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Abstract

The "equivalence" problem for shape descriptions is that a single three-dimensional shape may have several different descriptions. The Slant Theorem (Shafer[1]) for equivalent generalized cylinder descriptions was proven under the restrictions that the same radius function and the same axis be used for all the descriptions. A proof is given that the theorem still holds when the "same radius function" condition is removed. It does not hold when the "same axis" condition is removed. The ellipsoid is a counter-example.

Introduction

The equivalence problem for shape descriptions is that a single three-dimensional shape may have several different, equivalent descriptions. One way to deal with this problem is to use a method of generating descriptions which guarantees that the description produced is always a unique, canonical representation. The other approach is to permit alternate descriptions, but be able to tell when two descriptions are equivalent, i.e. describe the same shape.

Shafer[1] investigated this second approach for a class of generalized cylinders. After eliminating the trivial equivalences due to rotation, etc., Shafer gave theorems about some families of equivalent descriptions.

The Slant Theorem

Following Shafer[1], a generalized cylinder is Straight if its axis (or spine) is a straight line segment. It is Homogeneous if all its cross-sections have the same shape except for scale. A Straight Homogeneous Generalized Cylinder (SHGC) is given by the four-tuple (A,C,r,α) (see Figure 1).

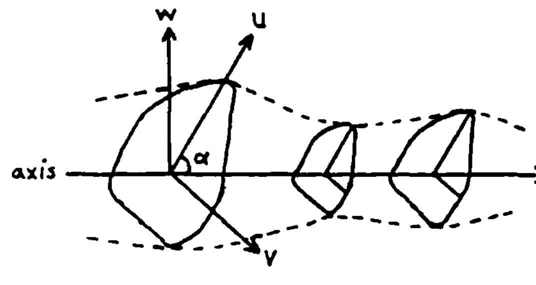

Figure 1: Straight Homogeneous Generalized Cylinder

A is a line segment in 3-space called the Axis or spine. It is parameterized in s, and an s-coordinate may be defined coinciding with the Axis. α is the (constant) angle of each cross-section plane to the Axis. C is the planar cross-section curve. Coordinates u and v may be defined for the cross-section plane, such that the u-coordinate coincides with the projection of the Axis onto the cross-section plane. C may then be parameterized in t: $C(t) = (u(t),v(t))$. r(s) is the radius function, which gives the scale of the cross-section at each point along the Axis (C gives the shape of the cross-section, r gives the scale). So in suv-space, each point on the surface is given in terms of parameters s and t by $(s, r(s)u(t), r(s)v(t))$. A mutually orthogonal set may be formed by replacing the u-coordinate with a w-coordinate perpendicular to the Axis and the v-coordinate. Then in swv-space, the point on the surface given by parameters s and t is $(s + r(s)u(t)\cos\alpha, r(s)u(t)\sin\alpha, r(s)v(t))$. In this paper, it is assumed that the Axis function A(s) is linear, and that the radius function r(s) and cross-section function C(t) are piecewise C^2.

An SHGC is a Right SHGC if its cross-section angle $\alpha = \pi/2$. Otherwise it is an Oblique SHGC. An SHGC is Linear if its radius function r(s) is linear. The Slant Theorem (Shafer[1], page 103 and Appendix E) states that:

> An Oblique Straight Homogeneous Generalized Cylinder (SHGC) has an equivalent Right SHGC if and only if the radius function of the Oblique SHGC is Linear. (Two otherwise equivalent descriptions which have differently sloped ends are regarded as equivalent for the purposes of this theorem).

The theorem was proven under the restricted conditions that the same radius function and same Axis be used for both the Oblique and the Right SHGCs. The question arises whether the theorem still holds when these conditions are relaxed.

The "same radius function" condition

The Slant Theorem still holds when the "same radius function" condition is removed. The "if" part of the theorem ("Linear radius function implies the existence of an equivalent Right SHGC") is already true from the restricted form of the theorem. So what must be proven is the following:

> Given an Oblique SHGC $G = (A,C,r,\alpha)$ where radius function r is non-linear, there does not exist any Right SHGC $G^* = (A,C^*,r^*,\pi/2)$ which has the same Axis as G (without restriction on the radius function r^* of G^*).

