Derived Categories of Moduli Spaces of Semistable Pairs over Curves

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ABSTRACT

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The context of this thesis is derived categories in algebraic geometry and geometric quotients. Specifically, we prove the embedding of the derived category of a smooth curve of genus greater than one into the derived category of the moduli space of semistable pairs over the curve. We also describe closed cover conditions under which the composition of a pullback and a pushforward induces a fully faithful functor. To prove our main result, we give an exposition of how to think of general Geometric Invariant Theory quotients as quotients by the multiplicative group.

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For Mama and Papa

1. Introduction

Derived categories were invented by Grothendieck in the early 1960's, although they were not put on paper until a few years later in the 1967 thesis of Verdier [26]. These objects have gone on to provide insight in fields as diverse as differential equations and the representation theory of Lie algebras.

In algebraic geometry, derived categories have become both a fundamental tool as well as interesting objects in their own right, acting as new invariants of schemes or even stacks. Applications to the minimal model program in birational geometry were investigated by Kawamata [15] and even the problem of the cubic fourfold has been attempted from this perspective by Kuznetsov [17]. In string theory, derived categories make a central appearance in Maxim Kontsevich's famous Homological Mirror Symmetry conjecture [16], and much recent work has been completed in this direction by Abouzaid [1; 2], Katzarkov [1; 13], Seidel [22], Smith [2], Kelly and Favero [9], and others. Also recently, Ballard/Favero/Katzarkov [4] and Halpern-Leistner [10] have made progress discussing derived categories in the setting of Geometric Invariant Theory.

The derived category $\mathcal{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$ is the localization of the homotopy category of complexes in $\mathcal{A}$ by quasi-isomorphisms between those complexes. In particular, $\mathcal{D}(\mathcal{A})$ will be a triangulated category. For a scheme X, the bounded derived category of coherent sheaves on X, denoted $\mathcal{D}^b(X)$, is nothing more than those objects in $\mathcal{D}(coh(X))$ with bounded cohomology. Of course, the category $\mathcal{D}^b(X)$ is not a strict invariant (if it were, there would be little to study), although it is a classical result of Bondal and Orlov [7] that if X and Y are smooth projective varieties, both with ample or anti-ample canonical bundle, and $\mathcal{D}^b(X) \cong \mathcal{D}^b(Y)$, then X is isomorphic to Y.

A semiorthogonal decompostion of a triangulated category $\mathcal{D}$, written as $\mathcal{D} =< \mathcal{T}_1, \ldots, \mathcal{T}_n >$,ss is a sequence of full admissible triangulated subcategories $\mathcal{T}_1, \ldots, \mathcal{T}_n$ which generate $\mathcal{D}$ such that there are no morphisms from right to left, i.e. for all $j > i$, we have that $\mathrm{Mor}(\mathcal{T}_j, \mathcal{T}_i) = 0$. It is often interesting to ask whether or not $\mathcal{D}^b(X)$ has a semiorthogonal decomposition, and if so what components it's comprised of. For example, a groundbreaking result of Beilinson [5] reveals that there is a semiorthogonal decomposition $\mathcal{D}^b(\mathbb{P}^n) =< \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2), \ldots, \mathcal{O}(n) >$. Another well-known result by Bondal and Orlov [6] states that if $\tilde{X}$ is the blow up of a smooth subvariety Y of codimension r in a smooth variety X, then there is a semiorthogonal decomposition of $\mathcal{D}^b(\tilde{X})$ into $r-1$ copies of $\mathcal{D}^b(Y)$ and one copy of $\mathcal{D}^b(X)$. There are certainly varieties whose derived category cannot be decomposed; in fact, Okawa has proven that the derived category of every smooth projective curve excluding $\mathbb{P}^1$ admits *no* semi-orthogonal decompositions [21].

In this thesis, we show that if M is the moduli space of σ-semistable pairs with fixed determinant of degree d over a curve C of genus $g > 1$, then when $d = 2g-1$ there is an admissible embedding of $\mathcal{D}^b(C)$ into $\mathcal{D}^b(M)$. In particular, this result implies a semiorthogonal decomposition of $\mathcal{D}^b(M)$ into $\mathcal{D}^b(C)$ and another component, although we do not explicitly compute what the complementary component is. Our main theorem which allows us to prove the embedding is the following:

Theorem 1.1. *(Formulation of Theorem 7.7) For $\sigma \in \{\frac{d}{2} + \mathbb{Z}\} \cap [0, \frac{d}{2}]$ (to ensure the moduli are non-empty), define $\sigma^+ = \sigma + \epsilon$ and $\sigma^- = \sigma - \epsilon$ for ϵ small. Let $M(\sigma^+)$, and respectively $M(\sigma^-)$, denote the moduli space of σ^+, and respectively σ^-, semistable pairs over a curve C of genus $g > 1$ with fixed determinant of degree $d \leqslant 2g-1$. Then there is a fully faithful functor $\mathcal{D}^b(M(\sigma^+)) \longrightarrow \mathcal{D}^b(M(\sigma^-))$.*

We also conjecture in Remark 8.3 that the bounded derived category of the curve $\mathcal{D}^b(C)$ embeds into $\mathcal{D}^b(M)$, where M is the usual moduli space of semistable bundles over C, again with fixed determinant of degree $2g-1$.

In Section 3, we verify this conjecture when $g=2$, and prove that the fully faithful functor is in fact given by the Fourier-Mukai transform associated to the universal bundle on $C \times M$.

In Section 4, we prove Proposition 4.2, which we will use to show Theorem 7.7. Although the Proposition's conditions are a bit technical, its statement is actually quite general.

In Section 5, we explain a trick to think of more general GIT quotients as $\mathbb{G}_m$ quotients. This trick is introduced by Thaddeus in his paper [25]. We will need this point of view, in addition to the algebraic construction of Section 6, to reduce the proof of Theorem 7.7 to a more manageable case.

In Section 7, we prove Theorem 7.7, using the preceding sections.

Finally, in Section 8, we deduce a couple of corollaries, including the fact that the derived category of the curve embeds into the derived categories of the moduli spaces of σ-semistable pairs with fixed determinant of degree $d=2g-1$.

2. Notation

For the duration of this paper, we fix k to be an algebraically closed field of characteristic zero. We also fix C to be a smooth curve, i.e. a dimension one integral scheme whose local rings are all regular and which is proper over k. Let g denote the genus of C, and note that for us g will always be greater than one. Fix Λ to be a line bundle over C of degree d.

In Section 3 only, M will denote the moduli space of rank two vector bundles over C, where $g = 2$ and $d = 1$, whose existence is guaranteed by Newstead [19].

In all later sections, $M(\sigma)$ denotes the moduli space of σ-semistable pairs (E, ϕ) over a curve C of any genus $g > 1$, where E is a rank two vector bundle over C and ϕ is a global section of E, for now with no restriction on d. The existence of these moduli spaces for a certain range of σ and d, to be discussed later, is guaranteed by Thaddeus [24].

3. The Genus 2 Case

Let C be a smooth genus two curve, and let M be the moduli space of stable rank two vector bundles over C of fixed degree three determinant. We denote by F the universal bundle on $C \times M$, and by Φ_F the associated Fourier-Mukai transform.

Proposition 3.1. *If $g = 2$ and $d = 3$, then $\Phi_F : \mathcal{D}^b(C) \to \mathcal{D}^b(M)$ is a fully faithful embedding.*

Proof. In Theorem 1.1 of [6], Bondal and Orlov prove criteria, amounting to orthogonality conditions on skyscraper sheaves, under which a Fourier-Mukai transform is fully faithful. Our goal is to show that for every point $x \in C$, the restriction of the universal bundle $F|_x$ is the spinor bundle on M associated to

x. The result then follows from the aforementioned theorem, combined with the proof of Theorem 2.7 in the same paper, during which they use their Theorem 1.1 to show the fully faithfulness of the Fourer-Mukai transform associated to a spinor bundle.

Let us first recall explicit constructions of M and F, following P.E. Newstead [19]. Since C has genus two, we have a double cover $C \to \mathbb{P}^1$ associated to the canonical bundle, ramified over six points. As shown by Newstead in [19], the moduli space M is then the intersection locus of a pencil of quadrics $\{Q_\lambda | \lambda \in \mathbb{P}^1\}$ in $\mathbb{P}^5$, which degenerates into a cone precisely at those 6 points. Each nondegenerate Q_λ carries two families of planes. Lifting the pencil to $\{Q_x | x \in C\}$, we have that each Q_x comes with a preferred family of planes H_x, where the two points lying over $\lambda \in \mathbb{P}^1$ give the two families on Q_λ. Now, for $\xi \in M$ and $x \in C$, consider $N_{\xi,x}$, the planes of H_x containing ξ. Via the Plücker embedding of the Grassmannian of planes in $\mathbb{P}^5$ into $\mathbb{P}^{19}$, we can realize $N_{\xi,x} \subset \mathbb{P}^{19}$ as a conic. It follows that

$$\mathrm{N} = \bigcup_{\xi \in M, x \in C} N_{\xi,x} = \{(x, \xi, [P]) | x \in C, \xi \in M, [P] \in N_{\xi,x}\} \subseteq C \times M \times Gr(3,6) \quad (3.1)$$

is a $\mathbb{P}^1$-bundle over $C \times M$. In fact, this $\mathbb{P}^1$-bundle N can be lifted to a rank 2 bundle F over $C \times M$, which Newstead proves is the universal bundle.

By a spinor bundle on M, we mean the restriction of a spinor bundle on a quadric $Q_\lambda \subset \mathbb{P}^5$, as defined by Addington [3] in the following way. Let the line bundle $\mathcal{E}$ be the ample generator of $\mathrm{Pic}(OG_\pm(3, Q_\lambda)) = \mathbb{Z}$, where $OG_\pm(3, Q_\lambda)$ is one of two connected components of the space of $\mathbb{P}^2$s on Q_λ. The square of $\mathcal{E}$ is the restricted Plücker line bundle embedding $Gr(3,6)$ into $\mathbb{P}^{19}$. Then a spinor bundle on Q_λ is defined as $p_* q^*(\mathcal{E})$, where $Q_\lambda \xleftarrow{p} I \xrightarrow{q} OG_\pm(3, Q_\lambda)$ is the incidence correspondence and .

Observe that if we restrict $I \xrightarrow{p} Q_\lambda$ to M, the restricted I equals $N|_x$ as subvarieties of $M \times \mathbb{P}^{19}$, where $x \in C$ is lying over $\lambda \in \mathbb{P}^1$. So to prove that $F|_x$ is indeed the spinor bundle on M associated to x, it suffices to show that $q^*(\mathcal{E}) = \mathcal{O}(1)$, where $\mathcal{O}(1)$ is the relative sheaf of the $\mathbb{P}^1$-bundle $I \xrightarrow{p} M$. But since the fibers of p are $N_{\xi,x}$, which is a conic in $\mathbb{P}^{19}$ under the Plucker embedding, it immediately follows that $q^*(\mathcal{E})$ is the relative sheaf of p.

□

4. A Technical Result

We now prove the abstract and technical Proposition 4.2, which we'll use in the proof of Theorem 7.7. It essentially says that if one wants to prove that the composition of a derived pushforward with a derived pullback is a fully faithful functor, and all involved schemes are over a scheme S, then it suffices to show it over any nice closed scheme cover of S.

