quad E_{Q_{n_0}}\left(H \,|\, \mathcal{G}_{T\wedge T_{n_0}}\right), \;\; P-\text{a.s.}$$

Equation (4.118) follows from Fatou's lemma. Without loss of generality, we can assume that $k \geq n_0$ and since $\mathcal{G}_{T\wedge T_{n_0}} \subseteq \mathcal{G}_{T\wedge T_k}$, and $Q_k = Q_{n_0}$ on $\mathcal{G}_{T\wedge T_{n_0}}$, we obtain equations (4.119) and (4.120). ■

The inequality that appears in equation (4.117) is a little bit unsatisfying, under additional assumptions, we can obtain an equality instead.

Lemma 4.5.9. *Let's assume H^* exists. Suppose $H^* \in L^1(dQ_n)$, $n \geq 1$, and H is bounded, then*

$$E_{Q_n}\left(H^* \,|\, \mathcal{G}_{T\wedge T_n}\right) \quad \geq \quad E_{Q_n}\left(H \,|\, \mathcal{G}_{T\wedge T_n}\right), \;\; P-\text{a.s.}, \tag{4.121}$$

for each n.

The assumption that H is bounded might appear limiting, but it is not in practice. The bound can be very large, hence the set of claims we consider is still a large subset of the set of $\mathcal{L}^{2+\Delta}-$ integrable claims despite the boundedness assumption (for some $\Delta > 0$).

Proof. Assuming H is bounded, we have that $E_{Q_k}\left(H \,|\, \mathcal{G}_{T\wedge T_k}\right)$ is also bounded for every $k \geq 1$. Then,

$$\begin{aligned}
E_{Q_{n_0}}\left(H^* \,|\, \mathcal{G}_{T\wedge T_{n_0}}\right) &= E_{Q_{n_0}}\left(\lim_{k\to\infty} E_{Q_k}\left(H \,|\, \mathcal{G}_{T\wedge T_k}\right) \,|\, \mathcal{G}_{T\wedge T_{n_0}}\right) \\
&\geq \limsup_{k\to\infty} E_{Q_{n_0}}\left(E_{Q_k}\left(H \,|\, \mathcal{G}_{T\wedge T_k}\right) \,|\, \mathcal{G}_{T\wedge T_{n_0}}\right) && (4.122) \\
&= \limsup_{k\to\infty} E_{Q_k}\left(E_{Q_k}\left(H \,|\, \mathcal{G}_{T\wedge T_k}\right) \,|\, \mathcal{G}_{T\wedge T_{n_0}}\right) && (4.123) \\
&= \limsup_{k\to\infty} E_{Q_k}\left(H \,|\, \mathcal{G}_{T\wedge T_{n_0}}\right) \\
&= \limsup_{k\to\infty} E_{Q_{n_0}}\left(H \,|\, \mathcal{G}_{T\wedge T_{n_0}}\right) && (4.124) \\
&= E_{Q_{n_0}}\left(H \,|\, \mathcal{G}_{T\wedge T_{n_0}}\right), \;\; Q_{n_0}-\text{a.s.} \\
&= E_{Q_{n_0}}\left(H \,|\, \mathcal{G}_{T\wedge T_{n_0}}\right), \;\; P-\text{a.s.}
\end{aligned}$$

where equation (4.122) follows from Fatou's lemma. Similarly as in the proof of Lemma 4.5.8, we can assume without loss of generality that $k \geq n_0$ and since $\mathcal{G}_{T\wedge T_{n_0}} \subseteq \mathcal{G}_{T\wedge T_k}$, and $Q_k = Q_{n_0}$ on $\mathcal{G}_{T\wedge T_{n_0}}$, we obtain equations (4.123) and (4.124). ■

The following lemma summarizes the results obtained above.

Lemma 4.5.10. *Under the same assumptions as in Lemma 4.5.8 and Lemma 4.5.9, we have*

$$E_{Q_n}(H^* \,|\, \mathcal{G}_{T\wedge T_n}) \quad = \quad E_{Q_n}(H \,|\, \mathcal{G}_{T\wedge T_n}), \;\; P - \text{a.s.}, \tag{4.125}$$

for each n.

