On Specifying The Parameters Of A Development Plan

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Abstract This paper reviews the recent literature on insensitivity and continuity of finite horizon optimal plans, in both aggregative as well as multi-sectoral models. Results on other comparative dynamics questions are also, briefly, discussed.

*This paper has been prepared for a volume in the memory of Sukhamoy Chakravarty, one of the true pioneers in development planning. I have benefitted from helpful conversations with Mukul Majumdar, Tapan Mitra and Itzhak Zilcha as well as the detailed comments of Tapan Mitra on an earlier draft on this paper.
1. Introduction

Much of Sukhamoy Chakravarty's work in the theory of development planning was directly inspired by his knowledge of the "real" problems that any actual planning exercise encounters and his deep desire to have theory and application mutually inform each other. In Chakravarty (1987) he writes of the general philosophy; "Theoretical understanding at a given point in time, based on the perception of objectives and constraints, led to the formulation of concrete action schemes or plan directives. In turn, these action schemes, with some delay, led to the emergence of conjunctures not always anticipated, which in turn led planners and policy-makers to rethink their objectives and strategies."

An important choice variable for any planning exercise is the length of the plan horizon (and the associated choice of a final period capital stock). There are well-known theoretical arguments that suggest that the relevant horizon, especially for national planning, should be infinite (Pigou (1920), Rawls (1971, pp 271 – 275))\(^1\). Chakravarty was clearly convinced of the theoretical content of this argument (see, for example, the discussion in his (1969) book and especially pages 19–21). Yet he also believed that, given the lack of information about technologies and preferences in the distant future as well as political considerations, "...for applied work on intertemporal planning, a finite horizon model with terminal capital-stock provision strikes one as the most acceptable (approach)...." (Chakravarty (1967, p160)). However, any particular choice of horizon is arbitrary (and indeed there is no logical manner in which to select it optimally without reverting to the infinite horizon problem). Hence, the question: (when) are optimal choices and maximized values robust to the actual specification of plan horizon (and terminal stock)? This was the focus of Chakravarty (1962b, 1966, 1969). Armed with the benefit of twenty five years of hindsight, that is also the focus of this paper.

There are really two related questions of interest: are finite horizon optimal
investment plans, and the associated level of maximized utility, close to each other for different specifications of horizon length? This I will call the insensitivity question. On the other hand, the continuity question is: is each finite horizon optimum plan close to some infinite horizon optimum? In turn, the insensitivity and continuity questions are examples of a broader set of questions in comparative dynamics— the changes in solutions to dynamic problems on account of changes in underlying parameters of the problem, like preferences or production relationships. In his (1969) survey, Chakravarty had this to say on the importance of these questions: "..the outstanding question would seem to be the question of sensitivity, especially if we are interested in making practical policy recommendations. Sensitivity...should also cover questions relating to changes in parametric representations of the utility or production relationships....Hence, we must be careful about distinguishing between relatively invariant properties of optimal consumption paths and merely accidental features" (op.cit. p252).

Chakravarty’s own work focussed only on the insensitivity question and, although he was appreciative of the general analytical problem, his discussion was limited to numerical simulations on some simple computable stationary aggregative examples (indeed this was also true of contemporaneous studies as those of Maneschi (1966) and Sen (1961)). The literature that I primarily review in this paper has since addressed both insensitivity and continuity questions, has done so analytically and (in some cases) in non—stationary and multi—sectoral models. Not surprisingly, some of the early results and intuitions have had to be modified, or even completely abandoned, but in some other cases Chakravarty’s preliminary results and intuitions have been richly rewarded. I will also discuss, more briefly, some recent results in comparative dynamics.

The plan of the paper is as follows. In Section 2, I develop the general model. Section 3 will review the results from the aggregative, or one—sector, model while Section 4 will do likewise for the multi—sectoral model. In Section 5, I collect together some recent
results on comparative dynamics. Throughout, technical details will be kept to a minimum and the emphasis will be on the intuition underlying the various results; in particular, no proofs of general theorems will be offered and the reader is invited to directly consult the relevant references for such proofs.

2. The Intertemporal Allocation Model

Let $\mathbb{R}^n$ be $n$-dimensional real space with $\| \cdot \|$ denoting the max norm on this space ($\mathbb{R}^n_+$ will denote the non-negative orthant). A correspondence, or set-valued mapping, $\Gamma$, from $X \subset \mathbb{R}^n$ to $Y \subset \mathbb{R}^n$ is said to be upper semicontinuous (usc) at $x \in X$, if $\Gamma(x) \neq \emptyset$ and for each sequence $x_n \to x$ and an associated sequence $y_n$, where $y_n \in \Gamma(x_n)$, we have a convergent subsequence whose limit $y \in \Gamma(x)$. $\Gamma$ is usc on $X$ if it is usc at each $x \in X$. Similarly, $\Gamma$ is said to be lower semicontinuous (lsc) at $x$, if $\Gamma(x) \neq \emptyset$ and for each sequence $x_n \to x$ and $y \in \Gamma(x)$, there is a sequence $y_n \to y$ with $y_n \in \Gamma(x_n)$. Again, $\Gamma$ is lsc on $X$ if it is lsc at all $x \in X$. If $\Gamma$ is both usc and lsc on $X$, it will be said to be continuous on $X$.

