Properties of Hamiltonian Torus Actions on Closed Symplectic Manifolds

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ABSTRACT

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In this thesis, we will study the properties of certain Hamiltonian torus actions on closed symplectic manifolds.

First, we will consider counting Hamiltonian T^n actions on closed, symplectic manifolds M^{2n} so that $\dim(H^2(M)) = 2$. In particular, all such manifolds are $\mathbb{C}P^r$ bundles over $\mathbb{C}P^s$ for some r, s. We use cohomological techniques to show that there is a unique toric structure if $r < s$. Furthermore, if $r > s$, we show that there is a finite number of toric structures on M that are compatible with some symplectic structure on M. Additionally, we show there is uniqueness in certain other cases, such as the case where (M, ω) is monotone.

Finally, we will be interested in the existence of symplectic, non-Hamiltonian circle actions on closed symplectic 6-manifolds. In particular, we will use J-holomorphic curve techniques to show that there are no such actions that satisfy certain fixed point conditions. This lends support to the conjecture that there are no such actions with a non-empty set of isolated fixed points.

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Chapter 1

Introduction

1.1 General Introduction

We say that (M^{2n}, ω) is a closed symplectic manifold if M^{2n} is a closed manifold and ω is a 2-form on M satisfying $d\omega = 0$ and $\omega^n > 0$. Now, consider a smooth action of S^1 on M. Such an action is determined by a vector field ξ which is obtained by differentiating the action. We say that the S^1 action is **symplectic** if

$$\mathcal{L}_\xi(\omega) = 0,$$

where $\mathcal{L}_\xi(\omega)$ denotes the Lie derivative of ω with respect to ξ. Applying Cartan's identity, this gives

$$0 = \mathcal{L}_\xi(\omega) = d\iota_\xi(\omega) + \iota_\xi(d\omega) = d\iota_\xi(\omega),$$

where the last equality holds since $d\omega = 0$. In particular, we calso say that an S^1 action is **symplectic** if $\iota_\xi(\omega)$ is closed. Furthermore, we will say an action is **Hamiltonian** if $\iota_\xi(\omega)$ is exact. In particular, there is a function H with $dH = \iota_\xi\omega$, and a specific choice of such an H is called a moment map for the action. Correspondingly, an action of T^k on (M^{2n}, ω) is called Hamiltonian if it can be written as an action of $S^1 \times \ldots \times S^1$ with each S^1 action being Hamiltonian. Such an action is determined by a moment map $\mu : M \to \mathbb{R}^k$ which comes from piecing together the moment maps on each coordinate action. Furthermore, if the action is effective, the sets $\mu^{-1}(x)$ are isotropic submanifolds for regular levels $x \in \mathbb{R}^k$. In particular, we can never have an effective Hamiltonian torus action T^k with $k > n$.

Hamiltonian torus actions are of particular interest because they can be used to study properties of the symplectic manifolds they act on. Of particular interest to us is the minimal dimensional case of a Hamiltonian S^1 action and the maximal dimensional case of a Hamiltonian T^n action. We will consider these two cases separately and will use different techniques to examine them.

First, we will consider the case of a T^n action on a closed symplectic manifold M^{2n}. Symplectic manifolds with such actions are called symplectic toric manifolds. In [10], Masuda and Suh discuss the topology of symplectic toric manifolds. A question they raise in this paper can be loosely paraphrased as follows:

Question 1.1.1. *Given a symplectic toric manifold (M, ω, T), to what extent does the cohomology ring $H^*(M)$ determine the toric manifold.*

One way to study this question is to consider how many different toric structures can be put on the same symplectic manifold, up to symplectomorphism. In particular, if there happens to be a unique toric structure, then we have reduced this question to asking how $H^*(M)$ determines the symplectic manifold. In [15], McDuff proved the following theorem related to this question.

Theorem 1.1.2. *Let $[\omega] \in H^2(M; \mathbb{Z})$ be given. Then up to equivariant symplectomorphism, there are at most finitely many toric symplectic manifolds (M, ω, T) for which there is a ring isomorphism Ψ taking the symplectic class to the fixed class $[\omega]$.*

In addition to the above version assuming $[\omega]$ is integral, Borisov-McDuff proved a version using more general ring coefficients that also fixes the cohomology classes of c_1 and c_2. In the first case, we conclude that the cohomology ring of the manifold together with the integral symplectic class determines that there are finitely many toric structures on a given symplectic manifold. In the same paper, McDuff proves the following theorem.

Proposition 1.1.3. *([15], Prop 1.8) Let $(M, \omega) = (\mathbb{C}P^r \times \mathbb{C}P^s, \omega_r \times \lambda\omega_s)$ with $\lambda > 0$. Then if either $r \geq s \geq 2$, or $r > s \geq 1$ and $\lambda \leq 1$, or $r = s = 1$ and $\lambda = 1$, there is a unique toric structure compatible with this symplectic structure. In all other cases, the toric structure is not unique.*

This proposition shows in particular that if we have $\mathbb{C}P^r \times \mathbb{C}P^s$ satisfying one of the above assumptions, then $H^*(\mathbb{C}P^r \times \mathbb{C}P^s)$ together with the symplectic class determines a unique toric structure. In the first half of the paper, we will generalize the above Proposition by considering these counting questions for manifolds M^{2n} which are $\mathbb{C}P^r$ bundles over $\mathbb{C}P^s$. We will prove several results about such bundles, and the results we will prove are outlined in Section 3.1.

In the second half of the paper, we will consider the case of an S^1 action on a closed symplectic manifold M^{2n}. A general question one can ask about such an action can be stated as follows.

Question 1.1.4. *Given a closed symplectic manifold (M, ω) and a symplectic S^1 action, what conditions on the action will guarantee that it is a Hamiltonian S^1 action?*

An obvious necessary condition is that the action should have some fixed points. In particular, if the action is Hamiltonian, we saw above it was determined by $\phi : M \to \mathbb{R}$. But then ϕ is a map from a closed manifold to $\mathbb{R}$, so it must have a maximum and a minimum, which would be critical points of ϕ and hence fixed points of the action. However, in [14], McDuff showed that this is not a sufficient condition. More specifically, she proved the following.

Theorem 1.1.5. *There is a closed symplectic manifold (M^6, ω) with a symplectic, non-Hamiltonian S^1 action which has fixed points and whose fixed point sets are 2-tori.*

The above theorem has led to the following conjecture.

Conjecture 1.1.6. *If (X^6, ω) has a symplectic S^1 action which has a non-empty set of isolated fixed points, then the action is Hamiltonian.*

In the second half of this thesis, we will rule out a certain class of counterexamples to the above conjecture using J-holomorphic curve techniques, which is a method that has not previously been applied to this conjecture.

1.2 Symplectic Toric Bundles of Projective Spaces

We say M is a symplectic bundle if M is an $\widetilde{M}$ bundle over $\widehat{M}$ so that $\widetilde{M}$ has a symplectic structure ω_0 and the structure group of the bundle is $\mathrm{Symp}(\widetilde{M})$. In particular, this implies that each fiber F_x over a point $x \in \widehat{M}$ has a symplectic structure ω_x so that $i^*(\omega_x) = \omega_0$ where i is the inclusion of the standard fiber.

If $H^1(\widehat{M}) = 0$, as it is if $\widehat{M} = \mathbb{C}P^s$, we can piece the forms ω_x together into a closed form τ on M so that τ is non-degenerate on the fibers of M. If also $(\widehat{M}, \widehat{\omega})$ is symplectic, then there is a closed form ω on M, defined by

$$\omega = \tau + K\pi^*(\widehat{\omega}),$$

where $\pi : M \to \widehat{M}$ is the projection and $K \in \mathbb{R}$. It is well known that ω is symplectic for sufficiently large K.

Now further assume that we have Hamiltonian torus actions $\widetilde{T}$, T, and $\widehat{T}$ on $\widetilde{M}$, M, and $\widehat{M}$ respectively, making them each symplectic toric manifolds. Then we say that M is a symplectic toric bundle if there is a short exact sequence

$$\widetilde{T} \to T \to \widehat{T},$$

such that $i : (\widetilde{M}, \widetilde{T}) \to (M, T)$ and $\pi : (M, T) \to (\widehat{M}, \widehat{T})$ are equivariant.

In the first half of this paper, we will consider toric structures on closed symplectic manifolds which have $\dim H^2(M) = 2$. By Lemma 3.3.2 below, any such manifold is a $\mathbb{C}P^r$ bundle over $\mathbb{C}P^s$. Furthermore, any such toric structure can be realized as the projectivization $\mathbb{P}(L_{-a_0} \oplus L_{-a_1} \oplus \cdots \oplus L_{-a_r})$ of a sum of complex line bundles L_{-a_i} over $\mathbb{C}P^s$ with the obvious action of the torus T^{r+s}, where L_c is the line bundle over $\mathbb{C}P^s$ with first Chern class c times a generator of $H^2(\mathbb{C}P^s)$. By tensoring with L_c where $c = -\max a_i$ we may assume that $a_0 = 0 \leq a_1 \leq \ldots \leq a_r$. Moreover, the symplectic form ω restricts to the standard Fubini-Study form on the fiber, and so may be normalized by requiring that $\omega(\ell) = r + 1$, where ℓ is the homology class of a line in the fiber. Since $H^2(M)$ has rank 2, the above normalization still leaves $[\omega]$ with one free parameter. We call this parameter κ, and it can be easily seen to be determined by $\mathrm{Vol}(M, \omega)$, as in Lemma 2.2.7. Thus from the above we see that the tuple $(\mathbf{a}; \kappa) := (a_1, \ldots, a_r; \kappa)$ determines a toric structure on a symplectic toric manifold (M, ω) with $\dim H^2(M) = 2$, where $0 \leq a_1 \leq \ldots \leq a_r$ are integers and κ is a real number related to the symplectic volume of M.

We denote the resulting toric manifold by $(M_{\mathbf{a}}, \omega_{\mathbf{a}}^{\kappa}, T_{\mathbf{a}})$. By Definition 2.2.1 and Lemma 2.2.3 below, for each tuple $\mathbf{a}$ there is a number $K_{\mathbf{a}}(s)$ such that $M_{\mathbf{a}}$ admits the structure described above for all $\kappa > K_{\mathbf{a}}(s) = \sigma_1(\mathbf{a}) - s$, where $\sigma_1(\mathbf{a})$ is the sum of the components of $\mathbf{a}$. Furthermore, we have the following fundamental result, which is proven in Section 3.3.

Theorem 1.2.1. *Let (M, ω, T) be a toric symplectic manifold with* $\dim H^2(M) = 2$. *Then there is a unique tuple $(\mathbf{a}; \kappa)$ with $0 \leq a_1 \leq \ldots \leq a_r$ such that (M, ω, T) is equivariantly symplectomorphic to $(M_{\mathbf{a}}, \omega_{\mathbf{a}}^{\kappa}, T_{\mathbf{a}})$.*

Thus, to count toric structures on closed symplectic manifolds with $\dim H^2(M) = 2$, it suffices to count toric structures on the manifolds $M_{\mathbf{a}}$.

The following result is based on Theorem 6.1 of [3] and is instrumental to the proofs of many of our results. Due to its important role in the rest of the paper, we will give the details of the proof using our notation in Section 3.2.

Proposition 1.2.2. *Let $M_{\mathbf{a}}$ and $M_{\mathbf{b}}$ be $\mathbb{C}P^r$ bundles over $\mathbb{C}P^s$ as above for some vectors* $\mathbf{a}$ *and* $\mathbf{b}$. *The following are equivalent:*

1. *$H^*(M_{\mathbf{a}};\mathbb{Z})$ is isomorphic to $H^*(M_{\mathbf{b}};\mathbb{Z})$ as a ring.*
2. *$\mathbb{P}(L_0 \oplus L_{-a_1} \oplus \cdots \oplus L_{-a_r})$ is isomorphic to $\mathbb{P}(L_0 \oplus L_{-b_1} \oplus \cdots \oplus L_{-b_r})$ as a projective vector bundle.*
3. *$M_{\mathbf{a}}$ is isomorphic to $M_{\mathbf{b}}$ as a symplectic bundle.*
4. *There is $C \in \mathbb{Z}$ such that*

   $$\sigma_i(C, C+\mathbf{a}) := \sigma_i(C, a_1+C, \dots, a_r+C) = \sigma_i(0, b_1, \dots, b_r) =: \sigma_i(0, \mathbf{b}), \quad 1 \le i \le \min\{r+1, s\},$$

   where σ_i denotes the ith elementary symmetric function.

It is natural to conjecture that if $(M_{\mathbf{a}}, \omega_{\mathbf{a}}^{\kappa})$ is isomorphic to $(M_{\mathbf{b}}, \omega_{\mathbf{b}}^{\kappa})$ as a symplectic bundle, i.e. there exists a diffeomorphism $\phi : M_{\mathbf{a}} \to M_{\mathbf{b}}$ preserving the fiberwise symplectic structure, then they are symplectomorphic for all $\kappa > \max(K_{\mathbf{a}}, K_{\mathbf{b}})$. However this is not yet known except when $s = 1$ or, as in Lemma 1.2.6 below, when $\kappa \gg 0$. In fact, we have the following theorem, proven in Section 4.1.

Theorem 1.2.3. *Let $(M_{\mathbf{a}}, \omega_{\mathbf{a}}^{\kappa})$ and $(M_{\mathbf{b}}, \omega_{\mathbf{b}}^{\kappa})$ be $\mathbb{C}P^r$ bundles over $\mathbb{C}P^1$ as above with $\kappa > \max(K_{\mathbf{a}}, K_{\mathbf{b}})$. Then $(M_{\mathbf{a}}, \omega_{\mathbf{a}}^{\kappa})$ is symplectomorphic to $(M_{\mathbf{b}}, \omega_{\mathbf{b}}^{\kappa})$ if and only if $(M_{\mathbf{a}}, \omega_{\mathbf{a}}^{\kappa})$ is isomorphic to $(M_{\mathbf{b}}, \omega_{\mathrm{bb}}^{\kappa})$ as a symplectic bundle.*

Since this is not known in the general case $s > 1$, we will consider the following weaker notion of equivalence.

Definition 1.2.4. *We say that two symplectic manifolds $(M, \omega), (M', \omega')$ are* **deformation equivalent** *if there is a diffeomorphism $\phi : M \to M'$ and a family $\omega_t, t \in [0, 1]$, of symplectic forms on M such that*

$$\phi^*([\omega']) = [\omega], \quad \omega_0 = \phi^*(\omega'), \quad \omega_1 = \omega.$$

Remark 1.2.5. In contrast to the usual definition of deformation equivalence, we have required $\phi^*([\omega']) = [\omega]$. Thus the deformation starts and ends in the same cohomology class, even if it leaves this class for some t.

The following lemma says that $(M_{\mathbf{a}}, \omega_{\mathbf{a}}^{\kappa})$ and $(M_{\mathbf{b}}, \omega_{\mathbf{b}}^{\kappa})$ are isomorphic as symplectic bundles if and only if they are deformation equivalent, and it will be proven in Section 4.2.

Lemma 1.2.6. *Let* $\mathbf{a} = (a_1, \ldots, a_r)$*,* $\mathbf{b} = (b_1, \ldots, b_r)$ *be non-negative integer vectors and* κ *a real number determine the bundles* $(M_{\mathbf{a}}, \omega_{\mathbf{a}}^{\kappa})$ *and* $(M_{\mathbf{b}}, \omega_{\mathbf{b}}^{\kappa})$*. Then* $M_{\mathbf{a}}$ *and* M_{bb} *are isomorphic as symplectic bundles if and only if* $(M_{\mathbf{a}}, \omega_{\mathbf{a}}^{\kappa})$ *and* $(M_{\mathrm{bb}}, \omega_{\mathrm{bb}}^{\kappa})$ *are deformation equivalent. Moreover, for* $\kappa \gg 0$*, we also have that* $(M_{\mathbf{a}}, \omega_{\mathbf{a}}^{\kappa})$ *and* $(M_{\mathrm{bb}}, \omega_{\mathrm{bb}}^{\kappa})$ *are symplectomorphic.*

Given the class of manifolds $(M_{\mathbf{a}}, \omega_{\mathbf{a}}^{\kappa})$, we can ask how many different toric structures we can put on the same deformation class. Given symplectic toric manifolds (M, ω, T) and (M', ω', T'), we recall that the toric structures are called **equivalent** if there is an equivariant symplectomorphism from one to the other, and are called **inequivalent** otherwise.

The following result uses the fact that two toric manifolds are equivalent if and only if their moment polytopes are affine equivalent, and is proven in Section 3.3.

Lemma 1.2.7. *Let* $\mathbf{a} = (a_1, \ldots, a_r)$ *and* $\mathbf{b} = (b_1, \ldots, b_r)$ *be integer vectors with* $0 \le a_1 \le \ldots \le a_r$ *and* $0 \le b_1 \le \ldots \le b_r$ *and let* κ *and* κ' *be real numbers. Then* $(M_{\mathbf{a}}, \omega_{\mathbf{a}}^{\kappa}, T_{\mathbf{a}})$ *is equivalent to* $(M_{\mathbf{b}}, \omega_{\mathbf{b}}^{\kappa'}, T_{\mathbf{b}}) \iff (\mathbf{a}; \kappa) = (\mathbf{b}; \kappa')$*.*

Using this we can now state the main question we will be considering in the first half of the paper.

Question 1.2.8. *Given a tuple* $(\mathbf{a}; \kappa)$*, what is* $N_n(\mathbf{a}; \kappa)$*, the number of inequivalent toric structures on the deformation class of* $(M_{\mathbf{a}}^{2n}, \omega_{\mathbf{a}}^{\kappa})$*? In particular, for fixed* $\mathbf{a}$ *and* n*, how does it depend on* κ*, and for which* $(\mathbf{a}; \kappa)$ *do we have* $N_n(\mathbf{a}; \kappa) = 1$*?*

If $\mathbf{a} = 0$, the manifold $M_{\mathbf{a}}$ is just a product $\mathbb{C}P^r \times \mathbb{C}P^s$ and this question was answered in [15] by Proposition 1.1.3 above. As such, we will focus on the case where $\mathbf{a} \neq 0$, and hence assume some $a_i \neq 0$.

We now briefly state two of the main results of the first half of the paper and give an idea of how they will be proven. There are several other smaller results that we will discuss and a complete list of results is given in Section 3.1, while the proofs of all the results are given in Section 5.1. The first main theorem is a uniqueness result.

Theorem 1.2.9. *Let* $(M^{2n}, \omega_{\mathbf{a}}^{\kappa})$ *be determined by* $\mathbf{a} = (a_1, \ldots, a_r)$ *and* κ*, as before. If* $r < s$ *and* $\mathbf{a} \neq 0$*, we have*

$$N_n(\mathbf{a}; \kappa) = \begin{cases} 0 & \text{if } \kappa \leq K_{\mathbf{a}}(s) \\ 1 & \text{if } \kappa > K_{\mathbf{a}}(s), \end{cases}$$

where $K_{\mathbf{a}}(s) := \sigma_1(\mathbf{a}) - s$.

This gives a complete characterization for $N_n(\mathbf{a}; \kappa)$ with $r < s$ and has the following obvious corollary.

Corollary 1.2.10. *Let* (M, ω, T) *and* (M', ω', T') *be non-trivial toric* $\mathbb{C}P^r$ *bundles over* $\mathbb{C}P^s$ *with* $r < s$*. If* (M, ω) *is deformation equivalent to* (M', ω')*, then* (M, ω, T) *is equivariantly symplectomorphic to* (M', ω', T')*.*

The case where $r \geq s$ is more complicated. We will give detailed results for the $r \geq s$ case in Section 3.1, but for now, we will state the main theorem for $r \geq s$, which describes the behavior of $N_n(\mathbf{a}; \kappa)$ for large κ.

Theorem 1.2.11. *Let* $\mathbf{a} = (a_1, \ldots, a_r)$ *and* $\mathbf{b} = (b_1, \ldots, b_r)$ *be as before and let* C *be an integer, as in Proposition 1.2.2. Furthermore, assume that*

$$\sigma_i(0, \mathbf{b}) = \sigma_i(C, C + \mathbf{a}), \quad i = 1, \ldots, s,$$

with $r \geq s \geq 2$*. Then we have*

$$-\tfrac{1}{r+1}\sigma_1(\mathbf{a}) \leq C < \tfrac{r-1}{r}\sigma_1(\mathbf{a}).$$

Moreover, this implies

$$\kappa_1, \kappa_2 \geq (r + 1 - \tfrac{1}{r})\sigma_1(\mathbf{a}) - s \Longrightarrow N_n(\mathbf{a}; \kappa_1) = N_n(\mathbf{a}; \kappa_2).$$

In particular, we have

$$N_n(\mathbf{a}; \infty) := \lim_{\kappa \to \infty} N_n(\mathbf{a}; \kappa) < \infty$$

Remark 1.2.12. This result is surprising at first glance. The condition $N_n(\mathbf{a};\infty)<\infty$ implies that there are at most finitely many toric structures which are compatible with an arbitrary symplectic structure on the given deformation class of $M_{\mathbf{a}}$. This is stronger than the finiteness result proven by Borisov and McDuff in [15], which relies on fixing a symplectic structure to get finiteness.

However, if $r=s=1$, this does not happen. Indeed, in that case, $\mathbf{a}=a$ and $\mathbf{b}=b$ are just numbers, and the manifolds $(M_{\mathbf{a}},\omega_{\mathbf{a}}^{\kappa})$ are the well known Hirzebruch surfaces. It is known for the Hirzebruch surfaces that if $b-a$ is even, then $(M_{\mathbf{a}},\omega_{\mathbf{a}}^{\kappa})$ is deformation equivalent to $(M_{\mathbf{b}},\omega_{\mathbf{b}}^{\kappa})$, which shows that for any $\mathbf{a}$, we have

$$\lim_{\kappa\to\infty} N_2(\mathbf{a};\kappa)=\infty.$$

We now briefly summarize the general techniques we will use to prove these theorems and the theorems in Section 3.1. The main technique we use combines Proposition 1.2.2 with Lemma 1.2.6 and Lemma 1.2.7. Namely, combining Proposition 1.2.2 with Lemma 1.2.6, we see that if we have positive integer vectors $\boldsymbol{a}=(a_1,\dots,a_r)$ and $\boldsymbol{b}=(b_1,\dots,b_r)$ and a positive integer C so that

$$\sigma_i(C,C+\boldsymbol{a}):=\sigma_i(C,a_1+C,\dots,a_r+C)=\sigma_i(0,b_1,\dots,b_r)=:\sigma_i(0,\boldsymbol{b}),\quad 1\le i\le \min\{r+1,s\},$$

where σ_i denotes the ith elementary symmetric function, then for any κ, $(M_{\boldsymbol{a}},\omega_{\boldsymbol{a}}^{\kappa})$ and $(M_{\boldsymbol{b}},\omega_{bb}^{\kappa})$ are isomorphic as symplectic bundles, and hence they are deformation equivalent. However according to Lemma 1.2.7, we know that $(M_{\boldsymbol{a}},\omega_{\boldsymbol{a}}^{\kappa},T_{\boldsymbol{a}})$ and $(M_{\boldsymbol{b}},\omega_{\boldsymbol{b}}^{\kappa},T_{\boldsymbol{b}})$ represent different toric structures if $\mathbf{a}\neq\mathbf{b}$. Thus, if we can find integers integers r, s, and C, and integer vectors $\boldsymbol{a}\neq\boldsymbol{b}$ satisfying the above equation, this will correspond to $(M_{\boldsymbol{a}},\omega_{\boldsymbol{a}}^{\kappa})$ having a non-unique toric structure. Additionally, if this never happens for any choice of C and $\mathbf{b}$, then $(M_{\boldsymbol{a}},\omega_{\boldsymbol{a}}^{\kappa})$ has a unique toric structure. In Section 5.1, we will consider possible solutions to this equation in order to prove our above results and those in Section 3.1.

1.3 Symplectic, non-Hamiltonian S^1 Actions

We now introduce the main topic of the second half of the paper. Consider a closed symplectic manifold (X^6,ω) with a symplectic S^1 action. The second half of this paper will focus on Conjecture 1.1.6.

In order to prove this, we will consider the case where we have a symplectic, non-Hamiltonian circle action. In this case, McDuff noticed that if the symplectic class is integral, the circle action is determined by a moment map with values in S^1. By perturbing the original symplectic form ω, we can assume that $[\omega]$ is rational so that a large multiple of $[\omega]$ is integral. In particular, we can always assume have such a moment map for a symplectic S^1 action on (X^6, ω) so that a critical point of the moment map corresponds to a fixed point of the S^1 action. Moreover, we also have no critical points of index or co-index 0 since the action is Hamiltonian, and we don't have any critical points of odd index since the action is symplectic. In particular, we only have critical points of index 2 and 4, and furthermore, as in Lemma 11 of [17], there is an equal number of each, so that there is an even total number of critical points.

Let Φ denote the above S^1 valued moment map. We can use this to define the notion of a reduced space.

Definition 1.3.1. *Let (M, ω) be a closed symplectic manifold with a symplectic S^1 action, and let Φ denote the S^1 valued moment map of the action. The for each $\lambda \in S^1$, we can form the **reduced space** $(M_\lambda, \omega_\lambda)$ as follows:*

$$(M_\lambda, \omega_\lambda) = \Phi^{-1}(\lambda)/S^1$$

The resulting space will in general be a four dimensional symplectic orbifold with orbifold singularities corresponding to non-trivial isotropy of the S^1 action. We will assume that all such orbifold singularities are isolated points. We will resolve these singularities by successive blowups, which adds curves of self intersection -2 or less. We denote the resulting space $\widetilde{M}_\lambda$, and we call it the resolution of M_λ. For more details on these resolutions, see Section 6.2.

We will use the following theorem of Cho, Hwang, and Suh from [2] to prove a special case of Conjecture 1.1.6 above.

Theorem 1.3.2. *Let (X^6, ω) be a closed symplectic manifold with a symplectic S^1 action which has non-empty fixed point set. If there exists a regular value λ of the S^1 moment map such that the reduced space M_λ satisfies $b_2^+(M_\lambda) = 1$, then the action is Hamiltonian.*

Remark 1.3.3. Since the blowup operation has no effect on b_2^+, it is sufficient for us to consider the resolutions $\widetilde{M}_\lambda$ instead of M_λ in order to show $b_2^+ = 1$ at some regular level. In particular, we always have $b_2^+(\widetilde{M}_\lambda) = b_2^+(M_\lambda)$.

We will prove a special case of Conjecture 1.1.6 where our S^1 actions have isolated fixed points which satisfy various technical conditions. In particular, since the action has isolated fixed points and the manifold X is closed, there will be a finite number of fixed points, which we call $x_1, \ldots, x_{2n}$. Some of the technical conditions give restrictions on the type of fixed points we are considering, namely they are what we call good fixed points. Some examples of good fixed points include fixed points with isotropy weights $\pm(p, q, -1)$, or more generally $\pm(p, q, -r)$ with $p > r$ or $q > r$. Another example of a good fixed point is $\pm(5, 4, -11)$, although fixed points of the form $\pm(p, q, -r)$ with $p < r$ and $q < r$ usually tend to fail to be good fixed points. A full definition of a good fixed point is given in Definition 6.3.9.

Theorem 1.3.4. *Let (X^6, ω) be a closed symplectic manifold. Then there does not exist a symplectic, non-Hamiltonian S^1 action with a non-empty set of isolated fixed points such that all fixed points are good, there is no codimension 2 isotropy, and such that there exists a fixed point with weights $(p, q, -1)$ with $p, q > 0$ such that the only other fixed points with isotropy weights either $\pm p$ or $\pm q$ are of one of the following 2 forms:*

1. $\pm(p, q', -1)$
2. $\pm(p', q, -1)$

Remark 1.3.5. This remark is based largely on a private communication with Sue Tolman. The above theorem can be independently proven using the Atiyah–Bott–Berline–Vergne localization theorem (see [1]) in all of the known cases where the set of fixed points satisfy the assumptions of Theorem 1.3.4 above. Namely, for dimensional reasons, the integral over X of the first equivariant chern class of (X^6, ω) with respect to the S^1 action must vanish. If each fixed point x_i has isotropy weights (p_i, q_i, r_i) for non-zero integers p_i, q_i, r_i, ABBV localization computes this integral in a different way to get the relation

$$\sum_{i=1}^{2n} \tfrac{p_i+q_i+r_i}{p_i q_i r_i} = 0. \tag{ABBV1}$$

In the case where all of the fixed points are of the form $\pm(p_i, q_i, -r_i)$ with $p_i, q_i, r_i > 0$ and either p_i or $q_i > r_i$, every term in this sum is a non-zero negative number, so that the above sum can never vanish. In particular, if all $r_i = 1$, this implies that there are no symplectic actions so that all fixed points are of the form $\pm(p, q, -1)$.

Now consider the S^1 action with the four fixed points $\pm(5, 4, -11)$ and $\pm(220, 219, -1)$. It can be shown that this set of fixed points does in fact satisfy the assumptions of Theorem 1.3.4 and furthermore, a simple computation shows that the above sum gives $\frac{1}{110} + \frac{1}{110} - \frac{1}{110} - \frac{1}{110} = 0$, so that this set of fixed point data satisfies Equation (ABBV1). However, $ABBV$ localization applied to equivariant K-theory also implies that

$$\sum_{i=1}^{2n} \frac{1}{(1-e^{p_i})(1-e^{q_i})(1-e^{r_i})} = 0. \qquad \text{(ABBV2)}$$

A simple computation shows that our above example does not satisfy Equation (ABBV2).

In fact, there is currently no known example of a set of fixed point data which satisfies the assumptions of Theorem 1.3.4 and also satisfies Equations (ABBV1) and (ABBV2). In this sense, the current version of the theorem should be thought of as a possible proof technique which could potentially be generalized to more cases, including some previously unknown cases. In Remark 7.2.2 at the end of the paper, we give some suggestions about how one might generalize some of the arguments in the paper to cover some previously unknown cases.

We now briefly summarize the main points of the argument. For definitions and further details, see sections 5 and 6.

Let λ_i be a critical value of the moment map with isotropy weights $(\pm p_i, \pm q_i, \mp r_i)$ describing a good fixed point, as in Definition 6.3.9. We will show that at λ_i, the reduced spaces M_λ of the S^1 action change by a (p_i, q_i)-weighted blowup. We will further show that if we resolve the corresponding orbifold singularities to form $\widetilde{M}_\lambda$, then this blowup produces two chains Z_i^1 and Z_j^2 of non-generic curves connected by a curve $\widetilde{E}$ which has $\widetilde{E}^2 = -1$ and is an exceptional divisor. In particular, $\widetilde{E}$ has a non-trivial Gromov invariant, and so this curve persists under perturbations. We then use holomorphic curve techniques on this curve to demonstrate that the reduced spaces must satisfy $b_2^+ = 1$, so that by Lemma 1.3.2, the action is Hamiltonian.

Remark 1.3.6. Using the above argument, we can easily recover the 6-dimensional case of [20] where the action is semifree and has isolated fixed points. Namely, in the semifree case, there are no orbifold points and all the blowups are standard smooth blowups. Thus, in this case we do not have the curves Z_i^j corresponding to the orbifold singularities and all of the curves that appear in blowups and blowdowns are exceptional divisors, which greatly simplifies the J-holomorphic curve arguments.

Chapter 2

Moment Polytopes and Bundles of Polytopes

In this section, we will discuss the notion of a moment polytope as well as the notion of a bundle of polytopes.

2.1 Moment Polytopes

Now, let $\mu : M \to \mathbb{R}^n$ be the moment map of a Hamiltonian T^n action on M. Then $\mu(M)$ is a polytope which we call the moment polytope of M. It can be shown that any moment polytope is a simple, smooth, rational polytope, where if $\dim(\Delta) = n$, **simple** means at each vertex exactly n facets meet, **rational** means that the conormal vectors to these facets are primitive integral vectors, and **smooth** means that these vectors form an integer basis of $\mathbb{Z}^n$. We call such polytopes Delzant polytopes. The well known Delzant theorem from [4] says the following.

Theorem 2.1.1. *(Delzant) For each Delzant polytope Δ, there is a symplectic toric manifold M_Δ with moment polytope Δ. Moreover, (M, ω, T) is equivariantly symplectomorphic to (M', ω', T') if and only if Δ_M and $\Delta_{M'}$ are equivalent under the affine group generated by translations and the action of $GL(n; \mathbb{Z})$.*

Let Δ be the moment polytope of a toric structure on some symplectic toric manifold (M, ω, T).

We can describe Δ as

$$\{x \in \mathbb{R}^n | \langle x, \eta_i \rangle \leq \kappa_i \text{ for all } i\}$$

where the η_i are the outward primitive integer conormals to the facets of Δ and the κ_i are support constants.

Example 2.1.2. The moment polytope of $\mathbb{C}P^n$ will be denoted Δ_n, and is a copy of the standard n-simplex when we choose $\eta_i = -e_i$ for $1 \leq i \leq n$, $\eta_{n+1} = (1, \ldots, 1)$ and all $\kappa_i = 1$. Notice that Δ_n has edges of length $n+1$ and has volume equal to

$$\mathrm{Vol}(\Delta_n) = \tfrac{1}{n!}(n+1)^n.$$

2.2 Bundles of Polytopes

We are most interested in the case where the manifold M is a symplectic toric $\widetilde{M}$ bundle over $\widehat{M}$. To study this, we will discuss the notion of a bundle of polytopes.

