A Spacetime Alexandrov Theorem

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Abstract

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Let Σ be an embedded spacelike codimension-2 submanifold in a spherically symmetric spacetime satisfying null convergence condition. Suppose Σ has constant null mean curvature and zero torsion. We prove that Σ must lie in a standard null cone. This generalizes the classical Alexandrov theorem which classifies embedded constant mean curvature hypersurfaces in Euclidean space. The proof follows the idea of Ros and Brendle. We first derive a spacetime Minkowski formula for spacelike codimension-2 submanifolds using conformal Killing-Yano 2-forms. The Minkowski formula is then combined with a Heintze-Karcher type geometric inequality to prove the main theorem. We also obtain several rigidity results for codimension-2 submanifolds in spherically symmetric spacetimes.

Contents

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To my family

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Chapter 1

Introduction

1.1 Alexandrov Theorem

The main goal of this work is to study the properties of *constant normalized null curvature* (CNNC) surfaces, a generalization of constant mean curvature (CMC) surfaces, in spherically symmetric spacetimes. We start by reviewing some history and notions of classical theory of CMC surfaces and general relativity which motivate this work. In this work, all submanifolds are assumed to be connected.

CMC hypersurfaces arise naturally as the stationary points of the isoperimetric problem in calculus of variations:

$$\inf \left\{ \mathcal{H}^{n-1}(\Sigma) : \Sigma = \partial\Omega, \Omega \subset \mathbb{R}^n \text{ is a smooth region with } \mathcal{L}^n(\Omega) = V \right\} \tag{1.1}$$

Here $\mathcal{H}^{n-1}$ and $\mathcal{L}^n$ denote the $(n-1)$-dimensional Hausdorff measure and the Lebesgue measure on $\mathbb{R}^n$.

Definition 1.1. Minimizers of the isoperimetric problem (1.1) are called the *isoperimetric hypersurfaces*. A closed CMC hypersurface Σ is *stable* if $\mathcal{H}^{n-1}(\Sigma)'' \geq 0$ for any variation that preserves the enclosed volume.

It is a natural question to characterize the minimizers and (stable) stationary points of (1.1). Besides the early work of Delauney on CMC surfaces of revolution in 19th century, the first breakthrough was made by Alexandrov in 1950's.

Alexandrov theorem. *[1] Let $\Sigma \subset \mathbb{R}^n$ be a closed (compact without boundary), embedded CMC hypersurface. Then Σ is a round sphere.*

Alexandrov theorem is remarkable in that it holds in all dimensions and requires no topological and stable assumptions for the hypersurface. The immersed stable stationary points were classified by Barbosa-do Carmo.

Theorem 1.2. *[2, Theorem 1.3] Let $\Sigma \subset \mathbb{R}^n$ be a closed, orientable immersed stable CMC hypersurface. Then Σ is a round sphere.*

There are many proofs of Alexandrov theorem nowadays. Here we present the one, due to A. Ros, that is the most relevant to us. We start with the classical Minkowski formula.

Theorem 1.3. *Let $\Sigma \subset \mathbb{R}^n$ be a closed immersed hypersurface. Let X, ν and H be the position vector, normal vector and the mean curvature of Σ. Then*

$$(n-1)\int_\Sigma d\mu = \int_\Sigma H\langle X, \nu\rangle d\mu. \tag{1.2}$$

The next step is a sharp geometric inequality.

Theorem 1.4. *Let $\Sigma \subset \mathbb{R}^n$ be a closed, embedded hypersurface with positive mean curvature. Then*

$$\int_\Sigma \frac{1}{H} d\mu \geq \int_\Sigma \langle X, \nu\rangle d\mu. \tag{1.3}$$

Moreover, the equality holds if and only if Σ is a round sphere.

(1.3) was proved in [26] by Reilly's formula. Later Ros-Montiel [22] gave another proof inspired by the paper of Heintze-Karcher.

For a CMC hypersurface, the Minkowski formula implies

$$(n-1)\int_\Sigma \frac{1}{H} d\mu = \int_\Sigma \langle X, \nu \rangle d\mu.$$

Hence the equality in (1.3) is achieved and Σ is a round sphere.

It turns out that Ros' idea generalizes to other rotationally symmetric manifolds. Consider the rotationally symmetric manifold (M, g) with the metric given by

$$g = \frac{1}{f(r)^2} dr^2 + r^2 g_{S^{n-1}}$$

where $f : (r_0, r_1) \to \mathbb{R}$, called the static potential, is a positive function with $\lim_{r \to r_0} f(r) = 0$ and $\lim_{r \to r_0} f'(r) > 0$. Here $r_0 > 0$ and r_1 can be taken to be ∞. The Minkowski formula is readily generalized to (M, g) thanks to the observation that the position vector of $\Sigma \subset \mathbb{R}^n$ can be replaced by the restriction of the (global) conformal Killing vector $r\frac{\partial}{\partial r}$ to Σ. Since (M, g) carries a conformal Killing vector $X = rf\frac{\partial}{\partial r}$, the same proof of Minkowski formula gives

$$(n-1)\int_\Sigma f d\mu = \int_\Sigma H\langle X, \nu \rangle d\mu.$$

On the other hand, Brendle was able to generalize (1.3) to a large class of rotationally symmetric manifolds.

Theorem 1.5. *[5] Let $\Sigma \subset (M, g)$ be a closed, embedded hypersurface with positive mean curvature. Suppose f satisfies*

$$(\Delta_g f)g - Hess_g f + fRic(g) \geq 0. \tag{1.4}$$

Then

$$\int_\Sigma \frac{f}{H} d\mu \geq \int_\Sigma \langle X, \nu \rangle d\mu. \tag{1.5}$$

Moreover, the equality holds if and only if Σ *is umbilical.*

An important class of rotationally symmetric manifolds arises in general relativity. For example, the *Schwarzschild manifold with mass* m given by

$$g^S = \frac{1}{1 - 2mr^{2-n}} dr^2 + r^2 S^{n-1}, \quad r \in (2m, \infty)$$

satisfies $(\Delta_{g^S} f) g^S - Hess_{g^S} f + f Ric(g^S) = 0$. As a consequence, Brendle proves the Alexandrov theorem for Schwarzschild manifolds.

Theorem 1.6. *[5] Let* Σ *be a closed embedded CMC hypersurface in Schwarzschild manifold. Then* Σ *is a sphere of symmetry.*

Another motivation for studying CMC hypersurfaces comes from general relativity. In general relativity, we study four (more generally, $n+1$) dimensional Lorentzian manifolds $(V, \bar{g})$ that satisfy the Einstein equation

$$\mathrm{Ric}(\bar{g}) - \frac{1}{2} R(\bar{g}) \bar{g} = 8\pi T, \tag{1.6}$$

where T is the stress-energy tensor of matter. When $T = 0$, (1.6) is called the vacuum Einstein equation and is equivalent to $Ric(\bar{g}) = 0$. Shortly after Einstein posed his equation, Schwarzschild discovered a solution to the vacuum Einstein equation that describes the gravitational field outside a static star. The metric of the *Schwarzschild spacetime with mass* m

$$\bar{g} = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{1}{1 - \frac{2m}{r}} dr^2 + r^2 g_{S^2}$$

is static and spherically symmetric. When $m = 0$, Schwarzschild spacetime reduces to

Minkowski spacetime. Note that Schwarzschild spacetime has an $(n+1)$-dimensional generalization (found by Tangherlini) with the metric given by

$$\bar{g} = -\left(1 - 2mr^{n-2}\right) dt^2 + \frac{1}{1 - 2mr^{n-2}} dr^2 + r^2 g_{S^{n-1}}.$$

Given a spacelike hypersurface $M \subset V$, the Gauss and Codazzi equation impose constraints on the induced metric g and the second fundamental form p of M, called the Einstein constraint equations:

$$\frac{1}{2}\left(R(g) + (\mathrm{tr}_g p)^2 - |p|_g^2\right) = \mu \tag{1.7}$$

$$\nabla^j \left(p_{ij} - (\mathrm{tr}_g p) g_{ij}\right) = J_i \tag{1.8}$$

where $\mu = T(\vec{n}, \vec{n})$, $J_i = T(\vec{n}, \partial_i)$ for the unit timelike normal $\vec{n}$ of M.

Definition 1.7. An *initial data set* consists of a manifold M, a Riemannian metric g and a symmetric (0,2)-tensor p on M that satisfy (1.7) and (1.8).

It is well-known that the Einstein equation admits an initial value formulation [30, Chapter 10]. Given an initial data set of the vacuum Einstein constraint equation ($\mu = J = 0$), there exists a spacetime $(V, \bar{g})$, called the maximal Cauchy development of (M, g, p), satisfying the vacuum Einstein equation ($T = 0$) and M is embedded in $(V, \bar{g})$ with induced metric g and second fundamental form p. It is thus natural to study the geometric and physical problems on the initial data set. For example, in the time-symmetric case, $p = 0$, the dominant energy condition reduces to $R(g) \geq 0$. Problems motivated by physics provide interesting geometric questions on Riemannian manifolds with nonnegative scalar curvature.

Recall that we call the $t = 0$ slice of Schwarzschild spacetime *Schwarzschild manifold with mass* m, (Sch, g^S). It is useful to express the metric in the conformally flat coordinates

$$g^S_{ij} = \left(1 + \frac{m}{(n-1)r^{n-2}}\right)^{\frac{4}{n-2}} \delta_{ij} \tag{1.9}$$

Bray initiated the study of isoperimetric surfaces in Schwarzschild manifolds. In [4, Theorem 8], he proved that in Schwarzschild manifold with $m \geq 0$, the spheres of symmetry are isoperimetric surfaces and any isoperimetric surface must be a sphere of symmetry.

Definition 1.8. An n-dimensional Riemannian manifold (M, g) is $\mathcal{C}^k$-asymptotic to Schwarzschild of mass m if there exists a bounded open set $U \subset M$ such that $M \setminus U \simeq \mathbb{R}^n \setminus B_{\frac{1}{2}(0)}$, and such that in the coordinates,

$$\sum_{l=0}^{k} r^{n-2+l} |\partial^l (g - g^S)_{ij}| \leq C \text{ for all } r \geq 1$$

where $r = \sqrt{\sum_{i=1}^n x_i^2}$ and g_{ij}^S is given in (1.9).

In their seminal paper [18], Huisken-Yau showed that outside a bounded set, a three-dimensional Riemannian manifold (M, g) that is $\mathcal{C}^4$-asymptotic to Schwarzschild of mass $m > 0$ is foliated by strictly stable CMC spheres. Moreover, the leaves of the foliation are the unique stable CMC spheres within a large class of surfaces. The result is strengthened in [25, 17, 21]. Finally, Eichmair and Metzger [14] proved the uniqueness of the stable CMC constructed by Huisken-Yau. More precisely,

Theorem 1.9. *[14, Theorem 1.1] Let (M, g) be an n-dimensional Riemannian manifold that is $\mathcal{C}^2$-asymptotic to Schwarzschild of mass $m > 0$. There exists $V_0 > 0$ such that for every $V \geq V_0$ the infimum in*

$$\inf\{\mathcal{H}_g^{n-1}(\partial\Omega) : \Omega \subset M \text{ is a smooth region with } \mathcal{L}_g^n(\Omega) = V\} \tag{1.10}$$

is achieved by a unique smooth minimizer (hence isoperimetric) $\Sigma_V = \partial\Omega_V$.

CMC surfaces play an important role in studying the conserved quantities of initial data sets. The first example contains two results on the behavior of the Hawking mass

$$m_H(\Sigma) = \sqrt{\frac{\mathcal{H}^2(\Sigma)}{16\pi}} \left(1 - \frac{1}{16\pi} \int_\Sigma H^2 d\mu\right) \tag{1.11}$$

in a three dimensional time-symmetric initial data set. Christodoulou-Yau [12] showed that the Hawking mass is nonnegative for stable CMC surfaces. In addition, Bray [4, Lemma 1] proved that the Hawking mass is non-decreasing along an isoperimetric foliation.

The second example concerns the center of mass.

Definition 1.10. For a three dimensional Riemannian manifold (M, g) that is asymptotic to Schwarzschild with mass $m > 0$, the *ADM center of mass* is defined by, for $\alpha = 1, 2, 3$,

$$\mathcal{C}^\alpha = \frac{1}{16\pi m} \lim_{r\to\infty} \left[\int_{|x|=r} \sum_{i,j} x^\alpha (g_{ij,j} - g_{ii,j}) \frac{x^j}{|x|} d\mathcal{H}^2_\delta - \int_{|x|=r} \sum_i \left(g_{i\alpha} \frac{x^i}{|x|} - g_{ii} \frac{x^\alpha}{|x|} \right) d\mathcal{H}^2_\delta \right] \tag{1.12}$$

Here $d\mathcal{H}^2_\delta$ denotes the area element with respect to the Euclidean metric.

Huisken-Yau defined a geometric center of mass using the CMC foliation.

Definition 1.11. Let $\{\Sigma_V\}_{V\geq V_0}$ be the CMC foliation constructed by Huisken-Yau. The Huisken-Yau center of mass is defined by, for $\alpha = 1, 2, 3$,

$$\mathcal{C}^\alpha_{HY} = \lim_{V\to\infty} \frac{1}{\mathcal{H}^2_\delta(\Sigma_V)} \int_{\Sigma_V} x^\alpha d\mathcal{H}^2_\delta. \tag{1.13}$$

The expression in (1.13) has the advantage that it is easy to compute once we have a CMC foliation. Moreover, in [16], Huang proved that the definition of Huisken-Yau coincides with that of ADM for a wide class of physical relevant asymptotics (see also [14, Theorem 6.1]).

1.2 Statement of the Main Theorem

Codimension-2 submanifolds play a special role in general relativity. Their null expansions are closely related to gravitation energy as seen in Penrose's singularity theorem [24] and the definition of quasilocal mass [32]. It is desirable to characterize when a codimension-2 submanifold lies in the null hypersurface generated by a round sphere.

Definition 1.12. A null hypersurface in a static spherically symmetric spacetime is called a *standard null cone* if it contains a sphere of symmetry in some static time slice.

The main result in this work is a spacetime Alexandrov-type theorem. The CMC condition for hypersurfaces is replaced by the constant null normalized curvature condition.

Definition 1.13. A spacelike codimension-2 submanifold Σ of an $(n+1)$-dimensional spacetime is said to have *constant normalized null curvature (CNNC)* if there exists a future null normal vector field $\tilde{L}$ such that $\langle \vec{H}, \tilde{L} \rangle = constant$ and $(D\tilde{L})^{\perp} = 0$.

We give a characterization of spacelike codimension-2 submanifolds in the standard null cones of the Schwarzschild spacetime.

Main Theorem. *Let Σ be a future incoming null smooth (see Definition 4.8) closed embedded spacelike codimension-2 submanifold in the Schwarzschild spacetime. Suppose Σ has $\langle \vec{H}, \underline{L} \rangle = constant$ and $(D\underline{L})^{\perp} = 0$ for its future incoming null normal $\underline{L}$. Then Σ lies in a standard null cone.*

The main theorem holds for a class of static spherically symmetric spacetimes (see Chapter 4 for precise statement). For simplicity, we state our results on Schwarzschild spacetimes.

We follow Ros' idea to combine Minkowski formula and a Heintze-Karcher type inequality. First of all, we derive a spacetime Minkowski formula using conformal Killing-Yano 2-forms which generalize conformal Killing vectors.

Definition 1.14. [19, Definition 1] Let Q be a 2-form on an $(n+1)$-dimensional pseudo-Riemannian manifold $(V, \langle , \rangle)$ with Levi-Civita connection D. Q is said to be a *conformal*

Killing-Yano 2-form if

$$(D_XQ)(Y,Z)+(D_YQ)(X,Z) = \frac{2}{n}\left(\langle X,Y\rangle\langle\xi,Z\rangle-\frac{1}{2}\langle X,Z\rangle\langle\xi,Y\rangle-\frac{1}{2}\langle Y,Z\rangle\langle\xi,X\rangle\right) \tag{1.14}$$

for any tangent vectors X, Y, and Z, where $\xi^\beta = (divQ)^\beta = D_\alpha Q^{\alpha\beta}$.