Proof: The basic idea is that at least one of the angled cross-sections of the Oblique SHGC will be on a non-linear bend in the radius function r(s). But the bend must be spread over a wider range of cross-sections in the Right SHGC, and there is no way for

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one radius function to consistently handle all of them.

Given an Oblique SHGC $G = (A,C,r,\alpha)$, the "zigzag" construction shall be defined as follows for a value of $s = s_2$ and values of $t = t_L$ and $t = t_M$ (see Figure 2). Call the point given by $s = s_2$ and $t = t_L$, L_2; similarly for M_2. Working in swv-space, the points may be displayed in a plot of w against s (see Figure 2). The coordinates of L_2 in swv-space are $(s_2 + r(s_2)u(t_L)\cos\alpha,\ r(s_2)u(t_L)\sin\alpha,\ r(s_2)v(t_L))$.

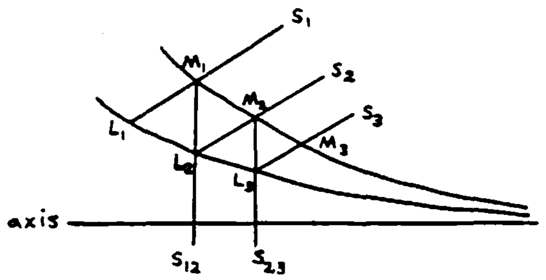

Figure 2: The zigzag construction

The set of points $(s + r(s)u(t_L)\cos\alpha,\ r(s)u(t_L)\sin\alpha,\ r(s)v(t_L))$ for all s, forms a curve in swv-space, call it "curve L" (likewise "curve M"). Take the plane in swv-space perpendicular to the Axis which contains L_2. Call the intersection of that plane with curve M, point M_1. Call the s-value for that point s_1. Now take the plane which is at an angle α to the Axis and contains M_1. Call the intersection of that plane with curve L, point L_1. Similarly, work in the other direction to define s_3, L_3, and M_3 (see Figure 2, which plots only the w and s coordinates).

(For some SHGCs and values of s and t, it may be that the intersection of the curve L and the plane in swv-space may include more than one point, or even a line (but not less than one point). In such cases, it is fairly easy to see that all the cross-sections perpendicular to the Axis cannot have the same shape, in which case no Right SHGC can be constructed which is homogeneous, and the theorem is satisfied. So in what follows it will be assumed that each point in the Oblique cross-section maps to exactly one point in the Right cross-section.)

Since r(s) is non-linear, there exists some value $s = s_2$, some $\epsilon > 0$, and some real value m, such that for all s in $(s_2 - \epsilon, s_2)$, the slope of r(s) is less (greater) than m, and for all s in $(s_2, s_2 + \epsilon)$, the slope of r(s) is greater (less) than m.

It is evident that by choosing t_L and t_M close enough together, L_2 and M_2 can be chosen with $u(t_L)$ and $u(t_M)$ close enough together so that the zigzag construction can be made with s_1 in $(s_2 - \epsilon, s_2)$, and s_3 in $(s_2, s_2 + \epsilon)$. Further, this can be done so that $u(t_L)$ and $u(t_M)$ are equal neither to each other nor to 0, because SHGC G is a closed figure. (As an example of a non-closed figure which does not satisfy this theorem, take as the cross-section a line segment parallel to the v-coordinate, and any reasonable non-linear radius function).

From the way in which this zigzag construction has been done, it is clear that the slopes of line L_1L_2 and line L_2L_3 are not equal (likewise for line M_1M_2 and line M_2M_3).

<u>Lemma</u>: The slopes of line L_1L_2 and line L_2L_3 are equal (likewise for line M_1M_2 and line M_2M_3) if and only if $r(s_1)/r(s_2) = r(s_2)/r(s_3)$.