The general definition of a polytope Δ being a $\widetilde{\Delta}$ bundle over $\widehat{\Delta}$ given as 3.10 of [16] is more complicated than we will need, so we instead summarize some key points. In particular, we only need the notion of a Δ_r bundle over Δ_s.

The basic idea is to develop a notion of bundles so that by the Delzant theorem above, a manifold (M, ω, T) is a symplectic toric $(\mathbb{C}P^r, \omega_r, T_r)$ bundle over $(\mathbb{C}P^s, \omega_s, T_s)$ if and only if Δ is a Δ_r bundle over Δ_s. At this point, we recall that $\Delta \subset \mathfrak{t}^*$, where $\mathfrak{t}$ is the Lie algebra of T, and similarly for Δ_r and Δ_s. Since the moment polytopes are then naturally subsets of the dual spaces to the Lie algebras of the torus actions, we should expect a Δ_r bundle over Δ_s to naturally be fibered by Δ_s over Δ_r, instead of the other way around. This motivates the following definition.

Definition 2.2.1. *We say that a polytope Δ is a Δ_r bundle over Δ_s if, for some choice of $(\boldsymbol{a}; \kappa)$ where $\boldsymbol{a} = (a_1, \ldots, a_r)$ are integers and $\kappa \in \mathbb{R}$ with $\kappa > K_{\boldsymbol{a}} := \sigma_1(\boldsymbol{a}) - s$, Δ is affine equivalent to the polytope $\Delta^\kappa_{\boldsymbol{a}}$, which is defined by setting*

$$\eta_i = \begin{cases} -e_i & \text{if } 1 \leq i \leq r \\ (1, \ldots, 1, 0, \ldots, 0) & \text{if } i = r+1 \\ -e_{i-1} & \text{if } r+2 \leq i \leq r+s+1 \\ (-a_1, \ldots, -a_r, 1, \ldots, 1) & \text{if } i = r+s+2, \end{cases} \tag{2.2.1}$$

with $\kappa_i = 1$ *for* $1 \leq i \leq r+s+1$, *and* $\kappa_{r+s+2} = \kappa$.

Remark 2.2.2. The polytope $\Delta_{\mathbf{a}}^{\kappa}$ naturally has the structure of a bundle with the base being a standard copy of Δ_r with fibers that are all rescaled copies of Δ_s. The vector $\mathbf{a}$ then has a natural interpretation as the slope of the increase of the rescaling as we move in the standard directions in Δ_r, while the number κ determines the rescaling over the origin. We will now take a few moments to show this more explicitly by computation.

We obtain relations on the coordinates x_i of an arbitrary point of $\Delta_{\mathbf{a}}^{\kappa}$ by computing $\langle x, \eta_i \rangle$ for each i, for $1 \leq i \leq r+s$. We get the inequalities

$$\begin{aligned} & x_i \geq -1, \forall i \\ & x_1 + \ldots + x_r \leq 1 \\ & x_{r+1} + \ldots + x_{r+s} \leq \kappa + a_1 x_1 + \ldots + a_r x_r. \end{aligned} \tag{*}$$

The first two lines of $(*)$ imply the first r coordinates of x, $(x_1, \ldots, x_r)$, form a standard copy of Δ_r, as described in Example 2.1.2. Also, the first and third lines of $(*)$ show that the last s coordinates of x, $(x_{r+1}, \ldots, x_{r+s})$, form a rescaled copy of Δ_s. Namely, they form a polytope $\Delta_s^{\kappa,x}$ described as a subset of $\mathbb{R}^s$ by the conormals

$$\eta_i = \begin{cases} -e_i, \forall 1 \leq i \leq s \\ (1, \ldots, 1), i = s+1 \end{cases}$$

with support constants $\kappa_i = 1$ for $1 \leq i \leq s$ and $\kappa_{s+1} = \kappa + a_1 x_1 + \ldots + a_r x_r$. Thus, $\Delta_s^{\kappa,x}$ is simply a standard simplex with edge length $s + \kappa + a_1 x_1 + \ldots + a_r x_r$.

Note also that the inequalities $(*)$ justify the restriction that $\kappa > \sigma_1(\mathbf{a}) - s$. Indeed, if we assume that $(x_1, \ldots, x_r) = (-1, \ldots, -1)$, then the third inequality of $(*)$ says that

$$x_{r+1} + \ldots + x_{r+s} \leq \kappa - a_1 - \ldots - a_r.$$

But on the other hand, the first line of $(*)$ implies that $x_i \geq -1$, so that

$$x_{r+1} + \ldots + x_{r+s} \geq -s.$$

Thus, to avoid contradiction, we must assume that $\kappa > \sigma_1(\mathbf{a}) - s$.

Also, note that in our case, we assumed all $a_i \geq 0$, so that in the inequality

$$x_{r+1} + \ldots + x_{r+s} \leq \kappa + a_1 x_1 + \ldots + a_r x_r,$$

the size of the right-hand side increases as the x_i increase. Thus, the Δ_s fiber of the point $(-1, \ldots, -1)$ is the smallest fiber if we assume $a_i \geq 0$.

We now have the following lemma, which gives the relation between $\mathbb{C}P^r$ bundles over $\mathbb{C}P^s$ and Δ_r bundles over Δ_s and discusses the effect of increasing κ on the symplectic form $\omega_{\mathbf{a}}^{\kappa}$.

Lemma 2.2.3. *Let (M, ω, T) be a symplectic toric manifold with moment polytope Δ. Then (M, ω, T) is a symplectic toric $(\mathbb{C}P^r, \omega_r, T_r)$ bundle over $(\mathbb{C}P^s, \omega_s, T_s)$ if and only if Δ is a Δ_r bundle over Δ_s equivalent to $\Delta_{\mathbf{a}}^{\kappa}$ for some $(\mathbf{a}; \kappa)$. Additionally, we have that $(M_{\mathbf{a}}, \omega_{\mathbf{a}}^{\kappa}, T_{\mathbf{a}})$ has moment polytope $\Delta_{\mathbf{a}}^{\kappa}$. Moreover, for a fixed pair $(\mathbf{a}; \kappa)$ and a real number $K > 0$, we have*

$$\omega_{\mathbf{a}}^{\kappa+K} = \omega_{\mathbf{a}}^{\kappa} + \tfrac{K}{s+1}\pi^*(\omega_s).$$

Proof. The first statement is discussed in detail in Remark 5.2 of [16], but is difficult to prove in much generality without the full definition of a bundle of polytopes, which we have omitted. The idea is to use the full definition of a bundle of polytopes to compute M as a complex manifold. In particular, to compute the moment polytope of $(M_{\boldsymbol{a}}, \omega_{\boldsymbol{a}}^{\kappa}, T_{\boldsymbol{a}})$, we notice that in Remark 5.2 of [16], it is concluded that if $\mathbf{a} = (a_1, \ldots, a_r)$ and M_Δ denotes the symplectic toric manifold corresponding to $\Delta_{\boldsymbol{a}}^{\kappa}$, then

$$M_\Delta = \mathbb{C}P^r \times_{\mathbb{C}^*} (\mathbb{C}^{s+1} \setminus \{0\})$$

for the following $\mathbb{C}^*$ actions. Let $[z_1 : \ldots : z_{r+1}]$ be standard coordinates on $\mathbb{C}P^r$. Then if $te^{i\theta}$ represents the standard polar form of a number in $\mathbb{C}^*$, the action on $\mathbb{C}P^r$ is described by

$$te^{i\theta} \cdot [z_1 : \ldots : z_r : z_{r+1}) = \Big[(te^{-ia_1\theta})z_1 : \ldots : (te^{-ia_r\theta})z_r : z_{r+1}\Big].$$

On $\mathbb{C}^{s+1} \setminus \{0\}$, the $\mathbb{C}^*$ action is described by

$$te^{i\theta} \cdot (z_1, \ldots, z_{s+1}) = \Big((te^{i\theta})z_1, \ldots, (te^{i\theta})z_{s+1}\Big),$$

which is the standard $\mathbb{C}^*$ action. In particular, this shows

$$M_\Delta = \mathbb{P}(L_0 \oplus L_{-a_1} \oplus \ldots \oplus L_{-a_r}) = M_{\boldsymbol{a}},$$

as claimed.

Furthermore, we have

$$(M_{\mathbf{a}}, \omega_{\mathbf{a}}^{\kappa}) = \Big(\mathbb{C}P^r \times_{\mathbb{C}^*} (\mathbb{C}^{s+1} \setminus \{0\}), \Omega_{\lambda(\kappa)} = \omega_r \times \lambda(\kappa)\omega_0\Big),$$

where ω_r is the standard form on $\mathbb{C}P^r$, scaled so that $\omega_r(\ell) = r+1$ with ℓ the homology class of a line, ω_0 is the standard form on $\mathbb{C}^{s+1}$, and $\lambda(\kappa)$ is a rescaling factor determined by κ.

We seek to compute $\omega_{\mathbf{a}}^{\kappa+K}$. As above, $M_{\mathbf{a}}$ is determined as a complex manifold by the relation

$$M_{\mathbf{a}} = \mathbb{C}P^r \times_{\mathbb{C}^*} (\mathbb{C}^{s+1} \setminus \{0\}).$$

Furthermore, $\omega_{\mathbf{a}}^{\kappa+K}$ is the reduction of $\Omega_{\lambda(\kappa+K)}$ by the $\mathbb{C}^*$ action. Then, an easy computation shows that

$$\omega_{\mathbf{a}}^{\kappa+K} - \omega_{\mathbf{a}}^{\kappa} = \tfrac{K}{s+1}\pi^*(\omega_s),$$

where ω_s is the standard form on $\mathbb{C}P^s$ normalized so that $\omega_s(\ell) = s+1$, as before. Reordering the terms, we get the desired result. □

The above has an obvious corollary.

Corollary 2.2.4. *Let (M, ω, T) be a symplectic toric $\mathbb{C}P^r$ bundle over $\mathbb{C}P^s$ with moment polytope Δ. Then there is some choice of $(\mathbf{a}; \kappa)$ so that (M, ω, T) is equivariantly symplectomorphic to $(M_{\mathbf{a}}, \omega_{\mathbf{a}}^{\kappa}, T_{\mathbf{a}})$.*

Proof. By Lemma 2.2.3 above, we know that Δ must be a Δ_r bundle over Δ_s, and hence is affine equivalent to some $\Delta_{\boldsymbol{a}}^{\kappa}$ for some choice of $(\mathbf{a}; \kappa)$. Lemma 2.2.3 then implies that (M, ω, T) is equivariantly symplectomorphic to $(M_{\boldsymbol{a}}, \omega_{\boldsymbol{a}}^{\kappa}, T_{\boldsymbol{a}})$. □

We now give a helpful condition for detecting when a polytope Δ is a Δ_r bundle over Δ_s. First, there is the notion of two polytopes being combinatorially equivalent.

Definition 2.2.5. *Two polytopes Δ and Δ' are said to be* combinatorially equivalent *if there is a bijection ϕ between the facets F_i of Δ and F_i' of Δ' with $\phi(F_i) = F_i'$ such that for each index set I*

$$\bigcap_{i \in I} F_i \neq \emptyset \Longleftrightarrow \bigcap_{i \in I} F_i' \neq \emptyset.$$

McDuff and Tolman prove the following lemma in [16].

Lemma 2.2.6. *([16] Lemma 4.10) Let Δ be a polytope which is smooth and combinatorially equivalent to $\Delta_r \times \Delta_s$. Then Δ is a Δ_r bundle over Δ_s or a Δ_s bundle over Δ_r.*

For the rest of the paper, we will only be interested in polytopes Δ which are Δ_r bundles over Δ_s for some choice of r, s, which as in Definition 2.2.1 are determined by pairs $(\mathbf{a}; \kappa)$.

Using the above presentation we see that the vector $\mathbf{a} = (a_1, \ldots, a_r)$ determines the underlying bundle structure of the corresponding manifold M, while the constant κ determines how much of the structure of the base $\widehat{M}$ is pulled back to the total space.

We now reinterpret κ in terms of the volume of the polytope to relate the above choice of $(\mathbf{a}; \kappa)$ to the choice given at the beginning of the paper. We have the following lemma.

Lemma 2.2.7. *Let r, s be positive integers, let $\mathbf{a} = (a_1, \ldots, a_r)$ be an integer vector and let κ be a real number so that $\kappa > K_{\mathbf{a}}(s)$. Then we have*

$$Vol(\Delta_{\mathbf{a}}^{\kappa}) = \tfrac{1}{r!}\tfrac{1}{s!}(r+1)^r(\kappa+s)^s.$$

Proof. Consider the polytope $\Delta_{\mathbf{a}}^{\kappa}$. As we saw before, this geometrically looks like a standard copy of Δ_r with a rescaled copy of Δ_s over each point. The point $(0, \ldots, 0)$ is the barycenter of the standard copy of Δ_r, and the copy of Δ_s over this point is the rescaled polytope Δ_s^{κ} discussed in Remark 2.2.2. We recall that it has the form of a standard s simplex with side length $s + \kappa$. Now, since the sizes of the side lengths of the rescaled copies of $\mathbb{C}P^s$ over the base copy of $\mathbb{C}P^r$ depend linearly on the coordinates in $\mathbb{C}P^r$ and Δ_s^{κ} is the Δ_s over the barycenter, we know that

$$Vol(\Delta_{\mathbf{a}}^{\kappa}) = Vol(\Delta_r)Vol(\Delta_s^{\kappa}).$$

However, a simple geometric argument shows that

$$Vol(\Delta_s^{\kappa}) = \tfrac{1}{s!}(\kappa+s)^s, \qquad Vol(\Delta_r) = \tfrac{1}{r!}(r+1)^r.$$

□

As the above shows, the tuple $(\mathbf{a}; \kappa)$ can be interpreted as $\mathbf{a}$ determining the bundle structure of $(M_{\mathbf{a}}, \omega_{\mathbf{a}}^{\kappa})$, while κ determines the volume of $(M_{\mathbf{a}}, \omega_{\mathbf{a}}^{\kappa})$.

Also, as we see in Section 2.4 of [15], we can restrict to the case where $a_i \geq 0$ for all i. To see this, recall from before that $\Delta_{\mathbf{a}}^{\kappa}$ has a standard copy of Δ_r with each point having a rescaled Δ_s over it. Also, we know that for any vertex of Δ, we can choose cordinates around that vertex so that the edge directions from that vertex are the standard vectors $e_1, \ldots, e_n$. If we choose coordinates for Δ around the point of Δ_r with the "smallest" copy of Δ_s, then by the interpretation of $-a_i$ as the slopes of linear changes in the standard coordinate directions, we have $-a_i \leq 0$ for all i, which means $a_i \geq 0$ for all i.

We now prove Lemma 1.2.7, which we recall states that

$$(M_{\mathbf{a}}, \omega^{\kappa}_{\mathbf{a}}, T_{\mathbf{a}}) \cong (M_{\mathbf{b}}, \omega^{\kappa}_{\mathbf{b}}, T_{\mathbf{b}}) \iff (\mathbf{a}; \kappa) = (\mathbf{b}; \kappa),$$

where $\cong$ denotes the relation of equivariant symplectomorphism.

Proof of Lemma 1.2.7. First, we notice that if $(\mathbf{a}; \kappa) = (\mathbf{b}; \kappa')$, then the manifolds are equivalent. It remains to show that if the manifolds are equivalent, then $(\mathbf{a}; \kappa) = (\mathbf{b}; \kappa')$. In particular, we show that if $\Delta^{\kappa}_{\mathbf{a}}$ is affine equivalent to $\Delta^{\kappa'}_{\mathbf{b}}$, then $(\mathbf{a}; \kappa) = (\mathbf{b}; \kappa')$.

Since $\Delta^{\kappa}_{\mathbf{a}}$ is affine equivalent to $\Delta^{\kappa'}_{\mathbf{b}}$ and affine equivalences preserve volume, $\mathrm{Vol}(\Delta^{\kappa}_{\mathbf{a}}) = \mathrm{Vol}(\Delta^{\kappa'}_{\mathbf{b}})$. Then, a simple application of Lemma 2.2.7 shows that $\kappa = \kappa'$. We now show that $\mathbf{a} = \mathbf{b}$.

As in Remark 2.2.2, the polytope $\Delta^{\kappa}_{\mathbf{a}}$ consists of a standard copy of Δ_r with a rescaled copy of Δ_s over each point. Furthermore, as we move in the direction e_i in the base copy of Δ_r, the edge lengths of the specific copy of Δ_s increase linearly with slope a_i. Thus, we have exactly $r+1$ different s-dimensional faces of Δ, which are all copies of Δ_s of various sizes sitting over the $r+1$ vertices of this Δ_r. In particular, combining Remark 2.2.2 with Lemma 2.2.7, we can easily compute that the volume of the smallest such Δ_s is $\frac{1}{s!}(\kappa + s - \sigma_1(\mathbf{a}))^s$, while the volumes of the other s faces will be $\frac{1}{s!}(\kappa + s - \sigma_1(\mathbf{a}) + a_i)^s$.

Similarly, in $\Delta^{\kappa}_{\mathbf{b}}$, there are $r+1$ different s-dimensional faces which are copies of Δ_s, and their volumes are given by $\frac{1}{s!}(\kappa + s - \sigma_1(\mathbf{b}))^s$ and $\frac{1}{s!}(\kappa + s - \sigma_1(\mathbf{b}) + b_i)^s$. Now, if there is an affine equivalence from $\Delta^{\kappa}_{\mathbf{a}}$ to $\Delta^{\kappa}_{\mathbf{b}}$, it would have to send the $r+1$ copies of Δ_s in $\Delta^{\kappa}_{\mathbf{a}}$ to the corresponding copies of Δ_s in $\Delta^{\kappa}_{\mathbf{a}}$ while preserving their volumes. In particular, by the above computations, this implies that $\sigma_1(\mathbf{a}) = \sigma_1(\mathbf{b})$ and furthermore that for each i, there is a j so that $a_i = b_j$. But the assumption that $0 \leq a_1 \leq \ldots \leq a_r$ and $0 \leq b_1 \leq \ldots \leq b_r$ implies that for each i, we have $a_i = b_i$. Thus, if the polytopes are affine equivalent, then $(\mathbf{a}; \kappa) = (\mathbf{b}; \kappa')$, as desired. $\square$

Chapter 3

Statements of Main Theorems and Technical Lemmas for Torus Actions

3.1 Statements of Main Theorems for Torus Actions

Recall that we are considering the following question.

Question 3.1.1. *Given a tuple $(\mathbf{a};\kappa)$, what is $N_n(\mathbf{a};\kappa)$, the number of inequivalent toric structures on the deformation class of $(M^{2n}_{\mathbf{a}},\omega^{\kappa}_{\mathbf{a}})$? In particular, for fixed $\mathbf{a}$ and n, how does it depend on κ, and for which $(\mathbf{a};\kappa)$ do we have $N_n(\mathbf{a};\kappa)=1$?*

Furthermore, recall that if we have $r<s$, then Theorem 1.2.9 says that the toric structure is always unique. However, the $r\geq s$ case is more complicated, as the next example shows.

Example 3.1.2. Let $r,s=3,2$ and take $\mathbf{a}=(1,4,4)$ and $\mathbf{b}=(2,2,5)$. For sufficiently large κ, both $(M_{\mathbf{a}},\omega^{\kappa}_{\mathbf{a}})$ and $(M_{\mathbf{b}},\omega^{\kappa}_{\mathbf{b}})$ describe $\mathbb{C}P^3$ bundles over $\mathbb{C}P^2$ which are deformation equivalent, by Proposition 1.2.2 and Lemma 1.2.6. However, these are obviously not the same toric manifold by Lemma 1.2.7 since $\mathbf{a}\neq\mathbf{b}$. In fact, we show in Example 5.1.1 that

$$N_5(\mathbf{a};\kappa)=\begin{cases}0 & \text{if } \kappa\leq 7\\ 2 & \text{if } \kappa>7,\end{cases}$$

so that there is no choice of κ for which either $N_5(\mathbf{a};\kappa)=1$ or $N_5(\mathbf{b};\kappa)$.

We will give more specific examples of the $r\geq s$ case in Section 5.2.

Even though there is not a general uniqueness theorem for the $r \geq s$ case, there are still some uniqueness results. In particular, by restricting the size of κ, we have the following theorem.

Theorem 3.1.3. *Let* $\mathbf{a} \neq 0$ *and* κ *be as before with the added assumption that* $\kappa \leq 1$. *We have*

$$N_n(\mathbf{a};\kappa) = \begin{cases} 0 & \text{if } \kappa \leq K_{\mathbf{a}}(s) \\ 1 & \text{if } K_{\mathbf{a}}(s) < \kappa \leq 1. \end{cases}$$

This has an interesting application. Recall that we say a symplectic manifold is monotone if we have $[\omega] = k[c_1(M)]$ for some positive constant k which is usually normalized to equal 1. Our notation is chosen so that we have the following, as in Remark 2.2(i) of [15].

Lemma 3.1.4. $(M_{\mathbf{a}}, \omega_{\mathbf{a}}^{\kappa})$ *is monotone* $\iff$ $\kappa = 1$.

In Question 1.11 of [15], McDuff conjectures that every monotone symplectic toric manifold has a unique toric structure. An obvious corollary of the above gives some support for this conjecture.

Corollary 3.1.5. *If* $(M_{\mathbf{a}}^{2n}, \omega_{\mathbf{a}}^{\kappa})$ *is monotone, then* $N_n(\mathbf{a};\kappa) = 1$.

Remark 3.1.6. Recall that a toric symplectic manifold (M, ω, T) is called Fano if there is a smooth family of T-invariant forms ω_t, $0 \leq t \leq 1$, with $\omega_0 = \omega$ and ω_1 monotone. In our case, we have that $(M_{\mathbf{a}}^{2n}, \omega_{\mathbf{a}}^{\kappa}, T_{\mathbf{a}})$ is Fano if and only if $K_{\mathbf{a}}(s) < 1$. We will call such vectors Fano vectors. In examples 5.2.1, 5.2.2, and 5.2.3 below, we will consider the Fano case in some specific examples.

Although we cannot yet compute $N_n(\mathbf{a};\kappa)$ for all κ, we can say that as a function of κ, $N_n(\mathbf{a};\kappa)$ is monotonic and locally constant, with the only jumps possible being at certain integer values of κ. Furthermore, if $r = s$, these jumps are of size at most 1. More specifically, we have the following theorem.

Theorem 3.1.7. *Let* $\mathbf{a} = (a_1, \ldots, a_r)$ *be a non-negative integer vector and let* K_M *be the minimum of the set of* κ *with* $N_n(\mathbf{a};\kappa) \neq 0$. *Then we have the following:*

1. *Let* $\ell \geq 0$ *be an integer, and let* κ_1, κ_2 *be real numbers. Then we have*

$$K_M + \ell(r+1) < \kappa_1, \kappa_2 \leq K_M + (\ell+1)(r+1) \Longrightarrow N_n(\mathbf{a};\kappa_1) = N_n(\mathbf{a};\kappa_2).$$

2. *If also* $r = s$ *and* $\kappa = K_M + \ell(r+1)$ *and* $0 < \epsilon \leq r+1$,

$$N_n(\mathbf{a};\kappa) \leq N_n(\mathbf{a};\kappa+\epsilon) \leq N_n(\mathbf{a};\kappa) + 1.$$

Notice that the number K_M above need not equal the number $K_{\mathbf{a}}(s)$ from before because there could be a different vector $\mathbf{b}$ so that $(M_{\mathbf{a}}, \omega_{\mathbf{a}}^{\kappa})$ is deformation equivalent to $(M_{\mathbf{b}}, \omega_{\mathbf{b}}^{\kappa})$ with $K_{\mathbf{b}}(s) < K_{\mathbf{a}}(s)$. Then, we would have $N_n(\mathbf{a};\kappa) = N_n(\mathbf{b};\kappa)$ for all κ, while $K_{\mathbf{b}}(s) < K_{\mathbf{a}}(s)$, which would obviously imply that $K_M \leq K_{\mathbf{b}}(s) < K_{\mathbf{a}}(s)$.

The above theorem then says that if K_M denotes the position of the first jump of $N_n(\mathbf{a};\kappa)$, then all subsequent jumps can only occur at the integers $K_M + l(r+1)$, and when $r = s$, these jumps are of size 0 or 1. An obvious corollary of the above two theorems is another uniqueness result.

Corollary 3.1.8. *Let K_M be as before, and assume that $r = s$. Further assume we have an $\mathbf{a} = (a_1, \ldots, a_r)$ so that $K_{\mathbf{a}} = K_M$. Then we have*

$$N_n(\mathbf{a};\kappa) = 1, \quad \forall\ K_M < \kappa \leq K_M + r + 1.$$

Lastly, recall that Theorem 1.2.11 says that for any $\mathbf{a}$, $N_{r+s}(\mathbf{a};\infty)$ is finite if $r \geq s \geq 2$. A natural question to ask is what happens if instead of allowing only κ to vary, we also allow $\mathbf{a}$ to vary. Namely, we can consider the quantity

$$\sup_{\mathbf{a}} N_{r+s}(\mathbf{a};\infty)$$

for fixed $r \geq s \geq 2$. We have the following conjecture.

Conjecture 3.1.9. *For any positive integers $r \geq s \geq 2$ we have*

$$\sup_{\mathbf{a}} N_{r+s}(\mathbf{a};\infty) = \infty$$

where the supremum is over all non-negative integer vectors $\mathbf{a}$.

Although we have not been able to verify this conjecture in full generality, we do have the following support for our conjecture.

Theorem 3.1.10. *For any integer $r \geq s = 2$ we have*

$$\sup_{\mathbf{a}} N_{r+2}(\mathbf{a};\infty) = \infty$$

where the supremum is over all non-negative integer vectors $\mathbf{a}$.

Theorem 1.2.11 above has the following interesting corollary.

Corollary 3.1.11. *Let* $\mathbf{a} = (a_1, \ldots, a_r)$ *be as before, and let* $r \geq s \geq 2$*. Then there is a constant* K *so that for all* $\kappa \geq K$*, the symplectomorphism class of* $(M_{\mathbf{a}}, \omega_{\mathbf{a}}^{\kappa})$ *has exactly* $N_n(\mathbf{a}; \infty)$ *inequivalent toric structures.*

Proof. By Theorem 1.2.11, we know that $N_n(\mathbf{a}; \infty)$ is a finite number. More specifically, for all $\kappa > (r + 1 - \frac{1}{r})\sigma_1(\mathbf{a}) - s$, $N_n(\mathbf{a}; \kappa) = N_n(\mathbf{a}; \infty)$. Also, as in Lemma 1.2.6, for each vector $\mathbf{b}$ so that $(M_{\mathbf{b}}, \omega_{\mathbf{b}}^{\kappa})$ is deformation equivalent to $(M_{\mathbf{a}}, \omega_{\mathbf{a}}^{\kappa})$ for some κ, there is a constant $C_{\mathbf{b}}$ so that for all $\kappa > C_{\mathbf{b}}$, $(M_{\mathbf{b}}, \omega_{\mathbf{b}}^{\kappa})$ is actually symplectomorphic to $(M_{\mathbf{a}}, \omega_{\mathbf{a}}^{\kappa})$. Furthermore, there is a finite number of such constants $C_{\mathbf{b}}$, and thus we can define the constant K as

$$K := \max\{C_{\mathbf{b}}, (r + 1 - \tfrac{1}{r})\sigma_1(\mathbf{a}) - s\}.$$

Then as above, for all $\kappa > K$, the symplectomorphism class of $(M_{\mathbf{a}}, \omega_{\mathbf{a}}^{\kappa})$ has exactly $N_n(\mathbf{a}; \infty)$ toric structures. □

3.2 Homological Lemmas

Now we will get into some of the more technical lemmas we will need for the proofs of our results.

Lemma 3.2.1. *Let* $r, s \geq 1$ *be integers with* $r > 1$ *and let* $\mathbf{a} = (a_1, \ldots, a_r)$ *be a non-negative integer vector with some* $a_i \neq 0$*. Assume* $H^*(M; \mathbb{Z})$ *is isomorphic to the graded ring generated by* α_0 *and* β_0 *of* $H^2(M)$ *with relations*

$$\alpha_0^{s+1} = 0, \qquad \beta_0 \prod_{i=1}^{r} (\beta_0 - a_i \alpha_0) = 0,$$

Then if there exist integers A, B *so that* $(A\alpha_0 + B\beta_0)^{s+1} = 0$*, we must have* $B = 0$*.*

Proof. This a slight restatement of Lemma 6.2 in [3]. We follow their proof closely. Since $(A\alpha_0 + B\beta_0)^{s+1} = 0$, $(A\alpha_0 + B\beta_0)^{s+1}$ must be a consequence of our other relations. Namely, there exists C, D so that

$$(A\alpha_0 + B\beta_0)^{s+1} - C\alpha_0^{s+1} = D\beta_0 \prod_{i=1}^{r} (\beta_0 - a_i \alpha_0),$$

where C is an integer and D is an integer polynomial in α_0 and β_0 of degree $s - r$ if $r \leq s$, and $D = 0$ if $r > s$.

If $r > s$, we then have $(A\alpha_0 + B\beta_0)^{s+1} - C\alpha_0^{s+1} = 0$, which gives $B = 0$ and $C = A^{s+1}$, as desired.

Consider now $r \leq s$. Suppose first that $A = 0$. Since the right hand side has no pure α_0 terms and $A = 0$, we must have $C = 0$ and the left hand side is only a β_0^{s+1} term. But some $a_i \neq 0$, so that the right hand side has a non-zero β_0^{r+1} term and a non-zero $\alpha_0\beta_0^r$ term, which is a contradiction. Thus, $A \neq 0$. Now, since the right hand side has no pure α_0 terms and $A \neq 0$, we must have $C = A^{s+1}$ to cancel the α_0^{s+1} term from the left hand side. If now $B \neq 0$, the remaining terms on the left hand side can be expressed as a polynomial in α_0 and β_0 with no more than 2 linear factors when optimally factored, while the right hand side has at least three linear factors since $r > 1$, so that the two polynomials can never be equal. We briefly describe the factorization of the LHS. First, let $A\alpha_0 = X$ and $B\beta_0 = Y$. Then, since $C = A^{s+1}$, the LHS can be expressed as

$$(X - Y)^{s+1} - X^{s+1},$$

and this has no more than 2 linear factors, as claimed. This contradiction establishes that $B = 0$. □

We will now prove Proposition 1.2.2, which we use heavily in the proofs of our main theorems.

Proof of Proposition 1.2.2. If $\mathbf{a} = \mathbf{b} = 0$, the result is obvious, thus one of $\mathbf{a}$ and $\mathbf{b}$ is nonzero. Without loss of generality, we assume some $a_i \neq 0$. We will prove that (1) ⇒ (2) ⇒ (4) ⇒ (1) and also (2) ⇔ (3).

First, we prove that (1) ⇒ (2). This is the hardest direction of the proof, and we will break it into three cases. First assume that $r > 1$. This proof is taken from Theorem 6.1 of [3]. The Stanley-Reisner presentation of $H^*(M_{\mathbf{a}}; \mathbb{Z})$ on $\Delta_{\mathbf{a}}$ gives generators α_0 and β_0 for $H^*(M_{\mathbf{a}}; \mathbb{Z})$ satisfying

$$\alpha_0^{s+1} = 0 \tag{α_0}$$

$$\beta_0 \prod_{i=1}^{r} (\beta_0 - a_i\alpha_0) = 0. \tag{β_0}$$

Similarly, from the polytope $\Delta_{\mathbf{b}}$, we get generators α and β of $H^*(M_{\mathbf{b}}; \mathbb{Z})$ with the relations

$$\alpha^{s+1} = 0 \tag{α}$$

$$\beta \prod_{i=1}^{r} (\beta - b_i\alpha) = 0. \tag{β}$$

Since $H^*(M_{\mathbf{a}})$ is isomorphic to $H^*(M_{\mathbf{b}})$, there exist integers A, B, C, D with $AD - BC = 1$ so that

$$\alpha_0 = A\alpha + B\beta, \qquad \beta_0 = C\alpha + D\beta.$$

Using $\alpha_0^{s+1} = 0$ and Lemma 3.2.1, we conclude that $B = 0$, so that $A = D = \pm 1$. Moreover we can arrange $A = D = 1$ by possibly changing the signs of both α and β. Now we substitute $\beta_0 = C\alpha + \beta$ and $\alpha_0 = \alpha$ into the relation (β_0), and since the relation (β_0) must equal the relation (β), we know that the two polynomials are equal as polynomials in β. Substituting the specific value $\beta = 1$, we obtain the relation

$$\prod_{i=0}^{r}(1 + (-a_i + C)\alpha) = \prod_{i=0}^{r}(1 - b_i\alpha), \tag{*}$$

where we assume that $a_0 = b_0 = 0$.

But the left hand side is just the total Chern class of the bundle

$$[L_0 \oplus L_{-a_1} \oplus \ldots \oplus L_{-a_r}] \otimes L_C,$$

while the right hand side is the total Chern class of the bundle

$$L_0 \oplus L_{-b_1} \oplus \ldots \oplus L_{-b_r}.$$

Thus, since these two bundles have the same total Chern class and are sums of line bundles, they are isomorphic as vector bundles, i.e.

$$[L_0 \oplus L_{-a_1} \oplus \ldots \oplus L_{-a_r}] \otimes L_C \cong (L_0 \oplus L_{-b_1} \oplus \ldots \oplus L_{-b_r}).$$

But the above shows that

$$\mathbb{P}(L_0 \oplus L_{-a_1} \oplus \ldots \oplus L_{-a_r}) = \mathbb{P}(L_0 \oplus L_{-a_1} \oplus \ldots \oplus L_{-a_r}),$$

as desired.