First we need the following lemma:

Lemma 4.1. *Let X be a regular Noetherian scheme, $\mathcal{F}$ a finite locally free sheaf on X, $\mathcal{E}^\bullet$ an object in $\mathcal{D}^b(X)$, and $g : \mathcal{F} \to \mathcal{E}^\bullet$ a map. If there exist closed subschemes H covering X such that $g : \mathcal{F} \otimes^{\mathbb{L}} \mathcal{O}_H \to \mathcal{E}^\bullet \otimes^{\mathbb{L}} \mathcal{O}_H$ is an isomorphism for all H, then $\mathcal{F} \cong \mathcal{E}^\bullet$ in $\mathcal{D}^b(X)$.*

Proof. Since X is regular we may assume all terms $\mathcal{E}^i$ of $\mathcal{E}^\bullet$ are finite locally free. Also, observe that since $\mathcal{F}$ is flat, the derived tensor product $\mathcal{E}^\bullet \otimes^{\mathbb{L}} \mathcal{O}_H \cong \mathcal{F} \otimes^{\mathbb{L}} \mathcal{O}_H = \mathcal{F} \otimes \mathcal{O}_H$ is a sheaf.

We first show that $\mathcal{E}^\bullet$ is concentrated in degree zero.

Let m be maximal with $\mathcal{H}^m(\mathcal{E}^\bullet) \neq 0$ and assume $m > 0$. Recall that for any morphism f of ringed spaces, we have the spectral sequence $E_2^{p,q} = L^p f^*(\mathcal{H}^q(\mathcal{E}^\bullet)) \Rightarrow L^{p+q} f^*(\mathcal{E}^\bullet)$. In our situation, letting $f : H \to X$ be the closed immersion, the terms $L^{p+q} f^*(\mathcal{E}^\bullet) = 0$ when $p + q \neq 0$ and $E_2^{p,q} = L^p f^*(\mathcal{H}^q(\mathcal{E}^\bullet) = 0$ when $q > m$ or $p > 0$. The latter assertion follows from replacing the coherent sheaf $\mathcal{H}^q(\mathcal{E}^\bullet)$ by its locally free resolution, which of course has no terms of positive degree. These two inequalities imply that $E_2^{0,m}$ survives to the infinity page, and so we obtain $\mathcal{H}^m(\mathcal{E}^\bullet) \otimes \mathcal{O}_H = f^*(\mathcal{H}^m(\mathcal{E}^\bullet)) = 0$. Because this holds for a a closed cover of X, by Nakayama's Lemma we conclude that $\mathcal{H}^m(\mathcal{E}^\bullet) = 0$. Thus we've shown that $\mathcal{E}^\bullet$ has no nonzero cohomology in degrees greater than zero.

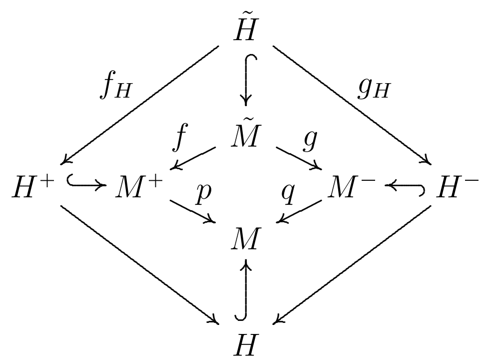
Now we show that $\mathcal{H}^0(\mathcal{E}^\bullet)$ is flat. Since $\mathcal{F} \otimes^{\mathbb{L}} \mathcal{O}_H \cong \mathcal{E}^\bullet \otimes^{\mathbb{L}} \mathcal{O}_H$ for closed subschemes H covering X, we also have that $\mathcal{F} \otimes^{\mathbb{L}} k(x) \cong \mathcal{E}^\bullet \otimes^{\mathbb{L}} k(x)$ for all closed points $x \in X$. Applying the above spectral sequence for the closed point cover, we see that in this case, since we have already shown that $\mathcal{E}^\bullet$ is concentrated in degrees less than zero, the terms $E_2^{p,q}$ are only nonzero in the third quadrant, and furthermore $E_2^{-1,0} = \mathcal{H}^{-1} i_x^*(\mathcal{H}^0(\mathcal{E}^\bullet)) = 0$ as well since there is no nontrivial map on any page involving this term. Unraveling the picture affine locally, the vanishing says that if $\tilde{M} = \mathcal{H}^0(\mathcal{E}^\bullet)$, then $\mathrm{Tor}_1(M, R/m) = 0$ for all maximal ideals $m \subset R = H^0(\mathcal{O}_X)$. Hence $\mathcal{H}^0(\mathcal{E}^\bullet)$ is flat.

Let m be minimal with $\mathcal{H}^m(\mathcal{E}^\bullet) \neq 0$ and $m < 0$. Then $E_2^{0,m}$ survives to the infinity page, and so $\mathcal{H}^m(\mathcal{E}^\bullet) \otimes \mathcal{O}_H = f^*(\mathcal{H}^m(\mathcal{E}^\bullet)) = 0$. Again by Nakayama's, we finally conclude that $\mathcal{H}^m(\mathcal{E}^\bullet) = 0$ and $\mathcal{E}^\bullet$ is concentrated in degree 0.

By truncation and quasi-isomorphism, we can choose the sheaf $\mathcal{H}^0(\mathcal{E}^\bullet)$ as a representative for $\mathcal{E}^\bullet$. So we have that $g : \mathcal{F} \to \mathcal{H}^0(\mathcal{E}^\bullet)$ induces an isomorphism $\mathcal{F} \otimes k(x) \to \mathcal{H}^0(\mathcal{E}^\bullet) \otimes k(x)$ for all closed points x (derived tensor is not necessary as both modules are flat). Then by Nakayama's, we have that g induces an isomorphism of stalks at the closed points of X. It follows that on the level of sheaves $g : \mathcal{F} \to \mathcal{H}^0(\mathcal{E}^\bullet)$ is an isomorphism, and hence $\mathcal{F} \cong \mathcal{E}^\bullet$ in $\mathcal{D}^b(X)$.

□

Suppose M, M^-, M^+ and M are varieties with maps f, g, p, and q satisfying commutativity as in the following diagram:



(4.1)

If H is a subvariety of M, we correspondingly define $H^+ = p^{-1}(H)$, $H^- = q^{-1}(H)$, and $\tilde{H} = f^{-1}(p^{-1}(H)) = g^{-1}(q^{-1}(H))$ to be the scheme theoretic inverse images.

Proposition 4.2. *We use the above notation as in diagram 4.1. Assume that $\tilde{M}$, M^-, and M^+ are smooth, and that M is covered by closed subvarieties H such that the corresponding $\tilde{H}$, H^-, and H^+ are smooth. If for all H in the closed cover, the functors $Rg_{H*} \circ Lf_H^* : \mathcal{D}^b(H^+) \longrightarrow \mathcal{D}^b(H^-)$ are fully faithful and the following two diagrams satisfy Tor independence*

$$\begin{array}{ccc} \tilde{H} & \xrightarrow{f_H} & H^+ \\ \downarrow & & \downarrow \\ \tilde{M} & \xrightarrow{f} & M^+ \end{array} \qquad \begin{array}{ccc} \tilde{H} & \xrightarrow{g_H} & H^- \\ \downarrow & & \downarrow \\ \tilde{M} & \xrightarrow{g} & M^-, \end{array} \tag{4.2}$$

then $Rg_* \circ Lf^* : \mathcal{D}^b(M^-) \longrightarrow \mathcal{D}^b(M^+)$ *is fully faithful.*

Proof. For simplicity we assume all pushforwards and pullbacks are derived, omitting the L's and R's.

Recall that the adjoint functor of $g_* \circ f^*$ is $f_* \circ g^!$. So it suffices to show the natural isomorphism $f_* g^! g_* f^* \simeq \mathbb{1}_{\mathcal{D}^b(M^-)}$. In fact, by [14], it suffices to show the adjunction map is an isomorphism for very negative powers of an ample line bundle.

We will show that for all H satisfying the conditions of the Proposition and all line bundles $\mathcal{L}$, the canonical map $\mathcal{L} \to f_* g^! g_* f^* \mathcal{L}$, obtained via the adjunction property $\mathrm{Hom}(\mathcal{L}, f_* g^! g_* f^*(\mathcal{L})) = \mathrm{Hom}(g_* f^* \mathcal{L}, g_* f^* \mathcal{L})$, induces isomorphisms $\mathcal{L} \otimes^{\mathbb{L}} \mathcal{O}_{H^-} \cong f_* g^! g_* f^*(\mathcal{L}) \otimes^{\mathbb{L}} \mathcal{O}_{H^-}$.

Since $g_{H*} \circ f_H^*$ are fully faithful, we have from the adjunction maps the canonical isomorphisms $\mathcal{L} \otimes^{\mathbb{L}} \mathcal{O}_{H^-} \cong f_{H*} g_H^! g_{H*} f_H^* (\mathcal{L} \otimes^{\mathbb{L}} \mathcal{O}_{H^-})$. It remains to show that the latter is naturally isomorphic to $f_* g^! g_* f^*(\mathcal{L}) \otimes^{\mathbb{L}} \mathcal{O}_{H^-}$.

Letting i denote inclusion, observe that by [23, Tag 08IB] and Tor-independence, we know that $i_{H^+}^* f_* = f_{H*} i_{\tilde{H}}^*$. Next, we claim that $i_{\tilde{H}}^* g^! = g_H^! i_{H^-}^*$. This is a consequence of the adjunction formula applied to

$$
\begin{aligned}
\mathrm{i}_{\tilde{H}}^* g^!(\mathcal{E}^\bullet) &= i_{\tilde{H}}^* g^*(\mathcal{E}^\bullet) \otimes i_{\tilde{H}}^* \omega_{\tilde{M}} \otimes i_{\tilde{H}}^* g^* \omega_{M^-}^\vee \\
&= g_H^*(i_{H^-}^* \mathcal{E}^\bullet) \otimes \omega_{\tilde{H}} \otimes \det(\mathcal{I}_{\tilde{H}}/\mathcal{I}_{\tilde{H}}^2) \otimes g_H^* \omega_{H^-}^\vee \otimes g_H^* \det(\mathcal{I}_{H^-}/\mathcal{I}_{H^-}^2) \\
&= g_H^*(i_{H^-}^* \mathcal{E}^\bullet) \otimes \omega_{\tilde{H}} \otimes g_H^* \omega_{H^-}^\vee \\
&= g_H^!(i_{H^-}^* \mathcal{E}^\bullet).
\end{aligned}
\tag{4.3}
$$

Using Tor-independence again, also have that $i_{H^+}^* g_* = g_{H*} i_{\tilde{H}}^*$, and all $i_{\tilde{H}}^* f^* = f_H^* i_{H^-}^*$.