Schwarzschild spacetime admits a conformal Killing-Yano 2-form $Q = rdr \wedge dt$ with $\xi = -n\frac{\partial}{\partial t}$.

We have

Theorem 1.15. *Let Σ be a closed immersed oriented spacelike codimension-2 submanifold in Schwarzschild spacetime. For any normal vector field $\underline{L}$ of Σ, we have*

$$-(n-1)\int_\Sigma\langle\frac{\partial}{\partial t},\underline{L}\rangle\,d\mu+\int_\Sigma Q(\vec{H},\underline{L})\,d\mu+\int_\Sigma Q(\partial_a,(D^a\underline{L})^\perp)\,d\mu=0. \tag{1.15}$$

Secondly, we show that there is a monotonicity formula, Proposition 4.1, when we evolve the surface along its incoming null hypersurface. The idea comes from Brendle's work. In particular, we learned a preliminary version of the monotonicity formula in Minkowski spacetime from Brendle. As a consequence of the monotonicity formula, we obtain a spacetime Heintze-Karcher inequality.

Theorem 1.16. *Let Σ be a future incoming null smooth closed spacelike codimension-2 submanifold in Schwarzschild spacetime. Suppose $\langle\vec{H},\underline{L}\rangle>0$ where $\underline{L}$ is a future incoming null normal. Then*

$$-(n-1)\int_\Sigma\frac{\langle\frac{\partial}{\partial t},\underline{L}\rangle}{\langle\vec{H},\underline{L}\rangle}d\mu-\frac{1}{2}\int_\Sigma Q(L,\underline{L})d\mu\geq 0, \tag{1.16}$$

for a future outgoing null normal L with $\langle L,\underline{L}\rangle=-2$. Moreover, the equality holds if Σ lies in a standard null cone.

The main theorem follows from the spacetime Minkowski formula and the Heintze-Karcher inequality. Suppose Σ satisfies $\langle H, \underline{L}\rangle = constant$ and $(D\underline{L})^\perp = 0$. Note that $\vec{H} = -\frac{1}{2}\langle \vec{H}, \underline{L}\rangle L - \frac{1}{2}\langle \vec{H}, L\rangle \underline{L}$ and the Minkowski formula implies

$$-(n-1)\int_\Sigma \frac{\langle \frac{\partial}{\partial t}, \underline{L}\rangle}{\langle \vec{H}, \underline{L}\rangle} d\mu - \frac{1}{2}\int_\Sigma Q(L, \underline{L}) d\mu = 0.$$

Hence the equality in the previous theorem is achieved and Σ lies in a standard null cone.

Another natural substitute of CMC condition for higher codimension submanifold is the notion of parallel mean curvature vector. Yau [33] and Chen[8] proved that a closed immersed spacelike 2-sphere with parallel mean curvature vector in Minkowski spacetime must be a round sphere. We are able to generalize their result to Schwarzschild spacetime.

Corollary 1.17. *Let Σ be closed embedded spacelike codimension-2 submanifold with parallel mean curvature vector in Schwarzschild spacetime. Suppose Σ is both future and past incoming null smooth. Then Σ is a sphere of symmetry.*

Now we describe the organization of this work. In Chapter 2, we set up the notations and derive the Gauss, Codazzi, and Ricci equations for spacelike codimension-2 submanifolds in Lorentzian manifolds. In Chapter 3, we derive two spacetime Minkowski formulae. We discuss how they recover classical Minkowski formulae. The first is the one needed in the proof of the spacetime Alexandrov theorem. The second one concerns the integral of null expansions which serves as a measure of gravitational energy. The main theorem is proved in Chapter 4. We first derive a monotonicity formula. Next we discuss the CNNC condition. In mean curvature gauge, it can be cast into a single equation. We then flow the submanifold into the totally geodesic slice where Brendle's result takes over. We thus get the spacetime Heintze-Karcher inequality and Alexandrov theorem. The characterization of submanifolds with parallel mean curvature vector would be a direct consequence. In the final Chapter 5, we discuss three rigidity results on codimension-2 submanifolds in spherically symmetric spacetimes. First of all, we show that codimension-2 submanifolds in the standard null cone is infinitesimally rigid in the sense of CNNC. The result is relevant to the construction of CNNC foliation in asymptotically Schwarzschild initial data set. Secondly, we show that a codimension-2 submanifold in the standard null cone with constant mean curvature vector norm must be a sphere of symmetry. In particular, our argument for the Minkowski spacetime provides a unified proof of the classical Liouville theorem (in 2-dimension) and Obata theorem (in higher dimension). At last, we show that a codimension-2 submanifold that has zero connection 1-form in the mean curvature gauge and satisfies a starshaped condition must lie in a totally geodesic slice. For all results, the energy condition comes in. It is reminiscent of how positivity of mass supports the uniqueness of CMC foliations.

Chapter 2

Preliminaries

Let $F : (\Sigma^{n-1}, \sigma) \to (V^{n+1}, \langle , \rangle)$ be a closed immersed oriented spacelike codimension-2 submanifold in an oriented (n+1)-dimensional Lorentzian manifold $(V^{n+1}, \langle , \rangle)$. We assume the normal bundle is also orientable. Fix a point $p \in \Sigma$. We choose a local orthonormal frame $e_1, e_2, \dots, e_n, e_{n+1}$ in V such that, when restricted to Σ, $e_1, \dots, e_{n-1}$ are tangent to Σ and e_n, e_{n+1} are normal to Σ with $\langle e_n, e_n \rangle = 1, \langle e_n, e_{n+1} \rangle = 0$, and $\langle e_{n+1}, e_{n+1} \rangle = -1$. Moreover, $e_1 \wedge e_2 \wedge \cdots \wedge e_{n-1}$, $e_n \wedge e_{n+1}$, and $e_1 \wedge e_2 \wedge \cdots \wedge e_{n-1} \wedge e_n \wedge e_{n+1}$ coincide with the orientation on $T\Sigma$, $N\Sigma$, and TV. We also choose a coordinate system $\{u^a | \ a = 1, 2, \cdots , n-1\}$ for Σ near p. We identify $\frac{\partial F}{\partial u^a}$ with $\frac{\partial}{\partial u^a}$, which is abbreviated as ∂_a. We use the following convention on the range of indices: $1 \le a, b, c, \ldots \le n-1, 1 \le \alpha, \beta, \gamma \ldots \le n+1$ and agree that repeated indices are summed over the respective ranges. Let D and ∇ denote the Levi-Civita connection of V and Σ respectively and let $\bar{R}_{\alpha\beta\gamma\delta}, \bar{R}_{\alpha\beta}$, and $\bar{R}$ (R_{abcd}, R_{ab}, and R respectively) denote the Riemann curvature tensor, Ricci curvature, and scalar curvature of V(Σ respectively). Let $h_\alpha = h_{\alpha ab} \equiv \langle D_a e_\alpha, \partial_b \rangle$ be the second fundamental form with respect to $e_\alpha, \alpha = n, n+1$.

We recall the Gauss, Codazzi, and Ricci equations.

Theorem 2.1. *Let $\sigma_2(h_\alpha)$ denote the second symmetric function of the eigenvalues of $h_\alpha, \alpha = n, n+1$. Let $\zeta_a = \langle D_a e_n, e_{n+1} \rangle$ be the connection one-form of the normal bundle with respect* *to the corresponding frame. We have*

$$R = \bar{R} - 2\bar{R}_{nn} + 2\bar{R}_{n+1,n+1} - 2\bar{R}_{n,n+1,n,n+1} + 2\sigma_2(h_n) - 2\sigma_2(h_{n+1}) \tag{2.1}$$

$$\nabla_a h_{nbc} - \nabla_b h_{nac} = \bar{R}_{abcn} + \zeta_b h_{n+1,ac} - \zeta_a h_{n+1,bc} \tag{2.2}$$

$$\nabla_a h_{n+1,bc} - \nabla_b h_{n+1,ac} = \bar{R}_{abc,n+1} + \zeta_b h_{nac} - \zeta_a h_{nbc} \tag{2.3}$$

$$\bar{R}_{ab,n+1,n} = (d\zeta)_{ab} + h_{na}{}^{c} h_{n+1,bc} - h_{nb}{}^{c} h_{n+1,ac} \tag{2.4}$$

Proof. For the Gauss equation, we compute

$$\begin{aligned}
\bar{R}_{abdc} &= \langle D_a D_b \partial_c, \partial_d \rangle - \langle D_b D_a \partial_c, \partial_d \rangle \\
&= \langle D_a(\nabla_b \partial_c - h_{nbc} e_n + h_{n+1,bc} e_{n+1}), \partial_d \rangle \\
&\quad - \langle D_b(\nabla_a \partial_c - h_{nac} e_n + h_{n+1,ac} e_{n+1}), \partial_d \rangle \\
&= R_{abdc} - h_{nbc} h_{nad} + h_{n+1,bc} h_{n+1,ad} + h_{nac} h_{nbd} - h_{n+1,ac} h_{n+1,bd}.
\end{aligned}$$

Taking trace twice with respect to the induced metric on Σ, we obtain

$$\begin{aligned}
R - 2\left(\sigma_2(h_n) - \sigma_2(h_{n+1})\right) &= \sigma^{ad} \sigma^{bc} \bar{R}_{abdc} \\
&= \sigma^{ad} (\bar{R}_{ad} - \bar{R}_{andn} + \bar{R}_{a,n+1,d,n+1}) \\
&= \bar{R} - \bar{R}_{nn} + \bar{R}_{n+1,n+1} - (\bar{R}_{nn} + \bar{R}_{n+1,n,n+1,n}) \\
&\quad + \bar{R}_{n+1,n+1} - \bar{R}_{n,n+1,n,n+1} \\
&= \bar{R} - 2\bar{R}_{nn} + 2\bar{R}_{n+1,n+1} - 2\bar{R}_{n,n+1,n,n+1}.
\end{aligned}$$

For the Codazzi equation, we derive

$$\nabla_a h_{nbc} = \langle D_a D_b e_n, \partial_c \rangle - \langle D_{\nabla_a \partial_b} e_n, \partial_c \rangle + \langle D_b e_n, (D_a \partial_c)^{\perp} \rangle$$

where $v^{\perp}$ denotes the normal component of the vector v. As $\langle D_b e_n, (D_a \partial_c)^{\perp} \rangle = \zeta_b h_{n+1,ac}$, anti-symmetrizing a, b, we obtain

$$\nabla_a h_{nbc} - \nabla_b h_{nac} = \langle \bar{R}(\partial_a, \partial_b)e_n, \partial_c \rangle + \zeta_b h_{n+1,ac} - \zeta_a h_{n+1,bc}.$$

(2.3) is derived similarly. The Ricci equation is derived as follows:

$$\begin{aligned}\bar{R}_{ab,n+1,n} &= \langle D_a D_b e_n, e_{n+1} \rangle - \langle D_b D_a e_n, e_{n+1} \rangle \\ &= \langle D_a ({h_{nb}}^c \partial_c - \zeta_b e_{n+1}), e_{n+1} \rangle - \langle D_b ({h_{na}}^c \partial_c - \zeta_a e_{n+1}), e_{n+1} \rangle \\ &= (d\zeta)_{ab} + {h_{na}}^c h_{n+1,bc} - {h_{nb}}^c h_{n+1,ac}.\end{aligned}$$

□

Given a spacelike codimension-2 submanifold Σ in a Lorentzian manifold, it is usually more convenient to take two null normals instead of one spacelike and one timelike normal. Let $L, \underline{L}$ be two future-directed null normals along Σ such that $\langle L, \underline{L} \rangle = -2$. Let $\chi_{ab} = \langle D_a L, \partial_b \rangle$ and $\underline{\chi}_{ab} = \langle D_a \underline{L}, \partial_b \rangle$ be the corresponding second fundamental form. Let $\zeta_{null}(X) = \frac{1}{2}\langle D_X L, \underline{L} \rangle$ be the torsion of Σ with respect to $L, \underline{L}$. When $L = e_{n+1} + e_n$ and $\underline{L} = e_{n+1} - e_n$, the torsion is the same as the connection one-form defined in Theorem 2.1. We omit the subscript *null* if there is no risk of confusion. We list the Gauss, Codazzi and Ricci equations in terms of the null frame without proof.

Theorem 2.2.

$$\begin{aligned}\bar{R} + \bar{R}_{L\underline{L}} + \frac{1}{2}\bar{R}_{\underline{L}L L\underline{L}} &= R + tr\chi tr\underline{\chi} - \chi_{ab}\underline{\chi}^{ab}, \\ \nabla_a \chi_{bc} - \nabla_b \chi_{ac} &= \bar{R}_{abcL} + \chi_{ac}\zeta_b - \chi_{bc}\zeta_a, \\ \nabla_a \underline{\chi}_{bc} - \nabla_b \underline{\chi}_{ac} &= \bar{R}_{abc\underline{L}} - \underline{\chi}_{ac}\zeta_b + \underline{\chi}_{bc}\zeta_a, \\ \frac{1}{2}\bar{R}_{ab\underline{L}L} &= (d\zeta)_{ab} + \frac{1}{2}\left({\chi_a}^c \underline{\chi}_{cb} - {\underline{\chi}_a}^c \chi_{cb} \right).\end{aligned}$$

To finish the preliminary, we recall the notion of mean curvature gauge from [32].

Definition 2.3. Let $\vec{H} = \sigma^{ab}(D_a\partial_b)^\perp$ denote the mean curvature vector of Σ. Suppose $\vec{H}$ is spacelike. We choose $e_n^H = -\frac{\vec{H}}{|\vec{H}|}$ and the complementing e_{n+1}^H as the orthonormal frame in the normal bundle. The connection form with respect to such frame is denoted by

$$\alpha_H(v) = \langle D_v e_n^H, e_{n+1}^H \rangle.$$

Chapter 3

Minkowski formulae

We recall the definition of conformal Killing-Yano 2-forms.

Definition 3.1. [19, Definition 1] Let Q be a two-form on an $n+1$-dimensional pseudo-Riemannian manifold $(V, \langle , \rangle)$ with Levi-Civita connection D. Q is said to be a conformal Killing-Yano 2-form if

$$\begin{aligned}&(D_X Q)(Y,Z) + (D_Y Q)(X,Z)\\&= \frac{2}{n}\left(\langle X,Y\rangle\langle \xi, Z\rangle - \frac{1}{2}\langle X,Z\rangle\langle \xi, Y\rangle - \frac{1}{2}\langle Y,Z\rangle\langle \xi, X\rangle \right)\end{aligned} \tag{3.1}$$

for any tangent vectors X, Y, and Z, where $\xi^\beta = (divQ)^\beta = D_\alpha Q^{\alpha\beta}$

In mathematical literatures, conformal Killing-Yano 2-forms were introduced by Tachibana [29], based on Yano's work on Killing forms. More generally, Kashiwada introduced the conformal Killing-Yano p-forms [20].

The main results of this section are the following two integral formulae.

Theorem 3.2. *Let Σ be a closed immersed oriented spacelike codimension-2 submanifold in an $(n+1)$-dimensional Riemannian or Lorentzian manifold V that possesses a conformal*

Killing-Yano 2-form Q. For any normal vector field $\underline{L}$ of Σ, we have

$$\frac{n-1}{n}\int_\Sigma \langle \xi, \underline{L}\rangle \, d\mu + \int_\Sigma Q(\vec{H}, \underline{L})\, d\mu + \int_\Sigma Q(\partial_a, (D^a \underline{L})^\perp)\, d\mu = 0, \tag{3.2}$$

where $\xi = divQ$ as in Definition 3.1.

