

GENERALIZED OMORI LAW FOR AFTERSHOCKS AND FORESHOCKS FROM A SIMPLE DYNAMICS

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**Abstract** A theory of the time dependence of earthquake foreshocks and aftershocks is presented. The theory involves the response to sudden forcing of a dynamics of self-driven acceleration to failure. The empirically observed Omori law, which says that the rate of aftershocks as a function of time falls as a power law in time, is derived theoretically. The exponent of the falloff in time is shown to generically give a value close to one, for rapidly accelerating dynamics. To see if the theory is consistent with other features of real data, foreshocks and aftershocks of small magnitude mainshocks are analyzed in a catalogue of real earthquakes. Results show that the spatial and temporal distribution of aftershocks is separable into a dependence on space and a dependence on time, that the spatial distribution of aftershocks is consistent with the hypothesis that stress changes from the mainshock cause aftershocks, and that the number of foreshocks approaches the number of aftershocks as the magnitude of the mainshock becomes smaller.

I. Introduction

Earthquakes do not occur in a purely random way. Quantifying and trying to understand the correlations between events has been an ancient task. One of the most robust and important empirical relations in seismology was obtained one hundred years ago by Omori [1894], who examined the rate of occurrence of smaller events as a function of time  $t$  following the largest event- the mainshock. He found that the rate  $R$  of aftershocks decayed like:

$$R(t) \sim 1/t \quad (1)$$

Later, Utsu [1961] introduced a generalization of Omori's law, that

$$R(t) \sim \frac{1}{(t + t_0)^p} \quad (2)$$

with  $t_0$  being a constant, and  $p$  being an exponent that can differ from 1, though typically a value near 1 is observed. Foreshocks preceding a mainshock have also been shown to satisfy this law, with  $t$  in this case being the time before the mainshock [Papazachos, 1975]. The timescales over which foreshocks and aftershocks occur- minutes to months- are too long to be explained by purely elastic processes, which change during ruptures on timescales of order seconds, and too short to be explained by tectonic loading, which produces a cycle of activity on a timescale of hundreds of years. In this paper, a simple dynamical explanation of the time dependence of foreshocks and aftershocks is given.

The rest of the paper is organized as follows. In section II, some previous explanations of Omori's law are briefly reviewed. In section III, the basic theory proposed in this paper is presented. In section IV, a series of measurements of real aftershocks and foreshocks is made from an earthquake catalogue, and compared with the theory. The main results are summarized in the conclusion, in section V.

II. Previous Work

Aftershocks and foreshocks are defined as follows. First, a cluster of events is identified which are believed to be causally connected to each other, and not independently triggered by tectonic loading. Then, the largest event in the cluster is defined as the mainshock, and all the events preceding it are defined as foreshocks, and those

following it are defined as aftershocks. There is general agreement that stress changes are the causal field connecting events. What is not agreed upon is the process that leads to a time delay which is much longer than the time over which the stress field is communicated.

Many explanations of Omori's law, using a variety of mechanisms of time delays, have been proposed. Mikumo and Miyatake [1979] assumed a distribution of viscoelastic relaxation times, which then numerically gave a power law decay in the rate of aftershocks. Yamashita and Knopoff [1987] assumed a power law distribution of crack sizes, and combined with a power law for the rate of growth of the cracks, obtained a power law decay in the rate of aftershocks.

A different kind of explanation was offered by Scholz [1968] and Das and Scholz [1982], who assumed an exponential stochastic process to get  $p = 1$ . The main way their work differs from the theory proposed in this paper is the underlying process which causes the time delay rests, in their work, on the fluctuations of some field- the temperature field- so that each stressed state has a distribution of time delays, while here, the time delay comes from the deterministic dynamical evolution of a field, so that each stressed state has a unique time delay.

Nur and Booker [1972] and Booker [1974] sought a dynamical explanation of time delays using the diffusion of pore fluid pressure. Nur and Booker, however, got  $p = 1/2$  when they coupled compressional stress changes to pore volume and pressure effects. Booker was able to get  $p = 1$ , but only along the fault surface that had previously ruptured during the mainshock.