Production relations in the intertemporal model are specified by (time-indexed) non-empty production correspondences, $(F_t)_{t \geq 0}: F_t: \mathbb{R}^n_+ \rightarrow \mathbb{R}^n_+$. $F_t(x)$ is the set of feasible outputs in period $t + 1$ that is consistent with an input $x$ in period $t$. Note that in a multi-sectoral model, the feasible output possibilities are better described by a correspondence, rather than a (single-valued) production function, since different combinations of the many commodities may be producible from the same input. Also note that it is not required that all commodities be essential for production. Finally, we take the production correspondences to be time-dependent so that (certain kinds of) technological progress can be accommodated.

A special case of the above framework is the aggregative or one-sector model. Traditionally, production relations in that model are described by (time-dependent)
production functions \( f_t(x) \) are the maximum output of the single commodity available in period \( t + 1 \), given an investment of \( x \) in period \( t \). Given a free disposal assumption (that I will make in the immediate sequel), a production correspondence can be derived in the aggregative case by writing \( F_t(x) = \{ 0 \leq y \leq f_t(x) \} \). Incidentally, (inelastically supplied) labor can be straightforwardly incorporated in both the aggregative as well as multi-sectoral models; indeed, since the production relations (as well as the utility functions that follow) are time-dependent, some patterns of growth in population are admissible as well.

The following assumptions on the production correspondences are standard. Each of the results that follow will employ some subset of these assumptions. Additional, less standard, assumptions will be introduced and discussed when needed.

(F0) (Null Production) No output is producible from zero inputs; \( F_t(0) = \{0\} \)

(F1) (Continuity) \( F_t \) is a continuous correspondence for all \( t \)

(F2) (Free Disposal) \( y \in F_t(x) \Rightarrow y' \in F_t(x') \) if \( x' > x, 0 \leq y' \leq y \)

(F3) (Convexity) The production possibility set \( \{(x,y) \in \mathbb{R}^n_+: y \in \Gamma(x)\} \), is convex for all \( t \).

(F4) (Boundedness) \( \exists \beta > 0 \) such that \( \| x \| > \beta \Rightarrow \| y \| \leq \| x \| \), for all \( y \in F_t(x) \) and for all \( t \)

The last two assumptions warrant brief comments. The convexity assumption rules out increasing returns to scale everywhere. Since increasing returns to scale is of central concern in a growth context, a number of authors have explored the basic questions without this assumption; consequently, several of the results that follow will not invoke (F3). The boundedness assumption asserts that strictly positive growth is, eventually, impossible; that the "marginal product" of capital is less than or equal to one for large capital stocks. (F0)—(F4) imply that in the aggregative model, the production functions \( f_t \)
satisfy $f_t(0) = 0$ and are increasing, continuous, concave and have a maximum sustainable stock.

A **finite horizon intertemporal allocation problem**, or planning problem, is characterized by a parameter triple $\xi = (x, a, T)$ where $x \in \mathbb{R}_+^n$ is the initial capital stock, $a \in \mathbb{R}_+^n$ is the target stock and $T$ is the plan horizon. Much of the analysis that follows will involve alternative specifications of the horizon $T$ for fixed $(x, a)$. Writing $c_t$ for the consumption in period $t$, we can define a $\xi$–feasible plan or program as $(x_t, c_t)_{t=0}^T$ satisfying

\begin{align}
    x_0 + c_0 &\in F_0(x) \quad (2.1) \\
    x_t + c_t &\in F_t(x_{t-1}), \ t = 1, \ldots, T \quad (2.2) \\
    x_T &\geq a \quad (2.3) \\
    x_t \geq 0, c_t \geq 0 &\quad t = 0, \ldots, T \quad (2.4)
\end{align}

An **infinite horizon feasible plan** is $x, c \equiv (x_t, c_t)_{t=0}^\infty$ such that (2.1), (2.2) and (2.4) are satisfied for all $t \geq 0$. Note that I shall refer to the investment (resp. consumption) of the $i$–th commodity in period $t$ as $x_t^i$ (resp. $c_t^i$).

The **preference structure** is defined by a sequence of time–dependent utility functions $(u_t)_{t \in \mathbb{N}}$, where $u_t : \mathbb{R}_+^n \to \mathbb{R}$. There are, of course, well known problems with defining a social welfare function that aggregates individual preferences in a "consistent" fashion (I refer here to the Arrow impossibility theorem and related results). Since, social choice issues are peripheral to the immediate concerns of development planning, I shall follow other writers in assuming that either a social welfare function can be defined or else that $u_t$ is some convex combination of individual utility functions. Utility is defined on consumption alone. The following **assumptions** are made on the utility functions:
(U1) **(Continuity)** \( u_t \) is a continuous function, for all \( t \).

(U2) **(Monotonicity)** \( c' \geq c \Rightarrow u_t(c') \geq u_t(c) \), for all \( t \).

(U3) **(Concavity)** \( u_t \) is a strictly concave function, for all \( t \).

The **finite horizon optimization problem** is to choose a \( \xi \)-feasible plan \((x_t^T(a),c_t^T(a))_{t=0}^T\) such that

\[
\sum_{t=0}^T u_t(c_t^T(a)) \geq \sum_{t=0}^T u_t(c_t)
\]

for all \( \xi \)-feasible \((x_t,c_t)_{t=0}^T\). Note that this optimization problem is trivially equivalent to maximizing the **average utility**, i.e. \( \max \frac{1}{T+1} \sum_{t=0}^T u_t(c_t) \) over \( \xi \)-feasible plans. Denote the maximized utilities or value functions \( V_T(x) \) (and \( v_T(x) \) for the average value).

There are several different ways in which **infinite horizon preferences** can be specified and each way of doing so is a response to the problem of defining an order on infinite utility streams. (A discussion of this problem is incidental to the objectives of this paper, but it is possibly worth pointing out that the second major capital theoretic contribution of Chakravarty's was precisely related to this question; see Chakravarty (1962a)). In this paper I confine attention to, and employ, two of the more popular alternatives.