Proof. Let $h_{ab} = \langle D_a \underline{L}, \partial_b\rangle$. Consider the one-form $\mathcal{Q} = Q(\partial_a, \underline{L}) du^a$ on Σ. We derive

$$\begin{aligned}
div_\Sigma \mathcal{Q} &= \partial^a \mathcal{Q}_a - Q(\nabla^a \partial_a, \underline{L}) \\
&= (D^a Q)(\partial_a, \underline{L}) + Q(\vec{H}, \underline{L}) + Q(\partial_a, D^a \underline{L}) \\
&= \frac{n-1}{n}\langle \xi, \underline{L}\rangle + Q(\vec{H}, \underline{L}) + h_{ab}Q^{ab} + Q(\partial_a, (D^a \underline{L})^\perp) \\
&= \frac{n-1}{n}\langle \xi, \underline{L}\rangle + Q(\vec{H}, \underline{L}) + Q(\partial_a, (D^a \underline{L})^\perp).
\end{aligned}$$

The assertion follows by integrating over Σ. □

The following is a generalization of the $k = 2$ Minkowski formula:

Theorem 3.3. *Let Σ be a closed immersed oriented spacelike codimension-2 submanifold in an $(n+1)$-dimensional Lorentzian manifold V that possesses a conformal Killing-Yano tensor Q. Then*

$$\begin{aligned}
\frac{n-2}{n}\int_\Sigma \langle \xi, \vec{J}\rangle d\mu = {}& 2\int_\Sigma \big(\sigma_2(h_n) - \sigma_2(h_{n+1})\big) Q_{n,n+1} d\mu \\
&+ \int_\Sigma \left(\bar{R}^{ab}{}_{an} Q_{b,n+1} - \bar{R}^{ab}{}_{a,n+1} Q_{bn}\right) d\mu \\
&+ \int_\Sigma \left(\bar{R}^{ab}{}_{n+1,n} - (d\zeta)^{ab}\right) Q_{ab} d\mu.
\end{aligned} \tag{3.3}$$

Proof. Consider the divergence quantity on Σ:

$$\nabla_a \left([\sigma_1(h_n)\sigma^{ab} - h_n{}^{ab}] Q_{b,n+1} - [\sigma_1(h_{n+1})\sigma^{ab} - h_{n+1}{}^{ab}] Q_{bn}\right). \tag{3.4}$$

From the Codazzi equations, we derive

$$\begin{aligned}\nabla_a(\sigma_1(h_n)\sigma^{ab} - {h_n}^{ab}) &= -\bar{R}^{ab}{}_{an} - \zeta^b\sigma_1(h_{n+1}) + \zeta_a {h_{n+1}}^{ab}\\ \nabla_a(\sigma_1(h_{n+1})\sigma^{ab} - {h_{n+1}}^{ab}) &= -\bar{R}^{ab}{}_{a,n+1} - \zeta^b\sigma_1(h_n) + \zeta_a {h_n}^{ab}.\end{aligned} \tag{3.5}$$

On the other hand,

$$\begin{aligned}\nabla_a Q_{b,n+1} &= (D_aQ)(\partial_b, e_{n+1}) - h_{nab}Q_{n,n+1} + Q_{bc}h_{n+1,a}{}^c - Q_{bn}\zeta_a\\ \nabla_a Q_{bn} &= (D_aQ)(\partial_b, e_n) + h_{n+1,ab}Q_{n+1,n} + Q_{bc}h_{na}{}^c - Q_{b,n+1}\zeta_a.\end{aligned} \tag{3.6}$$

Putting (3.5) and (3.6) together, we get

$$\begin{aligned}&\nabla_a\left((\sigma_1(h_n)\sigma^{ab} - {h_n}^{ab})Q_{b,n+1} - (\sigma_1(h_{n+1})\sigma^{ab} - {h_{n+1}}^{ab})Q_{bn}\right)\\ &= -\bar{R}^{ab}{}_{an}Q_{b,n+1}\\ &\quad + (\sigma_1(h_n)\sigma^{ab} - {h_n}^{ab})\left((D_aQ)(\partial_b, e_{n+1}) - h_{nab}Q_{n,n+1} + Q_{bc}h_{n+1,a}{}^c\right)\\ &\quad + \bar{R}^{ab}{}_{a,n+1}Q_{bn}\\ &\quad - (\sigma_1(h_{n+1})\sigma^{ab} - {h_{n+1}}^{ab})\left((D_aQ)(\partial_b, e_n) + h_{n+1,ab}Q_{n+1,n} + Q_{bc}h_{na}{}^c\right)\end{aligned} \tag{3.7}$$

From the definition of conformal Killing-Yano 2-forms,

$$\begin{aligned}&(\sigma_1(h_n)\sigma^{ab} - {h_n}^{ab})(D_aQ)(\partial_b, e_{n+1})\\ &\quad = \frac{1}{2}(\sigma_1(h_n)\sigma^{ab} - {h_n}^{ab})((D_aQ)(\partial_b, e_{n+1}) + (D_bQ)(\partial_a, e_{n+1}))\\ &\quad = \frac{1}{n}(\sigma_1(h_n)\sigma^{ab} - {h_n}^{ab})\langle\xi, e_{n+1}\rangle\sigma_{ab}\\ &\quad = \frac{n-2}{n}\langle\xi, \sigma_1(h_n)e_{n+1}\rangle.\end{aligned} \tag{3.8}$$

Similarly,

$$(\sigma_1(h_{n+1})\sigma^{ab} - {h_{n+1}}^{ab})(D_aQ)(\partial_b, e_n) = \frac{n-2}{n}\langle\xi, \sigma_1(h_{n+1})e_n\rangle$$

By the Gauss and Ricci equations, we have

$$
\begin{aligned}
&\nabla_a \left((\sigma_1(h_n)\sigma^{ab} - h_n{}^{ab})Q_{b,n+1} - (\sigma_1(h_{n+1})\sigma^{ab} - h_{n+1}{}^{ab})Q_{bn}\right) \qquad (3.9)\\
&= -\bar{R}^{ab}{}_{an}Q_{b,n+1} + \bar{R}^{ab}{}_{a,n+1}Q_{bn} + \frac{n-2}{n}\langle \xi, \sigma_1(h_n)e_{n+1} - \sigma_1(h_{n+1})e_n\rangle \\
&\quad - 2\sigma_2(h_n)Q_{n,n+1} + 2\sigma_2(h_{n+1})Q_{n,n+1} \\
&\quad + Q_{bc}(h_{n+1}{}^{ab}h_{na}^{\ \ c} - h_n{}^{ab}h_{n+1,a}^{\ \ c}) \\
&= \frac{n-2}{n}\langle \xi, \vec{J}\rangle - 2(\sigma_2(h_n) - \sigma_2(h_{n+1}))Q_{n,n+1} \\
&\quad - \bar{R}^{ab}{}_{an}Q_{b,n+1} + \bar{R}^{ab}{}_{a,n+1}Q_{bn} \\
&\quad - \left(\bar{R}^{ab}{}_{n+1,n} - (d\zeta)^{ab}\right)Q_{ab}
\end{aligned}
$$

The assertion follows by integrating over Σ. □

If we consider the divergence quantity

$$
\int_\Sigma \nabla_a \Big[\left(\mathrm{tr}\chi\sigma^{ab} - \chi^{ab}\right) Q(L, \partial_b) - \left(\mathrm{tr}\underline{\chi}\sigma^{ab} - \underline{\chi}^{ab}\right) Q(L, \partial_b)\Big],
$$

we obtain the Minkowski formula expressed in null frames.

Theorem 3.4. *Let Σ be a closed immersed oriented spacelike codimension-2 submanifold in an $(n+1)$-dimensional Lorentzian manifold V that possesses a conformal Killing-Yano tensor Q. Then*

$$
\begin{aligned}
\frac{2(n-2)}{n}\int_\Sigma \langle \xi, \vec{J}\rangle d\mu = &- \int_\Sigma \left(tr\chi tr\underline{\chi} - \chi_{ab}\underline{\chi}^{ab}\right)Q_{L\underline{L}}d\mu \qquad (3.10)\\
&+ \int_\Sigma \left(\bar{R}_b{}^a{}_{aL}Q_{\underline{L}b} - \bar{R}_b{}^a{}_{a\underline{L}}Q_{Lb}\right) d\mu \\
&+ \int_\Sigma \left(\bar{R}^{ab}{}_{\underline{L}L} - 2(d\zeta)^{ab}\right) Q_{ab}d\mu
\end{aligned}
$$

3.1 Important special cases

3.1.1 Static spherically symmetric spacetime

We consider the case of static spherically symmetric spacetime and show how (3.3) recovers a Minkowski formula proved by Brendle and Eichmair [7]. We start with the existence of conformal Killing-Yano 2-forms on static spherically symmetric spacetimes.

Lemma 3.5. *Let $(V, \bar{g})$ be an $(n+1)$-dimensional static spherically symmetric spacetime with the metric given by*

$$\bar{g} = -f^2(r)dt^2 + \frac{1}{f^2(r)}dr^2 + r^2 g_{S^{n-1}}. \tag{3.11}$$

where $g_{S^{n-1}}$ is the standard metric on S^{n-1}. Then the two-form $Q = rdr \wedge dt$ satisfies the conformal Killing-Yano equation (3.1) with $\xi = divQ = -n\frac{\partial}{\partial t}$.

Proof. The first assertion is proved in appendix A. The second assertion is verified by direct computation:

$$\xi_t = D^r Q_{rt} + D^a Q_{at} = nf^2$$
$$\xi_r = \xi_a = 0$$

Therefore, $\xi = -n\frac{\partial}{\partial t}$. □

Consider the manifold $M = I \times S^{n-1}$, where I is an interval, equipped with a Riemannian metric of the form $g = \frac{1}{f^2(r)}dr^2 + r^2 g_{S^{n-1}}$ where $g_{S^{n-1}}$ is the standard metric on S^{n-1}. (M, g) has a conformal Killing vector field $X = rf\frac{\partial}{\partial r}$. Let Σ be a hypersurface in M. Let e_n denote the unit normal to Σ, and let σ_p denote the p-th elementary symmetric polynomial in the principal curvatures of Σ. In [7, Proposition 8 and 9], Brendle and Eichmair derive a remarkable Minkowski formula

$$(n-2)\int_\Sigma f\sigma_1 = 2\int_\Sigma \langle X, e_n\rangle \sigma_2 + \int_\Sigma \mathrm{Ric}(X^\intercal, e_n) \tag{3.12}$$

where $X^\intercal$ denotes the tangential component of X on Σ.

Remark 3.6. For the space forms $\mathbb{R}^n, \mathbb{H}^n$ and $\mathbb{S}^n_+$, the upper hemisphere, $f(r) = 1, \cosh r$ and $\cos r$ respectively. For a hypersurface Σ in the space forms, (3.12) recovers the second Minkowski formula

$$(n-2)\int_\Sigma f\sigma_1 = 2\int_\Sigma \sigma_2 \langle X, \nu\rangle,$$

which can be found in [3].

In the rest of the section, we show that (3.3) recovers (3.12). More precisely, we consider M as a time slice of the spacetime $(V, \bar{g})$ with the metric $\bar{g} = -f^2(r)dt^2 + \frac{1}{f^2(r)}dr^2 + r^2 g_{S^{n-1}}$. Then Σ can be viewed as a codimension-2 submanifold in V. We show that (3.3) reduces to (3.12).

It is clear that M is a totally geodesic slice in V and thus $h_{n+1} = 0$. Let e_n be the unit normal of Σ in M.

First of all, note that the restriction of Q on a spherically symmetric hypersurface vanishes. Take $X = rf\frac{\partial}{\partial r}$, we claim that

$$Q_{n,n+1} = \langle X, e_n\rangle \text{ and } Q_{b,n+1} = \langle X, e_b\rangle.$$

By direct computation $Q_{n,n+1} = (rdr \wedge dt)(e_n, e_{n+1}) = \frac{1}{f}rdr(e_n)$. On the other hand $X = rf\frac{\partial}{\partial r}$ is dual to $\frac{1}{f}rdr$.

Taking Lemma 3.5 into account, (3.3) is reduced to

$$-(n-2)\int_\Sigma \langle \frac{\partial}{\partial t}, \vec{J}\rangle = 2\int_\Sigma \sigma_2(h_n) + \int_\Sigma \bar{R}^{ab}{}_{an} Q_{b,n+1}.$$

For the left hand side of (3.3), we have

$$-(n-2)\int_\Sigma \langle \frac{\partial}{\partial t}, \vec{J}\rangle = (n-2)\int_\Sigma fH.$$

For the right hand side, we compute

$$\begin{aligned}\bar{R}^{ab}{}_{an}Q_{b,n+1} &= \sum_{a,b=1}^{n-1} \bar{R}(e_a, e_b, e_a, e_n)\langle e_b, X\rangle \\ &= \sum_{a=1}^{n-1} \bar{R}(e_a, X^\intercal, e_a, e_n) \\ &= \mathrm{Ric}^M(X^\intercal, e_n),\end{aligned}$$

where the Gauss equation and $h_{n+1} = 0$ are used in the last equality.

Putting these together, we obtain

$$(n-2)\int_\Sigma fH = 2\int_\Sigma \sigma_2(h_n)\langle X, e_n\rangle + \int_\Sigma \mathrm{Ric}^M(X^\intercal, e_n)$$

We thus recover (3.12).

