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Individual Homelessness: Entries, Exits, and Policy

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Abstract:

Homelessness is part of the lives of many people. But almost no one is homeless for all or most of his or her life. The median shelter homeless spell is well under a month, and even "chronic homelessness" officially entails spells of a year or so. I model homelessness as part of people's lives in a dynamic stochastic framework in continuous time. I can explain many empirical regularities with a parsimonious model: for instance, why the last addresses of homeless people are concentrated in a few low-rent neighborhoods, why homeless entries are hard to predict, why recidivism is common and past homelessness is a good predictor of future homelessness, why some groups recidivate more often than others, why the hazard rate for shelter exit is single-peaked, why effective homelessness prevention programs do not alter the average length of homeless spells. I also examine policy. The optimal homelessness prevention program is Pigouvian and starkly simple. With an optimal prevention program in place, optimal shelter quality maximizes a simple and intuitive expression, and insurance programs always raise social welfare. Most of the previous economics literature about homelessness has been static, but most literature about homelessness outside economics has been dynamic. This paper tries to bring the two strands of literature closer together.

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Homelessness is part of the lives of many people. Link et al (1994) found that over 7 percent of the U.S. population had been literally homeless at some point in their lives. But almost no one is homeless for all of his or her life. The median shelter stay for a single adult in 2009 was 17 days; the median stay for a family with children was 36 days (U.S. Department of Housing and Urban Development, 2010, p. 36). Even “chronic homelessness” officially entails spells of a year or more. To understand homelessness, we have to understand how it fits into people’s lives.

In this paper, I model homelessness as part of people’s lives. The motivation is practical. Policy-makers have shown rising interest in homelessness prevention—intervening before people become homeless or early in a homeless spell. “Homelessness prevention and rapid rehousing” received one of the largest appropriations of any social service program in the 2009 economic stimulus package (the American Recovery and Reinvestment Act).

The important questions in designing and evaluating a homelessness prevention program are dynamic and stochastic: can homelessness be predicted? If so, what information predicts it? What interventions will affect the probability of becoming homeless and how will they affect behavior? How will interventions affect incentives to save and to consume before homelessness prevention programs kick in? These are questions I try to answer. The important questions in designing shelters and in deciding how difficult authorities should make living on the streets are also dynamic and stochastic: will better facilities draw too many people or cause them to linger too long? Or, on the other hand, is the insurance that good facilities and forgiving streets provide worth the moral hazard cost?

In a larger context, our questions are about the design of a social safety net. Income (Gottschalk and Moffitt 2009) and consumption (Gorbachev 2011) volatility have risen since 1980 (as has homelessness, although we are agnostic about a connection between the two trends), and shelters and homelessness prevention programs have become a major part of the social safety net. Few people have thought about how they fit in.

Most previous literature in economics about homelessness has been static. Theoretical models (e.g., O'Flaherty 1995) have asked about what determines the steady-state point in time (PIT) count of homeless people. Most of the empirical literature (e.g., Quigley, 1990; Honig and Filer, 1993; Early and Olsen, 1999, 2002; for a survey see O'Flaherty 2004) approaches the same question, looking for empirical determinants of PIT counts in cross-sections of cities. Four empirical papers (Cragg and O'Flaherty 1999; Culhane et al. 2003; O'Flaherty and Wu 2006, 2008) have followed large city shelter population over time, but the observations in these papers are PIT shelter populations; they do not follow individuals. Allgood, Moore and Warren (1997) and Allgood and Warren (2003) are among the few economics papers I am aware of that use longitudinal microdata; they concentrate on the length of time that people spend homeless, but do not include data on the length of time that people spend non-homeless. Hall and Freeman (1989) also look at durations of homeless spells.

By contrast, the literature outside economics abounds in longitudinal microdata, and many researchers study homelessness as part of the life-course. "How did you become homeless?" is a natural question for researchers to ask, even though it seems never to have occurred to economists. Shinn et al (1998), for instance, follow homeless families in New York City for a long period of time, both before and after their shelter experiences. Piliavin et al (1993) and Dworsky and Piliavin (2000) are among the many papers that follow homeless careers. The main way that interventions are studied is to follow individuals in a treatment group and a control group over an extended period of time; the papers that do this are too numerous to cite.

Thus in looking at homelessness as part of a dynamic stochastic process I am trying to move economics closer to psychiatry, social work, and public health.

In the positive part of this paper, I find that simple models of assets as an Ito process and homelessness as depletion of assets can explain parsimoniously many empirical regularities. For instance, why are the last addresses of homeless people are concentrated in a few low-rent neighborhoods? Because as a household depletes its assets and approaches homelessness, it reduces its housing consumption (although not to zero, and many households consume the same low quantity of housing even if they differ in assets). Why are homeless entries hard to predict? Because they are the result of stochastic processes, and the processes would not be stochastic if they could be predicted well. Why is recidivism common and past homelessness a good predictor of future homelessness? Because households leave homelessness with low net assets, and low assets make entry into homelessness more likely in the near future. Why do some groups recidivate more often than others? Because they use worse shelters and hence leave homelessness with less assets. Why is the hazard rate for shelter exit single-peaked? Because the distribution of homeless spell durations is Levy, and the hazard rate function associated with the Levy distribution is single-peaked. Why don't effective homelessness prevention programs alter the average length of homeless spells? Because the processes involved are Markovian: given that a household has just entered homelessness, the history of how it got there is irrelevant.

All of these empirical regularities have been explained separately and in ad hoc ways; the contribution of this paper is to provide a parsimonious unified explanation. To keep the explanations parsimonious, for instance, I do not study borrowing and saving constraints. Studying such constraints would be a worthwhile extension of this paper, but I suspect that savings constraints are more important in this problem than the borrowing constraints that are usually emphasized. Indeed, in many contexts in this paper, adding borrowing constraints would be socially desirable.

I also look at policies. I derive the optimal Pigouvian homelessness prevention program, which turns out to be remarkably simple: a flat subsidy to housed households contingent on being housed. If homelessness is bad, then the government should either tax homelessness or subsidize non-homelessness. Since taxing homelessness is cruel and impractical, subsidizing non-homelessness is the obvious policy, and the flat rate follows from the fact that all days non-homeless are equally valuable. With an optimal homelessness prevention program in place, many other questions have intuitive and straightforward solutions. The optimal shelter quality (and street enforcement regime) maximizes the instantaneous difference between benefits and costs, and any actuarily fair insurance that reduces volatility is welfare-improving. These are first-best results. I cannot derive easy formulas for the second-best results that obtain when an optimal homelessness prevention program is not in place, but I can indicate the directions in which they differ from first-best.

Section 1 describes the model of life outside homelessness, and derives some basic results about how and when households enter homelessness. Section 2 studies in more detail the process of entering homelessness. Section 3 describes the model of homelessness, and links it to the model of life outside homelessness. (The link between these two spheres is another contribution that this paper makes.) The last three sections are normative. Section 4 is about optimal homelessness prevention programs, section 5 is about insurance, and section 6 is about shelter quality. Proofs are collected in the appendix.

1. The Model and Behavior

Consider an eternally-lived household that consumes only one good, housing. At time *t*, the household has assets *A(t)*, and chooses consumption at rate *c(t)≥0.* Time is continuous, and the rate of housing consumption can be varied costlessly (for instance, by doubling up or allowing a furnace to deteriorate). The price of a unit of housing consumption is one per unit time.

Assets include financial assets like savings accounts; physical assets like cars or tools; human assets like health and skills; and social assets like employment contacts and the willingness of friends and relatives to help. We ignore the possibility of changes in household composition.

Luck manifests itself as unforeseen changes in assets. In particular, assets evolve according to a time-invariant Ito process

(1.1)

$$dA = (rA - c)dt + s(A)dw.$$

In (1.1), the rate of return on assets is *r*, *dw* is a standard Wiener process, and *s(A)* is a function that will vary to reflect different assumptions about the stochastic process under which assets evolve. I will discuss Ornstein-Uhlenbeck processes in section 4.4 below.

If assets fall to zero, the household becomes homeless and consumes zero housing. In a few special cases, the household may also consume zero housing even though it has positive assets. We call the two types of homelessness asset-homelessness (zero assets) and consumption-homelessness (zero consumption but positive assets) when there is a danger of confusion; otherwise homelessness will refer to asset-homelessness. The value of becoming (asset-) homeless is *P≥0.* (We will also relax this non-negativity constraint in the section 1.6 below.) In this section we will treat *P* as exogenous and homelessness as an absorbing state. We will endogenize *P* and allow escapes from homelessness in section 3.

Instantaneous utility is an increase, strictly concave, thrice continuously differentiable function *U(c).* I normalize *U(0)=0,* and *U'''(c)≥0.* I also assume

(A.1)

$$\lim_{c\to\infty}\frac{\int_0^c U'(c)ds}{\int_0^c U'(s)ds} < 1.$$

All standard utility functions satisfy this condition.

The household maximizes the discounted value of present and expected future utility flows. The discount rate is *r*, the same as the rate of return on assets. (Perhaps households discount at a rate higher than the return on financial assets, but other assets like health and friendship may have much higher rates of return than financial assets.) I assume that being homeless is worse than some level of consumption

$$rP < \lim_{c\to\infty} U(c).$$

A close variant of this problem has been studied in papers by Merton (1971), Lehoczky et al. (1983), and Karatzas et al. (1986). The household in those papers, however, can choose a level of risk to assume as well as a rate of consumption; there is a riskless asset and the household can insure itself completely. The risks that I am interested in cannot be insured against completely, and so for simplicity I assume that they cannot be insured against at all (or that all available insurance has already been purchased). The missing market in this model, then, is insurance. In section 6, I look at the welfare effects of allowing changes in volatility.

1.1 General results on behavior

Let *V(A)* denote the expected discounted value of present and future utility flows for a household with assets *A* that consumes optimally. The Hamilton-Jacobi-Bellman (HJB) equation for maximizing expected discounted lifetime utility is

(1.2)

$$rV(A) = \ sup_c \left[U(c) + (rA - c)V'(A) + \frac{1}{2}s^2(A)V''(A)\right].$$

(See Dixit and Pindyck, 1994, pp. 106-08 or Stokey, 2009, pp. 31-34 for a derivation.)

Let *c(A)* denote the value of *c* that attains the supremum in (1.2) when assets are *A,* and assume that this is interior. (I will discuss a case when the supremum is not interior in section 1.6 below.) The first-order condition is

(1.3)

$$U'\big(c(A)\big) = V'(A).$$

Differentiating this twice with respect to *A* gives

(1.4)

$$U''\big(c(A)\big)c'(A) = V''(A)$$

(1.5)

$$U''\big(c(A)\big)c''(A) + U'''\big(c(A)\big)c'(A) = V'''(A).$$

Applying the envelope theorem to (1.2) we obtain

$$rV'(A) = rV'(A) + (rA - c)V''(A) + s(A)s'(A)V''(A) + \frac{1}{2}s^2(A)V'''(A).$$

Substituting from (1.3), (1.4), and (1.5) gives

$$(c(A) - rA)U''(c)c'(A) = s(A)s'(A)U''(c)c'(A) + \frac{1}{2}s^2[U''(c)c''(A) + U'''(c)c'(A)]$$

Since *U(.)* is strictly concave, we can divide by *U''(c)* to obtain

(1.6)

$$(c - rA)c'(A) = s(A)s'(A)c'(A) + \frac{1}{2}s^2(A)\left[c''(A) + \frac{U'''(c)}{U''(c)}c'(A)\right].$$

Equation (1.6) is a second-order nonlinear differential equation in *c(A).* It will be our basic tool for describing how households arrange consumption.

As an initial condition, by definition,

$$V(0) = P.$$

Denote *c(0)*as c_0. Hence in (1.2):

(1.7)

$$rP = U(c_0) - c_0U'(c_0) + \frac{1}{2}s^2(0)U''(c_0)c'(0).$$

Denote $J(x) = U(x) - xU'(x)$. It is easy to prove that *J(x)* is always nonnegative for nonnegative *x,* and positive for positive *x,* and *J'(x)>0.*

For any *P*, let *h* solve

(1.8)

$$rP = J(rh).$$

Notice that if *P=0, h=0.* We can show:

Proposition 1.1.1. *If P>0, equation (1.8) has a positive solution.*

Proofs are gathered in the appendix.

We can prove some general results about how households behave as they approach homelessness.

Proposition 1.1.2. *(a) If either c'(0)=0 or s(0)=0, then* $c_0 = rh$.

(b) If not, then $c_0 > rh$.

(c) If P>0, then $c_0 > 0$.

Thus as long as *P>0*, the household has a positive minimum level of consumption, and at some point will spend above a long-run sustainable level.

1.2 Certainty

The easiest case to examine is certainty. Certainty implies no stochastic component: *s(A)=0* for all *A.* Studying certainty lets us understand what solutions look like, and it also shows that certainty is a bad approximation to actual homelessness.

Plugging *s(A)=0* into (1.6) and (1.7) we obtain

$$(c - rA)c'(A) = 0$$

$$rP = J(c_0).$$

Hence $c_0 = rh$, by definition. Since $(c_0 - r * 0) > 0$, if *P>0, c'(0)=0.* Hence the behavioral relationship between assets and consumption is

(1.9)

$$c(A) = r \min[h, A].$$

Figure 1.1 illustrates.

Figure 1.1 about here.

Hence

$$dA = r \min(A - h, 0).$$

Households that start with assets of *h* or more consume at the sustainable level and maintain their assets at the initial level. Households that start with assets below *h* consume at a constant unsustainable level until their assets are all gone and they become homeless. After this group becomes homeless, no one else becomes homeless. If *P=0*, no one ever becomes homeless. An outside observer who knows current consumption of a household can predict whether it will become homeless, but if the observer predicts it will become homeless, she cannot tell when unless she can observe assets.

1.3 General results under uncertainty.

Suppose that some uncertainty is present. Define

$$Q(c) = \frac{U'''(c)}{U''(c)}$$

$$B(A,c) = rA + s(A)s'(A) + \frac{1}{2}s^2(A)Q(c).$$

Then (1.6), the second order nonlinear differential equation for consumption, can be rewritten:

(1.10)

$$\left[c(A) - B\big(A, c(A)\big)\right]c'(A) = \frac{1}{2}s^2(A)Q\big(c(A)\big).$$

This form allows us to see some general properties of the consumption function *c(A)*, and to solve explicitly for a number of important special cases.

Consider the locus in *(A,c)* space:

$$c - B(A,c) = 0.$$

Suppose that this locus is linear and upward-sloping, and

$$c_0 - B(0, c_0) > 0.$$

Let A_0 solve

$$c_0 - B(A_0, c_0) = 0.$$

Then it is obvious that

(1.11a)

$$c(A) = c_0 = rh \quad if\ A \leq A_0$$

(1.11b)

$$c(A) - B\big(A, c(A)\big) = 0 \quad if\ A > A_0$$

solves equation (1.10).

Thus I define a combination of a utility function *U(.)* and an instantaneous variance function *s(.)* as <u>linear solvable</u> if the resulting locus

$$c - B(A, c) = 0$$

Is linear and upward-sloping, and

$$c_0 - B(0, c_0) > 0.$$

For the most part in this paper I will concentrate on linear solvable pairs.

In (1.11), I call the set $\{A | A \leq A_0\}$ the <u>lower segment</u>, and its complement the <u>upper segment.</u>

I have not argued that if a pair is linear solvable, (1.11) is the unique solution to (1.6). However, (1.11) is a solution that approaches the unique non-trivial solution to the problem with certainty as the *s(.)* function approaches the zero function.

I cannot say much about pairs that are not linear solvable, except that in most cases $c''(A)>0$ in the neighborhood of $A=0$. Fortunately, many frequently used pairs are linear solvable. I describe four such pairs in the next section.