Proof. The proof is a straightforward application of Lemma 4.5.8 and Lemma 4.5.9. ■

4.6 Insider's pricing under Local NFLVR

In this section, we are interested in what is the "optimal" price of a financial claim H. An insider has extra information modeled by $\mathbb{G}$. Under Local NFLVR, there exists a sequence of $\mathbb{G}-$ stopping times such that for each n, there exists Q_n, $\xi^{n\mathbb{G}}$, β_n and K^n

$$E_{Q_n}(H \,|\, \mathcal{G}_{T\wedge T_n}) \quad = \quad \beta_n + \int_0^{T\wedge T_n} \xi_t^{n\mathbb{G}} dS_t^{T_n} + K_T^n, \;\; Q_n - \text{a.s.}$$

and using Föllmer-Schweizer theory at time 0, if an insider uses the trading strategy $\xi^{n\mathbb{G}}$, then his or her minimum risk is the intrinsic risk represented by $E_{Q_n}(K_T^n)^2$. From our results, we have that

$$\begin{aligned} \xi^{(n+1)\mathbb{G}}|_{\mathcal{G}_{T\wedge T_n}} &= \xi^{n\mathbb{G}}, \;\; Q_{n+1}|_{\mathcal{G}_{T_n}} = Q_n \\ K^{(n+1)}|_{\mathcal{G}_{T\wedge T_n}} &= K^n, \;\; \beta_{n+1} = \beta_n = \beta. \end{aligned}$$

Definition 4.6.1. A price will be called *optimal* if on every $[0, T_n]$, The FS risk at time 0 is equal to the intrinsic risk $E_{Q_n}(K_T^n)^2$, $\xi^{n\mathbb{G}}$ is the risk-minimizing trading strategy and β_n the risk minimizing price, for each $n \geq 1$.

From our results, it follows that β is the optimal price at which the financial claim should be sold at, if we were to sell H at the $\mathbb{G}-$ stopping times $(T_n)_{n\geq 1}$. It is important to note that by definition, the $\mathbb{G}-$ stopping times $(T_n)_{n\geq 1}$ are visible to the insider since they are constructed out of his or her extra-information and the volatility of the stock. What if the insider is willing to sell the claim at a time that is not one of the T_n's? For instance, what is the price the insider will sell the claim at if he or she had to sell it at a fixed time $t \in [0.T)$?

Theorem 4.6.1. *Suppose the financial claim H is nonnegative and bounded, and $\sup_n E_{Q_n}(H^+) < \infty$, then at any time $t \in [0,T)$, the insider's optimal price for the claim H is β.*

Proof. Let fix a time $t_0 \in [0,T)$ and let

$$\Lambda_n = \{\omega : t_0 < T_n(\omega)\} \tag{4.126}$$

then, $\bigcup_{n\geq 1} \Lambda_n = \left\{t_0 < \overline{T}\right\}$, where $\overline{T} = \lim_{n\to\infty} T_n > T,\ P-$a.s.. Hence,

$$P\left(\bigcup_{n\geq 1} \Lambda_n\right) = P\left(t_0 < \overline{T}\right) = 1.$$

Consequently, there exists at least one Λ_n with $P(\Lambda_n) > 0$. Let's call one of such Λ_n, Λ_{n^*}. Therefore $P(\Lambda_{n^*}) > 0$.

Let's suppose that the insider sells the claim H at time t_0 and at a price γ such that $\gamma > \beta$. On Λ_{n^*}, $t_0 < T_{n^*}$, then

$$E_{Q_{n^*}}\left(H \,|\, \mathcal{G}_{t_0\wedge T_{n^*}}\right) = \beta + \int_0^{t_0\wedge T_{n^*}} \xi_t^{n^*\mathbb{G}} dS_t + K_{t_0\wedge T_{n^*}},\quad Q_{n^*}-\text{a.s.} \tag{4.127}$$

$$E_{Q_{n^*}}\left(H \,|\, \mathcal{G}_{t_0\wedge T_{n^*}}\right) = \beta + \int_0^{t_0\wedge T_{n^*}} \xi_t^{n^*\mathbb{G}} dS_t + K_{t_0\wedge T_{n^*}},\quad P-\text{a.s.} \tag{4.128}$$

$$E_{Q_{n^*}}\left(H \,|\, \mathcal{G}_{t_0}\right) = \beta + \int_0^{t_0} \xi_t^{n^*\mathbb{G}} dS_t + K_{t_0},\quad P-\text{a.s.} \tag{4.129}$$

The price corresponding to the minimum risk at t_0 is β.