Now, consider the case $r = s = 1$. Then $\mathbf{a} = (a)$, $\mathbf{b} = (b)$, and we know that $M_{\mathbf{a}}$ and $M_{\mathbf{b}}$ are just the Hirzebruch surfaces $H_{\mathbf{a}}$ and $H_{\mathbf{b}}$ respectively. Repeated application of Lemma 4.1.1 then implies that $H_{\mathbf{a}}$ is symplectomorphic to $H_{\mathbf{b}}$ if $b - a$ is even. A simple computation shows that $H_0 \not\cong H_1$, so that in fact $H_{\mathbf{a}}$ is symplectomorphic to $H_{\mathbf{b}}$ if and only if $b - a$ is even. In particular, $H^*(M_{\mathbf{a}}; \mathbb{Z}) \cong H^*(M_{\mathbf{b}}; \mathbb{Z})$ if and only if $b - a$ is even. But then $C = \frac{a-b}{2}$ is an integer. Now let $a_0 = b_0 = 0$ and let α be as before. In particular, $\alpha^2 = 0$ and a simple computation shows

$$\prod_{i=0}^{1}(1 + (-a_i + C)\alpha) = (1 + C\alpha)(1 + (C - a)\alpha) = 1 + (2C - a)\alpha = 1 - b\alpha = \prod_{i=0}^{1}(1 - b_i\alpha),$$

which implies condition (2) as above.

Lastly, consider the case $r = 1$, $s \geq 2$. This proof is taken from Theorem 6.1 of [3]. As before, $\mathbf{a} = (a)$ and $\mathbf{b} = (b)$. Using the Stanley-Reisner presentation, we get α_0, β_0, α and β as before, with integers A, B, C, and D with $AD - BC = 1$, and

$$\alpha_0 = A\alpha + B\beta, \qquad \beta_0 = C\alpha + D\beta.$$

Now, recall from equations (β_0) and (β) above that $\beta_0(\beta_0 - a\alpha_0) = 0$ and $\beta(\beta - b\alpha) = 0$. This implies that $\beta^2 = b\alpha\beta$. Substituting from the above, expanding, and simplifying, we get

$$\begin{aligned} 0 &= \Big(C\alpha + D\beta\Big)\Big((C\alpha + D\beta) - a(A\alpha + B\beta)\Big) \\ &= \Big(C(C - aA)\Big)\alpha^2 + \Big(C(D - aB) + D(C - aA)\Big)\alpha\beta + \Big(D(D - aB)\Big)\beta^2 \\ &= \Big(C(C - aA)\Big)\alpha^2 + \Big(C(D - aB) + D(C - aA) + b(D(D - aB))\Big)\alpha\beta. \end{aligned}$$

Also, since $s \geq 2$, equation (α) tell us that $\alpha^2 \neq 0$. Since also $\alpha\beta \neq 0$, it follows that

$$C(C - aA) = 0, \qquad C(D - aB) + D(C - aA) = -b(D(D - aB)).$$

$C(C - aA) = 0$ implies that either $C = 0$ or $C = aA$. If $C = 0$, then by $AD - BC = 1$, we know that $A = D = \pm 1$, where by changing signs of α and β if necessary, we can arrange $A = D = 1$. Substituting into the above, this tells us that

$$-a = -b(1 - aB),$$

so that b divides a.

Now, assume $C = aA$. Then $AD - BC = 1$ implies that $AD - aAB = 1$ so $A(D - aB) = 1$ which says that $A = D - aB = \pm 1$, where as before we can arrange $A = D - aB = 1$. Then substituting as before, we get

$$a = -bD,$$

so that again b divides a. Thus, in either case, we have b divides a. In particular, this implies $b \neq 0$, so by switching the roles of a and b, we clearly also have a divides b, so that $a = b$, which implies condition (2). Thus, we conclude that (1) $\Rightarrow$ (2).

We next prove (2) $\Rightarrow$ (4). Now, since we are assuming that $\mathbb{P}(L_0 \oplus L_{-a_1} \oplus \ldots \oplus L_{-a_r})$ is isomorphic to $\mathbb{P}(L_0 \oplus L_{-b_1} \oplus \ldots \oplus L_{-b_r})$ as a projective vector bundle, we know that there is some

C so that $(L_0 \oplus L_{-a_1} \oplus \ldots \oplus L_{-a_r}) \otimes L_C$ is isomorphic to $L_0 \oplus L_{-b_1} \oplus \ldots \oplus L_{-b_r}$ as vector bundles, which implies that they have the same total Chern class, which gives us the relation

$$\prod_{i=0}^{r}(1 + (-a_i + C)\alpha) = \prod_{i=0}^{r}(1 - b_i\alpha), \tag{*}$$

where we assume that $a_0 = b_0 = 0$ as before. Here, we have assumed that α is the standard generator of $\mathbb{C}P^s$, so that $L_{\mathbf{a}}$ is the line bundle over $\mathbb{C}P^s$ with first Chern class given by $a\alpha$. Since $\alpha^{s+1} = 0$, we know by expanding and comparing coefficients of α^i that the above equation is true if and only if:

$$\sigma_i(C, C - a_1, \ldots, C - a_r) = \sigma_i(0, -b_1, \ldots, -b_r) \quad 1 \leq i \leq \min\{r + 1, s\},$$

which in turn is true if and only if

$$\sigma_i(C, a_1 + C, \ldots, a_r + C) = \sigma_i(0, b_1, \ldots, b_r),$$

where we have replaced $-C$ by C as the arbitrary constant. This finishes the proof that (2) $\Rightarrow$ (4).

Next we show that (4) $\Rightarrow$ (1). By (4), we know as above that there exists a constant C so that

$$\prod_{i=0}^{r}(1 + (-a_i + C)\alpha) = \prod_{i=0}^{r}(1 - b_i\alpha), \tag{*}$$

where α is the standard generator of $H^2(\mathbb{C}P^s)$. As before, this implies condition (2), which implies condition (1).

It remains to show (2) $\Leftrightarrow$ (3). In both cases, the manifold M is a smooth $\mathbb{C}P^r$ bundle over $\mathbb{C}P^s$. The difference is that in (2), we are considering it as a projective vector bundle, so that the structure group of the bundle is $PU(r+1)$, whereas in condition (3), we are considering it as a symplectic bundle, so that the structure group of the bundle is $\mathrm{Symp}(\mathbb{C}P^r)$. Thus, the fact that (2) $\Rightarrow$ (3) follows immediately from the fact $PU(r+1) \subset \mathrm{Symp}(\mathbb{C}P^r)$.

It remains to show that (3) $\Rightarrow$ (2). However, as is shown in [18], there is a natural extension of the notion of Chern classes to symplectic bundles. Thus, since we have two isomorphic symplectic bundles, they have equal total Chern classes in the symplectic sense, which implies that they have equal total Chern class in the projective sense. Thus, there is a constant C so that the bundles $(L_0 \oplus L_{-a_1} \oplus \ldots \oplus L_{-a_r}) \otimes L_C$ and $(L_0 \oplus L_{-b_1} \oplus \ldots \oplus L_{-b_r})$ have the same total Chern class, which as before implies that they are isomorphic as vector bundles. This in turn implies the condition (2). $\square$

3.3 Lemmas about Moment Polytopes

Lastly, we need a couple more lemmas to characterize the possible moment polytopes of toric structures on symplectic toric bundles. First, we recall the following theorem from [19].

Lemma 3.3.1. *([19] Prop 1.1.1) Let Δ be a polytope of dimension n with $n+2$ facets. Then there exists k and m with $k+m=n$ so that Δ is combinatorially equivalent to $\Delta_k \times \Delta_m$.*

We use this to prove the following fundamental lemma.

Lemma 3.3.2. *If (M^{2n}, ω, T) is a symplectic toric manifold with $\dim H^2(M) = 2$, then M is a $\mathbb{C}P^r$ bundle over $\mathbb{C}P^s$, and hence is symplectomorphic to some $(M_{\mathbf{a}}, \omega_{\mathbf{a}}^{\kappa})$ as in Corollary 2.2.4. Moreover, if $\mathbf{a} \neq 0$, any other toric structure on M is a $\mathbb{C}P^r$ bundle over $\mathbb{C}P^s$ for the same r, s.*

Proof. This proof follows the proof of Corollary 6.3 in [3]. By assumption, $\dim H^2(M) = 2$, and therefore Δ_M has $\dim \Delta_M + \operatorname{rank}(H^2(M)) = n+2$ facets, which by Lemma 3.3.1 tells us it is combinatorially equivalent to some $\Delta_r \times \Delta_s$ with $r+s=n$. Since it is also smooth, Lemma 2.2.6 says Δ_M is a Δ_r bundle over Δ_s for some choice of r and s with $r+s=n$, which implies that M is a $\mathbb{C}P^r$ bundle over $\mathbb{C}P^s$ by Lemma 2.2.3. As in Definition 2.2.1, this bundle is determined by a pair $(\mathbf{a}; \kappa)$ where $\mathbf{a} = (a_1, \ldots, a_r)$ can be chosen so that $a_i \geq 0$.

Now, assume that some $a_i \neq 0$ and that we have some other toric structure generating a polytope Δ'. By the above, Δ' is a Δ_k bundle over Δ_m where $k+m=n$, and hence is determined by a pair $(\mathbf{b}; \kappa)$. Moreover, since $(\mathbf{b}; \kappa)$ determines the same toric structure, we know $(M_{\mathbf{a}}, \omega_{\mathbf{a}}^{\kappa})$ is deformation equivalent to $(M_{\mathbf{b}}, \omega_{\mathbf{b}}^{\kappa})$, so that

$$H^*(M_{\mathbf{a}}) \cong H^*(M_{\mathbf{b}})$$

We show that $k=r$ and $m=s$. Comparing information about Betti numbers, we can easily conclude that $r+s=m+k$ and $(1+r)(1+s) = (1+k)(1+m)$ so that $\{r,s\} = \{k,m\}$. We show that we can arrange $m=s$.

To see this, assume that $m=r$, so that $k=s$. If $r=s$, there is nothing to prove. First, assume $r<s$. Now, since M is a $\mathbb{C}P^k$ bundle over $\mathbb{C}P^m$, there is an element γ in $H^2(M;\mathbb{Z})$ so that $\gamma^{m+1} = 0, \gamma \neq 0$. But $\gamma^{m+1} = \gamma^{r+1}$, and $r<s$, therefore $\gamma^s = 0$. But M is a $\mathbb{C}P^r$ bundle over $\mathbb{C}P^s$ determined by the vector $\mathbf{a}$, so as in the proof of Proposition 1.2.2, we know that

$H^*(M;\mathbb{Z}) \cong H^*(M_{\mathbf{a}};\mathbb{Z})$ has generators α_0 and β_0 with relations

$$\alpha_0^{s+1} = 0, \qquad \beta_0 \prod_{i=1}^{r} (\beta_0 - a_i \alpha_0) = 0.$$

We recall now that we have assumed that some a_i is not zero. We claim that an element γ as above cannot exist. Indeed, if it did, then we would have $\gamma = A\alpha_0 + B\beta_0$ with $\gamma^s = 0$. But then we must have some degree $s - r$ polynomial in α_0 and β_0, D, so that

$$(A\alpha_0 + B\beta_0)^s = D\beta_0 \prod_{i=1}^{r} (\beta_0 + a_i \alpha_0).$$

Since the right hand side has no pure α_0 term, we must have $A = 0$, so that the left hand side is $B^s \beta_0^s$. However, we assumed some $a_i \neq 0$, so that regardless of the choice of D, the right hand side will have some terms containing α_0, so that the right hand side can never equal the left hand side for any choice of D. This contradiction implies that $m = s$ and $k = r$, as required.

Now, consider the case where $r > s$. Since $k = s$ and $m = r$, we then have $k < m$. There are two cases to consider. First, assume $\mathbf{b} \neq 0$. Thus, some b_i is non-zero, and we can run the above argument with the roles of $\mathbf{a}$, r, and s replaced by $\mathbf{b}$, k, and m to get the desired result.

Now, if $\mathbf{b} = 0$, then our Δ_k bundle over Δ_m is actually $\Delta_k \times \Delta_m = \Delta_m \times \Delta_k$ which is also a Δ_m bundle over Δ_k, hence a Δ_r bundle over Δ_s, as desired. □

Using this, we can now prove Theorem 1.2.1, which we recall said that any toric symplectic manifold (M, ω, T) with $\dim H^2(M) = 2$ is equivariantly symplectomorphic to the bundle $(M_{\mathbf{a}}, \omega_{\mathbf{a}}^{\kappa}, T_{\mathbf{a}})$ for a unique tuple $(\mathbf{a}; \kappa)$ with $0 \leq a_1 \leq \ldots \leq a_r$.

Proof of Theorem 1.2.1. Since $\dim H^2(M) = 2$, Lemma 3.3.2 implies that Δ_M is a Δ_r bundle over Δ_s determined by some tuple $(\mathbf{a}; \kappa)$ with $0 \leq a_1 \leq \ldots \leq a_r$, and in fact that (M, ω, T) is equivariantly symplectomorphic to $(M_{\mathbf{a}}, \omega_{\mathbf{a}}^{\kappa}, T_{\mathbf{a}})$. Lemma 1.2.7 implies that the tuple $(\mathbf{a}; \kappa)$ determined in this fashion is in fact uniquely determined. □

Chapter 4

Equivalence Relations on Toric Symplectic Manifolds

4.1 Lemmas about Symplectomorphisms

We now prove Theorem 1.2.3, which we recall said that if $(M_{\mathbf{a}}, \omega_{\mathbf{a}}^{\kappa})$ and $(M_{\mathbf{b}}, \omega_{\mathbf{b}}^{\kappa})$ are $\mathbb{C}P^r$ bundles over $\mathbb{C}P^s$ with $s = 1$ determined by vectors $(\mathbf{a}; \kappa)$ and $(\mathbf{b}; \kappa)$, then they are isomorphic as symplectic bundles if and only if they are actually symplectomorphic. First, we will prove a special case of this, which will act as a technical lemma.

Lemma 4.1.1. *Let $\mathbf{a} = (a_1, \ldots, a_r)$ be a non-negative integer vector and let $(\mathbf{a}; \kappa)$ determine the symplectic bundle $(M_{\mathbf{a}}, \omega_{\mathbf{a}}^{\kappa})$, as before, where we assume that $M_{\mathbf{a}}$ is a $\mathbb{C}P^r$ bundle over $\mathbb{C}P^1$. Now, assume that either $\mathbf{a}' = (a_1+1, \ldots, a_i+2, \ldots, a_r+1)$ for some i or that $\mathbf{a}' = (a_1, \ldots, a_i+1, \ldots, a_j-1, \ldots, a_r)$ for some i, j with $a_j - 1 \geq 0$. Then $(M_{\mathbf{a}}, \omega_{\mathbf{a}}^{\kappa})$ and $(M_{\mathbf{a}'}, \omega_{\mathbf{a}'}^{\kappa})$ are symplectomorphic.*

Proof. We will prove this theorem in two parts, corresponding to the cases where $\mathbf{a}' = (a_1 + 1, \ldots, a_i+2, \ldots, a_r+1)$ for some i or where $\mathbf{a}' = (a_1, \ldots, a_i+1, \ldots, a_j-1, \ldots, a_r)$. Both parts will use the same basic symplectomorphism technique, which we describe below.

Recall as in Definition 2.2.1 that Δ, a Δ_r bundle over Δ_1, has coordinates $(x_1, \ldots, x_r, z)$ where $(x_1, \ldots, x_r)$ are coordinates on the standard Δ_r and z will be thought of as the vertical direction, describing the copies of Δ_1 over various points of the base copy of Δ_r. Recall also there is a moment map, denoted $\Phi : M_\Delta \to \Delta$ which takes M_Δ to Δ. Let $\mathcal{H}$ be any hyperplane in Δ transverse to the z direction with conormal $\eta_{\mathcal{H}} = (b_1, \ldots, b_r, 1)$, with b_i integers. Consider the intersection of the hyperplane $\mathcal{H}$ with the polytope Δ. This gives a polytope $\Delta_{\mathcal{H}}$, which is still Delzant because the b_i are integers.

Since $\mathcal{H}$ is transverse to the vertical z direction, $\Delta_{\mathcal{H}}$ effectively splits the polytope Δ into a top half and a bottom half. The polytope Δ is then described by taking the top half and bottom half and gluing them together along $\Delta_{\mathcal{H}}$ by the identity. Now, consider an affine equivalence of the polytope $\Delta_{\mathcal{H}}$, which we will denote ϕ'. We can then define a polytope Δ' by taking the top half and bottom half, and gluing them together along $\Delta_{\mathcal{H}}$ by the affine equivalence ϕ' instead of the identity. Since the map ϕ' is an affine equivalence, Δ' is evidently still a Delzant polytope. An example of this is shown in Figure 1.

We briefly explain why M_Δ and $M_{\Delta'}$ are symplectomorphic. To do this, we redescribe the above process in a way that is the same symplectically, but not torically. Namely, we will look on the level of the manifolds, not the polytopes. First, we consider the hyperplane $Q = \Phi^{-1}(\mathcal{H})$ in M_Δ, and thicken it by taking $Q \times \{(0, \epsilon)\}$ and intersecting this with M_Δ. As before, this section of the manifold effectively divides M into a top and bottom half, with the attaching maps to $Q \times \{(0, \epsilon)\}$ at $Q \times \{0\}$ and $Q \times \{\epsilon\}$ being the identity. We can then symplectically isotop $Q \times \{(0, \epsilon)\}$ to a thickened hyperplane $Q' \times \{(0, \epsilon)\}$ by an isotopy Ψ where $Q' \times \{\epsilon\}$ is equal to $Q \times \{\epsilon\}$, while $Q' \times \{0\}$ is equivariantly symplectomorphic to $Q \times \{0\}$ by the map $\Phi^*\phi'$, the lift of the affine equivalence ϕ'. By doing this we can produce a manifold M' by letting $M' = M$ both above and below $Q \times \{(0, \epsilon)\}$, but replacing $Q \times \{(0, \epsilon)\}$ with $Q' \times \{(0, \epsilon)\}$, with the attaching map to the top half at $Q' \times \{\epsilon\}$ being the identity as before, while the attaching map to the bottom half at $Q' \times \{0\}$ is the map $\Phi^*\phi'$. M_Δ and M' are then isotopic, hence symplectomorphic, by the isotopy Ψ' which equals the identity on the top half and bottom half, and which isotops $Q \times \{(0, \epsilon)\}$ to $Q' \times \{(0, \epsilon)\}$ by the isotopy Ψ. However, by construction M' is symplectomorphic to $M_{\Delta'}$, which implies that M_Δ and $M_{\Delta'}$ are symplectomorphic, as desired.

To complete the proof, we need only show that we can choose the hyperplane $\mathcal{H}$ and affine equivalence of $\Delta_{\mathcal{H}}$ in such a way that we can obtain $\Delta' = \Delta^{\kappa}_{\mathbf{a}'}$, where $\mathbf{a}'$ is one of the vectors from before. Before we do this, we first notice that since Δ is a Δ_r bundle over Δ_1, if we take $\mathcal{H}$ transverse to the z direction and intersect it with Δ, then $\Delta_{\mathcal{H}}$ is simply a copy of Δ_r. We will label the vertices of the standard Δ_r as $v_0, \ldots, v_r$ where $v_0 = (-1, \ldots, -1)$ and $v_i = (-1, \ldots, n, \ldots, -1)$

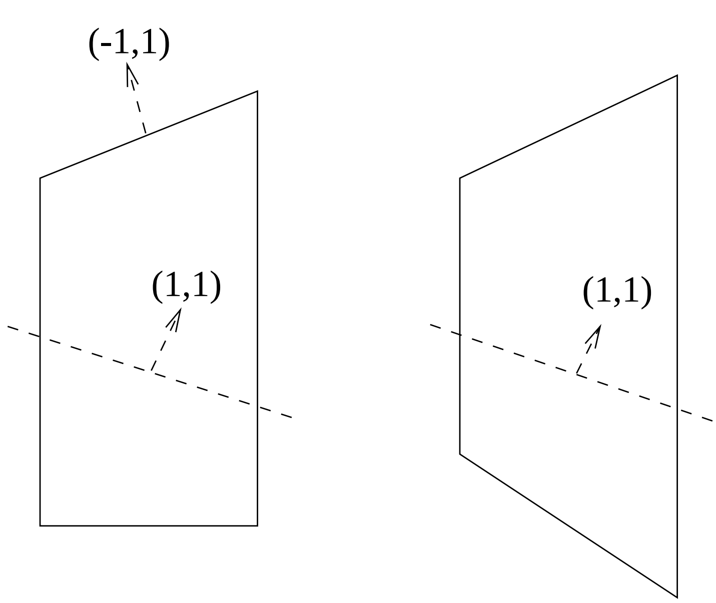

Figure 4.1: Example of Lemma 4.1.1 with $r = s = 1$, $\mathbf{a} = 1$, and $\eta_{\mathcal{H}} = (1, 1)$. The first figure is $\Delta^{\kappa}_{(1)}$, the dotted line in the first two figures represents the hyperplane $\mathcal{H}$, the second figure is Δ', and the third figure is $\Delta^{\kappa}_{(3)}$. Notice that Δ' is affine equivalent to $\Delta^{\kappa}_{(3)}$

where the n is in the i^{th} slot for $1 \leq i \leq r$.

First, we consider vectors of the form $\mathbf{a}' = (a_1+1, \ldots, a_i+2, \ldots, a_r+1)$. To show that $\Delta' = \Delta^{\kappa}_{\mathbf{a}'}$, we will consider the hyperplane with conormal vector $(1, \ldots, 1)$. Recall from Remark 2.2.2 that a Δ_r bundle over Δ_1 can be thought of as a copy of Δ_r fibered by vertical copies of Δ_1, where the value of a_i is the slope of increase of the sizes of Δ_1 along the edge from v_0 to v_i. Thus, to compute the value of a_i, we only need to know the size of the vertical edge over each vertex of Δ_r. It can then be easily computed that if we take the hyperplane described by $(1, \ldots, 1)$ as above and take an affine equivalence of $\Delta_{\mathcal{H}}$ which takes the vertex of $\Delta_{\mathcal{H}}$ over v_0 to the vertex of $\Delta_{\mathcal{H}}$ over v_i, then this shortens the vertical edge over v_0 by 1 unit, lengthens the vertical edge over v_i by 1, and fixes all other lengths. This corresponds exactly to changing $\mathbf{a}$ to $\mathbf{a}'$.

Consider now the vectors of the form $\mathbf{a}' = (a_1, \ldots, a_i + 1, \ldots, a_j - 1, \ldots, a_r)$ and take the hyperplane with conormal vector $(0, \ldots, -1, \ldots, 0, 1)$ where the -1 is in the j^{th} slot. Then as above, it can be easily computed that by taking an affine equivalence of $\Delta_{\mathcal{H}}$ which takes the vertex of Δ_H above v_j to the vertex of Δ_H above v_i, we shorten the vertical edge over v_j by 1 unit, lengthen the vertical edge over v_i by 1 unit, and fix all other lengths. Again, this corresponds exactly to changing $\mathbf{a}$ to $\mathbf{a}'$, which completes the proof. □

Remark 4.1.2. It can be shown that the above argument only works in the $s = 1$ case. Indeed, if we try to run the above argument in the $s > 1$ case, we will find that $\Delta_{\mathcal{H}}$ will correspond to a certain Δ_r bundle over Δ_{s-1} where $s - 1 > 0$. In the $s = 1$ case, we had $\Delta_{\mathcal{H}}$ as a Δ_r bundle over Δ_0, which is just a copy of Δ_r, which has plenty of affine symmetries. In fact, in Δ_r, there is an affine symmetry which swaps any two vertices. However, Δ_r bundles over Δ_{s-1} with $s-1 > 0$ have very few affine symmetries. The only time when $\Delta_{\mathbf{a}}^{\kappa}$ will have a symmetry is when some $a_i = 0$ or when some $a_i = a_j$. However, in our case it is easy to check that if we arrange our hyperplane $\mathcal{H}$ to have $\Delta_{\mathcal{H}}$ have one of these symmetries, then in fact the polytopes Δ and Δ' from before are affine equivalent. More specifically, the affine equivalence ϕ of $\Delta_{\mathcal{H}}$ could be extended to a global affine equivalence of either the top half or bottom half, which obviously would imply that Δ and Δ' are affine equivalent. In other words, if $s > 1$, this symplectomorphism technique only picks up the equivariant symplectomorphisms corresponding to coordinate changes on the polytope Δ.

We can now use Lemma 4.1.1 above to prove Theorem 1.2.3.

Proof of Theorem 1.2.3. First, we will assume that $(M_{\mathbf{a}}, \omega_{\mathbf{a}}^{\kappa})$ is symplectomorphic to $(M_{\mathbf{b}}, \omega_{\mathbf{b}}^{\kappa})$. If this is true, then $H^*(M_{\mathbf{a}}) \cong H^*(M_{\mathbf{b}})$, which by Proposition 1.2.2, implies that $M_{\mathbf{a}}$ and $M_{\mathbf{b}}$ are isomorphic as symplectic bundles. Note that the specific choice of symplectomorphism will in general have nothing to do with the choice of isomorphism of symplectic bundles.

Now assume that $M_{\mathbf{a}}$ is isomorphic to $M_{\mathbf{b}}$ as a symplectic bundle. By Proposition 1.2.2, there exists $C \in \mathbb{Z}$ so that $\sigma_1(C, \mathbf{a} + C) = \sigma_1(0, \mathbf{b})$, Thus, we have $\sigma_1(\mathbf{b}) = (r+1)C + \sigma_1(\mathbf{a})$ for some C. Without loss of generality, we assume $\sigma_1(\mathbf{a}) \leq \sigma_1(\mathbf{b})$. We show that any vector $\mathbf{b}$ can be reached from $\mathbf{a}$ by the following elementary moves. We will denote by $e_1(\mathbf{a})$ the elementary move described by

$$e_1(\mathbf{a}) = (a_1 + 2, a_2 + 1, \ldots, a_r + 1),$$

and by $e_{i,j}(\mathbf{a})$ the elementary move described by

$$e_{i,j}(\mathbf{a}) = (a_1, \ldots, a_i - 1, \ldots, a_j + 1, \ldots, a_r).$$

Lemma 4.1.1 then says that $(M_{\mathbf{a}}, \omega^{\kappa}_{\mathbf{a}})$ is symplectomorphic $(M_{e_1(\mathbf{a})}, \omega^{\kappa}_{e_1(\mathbf{a})})$ and to $(M_{e_{i,j}(\mathbf{a})}, \omega_{e_{i,j}(\mathbf{a})})$ for all i, j with $a_i - 1 \geq 0$. Thus, if we can reach $\mathbf{b}$ from $\mathbf{a}$ by the elementary moves e_1 and $e_{i,j}$, Lemma 4.1.1 would give a symplectomorphism from $(M_{\mathbf{a}}, \omega^{\kappa}_{\mathbf{a}})$ to $(M_{\mathbf{b}}, \omega^{\kappa}_{\mathbf{b}})$ as desired.

First, we recall that $\sigma_1(\mathbf{b}) = (r+1)C + \sigma_1(\mathbf{a})$, where by our assumption, $C \geq 0$. Thus, by repeatedly applying e_1, we can get a vector

$$\mathbf{a}' = e_1^C(\mathbf{a}) = (a_1 + 2C, a_2 + C, \ldots, a_r + C),$$

where $\sigma_1(\mathbf{a}') = \sigma_1(\mathbf{b})$. Next, repeatedly applying $e_{j,1}$ for $j \geq 2$, we can get a vector

$$\mathbf{a}'' = (\sigma_1(\mathbf{a}) + (r+1)C, 0, \ldots, 0) = (\sigma_1(\mathbf{b}), 0, \ldots, 0).$$

Notice that we can do this since each $a_j \geq 0$ and $C \geq 0$, so that $a_j + C \geq 0$ for all $j \geq 2$. Next we can get a vector $\mathbf{a}^1$ by repeatedly applying $e_{1,2}$

$$\mathbf{a}^1 = e_{1,2}^{\sigma_1(\mathbf{b}) - b_1}(\mathbf{a}'') = (b_1, \sigma_1(\mathbf{b}) - b_1, 0, \ldots, 0).$$

Notice that this is well defined because $b_1 \geq 0$ and $\sigma_1(\mathbf{b}) - b_1 = b_2 + \ldots + b_r \geq 0$ since all $b_i \geq 0$. Continuing on by induction, we get vectors $\mathbf{a}^i$, where

$$\begin{aligned} \mathbf{a}^i &= e_{i,i+1}^{\sigma_1(\mathbf{b}) - b_1 - \ldots - b_i}(\mathbf{a}^{i-1}) \\ &= (b_1, \ldots, b_i, \sigma_1(\mathbf{b}) - b_1 - \ldots - b_i, 0, \ldots, 0). \end{aligned}$$

Notice that this is well defined for all $1 \leq i \leq r-1$ because the b_i are all non-negative, and, for all i, $\sigma_1(\mathbf{b}) - b_1 - \ldots - b_i = b_{i+1}, \ldots b_r \geq 0$ since the b_i are all non-negative But then a straightforward computation shows that

$$\mathbf{a}^{r-1} = (b_1, \ldots, b_{r-1}, \sigma_1(\mathbf{b}) - b_1 - \ldots - b_{r-1}) = (b_1, \ldots, b_r) = \mathbf{b}$$

Thus, we have reached $\mathbf{b}$ from $\mathbf{a}$ by using the elementary moves e_1 and $e_{i,j}$, as desired. □

4.2 A Lemma about Deformation Equivalence

Lastly, we will say more about the deformation class of $(M_{\mathbf{a}}, \omega^{\kappa}_{\mathbf{a}})$. In particular, we will prove Lemma 1.2.6, which says that if $M_{\mathbf{a}}$ and $M_{\mathbf{b}}$ are both $\mathbb{C}P^r$ bundles over $\mathbb{C}P^s$, then they are isomorphic as symplectic bundles if and only if they are deformation equivalent. Furthermore, if the above conditions hold, we also have that $(M_{\mathbf{a}}, \omega_{\mathbf{a}}^{\kappa})$ is symplectomorphic to $(M_{\mathbf{b}}, \omega_{\mathbf{b}}^{\kappa})$ if $\kappa \gg 0$. This justifies our use of deformation equivalence as the equivalence relation on symplectic manifolds.

Proof of Lemma 1.2.6. First assume that $(M_{\mathbf{a}}, \omega_{\mathbf{a}}^{\kappa})$ is deformation equivalent to $(M_{\mathbf{b}}, \omega_{\mathbf{b}}^{\kappa})$. Then $M_{\mathbf{a}}$ and $M_{\mathbf{b}}$ are diffeomorphic, and in particular, $H^*(M_{\mathbf{a}})$ is isomorphic to $H^*(M_{\mathbf{b}})$, so that by Proposition 1.2.2, $M_{\mathbf{a}}$ is isomorphic to $M_{\mathbf{b}}$ as a symplectic bundle.

