The Optimal Income Tax Schedule

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Abstract

The principal conclusion of this paper is that the generic pattern for the optimal income tax schedule is that of a monononically rising marginal tax rate. The belief that the marginal rate should decline to zero at the upper end of the scale is not supported. The results hold for high and low elasticities of labor-leisure substitution, and for additive as well as strictly concave welfare functions. Differences between the conclusions of this paper and those of previous writers are due primarily to careful interpretation and analysis of the optimal solution, rather than differences between the models, although these exist.

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1. Introduction

Determining the income tax schedule is one of the most important public policy decisions made by modern governments, full of major political and economic consequences. Myths abound, yet the contributions of economists to date have tended to be incomplete or inconclusive, even confusing. The purpose of the present paper is to re-examine the problem, which has lately been neglected[1].

Mirrlees (1971), in his seminal paper on optimal income tax with labor-leisure substitution, concluded that the optimal marginal tax rate would be everywhere in the range (0,1) but that '...it is not possible to say in general whether marginal tax rates should be higher for high-income, low-income, or intermediate income groups'. For a specific model he choose as an example, he found the tax to be nearly linear, with marginal rates tending to fall rather than rise. Because of the relative ease of solution, much work has been done on properties of optimal *linear* income taxes. Examples include Sheshinski (1972), Itsumi (1974), Romer (1976), Stern (1976), Boskin and Sheshinski (1978), Helpman and Sadka (1978), but these throw only faint light on the shape of more general tax functions. However Sheshinski (1989) solved for an optimal two-bracket piecewise linear tax and showed that the marginal rate for the higher bracket was at least as great as that for the lower, thereby suggesting that nondecreasing marginal rate might be optimal in general.

However it has been argued that the marginal tax rate should be zero at the top end of an income distribution of finite range, a result clearly not consistent with a monotonic nondecreasing marginal rate. Versions of this argument appear in Phelps (1973), Sadka (1976), Stern (1976), Seade (1977), Cooter (1978), and Weymark (1987). This proposition is not supported by the results given here. Since belief in the zero marginal tax proposition is widely held[2], largely because of an appealing intuitive argument apparently in its favor, the Appendix to the paper is devoted to a brief analysis of why the argument fails to hold.

In the optimal income tax problem, the policymaker must set an income tax schedule $t(z)$ which optimizes his view of community welfare, given that individual households react to the schedule by choosing levels of taxable activity which optimize their private well-being. It is this problem of incentive compatibility that makes the determination of the optimal income tax inherently more difficult than that of choosing an optimal wealth or endowment tax, assuming full information about households in both cases. In addition to the compound optimization, the problem has other complications:

- Both levels of optimization are potentially constrained.
- The population has a "natural" distribution in terms of resources and/or

[1]The incentive to clarify the the situation arose from work by the author on optimal politically feasible tax-expenditure programs.

[2]Slemrod (1990), p.164, is a representative expression of both the belief and the discomfort with it.

skills as independent variable, but the solution is the tax as a function of income. The relationship between a household's optimal income and its endowment depends, however, on the tax function, which is the unknown.

- Since we are interested in the *shape* of the optimal tax schedule, we must avoid imposing prior restrictions that directly influence it[3]. Furthermore, it is not sufficient merely to determine the sign of the marginal tax rate[4] — we cannot consider the problem solved unless we can provide reasonable clues as to its direction of change.

2. Households

Households are assumed to have identical neoclassical utility functions $u(x, \zeta)$, where x is disposable market income made up of earned market income z less tax $t(z)$ on that income, plus any "grant" g from the government, and ζ is nonmarket income (including leisure). Only resource endowments ω, which are measured in efficiency units, vary between households. The endowments, which can include human capital and financial resources, can be transformed by the individual into either market or nonmarket income at a constant 1:1 rate of transformation. Thus $x = z - t(z) + g$ and $\zeta = \omega - z$.

This formulation is different and perhaps slightly more general than the more traditional one in which resources, measured in units such as time, do not vary between households, but there are *skill* or *wage* differences which determine the rate of transformation into market income. In the present formulation, higher skill is reflected in generating nonmarket income (leisure, home production, tax avoidance) as well as market income, and the shadow price of leisure is wealth-neutral.

The household is assumed to optimize its choice between market and non-market income, subject to a budget constraint which includes the tax function. Formally:

$$\max_{x,\zeta} u(x, \zeta), \text{ subject to: } x + \zeta \leq \omega + g - \tau(x),\ x \geq g,\ \zeta \geq 0 \tag{1}$$

where $\tau(x)$ is the tax as a function of x.

If $\tau(x)$ uniformly convex, the feasible set is convex. Since the maximand is strictly quasiconcave, there is a unique solution which can be fully interior or a corner solution. If $\tau(x)$ is not convex, the feasible set will not be convex and there is an interior solution only if the indifference curves are more curved than the boundary of the set (low substitutability between market and nonmarket income).

[3] Fair (1971), uses a nonlinear tax but restricts it to a specific form $t(z) = a \log(1 + z)$, with only one parameter.

[4] Shown to be positive or nonnegative over a wide class of models. See Mirrlees (1971), Phelps (1973), Röell(1985), Romer (1976), Seade (1982), among others.