Using γ, the insider has to hedge the position taken on H but since his or her market only satisfies Local NFLVR, as already noted above, the insider can recover only H^* not H. Under the assumptions of the theorem, H^* exists.

$$E_{Q_{n^*}}\left(H^* \,|\, \mathcal{G}_{t_0\wedge T_{n^*}}\right) = \gamma + \int_0^{t_0\wedge T_{n^*}} \overline{\xi}_t^{n^*\mathbb{G}} dS_t + \overline{K}_{t_0\wedge T_{n^*}},\quad Q_{n^*}-\text{a.s.} \tag{4.130}$$

$$E_{Q_{n^*}}\left(H^* \,|\, \mathcal{G}_{t_0\wedge T_{n^*}}\right) = \gamma + \int_0^{t_0\wedge T_{n^*}} \overline{\xi}_t^{n^*\mathbb{G}} dS_t + \overline{K}_{t_0\wedge T_{n^*}},\quad P-\text{a.s.} \tag{4.131}$$

$$E_{Q_{n^*}}\left(H \,|\, \mathcal{G}_{t_0}\right) = \gamma + \int_0^{t_0} \overline{\xi}_t^{n^*\mathbb{G}} dS_t + \overline{K}_{t_0},\quad P-\text{a.s.} \tag{4.132}$$

and just as above, the price corresponding to the minimum risk at t_0 is γ for the fictitious claim H^*.

Equation (4.132) follows from the assumptions on H and by Lemma 4.5.10. Consequently, by the uniqueness of the Kunita-Watanabe decomposition, we have $\beta = \gamma$. The same proof works in the case when $\gamma < \beta$. ■

We have been working with general admissible integrands but under Local NFLVR, an insider has no arbitrage with respect to admissible simple predictable integrands. Hence, let NAS be the set of admissible simple predictable trading strategies. Let's suppose we have a probability measure Q, then for every ϵ, since NAS is dense in the set of admissible predictable integrands we have:

$$\begin{aligned} H &= \beta + \int_0^T \xi_t dS_t + L_T^H, \ \ Q-\text{a.s.} \\ &= \beta + \int_0^T h_t^\epsilon dS_t + \int_0^T (\xi - h^\epsilon)_t \, dS_t + L_T^H, \ \ Q-\text{a.s.} \qquad (4.133) \\ &= \beta + \int_0^T h_t^\epsilon dS_t + \Sigma_T^\epsilon + L_T^H, \ \ Q-\text{a.s.} \qquad (4.134) \end{aligned}$$

where $h^\epsilon \in$ NAS and ξ is an admissible predictable trading strategy. Then Σ_T^ϵ is a risk that can be made as small as the trader wants. Indeed, it results from the fact that practically only buy and hold portfolios are achievable compared to the theoretical trading strategies represented in the literature by admissible predictable integrands. Therefore, we can choose h^ϵ in such a way that

$$\int_0^T (h_t^\epsilon - \xi_t)^2 d\overline{Q} < \epsilon, \qquad (4.135)$$

where $d\overline{Q}(d\omega, dt) = d\left[S,S\right]_t(\omega)Q(d\omega)$. L_T^H is the intrinsic risk that can not be controlled by the trader. Consequently, the total risk of the trader is measured by $\epsilon + E_Q(L_T^H)^2$.

In the case when the insider's market only satisfies Local NFLVR, for every $\epsilon^n > 0$, there exists $h^n \in$ NAS

$$E_{Q_n}\left(H \,|\, \mathcal{G}_{T\wedge T_n}\right) = \beta + \int_0^{T\wedge T_n} h_t^n dS_t^{T_n} + \Sigma_T^n + K_T^n, \ \ Q_n - \text{a.s.} \qquad (4.136)$$

such that

$$\int_0^{T\wedge T_n} (h_t^n - \xi_t^{n\mathbb{G}})^2 d\overline{Q}_n < \epsilon, \qquad (4.137)$$

where $d\overline{Q}_n(d\omega, dt) = d\left[S,S\right]_t(\omega)Q_n(d\omega)$. Moreover

$$\left(h^{(n+1)}\right)^{T_n} = h^n, \ \ Q_n - \text{a.s.}, \qquad (4.138)$$

and we could choose the ϵ^n such that $\lim_{n\to\infty}\epsilon^n = 0$. Consequently, the pricing theory developed above still applies except that the risk is the intrinsic risk plus ϵ^n.

Definition 4.6.2. A price will be called optimal if on every $[0, T_n]$, The FS risk at time 0 is equal to the intrinsic risk $E_{Q_n}(K_T^n)^2 + \epsilon^n$, h^n is the risk-minimizing admissible simple predictable trading strategy and β the risk minimizing price, for each $n \geq 1$.

