Where does smoothness count the most for two-point boundary-value problems?

Technical Report CUCS-026-97

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September 11, 1997

Abstract. We are concerned with the complexity of $2m$th order elliptic two-point boundary-value problems $Lu = f$. Previous work on the complexity of these problems has generally assumed that we had partial information about the right-hand side f and complete information about the coefficients of L, often making unrealistic assumptions about the smoothness of the coefficients of L. In this paper, we study problems in which f has r derivatives in the L_p-sense and for L having the usual divergence form

$$Lv = \sum_{0\le i,j\le m} (-1)^i D^i(a_{i,j} D^j v),$$

with $a_{i,j}$ being $r_{i,j}$-times continuously differentiable. We find that if continuous information is permissible, then the ε-complexity is proportional to $(1/\varepsilon)^{1/(\tilde{r}+m)}$, where

$$\tilde{r} = \min\{\, r, \min_{0\le i,j\le m} \{r_{i,j} - i\}\,\},$$

and show that a finite element method (FEM) is optimal. If only standard information (consisting of function and/or derivative evaluations) is available, we find that the complexity is proportional to $(1/\varepsilon)^{1/r_{\min}}$, where

$$r_{\min} = \min\{\, r, \min_{0\le i,j\le m} \{r_{i,j}\}\,\},$$

and we show that a modified FEM (which uses only function evaluations, and not derivatives) is optimal.

1. Introduction

We are concerned with the computational complexity of linear boundary value problems. These problems are specified by a linear elliptic operator L of given order $2m$ and a function f, so that we want to find the function u satisfying $Lu = f$, along with (say) homogeneous Dirichlet boundary conditions. In this paper, we will restrict our attention to the one-dimensional case, i.e., two-point boundary-value problems defined over a finite real interval. We shall study multidimensional problems, which involve additional technical difficulties, in a future paper.

Previous work on the complexity of such problems has usually assumed that we have complete information about the coefficients of the operator L and partial information about the right-hand side f. Moreover, it is generally assumed that these coefficients are as smooth as necessary; they are often assumed to even be infinitely differentiable. Under these assumptions, the complexity of such problems has long been well-known (see, e.g., [7, Chapter 5]). To be specific, suppose that we

This research was supported in part by the National Science Foundation under Grant CCR-95-00850.

measure error in the energy norm. Suppose f has r derivatives (in the L_2- sense). Then we know the following results:

(1) If continuous linear information is permissible, then the ε-complexity is proportional to $(1/\varepsilon)^{1/(r+m)}$, and a finite element method (FEM) is optimal.
(2) If standard information (function or derivative evaluations) is available, then the complexity is proportional to $(1/\varepsilon)^{1/r}$, and a modified FEM (or an FEM using quadrature) is optimal.

However in practice, the assumption that we have complete information about L is usually unrealistic (except in very special cases, such as a differential operator with constant coefficients). What can we say about the complexity of two-point boundary value problems for which we have only partial information about L and about f?

Some results along these lines were obtained in [8] and [9]. Suppose that the coefficients of L all have the same smoothness r as the right-hand side. (More precisely, the coefficients of L are r-times continuously differentiable and f has r derivatives in the L_p-sense.) We further suppose that only standard information is allowed. Then once again, we find that the complexity is proportional to $(1/\varepsilon)^{1/r}$, and an FEM using quadrature is optimal.

However, there is no reason to assume that the coefficients of L and the function f all have the same number of derivatives. Moreover, we would like to know the complexity for continuous information, as well as for standard information. In this paper, we investigate these topics.

As usual, we consider elliptic operators L in divergence form

$$Lv = \sum_{0\le i,j\le m} (-1)^i D^i(a_{i,j} D^j v).$$

We now assume that each coefficient $a_{i,j}$ being $r_{i,j}$-times continuously differentiable, and that f has r derivatives in the L_p-sense, where $p \in [2,\infty]$.

We first consider the case of continuous information. Let

$$\tilde{r} = \min\{\, r, \min_{0\le i,j\le m} \{r_{i,j} - i\}\,\}.$$

Then the following results hold:

- The ε-complexity is proportional to $(1/\varepsilon)^{1/(\tilde{r}+m)}$.
- If $k \ge 2m + \tilde{r} - 1$, then a FEM of degree k is optimal.

Next, we study standard information. Let

$$r_{\min} = \min\{\, r, \min_{0\le i,j\le m} \{r_{i,j}\}\,\}.$$

Then we have the following results:

- The ε-complexity is proportional to $(1/\varepsilon)^{1/r_{\min}}$.
- If $k \ge m + r_{\min} - 1$, then a modified FEM of degree k is optimal.

This modified FEM uses only function evaluations, even though both function and derivative evaluations are permissible standard information operations. Thus we find that standard information consisting of function evaluations alone is just as powerful as that which allows both function and derivative evaluations.

This allows us to see the effect of the smoothness of the various data making up our problem on the complexity. In particular, let us consider the relative strength of continuous vs. standard information. Previously, when we studied problems for which we had complete information about L, we found that the asymptotic penalty for using standard information instead of continuous information was unbounded, since the complexity of the former was proportional to $(1/\varepsilon)^{1/(r+m)}$, while that of the former was proportional to $(1/\varepsilon)^{1/r}$. However, when we allow each datum of the problem to have its own smoothness, no nontrivial relations hold between the complexities for continuous and for standard information. Indeed, let us consider two extreme examples:

(1) We first consider the case where

$$\min_{0\le i,j\le m}\{r_{i,j}-i\}\ge r.$$

Then the complexity for continuous information is proportional to $(1/\varepsilon)^{1/(r+m)}$, while the complexity for standard information is proportional to $(1/\varepsilon)^{1/r}$. So in this case, continuous information enjoys the same advantage over standard information as before.

(2) Next, we suppose that

$$r_{m,m}=\min_{0\le i,j\le m} r_{i,j}.$$

Then the complexity is proportional to $(1/\varepsilon)^{1/r}$ for both continuous and standard information. Hence continuous information is no more powerful than standard information.

These examples are the endpoints; anything in between can happen.

We close this Introduction by outlining the structure of the rest of this paper. In Section 2, we give a precise definition of the class of problems to be studied. In Section 3 we briefly recall the standard general techniques from [6] that will be useful in what follows. In Section 4, we establish our results for continuous information. Finally, in Section 5, we prove the results for standard information.

2. Problem Description

In what follows, we assume that the reader is familiar with the usual terminology and notations arising in the variational study of elliptic boundary value problems, such as Sobolev spaces, norms, seminorms, inner products, and the like. See [7, Chapter 5 and Appendix] for further details, as well as the references cited therein. For any ordered ring $\mathscr{X}$, we let $\mathscr{X}^+$ and $\mathscr{X}^{++}$ respectively denote the nonnegative and strictly positive elements of $\mathscr{X}$, this notation being used when $\mathscr{X}=\mathbb{R}$ or $\mathscr{X}=\mathbb{Z}$. The ball of the normed linear space X, centered at the origin and having radius R, will be denoted by $\mathscr{B}_R X$. All O-, Ω-, and Θ-relations will be independent of n and ε. We will use C as a generic constant, independent of n, ε, f, and A, whose value may change from one place to another. Where convenient, we shall use $\preccurlyeq$, $\succcurlyeq$, and $\asymp$ to respectively denote O-, Ω, and Θ-relations.