1.4 Some linear solvable pairs

1.4.1 Arithmetic Brownian motion

If assets evolve as arithmetic Brownian motion, then $s(A)=\sigma$, a constant. Then

$$B(A, c) = rA + \sigma^2 Q(c).$$

In particular

$$B(0, c) = \sigma^2 Q(c) \leq 0.$$

Hence

$$c_0 - B(0, c_0) > 0.$$

Thus $c''(A) \geq 0$ for A in the neighborhood of zero. I examine two special cases of linear solvability.

First, suppose utility is quadratic: $U(c)=c-bc^2$, where b is a positive constant. Then $Q(c)=0$, and $c - B(A, c) = c - rA$, which is linear and upward-sloping. The pair of arithmetic Brownian motion and quadratic utility is linear solvable. The relationship between assets and consumption is the same as the relationship under certainty, (1.9).

$$c(A) = r \min[h, A].$$

Assets evolve according to

$$dA = \sigma\, dw \quad if\ A \geq h$$

$$=-(h-A)r\, dt + \sigma\, dw \quad if\ A<h.$$

Households on the lower segment are losing assets on average, with poorer households losing faster, but they are not certain to reach homelessness. Households on the upper segment are maintaining their assets on average, but they are not certain to escape homelessness.

Next, consider constant absolute risk aversion (CARA). Let $U(c) = 1 - e^{-\lambda c}$, where λ is a positive constant. Then $Q(c)=-\lambda$ and

$$c - B(A, c) = c - rA + \frac{1}{2}\sigma^2 \lambda.$$

Let

$$A_0 = h^* = h + \frac{1}{2}\frac{\sigma^2 \lambda}{r}.$$

$$c(A) = rA - \frac{1}{2}\sigma^2 \lambda \quad if\ A \geq h^*$$

$$c(A) = rh\ if\ A < h^*.$$

Figure 1.2 illustrates.

Figure 1.2 around here.

Households with assets greater than *h** consume less than a sustainable level and so their assets increase on average. For these households, savings are precautionary in the sense that they increase with the volatility of assets. Households with assets between *h* and *h** also save on average. But households with assets less than *h* consume at an unsustainably high level and dissave on average. But with luck it is possible to move from one side of *h* to the other.

Assets are arithmetic Brownian motion with drift.

$$dA = \frac{1}{2}\sigma^2 \lambda\, dt + \sigma\, dw \quad if\ A \geq h^*$$

$$dA = -r(h - A)dt + \sigma\, dw \quad if\ A < h^*.$$

The drift is positive above *h*, negative below *h.*

1.4. 2. Geometric Brownian motion

If assets evolve according to geometric Brownian motion, $s(A)=\sigma A$, where σ is a positive constant. Then in (1.7), $s^2(0)=0$, and so $c_0=rh$. We also have

$$B(A,c) = rA + \sigma^2 A + \frac{1}{2}\sigma^2 Q(c).$$

Thus $c_0 - B(0, c_0) = c_0 > 0$. Hence *c''(0)≥0.*

Once again, I focus on two special utility functions.

If utility is quadratic, then

$$c - B(A, c) = c - (r + \sigma^2)A$$

And so

$$c(A) = \max(rh, (r + \sigma^2)A.$$

Let

$$h^{**} = \frac{rh}{r + \sigma^2}.$$

This is the dividing line between the upper and lower segments.

Figure 1.3 illustrates.

Figure 1.3 around here.

Thus households consume at unsustainably high levels no matter whether their assets are high or low; but they consume relatively more when assets are low. Consumption is geometric Brownian motion with negative drift when assets are high, constant when assets are low.

Assets are also geometric Brownian motion with negative drift when assets are high:

$$dA = -\sigma^2 A\, dt + \sigma A\, dw \quad if\ A \geq h^{**}$$

$$dA = r(A - h)\, dt + \sigma A\, dw \quad if\ A < h^{**}.$$

Notice that consumption (and asset depletion) rises when σ^2 rises; greater risk implies more spending.

Next assume that utility has constant relative risk aversion (CRRA). For $\eta \,\epsilon [0,1]$:

$$U(c) = \frac{c^{1-\eta}}{1-\eta}$$

Then

$$Q(c) = -\frac{\eta+1}{c}.$$

The pair of CRRA utility and geometric Brownian motion is linear solvable if and only if

(1.12)

$$(r+\sigma^2)^2 - 2\sigma^2(\eta+1) \geq 0.$$

I assume that this holds. Sufficient conditions for (1.12) to hold include $r \geq \frac{1}{2}(\eta+1)$ and $\sigma^2 \geq 2(\eta+1)$.

Then

$$c - B(A,c) = c - (r+\sigma^2)A + \frac{1}{2}\sigma^2 A^2 \frac{\eta+1}{c} = \left[c^2 - (r+\sigma^2)Ac + \frac{1}{2}\sigma^2 A^2(\eta+1)\right]\frac{1}{c}$$

$$= \frac{1}{c}(c - z_1(\sigma,r)A)\big(c - z_2(\sigma,r)\big),$$

where

$$z_1(\sigma,r) = \frac{1}{2}[(r+\sigma^2) - \sqrt{(r+\sigma^2)^2 - 2\sigma^2(\eta+1)}]$$

$$z_2(\sigma,r) = \frac{1}{2}[(r+\sigma^2) + \sqrt{(r+\sigma^2)^2 - 2\sigma^2(\eta+1)}]$$

and z_1 and z_2 are positive and real by (1.12). The second root violates second-order conditions; recall that (1.6) was derived as a first-order condition in a maximization problem. Thus

$$c(A) = \max[z_1(\sigma, r)A, rh].$$

Relative to a household with quadratic utility, a household with CRRA utility consumes less, but it is ambiguous whether such a household saves or dissaves on net on the upper segment. If $r > \frac{1}{2}(\eta + 1)$ or if $r > \sigma^2$, then $r > z(\sigma, r)$; otherwise $r \leq z(\sigma, r)$. Thus when r is high, absolutely or relative to σ^2, the household saves when its assets are high; when *r* is low, it dissaves.

The evolution of assets is similar to that under quadratic utility, although the drift of geometric Brownian motion will depend on whether the household is saving or dissaving.

1.5 Value functions

In all linear solvable problems, consumption is a linear function of assets on the upper segment, and constant with respect to assets on the lower segment. Thus from equation (1.4) *V''(A)=0* on the lower segment and *V''(A)<0* on the upper segment. The value function is concave. The value function is also continuously differentiable. This is because consumption is a continuous function of assets in all our cases, even where the upper and lower segments meet, and equation (1.3) states that the derivative of the value function equals marginal utility of consumption.

1.6 Severe homelessness

Although it is plausible that $P \geq 0$ in most actual applications, since people who enter homelessness almost always have an option to leave it eventually, studying the case *P<0* is helpful for understanding some aspects of optimal prevention in section 4. Suppose then that *P<0* and *U(.)* does not satisfy the Inada condition; so $U'(0) < \infty$. For simplicity, I assume that assets evolve according to arithmetic Brownian motion.

Since *J(.)* is nonnegative, the boundary condition $J(c_0)=rP$ cannot hold, but the HJB condition (1.2) must hold everywhere. Hence for small *A,* the supremum in the HJB equation must be attained at the boundary *c=0.* Thus the first-order condition must be that for some set of sufficiently small *A*:

(1.13)

$$U'(0) - V'(A) < 0.$$

Plugging *c=0* into the HJB equation and rearranging yields

(1.14)

$$\frac{1}{2}\sigma^2 V''(A) + rAV'(A) - rV(A) = 0.$$

This is a nonlinear homogeneous second-order differential equation. It does not have a closed form solution. Let $\hat{A}$ denote the greatest value for which (1.13) holds.

Then we have two boundary conditions on (1.14). First, when the supremum of the HJB condition moves to the interior:

(1.15)

$$V'(\hat{A}) = U'(0).$$

Second, since *V(0)=rP<0*, equation (1.14) when *A=0* implies

(1.16)

$$V''(0) = \frac{2rP}{\sigma^2} < 0.$$

Then we have:

Proposition 1.6. *The value function is strictly concave and increasing on (0, $\hat{A}$), and V($\hat{A}$)=0.*

Intuitively, when running out of assets is very bad—that is, *P<0*—households will reduce their consumption to avoid it. They will go so far as to reduce their consumption to zero when their assets are still positive. Consumption-homelessness without asset-homelessness may seem strange, and for most of this paper we will not need to discuss it. We will need it in discussing certain homelessness prevention programs in section 4.2.4. These are for the most part prevention programs when shelter and insurance conditions are far from optimal.

2. Predictions about entering homelessness

From these results about how assets evolve, we can derive expressions about the probability that a household ever enters homelessness, and the expected time before it does so. Everything else being equal, the smaller the assets a household has, the more likely it is to become homeless eventually, and the sooner it is likely to become homeless. Since housing consumption is correlated with assets, the less housing a household is currently consuming, the more likely it is to become homeless eventually and the sooner it is likely to become homeless. It is a well-known empirical finding that within a city, homeless households come disproportionately from a handful of neighborhoods (in the sense of their last address before becoming homeless), and that these neighborhoods have the cheapest and least desirable housing stock in the city (see, for instance, Culhane et al. 1996). This finding is a simple prediction of our model (and not any indication of a causal connection—the idea, for instance, that these neighborhoods "breed" homelessness).

Specifically, I will be concerned with $\psi(A, r)$, the expected discount factor for reaching homelessness when the discount rate is *r.* (When there is no risk of confusion, I will omit the argument *r.*) Let *A(t)* denote assets at time *t*. Let *T* denote the time at which assets first reach zero:

$$T = \inf[t|A(t) = 0].$$

This is a random variable. Then

$$\psi(A, r) = E(e^{-rT}).$$

There are two reasons why I am concerned with $\psi(A, r)$ rather than the expected time to homelessness. The first is my interest in prevention or delay. The benefit of some intervention or policy that changes a household's path to homelessness is the change in $\psi(A, r)$ times the cost of homelessness. Changes in $\psi(A, r)$ are our basic policy concern. Second, in many cases expected time to homelessness does not exist or is trivial. If there is a finite probability that a household will not become homeless, then expected time to homelessness does not exist. So $\psi(A, r)$ is both more relevant and more interesting than expected time to homelessness.

I will present results for both arithmetic and geometric Brownian motion, but concentrate on arithmetic.

2.1 Arithmetic Brownian motion

In the two cases of Brownian motion that we studied in section 1.4, assets evolved according to

$$dA = \mu\, dt + \sigma\, dw \quad if\ A\ \geq K$$

$$dA = r(A - h)dt + \sigma\, dw\ if\ A < K.$$

Specifically, with quadratic utility, μ=0 and *K=h.* With CARA utility, $\mu = \frac{1}{2}\sigma^2\lambda$ and *K=h*.*

Following Stokey (2009, p. 93), *ψ(.)* is the solution to the differential equation (2.1) and (2.2) subject to the four boundary conditions discussed below:

(2.1)

$$\frac{1}{2}\sigma^2\psi'' + \mu\psi' - r\psi = 0 \;\; if \; A \geq \; K$$

(2.2)

$$\frac{1}{2}\sigma^2\psi'' + r(A-h)\psi' - r\psi = 0 \;\; if \; A < \; K$$

The first boundary condition is *ψ(0)=1.* The expected discount factor for a household already homeless is one.

Second

(2.3)

$$\lim_{A\to\infty} \psi(A) \; = 0.$$

For any positive number, it is possible to imagine a household sufficiently rich that its expected discount factor is less than that number.

Third is the value-matching condition that the function *ψ(.)* is continuous at *K*:

$$\lim_{A\downarrow K} \psi(A) \; = \lim_{A\uparrow K} \psi(A) \, .$$

Fourth is the smooth-pasting condition that the derivative *ψ'(.)* is continuous at *K:*

$$\lim_{A\downarrow K} \psi'(A) \; = \lim_{A\uparrow K} \psi'(A) \, .$$

The intuition behind these last two requirements is the following. Consider two asset values, K^- slightly less than K, and K^+ slightly more than K, and how assets evolve over a small time Δt. The distributions of assets at time $(t+\Delta t)$ will be about the same no matter whether $A(t)= K^-$ or $A(t)= K^+$, which is the value-matching condition. Now increase K^- to $K^- +\Delta K$ and K^+ to $K^+ +\Delta K$, where ΔK is very small. This increase will cause a change in the $(t+\Delta t)$ distribution of assets that is approximately the same in both cases. This gives us the smooth-pasting condition.

Since (2.1) and (2.2) are a pair of second-order differential equations, and we have four boundary conditions, their solution subject to these conditions is unique generically.

Consider (2.1) first, since it has closed form solutions. In general, a solution to (2.1) will have the form

$$\psi(A) = k_1 e^{R_1 A} + k_2 e^{R_2 A}$$

where k_1 and k_2 are constants and

$$R_1 = \frac{-\mu - Z}{\sigma^2} < 0$$

$$R_2 = \frac{-\mu + Z}{\sigma^2} > 0$$

and

$$Z = \sqrt{\mu^2 + 2r\sigma^2}.$$

From the second boundary condition (2.3), k_2=0.

Since consumption on the lower segment is greater than it would be if the consumption rule on the upper segment were followed there, households become homeless sooner than they would if they followed the upper segment consumption rule always. If households followed the upper segment consumption rule, $\psi(A)$ on the upper segment would be $e^{R_1 A}$. Hence *k_1>1.* Thus for *A≥K,* $\psi(A)$ is a convex, decreasing function.

Equation (2.2) on the lower segment is more difficult, since it appears impossible to a find a closed form that characterizes all solutions. I can show, however, that the solution that satisfies the boundary conditions is decreasing and convex.

Proposition 2.1. *The solution to (2.1) and (2.3) that satisfies the four boundary conditions is decreasing and strictly convex if utility is quadratic or CARA.*

2.2 Comparative statics with arithmetic Brownian motion

There are two interesting comparative statics questions: the effect of greater volatility, and the effect of greater payoff from homelessness.

The latter is easier to think about. An increase in *P* operates through an increase in *h.* An increase in *h* has two effects, and they both increase *ψ(A)* and make homelessness more imminent. First, higher *h* increases consumption on the lower segment and so increases the drift toward homelessness. Second, it increases the extent of the lower segment.

The effect of volatility is more complex. Consider the case of quadratic utility first. In general, volatility makes homelessness more likely sooner. If there were no lower segment, this could be demonstrated easily: because $\partial R_1 / \partial \sigma^2 > 0$, an increase in volatility would raise ψ everywhere, except at zero, and make it flatter. Similarly, I can show (by methods similar to those in proposition 2.2 below) that if there were no upper segment, only a lower segment, increasing volatility would raise ψ and make homelessness more imminent.

In the extreme, in a model without volatility, $\psi(.)$ would be zero on the upper segment and would be a linear function of assets on the lower segment. Households above *h* would never become homeless, and households below *h* would be slowly moving toward homelessness (when $\sigma=0$, CARA and quadratic behavioral relations coincide). Only volatility can move a household on the upper segment to homelessness, and the more volatility present, the sooner homelessness is likely to occur. On the pure lower segment, the household is drifting toward homelessness, and greater volatility speeds the process on net.

When the upper and lower segments are considered together, as they must be, and the value-matching and smooth-pasting conditions must be met, I cannot prove that the effects are unambiguous. On the lower segment, greater volatility increases the probability that a household will escape to the upper segment soon, where it will not be drifting toward homelessness. On the upper segment, greater volatility increases the probability that the household will fall to the lower segment soon, but also raises the probability that after it falls to the lower segment it will regain the upper segment soon.