Nakanishi [1992] proposed a dynamical explanation based on the viscous postseismic relaxation of the crust. He obtained a power law decay in the rate of events following a mainshock. However,  $p$  differed from one, and he only obtained a power law when the time over which the decay of events occurred was comparable to the loading time.

The explanation that is closest to what is proposed in this paper is by Dieterich [1992]. Both his explanation, and the explanation proposed in this paper, seek to describe a deterministic dynamics for the nucleation of an event, with the dynamics being a self-driven acceleration to a critical threshold. Motivated by laboratory friction experiments, he proposes a dynamics described by a set of coupled nonlinear ordinary differential equations. I seek the simplest mathematical representation of the physical picture, and propose a dynamics described by a first order nonlinear differential equation.

III. Theory of the Triggering of Events

The theory proposed here concerns the nucleation of events. Before writing the equations that represent this process, it is useful to describe the physical picture of the earthquake cycle that is being used, and the context in which the nucleation occurs.

While the earth is complicated, and there may be many processes going on, we will be concerned here with the three processes that give the basic timescales in the problem: tectonic loading, nucleation, and rupture. With these processes, a cycle works as follows. Begin from the situation where the fault is stuck, so that nothing would happen if the fault were not loaded further. Slow tectonic loading uniformly raises the stress. Eventually, at some point on the fault, the stress becomes large enough so that a new process, nucleation, begins to dominate the time evolution of stress at that point. The physical process we have in mind for the nucleation phase is subcritical crack growth. During this phase, the crack length and stress grow slowly at first, then rapidly accelerate until a critical stress is reached. At that point, an event is triggered and unstable rupture occurs. The event occurs in a very short time, compared to the loading or nucleation times. The event relieves stress, and

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also redistributes stress to neighboring regions. It may be the case that the stress added to neighboring regions has loaded them enough that they will now be in the nucleation phase. If so, an event is soon triggered there. The process continues until all parts of the fault are below the stress threshold where subcritical crack growth is faster than tectonic loading. This is the end of a cluster. Tectonic loading reloads the fault until another point reaches the nucleation phase, and a new cluster occurs.

The distribution of time delays in the cluster, given by Omori's law, comes from a combination of two things: the acceleration of the stress during nucleation, and the fast redistribution of stress during an event. Consider a population of sites that have been suddenly loaded by a mainshock, some of which are then above the threshold for nucleation. The highest stressed sites will fail soonest, while those at lower stresses will take longer. The distribution of stresses in the nucleation regime, which was depleted before the mainshock, is, following the sudden loading of the mainshock, independent of the stress, on average. The fall in the rate of events with time occurs because, in the nucleation regime, a distribution uniform in stress gets stretched by the acceleration dynamics into a nonuniform distribution in time. This verbose description will become much clearer when we consider the equations that describe this nucleation process.

To represent the accelerating dynamics of subcritical crack growth, consider the equation

$$\frac{d\psi}{dt} = a\psi^n, \quad (3)$$

where the field  $\psi$  evolves with time  $t$ ,  $n$  is a constant exponent, and  $a$  is a constant. The magnitude of  $a$  can be scaled out by a rescaling of time  $t$ , so only its sign is important. The field  $\psi$  is related to the stress intensity factor; when the exponent  $n$  is positive,  $\psi$  can be taken to be the stress intensity factor, while for negative  $n$ , it is the difference between the stress intensity factor and the critical stress intensity factor for unstable rupture.

Suppose an event is triggered when the field  $\psi$  reaches some critical value  $\psi_c$ . The time  $\tau$  it takes the field to evolve from a value  $\psi$  to  $\psi_c$  is

$$\tau = \int_{\psi}^{\psi_c} \frac{d\psi}{d\psi/dt} \quad (4)$$

For a dynamics of  $\psi$  given by (3), this gives, for  $n \neq 1$ ,

$$\tau = \int_{\psi}^{\psi_c} \frac{d\psi}{a\psi^n} = \frac{1}{a(n-1)} (\psi^{-n+1} - \psi_c^{-n+1}), \quad (5)$$

or

$$\tau = \frac{1}{a(n-1)} \psi^{-n+1} - \tau_0, \quad (6)$$

with the constant  $\tau_0$  depending on  $\psi_c$ . In order to have  $\tau$  have a positive contribution from the  $\psi$  dependence in (6), the sign of  $a$  must be related to the value of  $n$  by