**Infinite—Sum Utility Functions** The obvious extension of finite horizon preferences is to define infinite horizon utility as

\[
U(x,c) = \sum_{t=0}^\infty u_t(c_t), \quad (x_t,c_t)_{t\geq0}
\]

The problem with (2.6) is of course that the infinite sum may not be well—defined, or finite, for all feasible programs. If it is, then optimality is defined in the usual manner and we shall denote the associated value function, \( V(x) \). The best—known example of
well-defined preferences under this criterion, is that of discounted utilities; \( u_t = \delta^t u \), where \( \delta \in [0,1) \).

**Catching-Up Preferences** Alternatively, one can define a binary order. We say that \((x^*, c^*)\) catches-up to another feasible plan \((x, c)\) if:

\[
\lim_{T \to \infty} \sum_{T=0}^{T} [u_t(c_t^*) - u_t(c_t)] \leq 0
\]

An optimal program is one that catches up to all other programs. Clearly, optimality under the first criterion implies optimality under the catching-up.

In the next two sections, I discuss the aggregative and multi-sectoral models respectively. It will be useful to distinguish between the two since the results, and the underlying intuition, will turn out to be quite different in the two cases.

3. **Sensitivity and Continuity in the Aggregative Model**

The one-sector planning model was the exclusive focus of the early literature on sensitivity analysis. For instance, Chakravarty (1962b) analyzed the following model: the utility function is time-independent and linearly homogenous, i.e. \( u_t(c) = u(c) = (1-v)^{-1} c^{1-v}, v \in [0,1) \) whereas the (time-independent) production function is linear, i.e. \( f_t(x) = f(x) = bx \) where \( b > 1 \). This specification of utility and production has the convenient feature that the optimal solutions can be explicitly computed for different specifications of the plan horizon and terminal capital stock (as Chakravaty indeed did do, in a continuous time framework). For the parameter values that he examined, he showed that consumption in early periods of the plan was more sensitive to the specification of horizon length and less sensitive to the specification of terminal capital stock. He surmised that these results would hold qualitatively for more general models. That some caution was
called for in arriving at such a conclusion was suggested by the computations of Manneschi (1966) who in the same model showed that the result on insensitivity to terminal capital specifications was overturned for parameter specifications other than those investigated by Chakravarty.

The inconclusiveness of the debate was largely explained by the fact that the analyses were based on numerical solutions. Furthermore, this early work strongly hinted at the need for a general analytical examination of the problem. Brock (1971) was the first to do so; he examined both the investment sensitivity and continuity issues in a general convex aggregative model.³

Brock proved two main results relevant to the insensitivity question. The first established a strong investment monotonicity property for finite horizon optimal plans when the terminal stock requirement is zero. To be precise, the result showed that if the length of the plan horizon is increased, say from $T$ to $T'$, but the size of initial capital stock remains unchanged, then the optimal plan for the $T'$ horizon maintains a higher investment level than the $T$ horizon optimal plan in every period between 0 and $T$. An immediate implication of this property is that consumption is initially lower under the $T'$—optimal plan, although it may be eventually higher. This monotonicity property is the critical intermediate result that implies insensitivity of optimal plans. Brock used it to show that if the horizons are appropriately long, then the investment (and consequently consumption) choices in the early periods would be quite similar, i.e. that optimal plans are (initially) insensitive to horizon specification; for example, the choices in the first three periods are approximately invariant over horizon $T$ or $T'$, provided both are "long enough". This last result is of great practical usefulness since a planner may not be sure at the outset of planning whether the "correct" horizon is $T$ or $T'$.

I present here Brock's results under hypotheses somewhat weaker than those
employed in his original discussion. These results are due to Mitra (1983). Recall that 
\((x_t^T(a), c_t^T(a))_{t=0}^T\) is the notation for an optimal plan from an initial stock \(x\) (which remains fixed throughout and therefore is suppressed in the notation), to a (feasible) terminal stock \(a\).

**Theorem 1** (Brock (1971), Mitra (1983)) Under (F0) - (F3) and (U1) - (U3), there is a unique optimal plan \((x_t^T(a), c_t^T(a))_{t=0}^T\) for every feasible \(\xi = (x, a, T)\). Moreover, these optimal choices satisfy

i) **Horizon monotonicity** \(x_t^{T+1}(0) \geq x_t^T(0)\), for all \(t = 0, \ldots, T\).

ii) **Horizon Insensitivity** For every \(t\), there is \(x_t = \lim_{T \to \infty} x_t^T(0)\). Consequently, for all \(\epsilon > 0\) and \(N\), there is \(T > N\) such that whenever \(T \geq T\) and \(T' \geq T\),

\[\|x_t^T(0) - x_t^{T'}(0)\| < \epsilon, \|c_t^T(0) - c_t^{T'}(0)\| < \epsilon, t = 0, \ldots, N,\] (3.1)

iii) **Terminal Stock and Horizon Insensitivity** There is a terminal stock \(\bar{a} > 0\), such that for every \(t\), \(x_t = \lim_{T \to \infty} x_t^T(a)\), whenever \(a \leq \bar{a}\).

Two additional comments are worth making. It is of some practical interest to ask how long the horizons need to be so that the investment and consumption choices are insensitive in the first three periods; equivalently, how long is \(T\) for any given \(N\)? For instance, if the choices in the first three periods are insensitive only when the horizons are at least three million periods long, such insensitivity would be of very limited practical significance. Unfortunately, such results on the "rate of convergence" are not currently available and seem likely to be very model specific. Secondly, the result in Theorem iii) shows that for a subset of terminal stocks, insensitivity can be established jointly in horizon and final stock. Clearly, the result states that for any terminal stock less than \(\bar{a}\), (3.1) can be established for an appropriate \(T\).