For future reference, we write the Minkowski formula (3.3) on Schwarzschild spacetime in terms of Q. In view of the curvature formula (B.1) on $(n+1)$-dimensional Schwarzschild spacetime, we have

$$\begin{aligned}\bar{R}^{ab}{}_{an} &= -\frac{nm}{r^{n+2}}Q^{ab}Q_{an} - \frac{n(n-2)m}{r^{n+2}}Q^{b}{}_{n+1}Q_{n,n+1}, \\ \bar{R}^{ab}{}_{a,n+1} &= -\frac{nm}{r^{n+2}}Q^{ab}Q_{a,n+1} - \frac{n(n-2)m}{r^{n+2}}Q^{b}{}_{n}Q_{n,n+1}, \\ \bar{R}^{ab}{}_{n+1,n} &= -\frac{n(n-1)m}{r^{n+2}}\left(\frac{2}{3}Q^{ab}Q_{n+1,n} - \frac{1}{3}Q^{a}{}_{n+1}Q_{n}{}^{b} - \frac{1}{3}Q^{a}{}_{n}Q^{b}{}_{n+1}\right).\end{aligned}$$

Proposition 3.7. *Let Σ be a closed immersed oriented spacelike codimension-2 submanifold in the $(n+1)$-dimensional Schwarzschild spacetime. Let $Q = r dr \wedge dt$ be the conformal*

Killing-Yano 2-form. Then

$$\frac{n-2}{n}\int_\Sigma \langle \xi, \vec{J}\rangle d\mu \tag{3.13}$$

$$= 2\int_\Sigma \big(\sigma_2(h_n) - \sigma_2(h_{n+1})\big)Q_{n,n+1}d\mu \tag{3.14}$$

$$+ \int_\Sigma \left(\frac{2nm}{r^{n+2}}Q^{ab}Q_{an}Q_{b,n+1} + \frac{n(n-2)m}{r^{n+2}}\left(-Q^b{}_{n+1}Q_{b,n+1} + Q^b{}_nQ_{bn}\right)\right)d\mu$$

$$- \int_\Sigma \left(\frac{n(n-1)m}{r^{n+2}}\left(\frac{2}{3}Q^{ab}Q_{n+1,n} - \frac{1}{3}Q^a{}_{n+1}Q_n{}^b - \frac{1}{3}Q^a{}_nQ^b{}_{n+1}\right) + (d\zeta)^{ab}\right)Q_{ab}d\mu.$$

3.1.2 The Kerr family

In Boyer-Lindquist coordinates, the metric is given by

$$\bar{g} = \left(-1 + \frac{2mr}{\rho^2}\right)dt^2 - \frac{2mra\sin^2\theta}{\rho^2}dtd\phi + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{2mra^2\sin^2\theta}{\rho^2}\right)d\phi^2 \tag{3.15}$$

where $\rho^2 = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2mr + a^2$. In [31], Walker and Penrose discovered a conformal Killing-Yano 2-form on the Kerr spacetime. In Boyer-Lindquist coordinates, it is of the form ([19, page 2907-2908] our choice of Y differs from theirs by -1)

$$Y = r\sin\theta d\theta \wedge [adt - (r^2 + a^2)d\phi] - a\cos\theta dr \wedge (dt - a\sin^2\theta d\phi).$$

The dual tensor $*Y$ is also a conformal Killing-Yano 2-form

$$*Y = a\cos\theta\sin\theta d\theta \wedge [adt - (r^2 + a^2)d\phi] + rdr \wedge (dt - a\sin^2\theta d\phi).$$

We have $divY = 0$ and $div(*Y) = -3\frac{\partial}{\partial t}$ [19, page 2908]. If we take $Q = *Y$ in our Minkowski formula, we obtain

Theorem 3.8. *Let Σ be a spacelike 2-surface in Kerr spacetime. Let $Q = a\cos\theta\sin\theta d\theta \wedge$*

$[adt-(r^2+a^2)d\phi]+rdr\wedge(dt-a\sin^2\theta d\phi)$ *be the conformal Killing-Yano 2-form. Then*

$$-\int_\Sigma \langle\frac{\partial}{\partial t},\vec{J}\rangle d\mu = \int_\Sigma (R+\bar{R}_{3434})Q_{34}d\mu - 8\pi m \\ - \int_\Sigma \left(\frac{1}{2}\bar{R}^{ab}{}_{34} + (d\zeta)^{ab}\right) Q_{ab}d\mu \tag{3.16}$$

Proof. For our choice of Q, $\xi = -3\frac{\partial}{\partial t}$. From the vacuum Einstein equation, we have

$$\begin{aligned}\bar{R}^{\mu\nu}{}_{43}Q_{\mu\nu} &= \bar{R}^{ab}{}_{43}Q_{ab} + 2\bar{R}^{4b}{}_{43}Q_{4b} + 2\bar{R}^{3b}{}_{43}Q_{3b} + 2\bar{R}^{43}{}_{43}Q_{43} \\ &= \bar{R}^{ab}{}_{43}Q_{ab} - 2\bar{R}^{ab}{}_{a3}Q_{4b} - 2\bar{R}^{ab}{}_{4a}Q_{3b} + 2\bar{R}_{3434}Q_{34} \\ &= \bar{R}^{ab}{}_{43}Q_{ab} + 2\bar{R}^{ab}{}_{a3}Q_{b4} - 2\bar{R}^{ab}{}_{a4}Q_{b3} + 2\bar{R}_{3434}Q_{34}.\end{aligned}$$

On the other hand, the Gauss equation implies

$$R = 2(\sigma_2(h_n) - \sigma_2(h_{n+1})) - 2\bar{R}_{3434}.$$

In view of these equations, the second term $\bar{R}^{ab}{}_{an}Q_{b,n+1} - \bar{R}^{ab}{}_{a,n+1}Q_{bn}$ in equation (3.3) can be replaced by other terms to get

$$-\int_\Sigma \langle\frac{\partial}{\partial t},\vec{J}\rangle d\mu = \int_\Sigma (R+\bar{R}_{3434})Q_{34}d\mu + \frac{1}{2}\int_\Sigma \bar{R}^{\mu\nu}{}_{43}Q_{\mu\nu}d\mu \\ - \int_\Sigma \left(\frac{1}{2}\bar{R}^{ab}{}_{34} + (d\zeta)^{ab}\right) Q_{ab}d\mu$$

Consider the two-form $\eta = \bar{R}^{\mu\nu}{}_{\alpha\beta}Q_{\mu\nu}dx^\alpha dx^\beta$. The conformal Killing-Yano equation, together with the vacuum Einstein equation and second Bianchi identity, imply $d\eta = d*\eta = 0$ [19, section 3.3]. Therefore, the integral $\int_\Sigma \bar{R}^{\mu\nu}{}_{43}Q_{\mu\nu}d\mu$ is the same for any two 2-surface bounding a 3-volume. The assertion follows by evaluating the integral on a sphere with $t=$constant and $r=$constant [19, equation (53)]. □

Remark 3.9. We may also arrange the terms to get

$$
\begin{aligned}
-\int_\Sigma \langle \frac{\partial}{\partial t}, \vec{J} \rangle d\mu = & \int_\Sigma R\, Q_{34} d\mu - 16\pi m \\
& + \int_\Sigma \left(\bar{R}^{ab}{}_{a4} Q_{b3} - \bar{R}^{ab}{}_{a3} Q_{b4} \right) d\mu - \int_\Sigma (d\zeta)^{ab} Q_{ab} d\mu
\end{aligned} \tag{3.17}
$$

When there is no angular momentum, $a = 0$, Kerr spacetime reduces to Schwarzschild spacetime. With the curvature formula (B.1), for a spacelike 2-surface Σ in 4-dimensional Schwarzschild spacetime, we have

$$
\begin{aligned}
-\int_\Sigma \langle \frac{\partial}{\partial t}, \vec{J} \rangle d\mu = & \, 2 \int_\Sigma \big(\sigma_2(h_n) - \sigma_2(h_{n+1}) \big) Q_{n,n+1} d\mu - \int_\Sigma (d\zeta)^{ab} Q_{ab} d\mu \\
& + \int_\Sigma \frac{2m}{r^5} Q^{ab} Q_{a,n+1} Q_{bn} d\mu \\
& + \int_\Sigma \frac{m}{r^5} \big(-3Q^b{}_{n+1} Q_{b,n+1} + 3Q^b{}_n Q_{bn} + 4Q^{ab} Q_{ab} \big) Q_{n,n+1} d\mu.
\end{aligned} \tag{3.18}
$$

Chapter 4

A Spacetime Alexandrov Theorem

4.1 A monotonicity formula

In this section, we assume that Σ is a spacelike codimension-2 submanifold with spacelike mean curvature vector in a Lorentzian manifold V that possesses a conformal Killing-Yano 2-form Q. We fix the sign of Q by requiring $\xi := divQ$ to be past-directed timelike. For example, we choose $Q = rdr \wedge dt$ on Schwarzschild spacetime. Let $\underline{L}$ be a future incoming null normal and L be the null normal vector with $\langle L, \underline{L} \rangle = -2$. Let $\zeta(X) := \frac{1}{2}\langle D_X L, \underline{L} \rangle$ be the connection one-form.

Define the functional

$$\mathcal{F}(\Sigma, [\underline{L}]) = \frac{n-1}{n} \int_\Sigma \frac{\langle \xi, \underline{L} \rangle}{\langle \vec{H}, \underline{L} \rangle} d\mu - \frac{1}{2} \int_\Sigma Q(L, \underline{L}) d\mu. \tag{4.1}$$

Note that $\mathcal{F}$ is well-defined in that it is invariant under the change $L \to aL, \underline{L} \to \frac{1}{a}\underline{L}$.

Let $\underline{C}_0$ denote the future incoming null hypersurface of Σ and extend $\underline{L}$ arbitrarily to a future-directed null vector field along $\underline{C}_0$, still denoted by $\underline{L}$. Consider the evolution of Σ

along $\underline{C}_0$, $F : \Sigma \times [0, T) \to \underline{C}_0$

$$\begin{cases} \frac{\partial F}{\partial s}(x,s) & = \varphi(x,s)\underline{L} \\ F(x,0) & = F_0(x). \end{cases} \tag{4.2}$$

for some positive function $\varphi(x,t)$.

Proposition 4.1. *Let $F_0 : \Sigma \to V$ be an immersed oriented spacelike codimension-2 submanifold in a Lorentzian manifold V that satisfies either one of the following assumption*

1. *V is vacuum (possibly with cosmological constant) and possesses a conformal Killing-Yano 2-form Q such that $\xi \neq 0$, or*

2. *V is static, spherically symmetric and we choose the conformal Killing-Yano 2-form obtained in Lemma 3.5. Moreover we assume V satisfies the null convergence condition, that is,*

$$Ric(L,L) \geq 0 \quad \textit{for any null vector}\, L. \tag{4.3}$$

Suppose that $\langle \vec{H}, \underline{L} \rangle > 0$ on Σ for some future-directed incoming null normal vector field $\underline{L}$. Then $\mathcal{F}(F(\Sigma, s))$ is monotone decreasing along the flow.

Proof. Suppose $D_{\underline{L}}\underline{L} = \underline{\omega}\underline{L}$ for a function $\underline{\omega}$. Let $\underline{\chi}_{ab} = \langle D_a \underline{L}, \partial_b \rangle$ be the null second fundamental form with respect to $\underline{L}$. The Raychadhuri equation [30, (9.2.32)] implies

$$\begin{aligned} \frac{\partial}{\partial s}\langle \vec{H}, \underline{L} \rangle &= \varphi \left(|\underline{\chi}|^2 + \underline{\omega}\langle \vec{H}, \underline{L} \rangle + Ric(\underline{L}, \underline{L}) \right) \\ &\geq \varphi \left(|\underline{\chi}|^2 + \underline{\omega}\langle \vec{H}, \underline{L} \rangle \right). \end{aligned} \tag{4.4}$$

On the other hand,

$$\frac{\partial}{\partial s}\langle \xi, \underline{L} \rangle = \varphi \left(\langle D_{\underline{L}}\xi, \underline{L} \rangle + \underline{\omega}\langle \xi, \underline{L} \rangle \right)$$

If V satisfies assumption (1), by [19, equation (19)], we have

$$\langle D_{\underline{L}}\xi, \underline{L}\rangle = \frac{n}{n-1} R_{a\underline{L}} Q^a{}_{\underline{L}} = 0.$$

If V satisfies assumption (2), $\langle D_{\underline{L}}\xi, \underline{L}\rangle$ also vanishes since ξ is a Killing vector. By the Cauchy-Schwartz inequality,

$$\begin{aligned}\frac{\partial}{\partial s}\int_\Sigma \frac{\langle \xi, \underline{L}\rangle}{\langle \vec{H}, \underline{L}\rangle} d\mu &= -\int_\Sigma \varphi \left[\frac{\langle \xi, \underline{L}\rangle}{(\mathrm{tr}\underline{\chi})^2} |\underline{\chi}|^2 - \langle \xi, \underline{L}\rangle \right] d\mu \\ &\leq -\frac{n}{n-1}\int_\Sigma \varphi \langle \xi, \underline{L}\rangle d\mu\end{aligned} \tag{4.5}$$

The evolution of $\int_\Sigma Q(L, \underline{L}) d\mu$ is given by

$$\begin{aligned}&\frac{\partial}{\partial s}\int_\Sigma Q(L, \underline{L}) d\mu \\ &= \int_\Sigma \left[\varphi \left(D_{\underline{L}} Q \right)(L, \underline{L}) + Q(D_{\partial_s} L, \underline{L}) + Q(L, D_{\partial_s}\underline{L}) - \varphi Q(L, \underline{L}) \langle \vec{H}, \underline{L}\rangle \right] d\mu.\end{aligned}$$

From the conformal Killing-Yano equation (3.1), we derive

$$\left(D_{\underline{L}} Q \right)(L, \underline{L}) = \frac{1}{n}\langle \xi, \underline{L}\rangle \langle L, \underline{L}\rangle = -\frac{2}{n}\langle \xi, \underline{L}\rangle.$$

On the other hand, we compute

$$\begin{aligned}\langle D_{\partial_s} L, \underline{L}\rangle &= -\langle L, \varphi \underline{\omega} \underline{L}\rangle \\ \langle D_{\partial_s} L, \partial_a\rangle &= -\langle L, D_a(\varphi \underline{L})\rangle = 2\nabla_a \varphi - \varphi \langle L, D_a \underline{L}\rangle.\end{aligned}$$

Putting these calculations together yields

$$\begin{aligned}
&Q(D_{\partial_s}L,\underline{L}) + Q(L, D_{\partial_s}\underline{L}) - \varphi Q(L,\underline{L})\langle \vec{H},\underline{L}\rangle \\
&\quad = 2\nabla^a\varphi Q(\partial_a,\underline{L}) + 2\varphi Q(\partial_a,(D_a\underline{L})^\perp) + 2\varphi Q(\vec{H},\underline{L}) \\
&\quad = 2\nabla^a\left(\varphi Q(\partial_a,\underline{L})\right) - \frac{2(n-1)}{n}\varphi\langle \xi,\underline{L}\rangle.
\end{aligned}$$

Consequently, we obtain

$$\frac{\partial}{\partial s}\int_\Sigma Q(L,\underline{L})d\mu = -2\int_\Sigma \langle \xi,\underline{L}\rangle d\mu. \tag{4.6}$$

The assertion follows from (4.5) and (4.6). □

4.2 A spacetime CMC condition

Hypersurfaces of constant mean curvature (CMC) provide models for soap bubbles, and have been studied extensively for a long time. A common generalization of this condition for higher codimension submanifolds is the parallel mean curvature condition. In general relativity, the most relevant physical phenomenon is the divergence of light rays emanating from a codimension-2 submanifold. This is called the null expansion in physics literature. We thus impose constancy conditions on the null expansion of codimension-2 submanifolds.

Definition 4.2. A codimension-2 submanifold of a Lorentz manifold is said to have *constant normalized null curvature (CNNC)* if there exists a future null normal vector field l such that $\langle \vec{H}, l\rangle$ is a constant and $(Dl)^\perp = 0$.

The CNNC condition can be written as a single equation on the connection one-form in mean curvature gauge.

Proposition 4.3. *Suppose the mean curvature vector field $\vec{H}$ of Σ is spacelike.*

1. *If* $\langle \vec{H}, L\rangle = c < 0$ *and* $(DL)^\perp = 0$ *for some future outward null normal* L *and some negative constant* c*, then* $\alpha_H = -d\log|\vec{H}|$.

2. *If* $\langle \vec{H}, \underline{L}\rangle = c > 0$ *and* $(D\underline{L})^\perp = 0$ *for some future inward null normal* $\underline{L}$ *and some positive constant* c*, then* $\alpha_H = d\log|\vec{H}|$

Proof. Recall that the dual mean curvature vector $\vec{J}$ is future timelike. For (1), the condition $\langle \vec{H}, L\rangle = c < 0$ is equivalent to

$$L = \frac{-c}{|\vec{H}|}\left(-\frac{\vec{H}}{|\vec{H}|} + \frac{\vec{J}}{|\vec{H}|}\right).$$

Choose $\underline{L} = \frac{|\vec{H}|}{-c}\left(\frac{\vec{H}}{|\vec{H}|} + \frac{\vec{J}}{|\vec{H}|}\right)$ such that $\langle L, \underline{L}\rangle = -2$. Since $(DL)^\perp = 0$, we have

$$\begin{aligned} 0 = \frac{1}{2}\langle D_a L, \underline{L}\rangle &= \frac{1}{2}\partial_a\left(\frac{-c}{|\vec{H}|}\right)\frac{|\vec{H}|}{-c}(-2) + \left\langle D_a\left(-\frac{\vec{H}}{|\vec{H}|}\right), \frac{\vec{J}}{|\vec{H}|}\right\rangle \\ &= \partial_a \log|\vec{H}| + \left\langle D_a\left(-\frac{\vec{H}}{|\vec{H}|}\right), \frac{\vec{J}}{|\vec{H}|}\right\rangle \end{aligned}$$

Hence $\alpha_H = -d\log|\vec{H}|$. (2) is proved similarly. □

When Σ lies in a totally geodesic time slice of a static spacetime, CNNC reduces to the CMC condition.