If the latter effect is not too big, then we can say that greater volatility raises ψ. Specifically, consider σ^2 and $\sigma^2+\Delta$, $\Delta>0$. Let hats denote solutions to (2.1) and (2.2) with $\sigma^2+\Delta$; absence of hats denotes solutions with σ^2. Then

Proposition 2.2. *If*

$$\hat{\psi}(K) = \widehat{k_1} e^{\widehat{R_1} K} \geq \psi(K) = k_1 e^{R_1 K},$$

then $\hat{\psi}(A) \geq \psi(A)$ *for all* A.

With CARA utility, increasing volatility also raises precautionary saving on the upper segment, and so there is another reason for ambiguity. In particular, ${\partial R_1}/{\partial \sigma^2} > 0$ if and only if $\sigma^2\lambda^2 < 8r$.

2.3 Geometric Brownian motion

In the two examples I presented where the behavioral relationship was closed-form and piecewise linear, optimal assets evolved according to

$$dA = \mu A\, dt + \sigma A\, dw \quad if\ A \geq K$$

$$dA = r(A-h)dt + \sigma A\, dw\ if\ A < K.$$

For quadratic utility $\mu = -\sigma^2$ and *K=h***. For CRRA utility, *μ=r-z* and $K = \frac{rh}{z_1(\sigma,r)}$.

Note that if

(2.6)

$$rh \geq \frac{1}{2}\sigma^2$$

fails, then households never become homeless because shocks disappear as assets approach zero. Once again, a certain amount of volatility must be present in order for entries into homelessness to be observed. For the rest of this section, I assume that (2.6) holds.

By the usual arguments, ψ obeys the differential equation

(2.7)

$$\frac{1}{2}\sigma^2 A\psi'' + \mu A\psi' - r\psi = 0\ \ if\ A \geq\ K$$

(2.8)

$$\frac{1}{2}\sigma^2 A\psi'' + r(A-h)\psi' - r\psi = 0 \;\; if\; A < \; K$$

subject to the same four boundary conditions we used with arithmetic Brownian motion.

On the upper segment, (2.7) is solved by

$$\psi(A) = kA^R$$

where

$$R = -\frac{1}{2}\left[\left(\mu - \frac{\sigma^2}{2}\right) + Z\right] < 0$$

$$Z = \sqrt{\left(\mu - \frac{\sigma^2}{2}\right)^2 - 2r\sigma^2}.$$

The lower segment has no complete set of closed-form solutions for $\sigma^2 > 0$. As with arithmetic Brownian motion, however, it can be proved that the solution subject to the boundary conditions is decreasing and convex.

Proposition 2.3. *The solution to equations (2.7) and (2.8) that satisfies the four boundary conditions is decreasing and convex with either of the two special cases of utility, if (2.6) holds.*

3.0 Homeless spells

Next we turn to a description of homeless spells. There are probably many different kinds of homeless spells, depending on many dimensions of the homeless experience, but I will try to work with a simple model that allows only a few policy variables.

3.1 Optimal behavior

A homeless household, in this model, pays no rent but receives instantaneous utility of *H*, which I take to be constant over time.

While homeless, a household's value of being housed evolves over time. Part of the reason for this evolution is that the assets the household has still change while the household is homeless, but the household's ability to use assets may also change. Let *V* denote the continuation value of being housed. While a household is homeless, I assume

(3.1)

$$dV = \mu_h \, dt + \sigma_h dw.$$

The parameters μ_h and σ_h are not necessarily connected to other parameters in the model, since living without a home can be a different experience from housed living. I do not commit myself on the sign of μ_h: perhaps it is positive because rent does not have to be paid, perhaps it is negative because health and skills deteriorate, perhaps it is zero.

Notice also that I have modeled value as evolving during homelessness, not assets. This is primarily for convenience, because value as we have seen is not a simple linear function of assets (and because I did not want to define a notional value of negative assets). Even if we think of the evolution of value as being driven solely by the evolution of assets, the assumption of constant instantaneous coefficients is a strong one, and so the assumption I am using here is in some sense no more demanding than the assumption that assets evolve according to arithmetic Brownian motion.

Since value is a concave function of assets, the assumption I am using might be interpreted as requiring larger instantaneous coefficients when assets and value are high. That would be like assuming that assets evolve according to geometric Brownian motion and value follows assets. But my approach is simpler.

Households want to leave homelessness in order to enjoy the flow utility of standard housing. To leave homelessness, a family must pay a one-time fixed cost *F*—search costs, perhaps a deposit, furniture, moving costs, learning about a new environment. So a household with small but positive assets may remain homeless, even though it would remain housed with the same assets if it started off housed. Fixed costs of leaving homelessness make history matter. I assume that *F* is measured in utility, and so can be subtracted directly from *V.*

Let $\Psi_h(V,P)$ denote the expected discount factor for hitting *V* when value begins at *P* and evolves according to (3.1). Then the household's expected utility when it first enters homelessness (at value *P*) is

$$\max_V \left\{ H\left(\frac{1-\Psi_h(V,P)}{r}\right) + \Psi_h(V,P)(V-F)\right\} = \max_V \left\{\frac{H}{r} + \Psi_h(V,P)(V-F-\frac{H}{r})\right\}.$$

The first term in this expression is the value of staying homeless forever, and the second is the value of the option to leave homelessness at the optimal time. See Dixit and Pindyck (1994, chapter 5). Finding the maximum is the problem of exercising a perpetual call option, and has been well studied (though more commonly with geometric than arithmetic Brownian motion).

It is easy to see that

$$\Psi_h(V,P) = e^{R(V-P)}$$

$$R = \frac{1}{\sigma_h^2}\left[-\mu_h - \sqrt{\mu_h^2 + 2r\sigma_h^2}\right] < 0.$$

Then first-order conditions imply that *V**, the optimal threshold value for leaving homelessness is

$$V^* = -\frac{1}{R} + F + \frac{H}{r}.$$

Notice that this is independent of *P.*

The threshold is higher if the cost of leaving or the instantaneous attractiveness of homelessness is greater, obviously, but a household does not exercise its option to leave precisely when housed value exceeds *(F+H/r).* The household does not give up its option unless housed value exceeds homeless value *(F+H/r)* by *(-1/R)>0.*

The option value of homelessness *(-1/R)* is greater when σ_h is greater: if value is more volatile, then waiting a little longer has a better chance of yielding a large gain. Volatility always improves the value of an option because the option does not have to be exercised if the realization is bad. On the other hand, the option value of homelessness is decreasing in μ_h. The faster the rate of growth of outside value, the lower the threshold at which a household wants to leave homelessness.

The optimal discount factor for exiting homelessness is

$$\Psi_h(V^*, P) = exp\left[R\left(F + \frac{H}{r} - P\right) - 1\right] < 1.$$

Higher exit costs, higher instantaneous utility while homeless, higher volatility, and slower outside improvement thus result in longer homeless spells.

Finally, maximized utility on entering homelessness is

$$\frac{H}{r} + \Psi_h(V^*, P)(V^* - F - \frac{H}{r})$$

$$= \frac{H}{r} - \frac{1}{R}\, exp\left[R\left(F + \frac{H}{r} - P\right) - 1\right] \triangleq K(P).$$

Since I defined *P* in section 1 as *V(0)*, continuation value on entering homelessness, the question is whether there exists a value $\pi < V^*$ that is a fixed point of *K(.).* The following proposition shows that the answer is yes:

Proposition 3.1. *There exists a unique P<V* such that K(P)=P.*

There is no closed-form solution for *P.* Comparative statics, however, are fairly straightforward. Any change that raises *K(π)* holding *π* constant raises (equilibrium) *P.*

It is easy to show that

$$\frac{\partial K}{\partial (H/r)} = 1 - \Psi_h(V^*, \pi) > 0$$

(3.2)

$$\frac{\partial K}{\partial F} = -\Psi_h(V^*, \pi) < 0$$

$$\frac{\partial K}{\partial R} = -\frac{1}{\mathrm{R}^3}\Psi_h(V^*, \pi)[V^* - \pi] > 0.$$

Hence greater instantaneous utility while homeless, lower fixed cost of exit, more volatility, and less positive drift of outside continuation value increase *P.*

The size of *P* is important for several reasons. It affects incentives to save and invest and the rate at which households enter homelessness as we saw in sections 1 and 2; it affects the lifetime expected utility of all households, but especially those at high risk of becoming homeless; and it affects the length of time that households stay homeless.

We consider the last effect here. As *P* rises, the distance *(V*-P)* that a household has to traverse before leaving homelessness goes down. *Ceteris paribus*¸ anything that raises *P* reduces (roughly speaking) the length of homeless spells. But the same parameters that affect *P* also affect *V**, and so the effects can be confounded.

By differentiation we obtain

(3.3)

$$\frac{\partial(V^* - P)}{\partial(H/r)} = 0$$

$$\frac{\partial(V^* - P)}{\partial F} = \frac{1}{1 - \Psi_h(V^*, \pi)} > 0.$$

These results are intuitive. Raising the instantaneous utility of being homeless increases *P* and *V** by the same amount, and so does not affect the length of homeless spells (once the incentive to become homeless is considered). Raising the fixed cost of leaving homelessness makes exits more difficult; it does not raise *P* and so the net effect is to lengthen homeless spells.

I have not been able to find unambiguous results on the effect of changes in *R* on the length of homeless spells.

3.2 Distribution of homeless spells

For many purposes, especially empirical work, it is useful to study the distribution of the lengths of homeless spells, not just their expected discount factor. The distribution of first hitting times is given by the Levy distribution. Let *f(t)* be the probability distribution (pdf) that a homeless spell is exactly *t* time long, and let *F(t)* be the (cdf) probability that a homeless spell is less than *t* time long. Then *F(t)* is the probability that arithmetic Brownian motion that starts at *P* reaches *V** before *t* time has passed.

The formulas are well-known (here Φ denotes the cdf of the standard normal distribution):

(3.4)

$$f(t) = \frac{V^* - P}{\sigma_h t^{3/2}\sqrt{2\pi}} exp\left[-\frac{(V^* - P - \mu_h t)^2}{2\sigma_h^2 t}\right].$$

(3.5)

$$F(t) = \Phi\left(\frac{-(V^* - P) + \mu_h t}{\sigma_h \sqrt{t}}\right) + exp\left[\frac{2\mu_h(V^* - P)}{\sigma_h^2}\right]\Phi\left(\frac{-(V^* - P) - \mu_h t}{\sigma_h \sqrt{t}}\right).$$

To understand these distributions, first let *t* go to infinity to find the probability of eventually leaving homelessness. Suppose $\mu_h > 0$. In (3.5) the first standard normal cdf approaches one, and the second approaches zero; hence the probability of leaving homelessness eventually is one. The same is true for $\mu_h = 0$. If $\mu_h < 0$, the reverse is true for the standard normal cdf's in (3.5), and so the probability of leaving homelessness eventually is

(3.6)

$$\lim_{t\to\infty} F(t) = exp\left[\frac{2\mu_h(V^* - P)}{\sigma_h^2}\right] < 1.$$

There is a positive probability of staying homeless forever.

Since we see almost no homeless spells of, say, twenty years' duration, the conclusion that $\mu_h \geq 0$ is tempting. But we have not included mortality in our model, and so evidence about very long homeless spells cannot be interpreted well. Homeless people are about three or four times more likely to die prematurely than is the general population (O'Connell 2005).

Second, consider the shape of the pdf. Differentiating (3.4) with respect to *t* yields

(3.7)

$$f'(t) = \frac{f(t)}{2t^2}\left[-\frac{\mu_h^2}{\sigma_h^2}t^2 - 3t + \frac{(V^* - P)^2}{\sigma_h^2}\right].$$

The quadratic in brackets has a maximum at a negative *t*, is positive for *t=0,* and is negative for *t* sufficiently large. Hence *f(t)* has a unique mode at some *T>0*; it is increasing below *T* and decreasing above *T.* For future reference, we denote by τ the unique positive root of the quadratic in brackets.

Since *F(t)* converges, and *f(t)* is decreasing for sufficiently large *t,*

(3.8)

$$\lim_{t\to\infty} f(t) = 0.$$

The hazard of exiting homelessness is often of empirical and practical concern. Consider the hazard function

$$\mathcal{H}(t) = \frac{f(t)}{1 - F(t)}.$$

The following proposition summarizes the properties of the hazard function.

Proposition 3.2. *(a)* $\mathcal{H}(t)$ *has a unique maximum at* $t^*>0$.

(b) *If* $\mu_h<0$, *then* $\lim_{t\to\infty}\mathcal{H}(t) = 0$, *and* $T<t^*\leq\tau$.

(c) *If* $\mu_h>0$, *then* $\lim_{t\to\infty}\mathcal{H}(t) = \frac{\mu_h^2}{2\sigma_h^2}$, *and* $t_1<t^*<\tau$, *where* t_1 *is the unique positive solution of*

$$t_1 = \frac{V^* - P}{3\sigma_h^2}.$$

The declining hazard rate for exit from homelessness after a certain period of time is sometimes interpreted in misleading ways: perhaps as evidence of heterogeneity among the population entering homelessness (some newly homeless people are different from others and so destined to become chronically homeless); perhaps as evidence of "shelterization" or some other ongoing process that causes deterioration in the skills, health, or attitudes of homeless people (that is, $\mu_h<0$). As we see now, the declining exit hazard rate after a certain amount of time is in fact evidence for neither of these hypotheses. We derived the declining hazard rate in a setting of homogeneity among homeless entrants –only assets matter and all homeless entrants have an identical stock of assets (zero). We also showed that the hazard rate eventually declined no matter what the sign of μ_h was. The exit hazard rate declines, eventually, even when homelessness improves outside prospects.

3.3 Recidivism

People who leave homelessness sometimes reenter it within a short time. Having recently been homeless is usually found in the literature to be a good predictor of future homelessness. See, e.g., Shinn et al 1998. Like the declining exit hazard rate, this empirical finding does not require either heterogeneity or harmful effects of homelessness. In our model, what matters are the assets that households have when they leave homelessness, and on the processes described in section 1 after they leave.

Let $v^{-1}(V)$ denote the inverse of the value function for non-homeless households that we studied in section 1. Then the expected discount factor for reentering homelessness for a household that has just left homelessness is $\psi(v^{-1}(V^* - F))$, where $\psi(.)$ is as in section 2.

Thus recidivism depends (although not entirely) on

$$V^* - F = -\frac{1}{R} + \frac{H}{r}.$$

Recidivism is common and quick if this difference is small, rare and delayed if this difference is large. Thus recidivism is more common if shelters are not very attractive (since people will leave in very poor circumstances), and if the option value of staying homeless is small (outside opportunities are not volatile). In many locations, for instance, shelters for single adults are less attractive than shelters for families with children. *Ceteris paribus,* single adults should recidivate more than families. More rigorously, since single adults and families with children differ in many other relevant aspects as well, it would be illuminating to examine plausibly exogenous changes in shelter (or street) quality, and see how they affect recidivism.

Notice that changes in fixed exit costs do not affect recidivism. There is an interesting symmetry here: fixed exit costs affect spell length but not recidivism; shelter quality affects recidivism but not spell length. These are sharp and probably testable predictions.

Households who recidivate are sometimes labeled "episodically homeless." Thus households are often divided into three "types": short-term, episodic, and chronic (see, e.g., Culhane et al, 2007). Ex post, there is nothing objectionable to attaching such labels to the realizations of a stochastic process. But attaching the labels to people makes a strong and probably misleading statement. Of any group of ex ante identical households entering homelessness, some will have short spells and some long; some will recidivate and others will not. There is no need to presume that there is something different ex ante about households ("different types") who will experience different realizations ex post, just as there is no need to presume today that dice that will roll a two tomorrow are somehow different from dice that will roll a three tomorrow.