Mathematically inclined readers will no doubt notice that ii) follows quite directly
from the monotonicity result i). Indeed, the assertion that investment levels are higher period by period if the horizon is longer, is a very strong assertion. It will be seen shortly that this is the critical property of the aggregative model which is fragile in that it is untrue in the multi-sectoral model under otherwise identical hypotheses. The horizon monotonicity result follows from yet another monotonicity result which says that for identical horizon length, an optimal plan to a higher capital stock maintains uniformly higher investment levels, i.e. if $a' \geq a$ then $x^T_t(a') \geq x^T_t(a)$ for $t = 0,...,T$. The intuition for this monotonicity result is the following: the convexity of the model implies that marginal valuations are increasing in terminal stock, i.e. for any $x'$, $x$ with, say, $x' > x$, the difference in continuation values, $V^T(x';a) - V^T(x;a)$, is increasing in $a$. Since optimal investment choices balance the marginal utility of immediate consumption (which is independent of terminal stock) against marginal valuations, it follows that investment levels are higher if terminal stocks are higher.

What of continuity, i.e. can one assert that the finite horizon optima are themselves close to any infinite horizon optimum? That this need not be so can be demonstrated quite easily by way of a well-known "cake-eating" example (which was first employed by Gale (1967) in a different context).

**Example 1** The production and utility functions satisfy (F0) — (F3) and (U1) — (U3). However, no finite horizon optimal plan is close to an infinite horizon optimal plan.

Details: Suppose that $f_t(x) = f(x) = x$ and $u_t(c) = u(c)$ is any function satisfying (U1)—(U3) and further that the terminal stock $a = 0$.

It can be shown that in this example, the optimal $T$-horizon consumption policy is to eat $1/T$ of the "cake" every period. (This relies on the fact that with a strictly concave utility a decision-maker strictly prefers to spread consumption over time). Clearly then, the finite horizon optimal plans involve smaller and smaller amounts of consumption each
period as $T$ increases, and in the limit involves zero consumption every period; evidently, an inoptimal plan under any specification of infinite horizon preferences.

However, this example does not settle the continuity issue since there is no infinite horizon optimal plan in this example under, e.g., the catching-up criterion. (The intuition behind this statement is the same as that driving the claim above). So the next question is: suppose that the infinite horizon problem does have a solution. Does continuity obtain in that instance? Brock (1971) (and in his generalization, Mitra (1983)) provided the following positive answer to the continuity question:

**Theorem 2** Suppose that (F0) — (F3) and (U1) — (U3) hold and suppose further that there is a catching-up optimal plan. Then, the plan $(x, c)$ defined as the limit of the finite horizon optima is precisely this infinite horizon optimum.

An equivalent statement of this result is that if an infinite horizon catching-up optimal plan exists, then every finite horizon optimal is close to this unique optimum. Again, the proofs of Brock and Mitra, which are different, both exploit critically the terminal stock monotonicity result that I have discussed above.

Note that Example 1 has also demonstrated that the insensitivity and continuity questions are distinct and a positive answer to the former does not imply a likewise positive answer to the latter. Shortly, we shall see that continuity does not imply insensitivity either.

Evidently, non-convexities caused on the production side by, for example, increasing returns to scale and on the consumption side by externalities are particularly important in a development context. The question I now turn to is: to what extent are the conclusions of Theorems 1 and 2 valid without the convexity assumptions, (F3) and (U3). Majumdar and Nermuth (1983), Mitra and Ray (1984) and Amir, Mirman and Perkins
(1991) have explored this issue; in each case, the authors relaxed production convexity while retaining consumption convexity. I report here a version of Theorem 1 proved by Mitra and Ray (1984).

Note that without convexity, albeit only on the production side, there is no longer uniqueness of optimal choice. A (weak) form of insensitivity is then: for every $T$—horizon optimal plan, is there a $T'$—horizon optimal plan close to it? Mitra and Ray prove just such a result after proving a (weak) monotonicity version of Theorem 3.1i). The intuition for the weak monotonicity is identical to that for the stronger version in the fully convex case of Theorem 3.1i), i.e. that the marginal valuation of capital is increasing in the size of the target stock (and therefore the length of the horizon).

Theorem 3 (Mitra and Ray (1984)) Suppose (F1) — (F2) and (U1) — (U3). Let $\xi \equiv (x, a, T)$ and $\xi' \equiv (x, a, T+1)$. Then,

i) **Weak Horizon Monotonicity** For every $\xi$—optimal plan $(x^T_0, c^T_0)_{t=0}^T$, there is a $\xi'$—optimal plan $(x^{T+1}_0, c^{T+1}_0)_{t=0}^{T+1}$ such that $x^T_0 \leq x^{T+1}_0$, $t = 0, \ldots, T$.

ii) **Weak Horizon Insensitivity** There is a feasible infinite horizon plan $(x, c)$, and a sequence of optimal finite horizon plans, $(x^T_0, c^T_0)_{t=0}^T$, $T \geq 0$, such that for every $t$, $x^T_t = \lim_{T \to \infty} x^T_t(0)$. Consequently, for all $\epsilon > 0$ and $N$, there is $T > N$ such that whenever $T \geq T$ and $T' \geq T$,

$$
\|x^T_0 - x^{T'}_0\| < \epsilon, \|c^T_0 - c^{T'}_0\| < \epsilon, \quad t = 0, \ldots, N \tag{3.2}
$$

iii) **Terminal Stock and Horizon Insensitivity** There is a terminal stock $\bar{a} \geq 0$, such that for every $t$, $x^T_t = \lim_{T \to \infty} x^T_t(a)$, whenever $a \leq \bar{a}$.