Now assume that $M_{\mathbf{a}}$ is isomorphic to $M_{\mathbf{b}}$ as a symplectic bundle. This implies that there is a diffeomorphism $\phi : M_{\mathbf{a}} \to M_{\mathbf{b}}$ so that

$$\phi^*(\omega_{\mathbf{b}}|_{F_{\phi(x)}}) = \omega_{\mathbf{a}}|_{F_x}$$

where F_x is the fiber over x and $F_{\phi(x)}$ is the fiber over $\phi(x)$. In other words, the diffeomorphism ϕ preserves the fiberwise symplectic structures of $M_{\mathbf{a}}$ and $M_{\mathbf{b}}$. We claim that since we also have the same κ, we must have $\phi^*([\omega_{\mathbf{b}}^{\kappa}]) = [\omega_{\mathbf{a}}^{\kappa}]$. To see this, recall that we have $\dim H^2(M_{\mathbf{a}}) = \dim H^2(M_{\mathbf{b}}) = 2$, and thus an element of either cohomology is determined by two parameters. Also, recall that our forms $\omega_{\mathbf{a}}^{\kappa}$ are determined by two parameters, where the first parameter determines the underlying bundle structure, while the second parameter determines the volume. However, by the above, $(M_{\mathbf{a}}, \omega_{\mathbf{a}}^{\kappa})$ and $(M_{\mathbf{b}}, \omega_{\mathbf{b}}^{\kappa})$ have the same underlying bundle structure and the same volume, so that in fact we must have

$$\phi^*([\omega_{\mathbf{b}}^{\kappa}]) = [\omega_{\mathbf{a}}^{\kappa}].$$

We wish to show that $M_{\mathbf{a}}$ and $M_{\mathbf{b}}$ are deformation equivalent. By the above, it suffices to show that there is a family of symplectic forms ω_t so that $\omega_0 = \omega_{\mathbf{a}}^{\kappa}$ and $\omega_3 = \phi^*(\omega_{\mathbf{b}}^{\kappa})$. We can produce such a family explicitly. Namely, if π is the map from $M_{\mathbf{a}}$ to $\mathbb{C}P^s$ and ω_s is the standard symplectic form on $\mathbb{C}P^s$, the deformation ω_t can be chosen explicitly as

$$\omega_t = \begin{cases} \omega_{\mathbf{a}}^{\kappa} + Kt\pi^*(\omega_s) & \text{if } 0 \leq t \leq 1 \\ (t-1)\phi^*(\omega_{\mathbf{b}}^{\kappa}) + (2-t)\omega_{\mathbf{a}}^{\kappa} + K\pi^*(\omega_s) & \text{if } 1 \leq t \leq 2 \\ \phi^*(\omega_{\mathbf{b}}^{\kappa}) + (3-t)K\pi^*(\omega_s) & \text{if } 2 \leq t \leq 3. \end{cases}$$

Recall that Lemma 2.2.3 says

$$\omega_{\mathbf{a}}^{\kappa+K} = \omega_{\mathbf{a}}^{\kappa} + \tfrac{K}{s+1}\pi^*(\omega_s)$$

For $0 \leq t \leq 1$, this implies that $\omega_t = \omega_{\mathbf{a}}^{(\kappa+(s+1)(tK))}$, and hence is non-degenerate. Also, if K is large enough, then ω_t for $1 \leq t \leq 2$ will all be non-degenerate. Now, recall that since ϕ is an isomorphism of symplectic bundles, $\pi \circ \phi = \pi$, and hence $\pi^*(\omega_s) = \phi^*(\pi^*(\omega_s))$. Using this and Lemma 2.2.3 as above, we have for $2 \leq t \leq 3$ that

$$\begin{aligned} \omega_t &= \phi^*(\omega_{\mathbf{b}}^{\kappa}) + (3-t)K\pi^*(\omega_s) \\ &= \phi^*(\omega_{\mathbf{b}}^{\kappa}) + (3-t)K\phi^*(\pi^*(\omega_s)) \\ &= \phi^*(\omega_{\mathbf{b}}^{\kappa} + (3-t)K\pi^*(\omega_s)) \\ &= \phi^*(\omega_{\mathbf{b}}^{\kappa+(s+1)((3-t)K)}), \end{aligned}$$

and hence ω_t is non-degenerate for $2 \leq t \leq 3$, so that ω_t is a family of symplectic forms with $\omega_0 = \omega_{\mathbf{a}}^{\kappa}$ and $\omega_3 = \phi^*(\omega_{\mathbf{b}}^{\kappa})$, while $[\omega_{\mathbf{a}}^{\kappa}] = \phi^*([\omega_{\mathbf{b}}^{\kappa}])$, so that $(M_{\mathbf{a}}, \omega_{\mathbf{a}}^{\kappa})$ is deformation equivalent to $(M_{\mathbf{b}}, \omega_{\mathbf{b}}^{\kappa})$, as desired.

Lastly, by the above, for any $\lambda > \kappa + K$ with K sufficiently large, $(M_{\mathbf{a}}, \omega_{\mathbf{a}}^{\lambda})$ is isotopic to $(M_{\mathbf{b}}, \omega_{\mathbf{b}}^{\lambda})$ by the linear isotopy $t\omega_{\mathbf{a}}^{\lambda} + (1-t)\omega_{\mathbf{b}}^{\lambda}$, and hence they are in fact symplectomorphic, as required. □

Chapter 5

Proofs and Examples for Torus Actions

5.1 Proofs of Main Theorems

We now give the proofs of the main theorems stated in the introduction. First we will prove Theorem 1.2.9, which we recall stated that if $(M_{\mathbf{a}}^{2n}, \omega_{\mathbf{a}}^{\kappa})$ is the $\mathbb{C}P^r$ bundle over $\mathbb{C}P^s$ determined by $(\mathbf{a}; \kappa)$, then $N_n(\mathbf{a}; \kappa) = 1$ when $r < s$.

Proof of Theorem 1.2.9. By Proposition 1.2.2, we know $M_{\mathbf{a}}$ and $M_{\mathbf{b}}$ are isomorphic as symplectic bundles if and only if there exists a $C \in \mathbb{Z}$ such that

$$\sigma_i(C, C + a_1, \dots, C + a_r) = \sigma_i(0, b_1, \dots, b_r) \quad 1 \leq i \leq \min\{r+1, s\} = r + 1,$$

where $\min\{r + 1, s\} = r + 1$ since $r < s$.

If $C = 0$, then $(C, C + \mathbf{a}) = (0, \mathbf{a})$. Therefore, $\sigma_i(0, \mathbf{a}) = \sigma_i(0, \mathbf{b})$ for all $1 \leq i \leq r + 1$, which implies $\mathbf{a} = \mathbf{b}$ up to reordering, as desired. If $C \neq 0$, then if $\sigma_{r+1}(C, C + \mathbf{a}) = \sigma_{r+1}(0, \mathbf{b}) = 0$, we must have some i where $C + a_i = 0$, so $C = -a_i < 0$. But then there is no way that $\sigma_i(C, C + \mathbf{a}) = \sigma_i(0, \mathbf{b})$, for all i since all $b_i \geq 0$ and $C < 0$. This contradiction finishes the proof of the theorem. □

We now focus on proving the theorems stated for the $r \geq s$ case. Before we do that however, we give an example of a vector $\mathbf{a}$ and constant κ so that $N_n(\mathbf{a}; \kappa) > 1$. We first note that by Proposition

1.2.2 and Lemma 1.2.7, we must only produce two vectors $\mathbf{a} = (a_1, \ldots, a_r)$ and $\mathbf{b} = (b_1, \ldots, b_r)$ and a number C so that

$$\sigma_i(C, C + \mathbf{a}) = \sigma_i(0, \mathbf{b}),\ 1 \le i \le \min\{r + 1, s\} = s$$

with $\mathbf{a} \neq \mathbf{b}$. Indeed, then by Lemma 1.2.7 they represent different toric structures since $\mathbf{a} \neq \mathbf{b}$, but by Proposition 1.2.2 and Lemma 1.2.6, we know that the underlying symplectic manifolds will be deformation equivalent. But $r \ge s$, so $\min(r + 1, s) = s$, which does not force $(C, C + \mathbf{a}) = (0, \mathbf{b})$. We have the following explicit example.

Example 5.1.1. Let $\mathbf{a} = (1, 4, 4)$, $\mathbf{b} = (2, 2, 5)$ describe $\mathbb{C}P^3$ bundles over $\mathbb{C}P^2$. Since $\sigma_1(\mathbf{a}) = \sigma_1(\mathbf{b})$, $K_{\mathbf{a}} = K_{\mathbf{b}}$, where we recall $K_{\mathbf{a}}$ is the number so that $\Delta^\kappa_{\mathbf{a}}$ is a bundle for all $\kappa > K_{\mathbf{a}}$. Thus, as we increase κ, the two toric structures $(M_{\mathbf{a}}, \omega^\kappa_{\mathbf{a}}, T_{\mathbf{a}})$ and $(M_{\mathbf{b}}, \omega^\kappa_{\mathbf{b}}, T_{\mathbf{b}})$ will both appear at the same time, so that the corresponding jump in $N_5(\mathbf{a}; \kappa)$, which occurs at $\kappa = 7$, will be a jump of size 2. Also, a fairly simple check will show that there is no other choice of vector $\mathbf{c}$ such that $(M_{\mathbf{c}}, \omega^\kappa_{\mathbf{c}})$ is deformation equivalent to $(M_{\mathbf{a}}, \omega^\kappa_{\mathbf{a}})$. More specifically, by Proposition 1.2.2, the only options would be vectors $\mathbf{c}$ that had $\sigma_1(\mathbf{c}) = 5, 1$, or $\sigma_1(\mathbf{c}) \ge 9$, corresponding to $C = -1$ or $C = -2$, or $C \ge 0$. The $C = -1$ and $C = -2$ cases can easily be checked not to work by hand. If $C = 0$, $(1, 4, 4)$ and $(2, 2, 5)$ are the only solutions, as a simple computation shows.

Now assume $C = 1$. If we take the vector $(1, 4, 4)$ and look for more examples with $C = 1$, we must compare the vector $(1, 2, 5, 5)$ to an arbitrary vector $(0, d_1, d_2, d_3)$. But $\sigma_1(1, 2, 5, 5) = 13$ and $\sigma_2(1, 2, 5, 5) = 57$, while the biggest that $\sigma_2(0, d_1, d_2, d_3)$ could be with $\sigma_1(0, d_1, d_2, d_3) = 13$ is when $(0, d_1, d_2, d_3) = (0, 4, 4, 5)$, which has $\sigma_2(0, 4, 4, 5) = 56$. Note that $(0, 4, 4, 5)$ is indeed the biggest because it is the vector which is closest to having all terms equal, which an exercise in calculus will confirm is the biggest. That there are no examples with $C \ge 2$ follows directly from Lemma 5.1.3. Hence, for the above choice of $\mathbf{a} = (1, 4, 4)$, we have

$$N_5(\mathbf{a}; \kappa) = \begin{cases} 0 & \text{if } \kappa \le 7 \\ 2 & \text{if } \kappa > 7. \end{cases}$$

We will now go back and prove the various theorems we stated for the case $r \ge s$, starting with Theorem 3.1.3, which says that $N_n(\mathbf{a}; \kappa) = 1$ if $\kappa \le 1$.

Proof of Theorem 3.1.3. If $\mathbf{a}$ defines a toric structure with $\kappa \le 1$, then we know that $\kappa > -s +$

$\sigma_1(\mathbf{a}) = K_{\mathbf{a}}$, so in particular,

$$0 < \sigma_1(\mathbf{a}) < s + \kappa \leq s + 1,$$

since $\kappa \leq 1$. Proposition 1.2.2 together with Theorem 1.2.9 implies that if $\mathbf{a}$ determines a bundle with a non-unique structure, then $r \geq s$ and there is a vector $\mathbf{b} = (b_1, \ldots, b_r)$ and a number C so that

$$\sigma_i(C, C + a_1, \ldots, C + a_r) = \sigma_i(0, b_1, \ldots, b_r) \quad 1 \leq i \leq s.$$

In our case, $\mathbf{a}$ and $\mathbf{b}$ satisfy $a_i \geq 0$ and $b_i \geq 0$, so that $\sigma_1(\mathbf{a}) \geq 0$ and $\sigma_1(\mathbf{b}) \geq 0$. If they are both to be valid toric structures with $\kappa \leq 1$, they must also satisfy $\sigma_1(\mathbf{a}) \leq s$ and $\sigma_1(\mathbf{b}) \leq s$, as we saw above. But $\sigma_1(C, C + \mathbf{a}) = (r + 1)C + \sigma_1(\mathbf{a})$, so putting this all together, we see that if $\sigma_1(C, C + \mathbf{a}) = \sigma_1(0, \mathbf{b}) = \sigma_1(\mathbf{b})$, then $C = 0$ and $\sigma_1(\mathbf{a}) = \sigma_1(\mathbf{b}) \leq s$.

Now assume $\mathbf{a} = (a_1, \ldots, a_r)$ with $a_i \geq 0$ and $\sigma_1(\mathbf{a}) = k$ with $1 \leq k \leq s$. Then since $a_i \in \mathbb{Z}$, the vector $\mathbf{a}$ has at most $k \leq s$ non-zero terms. Therefore, for any two vectors $\mathbf{a}$ and $\mathbf{b}$ as above, we must have $\sigma_i(\mathbf{a}) = \sigma_i(\mathbf{b}) = 0$ for all $i > s$. Therefore, if $\sigma_i(\mathbf{a}) = \sigma_i(\mathbf{b})$ for all $1 \leq i \leq s$, we actually have $\sigma_i(\mathbf{a}) = \sigma_i(\mathbf{b})$ for all i, which means that $\mathbf{a} = \mathbf{b}$ up to reordering, as required. □

We next prove Theorem 3.1.7, which is a simple consequence of the above machinery. Recall that Theorem 3.1.7 says first that the function $N_n(\mathbf{a}; \kappa)$ viewed as a function of κ is a step function which can only have jumps at the values $K_M + l(r + 1)$, and second that if $r = s$, then these potential jumps are all of size 1.

Proof of Theorem 3.1.7. We first prove statement (1). Recall that $K_M \in \mathbb{Z}$ is the largest number so that $N_n(\mathbf{a}; K_M) = 0$. Recall also that K_M need not equal $K_{\mathbf{a}}$ for every possible $\mathbf{a}$. However, by the definition of K_M, there is always some vector $\mathbf{b}$ so that $K_{\mathbf{b}} = K_M$. For convenience sake, we will assume that $K_{\mathbf{a}} = K_M$. We know from Proposition 1.2.2 and Lemma 1.2.6 that if there is another inequivalent toric structure on $M_{\mathbf{a}}$, there is a vector $\mathbf{b}$ so that $\mathbf{a} \neq \mathbf{b}$ and an integer C so that

$$\sigma_i(C, C + a_1, \ldots, C + a_r) = \sigma_i(0, b_1, \ldots, b_r) \quad 1 \leq i \leq s < r + 1$$

In particular, we know that $C \geq 0$, since any $\mathbf{b}$ determining a toric structure on $M_{\mathbf{a}}$ must have $K_{\mathbf{b}} \geq K_M = K_{\mathbf{a}}$ which implies $\sigma_1(\mathbf{b}) \geq \sigma_1(\mathbf{a})$. Thus, $\sigma_1(\mathbf{b}) = \sigma_1(\mathbf{a}) + C(r + 1)$ for some integer $C \geq 0$ and the value of $N_n(\mathbf{a}; \kappa)$ can only jump at the values of κ where

$$\kappa = K_{\mathbf{b}} = -s + \sigma_1(\mathbf{b}) = -s + \sigma_1(\mathbf{a}) + C(r + 1) = K_{\mathbf{a}} + \ell(r + 1) = K_M + \ell(r + 1),$$

where $\ell = C \geq 0$, which finishes the proof of statement (1).

We now prove statement (2). Assume that for some κ, there is a jump of size 2 or more. Then there exist two vectors $\mathbf{a} \neq \mathbf{b}$ with $K_{\mathbf{a}} = K_{\mathbf{b}}$ and a constant C so that

$$\sigma_i(C, C + a_1, \ldots, C + a_r) = \sigma_i(0, b_1, \ldots, b_r), \quad 1 \leq i \leq s < r + 1.$$

But $K_{\mathbf{a}} = K_{\mathbf{b}}$ implies $\sigma_1(\mathbf{a}) = \sigma_1(\mathbf{b})$, which implies that $C = 0$, which obviously implies

$$\sigma_i(\mathbf{a}) = \sigma_i(0, \mathbf{a}) = \sigma_i(0, \mathbf{b}) = \sigma_i(\mathbf{b}), \quad 1 \leq i \leq s = r.$$

This implies that $\mathbf{a} = \mathbf{b}$ up to reordering. This contradiction establishes statement (2). □

Next, we prove Theorem 1.2.11. Recall that this theorem first gave a bound on the value of C allowed in the equations

$$\sigma_i(0, b) = \sigma_i(C, C + \mathbf{a})$$

and further said that

$$\Big(\kappa_1, \kappa_2 > (r + 1 - \tfrac{1}{r})\sigma_1(\mathbf{a}) - s\Big) \Longrightarrow \big(N_n(\mathbf{a}; \kappa_1) = N_n(\mathbf{a}; \kappa_2)\big).$$

We first prove the bound on C in the below lemma.

Lemma 5.1.2. *Fix an $r \geq s \geq 2$. Assume we have non-negative integer vectors $\mathbf{a} = (a_1, \ldots, a_r)$ and $\mathbf{b} = (b_1, \ldots, b_r)$ as before, and a real number C so that*

$$\sigma_i(C, C + \mathbf{a}) = \sigma_i(0, \boldsymbol{b}), \quad \forall 1 \leq i \leq s < r + 1$$

Then if some $a_i \neq 0$,

$$-\tfrac{1}{r+1}\sigma_1(\mathbf{a}) \leq C < \tfrac{r-1}{r}\sigma_1(\mathbf{a})$$

Proof. First, notice that if $C < -\frac{1}{r+1}\sigma_1(\mathbf{a})$, then

$$\sigma_1(\mathbf{b}) = (r + 1)C + \sigma_1(\mathbf{a}) < 0$$

which contradicts the fact that $\mathbf{b}$ is a positive integer vector. Thus, we must have $C \geq -\frac{1}{r+1}\sigma_1(\mathbf{a})$.

It remains to show that $C \leq \frac{r-1}{r}\sigma_1(\mathbf{a})$. Since $s \geq 2$, it suffices to show that if $C > \frac{r-1}{r}\sigma_1(\mathbf{a})$, then any non-negative integer vector $\mathbf{b}$ with $\sigma_1(0, \mathbf{b}) = \sigma_1(C, C + \mathbf{a})$ satisfies

$$\sigma_2(0, \mathbf{b}) < \sigma_2(C, C + \mathbf{a}).$$

Indeed, for these values of C, there cannot exist a vector $\mathbf{b}$ with $\sigma_i(0,\mathbf{b}) = \sigma_i(C, C+\mathbf{a})$ for all i. To see this, we will consider the two polynomials

$$P_{\mathbf{a}}(C) := \sigma_2(C, C+\mathbf{a}), \quad P_{\mathbf{b}}(C) := \sigma_2(0, \tfrac{(r+1)C+\sigma_1(\mathbf{a})}{r}, \dots, \tfrac{(r+1)C+\sigma_1(\mathbf{a})}{r})$$

Notice that any vector $\mathbf{b}$ with $\sigma_1(0,\mathbf{b}) = \sigma_1(C, C+\mathbf{a})$, has $\sigma_2(\mathbf{b}) \le P_{\mathbf{b}}(C)$ as a consequence of basic calculus. Indeed, the quantity $\sigma_2(0, b_1, \dots, b_r)$ is maximized by $b_1 = \dots = b_r$, and the inequality follows from this fact. Thus, to prove the theorem, it only remains to show that $P_{\mathbf{a}}(C) - P_{\mathbf{b}}(C) > 0$ for all $C \ge \frac{r-1}{r}\sigma_1(\mathbf{a})$. We will do this by showing that

$$(P_{\mathbf{a}} - P_{\mathbf{b}})(\tfrac{r-1}{r}\sigma_1(\mathbf{a})) > 0, \quad (P_{\mathbf{a}} - P_{\mathbf{b}})'(\tfrac{r-1}{r}\sigma_1(\mathbf{a})) \ge 0, \quad (P_{\mathbf{a}} - P_{\mathbf{b}})''(C) > 0 \ \forall C.$$

Then since $P_{\mathbf{a}} - P_{\mathbf{b}}$ is a degree 2 polynomial, the desired result will follow. We show this by explicitly computing all three terms.

First, we see that

$$P_{\mathbf{a}}(C) = \sigma_2(C, C+\mathbf{a}) = \tbinom{r+1}{2}C^2 + \tbinom{r}{1}\sigma_1(\mathbf{a})C + \sigma_2(\mathbf{a}) = \tfrac{(r+1)(r)}{2}C^2 + r\sigma_1(\mathbf{a})C + \sigma_2(\mathbf{a}).$$

Next, after some rearranging, we see that

$$\begin{aligned} P_{\mathbf{b}}(C) = \sigma_2(0, \tfrac{(r+1)C+\sigma_1(\mathbf{a})}{r}, \dots, \tfrac{(r+1)C+\sigma_1(\mathbf{a})}{r}) &= \tbinom{r}{2}\left(\tfrac{r+1}{r}\right)^2 C^2 + 2\tbinom{r}{2}\tfrac{r+1}{r^2}\sigma_1(\mathbf{a})C + \tbinom{r}{2}\tfrac{\sigma_1(\mathbf{a})^2}{r^2} \\ &= \tfrac{r(r-1)(r+1)^2}{2r^2}C^2 + \tfrac{r(r-1)(r+1)\sigma_1(\mathbf{a})}{r^2}C + \tfrac{r(r-1)\sigma_1(\mathbf{a})^2}{2r^2} \\ &= \left(\tfrac{r^2-1}{r^2}\right)\left(\tfrac{(r+1)(r)}{2}C^2 + r\sigma_1(\mathbf{a})C\right) + \tfrac{r-1}{2r}\sigma_1(\mathbf{a})^2. \end{aligned}$$

A simple computation gives

$$(P_{\mathbf{a}} - P_{\mathbf{b}})(C) = \tfrac{r+1}{2r}C^2 + \tfrac{1}{r}\sigma_1(\mathbf{a})C + \sigma_2(\mathbf{a}) - \tfrac{r-1}{2r}\sigma_1(\mathbf{a})^2.$$

Also, taking the derivative of this, we get that

$$(P_{\mathbf{a}} - P_{\mathbf{b}})'(C) = \tfrac{r+1}{r}C + \tfrac{1}{r}\sigma_1(\mathbf{a}).$$

Finally, taking the derivative of this, we get

$$(P_{\mathbf{a}} - P_{\mathbf{b}})''(C) = \tfrac{r+1}{r} > 0 \ \forall C.$$

We then have the following computation:

$$\begin{aligned}(P_{\mathbf{a}} - P_{\mathbf{b}})(\tfrac{r-1}{r}\sigma_1(\mathbf{a})) &= \tfrac{(r+1)(r-1)^2}{2r^3}(\sigma_1(\mathbf{a}))^2 + \tfrac{2r^2-2r}{2r^3}(\sigma_1(\mathbf{a}))^2 + \sigma_2(\mathbf{a}) - \tfrac{r^3-r^2}{2r^3}\sigma_1(\mathbf{a})^2 \\ &= \tfrac{2r^2-3r+1}{2r^3}\sigma_1(\mathbf{a})^2 + \sigma_2(\mathbf{a}) = \tfrac{(2r-1)(r-1)}{2r^3}\sigma_1(\mathbf{a})^2 + \sigma_2(\mathbf{a}) > 0.\end{aligned}$$

where the last inequality follows since $a_i \geq 0$ and some $a_i \neq 0$ and $r \geq s > 1$, which implies that $(2r-1) > 0$ and $r - 1 > 0$. We also have

$$(P_{\mathbf{a}} - P_{\mathbf{b}})'(\tfrac{r-1}{r}\sigma_1(\mathbf{a})) = (\tfrac{r+1}{r}\tfrac{r-1}{r} + \tfrac{1}{r})\sigma_1(\mathbf{a}) \geq 0.$$

This computation completes the proof. □

The following similar lemma is useful in applications.

Lemma 5.1.3. *Fix an integer $C \geq 0$ and a non-negative integer vector $\mathbf{a} = (a_1, \ldots, a_r)$. Consider the inequalities*

$$\sigma_2(0, \mathbf{b}) \leq \sigma_2(C, C + \mathbf{a}), \tag{$*_0$}$$

and the inequalities

$$\sigma_2(0, \mathbf{b}) < \sigma_2(C + n, C + n + \mathbf{a}) \tag{$*_n$}$$

where $n \geq 1$, and in $(*_k)$ with $0 \leq k \leq n$, $\mathbf{b}$ ranges over all integer vectors with $\sigma_1(0, \mathbf{b}) = \sigma_1(C + k, C + k + \mathbf{a})$. Then

$$(*_0) \Longrightarrow (*_n)\ \forall\ n \geq 1.$$

Proof. An obvious induction shows that it suffices to prove the theorem in the case $n = 1$. Write $(r + 1)C + \sigma_1(\mathbf{a}) = kr + \ell$ for some integers k, ℓ, where $k \geq 0$ since $C \geq 0$ and $0 \leq \ell < r$. We will call $\mathbf{a}' = (a'_0, \ldots, a'_r) = (C, C + \mathbf{a})$. Then the integer vector $\mathbf{b}$ with $\sigma_1(\mathbf{b}) = \sigma_1(C, C + \mathbf{a})$ with largest value of $\sigma_2(0, \mathbf{b})$ is $\mathbf{b} = (k, \ldots, k, k+1, \ldots, k+1)$ with exactly $r - \ell$ entries equal to k and ℓ entries equal to $k+1$. Now, consider $(C+1, C+1+\mathbf{a})$. Then $\sigma_1(C+1, C+1+\mathbf{a}) = (k+1)r + \ell + 1$ and the vector $\mathbf{b}'$ with this σ_1 and the largest σ_2 is now $\mathbf{b}' = (k + 1, \ldots, k + 1, k + 2, \ldots, k + 2)$ where here there are $\ell + 1$ entries equal to $k + 2$ and $r - \ell - 1$ entries equal to $k + 1$. Then we have $(C + 1, C + 1 + \mathbf{a}) = (a'_0 + 1, \ldots, a'_r + 1)$ and $\mathbf{b}' = (b_1 + 1, \ldots, b_{r-\ell} + 2, \ldots, b_r + 1)$, since $b_{r-\ell} = k$ while $b'_{r-\ell} = k + 2$. A simple computation shows that

$$\sigma_2(C + 1, C + 1 + \mathbf{a}) = \sigma_2(\mathbf{a}') + r\sigma_1(\mathbf{a}') + \tbinom{r+1}{2}.$$

Another simple computation shows that

$$\begin{aligned}\sigma_2(\mathbf{b}') &= \sigma_2(\mathbf{b}) + r\sigma_1(\mathbf{b}) - b_{r-\ell} + \tbinom{r-1}{2} + 2r - 2 = \sigma_2(\mathbf{b}) + r\sigma_1(\mathbf{a}') - k + \tfrac{r^2-3r+2+4r-4}{2} \\ &= \sigma_2(\mathbf{b}) + r\sigma_1(\mathbf{a}') - k + \tfrac{r^2+r}{2} - 1 = \sigma_2(\mathbf{b}) + r\sigma_1(\mathbf{a}') + \tbinom{r+1}{2} + (-1-k) \\ &< \sigma_2(\mathbf{a}') + r\sigma_1(\mathbf{a}') + \tbinom{r+1}{2} = \sigma_2(C+1, C+1+\mathbf{a}),\end{aligned}$$

where the last inequality is true because $-1-k<0$ and $\sigma_2(\mathbf{b}) \le \sigma_2(\mathbf{a}')$ since we are assuming $*_0$ is satisfied. Thus, we have the desired result. □

Proof of Theorem 1.2.11. Lemma 5.1.2 above proves the first statement of the theorem. Thus it remains only to show that for a non-negative integer vector $\mathbf{a} = (a_1, \ldots, a_r)$,

$$\kappa_1, \kappa_2 \ge \left(r + 1 - \tfrac{1}{r}\right)\sigma_1(\mathbf{a}) - s \Longrightarrow N_n(\mathbf{a}; \kappa_1) = N_n(\mathbf{a}; \kappa_2).$$

First, by Theorem 1.2.9, it suffices to consider $r \ge s > 1$. We will show that for any non-negative integer vector $\mathbf{a}$, $N_n(\mathbf{a}; \kappa_1) - N_n(\mathbf{a}; \kappa_2) = 0$ when $\kappa_1, \kappa_2 \ge (r + 1 - \frac{1}{r})\sigma_1(\mathbf{a}) - s$. Without loss of generality, assume that $\kappa_1 > \kappa_2$. Now, if the result were false, we would have $N_n(\mathbf{a}; \kappa_1) - N_n(\mathbf{a}; \kappa_2) > 0$, which would mean there was some vector $\mathbf{b}$ with $\kappa_2 < K_{\mathbf{b}} \le \kappa_1$ and a corresponding number C so that

$$\sigma_i(C, C + a_1, \ldots, C + a_r) = \sigma_i(0, b_1, \ldots, b_r), \quad 1 \le i \le s < r + 1.$$

Also, since $K_{\mathbf{b}} > \kappa_2 \ge (r + 1 - \frac{1}{r})\sigma_1(\mathbf{a}) - s$ and $K_{\mathbf{b}} = -s + \sigma_1(\mathbf{b})$, we know that

$$\begin{aligned}\sigma_1(\mathbf{b}) = (r+1)C + \sigma_1(\mathbf{a}) \ge (r + 1 - \tfrac{1}{r})\sigma_1(\mathbf{a}) &\Longrightarrow (r+1)C \ge \tfrac{r^2-1}{r}\sigma_1(\mathbf{a}) \\ &\Longrightarrow C \ge \tfrac{r^2-1}{r(r+1)}\sigma_1(\mathbf{a}) = \tfrac{r-1}{r}\sigma_1(\mathbf{a}).\end{aligned}$$

But this is impossible by Lemma 5.1.2 above. □

We will now give a proof of Theorem 3.1.10, which says that for any $r \ge s = 2$,

$$\sup_{\mathbf{a}} N_{r+s}(\mathbf{a}; \infty) = \infty.$$

Proof of Theorem 3.1.10. First we show that the above can be reduced to the case $r = s$. A simple computation shows that for any vectors $\mathbf{a}$ and $\mathbf{b}$ and any real number κ,

$$\sigma_i(\mathbf{a}) = \sigma_i(\mathbf{b}), \quad \forall 1 \le i \le s, \Longrightarrow \sigma_i(\mathbf{a} + \kappa) = \sigma_i(\mathbf{b} + \kappa), \quad \forall 1 \le i \le s.$$

Next, consider a vector $\mathbf{a} = (a_1, \dots, a_r)$ with $N_{2r}(\mathbf{a}; \infty) = k$. Then there exist vectors $\mathbf{b}_1, \dots, \mathbf{b}_{k-1}$ with corresponding constants $C_1, \dots, C_{k-1}$ so that

$$\sigma_i(C_j, \mathbf{a} + C_j) = \sigma_i(0, \mathbf{b}_j), \quad \forall 1 \leq i \leq s,\ 1 \leq j \leq k-1.$$

Now consider some $r > s$. Then $r = s + \ell$ for some $\ell \geq 1$. We can then define the vectors

$$\mathbf{b}_j^\ell = (0, \dots, 0, C_{k-1} - C_j, C_{k-1} - C_j + \mathbf{b}_j)$$

where this vector has $\ell - 1$ entries equal to 0. Then, the above computation shows that

$$\sigma_i(0, \mathbf{b}_j^l) = \sigma_i(0, \dots, 0, C_{k-1}, C_{k-1} + \mathbf{a}), \quad \forall 1 \leq i \leq s,\ 1 \leq j \leq k-1,\ \ell \geq 1.$$

This shows that if the theorem holds for $r = s$, then it holds for any $r > s$.

We now consider the case where $r = s = 2$. We will show that for any k, there exists a vector $\mathbf{a}_k = (a_k, b_k)$ with $N_4(\mathbf{a}_k; \infty) = k$.

To see this, notice that any vector $\mathbf{a} = (a, b)$ and any vector $\mathbf{b}$ with $\sigma_1(0, \mathbf{b}) = (C, C + a, C + b)$ can be written as $\mathbf{b} = (C + a + x, 2C + b - x)$ for some integer x. Then

$$\sigma_2(0, C + a + x, 2C + b - x) = \sigma_2(C, C + a, C + b) \iff bx - ax + Cx - x^2 = bC + C^2.$$

We will look for solutions of the special form $C = \lambda x$. Now, substituting for C and solving for x gives

$$x = \tfrac{b - a - b\lambda}{\lambda^2 - \lambda + 1}.$$

Thus, any choice of a, b and λ such that x and C are both integers will result in a vector $\mathbf{b}$ with $\mathbf{b} \neq \mathbf{a}$, while $M_{\mathbf{a}}$ and $M_{\mathbf{b}}$ are isomorphic as symplectic bundles, by Proposition 1.2.2. We consider the family where $\lambda = \frac{1}{n}$. Substituting for λ, we have the following computation

$$x = \frac{b - a - \frac{b}{n}}{\frac{1}{n^2} - \frac{1}{n} + 1} = \frac{\frac{n-1}{n} b - a}{\frac{n^2 - n + 1}{n^2}} = \tfrac{n}{n^2 - n + 1}((n-1)b - na), \quad C = \tfrac{1}{n^2 - n + 1}((n-1)b - na).$$

Thus, if we can find a pair of integers a, b with $(n-1)b - na \equiv 0 \mod (n^2 - n + 1)$, then x and C will be integers as desired. More specifically, if for each k we can find integers a_k, b_k and $k - 1$ integers $n_1, \dots, n_{k-1}$ with

$$(n_i - 1)b_k - n_i a_k \equiv 0 \mod (n_i^2 - n_i + 1) \iff$$
$$- b_k \equiv n_i(a_k - b_k) \mod (n_i^2 - n_i + 1), \forall 1 \leq i \leq k-1,$$

then $\mathbf{a}_k = (a_k, b_k)$ would have $N_4(\mathbf{a}_k; \infty) \geq k$ for each k, as desired.

We will solve these equations using the Chinese Remainder Theorem. More specifically, we restrict our attention to vectors of the form $(K, c_k + K)$, for a fixed integer c_k which fixes the quantity $a_k - b_k = -c_k$. Then, plugging in and simplifying, we have reduced the problem to picking an integer K so that

$$K \equiv c_k(n_i - 1) \mod n_i^2 - n_i + 1,$$

for some collection of integers n_i. The Chinese Remainder Thereom then says that this system of equations will have a solution provided that the collection of integers $N_i := n_i^2 - n_i + 1$ can be chosen to be relatively prime. Thus, to complete the proof, we only need to produce a sequence $N_i = n_i^2 - n_i + 1$ such that $\gcd(N_i, N_j) = 1$ for all i, j. We will produce this sequence by induction. In particular, we will produce a sequence N_n such that if $i < j$, all prime factors of N_i are less than all prime factors of N_j. Such a sequence would obviously have $\gcd(N_i, N_j) = 1$.