$$a(n-1) > 0. \quad (7)$$

Now consider the distribution of event times  $R(\tau)$  given some distribution of initial values of the field  $R(\psi)$  following a sudden loading. Then, because of the deterministic dynamics,

$$R(\tau)d\tau = R(\psi)d\psi \frac{-a}{|a|}, \quad (8)$$

with the sign term  $-a/|a|$  coming from whether  $\tau$  is an increasing or decreasing function of  $\psi$ . Dropping constants, we have

$$R(\tau) \sim R(\psi) \frac{d\psi}{d\tau} \sim R(\psi) \psi^{\dagger n} \sim \frac{R(\psi)}{(\tau + \tau_0)^{\frac{n-1}{n}}}. \quad (9)$$

If  $R(\psi)$  is independent of  $\psi$ , then this is exactly the generalized Omori law, Eq. (2), with

$$p = \frac{n}{n-1}. \quad (10)$$

Note that as  $n$  gets large,  $p$  approaches 1. This relation works, in

fact, for all  $n$ , except  $n = 1$ ; thus, it is a general result for nonlinear self-driven growth.

To derive the generalized Omori law from (3), the only assumption needed was that the initial distribution  $R(\psi)$  following the mainshock is independent of  $\psi$ . Let us examine this assumption. We use the fact that the mainshock gives a sudden kick to the distribution of  $\psi$ , in a way that is independent of the value of  $\psi$ . The kick will, in general, be different for different positions along the fault. What is important, though, is that the kick does not depend on  $\psi$ . This leads to an  $R(\psi)$  that is roughly independent of  $\psi$ . Bigger mainshocks will push more sites into the nucleation regime, so  $R(\psi)$  will be bigger; but the that doesn't affect the lack of dependence on  $\psi$ . Even if there is some dependence in  $R$  on  $\psi$ , as long as it is less strong than  $\psi^{-n}$ , it will only add a minor correction to the time dependence. For example, suppose  $R(\psi) \sim \psi^m$ . Then  $p = \frac{n-m}{n-1}$ , which is still close to 1 for  $n$  large. This insensitivity to the exact form of  $R(\psi)$  means that the time dependence in the theory is insensitive to any long term changes in seismicity which may be occurring.

To get the generic Omori law exponent of 1 from the generalized Omori law, all we need is that the size of the exponent  $|n|$  be large. Experiments examining the rate of increase in crack length as a function of stress intensity factor during subcritical crack growth can be fit by power laws with large exponents— $n$  of order a few tens or more [Atkinson, 1979], or by exponentials [Wiederhorn and Bolz, 1970]. Having  $\psi$  grow exponentially, as in

$$\frac{d\psi}{dt} = ae^{b\psi} \quad (11)$$

also can be shown, in the same way (10) was derived, to give  $p = 1$ . Thus  $p \approx 1$  is a generic value for the rapidly self-accelerating dynamics.

A simple theory has been presented for the time dependence of aftershocks. Foreshocks can also be explained by this theory. Foreshocks can be understood as events that triggered an afterevent that happened to be bigger than the triggering event. With this hypothesis of the relationship of foreshocks to aftershocks, we can derive the time dependence of foreshocks from Eq. (2). There are two regimes where we can calculate things, depending on whether the number of foreshocks per mainshock is small or large compared to one. When the number is small compared to one, the time dependence for foreshocks is the same as Eq. (2), with time  $t$  measured as time before the mainshock. When the number is large compared to one, we need to consider the distribution in time of other events that were triggered by the initial event in the cluster. For the mainshocks occurring at time 0, and the initial event that triggered all the foreshocks occurring a time  $t'$  before; we can calculate the rate of events at time  $t$  before the mainshock. Integrating over all initial triggering times  $t'$  weighted by the joint probability that events occurred at time  $t'$ ,  $t$ , and 0 gives the rate at time  $t$ :

$$R(t) \sim \int_t^{\infty} \frac{dt'}{(t' - t + t_0)^p (t' + t_0)^p} \quad (12)$$

When  $p < 1$  and  $t \gg t_0$  we can scale out the  $t$  dependence to get

$$R(t) \sim t^{-2p+1} \quad (13)$$

In addition to their time dependence, there are other features of aftershocks and foreshocks that can be measured. In the next section, I examine aftershocks and foreshocks of small magnitude mainshocks to see if the theory is consistent with those other features as well.