Let $I=(0,1)$, and let m be a given nonnegative integer. For an $(m+1)\times(m+1)$ matrix $A=[a_{i,j}(\cdot)]_{0\le i,j\le m}$ of functions on I, we define a differential operator L_A by

$$L_A v=\sum_{0\le i,j\le m}(-1)^i D^i(a_{i,j}D^j v),$$

with D the derivative operator, and a bilinear form B_A on $H_0^m(I)\times H_0^m(I)$ by

$$B_A(v,w)=\sum_{0\le i,j\le m}\int_I a_{i,j}D^j v D^i w \qquad \forall\, v,w\in H_0^m(I).$$

In what follows, we will write

$$\langle v,w\rangle=\int_I vw$$

for any (generalized) functions v and w such that this integral exists.

We are interested in elliptic two-point boundary-value problems. The *classical formulation* of such a problem is to find, for $f: I \to \mathbb{R}$, a function $u: \overline{I} \to \mathbb{R}$ such that

$$\begin{aligned} L_A u &= f \quad \text{in } I, \\ (D^j u)(0) = (D^j u)(1) &= 0 \quad (0 \le j \le m-1), \end{aligned} \tag{2.1}$$

The *variational formulation* is to find, for $f \in W^{r,p}(I)$, an element $u \in H_0^m(I)$ such that

$$B_A(u, v) = \langle f, v \rangle \quad \forall v \in H_0^m(I). \tag{2.2}$$

Let us define a class $\mathscr{A}$ of coefficients for $2m$th order elliptic operators, depending on the following given parameters:

- ▷ positive constants γ and γ_0, with $\gamma \le \gamma_0$,
- ▷ a set $\{r_{i,j}\}_{0 \le i,j \le m}$ of nonnegative integers,
- ▷ a set $\{M_{i,j}\}_{0 \le i,j \le m}$ of positive reals, with $M_{i,j} \ge \gamma$ for $0 \le i, j \le m$.

Then we say that $A \in \mathscr{A}$ if the following conditions hold:

(1) The operator L_A is *strongly elliptic* in I, i.e.,

$$(-1)^m a_{m,m}(x) \ge \gamma_0 \qquad \forall x \in I.$$

(2) The bilinear form B_A is *uniformly weakly $H_0^m(I)$-coercive*, i.e., for any $v \in H_0^m(I)$, there exists nonzero $w \in H_0^m(I)$ such that

$$B_A(v, w) \ge \gamma \|v\|_{H^m(I)} \|w\|_{H^m(I)}. \tag{2.3}$$

(3) For any indices i and j, the coefficient $a_{i,j}$ of the operator L_A is bounded in the $C^{r_{i,j}}$ sense, i.e.,

$$\|a_{i,j}\|_{C^{r_{i,j}}(\overline{I})} \le M_{i,j}.$$

Roughly speaking, $A \in \mathscr{A}$ if (2.1) is a two-point elliptic boundary value problem, the only novelty being that we require a "uniformity condition."

We next let $r \ge -m$, $p \in [2, \infty]$, and $M > 0$ be given parameters. Our class of problem elements is then

$$F = \mathscr{B}_M W^{r,p}(I) \times \mathscr{A}.$$

We define a solution operator $S: F \to H_0^m(I)$ by letting $u = S([f; A])$ iff u satisfies (2.2), i.e., u is the variational solution to the Dirichlet problem (2.1). The operator S is nonlinear. However, $S([f; A])$ depends nonlinearly only on A, i.e., for any fixed A, the operator $S([\cdot; A])$ is a linear operator. Hence we may use the generalized Lax-Milgram Lemma ([2, pg. 112], [5, pg. 310]) to see that for any $[f; A] \in F$, there exists a unique solution $u \in H_0^m(I)$ to (2.2). Hence, the solution operator S is well-defined.

Next, we need to define information for our problem. Letting

$$\Lambda_{i,j} \subset [C^{r_{i,j}}(\overline{I})]^* \qquad (0 \le i, j \le m)$$

denote the set of permissible information functionals for the coefficient function $a_{i,j}$ and

$$\Lambda \subset [W^{r,p}(I)]^*,$$

denote the set of permissible information functionals for the right-hand side f, we let

$$\mathscr{L} = \Lambda \times \left[\bigotimes_{0 \le i,j \le m} \Lambda_{i,j} \right].$$

We say that N is *information (using $\mathscr{L}$)* if

$$N([f;A]) = y = [y_1, \dots, y_n],$$

with $n = n(y)$, where either

$$y_l = \lambda_l(f; y_1, \dots, y_{l-1}) \qquad \text{for some } \lambda_l(\cdot; y_1, \dots, y_{l-1}) \in \Lambda$$

or there exist indices $i, j \in \{0, \dots, m\}$ such that

$$y_l = \lambda_l(a_{i,j}; y_1, \dots, y_{l-1}) \qquad \text{for some } \lambda_l(\cdot; y_1, \dots, y_{l-1}) \in \Lambda_{i,j}.$$

Note that for any l,

- ▷ whether to terminate at the lth step,
- ▷ whether to evaluate a functional of the right-hand side f or of some coefficient $a_{i,j}$,
- ▷ which functional to evaluate

may all be determined adaptively, depending on the previously-calculated $y_1, \dots, y_{l-1}$.

The most important choices of information are the following:

(1) *Continuous information $\mathscr{L}^*$*. Here

$$\Lambda_{i,j} = [C^{r_{i,j}}(\overline{I})]^* \qquad (0 \le i, j \le m)$$

and

$$\Lambda = [W^{r,p}(I)]^*.$$

(2) *Standard information $\mathscr{L}^{\text{std}}$*. Here

$$\lambda \in \Lambda_{i,j} \text{ iff } \exists\, x \in \overline{I}, l \in [0, r_{i,j}] \text{ such that } \lambda(w) = (D^l w)(x) \qquad (0 \le i, j \le m)$$

and

$$\lambda \in \Lambda \text{ iff } \exists\, x \in \overline{I}, l \in [0, r - 1/p) \text{ such that } \lambda(w) = (D^l w)(x),$$

such λ being well-defined by the Sobolev embedding theorem. Note that although the solution operator is well-defined for any $r \ge -m$ and any nonnegative $\{r_{i,j}\}_{0 \le i,j \le m}$, standard information imposes additional restrictions on r and $\{r_{i,j}\}_{0 \le i,j \le m}$.