First, let $n_1 = 2$, so that $N_1 = 3$. Now, assume that we have such integers $N_1, \ldots, N_{k-1}$ with corresponding integers $n_1, \ldots, n_{k-1}$ so that $N_i = n_i^2 - n_i + 1$ and such that if $i < j$, all prime factors of N_i are less then all prime factors of N_j. Let p_k be the largest prime number dividing N_{k-1}. Since we have assumed $N_1 = 3$, such a number p_k will always exist for any k. Now, let $n_k = p_k!$ and let $N_k = n_k^2 - n_k + 1$. Then if q is any prime such that $q \leq p_k$, then by construction we have $N_k \equiv 1 \mod q$, and hence the only primes dividing N_k are bigger than p_k, as desired. This computation finishes the construction of the sequence N_n, and hence finishes the proof of the theorem. □

The above techniques show that to prove Conjecture 3.1.9 for any $r \geq s \geq 2$, it is enough to check it for any $r = s$. However, if $r = s \geq 3$, the equations involved are much more complicated than for the $s = 2$ case above, and it is not clear how to show directly that $\sup_{\mathbf{a}}(N_{2r}(\mathbf{a}; \infty)) = \infty$.

5.2 Interesting Examples

We conclude the paper with a few interesting examples which explore Theorem 1.2.11 in the Fano case. First, we explore the case $r = s = 2$.

Example 5.2.1. We claim that if $r = s = 2$ and $(M_{\mathbf{a}}, \omega_{\mathbf{a}}^{\kappa})$ is Fano, then $N_4(\mathbf{a}; \infty) = 1$. We recall from Remark 3.1.6 that if $\mathbf{a}$ is a Fano vector, we must have $K_{\mathbf{a}} < 1$. However, since $s = 2$, a simple computation shows that this implies that we must have $\sigma_1(\mathbf{a}) \le 2$, which gives us the 4 cases $\mathbf{a} = (0,0)$, $\mathbf{a} = (0,1)$, $\mathbf{a} = (1,1)$, and $\mathbf{a} = (0,2)$. Recall that Proposition 1.1.3, proven in [15], implies that if $\mathbf{a} = (0,0)$, then $N_4(\mathbf{a};\infty) = 1$. Thus, we only need to consider $(0,1)$, $(1,1)$, and $(0,2)$. The cases $(0,1)$ and $(1,1)$ are special cases of Example 5.2.2 below, and for $\mathbf{a} = (0,1)$ or $\mathbf{a} = (1,1)$, we get $N_4(\mathbf{a};\infty) = 1$ as well. Thus, it suffices to check that $N_4((0,2);\infty) = 1$.

Since $r = s = 2$, by Lemma 5.1.2 it suffices to show that there is no vector $\mathbf{b}$ and integer C such that

$$\sigma_i(0,\mathbf{b}) = \sigma_i(C, C, 2 + C), \quad i = 1, 2, \quad -\tfrac{2}{3} \le C < 1,$$

so that we only need check the $C = 0$ case. However, an obvious computation shows that if $C = 0$, $\mathbf{b} = \mathbf{a}$, so that $N_4((0,2);\infty) = 1$. Thus, if $r = s = 2$ and $(M_{\mathbf{a}}, \omega^\kappa_{\mathbf{a}})$ is Fano, then $N_4(\mathbf{a};\infty) = 1$.

Next, we look at some higher dimensional Fano examples.

Example 5.2.2. Consider vectors of the form $\mathbf{a} = (0,\dots,0,1,\dots,1)$ where $\sigma_1(\mathbf{a}) = k$. We will show that for such vectors, if $s \ge 2$, $N_{r+s}(\mathbf{a};\kappa) = 1$ for all $\kappa > k - s$, and in particular, $N_{r+s}(\mathbf{a};\infty) = 1$. Further, note that these vectors will be Fano whenever $s \ge k$.

If this is false, then by Proposition 1.2.2, there exists some $\mathbf{b}$ and C so that $\sigma_i(C, C+\mathbf{a}) = \sigma_i(0,\mathbf{b})$ where $i \ge 2$. We will show that for our specific choice of $\mathbf{a}$, this cannot happen.

First, notice that we cannot choose $C < 0$ because then $\sigma_1(\mathbf{a}) < 0$. Second, we cannot choose $C = 0$. Indeed, in that case, any non-negative integer vector $\mathbf{b}$ with $\sigma_1(\mathbf{b}) = k$ has $\sigma_2(\mathbf{b}) \le \sigma_2(\mathbf{a})$, with equality only if $\mathbf{a} = \mathbf{b}$. Furthermore, this implies that $\mathbf{a}$ satisfies the hypothesis of Lemma 5.1.3 with $C = 0$. Thus, if $C > 0$ and $\sigma_1(\mathbf{b}) = \sigma_1(C, C + \mathbf{a})$, Lemma 5.1.3 implies that $\sigma_2(\mathbf{b}) < \sigma_2(C, C + \mathbf{a})$. Since $s \ge 2$, the above and Proposition 1.2 then shows that $N_{r+s}(\mathbf{a};\infty) = 1$, as desired.

The next example shows that the general Fano case is not as nice, and in fact there are examples of Fano toric manifolds which have more than one toric structure.

Example 5.2.3. Let $\mathbf{a}$ be the vector $\mathbf{a} = (0,\dots,0,2)$ with $r \ge 3$. Note that here $r \ge 3$ is necessary, as is seen in Example 5.2.1. Notice that this vector is Fano with $s = 2$, since $K_{\mathbf{a}}(2) = \sigma_1(\mathbf{a}) - 2 = 2 - 2 = 0 < 1$. On the other hand, by choosing $C = 1$, we can consider the vector $(1, 1 + \mathbf{a}) = (1,\dots,1,3)$ with exactly $r \ge 3$ entries equal to 1. Then, consider the vector

$\mathbf{b} = (1, \ldots, 1, 2, 2, 2)$ with exactly 3 entries equal to 2. First, we see that

$$\sigma_1(\mathbf{b}) = r + 3 = \sigma_1(1, 1 + \mathbf{a}).$$

However, we also have

$$(1, 1 + \mathbf{a}) = (1, b_1, \ldots, b_{r-3}, b_{r-2} - 1, b_{r-1} - 1, b_r + 1).$$

Note that for the above to make sense, we must have a b_{r-2} term, which we do since $r \geq 3$. So, using the above substitution, an easy computation shows that

$$\begin{aligned}\sigma_2(1, 1 + \mathbf{a}) &= \sigma_2(0, \mathbf{b}) + \sigma_1(\mathbf{b}) - (\sigma_1(\mathbf{b}) - b_{r-2}) - (\sigma_1(\mathbf{b}) - b_{r-1}) + (\sigma_1(\mathbf{b}) - b_r) \\ &\quad - 1 - 1 + 1 + 1 - 1 - 1 = \sigma_2(\mathbf{b}) + b_{r-2} + b_{r-1} - b_r - 2 = \sigma_2(\mathbf{b}),\end{aligned}$$

so that $\sigma_i(\mathbf{b}) = \sigma_i(1, 1 + \mathbf{a})$ for $i = 1, 2$. Thus, Proposition 1.2.2 together with Lemma 1.2.6 implies that $(M_{\mathbf{a}}, \omega^{\kappa}_{\mathbf{a}})$ is deformation equivalent to $(M_{\mathbf{b}}, \omega^{\kappa}_{\mathbf{b}})$, while Lemma 1.2.7 implies that these give different toric structures since $\mathbf{a} \neq \mathbf{b}$. Also, by Lemma 5.1.2, we know that if C is to support a vector $\mathbf{b}$ with the desired properties, then

$$C < \tfrac{r-1}{r}\sigma_1(\mathbf{a}) = \tfrac{r-1}{r}2 < 2,$$

so that we cannot have $C \geq 2$. Finally, since we know $\sigma_1(\mathbf{b}) = r + 3$, we know that $K_{\mathbf{b}}(2) = r + 1$. The above finishes the proof of the following: if $r \geq 3$ and $\mathbf{a}$ is as above, we have

$$N_{r+2}(\mathbf{a}; \kappa) = \begin{cases} 0 & \text{if } \kappa \leq 0 \\ 1 & \text{if } 0 < \kappa \leq r + 1 \\ 2 & \text{if } r + 1 < \kappa. \end{cases}$$

Chapter 6

Definitions and Technical Lemmas for Circle Actions

In this section, we will build up the tools necessary to prove Theorem 1.3.4. We begin by giving a general discussion about orbifolds.

6.1 Orbifolds

We first give the definition of an orbifold. To do this, we first define a local uniformizing chart.

Definition 6.1.1. *Let M^4 be a topological space, and let $y \in M$ be a point. Then a C^∞* ***local uniformizing chart*** *at y is a 4-tuple $(U, \widetilde{U}, \Gamma, \phi)$ where U is a neighborhood of y in M, $\widetilde{U} \subset \mathbb{R}^4$, Γ is a finite group acting on $\widetilde{U}$ by diffeomorphisms, and $\phi : \widetilde{U} \to U$ is a continuous, equivariant map so that $\phi : \widetilde{U}/\Gamma \to U$ is a homeomorphism.*

Using this, we can now define an orbifold. Before we do this, we note that the below definition is not the standard definition of an orbifold. In fact, in Remark 6.1.5 below, we only consider 4-dimensional orbifolds with certain types of singularities.

Definition 6.1.2. *Let M^4 be a compact Hausdorff topological space and let $y_i \in M^4$, $i = 1, \dots N$ be points. Then M is a* ***smooth orbifold*** *if there are C^∞ local uniformizing charts $(U_i, \widetilde{U}_i, \Gamma_i, \phi_i)$ at y_i so that $U_i \cap U_j = \emptyset$ if $i \neq j$ and $M^4 \backslash \{y_1, \dots, y_N\}$ is locally Euclidean. Furthermore, if $(U_i, \widetilde{U}_i, \Gamma_i, \phi_i)$ are such local uniformizing charts, a* ***smooth orbifold structure*** *is given by a finite open cover*

$\mathcal{C}$ of M by C^∞ local uniformizing charts $(U_i, \widetilde{U}_i, \Gamma_i, \phi_i)$ for $i = 1, \ldots N' > N$ so that if $i > N$, Γ_i is the trivial group and so that if $U_i \cap U_j \neq \emptyset$ where i is anything and $j > N$, then

$$\phi_{ij} = \phi_j^{-1} \circ \phi_i : \phi_i^{-1}(U_i \cap U_j) \subset \widetilde{U}_i \longrightarrow \phi_j^{-1}(U_i \cap U_j) \subset \widetilde{U}_j$$

is a diffeomorphism.

Remark 6.1.3. One can define differential forms in this context in the usual way by defining them on each local uniformizing chart. In this fashion, it can be shown that all the usual theory of differential forms, including De Rham cohomology and Poincaré duality carries over to the smooth orbifold case. Additionally, one can define a symplectic orbifold in the obvious way.

This leads to the definition of an orbifold singularity

Definition 6.1.4. *Let M^4 be an orbifold. A point $y \in M$ will be called an **orbifold singularity** of **order** r and **type** (p,q) where $\gcd(p,r) = \gcd(q,r) = 1$ if there is a local uniformizing chart $(U, \widetilde{U}, \mathbb{Z}_r, \phi)$ near y so that $\mathbb{Z}_r$ acts on $\widetilde{U} \subset \mathbb{R}^4 = \mathbb{C}^2$ by*

$$\xi(z_1, z_2) \mapsto (\xi^p z_1, \xi^q z_2),$$

where $\xi = e^{\frac{2\pi i}{r}}$. Notice that this action is free away from the origin and has an isolated fixed point at the origin.

Remark 6.1.5. The above definitions are much simpler than the general definitions of an orbifold and an orbifold singularity. By the standard terminology, the above would be considered an isolated orbifold singularity. Additionally, it is not necessary in general to require that an order r singularity has type (p,q) with $\gcd(p,r) = \gcd(q,r) = 1$. There are certainly more general notions of orbifolds and orbifold singularities, but in our case, these are the only types of orbifolds we will encounter.

We now discuss what it means to say that two orbifolds are the same.

Definition 6.1.6. *Let M, M' be smooth orbifolds, where x_i are the orbifold points of M and y_j' are the orbifold points of N. Then if $i, j = 1, \ldots N$ and $(U_i, \widetilde{U}_i, \Gamma_i, \phi_i)$ is a local uniformizing chart for y_i while $(U_i', \widetilde{U}_i', \Gamma_i', \phi_i')$ is a local uniformizing chart for y_i', we will say that*

$$\phi : M \longrightarrow M'$$

*is a **diffeomorphism** if the following conditions are satisfied.*

1. *$\phi(y_i) = y'_i$, up to reordering.*

2. *$\phi : M \setminus \{y_1, \dots, y_N\} \to M' \setminus \{y'_1, \dots, y'_N\}$ is a diffeomorphism*

3. *The local uniformizing charts can be chosen so $\phi(U_i) = U'_i$. Also, ϕ lifts to*

$$\widetilde{\phi}_i : \widetilde{U}_i \longrightarrow \widetilde{U}'_i,$$

where $\widetilde{\phi}_i$ is an equivariant diffeomorphism. In other words, there is a group isomorphism $h : \Gamma_i \to \Gamma'_i$ so that

$$\widetilde{\phi}_i(\xi \cdot (z_1, z_2)) = h(\xi) \cdot \widetilde{\phi}_i(z_1, z_2),$$

for $\xi \in \Gamma_i$ and $(z_1, z_2) \in \widetilde{U}_i$.

Remark 6.1.7. Using the above definition, one can show that given a smooth orbifold M with orbifold points y_i for $i = 1, \dots N$ of orders r_i types (p_i, q_i) respectively where $\gcd(p_i, r_i) = \gcd(q_i, r_i) = 1$ is diffeomorphic to a smooth orbifold still called M where we can choose each y_i to have type $(1, c_i)$ instead. Specifically, if we let $c_i = -q_i\alpha_i$ where $\beta_i r_i - \alpha_i p_i = 1$, then letting ϕ be the identity gives such a diffeomorphism.

To see this, let $(U_i, \widetilde{U}_i, \mathbb{Z}_r, \Gamma)$ and $(U_i, \widetilde{U}_i, \mathbb{Z}_r, \Gamma')$ be local uniformizing charts for y_i, where Γ denotes the diagonal action of $\mathbb{Z}_r$ with weights $1, -q_i\alpha_i$ and Γ' denotes the action with weights p_i, q_i. Let $h_i : \mathbb{Z}_r \to \mathbb{Z}_r$ be given by $h(\xi) = \xi^{-\alpha}$. Then we have

$$\Gamma(\xi)(z_1, z_2) = (\xi z_1, \xi^{-q_i\alpha} z_2) = (\xi^{-p_i\alpha} z_1, \xi^{-q_i\alpha} z_2) = \Gamma'(\xi^{-\alpha})(z_1, z_2) = \Gamma'(h_i(\xi))(z_1, z_2),$$

where the second equality follows since $-p_i\alpha_i \equiv 1 \mod r$. This computation shows that we can always assume our orbifold singularities are of type $(1, c)$ for some c.

We finish this section by proving a lemma which discusses the extent to which the type of an order r orbifold singularity is preserved under such a diffeomorphism

Theorem 6.1.8. *Let M, M' be orbifolds, $\phi : M \to M'$ a diffeomorphism, and y, y' orbifold singularities of M, M' respectively so $\phi(y) = y'$. In particular, both y and y' have the same order, which we call r. Then if y is of type $(1, c)$, we must have y' of type $(1, c')$ where $\gcd(c, r) = \gcd(c', r) = 1$ and either $c' \equiv \pm c \mod r$ or $cc' \equiv \pm 1 \mod r$. Furthermore, if ϕ is orientation preserving, $c' \equiv c \mod r$ or $cc' \equiv 1 \mod r$*

Proof. As above, if we have such a diffeomorphism $\phi : M \to M'$ so $\phi(y) = y'$, then there are local uniformizing charts $(U, \widetilde{U}, \Gamma, \psi)$ and $(U', \widetilde{U}', \Gamma', \psi')$ at y, y' respectively so that $\phi(U) = U'$ and furthermore, there is an isomorphism $h : \Gamma \to \Gamma'$ and a lift $\widetilde{\phi}$ of ϕ so that

$$\widetilde{\phi}(\xi \cdot (z_1, z_2)) = h(\xi) \cdot \widetilde{\phi}(z_1, z_2),$$

for $\xi \in \Gamma$ and $(z_1, z_2) \in \widetilde{U}$ where by assumption Γ and Γ' are copies of $\mathbb{Z}_r$ acting diagonally with weights $(1, c)$ and $(1, c')$.

Consider the derivative of $\widetilde{\phi}$ at the origin. We can identify $T_0\widetilde{U}$ and $T_0\widetilde{U}'$ with copies of $\mathbb{C}^2$ in the standard way. Furthermore, the actions Γ and Γ' on $\widetilde{U}$ and $\widetilde{U}'$ give corresponding actions on $T_0\widetilde{U}$ and $T_0\widetilde{U'}$ under the identification to $\mathbb{C}^2$. In particular, we get a real linear diffeomorphism

$$\psi := d\widetilde{\phi} : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$$

so that

$$\psi(\xi \cdot (z_1, z_2)) = h(\xi) \cdot \psi(z_1, z_2),$$

for $\xi \in \Gamma$ and $(z_1, z_2) \in \mathbb{C}^2$. Additionally, ψ is orientation preserving if and only if $\widetilde{\phi}$ was orientation preserving. Now let $\mathbb{C}^2 = \mathbb{R}^4$, and let A, A' denote the linear symplectomorphisms $\xi\cdot$ and $h(\xi)\cdot$ determined by the actions of Γ and Γ', where ξ and $h(\xi)$ are generators of Γ and Γ'. Then A and A' have the matrices

$$A = \begin{pmatrix} R(\theta) & 0 \\ 0 & R(c\theta) \end{pmatrix} \quad A' = \begin{pmatrix} R(k\theta) & 0 \\ 0 & R(kc'\theta) \end{pmatrix},$$

where $\theta = \frac{2\pi}{r}$ and for any real number α, $R(\alpha)$ denotes the 2×2 rotation matrix with angle α. Thus, by the above we have a commutative square of real linear maps

$$\begin{array}{ccc} \mathbb{R}^4 & \xrightarrow{\psi} & \mathbb{R}^4 \\ {\scriptstyle A}\downarrow & & \downarrow{\scriptstyle A'} \\ \mathbb{R}^4 & \xrightarrow[\psi]{} & \mathbb{R}^4. \end{array}$$

Tensoring with $\mathbb{C}$ gives a corresponding commutative square of complex linear maps

$$\begin{array}{ccc} \mathbb{C}^4 & \xrightarrow{\psi_{\mathbb{C}}} & \mathbb{C}^4 \\ {\scriptstyle A_{\mathbb{C}}}\downarrow & & \downarrow{\scriptstyle A'_{\mathbb{C}}.} \\ \mathbb{C}^4 & \xrightarrow[\psi_{\mathbb{C}}]{} & \mathbb{C}^4. \end{array}$$

A simple computation then shows that the complex linear transformation $A_{\mathbb{C}}$ has eigenvalues $\pm\theta$ and $\pm c\theta$ with corresponding eigenvectors $v_\lambda^{\mathbb{C}}$. Multiplying $v_\lambda^{\mathbb{C}}$ by a complex number if necessary, the corresponding real vectors v_λ formed by taking the real part of $v_\lambda^{\mathbb{C}}$ will form a basis of $\mathbb{R}^4$. We denote v_i the basis of $\mathbb{R}^4$ corresponding to the ordering v_θ, $v_{-\theta}$, $v_{c\theta}$, $v_{-c\theta}$. Similarly, $A'_{\mathbb{C}}$ has eigenvalues $\pm k\theta$ and $\pm kc'\theta$ with corresponding eigenvectors $w_\lambda^{\mathbb{C}}$. As before, taking real parts gives us a corresponding basis of $\mathbb{R}^4$ denoted w_i corresponding to the ordering $w_{k\theta}$, $w_{-k\theta}$, $w_{kc'\theta}$, and $w_{-kc'_\theta}$.

Since $\psi_{\mathbb{C}}$ is complex linear and fits into our commutative square, $\psi_{\mathbb{C}}$ must preserve eigenvectors and eigenvalues. In particular, $\psi_{\mathbb{C}}(v_\theta^{\mathbb{C}}) = w_\lambda^{\mathbb{C}}$ for some $\lambda = \pm k\theta, \pm kc'\theta$. Thus, we must have that one of $k\theta$, $-k\theta$, $kc'\theta$, or $-kc'\theta$ equals θ mod 2π. However, $r\theta = 2\pi$, so if $\theta \equiv \pm k\theta \mod 2\pi$, then $k \equiv \pm 1 \mod r$ and thus, we must also have $c \equiv \pm c' \mod r$ from the other eigenvalues. Correspondingly, if θ equals $\pm kc'\theta$, then we must have $kc' \equiv \pm 1 \mod r$ and we must also have $c \equiv \pm k \mod r$ from the other eigenvalues. Thus, the only possibilities are $c' \equiv \pm c \mod r$ or $cc' \equiv \pm 1 \mod r$, as desired. It remains to show that if ϕ is orientation preserving, we have $c' \equiv c \mod r$ or $cc' \equiv 1 \mod r$.

To see this, notice that, since we know $\psi_{\mathbb{C}}$ preserves eigenvectors and eigenvalues, up to rescaling there is only a finite number of possibilities for $\psi_{\mathbb{C}}$. Namely, $\theta = \pm k\theta, \pm kc'\theta$ gives 4 choices, and for each choice, there is a corresponding choice of sign in what happens to the eigenvalues $\pm c\theta$. Thus, there are 8 total possibilities for the complex linear map ψ. An easy computation shows that exactly 4 of the choices for $\psi_{\mathbb{C}}$ correspond to an orientation preserving ψ on $\mathbb{R}^4$, where two of them correspond to $c' \equiv c \mod r$ and the other two correspond to $cc' \equiv 1 \mod r$. □

6.2 Resolutions and Almost Complex Structures

In this section, we will discuss resolutions of orbifold singularities. We begin by giving a nice reinterpretation of a local symplectic orbifold in terms of symplectic reduction.

Lemma 6.2.1. *Consider the symplectic manifold $\mathbb{C}^3$ with its standard symplectic structure and consider the standard diagonal circle action with weights $(p, q, -r)$ on $\mathbb{C}^3$ given by*

$$\lambda \cdot (z_1, z_2, z_3) = (\lambda^p z_1, \lambda^q z_2, \lambda^{-r} z_3),$$

where $\lambda = e^{2\pi i\lambda}$. Notice that this action has a Hamiltonian given by $H = p|z_1|^2 + q|z_2|^2 - r|z_3|^2$

and define $\overline{\mathbb{C}}^3_\lambda$ to be $H^{-1}(\lambda)/S^1$. Then $\overline{\mathbb{C}}^3_0$ is a symplectic orbifold with an orbifold singularity of order r and type (p,q) at the origin.

Proof. $H^{-1}(0)$ consists of all points (z_1, z_2, z_3) so that $p|z_1|^2 + q|z_2|^2 - r|z_3|^2 = 0$. In particular, $|z_3|^2 = \frac{p}{r}|z_1|^2 + \frac{q}{r}|z_2|^2$. Thus, there is a natural, embedding of $\mathbb{C}^2$ into $H^{-1}(0)$ given by

$$(z_1, z_2) \mapsto \left(z_1, z_2, \sqrt{\tfrac{p}{r}|z_1|^2 + \tfrac{q}{r}|z_2|^2}\right)$$

which is smooth away from $(0,0)$. Furthermore, for any (z_1, z_2, z_3) with $H(z_1, z_2, z_3) = 0$, there is a $\lambda \in S^1$ so that $z_3\lambda^{-r} = \sqrt{\frac{p}{r}|z_1|^2 + \frac{q}{r}|z_2|^2}$.

Thus, we can identify $H^{-1}(0)/S^1$ with the set $(z_1, z_2, \sqrt{\frac{p}{r}|z_1|^2 + \frac{q}{r}|z_2|^2})/\sim$, where

$$\left(z_1, z_2, \sqrt{\tfrac{p}{r}|z_1|^2 + \tfrac{q}{r}|z_2|^2}\right) \sim \left(w_1, w_2, \sqrt{\tfrac{p}{r}|z_1|^2 + \tfrac{q}{r}|z_2|^2}\right) \iff$$
$$\exists \lambda \in S^1 : \left(w_1, w_2, \sqrt{\tfrac{p}{r}|z_1|^2 + \tfrac{q}{r}|z_2|^2}\right) = \lambda \cdot \left(z_1, z_2, \sqrt{\tfrac{p}{r}|z_1|^2 + \tfrac{q}{r}|z_2|^2}\right).$$

However, this can only occur if $\lambda^{-r}(\frac{p}{r}|z_1|^2 + \frac{q}{r}|z_2|^2) = \frac{p}{r}|z_1|^2 + \frac{q}{r}|z_2|^2$, which in turn implies $\lambda^{-r} = 1$, so that $\lambda \in \mathbb{Z}_r \subset S^1$ generated by $\xi = e^{\frac{2\pi i}{r}}$.

Thus, using our embedding of $\mathbb{C}^2$ into $H^{-1}(0)$, we can identify $H^{-1}(0)/S^1$ with $\mathbb{C}^2/\mathbb{Z}_r$, where $\mathbb{Z}_r$ acts by $\xi \cdot (z_1, z_2) = (\xi^p z_1, \xi^q z_2)$, as desired. □

We now use this to show that any isolated orbifold singularity appearing in one of our reduced spaces M_λ has a local toric structure.

Proposition 6.2.2. *Consider the symplectic manifold $\mathbb{C}^3$ with its standard symplectic structure and consider the standard diagonal circle action with weights $(p, q, -r)$ on $\mathbb{C}^3$ given by*

$$\lambda \cdot (z_1, z_2, z_3) = (\lambda^p z_1, \lambda^q z_2, \lambda^{-r} z_3)$$

where $\lambda = e^{2\pi i\theta}$ for some angle θ, and let $\overline{\mathbb{C}}^3_\lambda$ be as before. Then $\overline{\mathbb{C}}^3_0$ has a toric structure given by a torus action $\overline{T}$ whose moment polytope is the wedge with outward conormals $(0,-1)$ and $(-r, -q\alpha)$, where $-\alpha p + \beta r = 1$ with $\alpha, \beta > 0$. Furthermore, $\overline{\mathbb{C}}^3_\epsilon$ with $\epsilon > 0$ has a toric structure given by $\overline{T}$ whose moment polytope is the wedge with outward conormals $(0,-1)$, $(-p, -q\beta)$, and $(-r, -q\alpha)$ where α and β are as before. In particular, $\overline{\mathbb{C}}^3_\epsilon$ has 1 orbifold singularity of type p and 1 orbifold singularity of type q.

Proof. $\overline{\mathbb{C}}^3_\epsilon$ for all $\epsilon \geq 0$ inherits a torus action $\overline{T}$ by taking the standard torus action T on $\mathbb{C}^3$ and quotienting by the diagonal S^1 action with weights $(p, q, -r)$. The moment polytope of this toric structure on $\overline{\mathbb{C}}^3_\epsilon$ has an embedding into $\mathfrak{t}^*$ by taking the moment polytope of the standard action, $\mathbb{R}^3_{\geq 0} \subset \mathfrak{t}^* \cong \mathbb{R}^3$ and restricting to the plane $px + qy - rz = \epsilon$. We will call this plane $\mathcal{H}^*(\epsilon)$, and we will denote its integer lattice in $\mathfrak{t}^*$ by $\mathcal{H}^*_{\mathbb{Z}}(\epsilon)$. This polytope is the piece of $\mathcal{H}^*(\epsilon)$ which has $x, y, z \geq 0$.

If $\epsilon = 0$, we have $\mathcal{H}^*(0)$ is the plane $px + qy - rz = 0$. If $z = 0$, we have $px + qy = 0$ which means $x = y = 0$. If $y = 0$, we have $px = rz$ which gives the ray starting at $(0, 0, 0)$ with direction $(r, 0, p)$. If $x = 0$, we have $qy = rz$ which gives the ray starting at $(0, 0, 0)$ with direction $(0, r, q)$. Our polytope is then clearly given by the wedge between $(r, 0, p)$ and $(0, r, q)$.

If $\epsilon > 0$, we have $px + qy = rz + \epsilon$. If $z = 0$, we have $px + qy = \epsilon$ which gives the line segment in the direction $(-q, p, 0)$ between $(\frac{\epsilon}{p}, 0, 0)$ and $(0, \frac{\epsilon}{q}, 0)$. If $y = 0$, we have $px = rz + \epsilon$ which gives the ray in the direction $(r, 0, p)$ starting at $(\frac{\epsilon}{p}, 0, 0)$. If $x = 0$, we have $qy = rz + \epsilon$ which gives the ray in the direction $(0, r, q)$ starting at $(0, \frac{\epsilon}{q}, 0)$. We denote this section of $\mathcal{H}^*(\epsilon)$ by $\Delta(\epsilon)$.

Furthermore, the moment polytope of the action of $\overline{T}$ also has an embedding into $\bar{\mathfrak{t}}^* \cong \mathbb{R}^2$. We similarly denote by $\bar{\mathfrak{t}}^*_{\mathbb{Z}}$ the integer lattice of this algebra.

We seek to produce an embedding of $\bar{\mathfrak{t}}^*$ into $\mathcal{H}^*(0)$ so that the wedge between $(1, 0)$ and $(-q\alpha, r)$ maps to the wedge between $(r, 0, p)$ and $(0, r, q)$ and furthermore so that the induced map from $\bar{\mathfrak{t}}^*_{\mathbb{Z}}$ to $\mathcal{H}^*_{\mathbb{Z}}(0)$ is an element of $\mathrm{GL}(2, \mathbb{Z})$ plus a translation. Similarly, we want an embedding of $\bar{\mathfrak{t}}^*$ into $\mathcal{H}^*(\epsilon)$ so that the wedge between $(1, 0)$ and $(-q\alpha, r)$ cut by the direction $(q\beta, -p)$ maps to $\Delta(\epsilon)$ and so that the induced map from $\bar{\mathfrak{t}}^*_{\mathbb{Z}}$ to $\mathcal{H}^*_{\mathbb{Z}}(\epsilon)$ is an element of $\mathrm{GL}(2, \mathbb{Z})$ plus a translation.

We claim that producing such embeddings would complete the proof. Indeed, the torus $\overline{T}$ is determined both as $\bar{\mathfrak{t}}/\bar{\mathfrak{t}}_{\mathbb{Z}}$ and as $\mathcal{H}(\epsilon)/\mathcal{H}_{\mathbb{Z}}(\epsilon)$. Thus, dualizing the embedding would give an embedding from $\mathcal{H}(\epsilon)$ into $\bar{\mathfrak{t}}$ so that $\mathcal{H}_{\mathbb{Z}}(\epsilon)$ maps by an element of $\mathrm{GL}(2, \mathbb{Z})$ plus a translation onto $\bar{\mathfrak{t}}_{\mathbb{Z}}$. In particular, this shows that the same torus action $\overline{T}$ is inducing these two moment polytopes, which then gives the desired result.

To produce such an embedding, we will complete $(p, q, -r)$ to an integer basis. Since $\gcd(p, r) =$

1, there exist integers α and β so that $-\alpha p + \beta r = 1$. Then we have

$$\det \begin{pmatrix} p & q & -r \\ 0 & 1 & 0 \\ \beta & 0 & -\alpha \end{pmatrix} = -\alpha p + \beta r = 1.$$

In particular, $(p, q, -r)$, $(0, 1, 0)$, and $(\beta, 0, -\alpha)$ is an integer basis of $\mathbb{Z}^3$. Using this, we can give a basis of $\mathcal{H}(0)$ by giving vectors e_1 and e_2 so that $e_1 \cdot (0, 1, 0) = e_1 \cdot (p, q, -r) = 0$ and $e_2 \cdot (p, q, -r) = e_2 \cdot (\beta, 0, -\alpha) = 0$. We choose $e_1 = (r, 0, p)$ and $e_2 = (q\alpha, 1, q\beta)$. Using this basis, we define a linear embedding Φ_0 from $\mathbb{R}^2$ to $\mathcal{H}^*(0)$ as follows:

$$\Phi_0(a, b) = ae_1 + be_2 = (ar + bq\alpha, b, ap + bq\beta).$$

By construction, Φ_0 is an element of $\mathrm{GL}(2, \mathbb{Z})$. Now notice that $\Phi_0(1, 0) = (r, 0, p)$, while

$$\Phi_0(-q\alpha, r) = (-rq\alpha + rq\alpha, r, -pq\alpha + rq\beta) = (0, r, q(-p\alpha + r\beta)) = (0, r, q).$$

Therefore, since Φ_0 is linear, the wedge between $(1, 0)$ and $(-q\alpha, r)$ maps to the wedge between $(r, 0, p)$ and $(0, r, q)$. Thus, M has a local toric structure given by the torus action $\overline{T}$ whose moment polytope is given by the wedge in $\mathbb{R}^2$ with conormals $(0, -1)$ and $(-r, -q\alpha)$, as desired.