Since $x = z - t(z) + g$, it is easily shown that τ' and τ'' have the same signs as t', t'', provided $t' < 1$, so uniform convexity of $t(z)$ (a nondecreasing marginal tax rate) is sufficient to guarantee a unique solution. If the tax function is nonconvex, problems may arise.

To obtain the most clearcut results, we want to investigate the optimal tax schedule for a continuum of households with identical utility functions but differing endowment levels. We want to separate, as far as possible, labor-leisure substitution effects from income and other effects. The general neoclassical function does not provide adequate separation of effects[5]. Desirable properties for the utility function are:

1. Strict quasiconcavity with at least weak concavity, for traditional reasons.
2. Homotheticity, so that there is no changing preference bias toward either market or nonmarket income as utility increases.
3. Linearity in income with prices constant, so that diminishing marginal utility of income effects, if any, appear as an assumption of the policy-maker.

We shall confine our analysis to a CES utility function[6], which meets the above specifications:

$$u = \left(a(z - t(z) + g)^{\frac{\sigma-1}{\sigma}} + (1-a)(\omega - z)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} \tag{2}$$

where $0 < a < 1$ and $\sigma > 0, \neq 1$.

Choice of the CES function, rather than the much simpler Cobb-Douglas, preempts queries as to whether the results depend on unit elasticity of substitution. However we will sometimes use the equivalent Cobb-Douglas form when it is desirable to show that a particular result holds also for unit elasticity.

Household Optimization

Households are assumed to choose z optimally, subject to $0 \leq z \leq \omega$, given g and the tax function $t(z)$. We have

$$z^*(\omega, g, t(z)) = \max\left(0, \frac{(1 - t'(z^*))^\sigma \omega - b^\sigma (g - t(z^*))}{(1 - t'(z))^\sigma + b^\sigma}\right) \tag{3}$$

where $b = ((1-a)/a)$. There will be an interior optimum for $\omega \geq (b/(1 - t'(0)))^\sigma g$, a lower boundary optimum otherwise. Note that an upper boundary optimum ($z = \omega$) would require $t = \omega + g > z$ and can be ruled out. The above relationships also hold when we set $\sigma = 1$.

[5] The first version of this paper was written in terms of such a function.

[6] Feldstein (1973) uses the same utility function in conjunction with a nonlinear transformation relationship, but reaches no well defined conclusion as to the general shape of the optimal tax function.

Note that a particular household's marginal choice is based on *two* properties of the tax function, the total tax $t(z^*)$ at its optimal income and the marginal tax rate $m(z^*) = t'(z^*)$ at that income. If there is a full interior equilibrium, the household's behavior near equilibrium z^* can be analyzed as if it faced a simple linear tax of the form $t(z^* + \delta z) = t(z^*) + m(z^*)\,\delta z$. For a household at income level z, the tax function can be seen as fully represented by the two variables (parameters, from its point of view), t and m.

The shape of the tax schedule elsewhere than at z^* is not entirely irrelevant, however, since it can determine whether there is a better choice than that given by the local marginal conditions. This is reflected in the appearance of m' in the second order condition. We have

$$\left(\frac{d^2u}{dz^2}\right)_{z=z*} = -\frac{a}{\sigma}\left((1-m)^2 + b^{-\sigma}(1-m)^{1+\sigma} + \sigma\, x\, m'\right) x^{-\frac{\sigma+1}{\sigma}}\, u^{\frac{1}{\sigma}} \qquad (4)$$

where $x = z - t + g$, so that $d^2u/dz^2 < 0$ for all $m' > 0$, but only for a limited range of $m' < 0$.

It is easy to show that $\partial z^*/\partial m < 0$, so that there is an upper limit to the value of m for which a household with endowment ω and grant g will generate any taxable income. Since $t(0) = 0$, this is given by

$$\hat{m}(g,\omega) = \max\left(0, 1 - b\left(\frac{g}{\omega}\right)^{\frac{1}{\sigma}}\right) \qquad (5)$$

Note that, for given g and given ω, $\hat{m}$ is smaller, the larger is σ. Also, since $\partial z^*/\partial\omega > 0$, there is a lower limit to the value of ω, given g, for which the household will generate market income even at $m = 0$. We will discuss this lower boundary optimum with $z = 0$ and $du/dz < 0$ at a later stage.

Given the tax structure $(g, t(z))$, there is a strictly monotonic one-to-one mapping between the endowment level ω and the optimal market income z, provided the household is at an interior optimum. Thus we can pick an endowment and determine the market income level which is optimal for that endowment, given the tax structure. This is the conventional approach. But we can just as well pick an income level and determine the endowment level for which that income is optimal, given the tax structure.

For reasons that will be obvious as the analysis progresses, z will be taken as the independent variable here. For an interior optimum, $\omega(z)$ is given by

$$\omega = \frac{b^\sigma(g-t)}{(1-m)^\sigma} + \left(1 + \frac{b^\sigma}{(1-m)^\sigma}\right) z \qquad (6)$$

with $0 < d\omega/dz < \infty$ for all $m < 1$.

Using the relationship $\omega(z)$, the utility function $u(g, t(z), \omega, z)$ becomes the indirect utility function $v(g, t(z), z)$. This can be written $v(g, t, m, z)$ because the tax function is embodied in the two parameters t, m for a given value of z. Here we have

$$v(g,t,m,z) = c\left(1+\frac{b^\sigma}{(1-m)^{\sigma-1}}\right)^{\frac{\sigma}{\sigma-1}}(z-t+g) \tag{7}$$

where $c = a^{\sigma/(\sigma-1)}$.