Let

$$Y = \bigcup_{[f;A] \in F} N([f;A])$$

denote the set of all possible information values. Then an *algorithm* using the information N is a mapping $\phi\colon Y \to H_0^m(I)$.

We want to solve this problem in the worst case setting. This means that the *cardinality* of information N is given by

$$\operatorname{card} N = \sup_{y \in Y} n(y),$$

and the *error* of an algorithm ϕ using N is given by

$$e(\phi, N) = \sup_{y\in Y} \sup_{[f;A]\in N^{-1}(y)} \|S([f;A]) - \phi(y)\|_{H^m(I)},$$

where

$$N^{-1}(y) = \{\, [f;A] \in F : N([f;A]) = y \,\}.$$

Our model of computation is the standard one studied in information-based complexity:

(1) There is a fixed positive constant c such that for permissible linear functional λ and any function v defined on I, the cost of calculating $\lambda(v)$ is c.
(2) Arithmetic operations and comparisons are done exactly, with unit cost.
(3) We are not charged for Boolean operations.
(4) Linear operations over $H_0^m(I)$ are done exactly, with cost $\mathbf{g}$.

For any information N and any algorithm ϕ using N, we shall let $\mathrm{cost}(\phi, N)$ denote the worst case cost of calculating $\phi\big(N([f;A])\big)$ over all $[f;A] \in F$.

We now say that

$$\mathrm{comp}(\varepsilon) = \inf\{\, \mathrm{cost}(\phi, N) : N \text{ and } \phi \text{ such that } e(\phi, N) \le \varepsilon \,\}$$

is the *ε-complexity* of our problem, and an algorithm ϕ using information N for which

$$e(\phi, N) \le \varepsilon \qquad \text{and} \qquad \mathrm{cost}(\phi, N) = \Theta\big(\mathrm{comp}(\varepsilon)\big)$$

is said to be an *optimal algorithm.*

3. Some general remarks

Recall that the goal of this paper is to find the complexity of our class of two-point boundary-value problems, as well as optimal algorithms, for the classes of continuous and standard information. We do this by applying the standard techniques described in, e.g., [6, Chapter 4], which we outline here for convenience.

First, we determine a lower bound on the complexity. We do this as follows. For information N, let

$$r(N) = \inf_{\phi} e(\phi, N)$$

denote the *radius of information*, i.e., the minimal error among algorithms using N. Since $r(N)$ is often hard to directly evaluate, we use the inequality

$$r(N) \le d(N) \le 2\, r(N), \tag{3.1}$$

where

$$d(N) = \sup_{y\in Y} \sup_{[f;A],[\tilde f;\tilde A]\in N^{-1}(y)} \|S([\tilde f;\tilde A]) - S([f;A])\|_{H^m(I)}$$

is the *diameter of information.* Let

$$r(n) = \inf\{\, r(N) : \mathrm{card}\, N \le n \,\}$$

denote the nth *minimal radius* (*of information*). If we let

$$m(\varepsilon) = \inf\{\, n \in \mathbb{Z}^+ : r(n) \le \varepsilon \,\},$$

denote the ε-*cardinality number*, then we immediately find that

$$\operatorname{comp}(\varepsilon) \geq c\, m(\varepsilon). \tag{3.2}$$

Moreover, even if we only establish a lower bound on $r(n)$ and, hence, on $m(\varepsilon)$, then (3.2) still holds.

Next, we determine an upper bound on the complexity that matches the lower bound determined previously. We do this by proposing, for each $n \in \mathbb{Z}^+$, an algorithm ϕ_n using information of cardinality n whose error is $O(r(n))$ and whose cost is $cn + \Theta(n)$. If we now choose $n = \Theta(m(\varepsilon))$, then the algorithm ϕ_n is computes an ε-approximation at cost $O(c\, m(\varepsilon))$. Hence this algorithm is optimal, and

$$\operatorname{comp}(\varepsilon) \asymp c\, m(\varepsilon).$$

One final remark before proceeding further: we allow ourselves one slight change in notation. Since we will be interested in the complexity for the classes $\mathscr{L} = \mathscr{L}^*$ and $\mathscr{L} = \mathscr{L}^{\mathrm{std}}$, we will explicitly show the dependence of the complexity and the minimal radius on the class $\mathscr{L}$ of permissible information functionals, writing $\operatorname{comp}(\varepsilon, \mathscr{L})$ and $r(n, \mathscr{L})$ in what follows.

4. Complexity for continuous information

In this section, we study the complexity of our problem when continuous information is permissible. We will show that the ε-complexity of our problem is proportional to $(1/\varepsilon)^{\tilde{r}+m}$, where

$$\tilde{r} = \min\{\, r, \min_{0 \leq i,j \leq m} \{r_{i,j} - i\}\, \}.$$

In addition, we define the finite element method (FEM) and give conditions that are necessary for the FEM to be optimal.

As mentioned in Section 3, we first prove a lower bound on the nth minimal radius.

THEOREM 4.1.

$$r(n, \mathscr{L}^*) \succcurlyeq \left(\frac{1}{n}\right)^{\tilde{r}+m}.$$

PROOF: Let N be information of cardinality at most n using information functionals from $\mathscr{L}^*$. In what follows, we will define $A \in \mathscr{A}$ as the constant matrix

$$a_{i,j} = \begin{cases} \gamma_0 & \text{if } i = j = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

We first show that

$$r(N) \succcurlyeq n^{-(r+m)}. \tag{4.2}$$

Indeed, let $f \equiv \frac{1}{2}M$. Write

$$N([f; A]) = [y_1, \dots, y_{\hat{n}}], \tag{4.3}$$

where $\hat{n} \leq n$. Let $\tilde{n} = \tilde{n}([f; A])$ be the number of evaluations of functionals of f in (4.3). Without essential loss of generality, we may assume that

$$y_l = \lambda_l(f) \qquad (1 \leq l \leq \tilde{n})$$

for functionals $\lambda_1, \dots, \lambda_{\tilde{n}} \in [W^{r,p}(I)]^*$. Note that the number and choice of these functionals may depend on f and on A.