Applying this series of relations, we obtain the desired equality $f_* g^! g_* f^*(\mathcal{L}) \otimes^{\mathbb{L}} \mathcal{O}_{H^-} \cong f_{H*} g_H^! g_{H*} f_H^*(\mathcal{L} \otimes^{\mathbb{L}} \mathcal{O}_{H^-})$. Hence we conclude, by the previous Lemma 4.1, that $g_* \circ f^*$ is fully faithful. □

5. General GIT Quotients as $\mathbb{G}_m$ Quotients

To prove the general case of our main theorem, we will need the following construction, described in [25], which allows us to view GIT quotients of a projective variety by an arbitrary reductive group as GIT quotients of a different variety by $\mathbb{G}_m$. A key facet of this construction is that we may start with linearizations with distinct underlying line bundles, but once we have switched to quotienting by $\mathbb{G}_m$, the new linearizations will all have the same underlying line bundle.

Let X be a projective variety over an algebraically closed field k and G be a reductive group scheme acting on X. Suppose L^+ and L^- are ample linearizations for this action, and let $L^0 = \frac{1}{2}(L^+ \otimes L^-)$ be the average linearization. Denote the three GIT quotients $X//^+G$, $X//^-G$, and $X//^0G$, and define $V = \mathbb{P}(L^+ \oplus L^-)$.

There is an obvious action of G on V with canonical linearization on $\mathcal{O}(1)$. There is also a natural action of $\mathbb{G}_m$ on V defined by $\lambda(x, y) = (\lambda x, \lambda^{-1} y)$ on $L^+ \oplus L^-$, which commutes with the G action. Now consider the following three linearizations on $\mathcal{O}(1)$ of the $\mathbb{G}_m$ action: the aforementioned natural linearization, denoted by a 0, the + linearization given by $\lambda(x, y) = (x, \lambda^{-2} y)$, and the $-$ linearization given by $\lambda(x, y) = (\lambda^2 x, y)$.

We claim that, with respect to all three linearizations, $V//\mathbb{G}_m = X$, and in all three cases, the residual G action has the corresponding L^+, L^-, and L^0 linearization. To see this, note that by definition the projective bundle $V = \mathrm{Proj}(\bigoplus_{i,j \in \mathbb{N}} H^0((L^+)^i \otimes (L^-)^j)$. So for the + linearization, the subalgebra of $\mathbb{G}_m$ invariants is $\bigoplus_{i \in \mathbb{N}} H^0(L^+)^i)$, for $-$ it is $\bigoplus_{i \in \mathbb{N}} H^0((L^+)^i)$, and for 0 it is $\bigoplus_{i\in\mathbb{N}} H^0((L^+)^i \otimes (L^-)^i)$. Since L^+ and L^- are ample line bundles on X, we have the three equalities $V//^+\mathbb{G}_m = V//^-\mathbb{G}_m = V//^0\mathbb{G}_m = X$ and, furthermore, the residual G action is linearized by, respectively, L^+, L^-, and L^0.

Thus we have the isomorphisms

$$(V//^0\mathbb{G}_m)//G = X//^0G$$

$$(V//^+\mathbb{G}_m)//G = X//^+G$$

$$(V//^-\mathbb{G}_m)//G = X//^-G$$

But the two group actions commute, so defining $W = V//G$, we have the desired set of equalities

$$W//^0\mathbb{G}_m = X//^0G$$

$$W//^+\mathbb{G}_m = X//^+G$$

$$W//^-\mathbb{G}_m = X//^-G,$$

and furthermore the linearizations of these GIT quotients of W all have the same underlying line bundle. In general, W may be singular. However, in our application of this trick, we require (and will show that) W is smooth at the desired point.

6. Étale Maps from Affine to Standard Flips

First we prove an algebra lemma which we will use repeatedly. Before we dive into the lemma, we will say what we mean by an algebraic action of a reductive linear algebraic group G on a finite type k-algebra R. We follow many of the conventions of Newstead's book [20]. Recall that throughout this paper, k is an algebraically closed field of characteristic zero.

Let G be a linear algebraic group, i.e. G is isomorphic to a closed subgroup of $\mathrm{GL}(n)$ for some n. A linear action of G on k^n is one arising from a rational representation $G \to \mathrm{GL}(n)$. We say G is geometrically reductive if for every linear action of G on k^n and invariant point $v \in k^n$, there exists an invariant homogeneous nonconstant polynomial f such that $f(v) \neq 0$. In fact by Mumford's Conjecture, this is equivalent to being reductive in the usual sense (see [20]).

Let R be a finite type k-algebra. A reductive linear algebraic group acts algebraically on R if we have a group homomorphism $G \to \mathrm{Aut}(R)$, and every element of R is contained in a finite dimensional G-invariant k vector space on which G acts as a linear action. A classic result, proven by Nagata in [18], is that in this situation the ring of invariants R^G is finitely generated.

Lemma 6.1. *Suppose A and B are finite type k-algebras, G is a reductive linear algebraic group acting algebraically on both A and B, and $A \to B$ is an étale G-homomorphism. If $\mathfrak{m}_A \subset A$ and $\mathfrak{m}_B \subset B$ are G-invariant maximal ideals such that $\mathfrak{m}_A$ is the inverse image of $\mathfrak{m}_B$, then $A^G_{\mathfrak{m}_A} \to B^G_{\mathfrak{m}_B}$ is also étale.*

Remark 6.2. It should be noted that although the action of G on A and B is algebraic, the action of G on the localizations is in general just a set action, although the invariants will still form a ring. For example, the algebraic action of $\mathbb{G}_m$ on $k[t]$ given by $t \mapsto \lambda t$ does not extend to an algebraic action on the localization at the invariant maximal ideal (t).

Proof. Since $A \to B$ is étale, of course $A_{\mathfrak{m}_A} \to B_{\mathfrak{m}_B}$ is as well.

In the case where the residue fields are isomorphic, a map of local rings $A_{\mathfrak{m}_A} \to B_{\mathfrak{m}_B}$ is étale if and only if the map of completion rings $\hat{A}_{\mathfrak{m}_A} \to \hat{B}_{\mathfrak{m}_B}$ is an isomorphism [SP]. Note that $\hat{A}_{\mathfrak{m}_A}$ is the inverse limit of the system $((A_{\mathfrak{m}_A})/\mathfrak{m}_A^n)$, which has a natural action of G since each quotient ring $(A_{\mathfrak{m}_A})/\mathfrak{m}_A^n$ does. Likewise, $\hat{B}$ has a natural action of G.

We claim that taking completions of local rings commutes with taking rings of invariants. This amounts to showing that $(A_{\mathfrak{m}_A}/\mathfrak{m}_A^n)^G = A_{\mathfrak{m}_A}^G/(\mathfrak{m}_A^G)^n$. Equivalently, we want to show that $A_{\mathfrak{m}_A}^G$ surjects onto $(A_{\mathfrak{m}_A}/\mathfrak{m}_A^n)^G$. To see this, observe the following commutative diagram:

$$\begin{array}{ccc} A_{\mathfrak{m}_A}^G & \longrightarrow & (A_{\mathfrak{m}_A}/\mathfrak{m}_A^n)^G \\ \Big\uparrow & & \Big\uparrow \\ A^G & \twoheadrightarrow & (A/\mathfrak{m}_A^n)^G \end{array} \tag{6.1}$$

The right vertical arrow is of course an isomorphism, and the bottom arrow is a surjection since for an algebraic action of a group on a ring, taking invariants is exact. Hence the top arrow is also a surjection, and we've proven that $(\hat{A}_{\mathfrak{m}_A})^G = \widehat{(A_{\mathfrak{m}_A}^G)}$.

We conclude that since $\hat{A}_{\mathfrak{m}_A} \to \hat{B}_{\mathfrak{m}_B}$ is an isomorphism, then $\widehat{A_{\mathfrak{m}_A}^G} \to \widehat{B_{\mathfrak{m}_B}^G}$ is as well, and thus the map of invariant rings $A_{\mathfrak{m}_A}^G \to B_{\mathfrak{m}_B}^G$ is also étale. $\square$

We consider a particular situation of an affine flip, as described by Thaddeus in [25], and construct commuting étale morphisms to a standard flip.

We first state the entire setup needed for our construction. Let k be an algebraically closed field, and let R be an integral finitely generated k-algebra with an algebraic action of $\mathbb{G}_m$. This is equivalent to a $\mathbb{Z}$-grading of $R = \bigoplus_{i \in \mathbb{Z}} R_i$, where $r \in R$ has degree i if it is acted on by $\mathbb{G}_m$ with weight i. Write $U = \mathrm{Spec}(R)$. Let $w \in U$ be a fixed smooth point of the action, and let $\mathfrak{m}_w \subset R$ be the corresponding homogeneous maximal ideal. We assume that $\mathbb{G}_m$ acts on $T_w U$ with weights

$-c$, 0, and c for some constant c. In fact, after regrading, we may assume that it acts on the tangent space with weights -1, 0, and 1.

We now describe the affine flip. Taking the canonical linearization 0 on $\mathcal{O}_U$ of $\mathbb{G}_m$, we have that $U//^0\mathbb{G}_m = \mathrm{Spec}(R^{\mathbb{G}_m}) = \mathrm{Spec}(R_0)$. To define other linearizations on $\mathcal{O}_U$, we need to define different $\mathbb{G}_m$-actions on $R[z]$ which are compatible with the action on R, i.e. $\mathbb{Z}$-gradings on $R[z]$ which agrees with the grading on R. Let the $+$ and $-$ linearizations be given by setting the degree of z equal to -1 and 1, respectively. Then $U//^+\mathbb{G}_m = \mathrm{Proj}((R[z])_0) = \mathrm{Proj}(\bigoplus_{i\in\mathbb{N}} R_i z^i) = \mathrm{Proj}(\bigoplus_{i\geqslant 0} R_i)$ and, similarly, $U//^-\mathbb{G}_m = \mathrm{Proj}(\bigoplus_{i\in\mathbb{N}} R_{-i} z^i) = \mathrm{Proj}(\bigoplus_{i\leqslant 0} R_i)$. By Theorem (1.9) in [25], we have the following commutative diagram, where all the arrows are blow-ups:

$$\begin{array}{ccc} & \tilde{U} & \\ \swarrow & & \searrow \\ U//^+\mathbb{G}_m & & U//^-\mathbb{G}_m \\ \searrow & & \swarrow \\ & U//^0\mathbb{G}_m & \end{array} \tag{6.2}$$

The variety $\tilde{U}$ at the top of the diamond is by definition the blow up of $\mathrm{Proj}(\bigoplus_{i\geqslant 0} R_i)$ at the ideal (R_d), where $d > 0$ is chosen such that $\bigoplus_{i\in\mathbb{Z}} R_{di}$ is generated in degrees 0, $-d$, and d. By Theorem 1.9 in [25], this blow up $\tilde{U}$ is naturally isomorphic to the blow up of $\mathrm{Proj}(\bigoplus_{i\leqslant 0} R_i)$ at the ideal (R_{-d}), and we obtain the above picture.

Now recall we have a $\mathbb{G}_m$-action on the tangent space $T_wU = \mathfrak{m}_w/\mathfrak{m}_w^2$ with weights -1, 0, and 1. Let $f_1, \ldots, f_{s_{-1}} \in R$ be a lift of a basis for the weight -1 part, and similarly define $g_1, \ldots, g_{s_0}$ as a lift for the 0 part, and $h_1, \ldots, h_{s_1}$ as a lift for the 1 part. These elements determine a morphism $\phi : U \to \mathbb{A}^{s_{-1}+s_0+s_1}$ which is étale on a neighborhood of w, so without loss of generality we may assume that ϕ is étale on U.