3.4 External costs of homelessness

Homelessness carries with it many costs that are not borne by homeless people themselves. For instance, sheltered homelessness imposes costs on shelter operators (and whoever subsidizes them), and street homelessness imposes external costs on passersby, mass transportation users, and probably whoever subsidizes emergency medical services. No one has studied how these costs vary with the length of a homeless spell. In the absence of better information, I will assume that external costs are incurred at the same rate γ per unit time throughout a homeless spell. Hence the expected discounted cost of a homeless spell is

$$\frac{\gamma}{r}\big(1 - \Psi_h(V^*, P)\big).$$

We must also account for the possibility of recidivism, however. Let Γ denote the expected discounted external cost of homelessness for a household entering homelessness now. Then

$$\Gamma = \frac{\gamma}{r}\big(1 - \Psi_h(V^*, P)\big) + \Psi_h(V^*, P)\,\psi\big(v^{-1}(V^* - F)\big)\Gamma.$$

Hence

(3.10)

$$\Gamma = \frac{\gamma}{r}\;\frac{1 - \Psi_h(V^*, P)}{1 - \Psi_h(V^*, P)\psi\big(v^{-1}(V^* - F)\big)}.$$

This expression will be useful for examining policy options.

3.5 Empirical consequences and tests

The results in this section are consistent with several empirical findings about homeless spells.

First, spell lengths seem to correspond with the fixed cost of exiting homelessness. Spells of single adults tend to be much shorter than spells of families with children. The cost of establishing a new home is usually much smaller for a single adult than it is for a family with children. Larger families also exit shelters in New York City more slowly than smaller families (Wong et al. 1997; Messeri et al. 2011).

Second, exit hazard rates, at least for sheltered homeless families in New York City, have a single peak. Messeri et al. (2011) calculate the empirical hazard function (net of several variables, including fixed effects for month of entry) for all families who entered New York City shelters from 2003 to 2008; they find a single peak. (Most studies of exits use Cox proportional hazard models and report only differences in exits by independent variable; they do not report the baseline survival probabilities, which are our interest here.)

Finally, two pieces of evidence support the proposition that the process under which households enter and leave homelessness is Markovian—that the history of why or how a household becomes homeless does not influence how long it stays homeless. Several high-quality studies of shelter exit propensities have been done, and their results suggest that the Markovian assumption with a homogeneous population is not seriously wrong. The strong implication of the Markovian assumption with a homogeneous population is that characteristics should not matter for shelter exit propensities. A weaker implication is that characteristics that predict a high probability of entry into homelessness should not systematically predict a low probability of shelter exit.

The strong implication seems to be supported most frequently for race and ethnicity. Although minorities, especially African-Americans, are considerably more likely to enter homelessness, several studies (Piliavin et al. 1993, Culhane and Kuhn 1998, Allgood and Warren 2003) find that race and ethnicity make no difference for length of shelter stays (Wong et al. 1997 and Poulin 2007 find that whites exit sooner). On gender, although men are much more likely than women to enter homelessness, no clear pattern emerges from the studies: Piliavin et al. 1993 find that gender makes no difference; Piliavin et al. 1996, Culhane and Kuhn 1998 for New York, and Allgood and Warren 2003 find that men stay longer; Culhane and Kuhn 1998 for Philadelphia and Poulin 2007 for New York find that women stay longer. Similarly for mental illness: although people who are mentally ill are more likely to become homeless than people who are not, no clear pattern emerges from the studies on whether mental illness prolongs shelter stays: Culhane and Kuhn 1998 and Allgood and Warren 2003 find no effect of serious mental illness; Piliavin et al. 1993 find that prehomeless psychiatric hospitalization leads to shorter spells; while McBride et al. 1998 find that mentally ill women stay longer.

Only on age are the studies consistent: older people stay longer (Piliavin et al. 1993, Culhane and Kuhn 1998 (but no age effect in Philadelphia), Wong et al. 1997, Allgood and Warren 2003, Poulin 2007). But older people are not more likely to enter shelters.

Thus the weak implication is strongly supported: characteristics that predict entry into homelessness do not predict longer homeless spells. Except for age, moreover, no characteristic has a consistent effect across studies on duration of homelessness. This is weak support for the strong implication.

On the other hand, Culhane and Kuhn (1998) find that higher order shelter stays tend to be longer (your third stay is longer than your first). This is not compatible with our Markovian framework unless the population is heterogeneous.

4. Homelessness prevention

4.0 General

The idea of preventing homelessness is intuitively appealing. Consider the case of certainty. A household with assets $(h\text{-}\varepsilon)$ is doomed to become homeless, and so cause external costs Γ. A household with assets h is destined never to become homeless (although it is only a tiny bit better off than the household with ε less in assets). For positive Γ, then, social welfare rises if the poorer household receives ε in assets. I call providing ε to this household a homelessness prevention program. The social benefits of such a program, if it were done by surprise, are obvious.

If the program is not a surprise, however, two problems arise, because it will alter behavior. Suppose the program is known and permanent. A household that began with h in assets—and so a household that will never become homeless—can increase its consumption for one unit of time to $(rh+\varepsilon)$, receive the asset injection, and return to consuming at rate rh again. This new consumption stream is clearly superior to the old. But the homelessness prevention program in this case is not preventing homelessness.

Similarly, consider a household that has received a grant from a permanent program. The optimal plan for such a household is to spend the grant immediately, return to its original level of assets, and get another grant. A permanent, naïve homelessness prevention program is a money pump.

On the other hand, a permanent anticipated homelessness prevention program has some advantages over a surprise program. Consider a household with initial assets $(h-\varepsilon-\eta)$, and establish a program to provide ε to every household with $(h-\varepsilon)$ in assets. On its face, this program does nothing for the household with $(h-\varepsilon-\eta)$ in assets, who is destined to become homeless. But for η sufficiently small, this household will save money, increase its assets to $(h-\varepsilon)$, get the grant, and escape homelessness.

Designing and analyzing homelessness prevention programs are thus challenging tasks. In this section I begin with surprise programs, then analyze anticipated programs that end the first time a household becomes homeless, and finally look at permanent anticipated programs. Section 4.1 and 4.2 are about programs that help households who are not currently homeless; section 4.3 is about programs that help homeless households, and how those programs interact with programs that help housed households.

4.1 Surprise programs

Surprise programs are useful to study to help build intuition, and to develop contrasts with anticipated programs. Often results are published about new and innovative programs; as far as their participants are concerned, these programs are probably surprises. Understanding both surprise and anticipated programs should help us interpret the published results and possibly extrapolate to how the anticipated version of these programs would work.

Consider a household with assets *A>0.* Absent a homelessness prevention program, the expected present value over eternity of the external costs of homelessness on the part of this household is $\psi(A)\Gamma$, which may be very small if assets are large and the probability of homelessness in the immediate future is small.

If a homelessness prevention program can observe *A* and gives this household *a>0* in additional assets, expected external costs of homelessness fall to $\psi(A+a)\Gamma$. Thus the outside world gains if

$$\psi(A)\Gamma - \psi(A+a)\Gamma > a.$$

On traditional cost-benefit terms, the program is a potential Pareto improvement if

$$[V(A+a) - V(a)] + [\psi(A)\Gamma - \psi(A+a)\Gamma] > a.$$

Define a (surprise) homelessness prevention program as a function $\alpha: \mathbb{R}^+ \rightarrow \mathbb{R}^+$that assigns to every nonnegative asset amount an increment, possibly zero. Let *W(A)=V(A)-* $\psi(A)\Gamma$*.* Clearly *W(.)* is differentiable and concave. A homelessness prevention program is *surprise-optimal (SO)* if for every *A* and *a'>0*

(4.1)

$$W\big(A+\alpha(A)\big) - \alpha(A) \geq W(A+a') - a'.$$

Moreover, for every *A*, $\alpha(A)$ is the largest number that satisfies (4.1).

Let $S = \{A | W'(A) \geq 1\}$, and let A^* denote the supremum of *S*, if *S* is not empty and has a supremum; positive infinity otherwise. Obviously if *A** is finite, *W'(A*)=1.* The surprise-optimal homelessness prevention program is obvious:

$$\alpha(A) = A^* - A \quad if\ A \epsilon S$$

$$\alpha(A) = 0 \quad otherwise.$$

Every household with assets below *A** is raised to *A*.* If *S* is empty, then the surprise-optimal homelessness prevention program is to do nothing.

4.2 Anticipated programs

4.2.1 Basic set-up

The above reasoning cannot apply to anticipated homelessness prevention programs. Suppose every household below *A** is raised to *A**in an S-O homelessness prevention program. Consider a household with assets *A<A*.* If it enters the program now, its assets rise to *A** immediately, and it can engage in optimal consumption starting with *A*.* Alternatively, it can increase its consumption now to a level greater than it would enjoy under *A*,* and let its assets fall (stochastically) to zero. Then It receives *A** and can engage in optimal consumption starting with *A*.* Clearly the strategy of consuming profligately and delaying program entry dominates the strategy of entering the program immediately, and so the costs of running anticipated programs are different from the costs of running surprise programs.

In this section I will model an anticipated homelessness prevention program as a change in the flow of income to a household, rather than a change in its stock of assets. Mathematically, this is easier, and a change in the flow of income can achieve the first-best optimum. Intuitively, once an asset injection has occurred it cannot affect subsequent behavior, but anticipated changes in the flow of income can do so.

Define a homelessness prevention program in this section as a function $g: \mathbb{R}^+ \to \mathbb{R}^+$that assigns to each asset amount a flow rate of subsidy; this flow is time-invariant. These grants could be in-kind as well as in cash. Let *V(A,g)*denote the optimal continuation value for a household with assets *A* when program *g* is in effect; and *ψ(A,g)* denote the corresponding expected discount factor for reaching homelessness. In this section I will assume that the homelessness prevention program operates for a household only until it becomes homeless for the first time. This restriction is purely for convenience in that it allows us to hold *P* and *Γ* fixed. In section 4.3 I will relax this restriction.

In this section, denote social welfare for a household with assets *A* as

$$W(A,g) = V(A,g) - \psi(A,g)\,\Gamma.$$

This is the net benefit that society will enjoy from this household's consumption .

Notice that I assume that the government can observe assets. This strong assumption makes the results more general, because the optimal program derived under this assumption does not require that the government observe assets.

I also assume that the grant emanates from a party external to the system described here (but whose welfare matters). This avoids difficulties that arise if homeless people cannot pay taxes.

4.2.2 Consumer behavior

Consider a household facing a homelessness prevention program *g(.).* The equation of motion for assets becomes

$$dA = \big(rA - c + g(A)\big)dt + s(A)dw.$$

Thus the Hamilton-Jacobi-Bellman equation is

$$rV(A) = sup_c \left[U(c) + (rA - c + g(A))V'(A) + \frac{1}{2}s^2(A)V''(A)\right].$$

Following the reasoning of section 1.1, the consumption function *c(A)* satisfies the differential equation

(4.2)

$$(c - rA - g(A))c'(A) = s(A)s'(A)c'(A) + \frac{1}{2}s^2(A)\left[c''(A) + \frac{U'''(c)}{U''(c)}c'(A)\right].$$

The initial condition for this differential equation is, as before,

(4.3)

$$rP = U(c_0) + (g(0) - c_0)U'(c_0) + \frac{1}{2}s^2(0)U''(c_0)c'(0).$$

In the special cases for which we have closed form solutions, *c'(0)=0*, and so (4.3) becomes

(4.4)

$$rP = U(c_0) + (g(0) - c_0)U'(c_0) = J(c_0) + g(0)U'(c_0).$$

Consider the right-hand side of (4.4). Denote

$$j(c) = U(c) + (g(0) - c)U'(c).$$

Since $\lim_{c\to\infty} J(c) > rP$, it follows that $\lim_{c\to\infty} j(c) > rP$.

It is easy to see that

$$j'(c) = -U''(c)\big(c - g(0)\big).$$

Hence *j(c)* is decreasing for $c<g(0)$, and increasing for $c>g(0)$; it has a unique minimum at $c=g(0)$. At this minimum, $j(g(0))=U(g(0))$. See figure 4.1.

Figure 4.1 around here

I will assume that $U(g(0))\leq rP$. I do not think of this as a substantive restriction, but as a definition of homelessness. By definition a household with zero assets is homeless and whatever assistance the government gives such a household is incorporated in *P*. There cannot be a separate assistance program for households with zero assets.

If $U(g(0))=rP$, $c_0 = g(0)$is the unique solution to (4.4).

If $U(g(0))<rP$, then (4.4) has two roots. At least one of them is positive, and one of them is greater than $g(0)$. Both roots are nonnegative if $g \geq {rP}/{U'(0)}$. The smaller root is less than $g(0)$: the household saves some of its regular income and part of the subsidy. The larger root is greater than $g(0)$: the household uses all of its regular income for consumption, and some of the subsidy.

It is easy to prove that with positive subsidies, $c_0<rh$ even for the larger root. The intuition is that the subsidy is a reward for not being homeless, and so the subsidy makes it relatively worse to become homeless.

Subsidies lengthen the time to homelessness in three different ways. First, for any given consumption rate, they reduce the depletion in assets that it causes. Second, on the lower segment, subsidies reduce consumption. Finally, they may reduce the upper bound on the lower segment at which the greatest rate of asset depletion commences.

4.2.3 Optimal prevention programs when $\Gamma\leq P$

Households are too likely to become homeless in this model because they do not internalize the external costs *Γ* of becoming homeless (which may include general social dismay about homelessness, as well as self-control costs). Thus they act to maximize *V(.),* not *W(.).* A simple Pigouvian tax on entry to homelessness would eliminate this distortion and achieve the first-best outcome; the amount of the tax would be *Γ*.

Such a tax, of course, is uncollectible and impractical—and probably cruel, too. The more practical equivalent of a Pigouvian tax on homelessness is Pigouvian subsidy for being housed. This subsidy is the first-best homelessness prevention program: it internalizes the external cost of homelessness, and by so doing maximizes *W(A,g)* for all *A.* The Pigouvian homelessness prevention program makes every household act exactly how a benevolent social planner would want it to act.

To derive the optimal homelessness prevention program, then, I start with the planner's problem. It is the same as the household's problem (with no subsidies), except that the payoff on entering homelessness is *(P-Γ)*, rather than *P.*

Thus if *c'(0)=0*, the boundary condition (1.7) becomes

$$r(P-\Gamma)=U(c_0)-c_0U'(c_0).$$

This can be rewritten as

(4.5)

$$rP=J(c_0)+r\Gamma.$$

I will denote by $c_0(\Gamma)$ the (unique) solution to (4.5).

Compare (4.5) with (4.4). A household will choose the same value of c_0 as a planner would only if

$$r\Gamma=g(0)U'(c_0(\Gamma)).$$

Hence in the optimal homelessness prevention program

(4.6)

$$g(0) = \frac{r\Gamma}{U'(c_0(\Gamma))}.$$

Now consider the differential equation that governs the consumption that the planner would order. It is simply (1.6), since *P* does not appear in this equation. Compare (1.6) with (4.2), the differential equation for the household's consumption with homelessness prevention program *g(.).*

Suppose *g'(A)=0* for all *A.* Then the right-hand side of (4.2) is identical to the right-hand-side of (1.6). Then for all *A,*

$$g(A) = g = \frac{r\Gamma}{U'(c_0(\Gamma))}.$$

On the left-hand side, consumption by a household with assets *A* under (1.6) is identical with consumption by a household with assets *(A+g/r)* under (4.1).