A major advantage of taking z as the independent variable is that, for given z, we can treat t, m as parameters rather than functions. Finding the effect of varying the tax parameters on $v(g,t,m,z)$ gives us:

$$\begin{aligned} v_g &= c\left(1+\frac{b^\sigma}{(1-m)^{\sigma-1}}\right)^{\frac{\sigma}{\sigma-1}} > 0 && (8)\\ v_t &= -v_g < 0 && (9)\\ v_m &= \sigma c\left(1+\frac{b^\sigma}{(1-m)^{\sigma-1}}\right)^{\frac{1}{\sigma-1}}\frac{b^\sigma}{(1-m)^\sigma}(z-t+g) > 0 && (10)\end{aligned}$$

provided $z - t + g \geq 0$. The same sign relationships are easily shown to hold for the Cobb-Douglas case ($\sigma = 1$).

Note that v_m is positive because, with z held constant, a higher marginal tax rate associates a higher endowment level with a given market income.

3. The Policy Maker

The policymaker's problem is to raise *per capita* revenue of g to be redistributed uniformly as a grant g in cash or any other form in which it is a perfect substitute for market goods. The revenue is to be obtained from a tax defined by a smooth continuous function $t(z) \geq 0$ with $t(z_0) = 0$ (where $z_0 \geq 0$ is the lowest household income), but otherwise unrestricted. The tax function is to be chosen so as to maximize a predetermined social welfare function. Two cases will be considered

1. The *per capita* grant g is given, and the income tax must raise the required revenue in an optimal way.
2. The tax-grant combination $(g, t(z))$ should solve the optimal redistribution problem, the value of g being part of the solution.

The first case might be described as a second best optimum, except that an income tax is itself a second best instrument of policy. The solutions must satisfy the individual optimization constraints $0 \leq z \leq \omega$ and any constraints imposed by the policymaker. The constraint placed on $t(z_0)$ is to prevent the tax function solving for optimal redistribution when a solution for nonoptimal g is sought.

The Welfare Function

The welfare function $G(v)$ adopted by the policymaker is assumed to be anonymous and based solely on the level of a household's optimized utility level. The policymaker is assumed to have full information as to household utility functions.

To separate effects due to welfare concavity from other effects, we shall consider two cases

1. An *additive* welfare function in which $G(v) \equiv v$.
2. A *strictly concave* welfare function with $G' > 0$ and $G'' < 0$.

The population of households is modelled as a one dimensional continuum. The "natural" index variable is the endowment ω, with distribution given by a smooth distribution function $\Phi(\omega)$ over $(0 \leq \omega_0, \omega_1 \leq \infty)$. The overall welfare criterion is the mean value of $G(v)$ taken over the whole population.

The Formal Optimizing Problem

If the problem was that of an optimal *wealth* tax, using ω as the independent variable for the problem would be straightforward, since the tax function would be defined over ω and households would it would react to it as a function of their wealth. But the tax function we seek must be perceived by households as a function of income alone.

The solution adopted here is to treat *income* as the index variable and take the income distribution as given. Under circumstances which give interior solutions for household optimization, there is a unique positive monotonic relation between ω and z, so that the final distribution of z, given the tax function, will have the same general pattern (of zeros, peaks, etc.) as the distribution of ω. Since the purpose of the present analysis is to derive *generic* properties of the tax function, not to solve for a given initial wealth distribution, this approach is acceptable. Boundary cases can also be handled, with care.

Taking the index variable to be z, with distribution defined by the density function $f(z)$ and range (z_0, z_1), the problem can be written as

$$\max \mathcal{W}(g) = \int_{z_0}^{z_1} G\left(v(g, t(z), m(z), z)\right) f(z)\, dz \tag{11}$$

where the maximand is the mean welfare of the population.

The solution[7] must satisfy the differential equations

$$t'(z) = m(z) \tag{12}$$

$$R'(z) = t(z) f(z) \tag{13}$$

[7] Kaneko (1981) proved the existence of a solution to the optimal tax problem for a quasi-concave utility function with a strictly concave conversion of leisure into income, and a positive monotonic welfare function, but not for a continuous distribution. Existence is assumed here.

and end-point conditions

$$\begin{aligned} t(z_0) &= 0 && (14)\\ t(z_1) & \quad \text{free} && (15)\\ R(z_0) &= 0 && (16)\\ R(z_1) &= g && (17)\end{aligned}$$

$R(z)$ is the contribution to net revenue per capita from taxes through income level z, with $R(z_1)$ is the mean revenue over all taxpayers.

The control variable is m, state variables are t, R, with the Hamiltonian

$$H(g, t(z), m(z), \lambda(z), \mu(z)) = G\left(v(g, t, m, z)\right) f(z) + \lambda(z)\, m(z) + \mu(z)\, t(z)\, f(z)$$

4. The Solution

Using standard maximum principle methods, the optimal trajectory for the state variables $t(z), R(z)$ and associated costate variables $\lambda(z), \mu(z)$, must satisfy the maximum condition

$$H(g, t^*(z), m^*(z), \lambda^*(z), \mu^*(z)) = \max_{m(z)} H(g, t(z), m(z), \lambda^*(z), \mu^*(z))$$

giving

$$H_m = G_m\, f + \lambda = 0 \tag{18}$$

Since $t'(z) = m(z)$ and $t(z_0) = 0$, the optimal solution is fully described by $m^*(z)$, the optimal marginal tax rate schedule. However the properties and economic interpretation of the optimal costate variables $\lambda^*(z)$ and $\mu^*(z)$ are critical in establishing the properties of the optimal tax schedule.