Define a $\mathbb{G}_m$ action on $\mathbb{A}^{s_{-1}+s_0+s_1} = \mathrm{Spec}(k[x_1, \ldots, x_{s_{-1}}, y_1, \ldots, y_{s_0}, z_1, \ldots, z_{s_1}])$ by $\lambda \cdot x_i = \lambda^{-1} x_i$, $\lambda \cdot y_i = y_i$, and $\lambda \cdot z_i = \lambda z_i$. Then ϕ is a $\mathbb{G}_m$-map, and we have a morphism $\phi_0 : U//^0\mathbb{G}_m \to \mathbb{A}^{s_{-1}+s_0+s_1}//^0\mathbb{G}_m$, where the GIT quotient of $\mathbb{A}^{s_{-1}+s_0+s_1}$ is taken with trivial linearization, i.e. $\mathbb{A}^{s_{-1}+s_0+s_1}//^0\mathbb{G}_m = \mathrm{Spec}(k[y_i])$. Lemma 6.1 tells us that ϕ_0 is also étale in a neighborhood of $\bar{w} \in U//^0\mathbb{G}_m$. Again replacing U by a smaller invariant affine open, we conclude that ϕ_0 is étale on the full domain $U//^0\mathbb{G}_m$.

Now consider the two different GIT quotients of the affine space $\mathbb{A} := \mathbb{A}^{s_{-1}+s_0+s_1}$ given by the $\pm$ linearizations, i.e. $\mathbb{A}//^+\mathbb{G}_m = \mathrm{Proj}(k[y_i, x_i z_j, z_i])$ and $\mathbb{A}//^-\mathbb{G}_m = \mathrm{Proj}(k[x_i, y_i, x_i z_j])$. There are natural maps $\mathbb{A}//^+\mathbb{G}_m \to \mathbb{A}//^0\mathbb{G}_m$ and $\mathbb{A}//^-\mathbb{G}_m \to \mathbb{A}//^0\mathbb{G}_m$. Note that, by symmetry, blowing up $\mathbb{A}//^+\mathbb{G}_m$ and $\mathbb{A}//^-\mathbb{G}_m$ at the ideals generated by $(x_i z_j)$ in both $k[y_i, x_i z_j, z_i]$ and $k[x_i, y_i, x_i z_j]$ results in isomorphic varieties, which we call $\tilde{\mathbb{A}}$. This $\tilde{\mathbb{A}}$ is identical to the one constructed as above in diagram (6.2) for a more general ring R.

Our next goal is to construct étale morphisms $\phi_\pm : U//^\pm\mathbb{G}_m \to \mathbb{A}//^\pm\mathbb{G}_m$ which commute with ϕ_0 and the contractions. We define ϕ_+ as the map associated to the obvious homomorphism of graded rings $k[y_i, x_i z_j, z_i] \to \bigoplus_{i \geqslant 0} R_i$ given by $y_i \mapsto g_i$, $x_i z_j \mapsto f_i h_j$, and $z_i \mapsto h_i$. Define ϕ_- analogously.

We need to show ϕ_+ and ϕ_- are well-defined on the entire domain, i.e. that they are morphisms, and of course we also need to show that they are étale. To do this, we first shrink U so that $V(h_1, \ldots, h_{s_1})$ and $V(f_1, \ldots, f_{s_{-1}})$ are irreducible. Since U is smooth at w, at most one irreducible component of each closed set goes through w, so this operation does not change the local geometry at w and we can assume $V(h_1, \ldots, h_{s_1})$ and $V(f_1, \ldots, f_{s-1})$ are irreducible.

Lemma 6.3. *The rational maps between quotients* $\phi_+ : U//^+\mathbb{G}_m \to \mathbb{A}//^+\mathbb{G}_m$ *and* $\phi_- : U//^-\mathbb{G}_m \to \mathbb{A}//^-\mathbb{G}_m$ *are in fact well defined on the whole domain.*

Proof. We prove this only for ϕ_+, as of course the proof is symmetric for ϕ_-.

The rational map ϕ_+ between Proj's comes from the homogeneous map of graded rings $k[y_i, x_iz_j, z_i] \to \bigoplus_{i \geqslant 0} R_i$ defined naturally as above. Suppose now that ϕ_+ fails to be defined at a proper homogeneous prime ideal $\mathfrak{p} \subset \bigoplus_{i \geqslant 0} R_i$. Then the inverse image of $\mathfrak{p}$ under the ring map must contain the irrelevant ideal of $k[y_i, x_iz_j, z_i]$, i.e. $\mathfrak{p}$ contains all the h_i. Furthermore, $\mathfrak{p}$ itself is not irrelevant, so there exists a positive degree element $a \in \bigoplus_{i>0} R_i$ such that $a \notin \mathfrak{p}$. We'll show this final deduction is in fact impossible.

Observe that because w is a smooth point, the local completion $\hat{\mathcal{O}}_{U,w}$ is isomorphic to $k[[x_i, y_i, z_i]]$, where the x_i are degree -1, the y_i are degree 0, and the z_i are degree 1. In particular, the z_i generate the positive degree elements in the completion ring, so the image of a in $\hat{\mathcal{O}}_{U,w}$ is in the ideal generated by the z_i. Now, since the h_i correspond to the z_i in $\hat{\mathcal{O}}_{U,w}$, our element a is also zero in the local completion $\hat{\mathcal{O}}_{V(h_1,\ldots,h_{s_1}),w}$. But $\mathcal{O}_{V(h_1,\ldots,h_{s_1}),w}$ is a subring of $\hat{\mathcal{O}}_{V(h_1,\ldots,h_{s_1}),w}$, so a is also zero in the stalk, and in particular $V(a) \supset V(h_1, \ldots, h_{s_1})$ after again shrinking U to a small neighborhood of w. Since $V(h_1, \ldots, h_{s_1})$ is irreducible, by Nullstellensatz $a^n \in (h_1, \ldots, h_{s_1}) \subset \mathfrak{p}$ for some $n \in \mathbb{N}$. On the other hand, $\mathfrak{p}$ is prime so $a \in \mathfrak{p}$, a contradiction.

□

Lemma 6.4. *The morphisms* ϕ_+ *and* ϕ_- *are étale morphisms.*

Proof. We prove this only for $\phi_+ : U//^+\mathbb{G}_m \to \mathbb{A}//^+\mathbb{G}_m$, as again the proof is symmetric for ϕ_-. The tactic is to show this on a principal affine cover of $U//^+\mathbb{G}_m$. Recall that the morphism ϕ_+ comes from the homogeneous map of graded rings $k[y_i, x_iz_j, z_i] \to \bigoplus_{i\geqslant 0} R_i$ as defined above. Note that the ring $k[y_i, x_iz_j, z_i]$ is the subring of non-negative degree elements in $k[x_i, y_i, z_i]$.

By construction, the morphism $\phi : U \to \mathbb{A}$ is étale, so the associated (graded) map of rings $k[x_i, y_i, z_i] \to \bigoplus_{i\in\mathbb{Z}} R_i$ is étale. Let $a \in k[x_i, y_i, z_i]$ be a positive degree homogeneous element, and let $a_R \in R$ be the image of a in $R = \bigoplus_{i\in\mathbb{Z}} R_i$. Then the localized map $k[x_i, y_i, z_i]_a \to R_{a_R}$ is also étale.

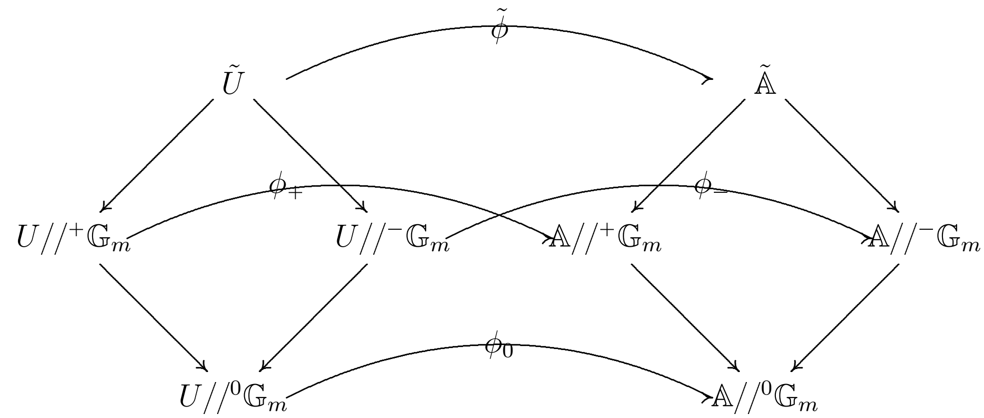
Note that $(\bigoplus_{i\in\mathbb{Z}} R_i)_{a_R} = (\bigoplus_{i\geqslant 0} R_i)_{a_R}$, since for b of negative degree, we have that $\frac{b}{a^i} = \frac{ba^j}{a^{i+j}} \in \bigoplus_{i\geqslant 0} R_i$ for large enough j. Similarly, $k[x_i, y_i, z_i]_a = k[y_i, x_iz_j, z_i]_a$. Thus we can deduce that the graded morphism $k[y_i, x_iz_j, z_i]_a \to (\bigoplus_{i\geqslant 0} R_i)_{a_R}$ is étale.

Now applying the algebra Lemma 6.1, and noting that taking quotient rings in this case amounts to taking degree zero components, we see that the map between homogeneous localizations $k[y_i, x_iz_j, z_i]_{(a)} \to (\bigoplus_{i\geqslant 0} R_i)_{(a_R)}$ is also étale. But as a varies, the open sets $\mathrm{Spec}(k[y_i, x_iz_j, z_i]_{(a)})$ cover $U//^+\mathbb{G}_m$. Hence $\phi_+ : U//^+\mathbb{G}_m \to \mathbb{A}//^+\mathbb{G}_m$ is an étale morphism. □

Since $\tilde{U}$ and $\tilde{\mathbb{A}}$ are defined by blowing up smooth loci which correspond to one another via ϕ_+ and ϕ_-, we also obtain an étale map $\tilde{\phi} : \tilde{U} \to \tilde{\mathbb{A}}$.

It is clear by the construction that we now have the following commutative diagram from a general affine flip to the standard affine flip, where all the curved arrows are étale.:

(6.3)



Remark 6.5. Observe that there are canonical line bundles (these are the sheaves denoted $\mathcal{O}(1)$) on $\mathbb{A}//^+\mathbb{G}_m$ and $U//^+\mathbb{G}_m$ stemming from the linearizations used to form the GIT quotients, and in fact ϕ_+ pulls the first back to the second. Of course the analogous statements hold for ϕ_- and ϕ_0.

7. General Case

We use the previous sections to tackle a more general case of Proposition 3.1. In particular, we prove Theorem 1.1.

Fix C a smooth projective curve of arbitrary genus g over an algebraically closed field k.