The rest of the pricing theory under Local NFLVR follows with $E_{Q_n}(K_T^n)^2$ being replaced by $E_{Q_n}(K_T^n)^2 + \epsilon^n$ when necessary.

4.7 Conclusion

In the present chapter, we have studied conditions under which the insider's risk is less than the market/liquidity trader's risk. Our results go a bit against the intuition that the extra information obtained by an insider should reduce the insider's risk. Even though those conditions might be not satisfied, hence the insider might face an intrinsic higher than the market/liquidity trader, the insider can still know better than a market trader whether a deal is a good or a bad one.

We have also introduced a new process called quasi-local martingale and we have proved that the discounted stock price is a quasi-local martingale if the insider's market satisfies Local NFLVR. Quasi-local martingales can be extended to a fixed time and by doing so, we obtained a Kunita-Watanabee process-wise representation.

As we have already mentioned in Chapter 3, the theory of derivatives pricing requires a risk neutral measure which is only ensured by the presence of NFLVR (see Delbaen and Schachermayer [19]). In this chapter, we have proved that even though the enlarged market does not satisfy NFLVR, the insider still has a notion of price which is the optimal price that minimizes the insider's risk. Hence, an insider can price his or her financial claims using the sequence of probability measures that are exhibited under local NFLVR.

As in Chapter 3, the most important assumption of the present chapter is the continuity of all $\mathbb{F}-$ local martingales. Although not critical in Chapter 3, the continuity assumption becomes very critical here because it simplifies the computation of quadratic covariation between processes that are adapted to different filtrations, facilates the definition of orthogonality between local martingales as well as the computation of compensators of quadratic variation processes. Consequently, the current work should be extended to the case when discounted price processes are càdlàg.

Chapter 5

Conclusion and future research

Insider trading, as understood by the public and investors, is a misconduct that affects the viability of financial markets. Indeed, it undermines the public's perception of the fairness of the securities markets. The multiple allegations of insider trading that investors read about everyday in the newpapers illustrates the importance of studying and understanding illegal insider trading. Our work builds on other works (see Kyle [43], Back [6], [7], Collin-Dufresne [17], Pikovsky & Karatzas [50], Imkeller [32], Amendinger [3], Fontana et al. [27] and many more) that have contributed to the modeling and understanding of illegal insider trading. Indeed, the present thesis has two main parts, in the first part we study the market of an informed trader while in the second, we focus on understanding the risk of such an informed trader.

In Chapter 3, we study conditions under which the additional knowledge of an insider gives him or her arbitrage opportunities and/or free lunches. Starting with a market or liquidity trader that has no free lunches, we show that under some conditions the insider's enlarged financial market satisfies no free lunch with vanishing risk as well. One of those conditions is the uniformly integrability condition of a certain sequence of random variables. Indeed, uniform integrability is helpful in proving that if NFLVR holds locally then it holds globally. Unfortunately, the uniform integrability condition cannot be removed or weakened because otherwise the three-dimensional Bessel process starting at 1 ($\mathrm{Bes}^3(1)$) provides a counterexample. In fact, Delbaen and Schachermayer [20] proved that the $\mathrm{Bes}^3(1)$ process satisfies the no arbitrage property with respect to simple integrands, hence with respect to admissible simple integrands while Delbaen and Schachermayer [21] proved that the $\mathrm{Bes}^3(1)$ does not satisfy NFLVR on $[0, T]$ for any fixed T. Therefore, the $\mathrm{Bes}^3(1)$ provides an example of a process that satisfies Local NFLVR but not NFLVR. Nevertheless, the three-dimensional Bessel process helps us make a connection between Local NFLVR and the no arbitrage property with respect to simple buy and hold trading strategies, which in some sense is enough since admissible simple predictable integrands are the only type of trading strategies that can be used in practice. In case the insider trader's market satisfies only local NFLVR, we prove that it automatically satisfies the no arbitrage property with respect to simple predictable integrands. Hence, the enlarged market is consistent with the requirements of a well-functioning financial market. Of course, local NFLVR does not give the insider a risk neutral measure that he or she could use to price his or her financial derivatives. However in Chapter 4, we show how using local NFLVR an insider can price her financial instruments using the sequence of consistent probability measures that are exhibited under local NFLVR.