#### IV. Real Aftershocks

Many researchers have examined the foreshocks and aftershocks of individual sequences. We will be interested in the average properties, and thus seek an ensemble of sequences that can be averaged over. The properties we will be examining include the spatial and temporal distribution of aftershocks, and the dependence on the magnitude of the mainshock.

The catalogue of events used is the USGS catalogue for Central and Northern California for magnitudes  $M_L \geq 1.5$  for the years

1969 – 1990. The mainshocks are found by searching through the catalogue, and checking whether, for each event, there has been a larger event within some distance in space and in either the preceding or following specified amount of time. If there has not been a larger event, then the event is considered as a mainshock. The distance is taken to be 50 kilometers, which is roughly three times the length scale of the brittle crust depth in California. The time period is chosen to be 100 days, and is chosen to be long enough so that any effect of previous events will not change much on the timescale of a few days which we consider, and short enough to obtain enough cases to get good statistics. This way of choosing mainshocks differs from standard declustering techniques, which use combined measures of spatial and temporal distances to decide what events belong together in a cluster. That is, however, exactly the kind of correlation we do not want to introduce into our data selection. Because we will be interested in the distances between events, it is desired that the mainshocks be small in spatial extent. They must be big enough, though, to produce an aftershock sequence significantly above the background level of activity. The errors for locating each event differ in the catalogue, but typical errors are of order 1 km.

To account for the variation in depth of seismicity the catalogue is cut further. Most events are concentrated in a band between the depths of 4 to 10 kilometers. Over this range of depths, the rate is roughly independent of depth. To avoid complicated depth-dependent corrections in the plots, only mainshocks lying in this band between 4 and 10 kilometers, and aftershocks of these mainshocks that also lie in this band, are used. Finally, to get good averaging, we do not want any single events to dominate the signal. Two events in the Mammoth lakes region, a magnitude 5.5 on 11-23-1984 and a magnitude 5.9 on 7-20-1986, produced large numbers of aftershocks. These two events have been left out of the analysis.

We first check Omori's law for the data that are considered. Figure 1 shows the rate of foreshocks and aftershocks per mainshock as a function of time from the mainshock. The three solid curves show the rate of aftershocks per mainshock, for different magnitude mainshocks. The lowest of the three curves is an average of 124 mainshocks having magnitudes between 3 and 4, the middle curve is an average of 84 mainshocks having magnitudes between 4 and 5, and the top curve is an average of 20 mainshocks with magnitudes between 5 and 6. The two dotted curves are the foreshocks of these

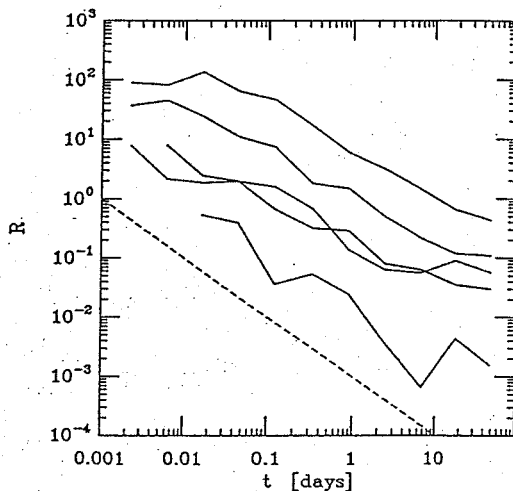


Fig. 1. Omori's law for real earthquakes. The rate of aftershocks and foreshocks per mainshock as a function of time  $T$  from the mainshock. The three solid aftershock curves are for mainshocks with magnitudes between 5 and 6 for the top curve, 4-5 for the middle curve, and 3-4 for the bottom curve. The bottom dotted foreshock curve is for only the largest magnitude foreshock, while the top dotted curve is all the foreshocks except the largest magnitude one. A long dashed line with slope  $-1$  has been added for comparison.

same 228 mainshocks with magnitudes between 3 and 6. In the lower dotted curve, only the largest foreshock of each mainshock is shown. In the upper dotted curve, all the other foreshocks except the largest one are shown. The division of the foreshock data was done so as to compare with the theory. Assuming the largest foreshock was the one that triggered the mainshock, it should have a distribution  $R(t) \sim t^{-p}$ , while the other foreshocks that it would have triggered would, by Eq. (13), have a distribution  $R(t) \sim t^{-2p+1}$ . A long dashed line with  $p = 1$  has been added to the figure for comparison.