In that sense, a simple subsidy given by (4.6) to every household every instant achieves efficiency. Every household always internalizes the external cost of homelessness; no Pareto improvement is possible. There is less homelessness than under the status quo with no subsidy, and less homelessness than if the government just awarded a lump sum *g/r* to every household. But homelessness is not eradicated, and it is possible that the government's total costs will be greater than they were at the status quo. The optimal subsidy scheme is not necessarily Pareto superior to the status quo.

(The optimal homelessness prevention program is not unique. For instance, with arithmetic Brownian motion and CARA utility with parameter λ, consider the subsidy

$$g(A) = g(0)exp\left\{r\lambda\, max\left[0, A - \left(\frac{c_0(\Gamma)}{r} + \frac{1}{2}\lambda\sigma^2\right)\right]\right\}.$$

In this expression, *g(0)* is given by (4.6). This makes

$$-g(A)c'(A) = g'(A)\frac{U'(c(A))}{U''(c(A))}.$$

So it reduces (4.2) to (1.6). This homelessness prevention program also induces efficient consumption and savings decisions.)

Although (4.6) gives the subsidy level that will induce terminal consumption at the desired level $c_0(\Gamma)$, the desired level of consumption is not necessarily the only level of consumption that can occur at low assets when this subsidy is in place. That is because (4.4) often has two roots, as I discussed in section 4.2.2. Only one of these roots is $c_0(\Gamma)$. Hence the government will often have to employ additional strategies to make this particular root salient. I discuss this in more detail below.

The optimal subsidy and the desired terminal consumption level $c_0(\Gamma)$ are both functions of the external costs of homelessness Γ. Differentiating (4.5) totally and rearranging implies

(4.7)

$$\frac{\partial c_0(\Gamma)}{\partial(r\Gamma)} = \frac{1}{c_0 U''(c_0)} < 0.$$

As the external costs of homelessness grow, terminal consumption falls: households should try harder to conserve their resources and avoid homelessness. Homelessness prevention programs should make housed households near homelessness spend less on housing consumption than they would ordinarily spend on their own.

Differentiating (4.6) and using (4.7) yields

$$\frac{\partial g}{\partial(r\Gamma)} = c_0\left[\frac{1}{U'(c_0)}\right]^2 [c_0U'(c_0) - r\Gamma].$$

This is positive for Γ close to zero: for small Γ, increases in the external cost of homelessness raise the subsidy and cut consumption. From (4.5)

(4.8)

$$c_0(\Gamma)U'\big(c_0(\Gamma)\big) - r\Gamma = U\big(c_0(\Gamma)\big) - rP.$$

Hence g reaches its maximum value when $c_0(\Gamma)$ falls far enough that $U\big(c_0(\Gamma)\big) = rP$. (For $\Gamma=0$, $J(c_0(0))=rP$, and so $U(c_0(0))>rP$.) Denote by Γ^* the value of Γ at which the maximum is attained: by definition, $U\big(c_0(\Gamma^*)\big) = rP$. For $\Gamma>\Gamma^*$, increases in external cost reduce the optimal subsidy (because they increase $U'\big(c_0(\Gamma)\big)$); increases in external cost still reduce terminal consumption.

From (4.6) and (4.8), at Γ^*

$$g = \frac{r\Gamma^*}{U'\big(c_0(\Gamma^*)\big)} = \frac{c_0(\Gamma^*)U'\big(c_0(\Gamma^*)\big)}{U'\big(c_0(\Gamma^*)\big)} = c_0(\Gamma^*).$$

Then for $\Gamma<\Gamma^*$, $c_0(\Gamma) > g$, since in this range $c_o(.)$ is decreasing in Γ and g is increasing. Conversely, it is easy to show that for $\Gamma>\Gamma^*$, $c_0(\Gamma) < g$. Since a household's income is $rA+g>g$ when it is not homeless, in this range savings on the lower segment are positive and so assets usually have positive drift. When the external costs of homelessness are high enough, households with proper incentives on the lower segment will save money in order to avoid it, and will end up homeless only because of unusual bad luck. The probability of never becoming homeless is positive, and the expected time to homelessness is infinite.

When $\Gamma=P$, it is obvious from (4.5) that $J(c_0)=0$, and so $c_0(\mathrm{P}) = 0$. When the external costs of homelessness are as large as the internal benefits, households will optimally decrease consumption to zero in order to avoid it. The lower segment will vanish.

Figure 4.2 illustrates and summarizes these results.

Figure 4.2 around here

The same subsidy *g* may arise from two different values of Γ, and be associated with two different values of $c_0(\Gamma)$. This is because (4.4) often has two roots as we have noted. To implement the optimal homelessness prevention program, the household has to choose the large root for c_o when $\Gamma<\Gamma^*$, and the small root when $\Gamma>\Gamma^*$. (The two roots are the same when $\Gamma=\Gamma^*$.) One way for the government to resolve this is to make the subsidy contingent on choosing a terminal consumption level either more or less than $c_0(\Gamma^*)$, or make shelter entry contingent on an appropriate level of pre-shelter consumption. Something like this process may be at work in the eligibility requirements of the New York City family shelter system, where the city conducts detailed checks of the accommodations available at the household's previous address.

This selection process may be particularly important when the smaller root is required for optimal prevention. For many utility functions (for instance, quadratic), the consumption function associated with the smaller root (with consumption less than the subsidy on the lower segment) is not continuous; with quadratic utility, consumption as a function of assets takes a discrete jump upwards at the arbitrary point where consumption begins to equal income. Forced saving may be an alternative selection mechanism in this case. Forced saving is often required of households in shelters—a certain portion of their weekly income goes to the shelter operator who accumulates it and returns it to them on shelter exit or some similar occasion. Homelessness prevention programs in New York City have begun to include budgeting sessions and requirements as part of their assistance, but mainly on an informal basis (Auwater, 2010).

4.3.4 Optimal prevention programs when $\Gamma > P$

The external costs of homelessness may be greater than the internal benefits, and so I need to consider this case. In section 6 I will argue that if weak conditions are met and shelters are managed optimally, external costs will not be greater than internal benefits, but the weak conditions may not be met and shelters may not be managed optimally. When external costs exceed internal benefits, naively applying the formulas from section 4.3.3 yields negative terminal consumption, which is impossible.

Start with a Pigouvian tax on homeless entry, and assume that the utility function does not satisfy the Inada condition. Since $V(0)=r(P-\Gamma)<0$, section 1.6 applies. For low assets, consumption is zero, and the first-order condition for choosing optimal consumption is a strict inequality. Households reduce their consumption to zero—probably briefly because asset-drift is positive—in order to avoid the large costs of having no assets.

An optimal homelessness prevention program has the same effect (by construction). The subsidy to having positive assets is so high that households reduce consumption to zero in order to avoid losing all assets. Mathematically, sections 1.6 and 4.3.3 fit together seamlessly.

The difficulty is that a program to prevent asset-homelessness in this case induces consumption-homelessness. The cause of the problem is that I have not recognized consumption-homelessness as having its own external costs distinct from those of asset-homelessness. That was because except in this section and in section 1.6, consumption-homelessness never occurred separate from asset-homelessness.

The remedy is obvious. Let $E: \mathbb{R}^+ \to \mathbb{R}^+$ denote a function that gives the external flow costs of being housed (or not) in a particular way. A household consuming *c* units of housing generates *E(c)* in external costs per unit time.

Probably the simplest external cost function would be

E(0)= w>0

E(c)= 0 otherwise.

This function, however, is discontinuous at zero, which makes using it mathematically challenging. Moreover, the idea that very poor qualities of housing generate no external costs is also unappealing; for over a century housing reform has been predicated on the idea that such costs are substantial.

Here I will assume that *E(.)* is a continuously differentiable weakly decreasing function, positive at zero and some neighborhood of zero.

The Pigouvian response then is to reduce the subsidy from *g* to *[g-E(c)]* for a household consuming *c* housing. With such a subsidy in place, consumption goes to zero only if

$$U'(0) - E'(0) - V'(A) < 0.$$

The largest *a* for which *c(a)>0* satisfies

(4.9)

$$U'(0) - E'(0) = V'(a).$$

Here *V(.)* is governed by (1.14), (1.16), *V(a)=0,* and (4.8).

Households will still become consumption-homeless, but not as often as they would if the subsidy were not reduced.

4.3 Shelter policies with optimal prevention

The previous two subsections treated shelter stays as a black box described by two scalars, *P* and *Γ*, and did not consider homelessness prevention programs for households who had already experienced homelessness. In this subsection, I will look at shelters more carefully, building on the results in section 3. I begin by showing that optimal subsidies when recidivism is possible are slightly different from the optimal subsidies of section 4.2. Then I show that optimal homelessness prevention programs do not need to be supplemented by shelter-based programs. Throughout this section I treat *H* and *c*, the flow rates of household utility and external cost while homeless respectively, as fixed. Section 5 will discuss optimal values for these variables.

4.3.1 The value of being housed

A key assumption in section 3 is that while a household is homeless, its value of being housed (if it were to be housed immediately) evolves as arithmetic Brownian motion. The parameters of that process were not tightly or mechanically linked to the other parameters in the model, but we might expect that adoption of a homelessness prevention program would in some way affect the process. However, the way in which a homelessness prevention program alters this process is unclear, and the effect of likely alterations to this process on homeless spells is also ambiguous. Hence I cannot make any strong statements about how a homelessness prevention program, operating through changes in the process governing the value of being housed, would change the distribution of homeless spells.

A homelessness prevention program essentially increases the slope of the value function *V(A),* since it does not directly affect *V(0)=P.* Thus if assets generally rise when a household is homeless, the drift μ_h

of the arithmetic Brownian motion will increase. But μ_h will decrease if assets generally fall during a homeless spell. Thus I cannot confidently say how a homelessness prevention program will affect μ_h. On the other hand, a homelessness prevention program will probably raise the instantaneous variance σ_h^2, since shocks to assets will have bigger impacts on value.

These parameters affect the distribution of homeless spells through *R*, the negative inverse option value of being homeless. Greater volatility raises *R,* but greater drift decreases *R.* If a homelessness prevention program reduces μ_h, then it unambiguously raises *R.* But if it increases μ_h, the effect on *R* cannot be signed.

But as we saw in section 3, the effect of *R* on Ψ_h, the expected discount factor leaving homelessness, is also ambiguous.

Hence for many purposes in the rest of this section, I will assume that the net effect of a homeless prevention program on the expected discount factor for leaving homelessness is zero.

4.3.2 Optimal subsidies with recidivists

Even if a homelessness prevention program has no effect on exits from homelessness, it does affect entries into homelessness, including entries by recidivists, if recidivists are part of the program. Authorities may not have enough information to exclude recidivists if they wanted to, and since recidivists too are overly likely to become homeless absent a prevention program, there is no reason to exclude them. Optimal subsidies in section 4.2 were calculated under the assumption that recidivists were excluded, so that the program would not affect the value of *Γ*.

In section 3.4 *Γ* was defined as

(3.10)

$$\Gamma = \frac{\gamma}{r}\ \frac{1-\Psi_h(V^*,P)}{1-\Psi_h(V^*,P)\psi\big(v^{-1}(V^*-F)\big)}.$$

Assume that a homelessness prevention program has no effect on the variables and functions describing the experience of homelessness: *V*, P, F,* and $\Psi_h(.)$. A Pigouvian homelessness prevention program, however, must affect the function v^{-1} and ψ.

Let $v^{-1}(.|\Gamma)$ denote the inverse value function when an optimal homelessness prevention program for external cost Γ is in place. Similarly, let $\psi(.|\Gamma)$ denote the expected discount factor function under these circumstances. Note that both of these are continuous functions of Γ.

Define

$$\mathrm{M}(\Gamma) = \frac{\gamma}{r}\ \frac{1-\Psi_h(V^*,P)}{1-\Psi_h(V^*,P)\psi(v^{-1}(V^*-F|\Gamma)|\Gamma)}$$

Then an optimal homelessness prevention program with recidivism must have a Γ that is a fixed point of *M(.)* and a subsidy derived from this Γ.

Assume $\Psi_h(V^*,P) < 1$; that is, some homeless spells are of greater than zero duration. Then *M(.)* is continuous and maps the following finite line segment into itself

$(\frac{\gamma}{r}(1-\Psi_h(V^*,P),\frac{\gamma}{r})$.

Hence *M(.)* has a fixed point, and at least one optimal homelessness prevention program with recidivism exists.

4.3.3 Are homeless spells too long?

Optimal homelessness prevention programs were derived by imitating a Pigouvian tax of Γ on entry to homelessness. The costs of homelessness, though, depend on how long a household stays homeless, not just on whether it enters homelessness. One might be tempted, then, to conjecture that homelessness prevention programs as outlined in section 4.2 are insufficient to control both exits and entries into homelessness and that a tax on homeless stays is needed in addition to the implicit tax on homeless entries.

This conjecture is wrong. Entries and exits from homelessness are not independent processes. A tax on homeless spells (which, of course, is largely impractical) is a substitute for a subsidy on housed spells, not a complement.

The reason is apparent in section 3.1. Homeless spell lengths depend on the difference V^*-P. Reducing the flow utility from being homeless H reduces both $V^*$and P by the same amount, and so it does not affect homeless spell lengths or exits from homelessness. Thus a Pigouvian tax of γ per unit time homeless (or any other amount) would not affect the distribution of homeless spells. Such a tax, however, would affect P and through P the rate of entry into homelessness. Thus it is equivalent to a subsidy on being housed, the subsidy we studied in section 4.2.

Another way of expressing this is to say that no matter what tax is or is not imposed on shelter stays, their distribution is the same as it would be if a Pigouvian tax of γ per unit time were imposed, and that distribution is efficient if the entry process is efficient.

A corollary is that when an optimal homelessness prevention program is in place, one-shot subsidies to induce shelter exits by reducing fixed exit costs F are not necessary and in fact move the system away from first-best. They shorten shelter spells but increase shelter entries, and since shelter spells were of optimal length anyway, the net effect cannot be welfare-enhancing.

One-shot exit subsidies—security deposits and furniture, for instance—are popular policies. In the absence of optimal prevention programs, whether some constraints on acceptable policies make exit subsidies part of a second-best program is an open question.

4.4 Digression: Ornstein-Uhlenbeck processes

In this paper I have concentrated on Brownian motions. But the algebra of grants with arithmetic Brownian motion turns out to be very similar to the algebra of Ornstein-Uhlenbeck processes (with or without grants).

Ornstein-Uhlenbeck processes are mean-reverting: there is some positive level of assets Λ and the drift of assets is toward Λ. Specifically, the equation of motion for assets with an Ornstein-Uhlenbeck process is

$$dA = \big(rA - c + \mu(\Lambda - A)\big)dt + \sigma dw$$

where μ>0. Rewrite this as

$$dA = ((r-\mu)A - c + \mu\Lambda)dt + \sigma dw.$$

Thus an Ornstein-Uhlenbeck process is the same as arithmetic Brownian motion with a lower interest rate—$(r-\mu)$ instead of *r*—and a grant of $\mu\Lambda$ per unit time. The analysis of section 4.2.2 carries over completely.

5. Insurance

The homelessness prevention programs studied in section 4 tried to change the drift of assets. In this section I will study programs that try to reduce the volatility of assets. Heretofore I have assumed implicitly that no one could do anything about volatility. The literature on consumption and investment (e.g., Merton 1971, Lehoczky et al 1983, Karatzas et al 1986), however, on which most of this paper is based, assumes that households can alter the volatility of their portfolios by varying the composition of their portfolios. My households do not have this ability. However, suppose that the government can suppose that the government can costlessly insure these households by trading the risky assets they actually hold for a safe asset with the same expected rate of return. In this section I show that such a trade almost always improves social welfare. Hence social welfare would also be improved by insurance that was less than actuarily fair by a sufficiently small amount.