Recall from [24] that for $\sigma \in \mathbb{R}$, a σ-semistable pair is a pair (E, ϕ), where E is a rank two vector bundle over C and $\phi \in H^0(E)$ is a nonzero section, satisfying the following two inequalities for all line bundles $L \subseteq E$:

$$\deg L \leqslant \frac{1}{2}\deg E - \sigma \text{ if } \phi \in H^0(L) \tag{7.1}$$

$$\deg L \leqslant \frac{1}{2}\deg E + \sigma \text{ if } \phi \notin H^0(L) \tag{7.2}$$

Let $M(\sigma)$ denote the moduli space of σ-semistable pairs where the bundle has fixed determinant Λ of degree d, as constructed in [24]. Note that in order for the moduli space to be nonempty, we must have $\sigma \leqslant d/2$ since otherwise the first condition stipulates $\deg(L) < 0$ for L with a nonzero global section ϕ. In fact, we can explicitly understand $M(\sigma)$ when σ is close to $d/2$ as follows.

Lemma 7.1. *For genus $g > 1$ and ϵ small, the moduli space $M(d/2-\epsilon)$ is isomorphic to $\mathbb{P}\mathrm{Ext}^1(\Lambda, \mathcal{O}_{\mathcal{C}}) \cong \mathbb{P}H^1(\Lambda^{-1})$.*

Proof. When $\sigma = d/2 - \epsilon$, the semistability conditions become

$$\deg L \leqslant 0 \text{ if } \phi \in H^0(L) \tag{7.3}$$

$$\deg L < d \text{ if } \phi \notin H^0(L) \tag{7.4}$$

This immediately implies that when $\phi \in H^0(L)$, we have $L = \mathcal{O}_{\mathcal{C}}$.

If E is a nonsplit extension $0 \to \mathcal{O}_{\mathcal{C}} \to E \to \Lambda \to 0$, and $\phi \in H^0(E)$ is given by the map $\mathcal{O}_{\mathcal{C}} \to E$, then we wish to show that E obeys the above inequalities. The first is trivial since in this case $L = \mathcal{O}_{\mathcal{C}}$ for $\phi \in H^0(L)$, so $\deg L = 0$. On the other hand, suppose M is a line bundle of degree at least d not containing ϕ as a section. Then we know that the composition $M \to E \to \Lambda$, where the right map is as in the extension, is nonzero since $\phi \notin H^0(M)$. So $\mathrm{Hom}(M, \Lambda) = H^0(\Lambda \otimes M^{-1})$ is nonzero, and hence $\deg M = d$ and $M = \Lambda$. But E is a nonsplit extension of Λ by $\mathcal{O}_{\mathcal{C}}$, so we have arrived at a contradiction.

Conversely, if E obeys both stability conditions, then the line bundle associated to ϕ is $L = \mathcal{O}_{\mathcal{C}}$. Consider the short exact sequence

$$0 \to L \to E \to Q \to 0 \tag{7.5}$$

where Q is the quotient line bundle. Taking exterior powers, we see the isomorphism $\Lambda \cong L \otimes Q$, i.e. we have $Q \cong \Lambda L^{-1} = \Lambda$. So it is immediate that E is an extension of Λ by $\mathcal{O}_{\mathcal{C}}$ and that ϕ gives the map $L \to E$. Finally, the extension cannot be split because if it were then Λ would be a subbundle of E, but Λ is a line bundle of degree d whose global sections do not include ϕ, a violation of the second inequality. □

On the other end of the spectrum, when $\sigma = 0$, the inequalities reduce to the usual semistability for vector bundles on a curve.

Henceforth we assume $\sigma \in [0, d/2)$. Now, we briefly recall the contruction of $M(\sigma)$, as done by Thaddeus in [24] and [25].

Let **Quot** be the Grothendieck Quot Scheme parametrizing flat quotients of $\mathcal{O}_C^{\chi}$ of degree d. Define $\mathbf{Quot}_{\boldsymbol{\Lambda}}$ to be the subset of rank 2 locally free quotients with determinant Λ. Note that being locally free is an open condition on **Quot** since it behaves well in families. If we consider the determinant map from this locus to $\mathrm{Pic}(C)$, we can also deduce that the sublocus with determinant Λ is closed. Finally, rank is an invariant of the Hilbert polynomial of a bundle and since **Quot** is stratified by Hilbert polynomials, we can conclude that $\mathbf{Quot}_{\boldsymbol{\Lambda}}$ is locally closed in **Quot**. Denote temporarily by V the open set of $\mathbf{Quot}_{\boldsymbol{\Lambda}}$ consisting of all quotients E inducing isomorphisms $k^{\chi} \to H^0(E)$. There is a natural action of $\mathrm{SL}(\chi)$ on V given by the action on the quotient maps $\mathcal{O}_C^{\chi} \twoheadrightarrow E$, and this extends to an action on $V \times \mathbb{P}\mathbb{C}^{\chi}$.

Lemma 7.2. *Let σ be in the interval $[0, d/2)$. There exists a compactification $\mathbf{Q}$ of $V \times \mathbb{P}\mathbb{C}^{\chi}$ and an ample linearization $\mathcal{L}$ such that GIT stability with respect to $\mathcal{L}$ is equivalent to σ stability as defined above. In fact, the linearization $\mathcal{L}$ is a power of $\mathcal{L}_1^{\chi+2\sigma} \otimes \mathcal{L}_2^{4\sigma}$, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are two fixed line bundles on $\mathbf{Q}$.*

Proof. First, we can assume $\deg\Lambda = d \gg \sigma$. For σ-semistable E, this guarantees that E is globally generated and $\chi := \chi(E) = h^0(E)$. The moduli spaces for d small will be contained in these since if (E, ϕ) is σ-semistable, then so is $(E(D), \phi(D))$ for any effective divisor D (see (1.9) in [24]). Furthermore, this gives an embedding of the moduli space of σ-semistable pairs with determinant Λ into the moduli space of σ-semistable pairs with determinant $\Lambda(D)$.

By [25], for d large enough we have an immersion

$$T : V \times \mathbb{P}\mathbb{C}^\chi \longrightarrow \mathbb{P}\mathrm{Hom}(\bigwedge^2 \mathbb{C}^\chi, H^0(\Lambda)) \times \mathbb{P}\mathbb{C}^\chi. \tag{7.6}$$

In [24], Thaddeus proves that σ semistability for bundles embedded by T is equivalent to GIT semistability in $\mathbb{P}\mathrm{Hom} \times \mathbb{P}\mathbb{C}^\chi$ with respect to the linearization given by a power of $\mathcal{O}(\chi + 2\sigma, 4\sigma)$. So defining $\mathbf{Q}$ to be the closure of $V \times \mathbb{P}\mathbb{C}^\chi$ in $\mathbb{P}\mathrm{Hom} \times \mathbb{P}\mathbb{C}^\chi$ and letting $\mathcal{L}$ be the restriction of a power of $\mathcal{O}(\chi + 2\sigma, 4\sigma)$, we deduce the conclusions of the Lemma. $\square$

It is shown in [25] that thus defined $\mathbf{Q}//\mathrm{SL}(\chi) = M(\sigma)$. We also note that by [24], $M(\sigma)$ is smooth outside of its strictly semistable points.

Observe that the stability condition changes precisely when σ crosses $\frac{d}{2} + \mathbb{Z}$, and in fact for numerical reasons, there only exist strictly semistable points when $\sigma \in \frac{d}{2} + \mathbb{Z}$. Fix $\sigma \in \frac{d}{2} + \mathbb{Z}$ and define $\sigma^- = \sigma - \epsilon$ and $\sigma^+ = \sigma + \epsilon$ for ϵ small.

Blowing up the preimages in $M(\sigma^\pm)$ of the singular locus in $M(\sigma)$, again by [24], we have the following commutative diagram:

$$
\begin{array}{ccccc}
 & & \tilde{M} & & \\
 & \overset{f}{\swarrow} & & \overset{g}{\searrow} & \\
M(\sigma^+) & & & & M(\sigma^-) \\
 & \searrow & & \swarrow & \\
 & & M(\sigma) & &
\end{array} \tag{7.7}
$$

Lemma 7.3. *Suppose that $g > 1$, $d \geqslant 3$, and $\sigma \in d/2 + \mathbb{Z}$. If $(E, \phi) \in \mathbf{Q}$ is strictly σ-semistable with closed orbit, then the pair splits as $(L \oplus M, (\phi, 0))$.*

Proof. As in the proof of Lemma 7.2, we may assume that $d \gg 0$, and so E is globally generated, and $\chi(E) = h^0(E)$, and we can work in $V \subseteq \mathbf{Quot}$.

By general GIT, we know that since (E, ϕ) is strictly semistable with closed orbit, it has nontrivial stabilizer in $\mathrm{SL}(\chi)$. This is equivalent to the pair (E, ϕ) having nontrivial automorphisms. So let T be a nontrivial automorphism of E fixing ϕ.

Defining L as the sub line bundle generated by ϕ, we have as in the above Lemma 7.1,

$$0 \to L \to E \to \Lambda \otimes L^{-1} \to 0 \tag{7.8}$$

Now, ϕ is in the kernel of $T - \mathrm{id_E}$, so L is as well. Thus the automorphism $T - \mathrm{id_E}$ descends to a morphism $E/L = \Lambda \otimes L^{-1} \to E$. This map must be nonzero, since otherwise $T - \mathrm{id_E}$ is zero everywhere, which contradicts T being a nontrivial automorphism of E. So $\Lambda \otimes L^{-1} \to E$ is nonzero, and we have a splitting $E = L \oplus (\Lambda \otimes L^{-1})$. Furthermore the now sub bundle $\Lambda \otimes L^{-1}$ has degree $d - \mathrm{degL} = \sigma + \mathrm{d}/2$, implying it is a sub bundle satisfying the second of the above equalities.

Note that since (E,ϕ) splits as $L \oplus M$ where $\phi \in H^0(L)$, then $\mathrm{degL} + \mathrm{degM} = \mathrm{d}$ and for σ determined by the degree of L, we have that

$$\mathrm{deg}L = \frac{1}{2}\mathrm{deg}E - \sigma \text{ and } \phi \in H^0(L) \tag{7.9}$$

$$\mathrm{deg}M = \frac{1}{2}\mathrm{deg}E + \sigma \text{ and } \phi \notin H^0(M) \tag{7.10}$$

Hence if (E,ϕ) is in a closed orbit, then both equalities in the semistability conditions are attained.

□

Remark 7.4. In fact, the strictly semistable locus has three strata. The points with closed orbit are split as $L \oplus M$, where L fails the first condition and M fails the second. The other two strata come from the nonzero extensions in $\mathrm{Ext}^1(M,L)$, which fail just the first inequality, and the nonzero extensions in $\mathrm{Ext}^1(L,M)$ with a lift $\tilde{\phi}$ of $\phi \in H^0(L)$, which fail just the second inequality. The intersection of these strata is exactly comprised of the split bundles failing both.

Lemma 7.5. *The stabilizer of a strictly σ-semistable point with closed orbit in* $\mathbf{Q}$ *is* $\mathbb{G}_m \subset \mathrm{SL}(\chi)$.

Proof. As in the proof of Lemma 7.2, we may assume that $d \gg 0$, and so E is globally generated, and $\chi(E) = h^0(E)$, and we can work in $V \subseteq \mathbf{Quot}$.

By Lemma 7.3, if E is a strictly σ-semistable vector bundle in a closed orbit, then it splits as $L \oplus M$ with $\phi \in H^0(L)$.