Define new information

$$\tilde{N} = [\lambda_1, \dots, \lambda_{\tilde{n}}].$$

Then

$$d(N) \geq \sup_{\substack{[f+z;A]\in F \\ \tilde{N}z=0}} \|S([f+z;A]) - S([f;A])\|_{H^m(I)}. \tag{4.4}$$

Note that if $z \in B_{M/2}W^{r,p}(I)$, then $[f+z;A] \in F$, and so

$$d(N) \geq \sup_{z\in B_{M/2}W^{r,p}(I)\cap\ker\tilde{N}} \|S([z;A])\|_{H^m(I)}.$$

Let

$$M^* = \max_{0\leq i,j\leq m} M_{i,j}. \tag{4.5}$$

Since

$$M^*\|S([z;A])\|_{H^m(I)}\|v\|_{H^m(I)} \geq |B_A(S[z;A],v)| = |\langle z,v\rangle| \qquad \forall\, v \in H_0^m(I),$$

we have

$$M^*\|S([z;A])\|_{H^m(I)} \geq \sup_{v\in H_0^m(I)} \frac{|\langle z,v\rangle|}{\|v\|_{H^m(I)}} = \|z\|_{H^{-m}(I)}. \tag{4.6}$$

Using (4.4) and (4.6), we have

$$\begin{aligned} d(N) &\geq \frac{1}{M^*} \sup_{z\in B_{M/2}W^{r,p}(I)\cap\ker\tilde{N}} \|z\|_{H^{-m}(I)} \\ &\geq \frac{M}{2M^*} d^{\tilde{n}}(BW^{r,p}(I), H^{-m}(I)), \end{aligned} \tag{4.7}$$

where $d^{\tilde{n}}$ denotes the usual Gelfand width, see, e.g., [4, pg. 401]. Since $p \geq 2$, we have

$$d^{\tilde{n}}(BW^{r,p}(I), H^{-m}(I)) \asymp d^{\tilde{n}}(BW^{r+m,p}(I), L_2(I)) \asymp \tilde{n}^{-(r+m)}, \tag{4.8}$$

see [4, Proposition 13.8.2]. From (3.1), (4.7), and (4.8), we have

$$r(N) \geq \tfrac{1}{2}d(N) \succcurlyeq \tilde{n}^{-(r+m)}.$$

Since $\tilde{n} \leq n$, it follows that (4.2) holds, as claimed.

We now claim that

$$r(N) \succcurlyeq n^{-(r_{i,j}-i+m)} \qquad (0 \leq i,j \leq m). \tag{4.9}$$

Indeed, choose $i,j \in \{0,\dots,m\}$. Let $I' \subset\subset I$, and choose $f \in BW^{r,p}(I)$ such that if $u = S([f;A])$ with A given by (4.1), then there exists $\alpha > 0$ such that $D^j u \equiv \alpha$ on I'. We write

$$N([f;A]) = [y_1, \dots, y_{\hat{n}}], \tag{4.10}$$

where $\hat{n} \leq n$. Let $\tilde{n} = \tilde{n}([f;A])$ be the number of evaluations of functionals of $a_{i,j}$ in (4.10). Without essential loss of generality, we may assume that

$$y_l = \lambda_l(f) \qquad (1 \leq l \leq \tilde{n})$$

for functionals $\lambda_1, \dots, \lambda_{\tilde n} \in [C^{r_{i,j}}(\overline{I})]^*$. Note that the number and choice of these functionals may depend on f and on A.

Define new information

$$\tilde N = [\lambda_1, \dots, \lambda_{\tilde n}].$$

For $z \in B_{M_{i,j}-a_{i,j}} C^{r_{i,j}}(\overline{I}) \cap \ker \tilde N$, define the matrix $\tilde A(z)$ as

$$\tilde a_{i',j'}(z) = \begin{cases} a_{i',j'} + z & \text{if } (i',j') = (i,j), \\ a_{i',j'} & \text{if } (i',j') \neq (i,j). \end{cases} \tag{4.11}$$

Then for any such z, we have

$$[f; \tilde A(z)] \in F \qquad \text{and} \qquad N([f;\tilde A(z)]) = N([f;A]).$$

Thus

$$d(N) \ge \sup_{z \in B_{M_{i,j}-a_{i,j}} C^{r_{i,j}}(\overline{I}) \cap \ker \tilde N} \|S([f;\tilde A(z)]) - S([f;A])\|_{H^m(I)}.$$

Let us momentarily fix a choice of $z \in B_{M_{i,j}-a_{i,j}} C^{r_{i,j}}(\overline{I}) \cap \ker \tilde N$, and let $\tilde u = S([f;\tilde A(z)])$. One may easily check that

$$B_{\tilde A(z)}(u - \tilde u, v) = \int_I z D^j u D^i v \qquad \forall\, v \in H_0^m(I), \tag{4.12}$$

and so

$$\begin{aligned} M^* \|u - \tilde u\|_{H^m(I)} \|v\|_{H^m(I)} &\ge |B_{\tilde A(z)}(u - \tilde u, v)| \\ &= \left| \int_I z D^j u D^i v \right| \qquad \forall\, v \in H_0^m(I). \\ &= |\langle D^i(zD^j u), v\rangle| \end{aligned}$$

Thus

$$\begin{aligned} M^* \|u - \tilde u\|_{H^m(I)} &\ge \sup_{v \in H_0^m(I)} \frac{|\langle D^i(zD^j u), v\rangle|}{\|v\|_{H^m(I)}} \\ &= \|D^i(zD^j u)\|_{H^{-m}(I)} \\ &\ge C \|zD^j u\|_{H^{-(m-i)}(I)} \end{aligned}$$

for some C, independent of z, $\tilde n$, and u. Since $z \in B_{M_{i,j}-a_{i,j}} C^{r_{i,j}}(\overline{I}) \cap \ker \tilde N$ is arbitrary, we thus have

$$\begin{aligned} d(N) &\ge \frac{C}{M^*} \sup_{z \in B_{M_{i,j}-a_{i,j}} C^{r_{i,j}}(\overline{I}) \cap \ker \tilde N} \|zD^j u\|_{H^{-(m-i)}(I)} \\ &= \frac{C(M_{i,j} - a_{i,j})}{M^*} \sup_{z \in BC^{r_{i,j}}(\overline{I}) \cap \ker \tilde N} \|zD^j u\|_{H^{-(m-i)}(I)} \\ &\ge \frac{\alpha C(M_{i,j} - a_{i,j})}{M^*} \sup_{\substack{z \in BC^{r_{i,j}}(\overline{I}) \cap \ker \tilde N \\ \text{support } z \subset I'}} \|z\|_{H^{-(m-i)}(I)} \\ &\succcurlyeq \sup_{z \in BC_0^{r_{i,j}}(\overline{I'}) \cap \ker \tilde N} \|z\|_{H^{-(m-i)}(I)}. \end{aligned}$$

Let $I'' \subset\subset I'$. For $g \in C^{r_{i,j}}(\overline{I''})$, we let $\Sigma g \in C^{r_{i,j}}(\overline{I})$ be an extension of g to all of I for which support $\Sigma g \subset I'$. This extension may be chosen so that $\Sigma\colon C^{r_{i,j}}(\overline{I''}) \to C^{r_{i,j}}(\overline{I})$ is a bounded operator. Now define continuous information $\tilde{\tilde{N}}$ of cardinality $\tilde{n}$ on $C^{r_{i,j}}(\overline{I''})$ by

$$\tilde{\tilde{N}} = \tilde{N} \circ \Sigma.$$

Then

$$\begin{aligned} d(N) &\succcurlyeq \sup_{z \in BC^{r_{i,j}}(\overline{I''}) \cap \ker \tilde{\tilde{N}}} \|z\|_{H^{-(m-i)}(I)} \\ &\geq d^{\tilde{n}}(BC^{r_{i,j}}(\overline{I''}), H^{-(m-i)}(I'')) \\ &\geq d^{\tilde{n}}(BC^{r_{i,j}-i+m}(\overline{I''}), L_2(I'')) \\ &\asymp \tilde{n}^{-(r_{i,j}-i+m)}, \end{aligned}$$

see, e.g., [4, Theorem 14.3.6]. From this inequality and (3.1), we find

$$r(N) \geq \tfrac{1}{2} d(N) \succcurlyeq \tilde{n}^{-(r_{i,j}-i+m)}.$$

Since $\tilde{n} \leq n$, we see that (4.9) holds, as claimed.