Henceforth it will be assumed that all variables are at optimal values unless it is clear otherwise, so the asterisks will be dropped except that $\mathcal{W}^*$ will always be used to identify the optimal redistribution solution.

The Costate Variables

The costate variables must satisfy the adjoint equations

$$\begin{aligned} \lambda' &= -H_t = -(G_t + \mu) f && (19)\\ \mu' &= -H_R = 0 \quad \Longrightarrow \mu \text{ constant} && (20)\end{aligned}$$

but we need further analysis to determine some important properties.

We can write

$$\begin{aligned} \mathcal{W}(g) &= \int_{z_0}^{z_1} [H - \lambda t' - \mu R']\, dz \\ &= \int_{z_0}^{z_1} [H + \lambda' t + \mu' R]\, dz + \lambda(z_0)\, t(z_0) - \lambda(z_1)\, t(z_1) - \mu g + \mu R(z_0)\end{aligned}$$

after integration by parts and substituting $g = R(z_1)$.

Varying the end values $t(z_0)$, $t(z_1)$, $R(0)$, gives:

$$\begin{aligned} \delta\mathcal{W}(g) &= \int_{z_0}^{z_1} [H_m \delta m + (H_t + \lambda')\delta t + (H_R + \mu')\,\delta R]\, dz \\ &\quad + \lambda(z_0)\,\delta t(z_0) - \lambda(z_1)\,\delta t(z_1) + \mu\,\delta R(0) \end{aligned} \tag{21}$$

Along the optimal trajectory the integral term vanishes so that

$$\frac{\partial\mathcal{W}(g)}{\partial t(z_0)} = \lambda(z_0) \tag{22}$$

$$\frac{\partial\mathcal{W}(g)}{\partial t(z_1)} = -\lambda(z_1) \tag{23}$$

$$\frac{\partial\mathcal{W}(g)}{\partial R(0)} = \mu \tag{24}$$

Since t is essentially unconstrained at z_1, $\partial\mathcal{W}(g)/\partial t(z_1)$ must vanish so that

$$\lambda(z_1) = 0 \tag{25}$$

However $t(z_0)$ is not unconstrained in this way, since g was specifically introduced so that we could fix $t(z_0) = 0$. Thus $\partial\mathcal{W}(g)/\partial t(z_0)$ need not vanish and there is no direct transversality restriction on $\lambda(z_0)$.

Interpretation of the Costates

The economic interpretation of the costate variables $\lambda(z)$ and μ is important in understanding the model and in unravelling the story told by the optimal conditions.

Since $\mu = \partial\mathcal{W}(g)/\partial R(0)$, it measures the effect on optimal mean welfare of a marginal variation in the starting value of the cumulated revenue $R(z)$. $R(z_1)$ is fixed at g, so that μ is the value in mean welfare terms of an exogenous addition of \$1 to the revenue "pot", enabling the tax function to be optimally reworked to collect a mean of \$1 less from taxpayers. The reason μ is constant is that only the final revenue is relevant, not the stage at which it is collected. It is obvious that μ is essentially positive.

The interpretation of $\lambda(z)$ is less straightforward. We have $\lambda(z_0) = \partial\mathcal{W}(g)/\partial t(z_0)$ so that $\lambda(z_0)$ measures the effect on mean welfare of raising the lowest tax bracket from zero to \$1 and then re-optimizing. Since other parameters (g in particular) are held constant, taxes will be reduced for at least some incomes above the lowest and the distribution will become marginally more regressive. If $t(z_0) = 0$ was itself an optimal outcome, then we would have $\lambda(z_0) = 0$, but since $t(z_0) = 0$ is an imposed constraint, the value (or even the sign) of $\lambda(z_0)$ is not immediate. We can, however, argue as follows:

1. If g is set at below the optimal redistribution level g^* and the constraint $t(z_0) = 0$ is removed, the program will optimally redistribute by making $t(z_0)$ negative. This implies $\lambda(z_0) < 0$ if $t(z_0) = 0$ and $g < g^*$.

2. If $g > g^*$ is above the optimal redistribution level, the opposite will be true and thus $\lambda(z_0) > 0$. As we will show in Property P2 below, we cannot have $\lambda(z_0) > 0$ and thus there is no regular solution for the case $g > g^*$. This is because we can only optimize mean welfare in this case by increasing $t(z_0)$ above zero, which the constraints do not permit.

The interpretation of $\lambda(z)$ for $z \neq z_0$ is that it represents the effect on mean welfare of an exogenously imposed change of $1 in the tax at income z, with re-optimization restricted to changes in taxes for households with incomes above z only. Note that one of the influences on the value of $\lambda(z)$ will be $f(z)$ since an increase in the tax on a sparsely populated income bracket will call for little readjustment elsewhere. We expect to find $\lambda(z) < 0$ except at the ends since, if we interrupt the program at z and restart to optimize only from that point on, the prior tax at z is now too high because it was designed to contribute to households with incomes lower than z.