4.3 A Heintze-Karcher type inequality

In this and the next sections, we study a class of static spacetimes in which the warped product manifolds considered in [6] are embedded as totally geodesic slices.

Assumption 4.4. We assume V is a spacetime that satisfies the null convergence condition

(4.3) and the metric $\bar{g}$ on $V = \mathbb{R} \times M$ is of the form

$$\bar{g} = -f^2(r)dt^2 + \frac{1}{f^2(r)}dr^2 + r^2 g_N. \tag{4.7}$$

where (N, g_N) is a compact n-dimensional Riemannian manifold. We consider two cases.

(i) $f : [0, r_1) \to \mathbb{R}$ with $f(0) = 1$, $f'(0) = 0$, and $f(r) > 0$ in the domain.

(ii) $f : [r_0, r_1) \to \mathbb{R}$ with $f(r_0) = 0$ and $f(r) > 0$ for $r > r_0$.

Here r_1 can be ∞.

In case (ii), V contains an event horizon $\mathcal{H} = \{r = r_0\}$.

Remark 4.5. Assumption 4.4 covers basic examples of static spherically spacetimes. Taking $f^2 = 1 + \kappa r^2$, we obtain the spacetimes with constant sectional curvature: Minkowski spacetime ($\kappa = 0$), anti de-Sitter spacetime ($\kappa > 0$), and de-Sitter spacetime ($\kappa < 0$). Taking $f^2 = 1 - \frac{2m}{r^{n-2}} + \frac{q^2}{r^{2n-4}}$, we obtain Reissner-Nordstrom spacetime with mass m and charge q.

Lemma 4.6. *Let (M, g) be a time slice in V. The null convergence condition of $(V, \bar{g})$ implies that*

$$(\Delta_g f)g - Hess_g f + f Ric(g) \geq 0 \tag{4.8}$$

on M.

Proof. O'Neill's formula in our case reduces to (see [13, Proposition 2.7])

$$\begin{aligned}
\mathrm{Ric}(\bar{g})(v, w) &= \mathrm{Ric}(g)(v, w) - \frac{\mathrm{Hess}_g f(v, w)}{f} \\
\mathrm{Ric}(\bar{g})(v, \frac{\partial}{\partial t}) &= 0 \\
\mathrm{Ric}(\bar{g})(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) &= -\frac{\Delta_g f}{f}\bar{g}(\frac{\partial}{\partial t}, \frac{\partial}{\partial t})
\end{aligned}$$

for any tangent vectors v and w on M. Given a unit tangent vector v on M, we form a null vector $L = \frac{1}{f}\frac{\partial}{\partial t} + v$ in the spacetime. Null convergence condition implies that

$$\begin{aligned} 0 &\leq \mathrm{Ric}(\bar{g})(L, L) \\ &= \frac{\Delta_g f}{f} + \mathrm{Ric}(g)(v, v) - \frac{\mathrm{Hess}_g f(v, v)}{f} \end{aligned}$$

as claimed. □

As in section 3.2, we denote the conformal Killing vector field X on (M, g) by $X = rf\frac{\partial}{\partial r}$. In [6], Brendle proves a Heintze-Karcher-type inequality for mean convex hypersurfaces in (M, g). In our context, it is as the follows:

Theorem 4.7. *[6] Let S be a smooth, closed, embedded, orientable hypersurface in a time slice of a spacetime V that satisfies Assumption 4.4 . Suppose that S has positive mean curvature $H > 0$ in the slice. Then*

$$(n-1)\int_S \frac{f}{H} d\mu \geq \int_S \langle X, \nu \rangle, \tag{4.9}$$

where ν is the outward unit normal of S in the slice and $X = rf\frac{\partial}{\partial r}$ is the conformal Killing vector field on the slice. Moreover, if equality holds, then S is umbilical.

Proof. We first remark that since S is embedded and orientable, S is either null-homologous or homologous to $\{r_0\} \times N$. Hence $\partial\Omega = S$ or $\partial\Omega = S - \{r_0\} \times N$ for some domain $\Omega \subset M$. Inequality (4.9) is equivalent to the one in Theorem 3.5 and the one in Theorem 3.11 of Brendle's paper in the respective cases. For the reader's convenience, we trace Brendle's argument leading to (4.9).

The assumptions on (M, g) are listed in page 248:

$$\mathrm{Ric}_N \geq (n-2)\rho g_N \tag{4.10}$$

and (H1)-(H3) (note that condition (H4) is not used in the proof of (4.9)). While Brendle writes the metric in geodesic coordinates

$$d\bar{r} \otimes d\bar{r} + h^2(\bar{r}) g_N,$$

it is equivalent to ours by a change of variables $r = h$ and $f = \frac{dh}{d\bar{r}}$. As explained in the beginning of section 2 (page 252), (H1) and (H2) are equivalent to our assumptions (i) and (ii) on f. In Proposition 2.1, (4.10) and (H3) together imply that (4.8) holds on (M, g).

The condition (4.8) turns out to be the only curvature assumption necessary to get (4.9). More precisely, (4.8) is used to prove the key monotonicity formula, Proposition 3.2 (page 256). Inequality (4.9) is a direct consequence of Proposition 3.2 up to several technical lemmata, Lemma 3.6 to Corollary 3.10, in which only assumptions (H1) and (H2) are used.

Finally, the inequalities appear in Theorem 3.5 and Theorem 3.11 are equivalent to (4.9) by divergence theorem. □

Before stating the spacetime Heintze-Karcher inequality, we define the notion of future incoming null smoothness and shearfree null hypersurface.

Definition 4.8. A closed, spacelike codimension-2 submanifold Σ in a static spacetime V is *future(past) incoming null smooth* if the future(past) incoming null hypersurface of Σ intersects a totally geodesic time-slice $\mathcal{M}_T = \{t = T\} \subset V$ at a smooth, embedded, orientable hypersurface S.

Definition 4.9. An incoming null hypersurface $\underline{\mathcal{C}}$ is *shearfree* if there exists a spacelike hypersurface $\Sigma \subset \underline{\mathcal{C}}$ such that the null second fundamental form $\underline{\chi}_{ab} = \langle D_a \underline{L}, \partial_b \rangle$ of Σ with respect to some null normal $\underline{L}$ satisfies $\underline{\chi}_{ab} = \varphi \sigma_{ab}$ for some function φ.

Note that being shearfree is a property of the null hypersurface. See [27, page 47-48]

Theorem 4.10. *Let V be a spacetime as in Assumption 4.4. Let $\Sigma \subset V$ be a future incoming null smooth closed spacelike codimension-2 submanifold with $\langle \vec{H}, \underline{L} \rangle > 0$ where $\underline{L}$ is a future* *incoming null normal. Then*

$$-(n-1)\int_\Sigma \frac{\langle \frac{\partial}{\partial t}, \underline{L}\rangle}{\langle \vec{H}, \underline{L}\rangle} d\mu - \frac{1}{2}\int_\Sigma Q(L,\underline{L}) d\mu \geq 0, \tag{4.11}$$

for a future outgoing null normal L with $\langle L, \underline{L}\rangle = -2$. Moreover, the equality holds if Σ lies in a shearfree null hypersurface.

Proof. We could arrange φ in (4.2) such that $\underline{\omega} \geq 0$ and that $F(\Sigma, 1) = S$, the smooth hypersurface defined in the previous definition. We first claim that $S \subset \mathcal{M}_T$ has positive mean curvature, $H > 0$. Recall that Raychadhuri equation implies

$$\begin{aligned} \frac{\partial}{\partial s}\langle \vec{H}, \underline{L}\rangle &= \varphi\left(|\underline{\chi}|^2 + \underline{\omega}\langle \vec{H}, \underline{L}\rangle + Ric(\underline{L}, \underline{L})\right) \\ &\geq \varphi\left(|\underline{\chi}|^2 + \underline{\omega}\langle \vec{H}, \underline{L}\rangle\right), \end{aligned} \tag{4.12}$$

and hence $\langle \vec{H}, \underline{L}\rangle > 0$ on S. We choose $\underline{L} = \frac{1}{f}\frac{\partial}{\partial t} - e_n$ on S, where e_n is the outward unit normal of S with respect to Ω, and compute

$$\left\langle \vec{H}, \frac{1}{f}\frac{\partial}{\partial t} - e_n \right\rangle = H.$$

The claim follows since the positivity of $\langle \vec{H}, \underline{L}\rangle$ is independent of the scaling of $\underline{L}$. Next we choose $L = \frac{1}{f}\frac{\partial}{\partial t} + e_n$ on S and compute

$$\begin{aligned} -\left\langle \frac{\partial}{\partial t}, \frac{1}{f}\frac{\partial}{\partial t} - e_n \right\rangle &= f \\ Q\left(\frac{1}{f}\frac{\partial}{\partial t} + e_n, \frac{1}{f}\frac{\partial}{\partial t} - e_n\right) &= 2\langle X, e_n\rangle. \end{aligned}$$

Since $(V, \bar{g})$ is static and satisfies the null convergence condition, the monotonicity formula, Corollary 4.1, together with (4.9) imply

$$
\begin{aligned}
&-(n-1)\int_\Sigma \frac{\langle \frac{\partial}{\partial t}, \underline{L}\rangle}{\langle \vec{H}, \underline{L}\rangle} d\mu - \frac{1}{2}\int_\Sigma Q(L,\underline{L}) d\mu \qquad (4.13)\\
&\geq -(n-1)\int_{F(\Sigma,1)} \frac{\langle \frac{\partial}{\partial t}, \underline{L}\rangle}{\langle \vec{H}, \underline{L}\rangle} d\mu - \frac{1}{2}\int_{F(\Sigma,1)} Q(L,\underline{L}) d\mu\\
&= (n-1)\int_S \frac{f}{H} d\mu - \int_S \langle X, e_n\rangle d\mu\\
&\geq 0.
\end{aligned}
$$

Moreover, if the equality holds, then S is umbilical. Hence the future incoming null hypersurface from Σ is shearfree. □

4.4 A Spacetime Alexandrov Theorem

We state our main result.

Theorem 4.11. *Let V be a spherically symmetric spacetime as in Assumption 4.4 and Σ be a future incoming null smooth, closed, embedded, spacelike codimension-2 submanifold in V. Suppose Σ has CNNC with respect to $\underline{L}$ and $\langle \vec{H}, \underline{L}\rangle > 0$. Then Σ lies in a shearfree null hypersurface.*

Proof. Write $\vec{H} = -\frac{1}{2}\langle \vec{H}, \underline{L}\rangle L - \frac{1}{2}\langle \vec{H}, L\rangle \underline{L}$. From CNNC assumption, $(D_a \underline{L})^\perp = 0$, the spacetime Minkowski formula (3.2) becomes

$$
-(n-1)\int_\Sigma \langle \frac{\partial}{\partial t}, \underline{L}\rangle d\mu - \frac{1}{2}\int_\Sigma \langle \vec{H}, \underline{L}\rangle Q(L,\underline{L}) = 0
$$

Again from CNNC assumption, $\langle \vec{H}, \underline{L}\rangle$ is a positive constant function and we can divide $\langle \vec{H}, \underline{L}\rangle$ on both sides to get

$$
-(n-1)\int_\Sigma \frac{\langle \frac{\partial}{\partial t}, \underline{L}\rangle}{\langle \vec{H}, \underline{L}\rangle} d\mu - \frac{1}{2}\int_\Sigma Q(L,\underline{L})\rangle d\mu = 0.
$$

Hence the equality is achieved in the spacetime Heintze-Karcher inequality (4.11) and we conclude that Σ lies in a shearfree null hypersurface. □

An example of the spacetime satisfying Assumption 4.4 is the exterior Schwarzschild spacetime for which the metric has the form

$$\bar{g} = -(1 - mr^{2-n})dt^2 + \frac{1}{1 - mr^{2-n}} dr^2 + r^2 S^{n-1}.$$

Since the spheres of symmetry are the only closed umbilical hypersurfaces in the totally geodesic time slice of Schwarzschild spacetimes [6, Corollary 1.2], we obtain

Theorem 4.12 (Theorem B)**.** *Let Σ be a future incoming null smooth closed embedded spacelike codimension-2 submanifold in Schwarzschild spacetime. Suppose Σ is CNNC with respect to $\underline{L}$ and $\langle \vec{H}, \underline{L} \rangle > 0$. Then Σ lies in a null hypersurface of symmetry.*

Corollary 4.13. *Let Σ be a closed embedded spacelike codimension-2 submanifold with parallel mean curvature vector in Schwarzschild spacetime. Suppose Σ is both future and past incoming null smooth. Then Σ is a sphere of symmetry.*

Proof. The condition of parallel mean curvature vector implies $|\vec{H}|$ is constant and α_H vanishes. The previous theorem implies Σ is the intersection of one incoming and one outgoing null hypersurface of symmetry. Therefore, Σ is a sphere of symmetry. □

Chapter 5

Rigidity Results

In this chapter, we prove various rigidity results. We consider $(n+1)$-dimensional static spherically symmetric spacetimes $(V, \bar{g})$ satisfying Assumption 4.4 and the null convergence condition.

We first discuss the equivalence of null convergence condition and the differential inequality

$$\frac{f^2 - 1}{r^2} - \frac{ff'}{r} \leq 0. \tag{5.1}$$

When the equality of (5.1) holds, we can solve the ODE to conclude that $(V, \bar{g})$ has constant sectional curvature. See Remark 4.5.

Lemma 5.1. *Assumption (5.1) is equivalent to null convergence condition.*

Proof. We consider the inequality $r^{n-1}ff' + r^{n-2}(1-f^2) \geq 0$, which is equivalent to (5.1). Since we have $r^{n-1}ff' + r^{n-2}(1-f^2) \geq 0$ at $r = 0$ or $r = r_0$, it suffices to check that the quantity has nonnegative derivative.

Let e_1 be a unit tangent vector on the sphere of symmetry. Null convergence condition implies that

$$0 \le \overline{Ric}\left(\frac{1}{f}\frac{\partial}{\partial t} + e_1, \frac{1}{f}\frac{\partial}{\partial t} + e_1\right) = (n-3)\frac{ff'}{r} + \frac{1}{2}(f^2)'' + (n-2)\frac{1}{r^2}(1-f^2).$$

On the other hand,

$$\left(r^{n-1}ff' + r^{n-2}(1-f^2)\right)' = r^{n-1}\left[(n-3)\frac{ff'}{r} + \frac{1}{2}(f^2)'' + (n-2)\frac{1}{r^2}(1-f^2)\right] \ge 0,$$

This completes the proof. □

Remark 5.2. Assumption (5.1) is equivalent to assumption (H4) in [6]. Indeed, for metric $g = \frac{1}{f^2}dr^2 + r^2 g_{S^2}$, one has $Ric(\nu,\nu) = -(n-1)\frac{ff'}{r}$ and $Ric(e_1, e_1) = (n-2)\frac{1-f^2}{r^2} - \frac{ff'}{r}$ where ν and e_1 are unit normal and unit tangent vector of the sphere of symmetry with areal radius r. Assumption (5.1) thus means that the Ricci curvature is smallest in the radial direction.

5.1 Infinitesimal Rigidity of CNNC surfaces

We first verify directly that surfaces in the standard null cone are CNNC surfaces and then show that CNNC condition is infinitesimally rigid.