Since (4.2) and (4.9) hold, we see that

$$r(N) \succcurlyeq n^{-(\tilde{r}+m)}.$$

But N is arbitrary continuous information of cardinality at most n, and so the theorem follows immediately. □

Having established a lower bound for the nth minimal radius, we want to find an algorithm using continuous information of cardinality n whose error matches this lower bound. We shall show that an appropriately-chosen finite element method (FEM) is optimal.

We first describe the FEM. Choosing $k \in \mathbb{Z}^{++}$, we let P_k denote the space of polynomials of degree at most k. For $\ell \in \mathbb{Z}^{++}$, we let

$$\Delta := \Delta_\ell = \{t_0, \dots, t_\ell\}$$

be a uniform partition of I, i.e.,

$$t_i = \frac{i}{\ell+1} \qquad (0 \leq i \leq \ell+1).$$

Let

$$\mathscr{S}_\Delta = \left\{ v \in H_0^m(I) : v\big|_{[t_i,t_{i+1}]} \in P_k \text{ for } 0 \leq i \leq \ell-1 \right\}$$

denote a spline space of dimension

$$n_\Delta = \dim \mathscr{S}_\Delta = (k+1)\ell - m(\ell+1).$$

Then the FEM of degree k using the partition Δ may be described as follows: for $[f;A] \in F$, find $u_\Delta \in \mathscr{S}_\Delta$ satisfying

$$B_A(u_\Delta, v) = \langle f, v \rangle \qquad \forall\, v \in \mathscr{S}_\Delta. \tag{4.13}$$

By the "inf-sup lemma" (see, e.g., [2, pg. 290 *ff.*]), there exists $\ell^* \in \mathbb{Z}^{++}$ such that if $\ell \geq \ell^*$, then u_Δ is well-defined. Moreover, there is a positive constant $C = C(\gamma_0, \gamma, M^*)$ such that

$$\|u - u_\Delta\|_{H^m(I)} \leq C \inf_{v \in \mathscr{S}_\Delta} \|u - v\|_{H^m(I)}. \tag{4.14}$$

We wish to express the FEM as an algorithm ϕ_n using continuous information N_n of cardinality n, where $n \asymp n_\Delta$. Before we can do this, we let

$$x_j = \frac{j}{n_\Delta + 1} \qquad (1 \leq j \leq n_\Delta), \tag{4.15}$$

and choose a basis $\{s_1, \ldots, s_{n_\Delta}\}$ for $\mathscr{S}_\Delta$ by the condition

$$s_i \in \mathscr{S}_\Delta \text{ satisfies } s_i(x_j) = \delta_{i,j} \qquad (1 \leq i, j \leq n_\Delta). \tag{4.16}$$

If we write

$$u_\Delta(x) = \sum_{j=1}^{n_\Delta} \alpha_j s_j(x),$$

then

$$G\alpha = \beta, \tag{4.17}$$

where

$$g_{i,j} = B_A(s_j, s_i) = \sum_{0 \leq i', j' \leq m} \langle D^{j'} s_j D^{i'} s_i, a_{i',j'} \rangle \qquad (1 \leq i, j \leq n_\Delta)$$

and

$$\beta_i = \langle f, s_i \rangle \qquad (1 \leq i \leq n_\Delta).$$

For any $i', j' \in \{0, \ldots, m\}$, let $\lambda_{1;i',j'}, \ldots, \lambda_{n_{i',j'};i',j'}$ be the nonzero linear functionals among $\{\, \langle D^{j'} s_j D^{i'} s_i, \cdot \rangle : 1 \leq i, j \leq n_\Delta \,\}$, and let

$$n = n^*(\ell) := \sum_{0 \leq i', j' \leq m} n_{i',j'} + n_\Delta.$$

Now the basis functions have "small supports," i.e., the number of overlapping supports of the basis functions is independent of n_Δ. This implies that the following hold:

(1) $n \asymp n_\Delta$.
(2) Each entry of G and β can be computed in constant time, independent of n.
(3) The linear system (4.17) is banded, the bandwidth depending only on k and m.

Using the uniform weak coercivity of B_A, it easily follows that for $\ell \geq \ell^*$, we can solve (4.17) in $\Theta(n_\Delta)$ arithmetic operations using banded Gaussian elimination without pivoting.

Allowing a slight abuse of notation, we now define information N_n by

$$N_n([f; A]) = [\langle f, s_1 \rangle, \ldots, \langle f, s_{n_\Delta} \rangle; N_n(A)],$$

with

$$N_n(A) = \{\, \lambda_{1;i',j'}(a_{i',j'}), \ldots, \lambda_{n_{i',j'};i',j'}(a_{i',j'}) : 0 \leq i', j' \leq m \,\}.$$

For $[f; A] \in F$ and $\ell \geq \ell^*$, we know that there exists a unique $u_\Delta \in \mathscr{S}_\Delta$ satisfying (4.13). Thus for $[f; A] \in F$ and $n \geq n^* = n^*(\ell^*)$, we may write

$$u_\Delta = \phi_n(N_n([f; A])).$$

The algorithm ϕ_n and the information N_n define the *finite elment method*. From the remarks in the preceding paragraph, it is clear that

$$\operatorname{cost}(\phi_n, N_n) = cn + \Theta(n),$$

the Θ-factor depending only on k and m.

The error of the FEM is given by

THEOREM 4.2. *Let*

$$k \geq 2m + \tilde{r} - 1, \tag{4.18}$$

where

$$\tilde{r} = \min\{\, r, \min_{0 \leq i,j \leq m} \{r_{i,j} - i\}\,\}.$$

For $n \geq n^*$, the error of the FEM *using finite element information of cardinality n satisfies satisfies*

$$e(\phi_n, N_n) \asymp \left(\frac{1}{n}\right)^{\tilde{r}+m}.$$

Before proving Theorem 4.2, we recall a standard approximation result for spline spaces. For $v \in H_0^m(I)$, let

$$(\Pi_\Delta v)(x) = \sum_{j=1}^{n_\Delta} v(x_j) s_j(x). \tag{4.19}$$

denote the $\mathscr{S}_\Delta$-*interpolant* of v.