Optimal Redistribution

The optimal solution to the redistribution problem is found by treating g as a control parameter and optimizing for it:

$$\mathcal{W}^* = \max_g \mathcal{W}(g)$$

Since $\mathcal{W}(g)$ is continuous in g for $g \leq g^*$, we can take the derivative from below to find the optimal condition

$$\frac{d\mathcal{W}(g)}{dg} = \int_{z_0}^{z_1} H_g \, dz - \mu = 0 \tag{26}$$

From the adjoint equation we have

$$H_g = G_g f = -G_t f = \lambda' + \mu f$$

so that

$$\int_{z_0}^{z_1} H_g \, dz = \lambda(z_1) - \lambda(z_0) + \mu$$

Since $\lambda(z_1) = 0$, $d\mathcal{W}(g)/dg = -\lambda(z_0)$ and so the optimal condition is

$$\lambda(z_0) = 0 \tag{27}$$

Note that this is consistent with the result given previously in the interpretation of $\lambda(z)$.

Lower Boundary Optima

From (3), $z_0 > 0$ only if $\omega_0 > \hat{\omega}(g) = b^\sigma g/(1 - m(z_0))^\sigma$, since $t(z_0) = 0$. If $\omega_0 < \hat{\omega}(g)$, households with endowments in the set $\hat{\Omega} = \{\omega | \omega_0 \le \omega \le \hat{\omega}(g)\}$ will not attain an interior optimum, so that $z = 0$ for all $\omega \in \hat{\Omega}$.

Since $\hat{\omega}(g)$ is increasing in g, the set is nonempty for sufficiently high levels of redistribution (values of g) and for all $g > 0$ if $\omega_0 = 0$. Since $z = 0$ for all $\omega \in \hat{\Omega}$, the mapping between the distributions of ω and z is not one-to-one in this range as it is for interior optima.

Provided $\omega_0 < \hat{\omega}(g)$, define β as the proportion of the population in $\hat{\Omega}$. Since β depends on the endowment distribution of the population as well as g, we shall treat it as part of the assumed market income distribution.

The tax schedule is irrelevant to households in $\hat{\Omega}$, the effect of redistribution policy being determined by g, which becomes their entire disposable income. Individual welfare is $u(g,\omega)$, (we do not use the v notation, since these are not interior optima). Write the mean welfare of the population in the set as $G^0(g)$.

The policymaker's optimizing problem now has the form

$$\max \mathcal{W}(g) = \int_{z_0}^{z_1} G\left(v(g, t(z), m(z), z)\right) f(z)\, dz + \beta G^0(g) \tag{28}$$

For given g, the last term is simply a constant and only the integral term is to be maximized. However there are three differences from the problem in the interior optimum case

1. $z_0 = 0$
2. $\int_0^{z_1} f(z)\, dz = 1 - \beta$
3. $R(z_1) = g/(1 - \beta)$

The last is because the taxpayers must accumulate enough revenue to distribute g over those in $\hat{\Omega}$ as well as themselves.

When we allow for the above changes and for the term in $G^0(g)$, the effect of varying g can be shown to become

$$\frac{d\mathcal{W}(g)}{dg} = -\lambda(0) - \beta\left(\frac{2-\beta}{1-\beta}\mu - \frac{dG^0(g)}{dg}\right) \tag{29}$$

Consider the second term on the right. dG^0/dg and μ are both positive comparable numbers, each measuring the effect of \$1 on mean welfare, dG^0/dg over households in $\hat{\Omega}$, μ over the remainder. Because of the large weight given to μ (between 2 and ∞), we expect this to dominate and the expression in parentheses to be positive. Thus for optimal redistribution ($d\mathcal{W}(g)/dg = 0$) we will have $\lambda(0) < 0$ rather than $= 0$ as in the interior optimum. Note that putting $\beta = 0$ gives the interior optimum results.

5. Properties of the Optimal Trajectory

The problem at this point is to translate the various necessary conditions for that trajectory into a meaningful description of the required tax function. It is first necessary to establish certain properties of the optimal solution.

P1. $\mu > 0$.

This follows directly from $\partial \mathcal{W}^*/\partial R(0) = \mu$. An exogenous increase in revenue will necessarily increase mean welfare.

P2. *For an interior optimum of the policymaker's problem, $\lambda(z) \leq 0$ for all z, and $\lambda(z) = 0$ only if $f(z) = 0$ or z is a boundary optimum for the household.*

For an interior optimum to the main problem $\lambda(z) = -G_m(z)f(z)$ from (18). An interior optimum for the household implies $G_m(z) = G'\, v_m > 0$, from (10), and $f(z) \geq 0$, so $\lambda(z) \leq 0$ certainly and $= 0$ only if $f(z) = 0$. At a boundary optimum for the household, $G_m(z) = 0$, so $f(z) = 0$ is no longer a necessary condition.

P3. $\lambda'(z) > 0$ (< 0) *only if (a)* $G_g > \mu$ $(< \mu)$ *and (b) either* $d(G_m)/dz < 0$ (> 0) *or* $f'(z) < 0$ (> 0)

The adjoint equation (19) can be expressed as

$$\lambda'(z) = (G_g(z) - \mu)f(z) \tag{30}$$

since $G_t = G'v_t = -G'v_g = -G_g(z)$, from (9). Condition (a) follows immediately, and (b) follows directly from the first order optimum condition $\lambda(z) = -G_m(z)f(z)$.