It is convenient to work in the Eddington-Finkelstein coordinates. Let $r^* = \int \frac{dr}{f^2}$ be the tortoise coordinate and $v = t + r^*, w = t - r^*$ be the advanced and retarded time. The metric in Eddington-Finkelstein coordinates is written as [15, page 153]

$$\bar{g} = -f^2 dv dw + r^2 g_{S^{n-1}}.$$

We compute the Christoffel symbols of $\bar{g}$:

$$\Gamma^b_{av} = \frac{1}{r}\frac{\partial r}{\partial v}\delta^b_a$$
$$\Gamma^v_{vv} = \frac{\partial \log f}{\partial v}$$

The outgoing standard null cones are defined by $w = constant$. For a spacelike hypersurface in the standard null cone, we write it as a graph over the sphere of symmetry. That is, the embedding of Σ is given by $F(\theta^a) = (v(\theta^a), w = constant, \theta^a)$ where θ^a are coordinates on S^{n-1}. Since $t = r^* + constant$ on standard null cones, we can write $r(x) = r(v(x))$ for points on standard null cones. From now on we view the restriction of coordinate functions v and r on Σ as functions on Σ.

Lemma 5.3. *Let Σ be a spacelike hypersurface in the outgoing standard null cone in $(V, \bar{g})$. Then Σ has CNNC, i.e. $\alpha_H = -d \log |\vec{H}|$. In particular, the intersection of a standard light one and a spacelike hypersurface has CNNC.*

Proof. The tangent vectors are given by $\frac{\partial F}{\partial \theta^a} = \frac{\partial v}{\partial \theta^a}\frac{\partial}{\partial v} + \frac{\partial}{\partial \theta^a}$. Let $L = \varphi(v)\frac{\partial}{\partial v}$ be a null normal of Σ where $\varphi(v) = \frac{2r}{f^2}$. We have

$$
\begin{aligned}
D_{\frac{\partial F}{\partial \theta^a}} L &= \varphi' \frac{\partial v}{\partial \theta^a}\frac{\partial}{\partial v} + \varphi\left(\frac{\partial v}{\partial \theta^a} D_{\frac{\partial}{\partial v}}\frac{\partial}{\partial v} + D_{\frac{\partial}{\partial \theta^a}}\frac{\partial}{\partial v}\right)\\
&= \varphi' \frac{\partial v}{\partial \theta^a}\frac{\partial}{\partial v} + \varphi \frac{\partial v}{\partial \theta^a}\frac{\partial \log f}{\partial v}\frac{\partial}{\partial v} + \varphi \frac{f^2}{2r}\frac{\partial}{\partial \theta^a}\\
&= \left(\varphi' + \varphi\frac{\partial \log f}{\partial v}\right)\frac{\partial v}{\partial \theta^a}\frac{\partial}{\partial v} + \frac{\partial}{\partial \theta^a}\\
&= \left(\frac{2\frac{\partial r}{\partial v} f^2 - 2r\frac{\partial f^2}{\partial v}}{f^4} + \frac{2r}{f^2}\frac{\partial \log f}{\partial v}\right)\frac{\partial v}{\partial \theta^a}\frac{\partial}{\partial v} + \frac{\partial}{\partial \theta^a}\\
&= \frac{2}{f^2}\frac{\partial r}{\partial v}\frac{\partial v}{\partial \theta^a}\frac{\partial}{\partial v} + \frac{\partial}{\partial \theta^a}\\
&= \frac{\partial F}{\partial \theta^a}.
\end{aligned}
$$

Hence $\langle \vec{H}, L\rangle = -2$. Let $\underline{L}$ be the null normal complement to L such that $\langle L, \underline{L}\rangle = -2$. It is straightforward to show that

$$
\underline{L} = \frac{1}{r}\left(2\frac{\partial}{\partial w} + f^2 \nabla v - \frac{f^2}{2}|\nabla v|^2 \frac{\partial}{\partial v}\right). \tag{5.2}
$$

Here ∇ denote the gradient on Σ. We have

$$\vec{H} = \underline{L} + \psi L$$
$$\vec{J} = \underline{L} - \psi L$$

where $|\vec{H}|^2 = -4\psi$. The connection one-form is given by

$$\begin{aligned}(\alpha_H)_a &= \langle D_a \left(-\frac{\vec{H}}{|\vec{H}|} \right), \frac{\vec{J}}{|\vec{H}|} \rangle \\ &= -\frac{1}{|\vec{H}|^2} \langle D_a(\underline{L} + \psi L), \underline{L} - \psi L \rangle \\ &= -\partial_a \log |\vec{H}|.\end{aligned}$$

This completes the proof of the lemma $\square$

Let Σ be a spacelike hypersurface in the standard null cone given by $F(\theta^a) = (v(\theta^a), w = constant, \theta^a)$ with null normals

$$\begin{aligned}L &= \frac{2r}{f^2} \frac{\partial}{\partial v} \\ \underline{L} &= \frac{1}{r} \left(2 \frac{\partial}{\partial w} + f^2 \nabla v - \frac{f^2}{2} |\nabla v|^2 \frac{\partial}{\partial v} \right).\end{aligned}$$

Consider the incoming null hypersurface $\underline{C}(\Sigma)$ of Σ. Extend $\underline{L}$ and L to $\underline{C}(\Sigma)$ such that $D_{\underline{L}} \underline{L} = 0$ and $\langle L, \underline{L} \rangle = -2$. Since CNNC is preserved for variation of Σ in the standard null cone, we focus on variations of Σ in $\underline{C}(\Sigma)$ when we discuss the infinitesimal rigidity of CNNC. Let $F(x,s) : \Sigma \times [0, \varepsilon) \to \underline{C}(\Sigma)$ be a variation of Σ. Let $\frac{\partial}{\partial s} = \frac{\partial F}{\partial s}(x, 0) = u(x)\underline{L}$ and $\underline{L}(s) = \psi(x,s)\underline{L}, L = \frac{1}{\psi} L$ be the null normals of $F(\Sigma, s)$.

Definition 5.4. We say $F(x,s)$ is an *infinitesimal CNNC variation* if

$$\begin{aligned} \frac{\partial}{\partial s}\Big|_{s=0} \partial_a \mathrm{tr}\chi = 0 \\ \frac{\partial}{\partial s}\Big|_{s=0} \langle D_a \underline{L}(s), L(s)\rangle = 0. \end{aligned} \tag{5.3}$$

We show that (5.3) can be written as a scalar equation of u. First of all, we compute the variation of L.

$$\langle D_{\partial_s} L(s), \partial_a\rangle\big|_{s=0} = -\langle L, D_a(u\underline{L})\rangle = 2\nabla_a u, \tag{5.4}$$

$$\langle D_{\partial_s} L(s), \underline{L}(s)\rangle\big|_{s=0} = -\langle L, \psi'\underline{L}\rangle = 2\psi', \tag{5.5}$$

where $\psi' = \frac{\partial\psi}{\partial s}(x,0)$. We have

$$\frac{\partial}{\partial s}\Big|_{s=0} \partial_a \mathrm{tr}\chi = \partial_a\left(-u\mathrm{tr}\underline{\chi} + 2\Delta u - 2\psi' + u\sigma^{ab}\bar{R}(\underline{L}, \partial_a, \partial_b, L)\right) \tag{5.6}$$

$$\frac{\partial}{\partial s}\Big|_{s=0} \langle D_a \underline{L}(s), L(s)\rangle = 2\underline{\chi}_{ab}\nabla^b u + u\bar{R}(\underline{L}, \partial_a, L, \underline{L}) - 2\nabla_a\psi' \tag{5.7}$$

By (B.2), we have

$$\sigma^{ab}\bar{R}(\underline{L}, \partial_a, \partial_b, L) = -2(n-1)\frac{ff'}{r} \quad \text{and} \quad \bar{R}(\underline{L}, \partial_a, L, \underline{L}) = \frac{4}{r}\nabla_a r\left(-(ff')' + \frac{ff'}{r}\right).$$

Equation (5.3) is equivalent to

$$\begin{aligned} -\nabla_a(u\mathrm{tr}\underline{\chi}) + 2\nabla_a\Delta u - 2(n-1)\nabla_a\left(\frac{ff'}{r}u\right) \\ - (n-1)\underline{\chi}_{ab}\nabla^b u - 2(n-1)u\frac{\nabla_a r}{r}\left((ff')' - \frac{ff'}{r}\right) = 0 \end{aligned} \tag{5.8}$$

When the lapse u is a constant multiple of r, it means that Σ is infinitesimally moved to neighboring standard null cones. Therefore we have the following definition.

Definition 5.5. A surface in the (outgoing) standard null cone is said to be *infinitesimally*

CNNC rigid if all the solutions of (5.8) are a constant multiple of r unless $(V, \bar{g})$ has constant sectional curvature and the deformation comes from a boost.

Theorem 5.6. *Let Σ be a spacelike hypersurface in the standard null cone of a static spherically symmetric spacetime satisfying Assumption 4.4 and null convergence condition. Then Σ is infinitesimally CNNC rigid.*

Proof. It is straightforward to show that r is a solution of (5.8). We replace u by $u \cdot r$ in (5.8) to get

$$\begin{aligned} -\nabla_a(ur\mathrm{tr}\underline{\chi}) - 2(n-1)\nabla_a\left(\frac{ff'}{r}ur\right) + 2\nabla_a(\Delta(ur)) \\ + 2(n-1)u\nabla_a r\left((ff')' - \frac{ff'}{r}\right) - (n-1)\underline{\chi}_{ab}\nabla^b(ur) = 0 \end{aligned} \tag{5.9}$$

It suffices to show that $u = \textit{constant}$ is the only solution of (5.9). On Σ, we have

$$\underline{\chi}_{ab} = -\frac{1}{r^2}\left((f^2 + |\nabla r|^2)\sigma_{ab} - 2r\nabla_a\nabla_b r\right), \tag{5.10}$$

$$\mathrm{tr}\underline{\chi} = \frac{1}{r^2}\left((n-1)\left(f^2 + |\nabla r|^2\right) - 2r\Delta r\right). \tag{5.11}$$

Combining (5.10) and (5.11) and the fact that r is a solution of (5.8), we have

$$\begin{aligned} 0 &= \frac{n-1}{r}(f^2 + |\nabla r|^2)\nabla_a u - 2(n-1)ff'\nabla_a u + 2\nabla_a(\Delta u \cdot r + 2\nabla_b u\nabla^b r) \\ &\quad + \frac{n-1}{r}\left((f^2 + |\nabla r|^2)\sigma_{ab} - 2r\nabla_a\nabla_b r\right)\nabla^b u \\ &= 2(n-1)\left(\frac{f^2 + |\nabla r|^2}{r} - ff'\right)\nabla_a u + 2\nabla_a(\Delta u \cdot r) + 4\nabla_a\nabla_b u\nabla^b r \\ &\quad + 4\nabla^b u\nabla_a\nabla_b r - 2(n-1)\nabla_a\nabla_b r\nabla^b u. \end{aligned}$$

Note that the induced metric of Σ is conformal to the standard metric on S^{n-1}: $\sigma = r^2\tilde{\sigma}$.

We rewrite the equation in terms of $\tilde{\sigma}$:

$$0 = (n-1)\left(\frac{f^2}{r} - ff'\right)\tilde{\nabla}_a u + \tilde{\nabla}_a\left(\frac{\tilde{\Delta}u}{r}\right) + (n-1)\tilde{\nabla}_a\tilde{\nabla}_b u\frac{\tilde{\nabla}^b r}{r^2}$$

Following the suggestion of Po-Ning Chen [9], we multiply the equation by $r^{n-1}\tilde{\nabla}^a u$ and integrate over Σ to get

$$\begin{aligned}
0 = &\int_\Sigma \Big[(n-1)r^{n-2}(f^2 - rff')|\tilde{\nabla}u|^2 + r^{n-2}\tilde{\nabla}_a\tilde{\Delta}u\tilde{\nabla}^a u - \frac{1}{n-2}\tilde{\Delta}u\tilde{\nabla}_a(r^{n-2})\tilde{\nabla}^a u \\
&+ \frac{n-1}{n-2}\tilde{\nabla}_a\tilde{\nabla}_b u\tilde{\nabla}^b(r^{n-2})\tilde{\nabla}^a u\Big]\sqrt{\tilde{\sigma}}dx \\
= &\int_\Sigma \left[(n-1)r^{n-2}\left(f^2 - 1 - rff'\right)|\tilde{\nabla}u|^2 - \frac{n-1}{n-2}r^{n-2}\left(|\tilde{\nabla}^2 u|^2 - \frac{1}{n-1}(\tilde{\Delta}u)^2\right)\right]\sqrt{\tilde{\sigma}}dx.
\end{aligned}$$

Note that $|\tilde{\nabla}^2 u|^2 - \frac{1}{n-1}(\tilde{\Delta}u)^2 = \left|\tilde{\nabla}^2 u - \frac{1}{n-1}(\tilde{\Delta}u)\tilde{\sigma}\right|^2$. Hence $u = \textit{constant}$ is the only solution unless $(V, \bar{g})$ has constant sectional curvature and $u = a + b\tilde{x}$ where $\tilde{x}$ is some first eigenfunction on $(S^{n-1}, \tilde{\sigma})$. □

5.2 A Generalization of Liouville Theorem and Obata Theorem

We first review the Liouville theorem in conformal geometry. The stereographic projection identifies $S^2 \subset \mathbb{R}^3$ and $\mathbb{C}$

$$z = \frac{x_1 + ix_2}{1 - x_3}.$$

Fractional linear transformations

$$w = \frac{az+b}{cz+d}, ad - bc = 1$$

on $\mathbb{C}$ provide a family of conformal transformations on S^2. Liouville theorem says that all conformal transformations on S^2 arise in this way.

Under a conformal transformation, the canonical metric on S^2 is changed by a conformal factor

$$\frac{1}{(1+|w|^2)^2}|dw|^2 = \frac{1}{(|cz+d|^2+|az+b|^2)^2}|dz|^2 = r^2\frac{1}{(1+|z|^2)^2}|dz|^2,$$

and the conformal factor satisfies

$$\begin{aligned}\frac{1}{r} &= \frac{|cz+d|^2+|az+b|^2}{1+|z|^2} \\ &= \frac{(|a|^2+|c|^2)|z|^2+|b|^2+|d|^2+Re\left(z(a\bar{b}+c\bar{d})\right)}{1+|z|^2} \\ &= (|a|^2+|b|^2+|c|^2+|d|^2)+Re(a\bar{b}+c\bar{d})x_1+Im(a\bar{b}+c\bar{d})x_2+(|a^2|+|c|^2-|b|^2-|d|^2)x_3.\end{aligned}$$

On the other hand, we have equation of constant Gauss curvature equation on S^2

$$\tilde{\Delta}w + Ee^{2w} = 1 \tag{5.12}$$

where E is a positive constant and $\tilde{}$ denotes the operator with respect to the standard metric on S^2. Let $r = e^w$ and (5.12) becomes

$$1 - \tilde{\Delta}\ln r = Er^2. \tag{5.13}$$

From the above discussion Liouville theorem is equivalent to saying all solutions of (5.13) are of the form $\frac{a}{1-\varphi}$ where a is some constant and φ is some first eigenfunction of $\tilde{\Delta}$.

We now present an analytical proof of this fact. Let $u = \frac{1}{r}$. The equation becomes

$$u^2 + u\tilde{\Delta}u - |\tilde{\nabla}u|^2 = E. \tag{5.14}$$

(5.14) is equivalent to

$$\tilde{\Delta}\left(u^2 + u\tilde{\Delta}u - |\tilde{\nabla}u|^2\right) = 0.$$

By the Bochner formula, we have

$$2u\tilde{\Delta}u + (\tilde{\Delta}u)^2 + u\tilde{\Delta}^2 u - 2|\tilde{\nabla}^2 u|^2 = 0,$$

which is equivalent to

$$u \cdot \tilde{\Delta}(\tilde{\Delta} + 2)u - (u_{11} - u_{22})^2 = 0.$$

The maximum principle implies that $(\tilde{\Delta} + 2)u$ doesn't have a local maximum unless it is a constant function.