LEMMA 4.1. *Let $v \in H_0^m(I) \cap H^s(I)$ and let Δ be a uniform partition of I. There exists a constant C, independent of v and Δ, such that*

$$\|v - \Pi_\Delta v\|_{H^m(I)} \leq C n^{-(\min\{k+1,s\}-m)} \|v\|_{H^s(I)}.$$

PROOF: See, e.g., [7, Lemma 5.4.3]. □

We are now ready to give the

PROOF OF THEOREM 4.2: Let $n \geq n^*$. From Theorem 4.1, it clearly suffices to establish that

$$e(\phi_n, N_n) \preccurlyeq n^{-(\tilde{r}+m)}. \tag{4.20}$$

Now let $[f; A] \in F$ and $u = S([f; A])$. From standard results on elliptic regularity theory (such as [3, Theorem 17.2]) we know that $u \in H^{2m+\tilde{r}}(I) \cap H_0^m(I)$, and there is a constant C such that

$$\|u\|_{H^{2m+\tilde{r}}(I)} \leq C \|f\|_{H^{\tilde{r}}(I)}. \tag{4.21}$$

Note that C depends (in a rather complicated manner) on γ, γ_0, r, and on $\{M_{i,j}, r_{i,j}\}_{0 \leq i,j \leq m}$, but is independent of $[f; A]$ and u.

Since (4.18) holds, we have

$$\min\{k+1, 2m+\tilde{r}\} - m = \tilde{r} + m. \tag{4.22}$$

Using (4.14), (4.22), Lemma 4.1 (with $v = u$, k satisfying (4.18), and $s = 2m + \tilde{r}$), and (4.21), we have

$$\begin{aligned} \|S([f; A]) - \phi_n(N_n([f; A]))\|_{H^m(I)} &\leq C \|u - \Pi_\Delta u\|_{H^m(I)} \\ &\leq C n^{-(\tilde{r}+m)} \|u\|_{H^{\tilde{r}+2m}(I)} \\ &\leq C n^{-(\tilde{r}+m)} \|f\|_{H^{\tilde{r}}(I)} \\ &\leq C M^* n^{-(\tilde{r}+m)}, \end{aligned}$$

the latter since $f \in \mathscr{B}_M W^{r,p}(I)$, $\tilde{r} \geq r$, and $p \geq 2$. Since $[f; A]$ is an arbitrary element of F, we immediately have (4.20), which establishes the desired result. □

Using Theorems 4.1 and 4.2 and the discussion in Section 3, we immediately have

COROLLARY 4.1. *Suppose that continuous information is permissible. Let*

$$\tilde{r} = \min\{\, r, \min_{0 \le i,j \le m} \{r_{i,j} - i\}\,\}.$$

(1) *The nth minimal radius is*

$$r(n, \mathscr{L}^*) \asymp \left(\frac{1}{n}\right)^{\tilde{r}+m}$$

.

(2) *The ε-complexity is*

$$\mathrm{comp}(\varepsilon, \mathscr{L}^*) \asymp \left(\frac{1}{\varepsilon}\right)^{1/(\tilde{r}+m)}.$$

(3) *If $k \ge 2m + \tilde{r} - 1$, then the* FEM *of degree k using finite element information of cardinality $n \asymp (1/\varepsilon)^{1/(\tilde{r}+m)}$ is optimal.* □

5. Complexity for standard information

In this section, we study the complexity of our problem when standard information is permissible. We will show that the ε-complexity of our problem is proportional to $(1/\varepsilon)^{r_{\min}}$, where

$$r_{\min} = \min\{\, r, \min_{0 \le i,j \le m} \{r_{i,j}\}\,\}. \tag{5.1}$$

In addition, we define a modified finite element method and give conditions that are necessary for the modified FEM to be optimal.

As mentioned in Section 3, we first prove a lower bound on the nth minimal radius.

THEOREM 5.1. $r(n, \mathscr{L}^{\mathrm{std}}) \succcurlyeq n^{-r_{\min}}$.

PROOF: Let N be information of cardinality at most n from $\mathscr{L}^{\mathrm{std}}$. We shall once again choose the special $A \in \mathscr{A}$ that was defined by (4.1).

We first claim that

$$r(N) \succcurlyeq n^{-r}. \tag{5.2}$$

Let $f \equiv \frac{1}{2}M$. Suppose that $N([f;A])$ contains $\tilde{n}$ evaluations of f (or its derivatives) at points in I. Without essential loss of generality, we may assume that these evaluations are of the form $(D^{l_1}f)(x_1), \dots, (D^{l_{\tilde{n}}}f)(x_{\tilde{n}})$ Define new information

$$\tilde{N}(w) = [(D^{l_1}w)(x_1) \dots (D^{l_{\tilde{n}}})w)(x_{\tilde{n}})] \qquad \forall\, w \in BW^{r,p}(I),$$

Then

$$d(N) \ge \sup_{\substack{[f+z;A] \in F \\ \tilde{N}z = 0}} \|S([f+z;A]) - S([f;A])\|_{H^m(I)}.$$

Note that if $z \in B_{M/2}W^{r,p}(I)$, then $[f+z;A] \in F$, and so

$$d(N) \ge \sup_{z \in B_{M/2}W^{r,p}(I) \cap \ker \tilde{N}} \|S([z;A])\|_{H^m(I)} = \tfrac{1}{2} d(\tilde{N}, S[\cdot;A])$$

where

$$d(\tilde{N}, S[\cdot;A]) = \sup_{y \in \tilde{N}(\mathscr{B}_M W^{r,p}(I))} \ \sup_{f, \tilde{f} \in \tilde{N}^{-1}(y)} \|S([f;A]) - S([\tilde{f};A])\|_{H^m(I)}$$

is the diameter of the information $\tilde{N}$ for the solution operator $S([\cdot;A])$ with problem element set $\mathscr{B}_M W^{r,p}(I)$. It is easy to see that [7, Theorem 5.5.1], which was only done for the case $p=2$, holds for arbitrary p as well; the same proof goes through with only minor changes. Hence

$$d(\tilde{N}, S[\cdot;A]) \succcurlyeq \tilde{n}^{-r} \succcurlyeq n^{-r}, \tag{5.3}$$

the latter holding because $\tilde{n} \le n$. Thus (5.2) holds, as claimed.

We now claim that

$$r(N) \succcurlyeq n^{-r_{i,j}} \qquad (0 \le i,j \le m). \tag{5.4}$$

Indeed, choose $i,j \in \{0,\dots,m\}$. Let $I' \subset\subset I$. Choose a function $u \in H_0^m(I)$ such that $D^j u > 0$ and $D^i u > 0$ in I'. Then there exists $\sigma > 0$ such that $D^j u \ge \sigma$ and $D^i u \ge \sigma$ in I'. Let $L_A u = f$. Without loss of generality, we may assume that $f \in \mathscr{B}_M W^{r,p}(I)$, since we can divide f by $\|f\|_{W^{r,p}(I)}$ otherwise. Suppose that $N([f;A])$ contains $\tilde{n}$ evaluations of $a_{i,j}$ (or its derivatives) at points in I'. Without essential loss of generality, we may assume that these evaluations are of the form $(D^{(l_1)}a_{i,j})(x_1),\dots,(D^{(l_{\tilde{n}})}a_{i,j})(x_{\tilde{n}})$.