We also investigate the possibility of an enlarged market to satisfy NFLVR using the NA1 property. We find, using an extended version of Jacod's condition [33] (Theorem 3.4.1, Chapter 3), a local martingale deflator for the discounted stock price process. As a consequence, the enlarged market satisfies no arbitrage of the First Kind. The local martingale deflator we obtained was also studied by Amendinger [3] who proved that it can be used to define an equivalent local martingale measure but only under the restrictive condition that the conditional distribution of the extra knowledge with respect to the regular trader's current set of information is equivalent to the distribution of the additional knowledge. We find a set of simple conditions under which the informed trader's financial market satisties NFLVR. We are also able to find the general structure of all local martingale deflators of the enlarged market. In addition, the risk neutral measure of the insider is different from the market/liquidity trader's risk neutral measure.

We next apply our results to some examples. One of the examples models a situation that has been talked about recently in the newspapers. That example can be used as a mathematical representation of Mr. Martoma's insider trading case (see Ben Protess and Matthew Goldstein's New York Times article [28], Alexandra Stevenson and Matthew

Goldstein's article in the NYT [61]). It results from the example that the informed investor has a risk neutral measure and hence did not have free lunches, but from Chapter 4, the informed investor probably took in less risk than a market/liquidity trader. We also study a stochastic volatility model with additional information. In the example, the insider has information that affects the volatility of the discounted price process. This type of extra information fits into our setting since it is not related to the path property of the traded discounted stock price. The type of inside information we consider in that example is in contrast to the type of extra knowledge that can be found in the literature. We prove that our results coincide with Amendinger's results [3] but still apply even when Amendinger's work is no longer applicable, for instance when the distribution of the additional knowledge has compact support.

An important assumption of Chapter 3 is the continuity of all $\mathbb{F}-$ local martingales. A possible extension of the current work is to consider discounted price processes which have jumps. We do not believe the consideration of such processes will be a big departure from our work since the presence of jumps will just require extra conditions to obtain Local NFLVR, then NAS, NA1 and NFLVR. It might nevertheless still be of interest to study that case and hence obtain a complete picture of the conditions under which an insider's market does not have arbitrage or free lunches.

In Chapter 3, we study how the inside information affects the risk of an insider and compare it to a market/liquidity trader's risk. Although in Chapter 3 it was not important to have markets be complete or incomplete, in Chapter 4 we specifically work in incomplete markets. Assuming the insider has a risk neutral measure, we call "risk" the remainder in the Kunita-Watanabe decomposition of any financial claim that cannot be perfectly hedge by both traders. We show that under certain conditions the risk undertaken by the insider is smaller than the market/liquidity trader's risk. Consequently, in some situations, the insider's intrinsic risk might be larger than the market trader's intrinsic risk; that raises the question of why do some investors engage in illegal insider trading? A possible answer to that question is even though the informed investor's intrinsic risk might be equal to or higher than the market/liquidity trader's intrinsic risk, the informed trader knows more accurately whether the current market price is a good or a bad deal.

We find that in the enlarged filtration and if using the risk neutral measure of the market, the insider's trading strategy should involve both the stock and the volatility of the stock. Kyle [43], Back [6] and Collin-Dufresne [17] using different models and assumptions, all observed that the optimal trading strategy for the insider should involve the volatility. Additionally, we also find the relationships among the different trading strategies; first of the market trader and the insider, second of the insider in his two enlarged markets; one governed by the risk neutral measure of the market, the other by the insider's own risk neutral measure.

In the second part of Chapter 4, the insider's enlarged market does not satisfy NFLVR but we assume it satisfies Local NFLVR. We answer the question of how an insider can price his or her financial claims under Local NFLVR. Moreover, we introduce a new type of process called a quasi-local martingale and prove that the discounted stock price is a quasi-local martingale if the insider's market satisfies Local NFLVR. Under some conditions, we extend quasi-local martingales to the whole trading horizon and we are therefore able to obtain a Kunita-Watanabe type decomposition. This new decomposition should not be understood in the sense of the usual Kunita-Watanabe decomposition but instead as one process equal to another. Finally, we define what the optimal price of a financial claim is for an insider and find the optimal price at which the insider should buy or sell his or her financial claims.

As in Chapter 3, an important assumption of Chapter 4 is the continuity of all $\mathbb{F}-$ local martingales. The continuity assumption was not critical in Chapter 3, but it is important in Chapter 4 as it simplifies the computation of quadratic covariation between processes that are adapted to different filtrations, facilates the definition of orthogonality between local martingales as well as the computation of compensators of quadratic variation processes. Consequently, a future research area could be the extension of our work to the case when the discounted stock price process has jumps.

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