5.1 First-best case

Suppose that an optimal homelessness prevention program is in place. Then every household acts as if it were maximizing the planner's value function *W* (.). This function is the solution to the usual Hamilton-Jacobi-Bellman equation and the associated initial condition:

(5.1)

$$rW(A) = \ sup_c \left[U(c) + (rA - c)W'(A) + \frac{1}{2}s^2(A)W''(A)\right]$$

(5.2)

$$P - \Gamma = W(0).$$

From a modest extension of the argument in section 1.5 and section 1.6, *W''(A)<0,* except on the lower segment, if it exists, where *W''(A)=0.*

Let *c′* denote the level of consumption that attains the supremum in (5.1). Consider a small decrease *(-δ)*, $\delta > 0$ in s^2(A). Holding *c′* constant, this decrease increases the expression in square brackets in (5.1) by $(-\delta W''(A))\geq 0$. Hence by the definition of a supremum

$$sup_c \left[U(c) + (rA - c)W'(A) + \frac{1}{2}[s^2(A) - \delta]W''(A)\right]$$

$$\geq . U(c') + (rA - c')W'(A) + \frac{1}{2}[s^2(A) - \delta]W''(A)$$

$$\geq U(c') + (rA - c')W'(A) + \frac{1}{2}s^2(A)W''(A) = W(A).$$

Thus reducing volatility always weakly improves the solution to the planner's problem.

In fact, a reduction of δ in $s^2(A)$ is still welfare-improving even if it is accompanied by a reduction of δ' in the rate of return on assets, provided

$$\delta' < -\frac{1}{2}\delta\frac{c'(A)U''(c')}{AU'(c')}.$$

This result is particularly interesting when $P\text{-}\Gamma\leq 0$. In that case, eliminating volatility eliminates homelessness. Thus if $P\text{-}\Gamma\leq 0$, no one should ever become homeless, and the combination of an optimal homelessness prevention program and optimal insurance assures that this is the case (see Karatzas et al., 1986).

5.2 Second-best

Reducing volatility will often be welfare-improving in the absence of an optimal homelessness prevention program, but not always.

Welfare is the sum of the household's value function *V(.)* and the government's expected cost of homelessness $\psi(.)\Gamma$. By an argument identical to that in section 5.1, but for *V(.)* instead of *W(.),* reducing volatility always increases *V(A).*

In section 2, I discussed at length the circumstances under which reducing volatility reduced $\psi(.)$. I gave some sufficient conditions, but did not find that it always occurred. In particular, in some cases the household offset reduced volatility by reducing precautionary savings.

6. Optimal shelter conditions

The final question is how to find optimal shelter conditions: how to set *H*, the flow utility while homeless, and γ, the flow external cost while homeless.

Note that I am using "shelter conditions" as shorthand for all the conditions that homeless people experience. For homeless people living on the street, for instance, "shelter conditions" include policing strategies, the availability of warm accommodations on subways overnight, and the dangers of violent attack.

6.1 First-best

As with insurance, it is easier to find optimal shelter conditions when an optimal homelessness prevention program is in place. Under those circumstances, welfare is given by the planner's value function *W(.),* and the only place where *H* and γ enter this value function is in the initial condition (5.2), since *P* and Γ depend on *H* and γ. Then optimal *H* and γ maximize $(P-\Gamma)$.

This maximization problem is constrained by the fact that *H* and γ cannot be set independently; how homeless people experience homelessness cannot be altered without changing the costs and perceptions of other people. The obvious relationship is that most improvements to shelter quality cost money. But some improvements to H may actually reduce γ. For instance, taxpayers may be more than willing to pay for improved shelters that keep homeless people off the streets or in less than degrading circumstances; or conventional living arrangements for chronically homeless people my cost less than the reductions in hospital emergency department use that they cause.

At an optimum all improvements to shelters that increase H and reduce γ should be done. We should be on the frontier of feasible (H, γ) combinations. On that frontier, H cannot be increased without increasing γ. Hence we can write $H=\eta(\gamma)$, where $\eta(.)$ is the inverse marginal cost of shelter quality and $\eta'>0$. If the various projects that could increase shelter quality are arrayed in an order where the most productive come first, then $\eta''<0$. The first-best problem is to maximize $(P-\Gamma)$, subject to the constraint $H=\eta(\gamma)$.

Now consider the version of the planner's problem where a Pigouvian tax is imposed on shelter stays. Then $\Gamma=0$ and so

$$P - \Gamma = P = \frac{H-\gamma}{r} + \Psi_h(V^*, P)\left(V^* - F - \frac{H-\gamma}{r}\right).$$

We discussed this expression in section 3.1, where we labeled the right-hand side $K(P)$, and showed that anything that increased $K(.)$ increased equilibrium P. From that section also, raising $(H\text{-}\gamma)$ raises $K(.)$ and hence P. Since neither H nor γ appears alone, but only in $(H\text{-}\gamma)$, maximizing $P=P\text{-}\Gamma$ is equivalent t o maximizing $(H\text{-}\gamma)$. Thus the first-order condition for optimal shelter quality is

$$\eta'(\gamma) = 1.$$

In the first-best situation, decisions about shelter quality can be based on straightforward standard instantaneous cost-benefit analysis.

6.2 Homelessness prevention and consumption-homelessness

Recall from section 4.3.4 that optimal homelessness prevention programs induced consumption-homelessness in order to avoid asset-homelessness when *P-Γ<0.* Now we see that shelter quality decisions should be based on making *(P-Γ)* large, and so avoiding the situation where this value is negative if at all possible. Since *H=0* is always feasible, a sufficient condition for the optimal *(P-Γ)* to be nonnegative is that there exists some policy strategy with sufficiently small instantaneous external cost. For instance, if it is possible for the rest of society to ignore homelessness entirely—that is, set γ=0—then optimal *(P-Γ)* is strictly positive.

Insurance programs also reduce the incidence of consumption-homelessness. In the limit, if insurance is perfect and *P-Γ≤0,* a household will never deplete its assets. If insurance is not perfect, a household might enter consumption-homelessness because of negative shocks, but when it does drift of assets is positive, and if volatility is small it is highly likely to leave consumption-homelessness very quickly.

Thus consumption-homelessness should be a rare phenomenon except when the government can run an optimal homelessness prevention program, but is otherwise incapable or unwilling to devise modestly effective insurance and shelter programs.

6.3 Second-best shelter quality

I do not have a general solution to the problem of shelter quality when an optimal homelessness prevention program is not in place. I am not even sure that the second-best optimum is on the feasible frontier of *(H, γ)* space. But I can point out general directions.

Suppose no homelessness prevention program at all is operating, but that the first-best shelter quality is in place (the shelter quality that would be optimal if an optimal homelessness prevention program were in place). Then behavior is governed by the standard HJB equation subject to the initial condition $V(0)=P$. If we were to choose H and γ *to* maximize $(P-\Gamma)$, then $V(0)$ would be greater than $(P-\Gamma)$. Homelessness would be inefficiently attractive, and the external costs of homelessness would be inefficiently high. Since $\eta'(\gamma)=1$ at the first-best shelter quality optimum, a small reduction δ in γ would be accompanied by an increase of δ in H, and social welfare would increase. Thus at the second-best optimum, the shadow price of external cost is greater than one; the second-best optimum maximizes $(H-n\gamma)$ for some $n>1$. Thus $\eta'(\gamma)=n>1$ at the optimum.

Optimal shelter quality is lower when a homelessness prevention program is not in place.

7. Conclusion

Many important elements of the stochastic processes that lead to and from homelessness have been omitted from this paper, even though it is extremely long. For instance, I have not explored capital market imperfections like borrowing constraints, savings constraints, and deviations from market discount rates; nor have I studied policies that seek to mitigate these imperfections. Since excess homelessness arises in this paper mainly from over-consumption, it seems that borrowing constraints are not a big problem, but that savings constraints might be. The question, however, should be explored rigorously.

Another major lacuna is the absence of explicit attention to institutions other than shelters—jails, prisons, hospitals, and substance abuse treatment facilities in particular. Implicitly entries to and exits from these institutions are shocks to assets (and possibly opportunities to save on rent), but the exact nature of these shocks should be considered explicitly. Because policies can affect how these institutions act, it is important to learn how they affect homelessness.

Similarly, I have not modeled various traditional housing assistance programs like public housing and housing choice vouchers. Expansions of these programs are a popular policy response to homelessness. But low-income housing assistance programs were designed long before homelessness was an issue. Households must usually wait a long time to enter them, and once in, they stay for a long time. We do not know how they act dynamically on an individual level.

I have also not explored the aggregate implications of the individual dynamics, or the effects of heterogeneity across households.

Finally, the paper is almost totally devoid of empirical work. This, too, can and should be remedied.

This paper, then, is only a first step. But, still, it is a step. Understanding homelessness requires understanding how it fits in a life course, how conditions while homeless affect conditions while not homeless, and vice versa. The paper is about the tools we need to do this.

Appendix: Proofs

Proposition 1.1.1. *If P>0, equation (1.8) has a positive solution.*

Proof: Suppose *U(.)* is bounded from above. I claim that

$\lim_{c\to\infty} J(c) = \lim_{c\to\infty} U(c)$.

Suppose not. Then there is some *k>0* such that

$$U(c) - J(c) = cU'(c) \geq k$$

for all *c* sufficiently large. Hence

$$U'(c) \geq {}^{k}/_{c}$$

and

$$U(c) \geq k' \ln c + K$$

Where *K* and *k'>0* are constants, for all *c* sufficiently large. Then *U(c)* is not bounded from above, which is a contradiction.

Since $\lim_{c\to\infty} U(c) > rP$,

(1.1.1-1)

$$\lim_{c\to\infty} J(c) > rP.$$

Thus since *J(0)=0,* and *J(.)* is continuous, equation (1.8) has a positive solution.

Now suppose that *U(.)* is not bounded from above. I claim that *J(.)* is not bounded from above either, and (1.1.1-1) holds. Suppose *J(.)*were bounded from above. Then

$\lim_{c\to\infty}\frac{J(c)}{U(c)}$=$\lim_{c\to\infty}[1-\frac{cU'(c)}{U(c)}]=1-\lim_{c\to\infty}\frac{\int_0^c U'(c)dx}{\int_0^c U'(x)dx}=1-k''=0$

where

$$k''=\lim_{c\to\infty}\frac{\int_0^c U'(c)dx}{\int_0^c U'(x)dx}.$$

This contradicts (1.2), *k''<1.*

Proposition 1.1.2. *(a) If either c'(0)=0 or s(0)=0, then* $c_0=rh$.

(b) If not, then $c_0>rh$.

(c) If P>0, then $c_0>0$.

Proof. (a) If either *c'(0)=0* or *s(0)=0,* then (1.7) becomes

$$rP=U(c_0)-c_0U'(c_0)=J(c_0).$$

By definition, $c_0=rh$ solves this equation.

(b) If not, then (1.7) becomes

$$rP=J(c_0)+\frac{1}{2}s^2(0)U''(c_0)c'(0)<J(c_0).$$

Since *J(.)* is an increasing function, $c_0>rh$.

(c) Immediate from (a) and (b).

Proposition 1.6. *The value function is strictly concave and increasing on (0, Â), and V(Â)=0.*

Proof. From (1.14), for all $A\epsilon(0, \hat{A})$,

$$V''(A) = \frac{2r}{\sigma^2}[V(A) - AV'(A)].$$

Differentiating this with respect to *A* yields

(1.17)

$$V'''(A) = \frac{2r}{\sigma^2}[-AV''(A)].$$

There are several implications. First, *V'''(0)=0.* Second, *V''(A)* and *V'''(A)* have opposite signs on [*0, Â).* By continuity, then, for sufficiently small $\varepsilon, V'''(\varepsilon) > 0$.

I claim that *V'''(A)*is positive and increasing on *[0, Â].* Since *V'''(.)* is increasing for sufficiently small *A*, it could be negative somewhere on this interval only if it reached a maximum somewhere on this interval, and such a maximum would have to be an internal maximum. A necessary condition then is that there be some $\tilde{A}\epsilon[0, \hat{A}]$ such that $V'''(\tilde{A}) > 0$ and $V''''(\tilde{A}) = 0$.

Differentiate (1.17):

$$V''''(\tilde{A}) = \frac{2r}{\sigma^2}[-V''(\tilde{A}) + AV'''(\tilde{A})].$$

Substitute from (1.17) to obtain:

$$V''''(\tilde{A}) = \frac{-2r}{\sigma^2}V''(\tilde{A})[1 + \frac{2r}{\sigma^2}\tilde{A}].$$

Hence $V''''(\tilde{A}) = 0$ only if $V''(\tilde{A}) = 0$. Suppose so. Then by (1.17), $V'''(\tilde{A}) = 0$. This contradicts the hypothesis that $V'''(\tilde{A}) > 0$. Hence *V'''(.)* has no internal maximum; hence *V'''(.)* is positive and increasing on *[0, Â].*

Since *V'''(A)* is positive, *V''(A)* is negative and increasing. Then *V'(A)* is positive, decreasing, and greater than *U'(0)* on *(0, Â].*

It remains to determine $\hat{A}$. Given $\hat{A}$, equations (1.14), (1.15), and (1.16) uniquely determine $V(\hat{A})$. I claim that $V(\hat{A}) = 0$.

For $A > \hat{A}$, the supremum in the HJB equation is internal, by definition, and so consumption and the value function are governed by differential equation (1.6) and a new boundary condition, analogous to (1.7), which is

(1.18)

$$rV(\hat{A}) = J\left(c(\hat{A})\right).$$

Suppose $V(\hat{A}) < 0$. Then (1.18) cannot be satisfied, since *J(.)* is nonnegative.

Suppose $V(\hat{A}) > 0$. Then by the same reasoning that established $c_0 > 0$ if *P>0,* we have $c(\hat{A}) > 0$. Hence by the envelope theorem (1.3),

$$V'(\hat{A}) = U'\left(c(\hat{A})\right) < U'(0).$$

So the value function fails to satisfy smooth-pasting. (For a discussion and motivation of smooth-pasting, see, e.g, Stokey (2009, pp. 123-4). Hence $V(\hat{A}) = 0$.

Proposition 2.1. *The solution to (2.1) and (2.3) that satisfies the four boundary conditions is decreasing and strictly convex if utility is quadratic or CARA.*

Proof. Clearly $\psi(A)$ is decreasing and convex on the upper segment.

Consider $\psi(A)$ on the lower segment. Let $0<A_1<A_2\leq K$. If a household falls from A_2 to 0, it must first fall to A_1. Hence $\psi(A_1) > \psi(A_2)$, and $\psi(A)$ is everywhere decreasing.

From the third and fourth boundary conditions, $\psi(K) > 0$ and $\psi'(K) < 0$.

Consider (2.2) and let $A=h$:

$$\frac{1}{2}\sigma^2\psi''(h) - r\psi(h) = 0.$$

Since $\psi(h) \geq \psi(K) > 0$, $\psi''(h) > 0$.

Differentiate (2.2) with respect to A to obtain

(2.4)

$$\frac{1}{2}\sigma^2\psi'''(A) = -r(A-h)\psi''(A)\,.$$

For $A<h$ then, $\psi'''(A)$ and $\psi''(A)$ have the same sign.

Suppose that for some $A<h$, $\psi''(A) \leq 0$. Let $\tilde{A}$ denote the largest such A. Specifically,

$$\tilde{A} = \sup\{A < h | \psi''(A) \leq 0\}.$$

Then from (2.4), $\psi'''(\tilde{A}) \leq 0$. Hence for some ε>0, $\psi''(\tilde{A}+\varepsilon) \leq \psi''(\tilde{A}) \leq 0$. This is a contradiction of the supremum hypothesis. Hence for all *A<h*, $\psi''(A) > 0$. Since we have already established the claim for *A=h*, this completes the proof for the lower segment for quadratic utility.