Majumdar and Nermuth impose the stronger assumption of differentiability on the production and utility functions. Correspondingly, they establish a stronger result; they prove Theorem 3i) for all finite horizon optimal plans, i.e. they prove that any $\xi$—optimal
plan has higher investment levels than any $\xi$-optimal plan. Consequently, they are able, like Brock (1971), to find a unique limiting behavior for finite horizon optimal plans, as the horizons become longer.

Note that the continuity question, i.e. whether or not an analog of Theorem 2 holds in the non-convex case, is still open. (It is, however, easy to see that the method of proof employed by Mitra (1983) on the continuity question implies the following result in the Mitra-Ray non-convex model: if there is a unique catching-up optimal plan, say $(x, z)$, then every convergent sequence of finite horizon optimal plans has $(x, z)$ as limit). Also it is not known which of these results would generalize to the fully non-convex case, i.e. when both production as well as utility functions can be non-concave. My conjecture would be that the monotonicity results, and hence the insensitivity results would not be robust to this generalization.

A brief recapitulation of the results for the aggregative model are in order. Chakravarty (1962a) had noted that investment and consumption levels in the early years of a plan were seemingly insensitive to the terminal capital requirement, although Manneschi then showed that sensitivity was reestablished for other terminal stock specifications. Chakravarty also conjectured that optimal choices appear to be more sensitive to the length of the horizon. Subsequent analytical investigations have identified the set of the terminal stocks on which insensitivity can be asserted and further shown that horizon insensitivity is more generally true, provided the horizons are appropriately long.

4. **Sensitivity and Continuity in the Multi-Sectoral Model**

The one-good model of intertemporal allocation is at once a convenient simplification and also a significant restriction. Its simplicity allows us to test intuitions and explicitly solve some examples. However, from a practical planning viewpoint, the
restriction to a single commodity is clearly unacceptable. The central issue is which of the conclusions of the aggregative model are robust to a multi-sectoral generalization. In this section I summarize recent results on multi-sectoral sensitivity and continuity. It is worth pointing out that the literature here is much smaller than that for the aggregative model; the papers I will refer to are Gale (1967), Radner (1967), Nermuth (1978), Amir (1991) and Dutta (1991).

From the perspective of sensitivity and continuity analysis, the multi-sectoral model turns out to be very different from the aggregative one. The principal reason for this is that the monotonicity results (Theorems 1i) and 3i)) are invalid in such a model. (Amir (1991) shows that monotonicity results can be established for the multi-sector model as well but under much stronger conditions). Since the insensitivity and continuity properties of the aggregative model were intimately predicated on the monotonicity results, they fail to generalize as well. We present an example to demonstrate this point.

Example 2  \( F_t \) and \( u_t \) satisfy (F0) – (F3) and (U1) – (U3) but period 0 investment is very sensitive to horizon length. In particular, \( \|x_0^{T+1} - x_0^T\| = 1 \), for all \( T \geq 1 \).

Details: Consider a two-sector model and denote investment (resp. consumption) in period \( t \) of the two commodities as \( x_t^1, x_t^2 \) (resp. \( c_t^1, c_t^2 \)). Suppose that:

\[
F_t(x_{t-1}^1, x_{t-1}^2) = \{(y_1^i, y_2^i) \in \mathbb{R}_+^2 : y_i^i \leq x_i^{t-1}, i=1,2\}, \quad t \geq 1
\]

\[
F_0(x^1, x^2) = \{(y_1^1, y_2^2) \in \mathbb{R}_+^2 : y_1 + y_2 \leq x_1 + x_2\}
\]

Let \( (m_t)_{t \geq 0} \) be a strictly increasing sequence, \( m_t > 0 \), for all \( t \). The preferences are defined as:

\[
u_t(c_1^i, c_2^i) = m_t c_i^i, \quad \text{if } t \text{ odd; } i=2, \text{ if } t \text{ even}
\]

\[
u_0(c_1^1, c_2^2) = 0
\]
Finally, let the initial stock \( x = (1/2, 1/2) \) and the terminal stock \( a = (0,0) \). It is easy to see from (4.1) and (4.2) that \((F0) - (F3)\) and \((U1) - (U3)\) are satisfied.

Claim: for \( T \) odd, the optimal period 0 investment is given by \( x_0^1 = 1, \, x_0^2 = 0 \) whereas for \( T \) even, \( x_0^1 = 0, \, x_0^2 = 1 \).

It is easy to see that, given (4.1) — (4.2), once \((x_0^1, x_0^2)\) has been determined, in any optimal policy the only consumption that takes place is at the terminal and penultimate dates; for instance, when \( T \) is odd \( c_T^1 = x_0^1, \, c_{T-1}^2 = x_0^2 \), and all other consumption is zero. But a unit of consumption yields greater utility in period \( T \) than in \( T-1 \). Hence, given the substitution possibilities in period 0, the claim follows.

A major role of convexity in the aggregative model (in production and especially in consumption) was to generate investment monotonicity. Since such monotonicity will not obtain, and should not be expected, in the multi-sector model even under convexity, the necessity for such assumptions is moot. In all of the arguments that follow in this section I will therefore drop the requirement of convexity and by so doing bring the theory arguably closer to the increasing returns and externality issues that are critical to the development context.