Choose $z \in C^\infty(I)$ satisfying

$$\begin{gathered} \operatorname{support} z \subseteq I', \\ z \ge 0, \\ \|z\|_{C^{r_{i,j}}(\overline{I})} \le M_{i,j} - |a_{i,j}|, \\ D^{(l_1)}z)(x_1) = \cdots = D^{(l_{\tilde{n}})}z)(x_{\tilde{n}}) = 0, \\ \int_{I'} z \ge C\tilde{n}^{-r_{i,j}}, \end{gathered} \tag{5.5}$$

where C is independent of n, see, e.g., [1, pp. 301–304]. Define the matrix $\tilde{A}(z)$ by (4.11), as in the proof of Theorem 4.1. It is easy to check that

$$[f;A], [f;\tilde{A}(z)] \in F$$

and

$$N([f;A]) = N([f;\tilde{A}(z)]).$$

Thus letting $\tilde{u} = S([f;\tilde{A}(z)])$, we have

$$d(N) \ge \|u - \tilde{u}\|_{H^m(I)}. \tag{5.6}$$

As in the proof of Theorem 4.1, we once again have (4.12), Since $\operatorname{support} z \subseteq I'$, we may use (4.12) with v replaced by u, the definition of M^* from (4.5), and (5.5) to see that

$$\begin{aligned} M^*\|u-\tilde{u}\|_{H^m(I)}\|u\|_{H^m(I)} \ge |B_{\tilde{A}(z)}(u-\tilde{u},u)| &= \left| \int_{I'} z D^j u D^i u \right| \\ &\ge \sigma^2 \left| \int_{I'} z \right| \ge C\sigma^2 \tilde{n}^{-r_{i,j}} \\ &\ge C\sigma^2 n^{-r_{i,j}} \end{aligned}$$

the latter since $\tilde{n} \le n$. Using this inequality, along with (3.1) and (5.6), we find that

$$r(N) \ge \tfrac{1}{2} d(N) \succcurlyeq n^{-r_{i,j}}.$$

Since N is arbitrary information of cardinality at most n and $i,j \in \{0,\dots,m\}$ are arbitrary, we see that (5.4) holds, as claimed.

Since (5.2) and (5.4) hold, we see that

$$r(N) \succcurlyeq n^{-r_{\min}}.$$

But N is arbitrary continuous information of cardinality at most n, and so the theorem follows immediately. □

Having established a bound for the nth minimal radius, we want to find an algorithm using standard information of cardinality n whose error matches this lower bound. We shall show that a properly-chosen modified FEM is optimal.

We first define the modified FEM. Recall that $\Delta = \Delta_\ell$ is an equidistant partition of I having meshsize $1/\ell$ and that $\mathscr{S}_\Delta$ is the corresponding spline space of degree k. For $A \in \mathscr{A}$, define a new bilinear form $B_{A,\Delta}$ on $H_0^m(I)$ by

$$B_{A,\Delta}(v,w) = \sum_{0\le i,j\le m} \int_I (\Pi_\Delta a_{i,j}) D^j v D^i w \qquad \forall\, v,w \in H_0^m(I), \tag{5.7}$$

where Π_Δ is the $\mathscr{S}_\Delta$-interpolation operator defined by (4.19). Then the *modified* FEM of is defined as follows: for $[f;A] \in F$, find $\tilde{u}_\Delta \in \mathscr{S}_\Delta$ satisfying

$$B_{A,\Delta}(\tilde{u}_\Delta, v) = \langle \Pi_\Delta f, v\rangle. \qquad \forall\, v \in \mathscr{S}_\Delta$$

Our main tool for analyzing the modified FEM is *Strang's Lemma* (see [7, pp. 310–312] for a proof of a version having slightly more restrictive hypotheses).

LEMMA 5.1. *Suppose that there exists $\ell^{**} \in \mathbb{Z}^{++}$ such that for any $\Delta = \Delta_\ell$ with $\ell \ge \ell^{**}$ and any $A \in \mathscr{A}$, we have*

$$|B_A(v,w) - B_{A,\Delta}(v,w)| \le \tfrac{1}{2}\gamma \|v\|_{H^m(I)} \|w\|_{H^m(I)} \qquad \forall\, v,w \in \mathscr{S}_\Delta, \tag{5.8}$$

where γ is as in (2.3). Then for any $\ell \ge \ell^{**}$ and any $[f;A] \in F$, there is a unique $\tilde{u}_\Delta \in \mathscr{S}_\Delta$ such that (5.7) holds. Moreover, there exists a positive constant C, such that if $u = S([f;A])$ is the solution to (2.2), then

$$\|u-\tilde{u}_\Delta\|_{H^m(I)} \le C \inf_{v\in\mathscr{S}_\Delta} \left[\|u-v\|_{H^m(I)} + \sup_{w\in\mathscr{S}_\Delta} \left(\frac{|B_A(v,w) - B_{A,\Delta}(v,w)|}{\|w\|_{H^m(I)}} + \frac{|\langle f,w\rangle - \langle \Pi_\Delta f, w\rangle|}{\|w\|_{H^m(I)}} \right)\right],$$

the constant C being independent of Δ and $[f;A]$. □

We can now show that the modified FEM is well-defined, and establish an error estimate:

THEOREM 5.2. *Let*

$$k \ge m + r_{\min} - 1,$$

where

$$r_{\min} = \min\{\, r, \min_{0\le i,j\le m}\{r_{i,j}\}\,\}.$$

(1) *There exists an index ℓ^{**} such that for any $\ell \ge \ell^{**}$ and any $[f;A] \in F$, there is a unique $\tilde{u}_\Delta \in \mathscr{S}_\Delta = \mathscr{S}_{\Delta_\ell}$ such that (5.7) holds.*

(2) *For $[f;A] \in F$, let $u = S([f;A])$ be the solution to (2.2). Then for any $\ell \ge \ell^{**}$, we have the error estimate*

$$\|u - \tilde{u}_\Delta\|_{H^m(I)} \le C n_\Delta^{-r_{\min}}, \tag{5.9}$$

where the constant C is independent of Δ and $[f;A]$. □

PROOF: We first prove (1). Note that for any $A \in \mathscr{A}$ and any partition Δ of I, we have

$$\begin{aligned}|B_A(v,w) - B_{A,\Delta}(v,w)| &= \left| \sum_{0\le i,j\le m} \int_I \big((a_{i,j} - \Pi_\Delta a_{i,j}) D^j v D^i w\big)\right| \\ &\le \left[\sum_{0\le i,j\le m} \|a_{i,j} - \Pi_\Delta a_{i,j}\|_{C^0(\overline{I})}\right] \|v\|_{H^m(I)} \|w\|_{H^m(I)}\end{aligned} \qquad \forall\, v,w \in H_0^m(I).$$