There are a couple of things to notice in Figure 1. First, all the aftershock curves have roughly the same slope, with a  $p$  value that appears to be slightly less than 1. Thus we see that  $p$  seems to be independent of the magnitude of the mainshock. The independence of  $p$  on mainshock magnitude is a feature required by the theory. A second thing to notice is that, within the uncertainty in the data, the  $p$  value of the foreshocks seems to be roughly the same as the  $p$  value of the aftershocks. Unfortunately, since the data is noisy and  $p$  is close to 1, we cannot test Eq. (13); the theory is consistent with the data, but not confirmed by it.

The second figure is plot of the temporal evolution of the spatial distribution of aftershocks. Figure 2 shows the results of an average over the same 84 mainshocks having magnitudes between 4 and 5; and 20 mainshocks between 5 and 6, as in the previous figure. The plot shows the number of events that occurred after the mainshock and before some time  $t$ , as a function of distance from the mainshocks. Distances are measured between hypocenters of events. The four solid curves in the figure show data for aftershocks following the magnitude 4-5 mainshocks, for all the events happening within 1/3, 1, 3, and 9 days following the mainshocks. The four dotted curves are the aftershocks following the magnitude 5-6 mainshocks, also for the same time cutoffs. The long dashed line is a theoretical curve that will be explained shortly. The peaks in the curves occur at a distance that corresponds to the radius of the mainshock rupture.

The main thing to notice from the figure is that the spatial distributions appear stationary in time; what changes is the rate at which events occur, not where they occur. We do not see an expanding pat-

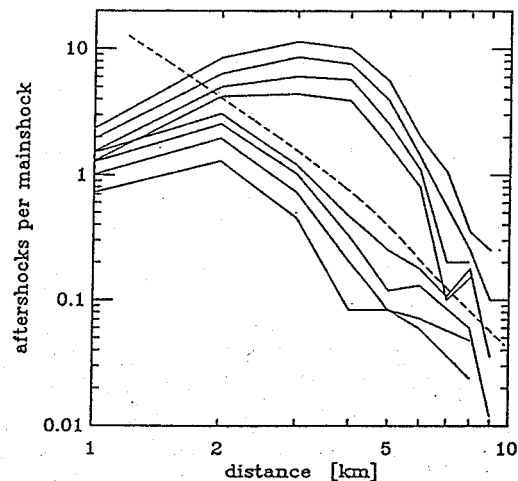


Fig. 2. The spatial distribution of aftershocks for real earthquakes. The number of aftershocks per mainshock as a function of distance is plotted, for events happening up to a given time following the mainshock. Distance is measured between hypocenters, in units of kilometers. The four solid curves are for mainshocks with magnitudes between 4 and 5, while the four dotted curves are for mainshock magnitudes between 5 and 6. The long dashed curve is a theoretical curve (see Eq.(15)). The times of the four curves in each set are all the events before 1/3, 1, 3, and 9 days. Observe that the spatial distribution appears stationary in time; it does not broaden, and only the rate of events changes. The finite size of the source can be seen in the drop in events at small distances.

tern moving out from the rupture surface. (Observations of an expanding aftershock pattern have been reported for large earthquakes [Tajima and Kanamori, 1985]. There are two effects that may cause the reported results, though. First, aftershocks of aftershocks tend to broaden the distribution with time, and their effect was not considered. Second, coupling of the unstably sliding seismogenic crust to the stably sliding crust below could cause an expanding pattern for large events.) Another way of saying this, is that the dependence on space and time is separable into a dependence on space and a dependence on time:

$$R(s, t) \sim R(s)R(t) \quad (14)$$

This separability is required by the theory proposed in this paper, and is an implicit assumption in writing down (3) as a differential equation with only time, and no space, dependence.