For CARA utility, consider $A\epsilon[h, h^*)$. Let

$$\tilde{A} = \inf\{A\epsilon[h, h^*) | \psi''(A) \leq 0\}.$$

Then $\psi'''(\tilde{A}) \geq 0$, and so for positive ε sufficiently small, $\psi''(\tilde{A}-\varepsilon) \leq \psi''(\tilde{A}) \leq 0$. This is again a contradiction of the infimum hypothesis. Hence $\psi''(A)$>0 on $[h, h^*)$.

Since *ψ′* is decreasing on the lower segment, and is continuous at the boundary of the lower and upper segments (by the smooth-pasting condition), and is decreasing on the upper segment, *ψ′* is decreasing everywhere and *ψ* is convex.

Proposition 2.2. *If*

$$\hat{\psi}(K) = \widehat{k_1} e^{\widehat{R_1} K} \geq \psi(K) = k_1 e^{R_1 K},$$

then $\hat{\psi}(A) \geq \psi(A)\ for\ all\ A$.

Proof. For the upper segment, the result is obvious because $\widehat{R_1} > R_1$ by calculus.

For the lower segment, let $d(A) = \hat{\psi}(A) - \psi(A)$. Then *d(0)=0,* and *d(K)≥0.* Moreover, for all *A* on the lower segment

(2.5)

$$\frac{1}{2}(\sigma^2 + \Delta)d'' + r(A-h)d' - rd = -\Delta\psi'' \leq 0.$$

The final inequality follows from proposition 2.1.

Suppose that for some subset *T* of *(0, K), d(A)<0, A ε T.* Then since *d(.)* is nonnegative at 0 and *K*, there is some *a ε T* such that *d(a)* is a local minimum. Thus

$$d''(a) > 0, d'(a) = 0, d(a) < 0.$$

Thus

$$\frac{1}{2}(\sigma^2 + \Delta)d''(a) + r(A - h)d'(a) - rd(a) > 0.$$

This contradicts (2.5). Hence *d(A)≥0* for all *A.*

Proposition 2.3. *The solution to equations (2.7) and (2.8) that satisfies the four boundary conditions is decreasing and convex with either of the two special cases of utility, if (2.6) holds.*

Proof. Clearly ψ is decreasing and convex on the upper segment.

On the lower segment, ψ is decreasing by the same reasoning as in the proof of proposition 2.1.

Let *A=0.* Then in (2.8)

$$-rh\psi'(0) - r\psi(0) = 0.$$

Since $\psi(0) = 1$,

$$\psi'(0) = -\frac{1}{h}.$$

Let *A=h.* Then in (2.8)

$$\frac{1}{2}\sigma^2 h\psi''(h) = r\psi(h).$$

If $h \leq K$, then $\psi(h) \geq \psi(K) > 0$, and so $\psi''(h) > 0$. Suppose $h > K$ and $\psi(h) = 0$. That is, *h* is on the upper segment. But from the solution to (2.7), if $\psi(h) = 0$, then $\psi(A) = \psi'(A) = 0$ for all *A* on the upper segment. In particular, $\psi(K) = 0$, and on the lower segment

$$\lim_{A \uparrow K} \psi'(A) = 0.$$

Thus for some $\varepsilon > 0$ arbitrarily small

$$K - \varepsilon > 0$$

$$\psi'(K - \varepsilon) > \psi'(0) = -\frac{1}{h}.$$

Thus for some $A \in (0, K - \varepsilon)$, $\psi''(A) > 0$. Hence in either case, for some $A \in (0, K)$, $\psi''(A) > 0$.

Now differentiate (2.8) with respect to *A* to obtain:

(2.9)

$$\frac{1}{2}\sigma^2 A\psi'''(A) = -\left[r(A - h) - \frac{1}{2}\sigma^2\right]\psi''(A).$$

Set $A = 0$. Hence

$$0 = \left[rh - \frac{1}{2}\sigma^2\right]\psi''(0).$$

There are two cases.

Case 1: Suppose $\left[rh - \frac{1}{2}\sigma^2\right] > 0$. Then $\psi''(0) = 0$. Let *a* solve $r(h - a) = \frac{1}{2}\sigma^2$. By construction, $a \in (0,h)$. Then $\psi'''(A)$ has the same sign as $\psi''(A)$ for $A < a$ and the opposite sign for $A > a$.

Suppose that ψ'' is negative somewhere on *(0,K)* and let

$$a' = \inf\{A|\psi''(A) < 0\}.$$

Suppose a'=0. Since we have already established that $\psi''(0)=0$ in this case, this is impossible.

Suppose $a' \epsilon\ [0, a)$. Thus ψ'' must reach a maximum in this interval, and be positive at that maximum. But then since ψ''' has the same sign at ψ'' on this interval, ψ''' is positive at the maximum. This is impossible, since ψ''' has to be zero at the maximum and negative for larger values.

Finally, suppose $a' \epsilon\ (a, K)$. Then $\psi''(a')=0$ and $\psi''(a'+\varepsilon)<0$ for ε positive and sufficiently small. Hence for some a'' between a' and $a'+\varepsilon$, we *have* $\psi''(a'')<0$, $\psi'''(a'')<0$ and $a''>a$. This is impossible because ψ'' and ψ''' have opposite signs in this range.

Hence $\psi''(A)\geq 0$ on the lower segment, and $\psi''(A)>0$ for $A>0$.

Case 2: Suppose $\left[rh - \frac{1}{2}\sigma^2\right] = 0$. Then from (2.9) for any $A>0$, $\psi''(A)$ and $\psi'''(A)$ have opposite signs. Hence either $\psi''(A)>0$ for all A, or $\psi''(A)<0$ for all A. But we have seen that $\psi''(A)>0$ for some A. Hence $\psi''(A)>0$ for all $A>0$.

The final step is to use smooth-pasting to assure that convexity on the lower and upper segments implies global convexity, as in the proof of proposition 2.1.

Proposition 3.1. *There exists a unique P<V* such that K(P)=P.*

Proof. For all π, $K(\pi)>H/r>0$. Hence $K(0)>0$.

Consider

$$K(V^*) = \frac{H}{r} - \frac{\Psi(V^* - V^*)}{R} = \frac{H}{r} - \frac{1}{R} < -\frac{1}{R} + F + \frac{H}{r} = V^*.$$

Consider the function *k(π)=K(π)-π*. This is continuous and differentiable on *(0,V*)*, with *k(0)>0* and *k(V*)<0*. Hence for some *π ε [0, V*], k(π)=0.* This implies *K(π)=π.* This establishes existence.

By algebra, *k'(π)<0* on *(0,V*]* with *k'(V*)=0*. This establishes uniqueness.

Proposition 3.2. *(a)* $\mathcal{H}(t)$ *has a unique maximum at t*>0.*

(b) *If* $\mu_h<0$*, then* $\lim_{t\to\infty}\mathcal{H}(t) = 0$, *and T<t*≤τ.*

(c) *If* $\mu_h>0$*, then* $\lim_{t\to\infty}\mathcal{H}(t) = \frac{\mu_h^2}{2\sigma_h^2}$, *and* $t_1<t^*<\tau$*, where* t_1 *is the unique positive solution of*

$$t_1 = \frac{V^* - P}{3\sigma_h^2}.$$

Proof. It is easy to see that $\mathcal{H}(t)$is continuous and continuously differentiable on $\mathbb{R}^+$. From (3.8) and (3.6) for $\mu_h<0$, it follows that

$$\lim_{t\to\infty}\mathcal{H}(t) = \lim_{t\to\infty}\frac{f(t)}{1-F(t)} = 0.$$

Now consider $\mu_h \geq 0$. Since in this case $\lim_{t\to\infty}(1 = F(t)) = 0$, apply L'Hopital's theorem:

$$\lim_{t\to\infty}\mathcal{H}(t) = \lim_{t\to\infty}\frac{f'(t)}{f(t)}.$$

From (3.7):

(3.9)

$$\frac{f'(t)}{f(t)} = -\frac{\mu_h^2}{2\sigma_h^2} - \frac{3}{2t} + \frac{V^* - P}{2\sigma_h^2 t^2} \triangleq q(t).$$

Hence

$$\lim_{t\to\infty} \mathcal{H}(t) = -\lim_{t\to\infty} \frac{f'(t)}{f(t)} = \frac{\mu_h^2}{2\sigma_h^2}.$$

Now consider $\mathcal{H}'(t)$. By algebra

$$\mathcal{H}'(t) = \mathcal{H}(t)[q(t) + \mathcal{H}(t)].$$

It is easy to establish that *q(.)* has the form shown in figure 3.1.

Figure 3.1 around here.

In particular, it is easy to see that

$$\lim_{t\downarrow 0} q(t) = \infty$$

$$q(T) = 0$$

$$q(t_1) = -\frac{\mu_h^2}{2\sigma_h^2}$$

$$\lim_{t\to\infty} q(t) = -\frac{\mu_h^2}{2\sigma_h^2}.$$

Moreover, τ is the unique minimum of *q(t)* on the positive reals.

Thus for all $t>t_1$, $q(t)<-\frac{\mu_h^2}{2\sigma_h^2}$.

Next, we have several useful lemmas about the location of maxima or minima of the hazard function, if there are any.

Lemma 3.1. *No maximum or minimum of the hazard function occurs at any t≤T.*

Proof. Since q(t)≥0 for t≤T, $\mathcal{H}(t) + q(t) > 0$, and so $\mathcal{H}'(t) > 0$.

Lemma 3.2. *No maximum of the hazard function occurs at any t>τ.*

Proof. Suppose a maximum of the hazard function occurs at *t>τ*. Then $\mathcal{H}'(t) = 0$ and so $\mathcal{H}(t) + q(t) = 0$. Consider *t+dt* for *dt* sufficiently small. Since $\mathcal{H}(t)$ is a maximum $\mathcal{H}(t + dt) \approx \mathcal{H}(t)$. Since *q(.)* is increasing at *t*, *q(t+dt)>q(t).* Hence $\mathcal{H}(t + dt) + q(t + dt) > H(t) + q(t) = 0$, and so $\mathcal{H}'(t + dt) > 0$. Similarly, $\mathcal{H}'(t - dt) < 0$. Thus a minimum occurs at t, not a maximum. This is a contradiction.

Lemma 3.3. *No minimum of the hazard function occurs in the interval (T, τ).*

Proof. The same as the proof of lemma 3.2, *mutatis mutandis,* since *q(.) is decreasing in this interval.*

The remainder of the proof has two different cases, depending on the drift of the housed option.

<u>Case 1: μ_h<0.</u>

Since $\mathcal{H}(t)$is nonnegative and $\mathcal{H}'(t) > 0$ for *t<T,* $\mathcal{H}(T) > 0$. Recall that in this case

$$\lim_{t\to\infty} \mathcal{H}(t) = 0.$$

Hence there must be at least one maximum of $\mathcal{H}(t)$ in the interval *[T, ∞).* By lemma 3.2, all maxima must be in the interval *[T, τ)*. Suppose there is more than one maximum in this interval. Then there must be at least one minimum in this interval, too (between any two maxima). But by lemma 3.3, there are no minima in this interval. Hence $\mathcal{H}(t)$ has precisely one maximum, and it is in the interval *[T, τ).*

Suppose there is a minimum in the interval *(τ,∞)*. Let *t* denote the largest such minimum. Since $\mathcal{H}(t') > 0$ for all *t'>t* by the hypothesis that *t* is the largest minimum, and the hazard rate approaches zero as *t'* goes to infinity, $\mathcal{H}(t) < 0$. This is impossible.

Hence in this case the only critical point of the hazard function is a unique maximum in the interval *[T, τ)*.

Case 2: $\mu_h \geq 0$.

We need the following lemma:

Lemma 3.4. $\mathcal{H}(t_1) > \frac{\mu_h^2}{2\sigma_h^2}$.

Proof. Suppose $\mathcal{H}(t_1) < \frac{\mu_h^2}{2\sigma_h^2}$. Then $\mathcal{H}(t_1) + q(t_1) < 0$. Then $\mathcal{H}'(t_1) < 0$. By lemma 3.3, no minimum of the hazard function occurs in the interval (t_1, τ), and so $\mathcal{H}'(t) < 0$ for this entire interval. Hence $\mathcal{H}(\tau) < H(t_1) < \frac{\mu_h^2}{2\sigma_h^2}$. Recall that in this case,

$$\lim_{t\to\infty} \mathcal{H}(t) = \frac{\mu_h^2}{2\sigma_h^2}.$$

Hence there must be some *t>τ* at which the hazard function reaches a minimum. Clearly

$$\mathcal{H}(t) < \frac{\mu_h^2}{2\sigma_h^2}.$$

But since $t>t_1$, $q(t)<-\frac{\mu_h^2}{2\sigma_h^2}$. Hence $q(t)+\mathcal{H}(t) < 0$, and so $\mathcal{H}'(t) < 0$. This contradicts the hypothesis that *t* was a minimum.

Suppose $\mathcal{H}(t_1) = \frac{\mu_h^2}{2\sigma_h^2}$. Then $\mathcal{H}(t_1) = 0$, and t_1 is a local maximum. Since there is no local minimum of the hazard function in the interval *(t_1, τ)*, $\mathcal{H}(\tau) < H(t_1) = \frac{\mu_h^2}{2\sigma_h^2}$. Then the reasoning of the previous paragraph applies.

Hence $\mathcal{H}(t_1) > \frac{\mu_h^2}{2\sigma_h^2}$.

Since $\mathcal{H}(t_1) > \frac{\mu_h^2}{2\sigma_h^2}$, and $\lim_{t\to\infty} \mathcal{H}(t) = \frac{\mu_h^2}{2\sigma_h^2}$, there must be at least one maximum in the interval *(t_1, ∞).* By lemma 3.2, all maxima must be in the interval *(t_1, τ).* Suppose there are two or more maxima in this interval. There must be at least one minimum. But lemma 3.3, there are no minima in this interval. Hence the hazard function has precisely one maximum, and it is in the interval *(t_1, τ).*

Suppose there is a minimum in the interval *(τ,∞)*. Let *t* denote the largest such minimum. Since $\mathcal{H}(t') > 0$ for all *t'>t* by the hypothesis that *t* is the largest minimum, and $\lim_{t\to\infty} \mathcal{H}(t) = \frac{\mu_h^2}{2\sigma_h^2}$, it follows that $\mathcal{H}(t) < \frac{\mu_h^2}{2\sigma_h^2}$. But since *t>$t_1$, q(t)<-* $\frac{\mu_h^2}{2\sigma_h^2}$. Thus $\mathcal{H}(t) + q(t) < 0$ and so *t* is not a minimum. This is a contradiction. Hence the hazard function has only one critical point..