Also, notice that $\mathcal{H}^*(\epsilon)$ can be formed from $\mathcal{H}^*(0)$ by the translation

$$\tau_\epsilon(x, y, z) = (x + \tfrac{\epsilon}{p}, y, z).$$

Thus, we can form an affine embedding Φ_ϵ from $\mathbb{R}^2$ to $\mathcal{H}^*(\epsilon)$ as $\tau_\epsilon \circ \Phi_0$ to get:

$$\Phi_\epsilon(a, b) = ae_1 + be_2 + (\tfrac{\epsilon}{p}, 0, 0) = (ar + bq\alpha + \tfrac{\epsilon}{p}, b, ap + bq\beta).$$

By construction, Φ_ϵ is an element of $\mathrm{GL}(2, \mathbb{Z})$ plus a translation. Also, as defined, we have

$$\Phi_\epsilon((a, b) + (c, d)) = \Phi_\epsilon(a, b) + \Phi_0(c, d).$$

Furthermore, $\Phi_\epsilon(0, 0) = (\frac{\epsilon}{p}, 0, 0)$ and

$$\begin{aligned} \Phi_\epsilon(-\tfrac{\epsilon\beta}{p}, \tfrac{\epsilon}{q}) &= (-\tfrac{\epsilon}{p}\beta r + \epsilon\alpha + \tfrac{\epsilon}{p}, \tfrac{\epsilon}{q}, -\epsilon\beta + \epsilon\beta) \\ &= (-\tfrac{\epsilon}{p}(\beta r - \alpha p) + \tfrac{\epsilon}{p}, \tfrac{\epsilon}{q}, 0) \\ &= (-\tfrac{\epsilon}{p} + \tfrac{\epsilon}{p}, \tfrac{\epsilon}{q}, 0) = (0, \tfrac{\epsilon}{q}, 0). \end{aligned}$$

Lastly, we see that

$$\Phi_0(-q\beta, p) = (-q\beta r + pq\alpha, p, -q\beta p + pq\beta) = (-q(-p\alpha + r\beta), p, 0) = (-q, p, 0).$$

Combining all this, we clearly see that the polytope with conormals $(0, -1)$, $(-p, -q\beta)$ and $(-r, -q\alpha)$ maps to $\Delta(\epsilon)$, as desired.

□

Next, we will prove a few lemmas relating the local toric structure of an orbifold singularity to the type of the orbifold singularity

Lemma 6.2.3. *Let $p, q, r > 0$ be positive integers so that $\gcd(p, r) = \gcd(q, r) = 1$ and let $\alpha > 0$ satisfy $\beta r - \alpha p = 1$. Consider the wedge in $\mathbb{R}^2$ determined by the outward conormals $(0, -1)$ and $(-r, -q\alpha)$. Then the corresponding symplectic toric orbifold has a unique orbifold point which can be chosen to be of type $(1, -q\alpha)$.*

Proof. Combining Lemma 6.2.5 and Proposition 6.2.2, we know that our given moment polytope arises as the moment polytope of a local toric structure near an orbifold singularity of order r and type (p, q). Hence, for geometric reasons, we know that this wedge corresponds to a symplectic toric orbifold with one orbifold singularity of order r and type (p, q). But then, as in Remark 6.1.7, we can instead choose the singularity to have type $(1, -q\alpha)$ where $\beta r - p\alpha = 1$. □

Lemma 6.2.4. *Consider the symplectic manifold $\mathbb{C}^3$ with its standard symplectic structure and consider the diagonal circle action with weights $(p, q, -r)$ on $\mathbb{C}^3$ given by*

$$\lambda \cdot (z_1, z_2, z_3) = (\lambda^p z_1, \lambda^q z_2, \lambda^{-r} z_3)$$

where $\lambda = e^{2\pi i\lambda}$ and $\gcd(p, r) = \gcd(q, r) = 1$, with $\beta r - \alpha p = 1$ for some $\alpha, \beta > 0$. Recall from Proposition 6.2.2 that for $\epsilon > 0$, $\overline{\mathbb{C}}^3_\epsilon$ has an orbifold singularity of order p and an orbifold singularity of order q, denoted y_p and y_q respectively. Then y_p is of type $(1, c_p)$ and y_q is of type $(1, c_q)$, where $c_p \equiv -q\beta \mod p$ and $-pc_q \equiv r \mod q$.

Proof. As in Proposition 6.2.2, $\mathbb{C}^3_\epsilon$ has a toric structure with outward conormals given by $(0, -1)$, $(-p, -q\beta)$, and $(-r, -q\alpha)$. In particular, the order p singularity is determined by the polytope with outward conormals $(0, -1)$ and $(-p, -q\beta)$ while the order q singularity is determined by the polytope with outward conormals $(-p, -q\beta)$ and $(-r, -q\alpha)$. In particular, Lemma 6.2.3 above implies that the order p singularity can be chosen to have type $(1, -q\beta)$, as desired. Furthermore, if we find an integer matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

so that $\det(A) = 1$, $A(-p, -q\beta) = (0, -1)$, and $A(-r, -q\alpha) = (-q, -x)$, then Lemma 6.2.3 above would imply that the type of the order q singularity would be $(1, -x)$. Thus, to complete the proof of the lemma, we only need to show that $-pc_q = px \equiv r \mod q$. To see this, notice that $A(-r, -q\alpha) = (-q, -x)$ implies that $cr + dq\alpha = x$, while $A(-p, -q\beta) = -1$ implies that $cp + dq\beta = 1$. Combining these, we have

$$px = cpr + dqp\alpha = r - rdq\beta + dq\alpha = r + q(d\alpha - rd\beta)$$

so that $px \equiv r \mod q$, as desired. This shows that the order q singularity has type $(1, c_q)$ where $-pc_q \equiv r \mod q$, which completes the proof of the lemma. □

We can also use Lemma 6.2.2 to give a local toric structure to one of our orbifold singularities.

Corollary 6.2.5. *Let M^4 be a symplectic orbifold with orbifold singularity y of order r and type (p, q). Then a neighborhood of y has a toric structure with moment polytope determined by the outward conormals $(0, 1)$ and $(r, -k)$ where $-p\alpha + r\beta = 1$, $k \equiv -q\alpha \mod r$ and $1 \leq k < r$.*

Proof. As in Lemma 6.2.1, a neighborhood of such an orbifold singularity y can be obtained as the reduced space $\overline{\mathbb{C}}^3_0$ at level 0 of the diagonal S^1 action with weights $(p, q, -r)$ on $\mathbb{C}^3$. Furthermore, Lemma 6.2.2 says that this has a toric structure with moment polytope determined by the conormals $(0, -1)$ and $(-r, q\alpha)$. Consider the following transformation

$$A = \begin{pmatrix} -1 & 0 \\ c & -1 \end{pmatrix}.$$

Then $A \cdot (0, -1) = (0, 1)$, and $A \cdot (-r, -q\alpha) = (r, q\alpha - rc)$. There is a unique choice of c so this equals $(r, -k)$ where $k \equiv -q\alpha \mod r$ and $1 \leq k < r$. This completes the proof. □

Remark 6.2.6. We can use the above theorem and the techniques of Fulton in [6] to resolve these singularities as follows. In [6], Fulton shows that a resolution of the polytope with outer conormals

$(0,1)$ and $(m,-k)$ with $0<k<m$ is given by a string of integers a_i so that

$$\frac{m}{k} = a_1 - \frac{1}{a_2 - \frac{1}{\dots - \frac{1}{a_n}}},$$

Then there is a resolution of this singularity by a series of blowups which produces a chain of classes Z_i so that

$$Z_i \cdot Z_j = \begin{cases} -a_i \text{ if } i = j \\ 1 \text{ if } |i-j| = 1 \\ 0 \text{ else} \end{cases}$$

Furthermore, Fulton shows there is a unique choice of the a_i so that $a_i \geq 2$ for all i. Hence, using the above theorem, we can apply these techniques with $m = r$ and $k \equiv -q\alpha \mod r$ with $0 < k < r$ as above to get a resolution of any isolated orbifold singularity of order r and type (p,q).

We can use the above to give the following definition.

Definition 6.2.7. *Let M^4 be an orbifold with singularities as in Definition 6.1.4. Then M has a finite set of isolated orbifold singularities, $y_1,\dots,y_N$. As in Remark 6.2.6 above, we can get a symplectic manifold $\widetilde{M}$, called the* ***resolution*** *of M by resolving each of these singularities separately.*

Remark 6.2.8. Using the above techniques, it is easy to see when two isolated singularities y, y' of orders r and types (p,q) and (p',q') respectively have the same resolution. In particular, Remark 6.1.7 implies that we can assume y, y' are of types $(1,-q\alpha)$ and $(1,-q'\alpha')$ respectively, where $\beta r - \alpha p = 1$ and $\beta' r - \alpha' p' = 1$. Now, if x is resolved as above by the integer string a_i, $i = 1,\dots,n$, then x' will have the same resolution only if x' is resolved by the same string a_i, or by the reversed string $\overline{a}_i$, where $\overline{a}_i = a_{n+1-i}$, $i = 1,\dots,n$. However, a simple induction shows that if

$$\frac{m}{k} = a_1 - \frac{1}{a_2 - \frac{1}{\dots - \frac{1}{a_n}}},$$

then

$$a_n - \frac{1}{a_{n-1} - \frac{1}{\dots - \frac{1}{a_1}}} = \frac{m}{k'},$$

where $kk' \equiv 1 \mod m$. Thus, y and y' will have the same resolution if and only if either $-q\alpha \equiv -q'\alpha' \mod r$ or $(-q\alpha)(-q'\alpha') \equiv 1 \mod r$, where $-\alpha p + \beta r = 1$ and $-\alpha' p' + \beta' r' = 1$.

Combining the above remark with Lemma 6.1.8 gives us the following useful lemma.

Lemma 6.2.9. *Let (X, ω) be a closed 6-dimensional manifold with an effective, symplectic S^1 action with no codimension 2 isotropy and only good fixed points, as in Definition 6.3.9, and consider the family M_λ of reduced spaces of this action. Let ϕ be any orientation preserving diffeomorphism*

$$\phi : M_\lambda \longrightarrow M_{\lambda'}.$$

Then ϕ lifts to a diffeomorphism

$$\widetilde{\phi} : \widetilde{M}_\lambda \longrightarrow \widetilde{M}_{\lambda'}.$$

Proof. First, notice that since we are assuming there is no codimension 2 isotropy we know that the reduced spaces M_λ will have only finitely many isolatedorbifold points. Furthermore, the assumption that the fixed points are good implies that if we have an orbifold singularity of order r and type (p, q), then we must have $\gcd(p, r) = \gcd(q, r) = 1$. In particular, the reduced spaces M_λ are orbifolds which only have orbifold singularities as in Definition 6.1.4. The proof of the lemma is then an immediate consequence of Lemma 6.1.8 and Remark 6.2.8. □

In the above discussion, we showed that given a symplectic orbifold (M^4, ω) with a finite number of orbifold singularities, there is a corresponding symplectic manifold $(\widetilde{M}^4, \widetilde{\omega})$ which is obtained from M by successive blowups near the singularities. Moreover, this implies that in $\widetilde{M}$, there are some homology classes with self intersection ≤ -2 which are represented by symplectically embedded spheres. We finish this section by discussing which almost complex structures on $\widetilde{M}$ can be blown down to almost complex structures on M. This discussion is largely based on [12]

More specifically, if M^4 has the singularities $y_1, \ldots, y_N$, there are classes $Z_{i,A}$ in $\widetilde{M}$ which are all represented by symplectically embedded spheres $C_{i,A}$ and which satisfy

$$Z_{i,A} \cdot Z_{k,B} = \begin{cases} -a_{i,A} \leq -2 & \text{if } i = k,\ A = B \\ 1 & \text{if } i = k,\ |A - B| = 1 \\ 0 & \text{else.} \end{cases}$$

Moreover, near a singularity y_i, we can blow up any almost complex structure J which is integrable near y_i to get an almost complex structure $\widetilde{J}$ defined in a neighborhood of $\cup_A C_{i,A}$ which by definition can be blown down to J in a neighborhood of y_i.

With the above in mind, we can give the following definition.

Definition 6.2.10. *Let $(\widetilde{M}^4, \widetilde{\omega})$ be the resolution of a symplectic orbifold with singularities y_i and a corresponding set $\mathcal{Z} = \{Z_{i,A}\}$ of homology classes that are represented by symplectically embedded spheres satisfying*

$$Z_{i,A} \cdot Z_{k,B} = \begin{cases} -a_{i,A} \leq -2 & \text{if } i = k,\ A = B \\ 1 & \text{if } i = k,\ |A - B| = 1 \\ 0 & \text{else.} \end{cases}$$

Then we define $\widetilde{\mathcal{J}}(\mathcal{Z}) := \mathcal{J}(\mathcal{Z}, \widetilde{\omega})$ to be the space of all $\widetilde{\omega}$-tame almost complex structures which arise as the blowup of an almost complex structure J which is integrable near each y_i.

Remark 6.2.11. The set $\widetilde{\mathcal{J}}(\mathcal{Z})$ is defined to be isomorphic to the set $\mathcal{J}(y_1, \dots, y_N; \omega)$ of ω-tame almost complex structures on M which are integrable near y_i. Namely, each $\widetilde{J} \in \widetilde{\mathcal{J}}(\mathcal{Z})$ corresponds to a unique almost complex structure $J \in \mathcal{J}(y_1, \dots, y_N; \omega)$ in the sense that J blows up to $\widetilde{J}$.

We finish this section by giving the definition of a J-holomorphic orbisphere

Definition 6.2.12. *Let (M, ω) be a symplectic orbifold with orbifold points $y_1, \dots, y_N$, let $J \in \mathcal{J}(y_1, \dots, y_N; \omega)$ and let $u : S^2 \to M$ be a map so that the points $x_1, \dots, x_m$ in S^2 satisfy $u(x_i)$ is an orbifold point of M so so that the x_i are the only points on S^2 which map to orbifold points. Then we say that u is a **J-holomorphic orbisphere** if the following conditions are satisfied:*

1. *$u : S^2 \setminus x_1, \dots, x_m \to M$ is a smooth, J-holomorphic embedded sphere.*

2. *For each x_i, there is a neighborhood $\mathcal{N}(x_i)$ and a local uniformizing chart $(U_i, \widetilde{U}_i, \Gamma_i, \phi_i)$ of $u(x_i)$ so that $u|\mathcal{N}(x_i)$ lifts to a map*

$$\widetilde{u} : \mathcal{N}(x_i) \longrightarrow \widetilde{U}_i,$$

so that $\phi_i \circ \widetilde{u} = u$, where we can choose the local uniformizing chart so that $\widetilde{u}(z) = z^{r_i}$ where r_i is the order of the singularity $u(x_i)$.

6.3 Weighted Blowups and Blowdowns

We will next discuss ellipsoid blowups. We will let $E(q,p) = \{(z_1, z_2) | \frac{|z_1|^2}{q} + \frac{|z_2|^2}{p} \leq 1\}$, where $p > q$ and $\gcd(p,q) = 1$.

Definition 6.3.1. *Let (M^4, ω) be a symplectic manifold, and let y be a point, and let p, q be integers with $\gcd(p,q) = 1$. Then the (p,q)* ***weighted blowup of size*** *ϵ* ***at*** *y, denoted $(\widetilde{M}, \widetilde{\omega})$, is given by removing*

$$\mathrm{Int}(\tfrac{\epsilon}{pq}E(q,p)) = \{(z_1,z_2) : \tfrac{|z_1|^2}{q} + \tfrac{|z_2|^2}{p} < \tfrac{\epsilon}{pq}\} = \{(z_1,z_2) : p|z_1|^2 + q|z_2|^2 < \epsilon\}$$

*and collapsing the resulting ellipsoid boundary along the characteristic flow to produce a curve C^E in the class E, called the (p,q)-**weighted divisor**. The form $\widetilde{\omega}$ can be chosen to be ω outside of $\frac{\epsilon}{pq}E(q,p)$ and to satisfy*

$$\int_{C^E} \widetilde{\omega} = \epsilon.$$

In general, this procedure will not result in a symplectic manifold, but rather in a symplectic orbifold which has two singularities, one of which has order p, and the other of which has order q. Furthermore, the (p,q)-weighted divisor E will intersect both of these singularities. To see an example of this, we can look at $\mathrm{Int}(E(q,p))$ in the toric picture in the case where y is a smooth point. Under the standard torus action of $\mathbb{C}^2$, $E(q,p)$ has the moment polytope $\Delta(q,p)$ given by a triangle determined by the conormals $(-1,0)$, $(0,-1)$, and $(-p,-q)$, which can be transformed to the triangle determined by $(-1,0)$, $(0,-1)$, and $(-q,-p)$. This obviously has a smooth vertex at $(0,0)$. Thus, we can give a neighborhood U of $\frac{\epsilon}{pq}E(q,p)$ a toric structure so that $\frac{\epsilon}{pq}E(q,p)$ maps to the corresponding rescaled triangle and the blowup corresponds to cutting out this triangle. In the polytope, this removes the smooth vertex $(0,0)$ which corresponded to the point y and replaces it with vertices $(p,0)$ and $(0,q)$ which represent orbifold singularities of orders q and p respectively.

Remark 6.3.2. In the case where y is a smooth point, we can resolve these two singularities using the techniques of Fulton, as in Remark 6.2.6. The result of this procedure is two families of classes, denoted Z_i^p and Z_i^q, each corresponding to resolving one of the singularities.

We cannot directly apply the techniques in [6] for either vertex, but up to some affine transformations we can apply the techniques. At the vertex $(0,q)$, we have the conormals $(-1,0)$ and $(-q,-p)$, which map to the conormals $(0,1)$ and $(p,-(p-q))$ under the transformation

$$\begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}$$

Hence, as in Remark 6.2.6, we get a chain of spheres Z_i^p where $(Z_i^p)^2 = -a_i$ where

$$\frac{p}{p-q} = a_1 - \frac{1}{a_2 - \frac{1}{\dots - \frac{1}{a_n}}}$$

Additionally, at the vertex $(p, 0)$, we have the conormals $(0, -1)$ and $(-q, -p)$, which map to the conormals $(0, 1)$ and $(q, -k)$ under the transformation

$$\begin{pmatrix} -1 & 0 \\ c & -1 \end{pmatrix}$$

where $p - cq = -k$, so that $k \equiv -p \equiv q - p \mod q$, with $1 \leq k < q$. Again, as in Remark 6.2.6, we get a chain of spheres Z_i^q where $(Z_i^q)^2 = -b_j$ where

$$\frac{q}{k} = b_1 - \frac{1}{b_2 - \frac{1}{\dots - \frac{1}{b_n}}}$$

Defined in this way, $Z_i^p \cdot Z_j^q = 0$. Also, as we will show in the below remark if y is a smooth point and the weighted divisor is given by a curve C^E in the class E, then the proper transform $\widetilde{C}^E$ in the class $\widetilde{E}$ will be an exceptional divisor in the usual sense. Furthermore, $\widetilde{C}^E \cdot Z_1^p = 1$, $\widetilde{C}^E \cdot Z_n^q = 1$, and $\widetilde{C}^E \cdot Z_i^l = 0$ for all other choices of i, l.

Remark 6.3.3. The above procedure produces a symplectic manifold $\widetilde{M}$ which is the resolution of the (p, q)-weighted blowup of a symplectic manifold M which is obtained by a sequence of blowups, the first of which is the (p, q) weighted blowup itself. However, as McDuff shows in Section 3 of [13], if y is a smooth point the same manifold $\widetilde{M}$ can be obtained from M by a sequence of standard blowups, the last of which corresponds to the (p, q) weighted blowup. We will demonstrate the general technique by showing how this works for $E(4, 7)$.

First, we write down a sequence of numbers according to the following rule. First, we let $q_1 = q$ and write down a_1 copies of q_1, where $a_1 q_1 \leq p < (a_1 + 1)q_1$. Next, we let $q_2 = p - a_1 q_1$ and write down a_2 copies of q_2 where $a_2 q_2 \leq q_1 < (a_2 + 1)q_2$. We continue this procedure inductively until there is an integer n so that $a_n q_n = q_{n-1}$. For $E(4, 7)$, this gives us the sequence $4, 3, 1, 1, 1$. We then cut the moment polytope of $\mathbb{C}^2$ successively a_1 times down from the vertical edge, a_2 times up from the horizontal edge, a_3 times down from the last of the a_1 blowups, a_4 times up from the last a_2 blowup and so on.

In our case, this gives us the cuts $(1, 1)$, $(1, 2)$, $(2, 3)$, $(3, 5)$, and $(4, 7)$, as in Figure 1.

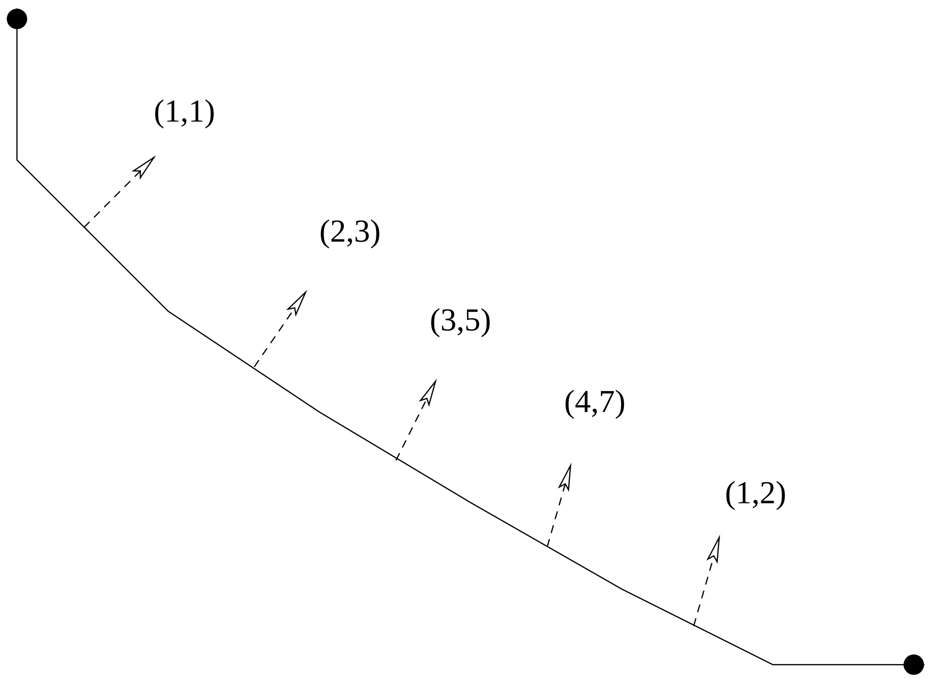

Figure 6.1: Resolution of $(4,7)$ weighted blowup

Remark 6.3.4. The above remarks deal with resolving weighted blowups of smooth points. However, as in lemma 6.4.1 below, we could certainly have (p,q)-weighted blowups of orbifold singularities. However, we will see in Section 7.2 that the assumptions of Theorem 1.3.4 guarantee that we will never have to consider the weighted blowup of an orbifold singularity in our arguments, so that the above remarks are sufficient for our case. We proceeded by noting that the weighted blowup of a smooth point y has a toric structure with moment polytope determined by the conormals $(-1,0)$, $(0,-1)$, and $(-q,-p)$. The key point now is to notice that we can interpret the resolution as arising from a series of smooth blowups of symplectic manifolds, which for example is how we prove Lemma 6.5.3 in Section 6.5.

We conclude this section by discussing weighted blowdowns of weighted exceptional divisors. To start off, we first must discuss exactly what we mean by a weighted exceptional divisor.

Definition 6.3.5. *Let (M^4,ω) be a symplectic orbifold and let y_p and y_q be orbifold singularities of order p and q and types $(1,c_p)$ and $(1,c_q)$ respectively with $1 \le c_p < p$ and $1 \le c_q < q$. We will say that a curve C^E in the class E is a (p,q)**-weighted divisor of type** r for positive integers p,q,r with $\gcd(p,q) = \gcd(p,r) = \gcd(q,r) = 1$ and $r < pq$ if the following conditions are satisfied:*

1. *C^E is a J-holomorphic orbisphere through x_p and x_q, as in Definition 6.2.12.*

2. $r \equiv -pc_q \mod q$

3. *If k is the smallest positive integer so $\frac{c_p - kp}{-q} := \beta$ is an integer and a, b are the smallest positive integers so that $b\beta - ap = 1$, then $r \equiv b \mod p$.*

Remark 6.3.6. Assume we have a symplectic orbifold (M, ω) with orbifold singularities y_p and y_q of types $(1, c_p)$ and $(1, c_q)$ respectively and assume that we have a (p, q)-weighted divisor of type r as above, denoted C^E. In particular, $\gcd(p, q) = \gcd(p, r) = \gcd(q, r) = 1$ and $r < pq$. Then since $r < pq$, we can use the Chinese Remainder Theorem to say that there is a unique r so that $r \equiv pc_q \mod q$ and $r \equiv b \mod p$. In particular, if $r < pq$, then the type of a (p, q)-weighted divisor is well defined.

Now assume we have a symplectic orbifold (M', ω') with an orbifold singularity of order r at the point y_r. Lemmas 6.4.1 and 6.2.2 together imply that if we take the (p, q)-weighted blowup of y_r, denoted $(\widehat{M}, \widehat{\omega})$, then $(\widehat{M}, \widehat{\omega})$ will have a local toric structure in a neighborhood of the corresponding weighted divisor whose moment polytope will have the conormals $(0, -1)$, $(-p, -q\beta)$, and $(-r, -q\alpha)$ where α and β are the smallest positive integers satisfying $r\beta - p\alpha = 1$, where the edge with conormal $(-p, -q\beta)$ corresponds to the weighted divisor. We know that $\widehat{M}$ has orbifold points y_p and y_q, and using the above toric model we can compute that they have types $(1, c_p)$ and $(1, c_q)$ respectively, where $c_p \equiv q\beta \mod p$ and $pc_q \equiv r \mod q$. Moreover, since $c_p \equiv q\beta \mod p$, we can compute the value of β, and then $r \equiv b \mod p$ where $b\beta - ap = 1$ for some a, b. Furthermore, if J is an almost complex structure on (M', ω') which is integrable near y_r, it can be pulled back to an almost complex structure $\widehat{J}$ which is integrable near the weighted divisor and for which the weighted divisor is a $\widehat{J}$-holomorphic orbisphere. In particular, this shows that we can choose coordinates near the weighted divisor so that it is exactly the curve C^E from before.

This shows that the definition of a weighted divisor given in Definition 6.3.5 agrees with the definition that appears in Definition 6.3.1, and moreover that any (p, q)-weighted divisor of type r can be blown down so that C^E is replaced by an order r orbifold singularity. We will call this process the (p, q)-weighted blowdown of type r. We notice that the above implies that as long as $r < pq$, the type of a (p, q)-weighted blowdown is well defined.

Definition 6.3.7. *Let (M^4, ω) be a symplectic orbifold with orbifold singularities y_p and y_q and let C^E be a (p, q)-weighted divisor of type r, as in Definition 6.3.5. Then, we will say that C^E is a (p, q)-**weighted exceptional divisor of type** r if the curve C^E in M^4 lifts to an exceptional divisor $C^{\widetilde{E}}$ in the resolution $\widetilde{M}$ of M.*

Remark 6.3.8. Given a weighted exceptional divisor C^E of type 1 as above, Remark 6.3.3 implies that we can successively blow down $\widetilde{C}^E$, Z_i^p and Z_j^q in $\widetilde{M}$ by smooth blowdowns of exceptional divisors to obtain a manifold $\widehat{M}$ which we call the (p, q) weighted blowdown of $\widetilde{M}$, or just the (p, q) weighted blowdown of M. Notice that this agrees with the above notion of the (p, q)-weighted blowdown of type 1.

We now will give a definition of a good fixed point, as seen in Theorem 1.3.4.

Definition 6.3.9. *Let (X^6, ω) be a closed symplectic manifold with a symplectic, non-Hamiltonian S^1 action, and let $x \in X$ be a fixed point of this action with isotropy weights $(\pm p, \pm q, \mp r)$ for $p, q, r > 0$. We will say that x is a good fixed point if the following are satisfied:*

1. $\gcd(p, q), \gcd(p, r), \gcd(q, r) = 1$.
2. *Any (p, q)-weighted divisor of type r is actually a (p, q)-weighted exceptional divisor of type r.*

We now give some examples of good fixed points. We start by stating a useful lemma, which was actually already proven in Remark 6.3.8, although we never stated it as a lemma.

Lemma 6.3.10. *Let $p, q, r, \alpha, \beta > 0$ be integers so that $\gcd(p, q) = \gcd(p, r) = \gcd(q, r) = 1$, $r < pq$, and $\beta r - \alpha p = 1$. Then any (p, q)-weighted divisor of type r has a local toric structure with outward conormals given by $(0, -1)$, $(-p, -q\beta)$, $(r, -q\alpha)$.*

Proof. As noted above, this was already proven in Remark 6.3.8. □

This lemma shows us the main way to figure out if a fixed point is good. Namely, assuming that the integers p, q, r satisfy the required numerical conditions, for a fixed point to be good, we only need to know that a (p, q)-weighted divisor C^E is actually a (p, q)-weighted exceptional divisor. For that, we need to compute the self intersection number of $\widetilde{C}^E$, the proper transform of C^E under the blowups forming the resolution. To compute this, we first see how we can use the above toric structure to compute the resolution.

Lemma 6.3.11. *Let $A_0, B_0 > 0$ have $\gcd(A_0, B_0) = 1$ and consider an orbifold singularity with a local toric structure with outward conormals $(0, -1)$ and $(-A_0, -B_0)$. Then the resolution of this singularity is given by blowups in the directions $(-A_i, -B_i)$, ending with $(-A_n, -B_n)$ where $A_n = 1$, and where A_i, B_i are the unique integers so that*

$$A_{i-1}B_i - B_{i-1}A_i = 1,$$

where $0 < A_i < A_{i-1}$ and $0 < B_i \leq B_{i-1}$, with $B_i = B_{i-1}$ if and only if $B_{i-1} = 1$.

Proof. By the definition of $(-A_i, -B_i)$, it is clear that the vertex between $(-A_{i-1}, -B_{i-1})$ and $(-A_i, -B_i)$ is a smooth vertex. Furthermore, as noted we also always have $A_i < A_{i-1}$ and either $B_i < B_{i-1}$ if $B_{i-1} > 1$ or $B_{i-1} = B_i = 1$. In particular, this implies that

$$\det \begin{pmatrix} A_{i-2} & B_{i-2} \\ A_i & B_i \end{pmatrix} = A_{i-2}B_i - B_{i-2}A_i > 1.$$

This follows since A_{i-1}, B_{i-1} are the smallest positive integers so that $A_{i-2}B_{i-1} - B_{i-2}A_{i-1} = 1$, whereas we have $A_i < A_{i-1}$, so that the above determinant cannot possibly be 1 and hence must be greater than 1. Thus, to see that the cuts $(-A_i, -B_i)$ form the resolution of the singularity, it remains only to be seen that some $A_n = 1$ so that the process terminates.

By the Euclidean algorithm, we clearly have that either some $A_n = 1$ or some $B_n = 1$. If $A_n = 1$, we are done, so assume $B_n = 1$ and $A_n > 1$. Then for $0 < k < A_n$,

$$(-A_{n+k}, -V_{n+k}) = (-(A_n - k), -1).$$

Thus, $(-A_{n+A_n-1}, -B_{n+A_n-1}) = (-1, -1)$ so that $A_{n+B_n-1} = 1$, as desired. □

Remark 6.3.12. Lemma 6.3.11 above gives a method for computing the self intersection number of $\widetilde{C}^E$ where C^E is a (p, q)-weighted divisor of type r as before. Namely, by Lemma 6.3.10, any such curve C^E has a toric structure with outward conormals $(0, -1)$, $(-p, -q\beta)$ and $(-r, -q\alpha)$ where $\beta r - \alpha p = 1$ where C^E corresponds to the edge $(-p, -q\beta)$. Then, using the above lemma, we can compute the outward conormals of the edges adjacent to $(-p, -q\beta)$ in the resolution, which we can use to compute the self-intersection of $\widetilde{C}^E$.

Example 6.3.13. If we have $r = 1$, then we already showed in Remarks 6.3.2 and 6.3.3 that a (p, q)-weighted divisor of type 1 is actually a (p, q)-weighted exceptional divisor, although we did not phrase it in that language. Additionally, using Lemma 6.3.11 above, one can show that if we have $p > r$ or $q > r$, then any (p, q)-weighted divisor of type r is a (p, q)-weighted exceptional divisor of type r. However, the proof of this fact is just a lengthy, straightforward computation, so we do not give it here.

As a specific example, we will verify that a $(5, 4)$-weighted divisor of type 11 is a weighted exceptional divisor, while a $(5, 3)$-weighted divisor of type 11 is not. This shows in particular that if $p < r$ and $q < r$, it can be quite tricky to tell when a fixed point is good or not.

First, consider the $(5, 4)$-weighted divisor. In this case, we have $p = 5$, $q = 4$, and $r = 11$. In particular, using our earlier language we can pick $\alpha = 2$ and $\beta = 1$, since $11 - 2 * 5 = 1$. In particular, our $(5, 4)$-weighted divisor of type 11 has a toric structure with polytope given by the outward normals $(0, -1)$, $(-5, -4)$, and $(-11, -8)$. A direct computation shows that the cut $(-1, -1)$ resolves the vertex determined by $(-5, -4)$ and $(0, -1)$ and that the cut $(-4, -3)$ resolves the vertex determined by $(-5, -4)$ and $(-11, -8)$. In particular, this shows that the self intersection of the proper transform of the divisor is given by the determinant

$$-\det \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix} = -1,$$

so that the $(5, 4)$-weighted divisor of type 11 is an exceptional divisor.