Note further that Example 2 need not be a cause for despair as far as multi-sectoral insensitivity is concerned. Asking for investment insensitivity, in the presence of the substitution possibilities that are opened up by a multi-sectoral specification is asking for too much in any case. Besides, from a planner's point of view, the relevant question would appear to be: is the level of \emph{maximized utility insensitive} to the specification of plan horizon? A partial intuition for a positive answer to this question is that a substantial wedge between the values for \( T \) and \( T+1 \) period plans would imply that we would be strictly better off by choosing one of the two plans in both cases.
Of course, for this question to be meaningful we have to normalize the sum of utilities appropriately, for different values of T. The most obvious normalization is to take averages. Recall that \( v_T(x) \) is the average utility from initial state \( x \). The value \textit{insensitivity} question I now analyze is: under what conditions are \( v_T(x) \) and \( v_{T'}(x) \) close, for long but distinct plan horizons \( T \) and \( T' \)?

I present now a positive result on value insensitivity. For this result I need three new assumptions and one additional piece of notation. For \( x, x' \in \mathbb{R}^n \), we say that \( x' > x \) if \( x' \geq x \) and \( x' \neq x \). Recall the productivity bound \( \beta \) which has been defined by (F4).

(F5) (Uniform Productivity) For all \( x \) such that \( 0 < x < (\beta, \ldots, \beta) \), there is \( y(x) \in F_t(x) \) satisfying \( x < y(x) \), for all \( t \).

(F6) (Limiting Technology) On the compact set \( \{ x \in \mathbb{R}^n : 0 \leq x \leq (\beta, \ldots, \beta) \} \), as \( t \to \infty \), the production correspondences \( F_t \) converge uniformly to a production correspondence \( F^* \).

(U4) (Limiting Preferences) On the compact set \( \{ c \in \mathbb{R}^n : 0 \leq c \leq (\beta, \ldots, \beta) \} \), the utility functions \( u_t \) converge uniformly to a function \( u^* \), as \( t \to \infty \).

(F5) is a standard assumption in intertemporal allocation models. The limiting assumptions, (F6) and (U4) are less standard (but trivially satisfied if the model is time-independent or discounted-stationary). If the average values are to satisfy some limiting behavior (as I will report that they do), it must be the case that the environments of planning (the production and utility relationships) also satisfy some limiting behavior. Indeed, in Example 3 below, I show that without (F5) and (U4) the value insensitivity result fails.

Consider any infinite horizon feasible plan \((x, c)\) and define its \textit{long-run average utility} as:
The long–run average value, for initial state \( x \), is then defined as \( v(x) = \sup u(x, c) \), where \((x, c)\) feasible from \( x \).

**Theorem 4** (Dutta (1991)) Suppose that (F0) – (F2) and (F4) – (F6) hold on the production side and (U1), (U2) and (U4) hold on the consumption side. Then, there is \( v^* \) such that

i) \( v^* = \lim_{T \to \infty} v_T(x), \quad 0 < x \leq (\beta, \ldots, \beta) \) \tag{4.4}

ii) \( v^* \) is the long–run average value, for all \( 0 < x \leq (\beta, \ldots, \beta) \). In particular, average values are insensitive to the length of the plan horizon (and initial non–zero stock) provided the horizon is sufficiently long; \( \forall \epsilon > 0 \) and \( 0 < x, x' \leq (\beta, \ldots, \beta) \), there is \( \hat{T} < \infty \) such that \( |v_T(x) - v_{T'}(x')| < \epsilon \), whenever \( \min (T, T') > \hat{T} \).

Theorem 4 implies that even if the investment and consumption choices from two different horizons, \( T \) and \( T' \), are very different, the associated values per period are very similar. This has the following useful implication for planning: suppose the planner is unsure of the exact horizon length but learns about this as time passes and consequently adjusts his investment levels appropriately. Such an "adaptive planning" framework yields average utilities that are approximately the same as those that would have been generated had the planner known the correct horizon with certainty at period zero.

I now present a brief example to show that the limiting assumptions (F6) and (U4) were necessary for Theorem 4.

**Example 3** Technology and preferences satisfy (F0) – (F5), (U1) – (U3) but average values are sensitive to the horizon.
Details: \[ f_t = f(x) = \begin{cases} 2x & x \leq 1/2 \\ 1 & x > 1/2 \end{cases} \] (4.5)

Let \( u_t \) be any equicontinuous sequence of functions that are individually continuous \((C^0 \text{ even})\), strictly increasing and strictly concave and which satisfy the following property

\[
\lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} u_t\left(\frac{1}{2}\right) > \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} u_t\left(\frac{1}{2}\right) \tag{4.6}
\]

Now define

\[
u_t(c) = \begin{cases} u_t(c) & c \leq 1/2 \\ u_t(1/2) & c > 1/2 \end{cases} \tag{4.7}
\]

It is clear that with initial state \( x = 1 \) (and hence \( f(x) = 1 \)), the optimal \( T \)-period plan is \( x_t = c_t = 1/2, \ t = 0, \ldots T \). But then (4.6) implies that average values are sensitive to the horizon. 

I turn now to the continuity question — is every finite horizon optimal plan (for some admissible set of terminal stocks) "close" to some infinite horizon optimal plan, for long but finite horizons? Similarly, is the value, or maximized utilities, continuous at horizon length infinity? From Example 1 we know that some conditions, in addition to the basic assumptions \((F0) - (F2)\) and \((U1) - (U2)\), will need to be placed in order to obtain affirmative answers to these questions. In the aggregative convex model, recall that it suffices to know that there exists an optimal plan in the infinite horizon problem. I now show that in the multi-sectoral model, one needs a somewhat stronger condition; one needs a condition which guarantees that an optimum exists in the infinite horizon problem.