The constant Gauss curvature equation has an interpretation in the geometry of Minkowski spacetime. Let $\Sigma \subset \mathbb{R}^{3,1}$ be a spacelike topological 2-sphere in the outgoing standard null cone. Suppose Σ is given by the embedding $F : S^2 \to \mathbb{R}^{3,1}$, $F(\theta, \phi) = (r(\theta, \phi), r(\theta, \phi), \theta, \phi)$. Then the induced metric of Σ is $r^2\tilde{\sigma}$ and the norm of the mean curvature vector of Σ is given by $|\vec{H}|^2 = \frac{1}{r^2}\left(1 - \tilde{\Delta}\ln r\right)$. Hence having constant Gauss curvature is the same as having constant mean curvature vector norm for surfaces in the standard null cone. Liouville theorem says those surfaces can only arise as the intersection of the standard null cone and hyperplanes.

We could generalize the Liouville theorem to Schwarzschild spacetime.

Theorem 5.7. *Let Σ be a spacelike topological 2-sphere in the standard null cone in Schwarzschild spacetime. Suppose the mean curvature vector of Σ has constant norm. Then Σ is a sphere of symmetry.*

Proof. The equation becomes

$$(1-\frac{2m}{r})-\tilde{\Delta}\ln r=Er^2 \tag{5.15}$$

The same argument applying to $u=\frac{1}{r}$ leads to

$$u\cdot\tilde{\Delta}(\tilde{\Delta}+2-3mu)u-6mu^2|\tilde{\nabla}u|^2-(u_{11}-u_{22})^2=0. \tag{5.16}$$

Maximum principle implies that u is a constant function. □

Po-Ning Chen [10] generalizes the above theorem to $(n+1)$-dimensional static spherically symmetric spacetimes.

Theorem 5.8. *Let Σ be a spacelike hypersurface in the standard null cone of a static spherically symmetric spacetime $(V,\bar{g})$ satisfying Assumption 4.4 and null convergence condition. Suppose the mean curvature vector of Σ has constant norm. Then Σ is a sphere of symmetry unless $(V,\bar{g})$ has constant sectional curvature and Σ is the intersection of a totally geodesic slice and standard null cone.*

Proof. With our choice of L and $\underline{L}$, $\mathrm{tr}\chi=n-1$. Since $|\vec{H}|^2=\mathrm{tr}\chi\mathrm{tr}\underline{\chi}$, the equation we want to investigate is

$$E=\frac{1}{r^2}\Big((n-1)(f^2+|\nabla r|^2)-2r\Delta r\Big) \tag{5.17}$$

for some constant E. As in the proof of infinitesimal CNNC rigidity, we express the equation with respect to the standard metric $\tilde{\sigma}$ on S^{n-1} and let $u=\frac{1}{r}$. We obtain

$$\begin{aligned}E&=\frac{1}{r^2}\Big((n-1)\left(f^2+\frac{|\tilde{\nabla}r|^2}{r^2}\right)-2r\left(\frac{\tilde{\Delta}r}{r^2}+(n-3)\frac{|\tilde{\nabla}r|^2}{r^3}\right)\Big)\\&=u^2\left((n-1)f^2-(n-1)\frac{|\tilde{\nabla}u|^2}{u^2}+\frac{2}{u}\tilde{\Delta}u\right).\end{aligned}$$

Taking $\tilde{\Delta}$ and using the Bochner formulas we get

$$0 = \tilde{\Delta}\big((n-1)f^2u^2\big) - 2(n-1)\left(|\tilde{\nabla}^2 u|^2 - \frac{1}{n-1}(\tilde{\Delta}u)^2\right) - (2n-6)\tilde{\nabla}\tilde{\Delta}u \cdot \tilde{\nabla}u$$
$$- 2(n-1)(n-2)|\tilde{\nabla}u|^2 + 2u\tilde{\Delta}^2 u.$$

We multiply u^{2-n} and integrate by parts over Σ to get

$$\begin{aligned} 0 = &\int_\Sigma \Bigg[\frac{(n-1)(n-2)}{u^{n-1}}\tilde{\nabla}_a\big(f^2u^2\big)\tilde{\nabla}^a u - \frac{2(n-1)}{u^{n-2}}\left(|\tilde{\nabla}^2 u|^2 - \frac{1}{n-1}(\tilde{\Delta}u)^2\right) \\ &- \frac{2(n-1)(n-2)}{u^{n-2}}|\tilde{\nabla}u|^2\Bigg]\sqrt{\tilde{\sigma}}dx \\ = &\int_\Sigma \Bigg[\frac{(n-1)(n-2)}{u^{n-1}}\left(-(f^2)' + \frac{2}{r}(f^2-1)\right)|\tilde{\nabla}u|^2 \\ &- \frac{2(n-1)}{u^{n-2}}\left|\tilde{\nabla}^2 u - \frac{1}{n-1}(\tilde{\Delta}u)\tilde{\sigma}\right|^2\Bigg]\sqrt{\tilde{\sigma}}dx \end{aligned}$$

Hence $u = \textit{constant}$ is the only solution unless $\frac{f^2-1}{r^2} - \frac{ff'}{r} = 0$ and $u = a + b\tilde{x}$ where $\tilde{x}$ is some first eigenfunction on $(S^{n-1}, \tilde{\sigma})$. □

We observe that the argument above gives a new proof of the Obata Theorem [23].

Theorem 5.9. *Suppose* (Σ^n, σ) *is a closed Einstein manifold with dimension* $n \geq 3$. *Let* $\bar{\sigma} = r^2\sigma$ *be a conformal metric with constant scalar curvature, where* r *is a positive smooth function. Then* r *must be constant unless* (Σ, σ) *is isometric to the standard sphere* (S^n, σ_c) *and*

$$r(x) = (c_1 + c_2 x \cdot a)^{-1}$$

for some constants c_1, c_2 *and point* $a \in S^n$.

Proof. Suppose $Ric(\sigma) = c\sigma$. Let $u = \frac{1}{r}$. The scalar curvature under conformal transformation satisfies [30, (D.9), page 446]

$$\bar{R} = u^2 \left(nc + 2(n-1)\left(\frac{\Delta u}{u} - \frac{|\nabla u|^2}{u^2} \right) - (n-1)(n-2)\frac{|\nabla u|^2}{u^2} \right).$$

Simplify the formula and we get

$$\frac{\bar{R}}{n-1} = \frac{nc}{n-1}u^2 + 2u\Delta u - n|\nabla u|^2.$$

By the assumption, $\bar{R} = constant$. Taking Laplacian to both sides and using the Bochner formula, we get

$$\begin{aligned} 0 &= \frac{nc}{n-1}(u\Delta u + |\nabla u|^2) + (\Delta u)^2 + 2\nabla u \cdot \nabla \Delta u + u\Delta^2 u - n(|\nabla^2 u|^2 + \nabla u \cdot \nabla \Delta u + c|\nabla u|^2) \\ &= -n(|\nabla^2 u|^2 - \frac{1}{n}(\Delta u)^2) + (2-n)\nabla u \cdot \nabla \Delta u + u\Delta^2 u + \frac{nc}{n-1}(u\Delta u + (2-n)|\nabla u|^2). \end{aligned}$$

Mutiplying by u^{1-n} and integrating by parts, we obtain

$$0 = \int_\Sigma -\frac{n}{u^{n-1}} \left| \nabla^2 u - \frac{1}{n}(\Delta u)\sigma \right|^2.$$

This completes the proof. □

5.3 Codimension-2 Submanifolds with Vanishing Connection 1-From

In [11], we show that a spacelike 2-surface with $\alpha_H = 0$ in 4-dimensional Minkowski spacetime must lie on a totally geodesic slice. Using the integral formula, we generalize the result to all dimension with additional starshaped assumption.

Theorem 5.10. *Let $\Sigma^{n-1} \subset \mathbb{R}^{n,1}$ be a spacelike codimension-2 submanifold with spacelike mean curvature vector. If $\alpha_H = 0$ and $Q(e_n^H, e_{n+1}^H) > 0$ on Σ, then Σ lies in a totally geodesic* *slice.*

Proof. We write e_n, e_{n+1} for e_n^H, e_{n+1}^H. Consider the divergence quantity

$$\nabla_a \left[\left(\sigma_1(h_{n+1})\sigma^{ab} - {h_{n+1}}^{ab} \right) Q_{nb} \right]$$

By the assumption $\alpha_H = 0$ and the Codazzi equation,

$$\nabla_a \left(\sigma_1(h_{n+1})\sigma^{ab} - {h_{n+1}}^{ab} \right) = 0.$$

By the conformal Killing-Yano equation,

$$\begin{aligned} \nabla_a(Q(e_n, \partial_b)) &= -\left(D_a Q(\partial_b, e_n) + h_{n+1,ab} Q_{n+1,n} + Q_{bc} {h_{na}}^c \right) \\ &= \sigma_{ab} \left\langle \frac{\partial}{\partial t}, e_n \right\rangle - h_{n+1,ab} Q_{n+1,n} - Q_{bc} {h_{na}}^c \end{aligned}$$

We have

$$Q_{bc} {h_{na}}^c {h_{n+1}}^{ab} = \frac{1}{2} Q_{bc} \left({h_{na}}^c {h_{n+1}}^{ab} - {h_{na}}^b {h_{n+1}}^{ac} \right) = 0$$

by the assumption $\alpha_H = 0$ and Ricci equation. Combining these facts together with $\sigma_1(h_{n+1}) = 0$, we get

$$\begin{aligned} 0 &= \int_\Sigma \nabla_a \left[\left(\sigma_1(h_{n+1})\sigma^{ab} - {h_{n+1}}^{ab} \right) Q_{nb} \right] d\mu \\ &= \int_\Sigma |h_{n+1}|^2 Q_{n+1,n} d\mu. \end{aligned}$$

Hence $h_{n+1} = 0$. The assertion follows from [33, Theorem 1, page 351]. $\square$

The result holds for the static spherically symmetric spacetimes satisfying the null convergence condition.

Theorem 5.11. *Let Σ^{n-1} be a spacelike codimension-2 submanifold with spacelike mean cur-*

vature vector in a static spherically symmetric spacetime satisfying Assumption 4.4 and null convergence condition. If $\alpha_H = 0$ and $Q(e_n^H, e_{n+1}^H) > 0$ on Σ, then the second fundamental form in e_{n+1}^H vanishes.

Proof. We write e_n, e_{n+1} for e_n^H, e_{n+1}^H. Consider the divergence quantity

$$\nabla_a \left[\left(\sigma_1(h_{n+1})\sigma^{ab} - {h_{n+1}}^{ab} \right) Q_{nb} \right]$$

By the assumption $\alpha_H = 0$, the Codazzi equation and the conformal Killing-Yano equation (3.1), we get

$$\nabla_a \left(\sigma_1(h_{n+1})\sigma^{ab} - {h_{n+1}}^{ab} \right) = -\bar{R}^{ab}{}_{a,n+1},$$

and

$$\nabla_a(Q(e_n, \partial_b)) = \sigma_{ab} \left\langle \frac{\partial}{\partial t}, e_n \right\rangle - h_{n+1,ab} Q_{n+1,n} - Q_{bc} {h_{na}}^c.$$

By the assumption $\alpha_H = 0$ and Ricci equation, we have

$$Q_{bc} {h_{na}}^c {h_{n+1}}^{ab} = \frac{1}{2} Q_{ba} \bar{R}^{ab}{}_{n+1,n}.$$

Using the curvature formula (B.2), we have

$$\begin{aligned}
\bar{R}^{ab}{}_{a,n+1} &= \frac{1}{r^2} \left((ff')' - \frac{2ff'}{r} - \frac{1-f^2}{r^2} \right) Q^{ab} Q_{a,n+1} \\
&\quad - \frac{1}{r^2}(n-2) \left(\frac{ff'}{r} + \frac{1-f^2}{r^2} \right) \left(-Q^{ab}_{a,n+1} + Q^b{}_n Q_{n,n+1} \right) \\
\bar{R}^{ab}{}_{n+1,n} &= \frac{1}{r^2} \left((ff')' - \frac{2ff'}{r} - \frac{1-f^2}{r^2} \right) \left(\frac{2}{3} Q^{ab} Q_{n+1,n} - \frac{1}{3} Q^a{}_{n+1} Q_n{}^b - \frac{1}{3} Q^a{}_n Q^b{}_{n+1} \right),
\end{aligned}$$

and

$$
\begin{aligned}
&-\bar{R}^{ab}{}_{a,n+1}Q_{nb}+\frac{1}{2}\bar{R}^{ab}{}_{n+1,n}Q_{ba}\\
&=\frac{1}{r^2}\Bigg\{\left(\frac{2}{3}Q^{ab}Q_{a,n+1}Q_{bn}+\frac{1}{3}Q^{ab}Q_{ab}Q_{n,n+1}\right)\left((ff')'-\frac{2ff'}{r}-\frac{1-f^2}{r^2}\right)\\
&\quad-(n-2)\left(\frac{ff'}{r}+\frac{1-f^2}{r^2}\right)\left(-Q^{ab}Q_{a,n+1}Q_{bn}+Q^{b}{}_{n}Q_{bn}Q_{n,n+1}\right)\Bigg\}\\
&=-\frac{n-2}{r^2}\left(\frac{ff'}{r}+\frac{1-f^2}{r^2}\right)\left(\frac{1}{2}Q^{ab}Q_{ab}Q_{n,n+1}+Q^{b}{}_{n}Q_{bn}Q_{n,n+1}\right)
\end{aligned}
$$

where we use (B.3) in the last equality. Combining these facts together, we obtain

$$
\begin{aligned}
0&=\int_\Sigma \nabla_a\left[\left(\sigma_1(h_{n+1})\sigma^{ab}-h_{n+1}{}^{ab}\right)Q_{nb}\right]d\mu\\
&=-\int_\Sigma\left(|h_{n+1}|^2+\frac{n-2}{r^2}\left(\frac{ff'}{r}+\frac{1-f^2}{r^2}\right)\left(\frac{1}{2}Q^{ab}Q_{ab}+Q^{b}{}_{n}Q_{bn}\right)\right)Q_{n,n+1}d\mu.
\end{aligned}
$$

From the remark after Lemma 5.1, null convergence condition implies that $\frac{ff'}{r}+\frac{1-f^2}{r^2}\geq 0$. Hence $h_{n+1}=0$. This completes the proof. $\square$

We note that when Σ lies in a standard null cone, $\alpha_H=0$ is equivalent to $|\vec{H}|=\mathit{constant}$. Hence Σ lies in a time slice unless $(V,\bar{g})$ has constant sectional curvature and Σ lies in a totally geodesic slice by Theorem 5.8.

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Appendices

Appendix A

The Existence of Conformal Killing-Yano Forms

In this appendix, we show the existence of conformal Killing-Yano form for a class of warped product manifold. We have the following equivalent definition of conformal Killing-Yano p-forms using the twistor equation [28, Definition 2.1].

Definition A.1. A $p-$form Q on an $n-$dimensional pseudo-Riemannian manifold (V, g) is said to be a conformal Killing-Yano form if Q satisfies the twistor equation

$$D_X Q - \frac{1}{n+1} X \lrcorner dQ + \frac{1}{n-p+1} g(X) \wedge d^* Q = 0 \tag{A.1}$$

for all tangent vector X.

The main result of the appendix is the following existence theorem.

Theorem A.2. *Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be two open sets. Let G be a warp-product metric on $U \times V$ of the form*

$$R^2(y)\sigma_{ab}(x)dx^a dx^b + g_{ij}(y)dy^i dy^j.$$

Then $Q = R^{n+1}(y)\sqrt{\det \sigma_{ab}}dx^1 \wedge \cdots \wedge dx^n$ and $*Q = R(y)\sqrt{\det g_{ij}}dy^1 \wedge \cdots \wedge dy^m$ are both conformal Killing-Yano forms.