Next, consider the $(5, 3)$-weighted divisor. As before, we can choose $\alpha = 2$ and $\beta = 1$. In this case, that gives us the polytope $(0, -1)$, $(-5, -3)$, and $(-11, -6)$. According to Lemma 6.3.11, the first cut in the resolution from $(-5, -3)$ in the direction of $(0, -1)$ is given by $(-3, -2)$, whereas it can be shown that the first cut from $(-5, -3)$ to $(-11, -6)$ is given by $(-7, -4)$. In particular, we could see this by using an affine transformation to transform the vertex between $(-11, -6)$ and $(-7, -4)$ to the vertex between $(0, -1)$ and $(-3, x)$, then applying Lemma 6.3.11, then applying the inverse affine transformation. Using the above, we see that the self intersection of the proper transfrom of the divisor is given by the determinant

$$\det \begin{pmatrix} 7 & 4 \\ 3 & 2 \end{pmatrix} = -2,$$

so that the $(5, 3)$-weighted divisor of type 11 is not an exceptional divisor.

We now say more about almost complex structures. Specifically, we want to discuss which almost complex structures on M or $\widetilde{M}$ can be blown down in the above sense. The following theorem is based on Theorem 1.2.5 and Remark 1.2.6 of [12], and describes a certain set of almost complex structures.

Theorem 6.3.14. *Let (M^4, ω) be a symplectic orbifold with singularities at $y_1, \ldots, y_N$, and let $(\widetilde{M}^4, \widetilde{\omega})$ be its resolution. In particular, we have homology classes $Z = \{Z_{i,A}\}$ for $i = 1, \ldots, N$ on $\widetilde{M}$ so that*

$$Z_{i,A} \cdot Z_{k,B} = \begin{cases} -a_{i,A} \leq -2 & \text{if } i = k,\ A = B \\ 1 & \text{if } i = k,\ |A - B| = 1 \\ 0 & \text{else.} \end{cases}$$

Let $\widetilde{\mathcal{J}}(\mathcal{Z}, \widetilde{\omega})$ and $\mathcal{J}(y_1, \ldots, y_N; \omega)$ be defined as in Definition 6.2.10 and Remark 6.2.11. Also, let $\widetilde{\mathcal{A}}$ be a finite, disjoint subset of $\widetilde{\mathcal{E}} \subset H_2(\widetilde{M}; \mathbb{Z})$, the collection of all standard exceptional classes on $\widetilde{M}$. Further assume that for $\widetilde{A} \in \widetilde{\mathcal{A}}$, we have $\widetilde{A} \cdot Z_i \geq 0$ for all i and $\widetilde{A} \cdot \widetilde{E} \geq 0$ for all $\widetilde{E} \in \widetilde{\mathcal{E}} \setminus \{\widetilde{A}\}$.

Then, under these assumptions, there is a subset $\widetilde{\mathcal{J}}(\mathcal{Z}; \widetilde{\mathcal{A}})$ of $\widetilde{\mathcal{J}}(\mathcal{Z}, \widetilde{\omega})$ which is path connected and residual in the sense of Baire so that for all $\widetilde{J} \in \widetilde{\mathcal{J}}(\mathcal{Z}; \widetilde{\mathcal{A}})$, all the classes $\widetilde{A}$ and Z_i are represented by embedded, $\widetilde{J}$-holomorphic spheres so that all intersections are positive and transverse, and a corresponding subset $\mathcal{J}(y_1, \ldots, y_N; \mathcal{A})$ of $\mathcal{J}(y_1, \ldots, y_N; \omega)$ which is also path connected and residual in the sense of Baire.

Remark 6.3.15. Let M^4 be a symplectic orbifold with singularities at y_p and y_q and let C^E be a (p, q)-weighted exceptional divisor of type r, as defined in Definition 6.3.7. Then, in particular, we have classes $Z_{i,A}$ obtained from resolving y_p and y_q, as well as an exceptional divisor $\widetilde{E}$ so that $\widetilde{E} \cdot Z_{i,A} \geq 0$ for all i. Then, given any $\widetilde{J} \in \widetilde{\mathcal{J}}(\mathcal{Z}, \widetilde{E})$, there is a corresponding $J \in \mathcal{J}(y_p, y_q; E)$, and furthermore, there is also an almost complex structure $\widehat{J}$ on $\widehat{M}$, the (p, q)-weighted blowdown of C^E. In other words, any such $\widetilde{J}$ and J can be blown down in the (p, q)-weighted sense described above. Note that since we will only be considering good fixed points, we will not need to consider blowdowns of weighted divisors that are not weighted exceptional divisors.

6.4 Topology of Reduced Spaces

Now, let (X, ω) be a closed, 6-dimensional symplectic manifold with a symplectic S^1 action. We will consider the resulting reduced spaces M_λ, which form a family of closed symplectic orbifolds. In particular, as we move λ counterclockwise around the circle, we will examine how the topology of M_λ changes. The below theorem shows how the reduced spaces change as we move through a critical level. The statement and proof are based on Theorem 6.1 of [7]

Lemma 6.4.1. *Let (X, ω) be a closed symplectic manifold with a symplectic S^1 action which has an isolated fixed point at $x_0 \in X$ with isotropy weights $(p, q, -r)$ at the moment map level λ_0 with* $\gcd(p, q) = \gcd(p, r) = \gcd(q, r) = 1$. *Then $M_{\lambda_0 - \epsilon}$ has an orbifold singularity of order r for all $\epsilon \geq 0$ and $M_{\lambda_0+\epsilon}$ is the (p, q) weighted blowup of size $\frac{\epsilon}{pq}$ of M_{λ_0} at the corresponding order r singularity.*

Proof. Since the S^1 action has isolated fixed points, there is a neighborhood of the $(p, q, -r)$ fixed point which maps equivariantly to $\mathbb{C}^3$ with the action

$$e^{2\pi i\lambda} \cdot (z_1, z_2, z_3) = (z_1 e^{2\pi p i\lambda}, z_2 e^{2\pi q i\lambda}, z_3 e^{-2\pi r i\lambda}).$$

A moment map for this action is given by

$$H = p|z_1|^2 + q|z_2|^2 - r|z_3|^2$$

Clearly, 0 is the only critical value of the moment map. For $\epsilon > 0$, we examine the structure of the reduced spaces $\overline{\mathbb{C}}^3_{-\epsilon}$ and $\overline{\mathbb{C}}^3_{\epsilon}$.

First, consider $\overline{\mathbb{C}}^3_{-\epsilon}$ for $\epsilon \geq 0$. For $\epsilon = 0$, Lemma 6.2.1 gives that $\overline{\mathbb{C}}^3_0 \cong \mathbb{C}^2/\mathbb{Z}_r$ where $\mathbb{Z}_r$ acts in the standard way with weights p, q. In particular, this is a symplectic orbifold with a unique orbifold singularity of order r and type (p, q) at the origin. For $\epsilon > 0$, the same argument works by using the embedding of $\mathbb{C}^2$ into $H^{-1}(-\epsilon)$ given by

$$(z_1, z_2) \mapsto \left(z_1, z_2, \sqrt{\tfrac{p}{r}|z_1|^2 + \tfrac{q}{r}|z_2|^2 + \tfrac{\epsilon}{r}}\right).$$

Now, consider $\overline{\mathbb{C}}^3_{\epsilon}$. Recall from Lemma 6.2.1 that there is a moment map for our S^1 action given by $H = p|z_1|^2 + q|z_2|^2 - r|z_3|^2$, and therefore, $\overline{\mathbb{C}}^3_{\epsilon}$ can be computed by taking the manifold

$$r|z_3|^2 + \epsilon = p|z_1|^2 + q|z_2|^2$$

and quotienting by the S^1 action. Thus, $H^{-1}(\epsilon)$ consists of all points (z_1, z_2, z_3) satisfying

$$\tfrac{|z_1|^2}{q} + \tfrac{|z_2|^2}{p} = \tfrac{1}{pq}(r|z_3^2| + \epsilon).$$

Reordering terms we see there is an embedding from $\mathbb{C}^2 \setminus (\mathrm{Int}\, \frac{\epsilon}{pq} E(q, p))$ into $H^{-1}(\epsilon)$ defined as follows:

$$(z_1, z_2) \mapsto (z_1, z_2, \sqrt{\tfrac{p}{r}|z_1|^2 + \tfrac{q}{r}|z_2|^2 - \tfrac{\epsilon}{r}}).$$

Now, consider $H^{-1}(\epsilon)/S^1$. As in Lemma 6.2.1, using the above embedding, we can identify this with the set of points

$$\{(z_1, z_2, \sqrt{\tfrac{p}{r}|z_1|^2 + \tfrac{q}{r}|z_2|^2 - \tfrac{\epsilon}{r}})\}/\sim \;\; \cong \;\; [\mathbb{C}^2 \setminus (\mathrm{Int}\, \tfrac{\epsilon}{pq} E(q,p))]/\sim,$$

where $(z_1, z_2) \sim (w_1, w_2)$ if and only if there is $\lambda \in S^1$ so that

$$\lambda^{-r}(p|z_1|^2 + q|z_2|^2 - \epsilon) = p|w_1|^2 + q|w_2|^2 - \epsilon.$$

This gives two cases: either $p|z_1|^2 + q|z_2|^2 - \epsilon = 0$ or $p|z_1|^2 + q|z_2|^2 - \epsilon > 0$. With respect to our earlier embedding, $p|z_1|^2 + q|z_2|^2 - \epsilon = 0$ corresponds to the boundary of $\mathbb{C}^2 \setminus (\mathrm{Int}\, \frac{\epsilon}{pq} E(q,p))$, while $p|z_1|^2 + q|z_2|^2 - \epsilon > 0$ corresponds to the interior.

If $p|z_1|^2 + q|z_2|^2 - \epsilon > 0$, then as in Lemma 6.2.1we must have $\lambda \in \mathbb{Z}_r$.

Now consider $p|z_1|^2 + q|z_2|^2 - \epsilon = 0$. In this case, any $\lambda \in S^1$ preserves this, since 0 is a fixed point of the S^1 action. In particular, along this ellipsoid boundary, we collapse the entire S^1 action. However, our S^1 action restricted to this ellipsoid boundary is exactly the action which generates the characteristic flow. Combining this with the above, we see that $\overline{\mathbb{C}}^3_\epsilon$ is formed from $\overline{\mathbb{C}}^3_0$ by removing the interior of $\frac{\epsilon}{pq} E(q,p)$, quotienting by the action of $\mathbb{Z}_r$ with weights (p,q), and collapsing the boundary along its characteristic flow. However, the action of $\mathbb{Z}_r$ is free on $\mathbb{C}^3_0 \setminus \mathrm{Int}(\frac{\epsilon}{pq} E(q,p))$ since we have removed the only fixed point, so that

$$\mathrm{Int}(\mathbb{C}^2 \setminus (\mathrm{Int}\, \tfrac{\epsilon}{pq} E(q,p)))/\mathbb{Z}_r \cong \mathrm{Int}(\mathbb{C}^2 \setminus (\mathrm{Int}\, \tfrac{\epsilon}{pq} E(q,p))),$$

so that up to isomorphism, we can choose not to quotient by the $\mathbb{Z}_r$ action. In particular, the above procedure gives an orbifold which is isomorphic to the (p,q) weighted blowup of size ϵ at the origin of $\overline{\mathbb{C}}^3_0$, as claimed.

This all shows that there are open sets $\mathcal{N}(\lambda)$ in M_λ so that $\overline{x}_0 \in \mathcal{N}(\lambda_0)$ and furthermore, $\mathcal{N}(\lambda_0 + \epsilon)$ is the (p,q) weighted blowup of size $\frac{\epsilon}{pq}$ of $\mathcal{N}(\lambda_0)$ at $\overline{x}_0 \in \mathcal{N}(\lambda_0)$. In particular, we have a blowdown map $\rho_\epsilon : \mathcal{N}(\lambda_0 + \epsilon) \to \mathcal{N}(\lambda_0)$ so that

$$\rho_\epsilon : \mathcal{N}(\lambda_0 + \epsilon) \setminus \{C^E_{\lambda_0+\epsilon}\} \longrightarrow \mathcal{N}(\lambda) \setminus \{\overline{x}_0\}$$

is a diffeomorphism, where C^E_λ is a representative of the weighted divisor. Thus, Lemma 6.4.4 below shows that we can extend ρ_ϵ to a map

$$\rho_\epsilon : M_{\lambda_0+\epsilon} \longrightarrow M_{\lambda_0}$$

so that the restriction

$$\rho_\epsilon : M_{\lambda_0+\epsilon} \setminus C^E_{\lambda_0+\epsilon} \longrightarrow M_{\lambda_0} \setminus \{\overline{x}_0\}$$

is a diffeomorphism. The map ρ_ϵ then clearly identifies $M_{\lambda_0+\epsilon}$ as the (p,q)-weighted blowup of M_{λ_0} at $\overline{x}_0$, as desired. □

Remark 6.4.2. An exactly analogous computation for a fixed point with isotropy weights $(-p,-q,r)$ would show that locally, we have M_{λ_0} is the (p,q) weighted orbifold blowdown of type r of $M_{\lambda_0-\epsilon}$. Equivalently we could read the argument backwards to get that $M_{\lambda_0-\epsilon}$ is the (p,q) weighted orbifold blowup of M_{λ_0}.

The above discusses how the reduced spaces change when we move across a critical point of the moment map. The following theorem says that if we move across an interval without critical points, then we do not change the reduced spaces. This theorem is proven in the introduction of [5].

Lemma 6.4.3. *Let (X,ω) be a 6-dimensional symplectic manifold with an effective, Hamiltonian S^1 action with a proper moment map H. Consider the family M_λ of reduced spaces of this action. Then if λ_0, λ_1 lie inside of an interval of regular values of the moment map, there is an orientation-preserving diffeomorphism*

$$\phi : M_{\lambda_0} \to M_{\lambda_1}.$$

Using this, we can prove the following technical diffeomorphism extension lemma which we used in Lemma 6.4.1 and which we will use below.

Lemma 6.4.4. *Let (X,ω) be a 6-dimensional symplectic manifold with an effective, Hamiltonian S^1 action which has isolated fixed points and a moment map H with image $[\lambda-\epsilon,\lambda+\epsilon]$. Further assume that λ is the only interior critical value, which corresponds to a fixed point $x_\lambda \in X$ and further corresponds to a point $\overline{x}_\lambda$ in the reduced space M_λ.*

Now, assume that we have open sets $\mathcal{N}(t)$ in $M_{\lambda+t}$ for all $t \in (-\epsilon,\epsilon)$ so that $\overline{x}_\lambda \in \mathcal{N}(0)$. Furthermore, assume that there is some (possibly empty) closed set $\overline{U}(t) \subset \mathcal{N}(t)$ for $t \in (-\epsilon,\epsilon)$ and diffeomorphisms

$$\phi_t : \mathcal{N}(t) \setminus \overline{U}(t) \longrightarrow \mathcal{N}(0) \setminus \overline{U}(0).$$

Then by possibly shrinking $\mathcal{N}(t)$ in a way so $\overline{x}_0$ is still in $\mathcal{N}(0)$, there are extensions

$$\psi_t : M_{\lambda+t} \setminus \overline{U}(t) \longrightarrow M_\lambda \setminus \overline{U}(0)$$

so that ψ_t is a diffeomorphism.

Proof. We can define the manifold $X(\lambda, \epsilon)$ by taking

$$X(\lambda, \epsilon) = H^{-1}\left(\bigcup_{t\in[-\epsilon,\epsilon]} M_{\lambda+t} \setminus \overline{U}(t)\right)$$

Then $X(\lambda, \epsilon)$ is a compact, symplectic, 6-dimensional manifold with a Hamiltonian circle action with moment map H. Further, H is proper since $X(\lambda, \epsilon)$ is compact. Also, since we assumed x_λ was the only fixed point an that $\overline{x}_\lambda \in \mathcal{N}(0)$, the moment map of $X(\lambda, \epsilon)$ has no interior critical points. Thus, by Lemma 6.4.3, for all $t \in (-\epsilon, \epsilon)$, we know that there is a diffeomorphism

$$\phi_t' : M_{\lambda+t} \setminus \mathcal{N}(t) \longrightarrow M_\lambda \setminus \mathcal{N}(0)$$

Extrapolating between ϕ_t' and ϕ_t gives diffeomorphisms

$$\psi_t : M_{\lambda+t} \setminus \overline{U}(t) \longrightarrow M_\lambda \setminus \overline{U}(0)$$

Possibly shrinking the size of $\mathcal{N}(t)$, we can further assume that ψ_t restricts to ϕ_t, as desired. □

We can use this diffeomorphism extension lemma to prove the following.

Lemma 6.4.5. *Let (X, ω) be a closed, 6-dimensional symplectic manifold with an effective, symplectic S^1 action with moment map Φ and isolated fixed points. Then if λ_0 and/or λ_1 are the only critical values in $[\lambda_0, \lambda_1]$ and M_{λ_0} does not differ from M_{λ_1} by a weighted blowup as in Lemma 6.4.1, we have an orientation preserving diffeomorphism*

$$\phi : M_{\lambda_0} \longrightarrow M_{\lambda_1}.$$

Furthermore, for any $\lambda, \lambda' \in [\lambda_0, \lambda_1]$ where $[\lambda_0, \lambda_1]$ is an interval as above, there is a diffeomorphism

$$\widetilde{\phi} : \widetilde{M}_\lambda \longrightarrow \widetilde{M}_{\lambda'}.$$

Proof. To prove the first half of the statement, note that if $[\lambda_0, \lambda_1]$ has λ_0 and/or λ_1 as the only critical values and the reduced spaces do not differ by weighted blowups as in Lemma 6.4.1, then as in the proof of Lemma 6.4.1, there is an ϵ and neighborhoods $\mathcal{N}(t)$ inside of M_{λ_0+t} for $t \in [0, 2\epsilon]$ such that the fixed point at λ_0 is in $\mathcal{N}(0)$, and $\mathcal{N}(0)$ is diffeomorphic to $\mathcal{N}(t)$ for all t. Then choosing the closed sets $\overline{U}(t) = \emptyset$ for all t, Lemma 6.4.4 above implies the existence of orientation preserving diffeomorphisms

$$\phi_t^0 : M_{\lambda_0} \longrightarrow M_{\lambda_0+t}.$$

By a similar argument near λ_1, there is an orientation preserving diffeomorphism

$$\phi_t^1 : M_{\lambda_1-t} \longrightarrow M_{\lambda_1}.$$

Then, by assumption, $[\lambda_0 + 2\epsilon, \lambda_1 - 2\epsilon]$ has no critical values, so by Lemma 6.4.3, there is an orientation-preserving diffeomorphism

$$\phi' : M_{\lambda_0+\epsilon} \longrightarrow M_{\lambda_1-\epsilon}.$$

Defining $\phi := \phi_\epsilon^1 \circ \phi' \circ \phi_\epsilon^0$, we get the desired diffeomorphism, which finishes the proof of the first statement.

To prove the second statment, we notice that by the above statement and Lemma 6.4.3, there is an orientation preserving diffeomorphism

$$\phi : M_\lambda \longrightarrow M_{\lambda'}$$

which then lifts to

$$\widetilde{\phi} : \widetilde{M_\lambda} \longrightarrow \widetilde{M_{\lambda'}}$$

by Lemma 6.2.9. □

6.5 Some Intersection Theory

In this section, we will prove some useful results pertaining to intersection theory. First, we will give a useful criterion for determining when a closed, symplectic 4 manifold has $b_2^+ = 1$. We recall Theorem 1.4 from [11].

Theorem 6.5.1. *Let (M, ω) be a closed symplectic 4-manifold and assume that there exists a symplectically immersed 2-sphere C with only positively oriented transverse double points. Then if $c_1(C) \geq 2$, (M, ω) is rational or ruled. In particular, $b_2^+ = 1$.*

We can use the above theorem to prove the following lemma.

Lemma 6.5.2. *Let (M, ω) be a closed symplectic 4 manifold. If M contains two embedded J-holomorphic -1 spheres C_1^E and C_2^E with $E_1 \cdot E_2 = k \geq 1$, then $b_2^+ = 1$.*

Proof. To prove this, we resolve exactly 1 of the intersection points of C_1^E and C_2^E to get a single sphere C in the class $E_1 + E_2$ which is immersed with $k - 1$ positive transverse double points. Notice that the immersion points of C come from unresolved intersections of C_1^E and C_2^E which are all positive transverse intersections by positivity of intersections in dimension 4. Thus, it remains to show that $c_1(C) \geq 2$. In fact, we have

$$c_1(C) = c_1(E_1 + E_2) = c_1(E_1) + c_1(E_2) = 1 + 1 = 2$$

as desired. □

We now prove a similar lemma in the case where $(\widetilde{M}, \widetilde{\omega})$ is the resolution of an orbifold (M, ω).

Lemma 6.5.3. *Let $\widetilde{M}^4$ be a symplectic manifold which is the resolution of M^4, a symplectic orbifold. Recall from Definition 6.3.5 that if M has a (p, q)-weighted divisor of type 1, C^E, then $\widetilde{M}^4$ has classes $\widetilde{E}$ and Z_i^j for $j = 1, 2$ represented by curves $\widetilde{C}^E$ and $\widetilde{C}_i^j$ so that $\widetilde{E}$ is the class of an exceptional divisor, $\widetilde{E} \cdot Z_1^j = 1$, $\widetilde{E} \cdot Z_l^j = 0$ if $l > 1$, and*

$$Z_i^l \cdot Z_j^m = \begin{cases} -a_i^l \leq -2 \text{ if } i = j, \ l = m \\ 1 \text{ if } l = m, \ |i - j| = 1 \\ 0 \text{ else.} \end{cases}$$

Then if $\widetilde{M}$ has an exceptional divisor $\widetilde{C}^{E'}$ in a class $\widetilde{E}' \neq \widetilde{E}$ so that either $\widetilde{E'} \cdot \widetilde{E} \neq 0$ or $\widetilde{E'} \cdot Z_{i_0}^{j_0} \neq 0$ for some i_0, j_0, $b_2^+(M) = 1$.

Proof. First, notice that Lemma 6.5.2 above implies that $b_2^+(\widetilde{M}) = 1$ if $\widetilde{E'} \cdot \widetilde{E} \neq 0$. Since $\widetilde{M}$ differs from M by a sequence of blowups, this implies that $b_2^+(M) = 1$ as well. Thus, we can assume that there is some (i_0, j_0) so that $\widetilde{E'} \cdot Z_{i_0}^{j_0} \neq 0$.

Now recall that since our divisor C^E is of type 1, Remark 6.3.8 says that the collection $\widetilde{C}^E$ and $\widetilde{C}_i^j$ can be successively blown down by smooth blowdowns of exceptional divisors starting with $\widetilde{C}^E$ to form the (p,q)-weighted blowdown of the weighted exceptional divisor C^E.

Thus, if we begin performing these successive blowdowns, there will be some intermediate stage where we have a closed symplectic 4-manifold $\widehat{M}$ so that the proper transform of $C_{i_0}^{j_0}$ to $\widehat{M}$ is an exceptional divisor, which we denote $\overline{C}_{i_0}^{j_0}$. Then by our assumptions $\widetilde{C}^{E'}$ has a proper transform to a curve $\overline{C}^{E'}$ so that $\overline{C}^{E'} \cap \overline{C}_{i_0}^{j_0} \neq 0$. Furthermore, if we assume that (i_0, j_0) is the first pair of indices so that this intersection is non-zero, $\overline{C}^{E'}$ will be an exceptional divisor as well.

But then $\widehat{M}$ has two intersecting exceptional divisors, so that by Lemma 6.5.2 above, $b_2^+(\widehat{M}) = 1$. Now, since $\widehat{M}$ differs from M by a series of blowups and blowdowns, this implies that $b_2^+(M) = 1$ as well. □

Chapter 7

Proof of Main Theorem for Circle Actions

7.1 Generalized Bundles

In this section, we will give a technical definition which will be useful to our proof of Theorem 1.3.4.

Definition 7.1.1. *Let $\{V_\alpha\}_{\alpha \in A}$ be a finite open cover of S^1 by intervals so that all triple intersections are empty. Furthermore, assume that A has a partial ordering so that if $V_{\alpha\beta} := V_\alpha \cap V_\beta \neq \emptyset$, then either $V_\alpha < V_\beta$ or $V_\beta < V_\alpha$. Then a* ***generalized bundle*** *over S^1 is given by topological spaces $\mathcal{F}_\alpha$ with projections $\pi_\alpha : \mathcal{F}_\alpha \to S^1$ such that if $V_\alpha \cap V_\beta \neq \emptyset$ and $V_\alpha < V_\beta$, there is a fiberwise inclusion*

$$\phi_{\alpha\beta} : \pi_\alpha^{-1}(V_{\alpha\beta}) \longrightarrow \pi_\beta^{-1}(V_{\alpha\beta}).$$

Furthermore, a ***section*** *of a generalized bundle is a collection of maps $s_\alpha : V_\alpha \to \mathcal{F}_\alpha$ satisfying $s_\alpha = s_\beta \circ \phi_{\alpha\beta}$ whenever $V_{\alpha\beta} \neq \emptyset$ and $V_\alpha < V_\beta$.*

Remark 7.1.2. This definition differs from the standard definition of a bundle primarily in the fact that the fiber F_x over a point $x \in S^1$ is allowed to change its topological type as we change x. However, a section of a generalized bundle still gives us a notion of a smoothly varying family of elements of the $\mathcal{F}_\alpha$, with one for each $x \in S^1$. This notion of section is the main reason we gave this definition.

Example 7.1.3. The family of reduced spaces corresponding to a symplectic S^1 action on (X, ω) gives a trivial example of a generalized bundle. Namely, we can consider the cover of S^1 just given by all of S^1, and we can let

$$\mathcal{F}_{S^1} := X/S^1.$$

We will now show how one could put a more complicated generalized bundle structure on the family of reduced spaces.

Example 7.1.4. Let (X^6, ω) be a closed symplectic manifold with a symplectic, non-Hamiltonian S^1 action. Then as before, this has an S^1 valued moment map and a family of reduced spaces M_λ for $\lambda \in S^1$. By our earlier assumptions, the fixed point set of this action is a finite set of isolated fixed points, which, perturbing ω if necessary, we can assume all happen at different moment map levels. We denote these levels $\lambda_1, \ldots, \lambda_{2n}$.

Define $U_i = (\lambda_i, \lambda_{i+1})$ for $i = 1, \ldots 2n-1$, and $U_{2n} = (\lambda_{2n}, \lambda_1)$. Also, define $I_i = (\lambda_i - \epsilon, \lambda_i + \epsilon)$, and assign the partial ordering $I_i < U_i$ for all $i = 1, \ldots n$, $I_i < U_{i-1}$ if $i = 2, \ldots 2n$, and $I_1 < U_{2n}$. This cover gives the reduced spaces the structure of a generalized bundle. Indeed, we can define

$$\mathcal{F}_{U_i} := \bigcup_{\lambda \in U_i} M_\lambda, \quad \mathcal{F}_{I_j} := \bigcup_{\lambda \in I_j} M_\lambda$$

Then Lemma 6.4.1 and Lemma 6.4.3 give that the spaces $\mathcal{F}_{U_i}$ and $\mathcal{F}_{I_j}$ are topological spaces which are fibered over λ by smooth orbifolds, while on all overlaps they are equal to each other, so that the fiberwise inclusions can just be chosen to be the identity.

Remark 7.1.5. The generalized bundle $\widetilde{\mathcal{J}}$ that we eventually construct in the below proof will be very similar to the above example. In particular, it will use the same cover U_i and I_j with the same ordering. However, $\widetilde{\mathcal{F}}_{U_i}$ and $\widetilde{\mathcal{F}}_{I_j}$ will not be fibered by M_λ, but rather they will be fibered by carefully chosen spaces of almost complex structures of $\widetilde{M}_\lambda$.

7.2 Proof of Main Result

We will now prove our main result, which we restate here for convenience.

Proposition 7.2.1. *Let (X^6, ω) be a closed symplectic manifold. Then there does not exist a symplectic, non-Hamiltonian S^1 action with a non-empty set of isolated fixed points such that all* *fixed points are good, there is no codimension 2 isotropy, and such that there exists a fixed point with weights $(p, q, -1)$ with $p, q > 0$ such that the only other fixed points with isotropy weights either $\pm p$ or $\pm q$ are of one of the following 2 forms:*

1. $(\pm p, \pm q', \mp 1)$
2. $(\pm p', \pm q, \mp 1)$.

Proof. We will assume that we do have a symplectic, non-Hamiltonian action with such a set of isolated fixed points and derive a contradiction. Recall from before that since we have a symplectic circle action which is not Hamiltonian, we can assume we have an S^1 valued moment map and that we can form the corresponding reduced spaces M_λ for $\lambda \in S^1$.

Furthermore, as in Example 7.1.4 above, our moment map can be assumed to have $2n$ critical levels which correspond to the isolated fixed points. We can define the sets U_i and I_j as in Example 7.1.4.

Also, since we assumed that the original S^1 action has no codimension 2-isotropy, we know that each reduced space M_λ is a symplectic orbifold with a finite number of isolated orbifold singularities, denoted y_i^λ. We then have $\widetilde{M_\lambda}$, the unique resolution of these singularities as in Definition 6.2.7. As in Remark 6.2.6, there are homology classes $Z_{i,A}^\lambda$ coming from the blowups used to resolve the singularities y_i^λ. We have

$$Z_{i,A}^\lambda \cdot Z_{j,B}^\lambda = \begin{cases} -k_{i,A}^\lambda \leq -2 & \text{if } i = j,\ A = B \\ 1 & \text{if } i = j,\ |A - B| = 1 \\ 0 & \text{else.} \end{cases}$$

We will let $\mathcal{Z}_\lambda$ denote the union of all these classes over i, j.

By Theorem 1.3.2, if for some regular level λ we have $b_2^+(M_\lambda) = 1$, the action is Hamiltonian which is a contradiction. Hence, $b_2^+(M_\lambda) > 1$ for all regular levels λ, and thus also $b_2^+(\widetilde{M_\lambda}) > 1$ for regular levels λ.

We will use this to derive a contradiction in 4 steps. In steps 1 and 2, we will construct specific families of almost complex structures $\widetilde{J}(\lambda)$ and $J(\lambda)$ on $\widetilde{M_\lambda}$ and M_λ respectively. In step 3, we will use Lemma 6.4.1 to construct certain $\widetilde{J}(\lambda)$-holomorphic curves on $\widetilde{M_\lambda}$. In step 4, we will then use J-holomorphic curve techniques with the curves constructed in step 3 to argue that for some λ_0, we must have $b_2^+(\widetilde{M}_{\lambda_0} = 1$, which contradicts Lemma 1.3.2.

First, we will use the language of generalized bundles to find a preferred family $\widetilde{J}(\lambda)$ of almost complex structures on $\widetilde{M}_\lambda$.

Step 1 (**Constructing the generalized bundle** $\widetilde{\mathcal{J}}$). In this step, we will prove the following claim.

Claim 1. *There is a generalized bundle $\widetilde{\mathcal{J}}$ over S^1 with the open cover U_i and I_j so that the fiber over λ is a certain nice set of almost complex structures on $\widetilde{M}_\lambda$.*

First, consider U_k. We will first define a set $\widetilde{J}(\lambda)$ of almost complex structures on $\widetilde{M}_\lambda$ by using Lemma 6.3.14. By Lemmas 6.4.1, 6.4.3, and 6.2.9, there is a $\lambda' \in U_k$ and a smooth family of orientation-preserving diffeomorphisms

$$\phi_\lambda : M_{\lambda'} \longrightarrow M_\lambda,$$

for all $\lambda \in U_k$ which have lifts to diffeomorphisms

$$\widetilde{\phi}_\lambda : \widetilde{M}_{\lambda'} \longrightarrow \widetilde{M}_\lambda.$$

Recall from Definition 6.2.10 that we have the set $\widetilde{\mathcal{J}}(\mathcal{Z}_\lambda)$ defined so that any $\widetilde{J} \in \widetilde{\mathcal{J}}(\mathcal{Z}_\lambda)$ satisfies the property that for any i, A, $Z^\lambda_{i,A}$ is represented by an embedded, $\widetilde{J}_\lambda$-holomorphic sphere, denoted $C^\lambda_{i,A}$. Furthermore, up to a reordering of the i indices,

$$\widetilde{\phi}^*_\lambda(Z^\lambda_{i,j}) = Z^{\lambda'}_{i,j}.$$

Since U_k contains no critical values, we know by Theorem 1.3.2 that for all $\lambda \in U_k$, $b_2^+(\widetilde{M}_\lambda) > 1$. This implies that the set $\widetilde{\mathcal{E}}_\lambda$ of homology classes of exceptional divisors on $\widetilde{M}_\lambda$ is finite. Furthermore by Lemma 6.5.2, if $\widetilde{E} \neq \widetilde{E}' \in \widetilde{\mathcal{E}}$, then $\widetilde{E} \cdot \widetilde{E}' = 0$. Consider the finite subset $\widetilde{\mathcal{A}}_\lambda \subset \widetilde{\mathcal{E}}_\lambda$ defined by the property that any $\widetilde{A} \in \widetilde{\mathcal{A}}_\lambda$ satisfies $\widetilde{A} \cdot Z^\lambda_{i,j} \geq 0$. Then, as in Theorem 6.3.14, there is a subset $\widetilde{\mathcal{J}}(\mathcal{Z}_\lambda, \widetilde{\mathcal{A}}_\lambda) \subset \widetilde{\mathcal{J}}(\mathcal{Z}_\lambda)$ which is path connected and residual in the sense of Baire so that for any $\widetilde{J} \in \widetilde{\mathcal{J}}(\mathcal{Z}_\lambda, \widetilde{\mathcal{A}}_\lambda)$, $\widetilde{A} \in \widetilde{A}_\lambda$ is represented by a smooth, embedded $\widetilde{J}$-holomorphic sphere which intersects each curve $C^\lambda_{i,j}$ transversally in $\widetilde{A} \cdot C^\lambda_{i,j}$ distinct points. We define

$$\widetilde{\mathcal{J}}(\lambda) := \widetilde{\mathcal{J}}(\mathcal{Z}_\lambda, \widetilde{\mathcal{A}}_\lambda).$$

Also, each $\widetilde{J} \in \widetilde{\mathcal{J}}(\lambda)$ is pulled back from an almost complex structure J on M_λ, so that we get a corresponding family $\mathcal{J}(\lambda)$ in this fashion.