Elements of $\mathrm{SL}(\chi)$ stabilizing $(L \oplus M, (\phi, 0))$ are in one-to-one correspondence with automorphisms of $L \oplus M$ which fix ϕ. Since $\phi \in H^0(L)$, this is equivalent to homomorphisms from M to $L \oplus M$ inducing an automorphism of the vector bundle. But

$$\text{(7.11)} \qquad \mathrm{Hom}(M, L \oplus M) = \mathrm{Hom}(M, L) \oplus \mathrm{Hom}(M, M) = \mathrm{Hom}(M, M) = \mathbb{C}$$

since $\deg(L) - \deg(M) = -2\sigma < 0$. Of course excluding the zero homomorphism, we conclude that the stabilizer is $k^\times = \mathbb{G}_m$. □

Lemma 7.6. *Let $q \in \mathbf{Q}$ be a split strictly σ-semistable point. Then the stabilizer $\mathbb{G}_m$ acts on the tangent space $T_q\mathbf{Q}$ with weights $-c$, 0, and c for some nonzero constant c.*

Proof. As in the proof of Lemma 7.2, we may assume that $d \gg 0$, and so E is globally generated, and $\chi(E) = h^0(E)$, and we can work in $V \subseteq \mathbf{Quot}$.

Again, if E is a strictly σ-semistable vector bundle with closed orbit in $\mathbf{Q}$, then E splits as $(L \oplus M, (\phi, 0))$. The tangent space $T_q\mathbf{Q}= T_E\mathbf{Quot}_\Lambda \times T_\phi\mathbb{P}\mathbb{C}^\chi$ and our action respects this product, so we deal with each factor individually.

First observe that matrices in the stabilizer $\mathbb{G}_m \subset \mathrm{SL}(\chi)$ correspond to block matrices g acting as $g\psi = z^a\psi$ for a section $\psi \in H^0(L)$ and $g\psi = z^b\psi$ for a section $\psi \in H^0(M)$, where a and b are fixed nonzero integers satisfying the equation $ah^0(L) + bh^0(M) = 0$ (to guarantee the matrix has determinant 1) and z parametrizes the group $\mathbb{G}_m$.

Recall that if our strictly σ-semistable bundle E corresponds to the exact sequence

$$\text{(7.12)} \qquad 0 \to K \to \mathcal{O}_C^\chi \to E \to 0$$

then the tangent space T_E $\mathbf{Quot}_\Lambda$= Hom(K, E).

Since the surjection $\mathcal{O}_C^\chi \to E$ is determined by global sections and E splits, the short exact sequence 7.12 itself splits as:

$$0 \to K_L \to H^0(L) \otimes \mathcal{O}_C \to L \to 0$$

$$\oplus \tag{7.13}$$

$$0 \to K_M \to H^0(M) \otimes \mathcal{O}_C \to M \to 0$$

In particular, the kernel $K = K_L \oplus K_M$ splits.

Returning to the tangent space, we then have the weight decomposition

$$T_E\mathbf{Quot}_\Lambda = \mathrm{Hom}(K_L, L) \oplus \mathrm{Hom}(K_M, M) \oplus \mathrm{Hom}(K_L, M) \oplus \mathrm{Hom}(K_M, L) \tag{7.14}$$

Considering the matrix structure outlined above, we see that $\mathbb{G}_m$ acts on $\mathrm{Hom}(K_L, L)$ and $\mathrm{Hom}(K_M, M)$ with weight 0, on $\mathrm{Hom}(K_L, M)$ with weight $b - a$, and on $\mathrm{Hom}(K_M, L)$ with weight $a - b$.

Finally, recall that the tangent space $T_\phi\mathbb{P}\mathbb{C}^\chi = \mathrm{Hom}(\mathbb{C}\phi, \mathbb{C}^\chi/\mathbb{C}\phi)$. The vector space $\mathbb{C}^\chi/\mathbb{C}\phi = H^0(E)/\mathbb{C}\phi$ naturally splits as $H^0(L)/\mathbb{C}\phi \oplus H^0(M)$. So

$$T_\phi\mathbb{P}\mathbb{C}^\chi = \mathrm{Hom}(\mathbb{C}\phi, H^0(L)/\mathbb{C}\phi) \oplus \mathrm{Hom}(\mathbb{C}\phi, H^0(M)) \tag{7.15}$$

and we see that $\mathbb{G}_m$ acts on $T_\phi\mathbb{P}\mathbb{C}^\chi$ with weights 0 and $b - a$. □

Theorem 7.7. *For $\sigma \in \{\frac{d}{2} + \mathbb{Z}\} \cap [0, \frac{d}{2}]$ (to ensure the moduli are non-empty) and $\frac{d-1}{2} - g \leqslant 3\sigma$, the functor $Rg_* \circ Lf^* : \mathcal{D}^b(M(\sigma^+)) \longrightarrow \mathcal{D}^b(M(\sigma^-))$ is fully faithful.*

Proof. To simplify the notation, we assume all pullbacks and pushforwards are derived, and omit the L's and R's. Furthermore, we fix σ and write $M = M(\sigma)$, $M^+ = M(\sigma^+)$, and $M^- = M(\sigma^-)$.

The idea is to show that the above commutative diagram 7.7 is étale-locally isomorphic to a product of the standard flip with a trivial part, and in this situation the result follows from how derived categories behave under standard flips.

Recall that the right adjoint of a pullback h^* between projective varieties is the shriek functor $h^!$ defined by $h^!(\mathcal{E}^\bullet) = h^*\mathcal{E}^\bullet \otimes^{\mathbb{L}} \omega_h$, and so the right adjoint of g_*f^* is $f_*g^!$. Thus it suffices to show that the natural transformation of functors $id \to f_*g^!g_*f^*$ is an isomorphism. By a result of Kawamata [14], it is enough to prove that for all powers $\mathcal{L}$ of a fixed ample line bundle, the canonical maps $\mathcal{L} \to f_*g^!g_*f^*\mathcal{L}$ are isomorphisms. We will do this étale-locally.

Note that away from the singular locus of M, f and g are isomorphisms, so there is nothing to check. Hence we restrict our attention to finding etale neighborhoods of the singular locus in M, over which we will show the diagram essentially simplifies to a standard flip.

The first step is to view M, M^+, and M^- as GIT quotients of a single variety acted on by $\mathbb{G}_m$. For this we employ the trick outlined in Section 5 with respect to the ample linearizations L^0, L^+, and L^- on $\mathbf{Q}$, defined as in the statement of Lemma 7.2, corresponding to M, M^+, and M^-. Hence we obtain a projective variety W with line bundle L and three linearizations on L, denoted by zero, plus, and minus, such that $W//^0\mathbb{G}_m = M$, $W//^+\mathbb{G}_m = M^+$, and $W//^-\mathbb{G}_m = M^-$.

Let $q \in \mathbf{Q}$ be a lift of a singular point in M corresponding to a strictly σ-semistable vector bundle over C with closed orbit. Define $v \in V = \mathbb{P}(L^+ \oplus L^-)$

to be the point in the fiber over q corresponding to $(1:1)$, and let $w \in W$ be the image of v.

We claim w is a fixed point for the $\mathbb{G}_m$ action on W. Since $q \in \mathbf{Q}$ is split strictly semistable, it has stabilizer $S = \mathbb{G}_m \subset \mathrm{SL}(\chi)$ (Lemma 7.5). Recall from the proof of Lemma 7.6 that S acts on the sections of $q = L \oplus M$ with weight space decomposition $H^0(L) \oplus H^0(M)$ and the weights are of opposite sign, since our matrices live in $\mathrm{SL}(\chi)$. Furthermore, by Remark 7.4, q is at the intersection of $\mathrm{Ext}^1(L, M)$ and $\mathrm{Ext}^1(M, L)$, and in fact note that all the bundles in $\mathrm{Ext}^1(L, M)$ are σ^--stable since the violating subbundle L obeys a strict inequality after replacing σ by σ^-, and similarly those in $\mathrm{Ext}^1(M, L)$ are σ^+-semistable. The stabilizer $S = \mathbb{G}_m$ acts on these two Ext groups by pushing towards and pulling away from q. Hence the group $S \subset \mathrm{SL}(\chi)$ acts on the fiber over q by pushing or pulling from the endpoints determined by L^+ and L^-. So after quotienting by $\mathrm{SL}(\chi)$, the image of v will be fixed by the action of $\mathbb{G}_m$ on W.

Furthermore, w is a smooth point of W. The variety W arises as the quotient of V by $\mathrm{SL}(\chi)$ and w is the image of v under this quotient. But since $S \subset \mathrm{SL}(\chi)$ moves v, the point v has trivial stabilizer in $\mathrm{SL}(\chi)$, and hence its image point w is smooth in the quotient W.

Next we claim that T_wW is acted on by $\mathbb{G}_m$ with weights $-c$, 0, and c where c is as in Lemma 7.6. To see this, consider the following maps:

$$\begin{array}{ccc} T_v\mathbb{P}(L^+ \oplus L^-) & \twoheadrightarrow & T_wW \\ \downarrow & \overset{\exists\, \phi}{\nearrow} & \\ T_q\mathbf{Q} & & \end{array} \tag{7.16}$$

The top arrow is a surjection since the quotient $\mathbb{P}(L^+ \oplus L^-) \to W$ by $\mathrm{SL}(\chi)$ is smooth at $w \in W$. The left arrow comes from a $\mathbb{P}^1$-bundle map and the stabilizer $\mathbb{G}_m$ of q in $\mathrm{SL}(\chi)$ acts nontrivially on the fiber over q, so that the line collapses under the quotient to W. This ensures that there exists ϕ as in the picture making the diagram commute. In particular, ϕ is a surjective $\mathbb{G}_m$ map. Since by Lemma 7.6 $\mathbb{G}_m$ acts on $T_q\mathbf{Q}$ with weights $-c$, 0, and c, we conclude that it acts on T_wW with these same weights.

Fix $\mathcal{L}$ on M^+ to be any integral power of the ample line bundle underlying the linearization coming from our GIT quotients of W. It suffices to show the isomorphisms $\mathcal{L} \to f_*g^!g_*f^*\mathcal{L}$ on any neighborhood of w^+, where w^+ is the image of w in $M^+ = W//^+\mathbb{G}_m$. We now construct étale morphisms from the commutative diagram 7.7, restricted to a neighborhood of w, to the diagram of a flip.