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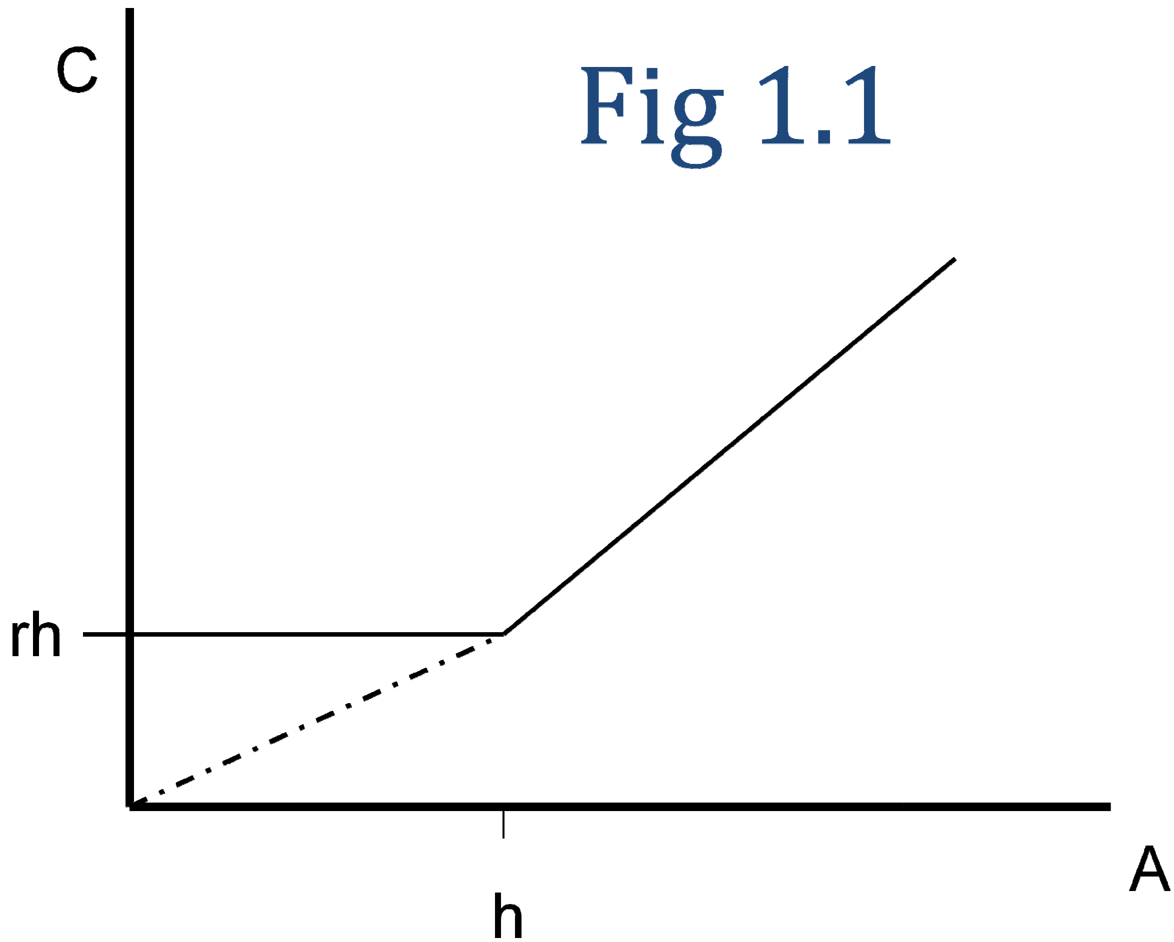
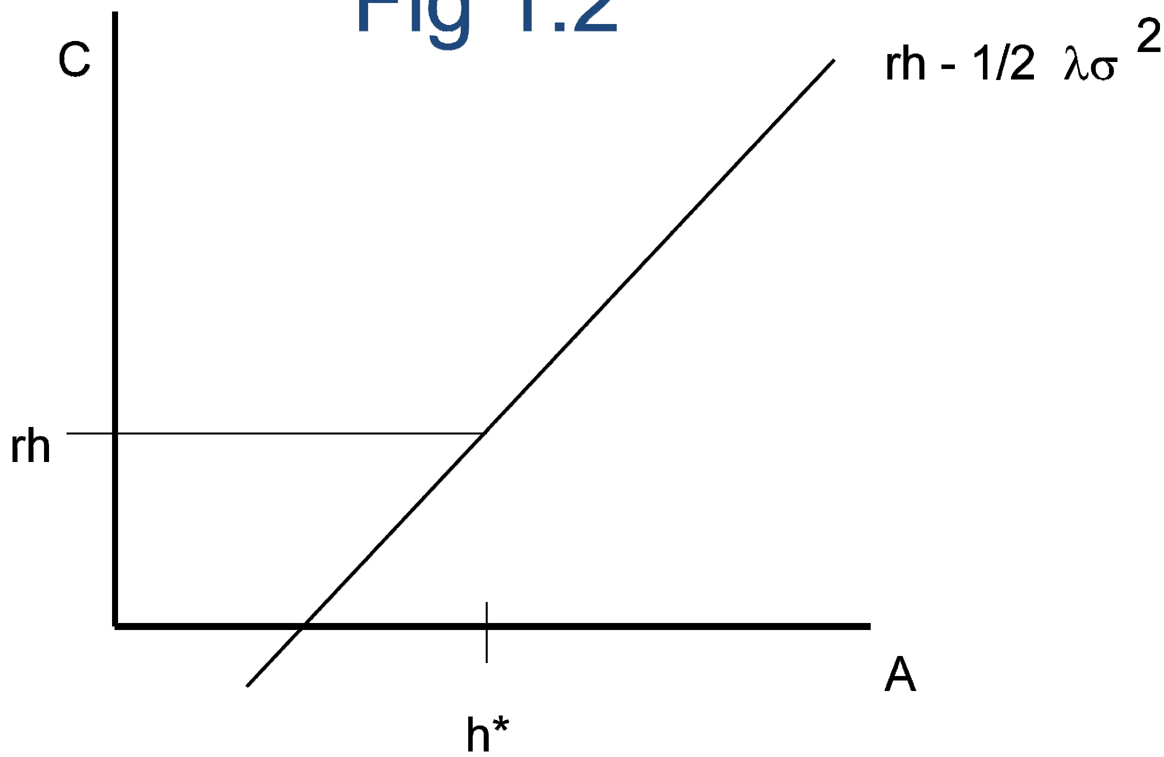


Fig 1.1

Fig 1.2



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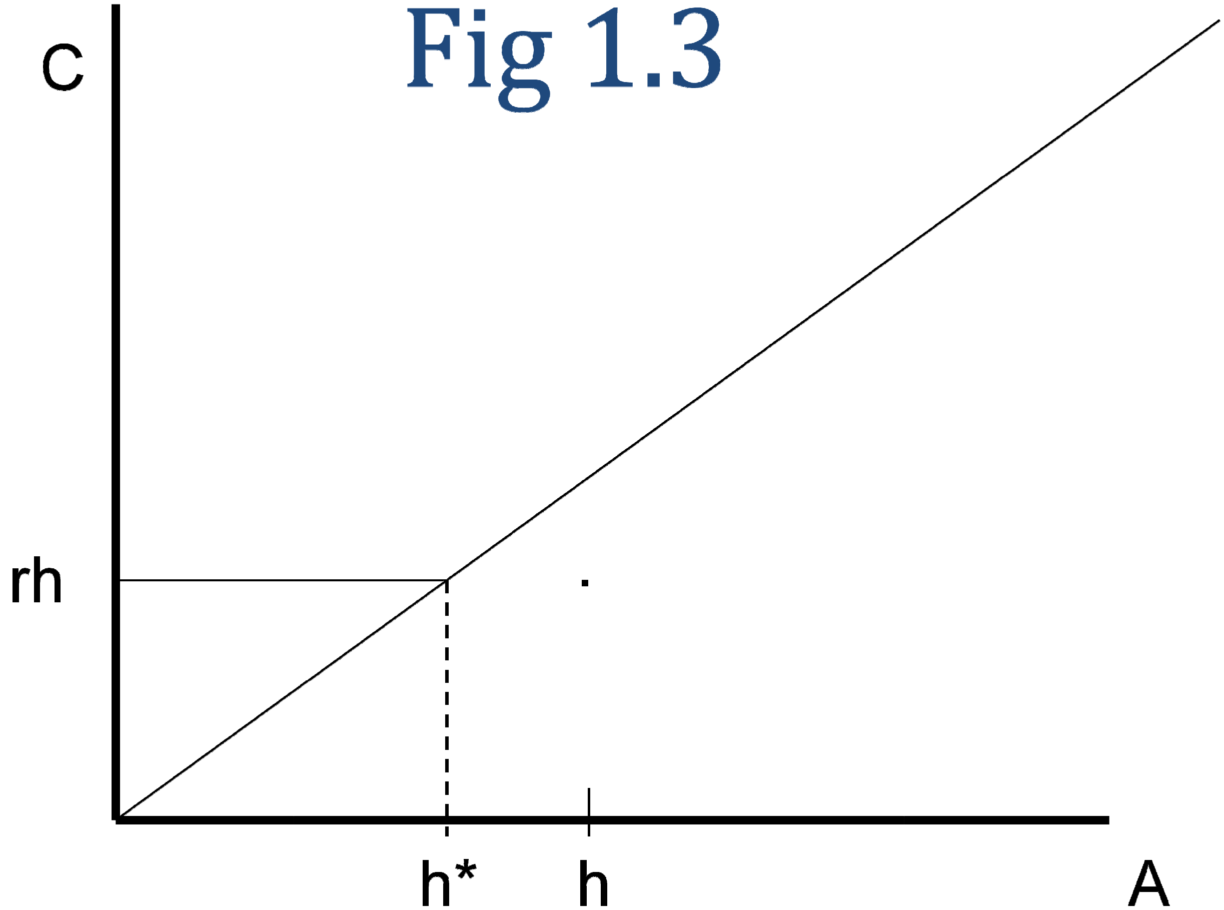

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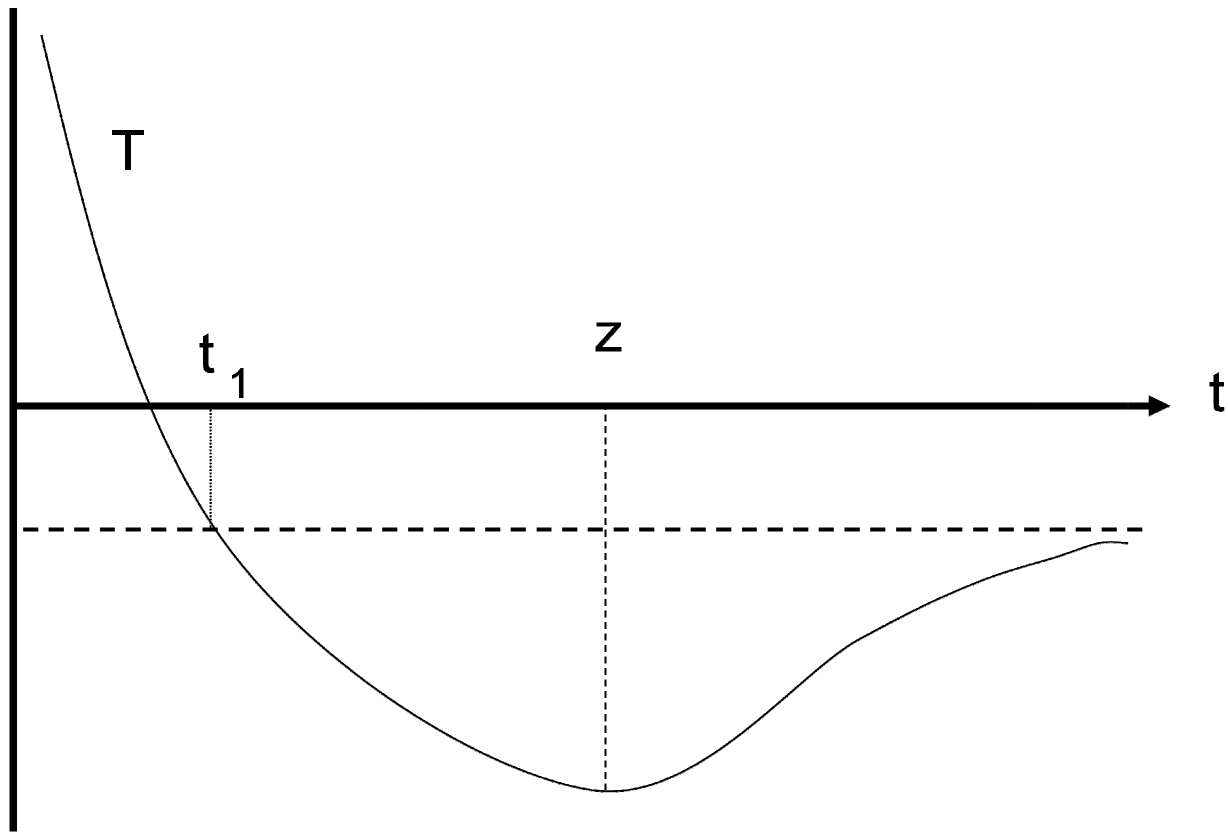

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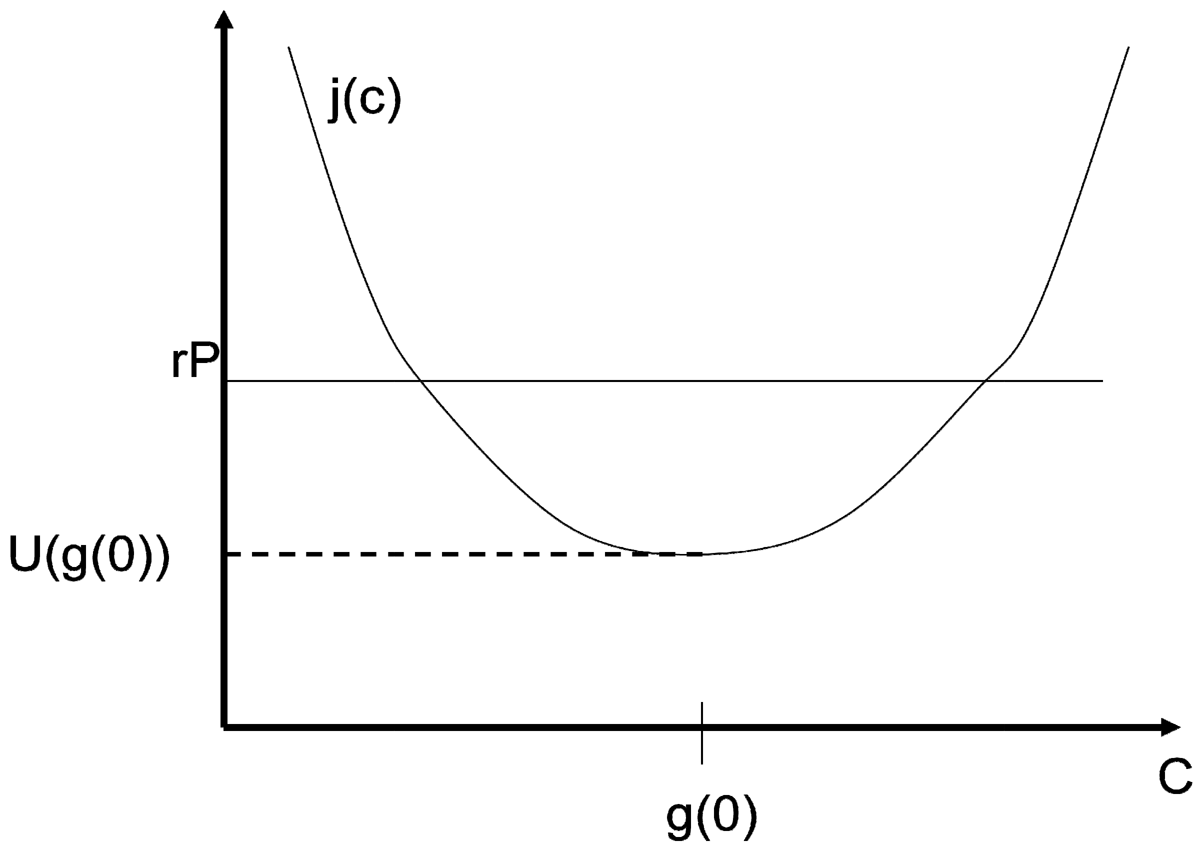
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Figure 4.2