Proof. By [28, Lemma 2.3], the Hodge star-operator $*$ maps conformal Killing-Yano $p-$form into conformal Killing-Yano $(n+m-p)-$form. It suffices to verify that Q satisfies the twistor equation. Let $\omega^\alpha, \alpha = 1, \ldots, n+m$ be a local orthonormal coframe for G such that $\omega^1, \ldots, \omega^n$ is an orthonormal coframe for $R^2(y)\sigma_{ab}(x)dx^a dx^b$ on each slice $U \times \{c_1\}$ and $\omega^{n+1}, \ldots, \omega^{n+m}$ is an orthonormal coframe for $g_{ij}(y)dy^i \wedge dy^j$ on each slice $\{c_2\} \times V$. Let E_α be the dual frame to ω^α. If we write $\Omega = \omega^1 \wedge \cdots \wedge \omega^n$, then $Q = R\Omega$. From the structure equations

$$\begin{aligned} d\omega^a &= -\omega^a{}_b \wedge \omega^b - \omega^a{}_{n+i} \wedge \omega^{n+i} = dR \wedge \sigma^a - R\gamma^a{}_b \wedge \sigma^b \\ d\omega^{n+i} &= -\omega^{n+i}{}_b \wedge \omega^b - \omega^{n+i}{}_{n+j} \wedge \omega^{n+j}, \end{aligned}$$

we solve for the connection 1-forms

$$\omega^a{}_{n+i} = \frac{E_{n+i}(R)}{R}\omega^a, \quad \omega^a{}_b = \gamma^a{}_b$$

where $\gamma^a{}_b$ are the connection 1-forms with respect to the metric $\sigma_{ab}(x)dx^a dx^b$.

We compute each term in the twistor equation.

$$D_X Q = X(R)\Omega + R\nabla_X\Omega = X(R)\Omega - \sum_{i=1}^{m} E_{n+i}(R)\omega^{n+i} \wedge (X \lrcorner \Omega).$$

$$\begin{aligned}
X \lrcorner d\Omega &= X \lrcorner \sum_{i=1}^{m} \left(-\omega^1{}_{n+i} \wedge \omega^{n+i} \wedge \omega^2 \wedge \cdots \wedge \omega^n + \omega^1 \wedge \omega^2{}_i \wedge \omega^{n+i} \wedge \cdots \wedge \omega^n - \cdots\right) \\
&= X \lrcorner \sum_{i=1}^{m} n \left(\frac{E_{n+i}(R)}{R}\omega^{n+i} \wedge \Omega\right) \\
&= n \sum_{i=1}^{m} \frac{E_{n+i}(R)}{R} X \lrcorner (\omega^{n+i} \wedge \Omega) \\
&= n \sum_{i=1}^{m} \frac{E_{n+i}(R)}{R} \left(\omega^{n+i}(X)\Omega - \omega^{n+i} \wedge (X \lrcorner \Omega)\right)
\end{aligned}$$

This implies that

$$\begin{aligned}
X \lrcorner dQ &= X \lrcorner (dR \wedge \Omega + R d\Omega) \\
&= X(R)\Omega - dR \wedge (X \lrcorner \Omega) + n \sum_{i=1}^{m} E_{n+i}(R) \left(\omega^{n+i}(X)\Omega - \omega^{n+i} \wedge (X \lrcorner \Omega)\right).
\end{aligned}$$

On the other hand, $d^*Q = 0$ since R only depends on y. Putting these facts together, we verify that Q satisfies the twistor equation

$$\begin{aligned}
&D_X Q - \frac{1}{n+1} X \lrcorner dQ + \frac{1}{m+1} g(X) \wedge d^*Q \\
&= X(R)\Omega - \sum_{i=1}^{m} E_{n+i}(R)\omega^{n+i} \wedge (X \lrcorner \Omega) \\
&\quad - \frac{1}{n+1}\Big(X(R)\Omega - dR \wedge (X \lrcorner \Omega) - n \sum_{i=1}^{m} E_{n+i}(R) \left(\omega^{n+i}(X)\Omega - \omega^{n+i} X \lrcorner \Omega\right)\Big) \\
&= \frac{n}{n+1} X(R)\Omega - \frac{1}{n+1} E_{n+i}(R)\omega^{n+i} \wedge (X \lrcorner \Omega) \\
&\quad + \frac{1}{n+1} dR \wedge (X \lrcorner \Omega) - \frac{n}{n+1} E_{n+i}(R)\omega^{n+i}(X)\Omega \\
&= 0.
\end{aligned}$$

We use the fact that R only depends on y in the last equality. □

We have the following existence result, generalizing the fact that $rdr \wedge dt$ is a conformal Killing-Yano 2-form on the Minkowski and Schwarzschild spacetime.

Corollary A.3. *Let (V, g) be a warped product manifold with*

$$g = g_{tt}(t,r)dt^2 + 2g_{tr}(t,r)dtdr + g_{rr}(t,r)dr^2 + r^2(g_N)_{ab}dx^a dx^b \tag{A.2}$$

where (N, g_N) is an $(n-1)-$dimensional Riemannian manifold. Then the two-form

$$Q = r\sqrt{\left|\det\begin{pmatrix} g_{tt} & g_{tr} \\ g_{rt} & g_{rr} \end{pmatrix}\right|}\; dr \wedge dt$$

is a conformal Killing-Yano 2-form on (V, g).

Appendix B

Curvature tensors in terms of Conformal Killing-Yano Tensor

We consider $(n+1)$-dimensional (exterior) Schwarzschild spacetime. The metric is given by

$$\bar{g} = -\left(1 - \frac{2m}{r^{n-2}}\right) dt^2 + \frac{1}{1 - \frac{2m}{r^{n-2}}} dr^2 + r^2 g_{S^{n-1}}$$

The spacetime admits a conformal Killing-Yano tensor

$$Q = r dr \wedge dt$$

Let Q^2 be the symmetric 2-tensor given by

$$(Q^2)_{\alpha\beta} = Q_\alpha{}^\gamma Q_{\gamma\beta}$$

Lemma B.1. *The curvature tensor of Schwarzschild spacetime can be expressed as*

$$\begin{aligned}\bar{R}_{\alpha\beta\gamma\delta} = &\frac{2m}{r^n}\left(\bar{g}_{\alpha\gamma}\bar{g}_{\beta\delta} - \bar{g}_{\alpha\delta}\bar{g}_{\beta\gamma}\right) - \frac{n(n-1)m}{r^{n+2}}\left(\frac{2}{3}Q_{\alpha\beta}Q_{\gamma\delta} - \frac{1}{3}Q_{\alpha\gamma}Q_{\delta\beta} - \frac{1}{3}Q_{\alpha\delta}Q_{\beta\gamma}\right) \\ &- \frac{nm}{r^{n+2}}\left(\bar{g} \circ Q^2\right)_{\alpha\beta\gamma\delta}\end{aligned} \tag{B.1}$$

where $(\bar{g} \circ Q^2)_{\alpha\beta\gamma\delta} = \bar{g}_{\alpha\gamma}(Q^2)_{\beta\delta} - \bar{g}_{\alpha\delta}(Q^2)_{\beta\gamma} + \bar{g}_{\beta\delta}(Q^2)_{\alpha\gamma} - \bar{g}_{\beta\gamma}(Q^2)_{\alpha\delta}$

Proof. Denote $f^2 = 1 - \frac{2m}{r^{n-2}}$. Let $E_1, E_2, \ldots, E_{n+1}$ be the orthonormal frames for $\bar{g}$ with $E_{n+1} = \frac{1}{f}\frac{\partial}{\partial t}, E_n = f\frac{\partial}{\partial r}$ and $E_i, i = 1, \ldots, n-1$ tangent to the sphere of symmetry. We have

$$\begin{aligned}\bar{R}(E_{n+1}, E_n, E_{n+1}, E_n) &= -\frac{m(n-1)(n-2)}{r^n} \\ \bar{R}(E_{n+1}, E_i, E_{n+1}, E_j) &= \frac{m(n-2)}{r^n}\delta_{ij} \\ \bar{R}(E_n, E_i, E_n, E_j) &= -\frac{m(n-2)}{r^n}\delta_{ij} \\ \bar{R}(E_i, E_j, E_k, E_l) &= \frac{2m}{r^n}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})\end{aligned}$$

Except for the symmetries of the curvature tensors, the other components are zero.

On the other hand, we have $Q(E_n, E_{n+1}) = r, (Q^2)(E_{n+1}, E_{n+1}) = -r^2$, and $(Q^2)(E_n, E_n) = r^2$. Let $b(Q) = \frac{2}{3}Q_{\alpha\beta}Q_{\gamma\delta} - \frac{1}{3}Q_{\alpha\gamma}Q_{\delta\beta} - \frac{1}{3}Q_{\alpha\delta}Q_{\beta\gamma}$. The following table lists the nonzero components for the $(0,4)$-tensors involved.

T	$\bar{g}_{\alpha\gamma}\bar{g}_{\beta\delta} - \bar{g}_{\alpha\delta}\bar{g}_{\beta\gamma}$	$b(Q)$	$(\bar{g} \circ Q^2)_{\alpha\beta\gamma\delta}$
$T(E_{n+1}, E_n, E_{n+1}, E_n)$	-1	r^2	$-2r^2$
$T(E_{n+1}, E_i, E_{n+1}, E_j)$	$-\delta_{ij}$	0	$-r^2\delta_{ij}$
$T(E_n, E_i, E_n, E_j)$	δ_{ij}	0	$r^2\delta_{ij}$
$T(E_i, E_j, E_k, E_l)$	$\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$	0	0

Suppose $\bar{R}_{\alpha\beta\gamma\delta} = A\frac{m}{r^n}\left(\bar{g}_{\alpha\gamma}\bar{g}_{\beta\delta} - \bar{g}_{\alpha\delta}\bar{g}_{\beta\gamma}\right) + B\frac{m}{r^{n+2}}\left(\frac{2}{3}Q_{\alpha\beta}Q_{\gamma\delta} - \frac{1}{3}Q_{\alpha\gamma}Q_{\delta\beta} - \frac{1}{3}Q_{\alpha\delta}Q_{\beta\gamma}\right) + C\frac{m}{r^{n+2}}(\bar{g} \circ Q^2)_{\alpha\beta\gamma\delta}$. We can solve for $A = 2, B = -n(n-1)$, and $C = -n$. □

The same proof applies to any $(n+1)$-dimensional static spherically symmetric spacetime $(V, \bar{g})$ with the metric given by

$$\bar{g} = -f^2dt^2 + \frac{1}{f^2}dr^2 + r^2g_{S^{n-1}}$$

The conformal Killing-Yano tensor has the same form

$$Q = rdr \wedge dt$$

As above, let $E_1, E_2, \ldots, E_{n+1}$ be the orthonormal frames for $\bar{g}$ with $E_{n+1} = \frac{1}{f}\frac{\partial}{\partial t}, E_n = f\frac{\partial}{\partial r}$ and $E_i, i = 1, \ldots, n-1$ tangent to the sphere of symmetry. Denote $\frac{df}{dr}$ by f'. We compute

$$\begin{aligned}
\bar{R}(E_{n+1}, E_n, E_{n+1}, E_n) &= (ff')' \\
\bar{R}(E_{n+1}, E_i, E_{n+1}, E_j) &= \frac{ff'}{r}\delta_{ij} \\
\bar{R}(E_n, E_i, E_n, E_j) &= -\frac{ff'}{r}\delta_{ij} \\
\bar{R}(E_i, E_j, E_k, E_l) &= \frac{1-f^2}{r^2}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})
\end{aligned}$$

Comparing with the table we obtain

Lemma B.2. *The curvature tensor of $(V, \bar{g})$ can be expressed as*

$$\begin{aligned}
\bar{R}_{\alpha\beta\gamma\delta} = \frac{1}{r^2}\Bigg(& (1-f^2)\left(\bar{g}_{\alpha\gamma}\bar{g}_{\beta\delta} - \bar{g}_{\alpha\delta}\bar{g}_{\beta\gamma}\right) \\
& + \left((ff')' - \frac{2ff'}{r} - \frac{1-f^2}{r^2}\right)\left(\frac{2}{3}Q_{\alpha\beta}Q_{\gamma\delta} - \frac{1}{3}Q_{\alpha\gamma}Q_{\delta\beta} - \frac{1}{3}Q_{\alpha\delta}Q_{\beta\gamma}\right) \\
& - \left(\frac{ff'}{r} + \frac{1-f^2}{r^2}\right)\left(\bar{g} \circ Q^2\right)_{\alpha\beta\gamma\delta}\Bigg)
\end{aligned} \tag{B.2}$$

The following lemma is useful in simplifying $\bar{R}^{ab}{}_{a,n+1}$ and $\bar{R}^{ab}{}_{n,n+1}$.

Lemma B.3.

$$Q^{ab}Q_{a,n+1}Q_{bn} = -\frac{1}{2}Q^{ab}Q_{ab}Q_{n,n+1} \tag{B.3}$$

Proof. Let $E_1, E_2, \ldots, E_n = f\frac{\partial}{\partial r}, E_{n+1} = \frac{1}{f}\frac{\partial}{\partial t}$ be the standard frame of $(V, \bar{g})$ where $E_1, \ldots, E_{n-1}$ are tangent to the sphere of symmetry. Let $\omega^1, \ldots, \omega^n$ be its dual coframe. Let

$$e_\alpha = \sum_{\beta=1}^{n+1} \Lambda_{\alpha\beta} E_\beta.$$

Since $Q = r\omega^n \wedge \omega^{n+1}$ only sees the component in E_n and E_{n+1} directions, we have

$$\begin{aligned} Q^{ab}Q_{a,n+1}Q_{bn} &= r^3 Q_{ab}(\Lambda_{an}\Lambda_{n+1,n+1} - \Lambda_{a,n+}\Lambda_{n+1,n})(\Lambda_{bn}\Lambda_{n,n+1} - \Lambda_{b,n+1}\Lambda_{nn}) \\ &= r^3 Q_{ab}(-\Lambda_{an}\Lambda_{b,n+1}\Lambda_{n+1,n+1}\Lambda_{nn} - \Lambda_{a,n+1}\Lambda_{bn}\Lambda_{n+1,n}\Lambda_{n,n+1}) \end{aligned}$$

where in the last equality we use the antisymmetry of a, b. On the other hand,

$$Q_{n,n+1} = r(\Lambda_{nn}\Lambda_{n+1,n+1} - \Lambda_{n,n+1}\Lambda_{n+1,n}), Q_{ab} = r(\Lambda_{an}\Lambda_{b,n+1} - \Lambda_{a,n+1}\Lambda_{bn})$$

Using the antisymmetry of a, b again, we get

$$\begin{aligned} Q^{ab}Q_{a,n+1}Q_{bn} &= \frac{1}{2}r^3 Q_{ab}\left(-\Lambda_{an}\Lambda_{b,n+1}\left(\frac{1}{r}Q_{n,n+1} + \Lambda_{n,n+1}\Lambda_{n+1,n}\right) - \Lambda_{a,n+1}\Lambda_{bn}\Lambda_{n+1,n}\Lambda_{n,n+1}\right) \\ &\quad + \frac{1}{2}r^3 Q_{ab}\left(-\Lambda_{an}\Lambda_{b,n+1}\Lambda_{n+1,n+1}\Lambda_{nn} + \Lambda_{a,n+1}\Lambda_{bn}\left(\frac{1}{r}Q_{n,n+1} - \Lambda_{nn}\Lambda_{n+1,n+1}\right)\right) \\ &= -\frac{r^2}{2}Q_{ab}\left(\Lambda_{an}\Lambda_{b,n+1} - \Lambda_{a,n+1}\Lambda_{bn}\right)Q_{n,n+1} \\ &= -\frac{1}{2}Q^{ab}Q_{ab}Q_{n,n+1} \end{aligned}$$

□