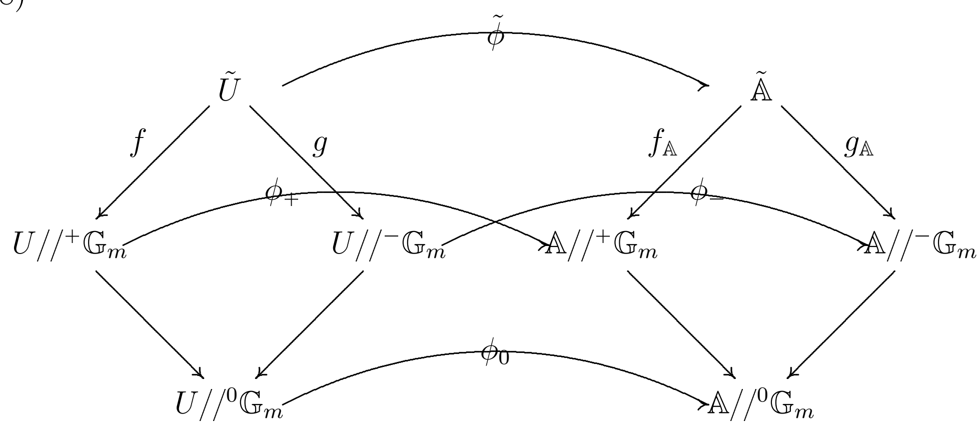
By (3.2) in [25], there is a $\mathbb{G}_m$-invariant affine $w \in U \subseteq W$ so that we have the following commutative diagram, where the horizontal arrows are open immersions:

$$\begin{array}{ccc} U//^{\pm}\mathbb{G}_m & \hookrightarrow & W//^{\pm}\mathbb{G}_m \\ \downarrow & & \downarrow \\ U//^{0}\mathbb{G}_m & \hookrightarrow & W//^{0}\mathbb{G}_m \end{array} \tag{7.17}$$

Write $U = \mathrm{Spec}(R)$, where $R = \bigoplus_{i\in\mathbb{Z}} R_i$ is the grading induced by the $\mathbb{G}_m$-action. Then, again by results in [25], $U//^0\mathbb{G}_m = \mathrm{Spec}(R_0)$, $U//^+\mathbb{G}_m = \mathrm{Proj}(\bigoplus_{i\geqslant 0} R_i)$, and $U//^-\mathbb{G}_m = \mathrm{Proj}(\bigoplus_{i\leqslant 0} R_i)$. Furthermore, since w is a fixed point of the $\mathbb{G}_m$-action corresponding to a split strictly semistable point, by Lemma 7.6 we have an induced action on the tangent space R_w with weights $-c$, 0, and c. After regrading, we can assume it has weights -1, 0, and 1.

Hence we can use Section 6 and construct a commutative diagram as follows

(7.18)



,

where $\tilde{U}$ and the $\mathbb{A}$ notation is defined as in Section 6, and all the curved arrows are étale. We write $f_{\mathbb{A}}$ and $g_{\mathbb{A}}$ for the top two blow-ups in the right diamond. By Remark 6.5, there exists a line bundle $\mathcal{M}$ on $\mathbb{A}//^+\mathbb{G}_m$, which is a power of the relative sheaf $\mathcal{O}(1)$, such that $\phi_+^*\mathcal{M} = \mathcal{L}$.

We claim that it suffices to show that the canonical map $\mathcal{M} \to f_{\mathbb{A}*}g_{\mathbb{A}}^!g_{\mathbb{A}*}f_{\mathbb{A}}^*\mathcal{M}$ is an isomorphism. If so, then $\phi_+^*\mathcal{M} \to \phi_+^*f_{\mathbb{A}*}g_{\mathbb{A}}^!g_{\mathbb{A}*}f_{\mathbb{A}}^*\mathcal{M}$ is as well. We want to show that we can shift the ϕ through the expression on the right side of the arrow, so that in fact this pulled back adjunction map is the same as the canonical map $\mathcal{L} \to f_*g^!g_*f^*\mathcal{L}$. Certainly, we immediately have that $\phi_+^*\mathcal{M} = \mathcal{L}$.

Since ϕ_+ is étale, it's flat, and hence $\tilde{\mathbb{A}}$ and $U//^+\mathbb{G}_m$ are Tor-independent over $\mathbb{A}//^+\mathbb{G}_m$. Thus we can apply [23, Tag 08IB] to deduce that $\phi_+^*f_{\mathbb{A}*} = f_*\tilde{\phi}^*$.

Recall that $g_{\mathbb{A}}^!(\mathcal{E}) = g_{\mathbb{A}}^*(\mathcal{E}) \otimes \omega_{g_{\mathbb{A}}}$ where $\omega_{g_{\mathbb{A}}} = \omega_{\tilde{\mathbb{A}}} \otimes g_{\mathbb{A}}^*\omega_{\mathbb{A}//^-\mathbb{G}_m}^\vee$. In general, if we have an étale map $\psi : X \to Y$, then $\Omega_{X/Y} = 0$, and so (e.g from [11]) we have a surjection $\psi * \Omega_{Y/k} \to \Omega_{X/k}$. But these bundles are of the same dimension, so in fact we obtain an isomorphism, implying that, after taking highest powers, ψ pulls back ω_Y to ω_X. So, applying this in our case, we see that

(7.19)

$$\tilde{\phi}^*(\omega_{g_{\mathbb{A}}}) = \tilde{\phi}^*(\omega_{\tilde{\mathbb{A}}} \otimes g_{\mathbb{A}}^* \omega_{\mathbb{A}//^{-}\mathbb{G}_m}^\vee) = \tilde{\phi}^* \omega_{\tilde{\mathbb{A}}} \otimes \tilde{\phi}^* g_{\mathbb{A}}^* \omega_{\mathbb{A}//^{-}\mathbb{G}_m}^\vee = \omega_{\tilde{U}} \otimes g^* \phi_-^* \omega_{\mathbb{A}//^{-}\mathbb{G}_m}^\vee = \omega_g,$$

i.e. $\tilde{\phi}$ pulls relative canonical sheaf back to relative canonical sheaf. We can use this and the commutativity of the above diagram to show

$$\tilde{\phi}^* g_{\mathbb{A}}^!(\mathcal{E}) = \tilde{\phi}^*(g_{\mathbb{A}}^* \mathcal{E} \otimes \omega_{g_{\mathbb{A}}}) = g^* \phi_-^* \mathcal{E} \otimes \omega_g = g^! \phi_-^*(\mathcal{E}). \tag{7.20}$$

The identical Tor-independence argument done for f gives us the equality of functors $\phi_{-\,*} g_{\mathbb{A}*} = g_* \tilde{\phi}^*$.

Finally, the equation $\tilde{\phi}^* f_{\mathbb{A}}^* = f^* \phi_+^*$ is a direct consequence of the commutativity of the above diagram.

Combining all these equalities in order, we conclude that the pulled back morphism $\phi_+^* \mathcal{M} \to \phi_+^* f_{\mathbb{A}*} g_{\mathbb{A}}^! g_{\mathbb{A}*} f_{\mathbb{A}}^* \mathcal{M}$ is in fact the canonical map $\mathcal{L} \to f_* g^! g_* f^* \mathcal{L}$ derived from adjunction. Thus, it suffices to show that $\mathcal{M} \to f_{\mathbb{A}*} g_{\mathbb{A}}^! g_{\mathbb{A}*} f_{\mathbb{A}}^* \mathcal{M}$ is an isomorphism.

To prove $g_{\mathbb{A}*} f_{\mathbb{A}}^*$ is fully faithful, we use Proposition 4.2, which requires a closed cover of $\mathbb{A}//^0\mathbb{G}_m$ obeying certain Tor-independence conditions. Recall from Section 6 that $\mathbb{A} = \operatorname{Spec}(k[x_1, \ldots, x_{s-1}, y_1, \ldots, y_{s_0}, z_1, \ldots, z_{s_1}])$ and that $\mathbb{G}_m$ acts on $\mathbb{A}$ with weight -1 on x_i, weight 0 on y_i, and weight 1 on z_i. Thus the quotient $\mathbb{A}//^0\mathbb{G}_m = \operatorname{Spec}(k[y_i, x_i z_j]) = \mathbb{A}^{s_0} \times \operatorname{Spec}(k[x_i z_j])$. We parametrize the closed cover of $\mathbb{A}//^0\mathbb{G}_m$ with $\mathbb{A}^{s_0}$, so that each closed variety H in the cover is isomorphic to $\operatorname{Spec}(k[x_i z_j])$. Let $H^\pm$ be the inverse image scheme of H in $\mathbb{A}//^\pm\mathbb{G}_m$ and let $\tilde{H}$ be the inverse image in $\tilde{\mathbb{A}}$, so that we have the following familiar picture:

$$
\begin{array}{ccccc}
 & & \tilde{H} & & \\
 & \overset{p}{\swarrow} & & \overset{q}{\searrow} & \\
H^+ & & & & H^- \\
 & \searrow & & \swarrow & \\
 & & H & &
\end{array}
\tag{7.21}
$$

The maps p and q are the restrictions of $f_{\mathbb{A}}$ and $g_{\mathbb{A}}$.

To satisfy the conditions of Proposition 4.2, we need to check that $H^{\pm}$ and $\tilde{\mathbb{A}}$ are Tor-independent over $\mathbb{A}//^{\pm}\mathbb{G}_m$. By a theorem from commutative algebra (e.g. see the Koszul Homology chapter in Eisenbud [8]), if R is a Noetherian ring, M is an R-module, and $a_1, \ldots, a_n$ is both R-regular and M-regular, then $\mathrm{Tor}_i^R(R/I, M) = 0$ for all $i > 0$. Well, if $H = (t_1, \ldots, t_{s_0}) \times \mathrm{Spec}(k[x_i z_j])$, then it is cut out by the regular sequence $y_i - t_i$. Furthermore, as discussed in Section 6, $\tilde{\mathbb{A}}$ is the blow up at the loci cut out by $(x_i z_j)$ in $\mathbb{A}//^{\pm}\mathbb{G}_m$, so the sequence $y_i - t_i$ is also regular on $\tilde{\mathbb{A}}$. Thus the Tor groups at the stalks vanish, and $H^{\pm}$ and $\tilde{\mathbb{A}}$ are Tor-independent over $\mathbb{A}//^{\pm}\mathbb{G}_m$.

Of course the most important condition in Proposition 4.2 is that for each H, the functor $q_* p^* : \mathcal{D}^b(H^+) \to \mathcal{D}^b(H^-)$ is fully faithful. We claim this follows from Proposition 11.23 in [12]. From the explicit construction in Section 6, we know that $\mathbb{A}//^{+}\mathbb{G}_m = \mathrm{Proj}(k[y_i, x_i z_j, z_i])$ and so $H^+ = \mathrm{Proj}(k[x_i z_j, z_i])$ and $\tilde{H}$ is the blow up of H^+ in the vanishing locus of the ideal $(x_i z_j)$. Quotienting the ring by $(x_i z_j)$, we see that we are blowing up at a subvariety isomorphic to $\mathrm{Proj}(k[z_i]) = \mathbb{P}^{s_1 - 1}$, and in fact the normal bundle of $\mathbb{P}^{s_1-1}$ in H^+ is isomorphic to $\mathcal{O}(-1)^{\oplus s_{-1}}$. Hence the exceptional divisor of this blow up is isomorphic to $\mathbb{P}(\mathcal{O}(-1)^{\oplus s_{-1}}) \cong \mathbb{P}^{s_{-1}-1}$. The contraction of the exceptional divisor in the other direction is precisely isomorphic to H^-.

Thus the pair of blowups $H^+ \leftarrow \tilde{H} \rightarrow H^-$ is a standard flip (defined as in Huybrechts [12]), and the last condition we need to check to apply Proposition 11.23 of [12] is that $s_{-1} \leqslant s_1$. To see this, note that by [24], in our original picture $\tilde{M}$ is the blow up of $M(\sigma^+)$ at the projectivization of a rank $g + 2\sigma$ bundle and it is simultaneously the blow of $M(\sigma^-)$ at the projectivization of a rank $\frac{d-1}{2} - \sigma$ bundle. Of course étale localization doesn't change the rank of these bundles, and so we deduce that $s_{-1} = \frac{d-1}{2} - \sigma - 1$ and $s_1 = g + 2\sigma - 1$. Then $s_{-1} \leqslant s_1$ if an only if $\frac{d-1}{2} - \sigma \leqslant g + 2\sigma$, i.e. if and only if $\frac{d-1}{2} - g \leqslant 3\sigma$. Since we have set $\frac{d-1}{2} - g \leqslant 3\sigma$, it follows that $s_{-1} \leqslant s_1$ and so by [12], we conclude that $q_* p^* : \mathcal{D}^b(H^+) \rightarrow \mathcal{D}^b(H^-)$ is fully faithful.