P4. *Along any optimal trajectory in which all households are at interior optima, $\bar{G}_g \geq \mu$, where $\bar{G}_g$ is the frequency weighted average of G_g along the trajectory. Unless $G_g(z)$ is constant,* $\max G_g(z) > \mu > \min G_g(z)$. *However if there is a lower boundary optimum (the set $\hat{\Omega}$ is nonempty), it is possible to have $G_g(z) \geq \mu$ for all z.*

From (30) above, we obtain

$$\begin{aligned} \int_{z_0}^{z_1} (G_g(z) - \mu)\; f(z)\, dz &= \lambda(z_1) - \lambda(z_0) \\ \bar{G}_g - \mu &= -\lambda(z_0) \geq 0 \end{aligned} \tag{31}$$

since $\lambda(z_1)$ vanishes and $\lambda(z_0) \leq 0$ (from P2).

Now $\mu = \partial \mathcal{W}(g)/\partial R_0$ is the increase in mean social welfare which would result from an exogenous increase of \$1 in per capita revenue, after optimally redistributing the resulting saving in taxes. $G_g(z)$ measures the social valuation of the effect of \$1 on a single household with income z.

For an interior optimum, \$1 saved in taxes is equivalent to \$1 increase in g. It follows that we must have $\mu > G_g(z)$ for some z, since μ is the *optimal* mean welfare from an increase of \$1 per capita. On the other hand, we must have $G_g(z) > \mu$ for some z, otherwise the optimum would be a Pareto improvement, which is impossible for a pure redistribution without externalities.

For a lower boundary optimum, however, \$1 saved in taxes is equivalent to only \$ $(1-\beta)$ increase in g for the taxpayers, due to the payments to those in the set $\hat{\Omega}$. While it is true that $\mu > (1-\beta)G_g(z)$ for some z by the same argument as above, this is consistent with $G_g(z) \geq \mu$ everywhere for a large enough $\beta \in (0,1)$.

P5. *$dG_g(z)/dz \leq 0$ unless $m'(z) > 0$. If $G'' = 0$, $m'(z) > 0$ implies $dG_g(z)/dz > 0$, but if $G'' < 0$, $dG_g(z)/dz > 0$ only if $m'(z)$ is sufficiently large relative to $|G''/G'|$.*

We have

$$\frac{dG_g}{dz} = G'(u)\left(\frac{dv_g}{dz} + \frac{G''(u)}{G'(u)}\frac{dv}{dz}\right) \tag{32}$$

The second term is negative and its magnitude depends on the degree of concavity $|G''(u)/G'(u)|$. If the welfare function is additive, the term vanishes and $dG_g/dz = dv_g/dz$.

From (8) derive

$$\frac{dv_g}{dz} = \sigma\, b^\sigma\, c\left(1 + \frac{b^\sigma}{(1-m)^{\sigma-1}}\right)^{\frac{1}{\sigma-1}} \frac{m'}{(1-m)^\sigma} \tag{33}$$

so that dv_g/dz has the same sign as m'. Note that this result holds for both $\sigma > 1$ and $0 < \sigma < 1$, and a similar result can be shown to hold for $\sigma = 1$.

P6. *$dG_m(z)/dz > 0$ unless $m' < 0$ or $G'' < 0$.*

Using (10):

$$\frac{dv_m}{dz} = \sigma\, c\, \frac{b^\sigma Q^{\frac{1}{\sigma-1}}}{(1-m)^{\sigma-1}}\left[1 + \left(1 + \frac{b^\sigma}{b^\sigma + (1-m)^{\sigma-1}}\right)\frac{(z-t+g)m'}{(1-m)^2}\right] \tag{34}$$

where

$$Q = 1 + \frac{b^\sigma}{(1-m)^{\sigma-1}}$$

so that v_m is certainly increasing when m is nondecreasing, although it may not be decreasing when m is decreasing.

The relationship between dG_m/dz and dv_m/dz is essentially the same as was shown in (32) above between dG_g/dz and dv_g/dz. That is, G_m and v_m move in the same direction with additive welfare, but welfare concavity introduces a downward bias to dG_m/dz which increases with the degree of concavity.

P7. *Either $m(z) < 1$ and $t(z) < z$ for all $z > 0$ or $t(z) = 0$ for all z.*

From (5), $z > 0$ only if $m(z) \leq \hat{m}$, and $\hat{m} < 1$ if $g > 0$. But $t(z) \geq 0$ implies $g = 0$ if and only if $t(z) = 0$ everywhere. Since $t(z_0) = 0$, $m < 1$ everywhere implies $t(z) < z$ everywhere.