We further define

$$\widetilde{\mathcal{F}}_{U_k} := \bigcup_{\lambda \in U_k} \widetilde{\mathcal{J}}(\lambda).$$

We can use the isomorphisms $\widetilde{\phi}_\lambda$ to identify $\widetilde{\mathcal{F}}_{U_k}$ with an open subset of the set of almost complex structures $\widetilde{J}$ so that $\widetilde{J} \in \widetilde{\mathcal{J}}(\lambda', \widetilde{\omega}_t)$ for some smooth path of symplectic forms $\widetilde{\omega}_t$ on $\widetilde{M}_{\lambda'}$, which is a topological space. Hence, $\widetilde{\mathcal{F}}_{U_k}$ is also a topological space, as desired.

Consider now $I_k = (\lambda_k - \epsilon, \lambda_k + \epsilon)$ for some ϵ. To define $\widetilde{\mathcal{F}}_{I_k}$, we will first construct an explicit family $\widetilde{J}(\lambda)$ of almost complex structures on $\widetilde{M}_\lambda$ for all $\lambda \in (\lambda_k - 2\epsilon, \lambda_k + 2\epsilon)$ and then let $\widetilde{\mathcal{F}}_{I_k}$ be this path restricted to I_k.

By Lemma 6.4.1, and Remark 6.4.2 we know that if λ_k has isotropy weights $(p_k, q_k, -r_k)$, then M_{λ_k+t} is the (p_k, q_k)-weighted blowup of M_{λ_k} at an orbifold point of order r_k for all $t \in (0, 2\epsilon]$, while if λ_k has isotropy weights $(-p_k, -q_k, r_k)$, the same is true with the signs reversed. We will first assume that $r_k = 1$, so that we are doing a (p_k, q_k)-weighted blowup at a smooth point which we will denote y^{λ_k}.

Recall from Lemma 6.4.5 that we have orientation-preserving diffeomorphisms

$$\phi_\lambda : M_{\lambda_k} \to M_\lambda, \quad \lambda \in (\lambda_k - 2\epsilon, \lambda_k],$$

as well as corresponding lifts

$$\widetilde{\phi}_\lambda : \widetilde{M}_{\lambda_k} \to \widetilde{M}_\lambda$$

where $\phi_{\lambda_k} = \mathrm{id}$ and $\widetilde{\phi}_{\lambda_k} = \mathrm{id}$. Furthermore, since M_{λ_k} has isolated orbifold singularities, we have neighborhoods in M_{λ_k} denoted $\mathcal{N}(y_i^{\lambda_k})$ and $\mathcal{N}(\lambda_k)$ of the orbifold points $y_i^{\lambda_k}$ and of y^{λ_k} respectively so that $\mathcal{N}(\lambda_k) \cap \mathcal{N}(x_i^{\lambda_k}) = \emptyset$ for all i.

Now, consider the resolution $\widetilde{M}_{\lambda_k}$. This resolution has its corresponding set of homology classes $\mathcal{Z}_{\lambda_k}$. Notice that since y^{λ_k} stays away from $y_i^{\lambda_k}$, there is a corresponding point $\widetilde{y}^{\lambda_k} \in \widetilde{M}_{\lambda_k}$. Consider the set $\widetilde{\mathcal{J}}(\mathcal{Z}_{\lambda_k}, \widetilde{\omega}_{\lambda_k})$ as above.

For all $t \in [0, 2\epsilon)$, we can define a smooth family of symplectic forms on $\widetilde{M}_{\lambda_k}$ by

$$\widetilde{\omega}_t = \widetilde{\phi}^*_{\lambda_k - t}(\widetilde{\omega}_{\lambda_k - t}),$$

so that we can form $\widetilde{\mathcal{J}}(\mathcal{Z}_{\lambda_k}, \widetilde{\omega}_t)$.

Now choose a $\widetilde{J} \in \widetilde{\mathcal{J}}(\mathcal{Z}_{\lambda_k}, \widetilde{\omega}_{\lambda_k})$ so that $\widetilde{J}$ equals J_0 near y^{λ_k} and so that there is a neighborhood $\mathcal{N}(y^{\lambda_k})$ so that no $\widetilde{J}_\lambda$ holomorphic exceptional divisors intersect $\mathcal{N}(y^{\lambda_k})$, which we can do since there are finitely many exceptional classes. Then since the taming condition is open, we can choose ϵ depending on $\widetilde{J}$ small enough so that for all $t \in [0, 2\epsilon)$, $\widetilde{J} \in \widetilde{\mathcal{J}}(\mathcal{Z}_{\lambda_k}, \widetilde{\omega}_t$. Thus, for each $t \in [0, 2\epsilon)$, we can push $\widetilde{J}$ forward by $\widetilde{\phi}_{\lambda_k - t}$ to an almost complex structure $\widetilde{J}(\lambda_k - t)$ to get a family

$$\widetilde{J}(\lambda) \in \widetilde{\mathcal{J}}(\mathcal{Z}_\lambda, \widetilde{\omega}_\lambda), \quad \lambda \in (\lambda_k - 2\epsilon, \lambda_k].$$

Also, we can choose $\widetilde{J}(\lambda_k)$ so that

$$\widetilde{J}(\lambda) \in \widetilde{\mathcal{J}}(\lambda), \quad \lambda \in (\lambda_k - 2\epsilon, \lambda_k),$$

where $\widetilde{\mathcal{J}}(\lambda)$ is as before. Furthermore, there is a family

$$J(\lambda) \in \mathcal{J}(y_1^\lambda, \ldots, y_N^\lambda; \omega_\lambda)$$

of almost complex structures on M_λ so that $J(\lambda_k) = J_0$ near y^{λ_k} and so that $J(\lambda) \in \mathcal{J}(\lambda)$ for $\lambda \in (\lambda_k - 2\epsilon, \lambda_k)$.

Now, since for each $t \in (0, 2\epsilon)$, $M_{\lambda_k + t}$ is equal to the (p_k, q_k) weighted blowup of M_{λ_k} at the point y^{λ_k} and $J(\lambda_k)$ equals J_0 near y^{λ_k}, we get corresponding almost complex structures $J_{\lambda_k + t}$ which are integrable near the (p_k, q_k) weighted exceptional divisor. Also, any orbifold point on $M_{\lambda_k + t}$ either corresponds to some $y_i^{\lambda_k}$ on M_{λ_k}, or lies on the weighted exceptional divisor. Thus, $J_{\lambda_k + t}$ is integrable near all the orbifold points $y_i^{\lambda_k + t}$, and we have

$$J(\lambda_k + t) \in \mathcal{J}(y_1^{\lambda_k + t}, \ldots, y_N^{\lambda_k + t}; \omega_{\lambda_k + t}).$$

Also, as before, we can choose $J(\lambda_k)$ so that $J(\lambda)$ defined in this way satisfies $J(\lambda) \in \mathcal{J}(\lambda)$. Thus, we can blow these almost complex structures up to get almost complex structures $\widetilde{J}(\lambda) \in \widetilde{\mathcal{J}}(\lambda)$ for all $\lambda \in (\lambda_k, \lambda_k + 2\epsilon)$.

In particular, for all $\lambda \in (\lambda_k - 2\epsilon, \lambda_k + 2\epsilon) \supset I_k$, we have constructed a family $\widetilde{J}(\lambda)$ of almost complex structures on $\widetilde{M}_\lambda$ such that if $\lambda \neq \lambda_k$, $\widetilde{J}(\lambda) \in \widetilde{\mathcal{J}}(\lambda)$, in the case where $r_k = 1$.

If now $r_k > 1$, we can run a very similar argument. In particular, as before we can assume we have isotropy weights $(p_k, q_k, -r_k)$. However, now that $r_k > 1$, we have that $\widetilde{M}_{\lambda_k + t}$ is the (p_k, q_k)-weighted blowup of an order r_k orbifold singularity. Up to reordering, we can assume this orbifold singularity is $y_1^{\lambda_k}$. Now, letting $y_1^{\lambda_k}$ play the role of y_k^λ and ignoring the added assumptions about exceptional divisors on $\widetilde{M}_\lambda$, we can choose a family $\widetilde{J}(\lambda)$ of almost complex structures on $\widetilde{M}_\lambda$ so that if $\lambda \neq \lambda_k$, we have

$$\widetilde{J}(\lambda) \in \widetilde{\mathcal{J}}(\lambda).$$

In particular, regardless of what $(p_k, q_k, -r_k)$ is, we will always have such a family of almost complex structures $\widetilde{J}(\lambda)$.

We now define

$$\widetilde{\mathcal{F}}_{I_k} := \bigcup_{\lambda \in I_k} \widetilde{J}(\lambda).$$

Then $\widetilde{\mathcal{F}}_{I_k}$ is diffeomorphic to an open interval, hence it is obviously a topological space. Furthermore, since for $\lambda \neq \lambda_k$ we have $\widetilde{J}(\lambda) \in \widetilde{\mathcal{J}}(\lambda)$ there is a natural fiberwise inclusion from the piece of $\widetilde{\mathcal{F}}_{I_k}$ over $I_k \cap U_l$ into $\widetilde{\mathcal{F}}_{U_l}$, whenever $I_k \cap U_l \neq \emptyset$.

This completes the construction of a generalized bundle over S^1 which we will denote $\widetilde{\mathcal{J}}$.

Step 2 (Showing that the generalized bundle $\widetilde{\mathcal{J}}$ has a non-zero section). In this step, we will prove the following claim.

Claim 2. *The generalized bundle $\widetilde{\mathcal{J}}$ has a non-zero section. In particular, there is a consistent choice of $\widetilde{J}(\lambda) \in \widetilde{\mathcal{J}}(\lambda)$.*

We will prove the claim by taking sections on each I_j and patching them together over the U_k.

First, consider $I_j = (\lambda_j - \epsilon, \lambda_j + \epsilon)$. Recall from the definition of $\mathcal{F}_{I_j}$ above that for each $\lambda \in (\lambda_j - 2\epsilon, \lambda_j + 2\epsilon) \supset I_j$, we have an almost complex structure $\widetilde{J}(\lambda)$ on $\widetilde{M}_\lambda$ so that if $\lambda \neq \lambda_k$, $\widetilde{J}(\lambda) \in \widetilde{\mathcal{J}}(\lambda)$. In particular, this defines a section on I_j which has already been extended a little past I_j.

Next consider U_k. We seek to find a section of $\widetilde{\mathcal{J}}$ over U_k which equals $\widetilde{J}(\lambda)$ on $U_k \cap I_j$ whenever this intersection is non-empty. Using the diffeomorphisms $\widetilde{\phi}_\lambda$ from before, we can define

$$\widetilde{\mathcal{J}}(\lambda) \cong \widetilde{\phi}_\lambda^* \widetilde{\mathcal{J}}(\lambda) =: \widetilde{\mathcal{J}}(\lambda'; \lambda).$$

Notice that any $\widetilde{J} \in \widetilde{\mathcal{J}}(\lambda'; \lambda)$ is an almost complex structure on $M_{\lambda'}$ so that

$$(\widetilde{\phi}_\lambda)_*(\widetilde{J}) \in \widetilde{\mathcal{J}}(\lambda)$$

Thus, to find a family $\widetilde{J}(\lambda) \in \widetilde{\mathcal{J}}(\lambda)$ over U_k, it suffices to find a path $\widetilde{J}(\lambda) \in \widetilde{\mathcal{J}}(\lambda_0; \lambda)$.

Now, for $\lambda \in (\lambda_k, \lambda_k + 2\epsilon) \cup (\lambda_{k+1} - 2\epsilon, \lambda_{k+1})$, we already have a choice of $\widetilde{J}(\lambda)$ on $\widetilde{M}_\lambda$, which as above gives us a choice of $\widetilde{J}(\lambda) \in \widetilde{\mathcal{J}}(\lambda'; \lambda)$. Consider the interval

$$U_k \setminus \{(I_k \cup I_{k+1}) \cap U_k\} = [\lambda_k + \epsilon, \lambda_{k+1} - \epsilon]$$

and define the set

$$\widetilde{\mathcal{J}}_k := \bigcup_{\lambda \in [\lambda_k+\epsilon, \lambda_{k+1}-\epsilon]} \widetilde{\mathcal{J}}(\lambda'; \lambda).$$

This set is obviously fibered over $[\lambda_k + \epsilon, \lambda_{k+1} - \epsilon]$ by $\widetilde{\mathcal{J}}(\lambda'; \lambda)$, which is a path connected set of $\widetilde{\phi}_\lambda^*(\widetilde{\omega}_\lambda)$-tame almost complex structures on $\widetilde{M}_{\lambda_0}$. Thus, since the taming condition is open, the set $\widetilde{\mathcal{J}}_k$ defined in this way is path connected. Also, as pointed out before, we already have two almost complex structures $\widetilde{J}(\lambda_k + \epsilon)$ and $\widetilde{J}(\lambda_{k+1} - \epsilon)$ defined on $\widetilde{\mathcal{J}}_k$, so that we can choose a path $\widetilde{J}(\lambda)$ connecting them so that

$$\widetilde{J}(\lambda) \in \widetilde{\mathcal{J}}(\lambda'; \lambda),$$

which we can then push forward to a family

$$\widetilde{J}(\lambda) \in \widetilde{\mathcal{J}}(\lambda),$$

for all $\lambda \in [\lambda_k + \epsilon, \lambda_{k+1} - \epsilon]$.

In particular, this gives a path $\widetilde{J}(\lambda)$ on U_k which agrees with the previous choice of $\widetilde{J}(\lambda)$ on I_j whenever $U_k \cap I_j \neq \emptyset$ as desired.

For the rest of the proof, for $\lambda \in S^1$, let $\widetilde{J}(\lambda)$ denote a specific choice of a section of $\widetilde{\mathcal{J}}$, and $J(\lambda)$ the corresponding family of almost complex structures on M_λ which pull back to $\widetilde{J}(\lambda)$ under the blowup maps. To derive a contradiction, we will produce specific exceptional divisors on the spaces $\widetilde{M}_\lambda$ and use J-holomorphic curve techniques using the family $\widetilde{J}(\lambda)$ above.

Step 3 (Constructing exceptional divisors on U_k). In this step, we will first prove the following claim.

Claim 3. *If x_k is a fixed point with isotropy weights $(p_k, q_k, -r_k)$ and moment map level λ_k, then for all $\lambda \in U_k$, $\widetilde{M}_\lambda$ has an exceptional class $\widetilde{E}_k^{\lambda,+}$ so that $\widetilde{\omega}_\lambda(\widetilde{E}_k^{\lambda,+})$ is increasing with λ.*

Recall that we have the resolution $\widetilde{M}_\lambda$ with its corresponding set of homology classes

$$\mathcal{Z}_\lambda = \bigcup_{i,A} Z_{i,A}^\lambda$$

Now consider I_k. As in Lemma 6.4.1, we can choose ϵ small enough so that the interval $I_k = (\lambda_k - \epsilon, \lambda_k + \epsilon)$ satisfies that given any $\lambda \in (\lambda_k, \lambda_k + \epsilon)$, M_λ is the (p_k, q_k) weighted blowup of size $\lambda - \lambda_k$ of M_{λ_k} at either a smooth point y^{λ_k} if $r_k = 1$ or at an orbifold point of order r_k $y_1^{\lambda_k}$ if

$r_k > 1$. In either case the condition that the fixed point is good implies that for all $\lambda \in (\lambda_k, \lambda_k + \epsilon)$, there is a (p_k, q_k) weighted exceptional divisor in the class $E_k^{\lambda,+}$ which passes through two isolated orbifold singularities of M_λ. Thus, as in Remark 6.3.2 there is an ordering of the classes $Z_{i,A}^{\lambda}$ from step 1 and a choice of indices $i_k^1 = 1, m_1$ and $i_k^2 = 1, m_2$ where $Z_{i,A}$ has $i = 1, \ldots, m_A$, and a class $\widetilde{E}_k^{\lambda,+}$ satisfying the following properties.

1. $\widetilde{E}_k^{\lambda,+}$ is an exceptional class in $\widetilde{M}_\lambda$ which is the pullback of $E_k^{\lambda,+}$ under the natural projection from $\widetilde{M}_\lambda \to M_\lambda$.

2. For $A = 1, 2$, $\widetilde{E}_k^{\lambda,+} \cdot Z_{i_k^A,1}^{\lambda} = 1$.

3. $\widetilde{E}_k^{\lambda,+} \cdot Z_{i,A}^{\lambda} = 0$ for all other i, A.

Furthermore, as $\lambda \in (\lambda_k, \lambda_k + \epsilon)$ increases, $\widetilde{\omega}_\lambda(\widetilde{E}_k^{\lambda,+})$ also increases, while $\widetilde{\omega}_\lambda(Z_{i,A}^{\lambda})$ can be fixed to be as small as desired for all i, A and λ. Furthermore, the classes $\widetilde{E}_k^{\lambda,+}$ and $Z_{i,A}^{\lambda}$ all correspond to $\widetilde{E}_k^{\lambda',+}$ and $Z_{i,A}^{\lambda'}$ under the diffeomorphisms $\widetilde{\phi}_\lambda$ from Step 1. As such, we will omit the λs from the notation, and simply refer to the classes as $\widetilde{E}_k^{+}$ and $Z_{i,A}^{k}$. We can also use these diffeomorphisms to extend the classes $\widetilde{E}_k^{+}$ as being defined over all of U_k, and we will still have that $\widetilde{\omega}_\lambda(E_k^{+})$ increases with λ while $\widetilde{\omega}_\lambda(Z_{i,A}^{k})$ can be fixed as small as desired. This completes the proof of the claim.

Now, consider a fixed point x_l which has isotropy weights $(-p_l, -q_l, r_l)$. By a similar argument, we can produce classes $\widetilde{E}_l^{-}$ and indices $i_l^A = 1, m_A$ satisfying properties (1) and (2) above and so that $\widetilde{\omega}_\lambda(\widetilde{E}_l^{-})$ decreases with λ.

Step 4 (**Deriving a contradiction**). Let λ_1 correspond to a fixed point of the form $(p, q, -1)$ satisfying the assumptions of the theorem. Namely, the only other fixed points with isotropy containing either p or q is of the form $(\pm p, \pm q', \mp 1)$ or $(\pm p', \pm q, \mp 1)$, where here we could have $p' = p$ or $q' = q$. Consider the class $\widetilde{E}_1^{+}$ as above. We will show the following.

Claim 4. *For all $\lambda \in S^1$, the exceptional class $\widetilde{E}_1^{+}$ has a representative $\widetilde{C}_\lambda^{E}$ by a smooth, embedded $\widetilde{J}(\lambda)$-holomorphic sphere such that $\widetilde{\omega}_\lambda(\widetilde{C}_\lambda^{E})$ is an increasing function of λ as λ moves counterclockwise around S^1.*

Before we prove this claim, we show how we can use this claim to prove the theorem. Picking a base point $\lambda_0 \in S^1$ and using the claim repeatedly, we would obtain exceptional spheres $\widetilde{C}_{\lambda_0+2\pi i}^{E}$

for all i. Also, since $\widetilde{\omega}_\lambda(\widetilde{C}^E_\lambda)$ is an increasing function of λ, we would have

$$\widetilde{\omega}_{\lambda_0}(\widetilde{C}^E_{\lambda_0}) < \widetilde{\omega}_{\lambda_0}(\widetilde{C}^E_{\lambda_0+2\pi}) < \ldots < \widetilde{\omega}_{\lambda_0}(\widetilde{C}^E_{\lambda_0+2k\pi}) < \ldots$$

so that all these exceptional spheres would represent different homology classes. In particular, this would imply that the set $\mathcal{E}_{\lambda_0}$ of exceptional classes in $(\widetilde{M}_{\lambda_0}, \widetilde{\omega}_{\lambda_0})$ would be infinite where by all our previous assumptions, $\widetilde{M}_{\lambda_0}$ is a closed, symplectic 4 manifold with $b_2^+ > 1$, and thus has a finite number of exceptional classes. This contradiction would then finish the proof of the theorem.

Thus, it only remains to prove the claim. We will split this up into cases. Namely, we claim the following, which implies Claim 4.

Claim 5.
- *If $\widetilde{E}_1^+$ is represented by an embedded $\widetilde{J}(\lambda)$-holomorphic sphere in $\widetilde{M}_{\lambda'}$, for some $\lambda' \in U_k$, then the same is true for all $\lambda \in U_k$.*
- *If $\widetilde{E}_1^+$ is represented by an embedded $\widetilde{J}(\lambda)$-holomorphic sphere in $\widetilde{M}_{\lambda'}$, for some $\lambda' \in I_k$, then the same is true for all $\lambda \in I_k$.*

We first prove Statement 1. In particular, for some $\lambda' \in U_k$, $\widetilde{E}_1^+$ is represented by an embedded $\widetilde{J}(\lambda)$ holomorphic sphere $\widetilde{C}^E_{\lambda'}$. By Lemma 6.4.5, we have diffeomorphisms

$$\phi_\lambda : M_{\lambda'} \longrightarrow M_\lambda, \quad \widetilde{\phi}_\lambda : \widetilde{M}_{\lambda'} \longrightarrow \widetilde{M}_\lambda$$

for all $\lambda \in U_k$. Thus, if $\widetilde{E}_1^+$ is represented by an embedded, $\widetilde{J}(\lambda')$ holomorphic sphere $\widetilde{C}^E_{\lambda'}$, we can push forward by $\widetilde{\phi}_\lambda$ to obtain embedded, $\widetilde{J}(\lambda)$-holomorphic sphere $\widetilde{C}^E_\lambda$ representing $\widetilde{E}_1^+$, as desired.

We now prove Statement 2. In particular, for some $\lambda' \in I_k$, $\widetilde{E}_1^+$ is represented by an embedded $\widetilde{J}(\lambda)$ holomorphic sphere $\widetilde{C}^E_{\lambda'}$. Without loss of generality, assume the fixed point corresponding to I_k has isotropy weights $(p_k, q_k, -r_k)$. Lemma 6.4.5 implies that there are diffeomorphisms

$$\phi_\lambda : M_\lambda \longrightarrow M_{\lambda_k}, \quad \widetilde{\phi}_\lambda : \widetilde{M}_\lambda \longrightarrow \widetilde{M}_{\lambda_k}$$

for all $\lambda \in (\lambda_k - \epsilon, \lambda_k]$. Thus, pushing forward by $\widetilde{\phi}_\lambda$, we see that $\widetilde{E}_1^+$ is represented by an embedded $\widetilde{J}(\lambda)$ holomorphic sphere $\widetilde{C}^E_\lambda$ in $\widetilde{M}_\lambda$ for all $\lambda \in (\lambda_k - \epsilon, \lambda_k]$ if and only if it is represented by an embedded, $\widetilde{J}(\lambda_k)$-holomorphic sphere $\widetilde{C}^E_{\lambda_k}$ in $\widetilde{M}_{\lambda_k}$. Thus, to prove statement (2), it suffices to show we have spheres $\widetilde{C}^E_\lambda$ as desired for all $\lambda \in (\lambda_k, \lambda_k + \epsilon)$ if and only if we have a sphere $\widetilde{C}^E_{\lambda_k}$ as desired for λ_k.

Since we assumed our critical point is of the form $(p_k, q_k, -r_k)$, we know that for all $\lambda \in (\lambda_k, \lambda_k + \epsilon)$, M_λ is the (p_k, q_k)-weighted blowup of M_{λ_k} at a point y^{λ_k} which is an orbifold point of order r_k, or is smooth if $r_k = 1$. In particular, if $C^{E,+}_{\lambda,k}$ is a curve representing the weighted exceptional divisor E^+_k as a $J(\lambda)$ holomorphic weighted exceptional divisor, there is a map

$$\rho_\lambda : M_\lambda \longrightarrow M_{\lambda_k}$$

so that the restriction

$$\rho_\epsilon : M_\lambda \setminus C^{E,+}_{\lambda_k} \longrightarrow M_{\lambda_k} \setminus \{y^{\lambda_k}\}$$

is an orientation preserving diffeomorphism for all $\lambda \in (\lambda_k, \lambda_k + \epsilon)$. This means we have to show two things. In particular, if we have $C^E_{\lambda_k}$ defined as desired, we need to show that $y^{\lambda_k} \notin C^E_{\lambda_k}$ while if we can define C^E_λ for all $\lambda \in (\lambda_k, \lambda_k + \epsilon)$, we must show that $C^E_\lambda \cap C^{E,+}_{\lambda,k} = \emptyset$. Indeed, this would show that we can define C^E_λ along all of I_k, so that by taking the resolution, we can define $\widetilde{C}^E_\lambda$ along all of I_k, thus proving statement 2.

According to the assumptions of our theorem, there are now 2 cases to consider. Namely, either $r_k > 1$ and none of p_k, q_k, or r_k are equal to either p or q, or $r_k = 1$. We will consider these cases separately.

First, assume $r_k > 1$, so that none of p_k, q_k, or r_k equals p or q. Now assume we have $C^E_{\lambda_k}$ defined as desired. In this case, the blowup point y^{λ_k} occurs at an orbifold point of order r_k with $r_k \neq p$ and $r_k \neq q$. In particular, since $C^E_{\lambda_k}$ only intersects two orbifold points, one of order p and one of order q, we clearly have $y^{\lambda_k} \notin C^E_{\lambda_k}$, as desired.

Now, assume we have C^E_λ defined for all $\lambda \in (\lambda_k, \lambda_k + \epsilon)$. In order to show $C^E_\lambda \cap C^{E,+}_{\lambda,k} = \emptyset$, we must consider two cases. Namely, either these two curves intersect at an orbifold point or they intersect at a smooth point. However, the only orbifold points on C^E_λ have orders p and q while the only orbifold points on $C^{E,+}_{\lambda,k}$ have orders p_k and q_k, and we assumed that neither p_k nor q_k equals either p or q, so that clearly these curves do not intersect at an orbifold point. Furthermore, if they intersect at a smooth point, then in the resolution $\widetilde{M}_\lambda$, the corresponding exceptional divisors $\widetilde{C}^E_\lambda$ and $\widetilde{C}^{E,+}_{\lambda,k}$ would satisfy

$$\widetilde{C}^E_\lambda \cap \widetilde{C}^{E,+}_{\lambda,k} \neq \emptyset$$

which, by Lemma 6.5.2, contradicts the fact that $b^+_2(\widetilde{M}_\lambda) > 1$. In particular, we must have $C^E_\lambda \cap C^{E,+}_{\lambda_k} = \emptyset$ as desired.

Next, consider the case where $r_k = 1$. Since in this case $r_k = 1$, we know that y^{λ_k} is a smooth point which does not intersect any orbifold points and hence corresponds to a point $\widetilde{y}^{\lambda_k}$ in the resolution $\widetilde{M}_{\lambda_k}$. Thus, the map ρ_ϵ lifts to a diffeomorphism

$$\widetilde{\rho}_\epsilon : \widetilde{M}_{\lambda_k+\epsilon} \setminus S_k^{\epsilon,+} \longrightarrow \widetilde{M}_{\lambda_k} \setminus \{\widetilde{y}^{\lambda_k}\}$$

where $S_k^{\epsilon,+}$ is the nodal curve in $\widetilde{M}_{\lambda_k+\epsilon}$ formed by taking the resolution of $C_{\lambda,k}^{E,+}$ as in 6.3.2. This breaks the proof of this case into 2 subcases. Namely, if we can define $\widetilde{C}_{\lambda_k}^E$ as desired, we must show that $\widetilde{y}^{\lambda_k} \notin \widetilde{C}_{\lambda_k}^E$, while if we can define $\widetilde{C}_\lambda^E$ as desired for all $\lambda \in (\lambda_k, \lambda_k + \epsilon)$, we must show that $\widetilde{C}_\lambda^E \cap S_k^{\lambda,+} = \emptyset$.

Consider first the case where we have $\widetilde{C}_{\lambda_k}^E$ defined as desired. Recall from step 1 that the almost complex structure $\widetilde{J}(\lambda)$ was chosen so that $\widetilde{y}^{\lambda_k}$ does not intersect any exceptional spheres so that in particular, $\widetilde{y}^{\lambda_k} \notin \widetilde{C}_{\lambda_k}^E$, as desired.

Next, consider the case where we have $\widetilde{C}_\lambda^E$ defined as desired for all $\lambda \in (\lambda_k, \lambda_k + \epsilon)$. Now, unless we have $p_k = p$ and $q_k = q$, we will obviously have that the exceptional classes $\widetilde{E}_\lambda$ and $\widetilde{E}_{\lambda,k}^+$ are different classes. Then, since $b_2^+(M_\lambda) > 1$ for all λ and since in this case $r_k = 1$, Lemma 6.5.3 implies that $\widetilde{C}_{\lambda,k}^{E,+} \cap \widetilde{C}_\lambda^E = \emptyset$ and that $\widetilde{C}_{i,A}^\lambda \cap \widetilde{C}_\lambda^E = \emptyset$, where $\widetilde{C}_{i,A}^\lambda$ are representatives of the resolution curves of $C_{\lambda,k}^{E,+}$ as in 6.3.2. Thus, combining these facts we get $\widetilde{C}_\lambda^E \cap S_k^{\lambda,+} = \emptyset$ as desired.

The case where $p_k = p$ and $q_k = q$ and we have $\widetilde{C}_\lambda^E$ defined as desired for all $\lambda \in (\lambda_k, \lambda_k + \epsilon)$ can safely be ignored since in this case, we would also have $\lambda \in U_k$ and we could apply the arguments of that case to get $\widetilde{C}_\lambda^E$ defined for all λ in the interval $(\lambda_{k+1} - \epsilon, \lambda_{k+1})$ and then use the previous techniques in I_{k+1}. There are no such difficulties in the case of $(-p_k, -q_k, 1)$, since in this case the exceptional classes $\widetilde{E}_\lambda$ and $\widetilde{E}_{\lambda,k}^-$ are obviously different since as is shown below, $\widetilde{\omega}(\lambda)$ is increasing on the first class while it is obviously decreasing on the second class.

Lastly, we will use Statements 1 and 2 to prove Claim 4. Since we assumed λ_1 has isotropy weights $(p, q, -1)$, we know from step 3 that for each $\lambda \in U_1$ there is a $J(\lambda)$ holomorphic weighted exceptional divisor C_λ^E in the class E_1^+. Resolving C_λ^E as in Remark 6.3.2 gives in particular an embedded $\widetilde{J}$-holomorphic sphere $\widetilde{C}_\lambda^E$ in the class $\widetilde{E}_1^+$ so that $\widetilde{\omega}_\lambda(\widetilde{C}_\lambda^E)$ increases as λ moves counterclockwise around S^1. Then, since $U_1 \cap I_2 \neq \emptyset$, we can extend this family to I_2. Similarly, $I_2 \cap U_2 \neq \emptyset$, so we can further extend the family to U_2. A simple induction shows that we can define $\widetilde{C}_\lambda^E$ for all $\lambda \in S^1$. Furthermore, since it comes from a blowup at λ_1, we will still have $\widetilde{\omega}_\lambda(\widetilde{C}_\lambda^E)$

increases as λ moves counterclockwise around S^1, as required. This completes the proof of Claim 4 which in turn completes the proof of the theorem.

□

Remark 7.2.2. Note that the above argument only works under our assumptions that the fixed points are all good and that there is fixed point $(p, q, -1)$ so that any other fixed point $(\pm p_i, \pm q_i, \mp r_i)$ with an isotropy weight equal to p or q has $r_i = 1$. First, the assumption that we never have $r_i = \pm p$ or $\pm q$ implies that we never have to deal with the case of blowup up an orbifold point that lies on a weighted divisor. In particular, the good assumption implies that all weighted divisors arising from fixed points are weighted exceptional divisors, which we use in step 4 to rule out the possibility of two weighted divisors intersecting at a smooth point. Also, the assumption that $r_i = 1$ if there is a shared isotropy weight is important because presently the only method we have to rule out weighted divisors intersecting at an orbifold point is Lemma 6.5.3, which requires that $r_i = 1$.

There are several possible ways to generalize these techniques. Namely, one could try to come up with more general intersection theory techniques to come up with new reasons why a chosen weighted exceptional divisor can not intersect various other weighted divisors either at a smooth point or an orbifold point. Additionally, one could try to compute conditions under which the blowup up an orbifold singularity on a weighted exceptional divisor still results in an exceptional divisor in the resolution. Additionally, one could try to consider weighted divisors on M_λ instead of lifts of weighted exceptional divisors on $\widetilde{M}_\lambda$ if one were to learn more about orbifold Gromov-Witten theory and corresponding facts about intersection theory.

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