But for any indices $i,j \in \{0,\dots,m\}$, our choice of k and Lemma 4.1 imply that

$$\|a_{i,j} - \Pi_\Delta a_{i,j}\|_{C^0(\overline{I})} \le C n_\Delta^{-\min\{k+1,r_{i,j}\}} \|a_{i,j}\|_{C^{r_{i,j}}(\overline{I})} \le C M_{i,j} n_\Delta^{-r_{i,j}}.$$

Thus

$$|B_A(v,w) - B_{A,\Delta}(v,w)| \le C n_\Delta^{-r_{\min}} \|v\|_{H^m(I)} \|w\|_{H^m(I)} \qquad \forall\, v,w \in H_0^m(I). \tag{5.10}$$

Choosing ℓ^{**} such that $C n_{\Delta_{l^{**}}}^{-r_{\min}} = \frac{1}{2}\gamma$, we see that condition (5.8) in Strang's Lemma holds. Thus for any $\ell \ge \ell^{**}$ and any $[f;A] \in F$, there is a unique $\tilde{u}_\Delta \in \mathscr{S}_\Delta$ such that (5.7) holds. Thus (1) holds, as claimed.

We now prove (2). Let $\ell \ge \ell^{**}$, and let $[f;A] \in F$. We again let $u = S([f;A])$ and $\Delta = \Delta_\ell$. By our choice of k and Lemma 4.1, we find

$$\begin{aligned}\|u - \Pi_\Delta u\|_{H^m(I)} &\le C n_\Delta^{-\min\{k+1-m, r_{\min}\}} \|u\|_{H^{m+r_{\min}}(I)} \\ &\le C n_\Delta^{-r_{\min}} \|f\|_{H^{r_{\min}-m}(I)} \\ &\le C M n_\Delta^{-r_{\min}}.\end{aligned} \tag{5.11}$$

One more use of Lemma 4.1 yields that

$$\begin{aligned}|\langle f,w\rangle - \langle \Pi_\Delta f, w\rangle| &\le \|f - \Pi_\Delta f\|_{L_2(I)} \|w\|_{L_2(I)} \\ &\le C n_\Delta^{-\min\{k+1,r\}} \|f\|_{H^r(I)} \|w\|_{L_2(I)} \qquad \forall\, w \in \mathscr{S}_\Delta. \\ &\le C M n_\Delta^{-r} \|w\|_{H^m(I)}.\end{aligned} \tag{5.12}.$$

Next, note that (5.11) and the fact that $u = S([f;A])$ for $[f;A] \in F$ imply that

$$\|\Pi_\Delta u\|_{H^m(I)} \le C\|u\|_{H^m(I)} \le C \tag{5.13}$$

for constants C, independent of u and Δ. Letting $v = \Pi_\Delta u$ in (5.10), we find that

$$|B_A(\Pi_\Delta u, w) - B_{A,\Delta}(\Pi_\Delta u, w)| \le C n_\Delta^{-r_{\min}} \|w\|_{H^m(I)} \qquad \forall\, w \in H_0^m(I). \tag{5.14}$$

Using (5.11), (5.12) and (5.14) in Lemma 5.1, we immediately have

$$\|u - \Pi_\Delta u\| \le C n_\Delta^{-r_{\min}},$$

as required. □

What information do we need to calculate the modified FEM? For $[f;A] \in F$ and sufficiently-fine Δ, we write

$$u_\Delta(x) = \sum_{j=1}^{n_\Delta} \alpha_j s_j(x),$$

where the points $x_1, \dots, x_{n_\Delta}$ and the basis functions $s_1, \dots, s_{n_\Delta}$ are given by (4.15) and (4.16), respectively. Then

$$G\alpha = \beta, \tag{5.15}$$

where

$$g_{i,j} = B_{A,\Delta}(s_j, s_i) = \sum_{\substack{0 \le i',j' \le m \\ 1 \le l' \le n_\Delta}} a_{i',j'}(x_l') \langle s_{l'}, D^{j'} s_j D^{i'} s_i \rangle \qquad (1 \le i,j \le n_\Delta)$$

and

$$\beta_i = \langle \Pi_\Delta f, s_i \rangle = \sum_{1 \le l' \le n_\Delta} f(x_{l'}) \langle s_{l'}, s_i \rangle \qquad (1 \le i \le n_\Delta).$$

Allowing a slight abuse of notation, we now define information $\tilde{N}_n$ of cardinality n by

$$\tilde{N}_n([f;A]) = [f(x_1), \dots, f(n_\Delta); \tilde{N}_n(A)],$$

with

$$\tilde{N}_n(A) = \{ \, (a_{i',j'})(x_l') : 1 \le l' \le n_\Delta, 0 \le i', j' \le m \, \}.$$

Since the basis functions have small supports, we find that the following hold:

(1) $n \asymp n_\Delta$.
(2) Each entry of G and β can be computed in constant time, independent of n.
(3) The linear system (5.15) is banded, the bandwidth depending only on k and m.

Using (1) of Lemma 5.1, it easily follows that for $\ell \ge \ell^{**}$, we can solve (5.15) in $\Theta(n_\Delta)$ arithmetic operations using banded Gaussian elimination without pivoting.

For $[f;A] \in F$ and sufficiently-fine Δ, it is clear that $\tilde{u}_\Delta$ depends on $[f;A]$ only through the information $\tilde{N}_n([f;A])$. Hence we may write

$$\tilde{u}_\Delta = \tilde{\phi}_n(\tilde{N}_n([f;A])),$$

where $\tilde{\phi}_n$ is an algorithm using $\tilde{N}_n$. The algorithm $\tilde{\phi}_n$ and the information $\tilde{N}_n$ define the *modified finite elment method.* From the remarks in the previous paragraph, it follows that

$$\operatorname{cost}(\tilde{\phi}_n, \tilde{N}_n) = cn + \Theta(n),$$

the Θ-factor depending only on k and m.

Hence, using Theorems 5.1 and 5.2, we immediately have

COROLLARY 5.1. *Suppose that standard information is permissible. Let*

$$r_{\min} = \min\{ \, r, \min_{0 \le i,j \le m} \{r_{i,j}\} \, \}.$$

(1) *The nth minimal radius is*

$$r(n, \mathscr{L}^{\text{std}}) \asymp \left(\frac{1}{n}\right)^{r_{\min}}$$

.

(2) *The ε-complexity is*

$$\operatorname{comp}(\varepsilon, \mathscr{L}^{\text{std}}) \asymp \left(\frac{1}{\varepsilon}\right)^{1/r_{\min}}.$$

(3) *If $k \ge m + r_{\min} - 1$, then the modified* FEM *of degree k of cardinality $n \asymp (1/\varepsilon)^{1/r}$ is optimal.* □

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