I present here only one positive result and the reader can consult Dutta (1991) for other results. The common intuition for these results is the following: think of the finite and infinite horizon planning problems as special cases of the same family of problems —
differentiated only by the fact that a relevant parameter, the plan horizon, varies. As the parameter varies in a continuous fashion, the horizon goes from finite to infinite, under some restrictions on technology and preferences, the associated optimal choices should vary continuously as well.

**Theorem 5** (Dutta (1991)) Suppose that (F1) - (F2) and (U1) - (U2) hold and further that on the set of feasible infinite horizon plans from initial state $x$, $\sum_{t=0}^{\infty} u_t(c_t)$ is finite and upper semi-continuous (with respect to the product topology). Then,

i) **(Value Continuity)** As $T \to \infty$, $V_T(x) \to V(x)$ the infinite horizon value function.

ii) **(Investment-Consumption Continuity)** If the horizon is appropriately long, each finite horizon optimal plan has an infinite horizon optimal plan close to it, i.e. for all $\epsilon > 0$ and $T < \omega$, there is $T'$ such that whenever $T' \geq T$, for any $T'$-optimal program to target stock zero, $x_{T'}^T(0), c_{T'}^T(0)$ there is an infinite horizon optimal plan $(x^*, c^*)$ satisfying

$$ \| x_t(T') - x_t^* \| < \epsilon, \ t = 0, ..., T $$
$$ \| c_t(T') - c_t^* \| < \epsilon, \ t = 0, ..., T $$

**Remark** Nermuth (1978) sought to prove the same theorem under a considerably stronger condition on infinite horizon preferences but for a larger set of terminal stocks. Although the theorem is not true under the hypotheses he examined (see Dutta (1991), Example 4.3), he did however pioneer an analytical approach to the continuity problem which has proved very useful in general. It should also be noted that an early value-continuity result is to be found in Radner (1967). In that paper, Theorem 5i) was shown to hold in a multi-sectoral model with continuity-monotonicity assumptions much like the ones employed here and the additional assumption that the model is discounted stationary ($F_t \equiv F$ and $u_t \equiv \delta^t u$). (Some additional technical restrictions were also placed).
The discounting feature implies that the distant future is (relatively) unimportant and drives the conclusion of continuity in that paper.

Finally, I present an example to show that even if it is known that there is an infinite horizon optimum, under the catching-up criterion, it does not follow that finite horizon optima are close to it even if the finite horizons are "long". (Contrast this with the Brock result for the aggregative case reported in Theorem 2).

Example 4 There is a unique catching-up optimal plan \((x^*, c^*)\) and unique finite horizon optimal plans \((x^T(0), c^T(0))\). However, no finite horizon optimal plan is close to the infinite horizon optimum. In particular, \(\|c^T(0) - c^*_0\| = 1\), for all \(T\).

Details \(n = 2\). Let \(x = (1/2, 1/2)\)

\[
F_0(x) = \{y^1 + y^2 \leq x^1 + x^2\}
\]

\[
F_t(x) = \{(y^1, y^2) : y^1 \leq f(x^1), y^2 \in Q_t(x^2)\}
\] (4.8)

\[
f(x^1) = \begin{cases} 
2x^1 & x^1 \leq 1 \\
2 & x^1 > 2 
\end{cases}
\] (4.9)

\[
Q_t(x^2) = \begin{cases} 
0 & x^2 < Q_{t-1} \\
[0, Q_t] & x^2 \geq Q_{t-1} 
\end{cases}
\] (4.10)

and \(Q_{t-1}\) is an increasing sequence such that \(Q_t > t, Q_{-1} = 1\). Finally,

\[
u_t(c^1, c^2) = c^1 + c^2
\] (4.11)

Essentially the two commodities are perfectly substitutable in production in period zero and thereafter follow totally independent processes. Moreover, for commodity 2's production to get off the ground, the sum of the commodities has to be used in the second production process. So the choices are: a) only produce commodity 1 from period 1 onwards (and then the catching-up optimal policy is \(x_t = c_t = 1\), for all \(t\)) or b) switch to
commodity 2 and the discrete alternatives are $x = (1, Q_0, Q_1, \ldots Q_T, 0, 0\ldots)$ with an associated $c = (0, 0, \ldots 0, Q_T, 0)$. Since $Q_t > t$, the finite horizon optimum is $b_t$, for $T = T'$. But clearly the unique catching-up optimum is $c_t, c_t = 1, 0$, for all $t$.

To summarize, I have argued that some of the strong results regarding investment monotonicity do not carry over from the aggregative to the multi-sectoral models given the substitution possibilities inherent in the latter. However, value insensitivity and investment-consumption continuity can still be demonstrated under quite general conditions on production and preference. It is worth pointing out that that in the stationary, convex multi-sectoral model, an early result of Gale (1967) elegantly demonstrated that both insensitivity and continuity did hold. However, his approach relied very heavily on the underlying convexities.

5. Other Comparative Dynamics Topics

In this section, I will briefly discuss some other comparative dynamics questions that arise in the theory of intertemporal allocation. A planner is typically unsure about preferences and technologies in the future. This is particularly so, the further away are the relevant periods. On the technological side, the lack of information relates to: how much of technological progress will there be and how fast, what will be the likely menu of commodities available in the future, what are the prospects for future resource discoveries etc. On the preference side, a current planner can only approximate actual social preferences in the future. All of this is self-evident and so is the first theoretical query that is suggested by it: how robust are the qualitative features of optimal plans to different specifications of technology and preferences?