Having checked all the conditions of Proposition 4.2, we apply it to deduce that $g_{\mathbb{A}*} f_{\mathbb{A}}^* : \mathcal{D}^b(\mathbb{A}//^+\mathbb{G}_m) \longrightarrow \mathcal{D}^b(\mathbb{A}//^-\mathbb{G}_m)$ is fully faithful. As discussed above, because it suffices to show our theorem étale-locally, we may finally conclude that $g_* f^* : \mathcal{D}^b(M(\sigma^+)) \longrightarrow \mathcal{D}^b(M(\sigma^-))$ is a fully faithful functor.

□

Remark 7.8. It is crucial that $\frac{d-1}{2} - g \leqslant 3\sigma$ in our proof. Proposition 11.23 from [12] requires that the dimension of the projective space blown up first is at least that of the one contracted to - otherwise the pull-push functor fails to be fully faithful in general. It is evident from our proof that to guarantee this inequality, we must have $\frac{d-1}{2} - g \leqslant 3\sigma$.

8. Corollaries

We now discuss a few consequences of Theorem 7.7. As usual, we fix a line bundle Λ over C of degree $d \geqslant 3$. First off, we claim that concatenating the triangular roofs in diagram 7.7 we have, as in [24], the following picture,

(8.1)

$$
\begin{array}{ccccccccc}
 & \tilde{M}_{d/2-2} & & \tilde{M}_{d/2-3} & & \tilde{M}_{d/2-4} & & \tilde{M}_{1-(d/2-[d/2])-\epsilon} & \\
\swarrow & & \searrow\ \swarrow & & \searrow\ \swarrow & & \searrow\ \swarrow & & \searrow \\
M_{d/2-1-\epsilon} & & M_{d/2-2-\epsilon} & & M_{d/2-3-\epsilon} & & \cdots & & M_{1-(d/2-[d/2])-\epsilon} \\
\downarrow & & & & & & & & \downarrow \\
M_{d/2-\epsilon} & & & & & & & & M
\end{array}
$$

where the notation is $M_\sigma = M(\sigma)$, and $\tilde{M}_{d/2-i}$ is as in diagram 7.7 where we take $\sigma^- = d/2 - i - \epsilon$. The left vertical arrow is the degeneration of a roof, and in fact it is a blow up of an embedding of C (see [24]). The bottom right space M is the moduli space of semistable vector bundles over C, without σ and without the section. Note that regardless of the parity of d, the moduli space $M(1-(d/2-[d/2])-\epsilon)$ is the same as $M(0)$, which as discussed above, reduces the inequalities to the ordinary semistability inequalities. So $M(1-(d/2-[d/2])-\epsilon)$ is the moduli space of semistable *pairs* over C, and the right vertical arrow is a birational map whose fibre over E is $\mathbb{P}H^0(E)$.

In order for the functor $\mathcal{D}^b(M_i) \to \mathcal{D}^b(M_{i-1})$ to be fully faithful for all the moduli spaces in the diagram, i.e. for all possible $\sigma \in [0, d/2)$, we require that $\frac{d-1}{2} \leqslant g$, which holds as long as $d \leqslant 2g-1$. Note that $2g-1$ is the minimum d can be in order for the machinery (and diagrams) from [24] and [25] to hold.

Our Theorem 7.7 immediately implies:

Corollary 8.1. *Fix $d \leqslant 2g-1$ and as always let $M(\sigma)$ denote the moduli space of σ-semistable pairs over C with fixed determinant Λ of degree d. Suppose we have $\sigma, \tau \in [0, d/2)$ and $\sigma, \tau \notin d/2 + \mathbb{Z}$. Then when $\sigma < \tau$, there is an embedding of derived categories $\mathcal{D}^b(M(\tau)) \longrightarrow \mathcal{D}^b(M(\sigma))$.*

We can also show:

Corollary 8.2. *Fix $d \leqslant 2g-1$, and as always let $M(\sigma)$ denote the moduli space of σ-semistable pairs over C with fixed determinant Λ of degree d, and fix $\sigma \in [0, d/2)$ and $\sigma \notin d/2+\mathbb{Z}$. Then there is an embedding of derived categories $\mathcal{D}^b(C) \longrightarrow \mathcal{D}^b(M(\sigma))$.*

Proof. By (3.19) in [24], the leftmost vertical arrow in diagram 8.1 is a blow up at a subvariety isomorphic to C. But it is well-known (e.g. see [6]) that the pullback of a blow up of a variety at a smooth subvariety induces an embedding of derived categories. Hence by Corollary 8.2, we obtain an embedding of $\mathcal{D}^b(C)$ into $\mathcal{D}^b(M(\sigma))$ for every $\sigma \in [0, d/2) \cap (d/2+\mathbb{Z})^c$. □

Remark 8.3. In Section 3, we proved that when M is the moduli space of semistable bundles over a genus 2 curve C, the Fourier-Mukai transform associated to the universal bundle on $C \times M$ gives an embedding of $\mathcal{D}^b(C)$ into $\mathcal{D}^b(M)$.

We conjecture that the same should be true for higher genus C. Indeed, we believe, but have not checked, that the fully faithful functors $\mathcal{D}^b(C) \longrightarrow \mathcal{D}^b(M(\sigma))$ we've constructed when $d \leqslant 2g-1$ are in fact also twists of the Fourier-Mukai transforms given by the associated universal bundles. By twist of a Fourier-Mukai transform, we mean either the Fourier-Mukai tranform associated to the tensor product of the original kernel by a line bundle, or, equivalently, the original functor composed with tensoring by a line bundle. Furthermore, in the case where $d = 2g-1$, the very rightmost map of diagram 8.1 is actually birational [24], and it pulls the universal bundle on M (the moduli space of semistable bundles over C) back to a twist of the universal bundle on $M(1-(d/2-[d/2])-\epsilon)$. Hence we have the following commutative diagram

$$
\begin{array}{ccc}
 & & \mathcal{D}^b(M_{1-(d/2-[d/2])-\epsilon}) \\
 & \overset{\Phi}{\nearrow} & \uparrow \pi \\
\mathcal{D}^b(C) & & \\
 & \underset{\Phi'}{\searrow} & \\
 & & \mathcal{D}^b(M)
\end{array}
\tag{8.2}
$$

where Φ and Φ' are Fourer-Mukai transforms with respect to the appropriate universal bundles, possibly twisted by a line bundle. So if we know Φ is fully faithful, it follows that Φ' is as well, and we obtain an embedding of $\mathcal{D}^b(C)$ into $\mathcal{D}^b(M)$ for all curves C of genus $g > 1$.

References

[1] Abouzaid, M., Auroux, D., Efimov, A. I., Katzarkov, L., and Orlov, D. Homological mirror symmetry for punctured spheres. *J. Amer. Math. Soc. 26*, 4 (2013), 1051–1083.

[2] Abouzaid, M., and Smith, I. Homological mirror symmetry for the 4-torus. *Duke Math. J. 152*, 3 (2010), 373–440.

[3] Addington, N. *Spinor sheaves and complete intersections of quadrics.* ProQuest LLC, Ann Arbor, MI, 2009. Thesis (Ph.D.)–The University of Wisconsin - Madison.

[4] Ballard, M., Favero, D., and Katzarkov, L. Variation of geometric invariant theory quotients and derived categories.

[5] Beĭlinson, A. A. The derived category of coherent sheaves on $\mathbf{P}^n$. *Selecta Math. Soviet. 3*, 3 (1983/84), 233–237. Selected translations.

[6] Bondal, A., and Orlov, D. Semiorthogonal decompositions for algebraic varieties, 1995.

[7] Bondal, A., and Orlov, D. Reconstruction of a variety from the derived category and groups of autoequivalences. *Compositio Math. 125*, 3 (2001), 327–344.

[8] Eisenbud, D. *Commutative algebra*, vol. 150 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[9] Favero, D., and Kelly, T. L. Toric mirror constructions and derived equivalence.

[10] Halpern-Leistner, D. S. *Geometric invariant theory and derived categories of coherent sheaves.* ProQuest LLC, Ann Arbor, MI, 2013. Thesis (Ph.D.)–University of California, Berkeley.

[11] Hartshorne, R. *Algebraic geometry.* Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[12] Huybrechts, D. *Fourier-Mukai transforms in algebraic geometry.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2006.

[13] Katzarkov, L. Generalized homological mirror symmetry and rationality questions. In *Cohomological and geometric approaches to rationality problems*, vol. 282 of *Progr. Math.* Birkhäuser Boston, Inc., Boston, MA, 2010, pp. 163–208.

[14] Kawamata, Y. Equivalences of derived categories of sheaves on smooth stacks. *Amer. J. Math. 126*, 5 (2004), 1057–1083.

[15] Kawamata, Y. Derived categories and birational geometry. In *Algebraic geometry—Seattle 2005. Part 2*, vol. 80 of *Proc. Sympos. Pure Math.* Amer. Math. Soc., Providence, RI, 2009, pp. 655–665.

[16] Kontsevich, M. Homological algebra of mirror symmetry. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)* (1995), Birkhäuser, Basel, pp. 120–139.

[17] Kuznetsov, A. Derived categories of cubic fourfolds. In *Cohomological and geometric approaches to rationality problems*, vol. 282 of *Progr. Math.* Birkhäuser Boston, Inc., Boston, MA, 2010, pp. 219–243.

[18] Nagata, M. Invariants of a group in an affine ring. *J. Math. Kyoto Univ. 3* (1963/1964), 369–377.

[19] Newstead, P. E. Stable bundles of rank 2 and odd degree over a curve of genus 2. *Topology 7* (1968), 205–215.

[20] Newstead, P. E. *Introduction to moduli problems and orbit spaces*, vol. 51 of *Tata Institute of Fundamental Research Lectures on Mathematics and Physics.* Tata Institute of Fundamental Research, Bombay; by the Narosa Publishing House, New Delhi, 1978.

[21] Okawa, S. Semi-orthogonal decomposability of the derived category of a curve. *Adv. Math. 228*, 5 (2011), 2869–2873.

[22] Seidel, P. Homological mirror symmetry for the genus two curve. *J. Algebraic Geom. 20*, 4 (2011), 727–769.

[23] StacksProjectAuthors. Stacks project, 2015.

[24] Thaddeus, M. Stable pairs, linear systems and the Verlinde formula. *Invent. Math. 117*, 2 (1994), 317–353.

[25] Thaddeus, M. Geometric invariant theory and flips. *J. Amer. Math. Soc. 9*, 3 (1996), 691–723.

[26] VERDIER, J.-L. Des catégories dérivées des catégories abéliennes. *Astérisque*, 239 (1996), xii+253 pp. (1997). With a preface by Luc Illusie, Edited and with a note by Georges Maltsiniotis.