6. Some Basic Propositions

Proposition I. *Any solution to the optimal redistributive income tax problem in which all households are at interior optima will be characterized by a rising marginal tax rate, the rate of rise being greater, the greater the degree of concavity of the welfare function. The marginal rate will always remain strictly less than unity, however. The class of distributions for which such a solution exists is characterized by falling income density at the upper end and $f(z) \to 0$ as $z \to z_1$. For a sufficiently concave welfare function, no solutions exist with all households at interior optima. The results are independent of the elasticity of substitution between market and nonmarket income.*

For optimal redistribution, $g = g^*$ and $\lambda(z_0) = \lambda(z_1) = 0$, as shown earlier. Now $\lambda(z) < 0$ for $z_0 < z < z_1$ from P2 and the assumptions on $f(z)$, so that the graph of $\lambda(z)$ must look like a slack clothesline, first falling from the zero level then rising back to it. Thus $\lambda'(z)$ must first be negative, then positive. From P3, this requires $G_g(z) < \mu$ initially, then $> \mu$, so $G_g(z)$ must be rising. From P5, a necessary condition for this is $m'(z) > 0$, whatever the elasticity of substitution. This condition also sufficient if the welfare function is additive, but if it is strictly concave $m'(z)$ must not only be positive, but of sufficient magnitude to outweigh the negative concavity term. However, $m(z) < 1$ even at z_1, from P7.

From P3, $\lambda'(z) > 0$, which characterises the latter part of the trajectory, requires either $f'(z) < 0$ or falling G_m at that stage. But G_m is rising since $m'(z) > 0$, from P6. $f'(z) < 0$ is consistent with the transversality requirement $\lambda(z_1) = 0$, which can only be satisfied if $f(z_1) = 0$ from P2.

If the concavity of the welfare function is such that G_g is falling throughout, even if $m' > 0$, then it must be true that $G_g > \mu$ at the beginning and $< \mu$ at the end, so that λ is first rising, then falling. But an interior optimum requires λ first falling, then rising, and so cannot be attained with this degree of concavity.

Although the pattern of the optimal tax is independent of the value of σ, the value of $\hat{m}(g, \omega)$, the upper bound to m, varies inversely with σ (from (5)). Thus there is a *presumption* (but not a proof) that the optimal path of $m(z)$ will be lower, the higher the elasticity of substitution between market and nonmarket income.

Proposition Ia. *The conclusions of Proposition I hold for the problem of optimizing the tax schedule given the level of g, provided $g \leq g^*$.*

If $g < g^*$, $\lambda(z_0) < 0$ rather than $= 0$. An interior optimum requires $\lambda'(z) > 0$ and thus $G_g > \mu$ in the final phase, while the phase with $G_g < \mu$ must precede it. Thus the pattern is essentially the same as in optimal redistribution. As already pointed out, there is no interior optimum solution if $g > g^*$.

Proposition II. *If the degree of welfare concavity is too high to give an interior optimum, optimal redistribution will give a lower boundary solution. It will be optimal for those with the lowest endowments to produce no market income*[8]*. Like the interior optimum, this will be characterized by a rising marginal tax rate.*

If the degree of welfare concavity is so high that $G_g(z)$ is falling, then an interior optimum does not exist because this implies $G_g(z) < \mu(z)$ and thus falling $\lambda(z)$ near z_1, where $\lambda(z)$ must be rising. But from P4 it is possible to have $G_g(z) \geq \mu$ everywhere for a lower boundary optimum ($\beta > 0$), so that $G_g(z) \geq \mu$, hence $\lambda(z)$ rising near z_1, is consistent with falling $G_g(z)$.

Since we must have $\min G_g < \mu/(1-\beta)$, the condition $\min G_g \geq \mu$ can always be satisfied for large enough $\beta \in (0, 1)$. However there is an efficiency loss at the lower boundary because the households are not able to reach an interior optimum, so it is clear that the optimal solution will keep β as small as possible. Since G_g is falling, this implies the rate of fall should kept small. From P5, this implies $m(z)$ rising.

Since $G_g(z) \geq \mu$ throughout, $\lambda(z)$ will be rising throughout. This is consistent with $\lambda(z_0) < 0$ and $\lambda(z_1) = 0$.

Thus the optimal trajectory for high welfare concavity will be a lower boundary optimum with the least endowed households receiving g as their only disposable income, generating no market income of their own but using all their resources for nonmarket income. As in the interior optimum case, the marginal tax rate will be rising.

High concavity implies $G_m(z)$ will be falling throughout, which is consistent with $\lambda(z)$ falling thoughout (P3), so no restrictions on $f(z)$ are necessary. It must still be true that $\lambda(z_1) = 0$, however.

7. Concluding Remarks

The principal conclusion of this paper is that the generic pattern for the optimal income tax schedule is that of a mononononically rising marginal tax rate. This is a kind of dejà vue result, since it is what would have been expected *prior* to the optimal tax literature of the 1970's and 1980's. That literature came to be dominated by the top end zero marginal rate proposition, even though it often seemed to be inconsistent with what the main optimization was indicating. The proposition, which seemed to rule out precisely what this paper concludes, was due to a misinterpretation of the formal results of optimization. Indeed, the special contribution of this paper lies most of all in the careful interpretation

[8] A somewhat similar solution appears in Mirrlees (1971)

of the optimum conditions, a somewhat Sherlock Holmesian process in which every single mathematical property of the optimal trajectory, together with its economic interpretation, is fitted into its place in the puzzle.

The generic pattern holds for all positive elasticities of substitution between market and nonmarket income (or consumption and leisure), although we can expect (this is not formally proved in the paper) that the optimal marginal tax rate will be lower throughout the trajectory when the elasticity is higher. The pattern holds for simple additive welfare functions as well as those with strict concavity, but increased concavity will be associated with a more rapidly rising marginal rate. Sufficiently concave welfare functions, and endowment distributions with sufficient concentration at the low end, will make it optimal to have bottom end households generate no market income.