The literature on comparative dynamics with respect to production and utility is limited. Feldman and Mclennan (1990) and Dutta, Majumdar and Sundaram (1991) are
two recent papers which address aspects of this problem. Both of these papers are set in a framework much more general than the intertemporal allocation problem that I have discussed here. Within a very general dynamic programming problem, Dutta, Majumdar and Sundaram establish conditions under which optimal choices and value functions will vary continuously with the underlying parameters that index technology and preferences. Feldman and McLennan are interested in differentiable changes in optimal choices (under correspondingly stronger restrictions on the dependence of technology and preferences on unknown parameters).

One particular aspect of intertemporal preference, whose effect on optimal choices has been examined in great detail, is the discount factor. In our discussion above, if we take \( u_t = \delta^t u \), for some discount factor \( \delta \in [0,1) \) and the horizon \( T = \infty \), then we are in the standard infinite horizon—discounted model. A greater value of the discount factor places a bigger weight on the utilities of future consumption, and in this sense implies an increase in "patience". Several authors (Becker (1983, 1985), Dutta (1987), Amir, Mirman and Perkins (1991)) have explored the following question in the context of the aggregative model: does an increase in patience imply higher investment out of a given capital stock? I present a result from Amir, Mirman and Perkins. The result establishes capital deepening along the optimal discounted plans by exploiting the fact that the marginal continuation valuations are increasing in the rate of patience, for every fixed level of capital stock.

Theorem 6  (Amir, Mirman and Perkins (1991)) Suppose that (F0) — (F2), (F4) and (U1) — (U3) hold and suppose further that \( \delta' \geq \delta \). Then, for every \( \delta \)-optimal plan \( (x_t(\delta), c_t(\delta))_{t=0}^\infty \), there is a \( \delta' \)-optimal plan \( (x_t(\delta'), c_t(\delta'))_{t=0}^\infty \) such that \( x_t(\delta) \leq x_t(\delta') \), \( t \geq 0 \).

A different aspect of the comparative dynamics of discounting is the following question: under what circumstances can we treat the two cases of discounting and no discounting (\( \delta = 1 \)) as special instances of the same general problem? In particular, do the
optimal solutions under discounting converge to optimal solutions under no discounting? In general, the undiscounted case creates many problems for the definition of infinite horizon preferences and the establishment of the existence of optima. Dutta (1991a) has recently shown that under some general conditions like convexity and in the presence of uncertainty, in many economic models the discounted and undiscounted cases can indeed be analyzed as special case of the same unified problem.

6. Conclusions

This paper reported some recent results on a research question which Sukhamoy Chakravarty pioneered and considered to be a theoretical question of central importance for development planning. For both sensitivity and continuity questions, strong characterizations are now available in the one–sector model when both preferences and technologies are convex. A modified version of the insensitivity result also holds when preferences are convex (although technologies need not be so). In the multi–sectoral, non–convex model, a framework of particular interest in development planning, the results are necessarily less striking. However, even here value insensitivity and investment continuity hold under reasonably general conditions.

Many interesting questions remain to be explored. Clearly, uncertainty, particularly in production, is an important feature of any planning problem. Majumdar and Zilcha (1987) have investigated the sensitivity question in the aggregative model with production uncertainty; an inquiry in the multi–sectoral model remains to be done. Furthermore, all of the analyses deal with technological progress in virtually an exogenous manner; the production correspondence and utility functions have arbitrary time–dependence. However, not all forms of technological growth are feasible or even desirable. Moreover, such growth is itself determined by the rate and composition of capital accumulation and hence needs to be substantively endogenous to the model.
Footnotes

1Pigou and Rawls argued against a finite horizon for social planning problems because it reflects a bias against the consumption of future generations. Indeed, Pigou dismissed the time preference exhibited by individuals as "habitual myopia" motivated perhaps by the finiteness of lifetimes. Since, the lifetime of a nation has no logical terminal date and since the interests of current and future generations should be a priori identical from a social welfare viewpoint, the Pigou–Ramsey approach advocated an infinite horizon–zero discount rate framework for planning.

2An exception is Radner (1967). See the discussion in Section 4.

3Although Brock was the first to analytically discuss investment insensitivity, an earlier discussion of value insensitivity is contained in Radner (1967). Since Radner's framework accommodates the multi-sectoral model, his result is discussed in detail in Section 4.

4Actually's Mitra's result only requires that there exist a "weakly maximal" plan; any catching-up optimal plan is definitionally also weakly maximal. (For a definition of the weak maximality criterion, see Mitra (1983)).

5An alternative definition of value insensitivity was proposed by Mirrlees and explored in Hammond and Mirrlees (1973) and Hammond (1975) – see also Mckenzie (1974). For a fixed infinite horizon plan \((x,c)\) and an initial length of time \(N\), let \(V_T(x;N)\) denote the maximum utility that can be generated in a \(T\)-period planning problem, starting from initial stock \(x\), if in the first \(N\) periods investment (and consumption) has to be identical to that specified by \((x,c)\). The finite horizon optimum plans are said to be value-insensitive, agreeable in the terminology of these papers, if there is a plan \((x,c)\) and a sufficiently long horizon \(T\) such that \(V_T(x;N)\) and \(V_{T'}(x;N)\) are appropriately close to each other whenever \(T\) and \(T'\) are greater than \(T\) (and this is true for all \(N\)). This concept of value insensitivity is interesting; however, the literature was largely inconclusive in that the authors were able to derive results on questions like the existence of agreeable plans, their relation to optimality etc, only in very special cases. This literature is not detailed in this survey.
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