Is the spatial dependence of aftershocks consistent with stress changes being the cause? The Green's function for the stress change due to a point source falls off as  $1/r^3$ , where  $r$  is the distance from the source. Averaging over mainshocks uniformly distributed at depth on the strip of width  $w$  gives a distribution:

$$N(r) \propto \begin{cases} \frac{1}{r^2} [\frac{w}{2} - r] & r \leq w \\ \frac{1}{r^2} [w \sin^{-1} \frac{w}{r} - r(1 - \sqrt{1 - \frac{w^2}{r^2}})] & r \geq w \end{cases} \quad (15)$$

The dashed curve in figure 2 is a plot of this expression, with  $w = 6$  km, since we are only using events between 4 and 10 kilometers in depth. Note that there is an overall multiplicative constant that is left to be set. The aftershock data is quite consistent with this curve. While the hypothesis that changes in stress are what cause aftershocks is generally believed in the seismological community, Figure 2 is an important plot is a new quantitative confirmation of this hypothesis. This spatial distribution is consistent with what would be expected by the theory, when the dynamical field  $\psi$  in the nucleation process is identified with the stress intensity factor.

The final plot using the real earthquake data is the number of aftershocks and foreshocks per mainshock, as a function of mainshock magnitude, shown in Figure 3. Note that the number of foreshocks approaches the number of aftershocks as the mainshock magnitude becomes smaller. This is what would be expected for the connection between foreshocks and aftershocks proposed in the previous section.

#### V. Conclusion

In this paper, the dynamics of self-drive acceleration to failure has been proposed as an explanation of earthquake foreshocks and aftershocks. The generalized Omori law was derived analytically

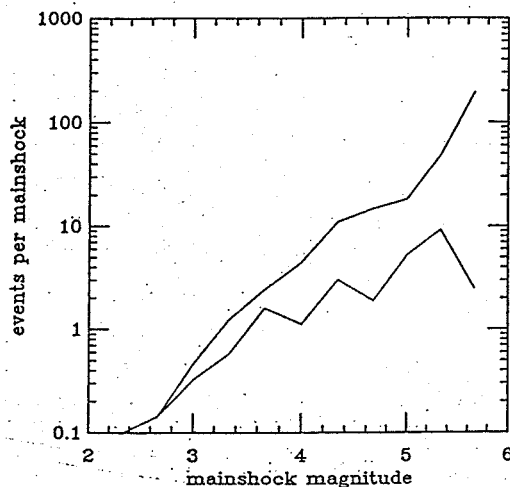


Fig. 3. The number of aftershocks and foreshocks per mainshock, as a function of mainshock magnitude, for real earthquakes. The solid line is the aftershocks, while the dotted line is the foreshocks. Note that the number of foreshocks approaches the number of aftershocks as the magnitude of the mainshock becomes smaller.

from the proposed dynamics. The exponent of the falloff was shown to have a generic value close to one for a dynamics which accelerated strongly. Foreshocks were explained as events which had an afterevent which happened to be bigger than the triggering event. With this understanding, the time dependence of foreshocks was derived from the time dependence of aftershocks.

To test the theory, an analysis of aftershock and foreshocks in a real earthquake catalogue was made. An average over aftershocks and foreshocks of small magnitude mainshocks showed: the distribution of aftershocks in space and time is separable into a dependence on space and a dependence on time; the spatial distribution of aftershocks is consistent with stress changes from the mainshock being what causes aftershocks; the exponent of the decay of the rate of aftershocks in time was seen to be independent of mainshock magnitude; the number of foreshocks approaches the number of aftershocks as the mainshock magnitude goes to zero. All these observations are consistent with what is expected by the theory.

While the discussion in this paper has centered around clustering in earthquakes, the accelerating failure dynamics proposed in this paper is quite general, and seems likely to hold in other physical systems. The basic result is that a system which undergoes self-driven acceleration to a threshold where an event happens, when kicked suddenly, responds with a power law decay in time in the rate of events. Work is being done to find other systems where this result might apply.

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