The analysis has been confined to interior optima except at the bottom end of the distribution, and is thus restricted to endowment (hence income) distributions in which density tails off to zero at the upper end. While this conforms to normal expectations concerning such distributions, it does leave a loose end lying about. And there are others.

Appendix

The Top End Marginal Rate

The proposition that the marginal tax rate should be zero on the highest income is not supported by the present analysis. This appendix is designed to show why common arguments do not hold for smooth neoclassical utility functions and smooth continuous tax functions.

The key is our property P2, that top end households cannot be at interior equilibria unless $f(z_1) = 0$. We need confine our attention to interior equilibria only, since all standard models require a one-to-one relationship between the index variable (endowment, skill, or wage rate) and market income.

We shall consider the informal intuitive argument, and two formal arguments.

The Informal Argument

This can be stated as follows[9]: If $\bar{z}$ is the highest market income under a tax system in which $t(\bar{z}) = \bar{t}$ and $m(\bar{z}) = \bar{m} > 0$, there is no welfare loss, and there may be a gain, from changing the upper end tax schedule to $t(z \geq \bar{z}) = \bar{t}$, so that $m(\bar{z}) = 0$. The reasoning is that the individual at $\bar{z}$ retains his original choice, but has an expanded opportunity because of the lowered marginal tax rate, so he cannot lose and may gain, while his tax contribution remains at $\bar{t}$ so that other taxpayers are left unaffected even if he gains.

[9] There are several variations on this argument, including a geometrical version in Seade (1977). The version here is the simplest

There is a counterargument, however. If the upper end individual has an actual gain, it is because he can move to a preferred position with income $\tilde{z} > \bar{z}$, and this will always be true for a smooth neoclassical utility function and an *interior* equilibrium. Consider the situation after the change, and now impose tax on income above $\bar{z}$ at a marginal rate m in the range $0 < m < \bar{m}$. The top end individual will be worse off that with $m = 0$, but his optimal income will be greater than $\bar{z}$, so that he will pay more than $\bar{t}$ and there is a welfare gain to other taxpayers. Is overall welfare increased or decreased? It is precisely the function of the optimizing program to determine this. Thus the upper end marginal rate may or may not be zero, but this can be determined only from the total trajectory, not from an informal argument concerning the top end in isolation.

In any case, $f(z_1) = 0$ implies that the top end is of negligible importance in the overall welfare picture.

Phelps (1971)

Phelps is concerned with the optimal redistributive income tax under a *Rawlsian* (maximin), rather than utilitarian, criterion. His *Model B*, in which the incentive effects fall on leisure (rather than education as in Model A) is closest to that given here. Furthermore, Phelps' Appendix reworks the problem in control theory form, making the comparison even easier.

The relevant expressions are the first order conditions set out as (ii) and (iii) in the Appendix:

$$
\begin{aligned}
\frac{-\partial H}{\partial t} &= q^* = m\, f\, \phi_t \\
\frac{\partial H}{\partial m} &= 0 = 1 - F - mf\phi_m + q
\end{aligned}
$$

Phelps' q is the costate associated with t, thus corresponding to our λ, and $\phi(t, m, y; g)$ $(= n)$ is the inversion of the optimal income function $y^* = \psi(t, m, n; g)$, where y is earnings and n is the skill index. Variables m, f $(= F')$ are essentially the same as ours.

Phelps shows that $\phi_t < 0$, $\phi_m > 0$, so that $q(y) \leq 0$, the same as our λ. Since $t(y_{max})$ is taken to be unconstrained, the transversality condition $q(y_{max}) = 0$ must be satisfied. From the second equation, therefore, we must have $mf\phi_m = 0$ at y_{max}. However we cannot conclude that m is necessarily zero, since the relationship is also satisfied by $f = 0$ or $\phi_m = 0$ (boundary optimum), consistent with our findings.

Cooter (1978)

Cooter optimizes an additive social welfare function similar to ours, and uses a control theory formulation, but his analysis is very different in many respects.

We shall accept his analysis at face value and concentrate on his argument that the marginal tax rate be zero at the top end.

Cooter derives ((iii) on p. 759)

$$T' = \frac{-\mu\{\cdot\}}{\lambda f g_2}$$

where λ is the costate variable associated with aggregate tax revenue (corresponding to our μ, and essentially positive), μ here is the costate associated with v, the indirect utility as a state variable, and

$$\{\cdot\} = \frac{\partial}{\partial x}\left(\frac{u_1 g}{n^2}\right)$$

with

$$g = g(v, x, n) = \arg\max_z u(-z/n, x)$$

The independent variable is the skill index n.

Note that Cooter uses x (final consumption) as the control, rather than the tax $T = z - x$ itself. He then argues that the transversality condition $\mu = 0$ necessarily implies $T' = 0$.

However the derivation given is incomplete, since $g_2 = dz/dx = 1/(1 - T')$. Fully solving for T' gives

$$T' = \frac{\mu\{\cdot\}}{-\lambda f + \mu\{\cdot\}}$$

Then if $f \neq 0$, $\mu = 0$ implies $T' = 0$, but $f = 0$ implies $T' = 1$ if $\mu \neq 0$ and is consistent with any value of T' for $\mu = 0$. Again, the argument that the transversality condition *necessarily* implies $T' = 0$ fails to hold, and the actual result is consistent with the current paper.

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