<u>Proof</u>: Using (s,w) coordinates (and ignoring the v-coordinate), it can be seen from the way in which the zigzag construction is done that

$$L_1 = (s_1 + r(s_1)u(t_L)\cos\alpha,\ r(s_1)u(t_L)\sin\alpha).$$

$$M_1 = (s_1 + r(s_1)u(t_M)\cos\alpha,\ r(s_1)u(t_M)\sin\alpha).$$

$$L_2 = (s_2 + r(s_2)u(t_L)\cos\alpha,\ r(s_2)u(t_L)\sin\alpha).$$

But also

$$L_2 = (s_1 + r(s_1)u(t_M)\cos\alpha,\ r(s_2)u(t_L)\sin\alpha).$$

So the slope of line L_1L_2

$$= [r(s_1)u(t_L)\sin\alpha - r(s_2)u(t_L)\sin\alpha] \,/\, [(s_1 + r(s_1)u(t_L)\cos\alpha) - (s_1 + r(s_1)u(t_M)\cos\alpha)]$$

$$= \tan\alpha\,[u(t_L)/(u(t_L)-u(t_M))]\,[1 - (r(s_2)/r(s_1))]$$

Likewise it can be shown that the slope of line L_2L_3

$$= \tan\alpha\,[u(t_L)/(u(t_L)-u(t_M))]\,[1 - (r(s_3)/r(s_2))]$$

And the result follows (with the same argument for M_1, M_2, and M_3).

Now using the Lemma, we get the result that $r(s_1)/r(s_2) \neq r(s_2)/r(s_3)$.

But if G^* were a valid SHGC, with its radius and cross-section functions $r^*(s)$ and $C^*(t)$, the following would hold:

$$r(s_1)u(t_M)\sin\alpha \,/\, r(s_2)u(t_L)\sin\alpha$$

$$= r^*(s_{12})u^*(t_M^*) \,/\, r^*(s_{12})u^*(t_L^*)$$

$$= r^*(s_{23})u^*(t_M^*) \,/\, r^*(s_{23})u^*(t_L^*)$$

$$= r(s_2)u(t_M)\sin\alpha \,/\, r(s_3)u(t_L)\sin\alpha$$

where the middle equality is due to the "Homogeneous" part of "SHGC" (all cross-sections must have the same shape, up to scale). These equalities would imply that $r(s_1)/r(s_2) = r(s_2)/r(s_3)$. Since this has been shown to be false, no equivalent Right SHGC can exist.

<u>The "same axis" condition</u>

The theorem does not hold if the "same Axis" restriction is removed. If different axes are permitted, then there are non-Linear SHGCs that have different, equivalent SHGC descriptions. The sphere is a trivial counter-example that will not be considered, since its alternate descriptions differ only by rotation.

But there are non-trivial counter-examples: Consider a right ellipsoid with center at the origin in Cartesian 3-space. It can be represented in equation form as:

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$$

Thinking in terms of generalized cylinders, and taking the x-axis as the Axis, we have a Right Non-linear SHGC, with elliptical cross-sections.

Now suppose we slant the Axis by an angle α in the x-y plane, but leave the elliptical cross-sections parallel to the y-z plane (a kind of skew transformation). This "oblique" figure is clearly an Oblique Non-linear SHGC, again with elliptical cross-sections. This "slant" transformation can be carried out in the equation representation by replacing y with $y - x \tan \alpha$ and rearranging to get:

$$x^2(1/a^2 + \tan^2\alpha / b^2) - xy(2 \tan \alpha / b^2) + y^2/b^2 = 1 - z^2/c^2$$

Analytic geometry texts show that the left side is the equation of a family of ellipses that have been rotated in the x-y plane by an angle

$$\beta = (1/2) \arctan[2 \tan \alpha / (1 - b^2/a^2 - \tan^2\alpha)]$$

These ellipses are centered on the z-axis, and it is easy to show that their orientation and eccentricity is independent of the value of z. They all have the same shape. So this "oblique" figure may be represented as a Right Non-Linear SHGC, with Axis on the z-axis, and elliptical cross-sections.

This type of result is not limited to ellipsoids. But the ellipsoid has this additional property: the "oblique" figure is simply another right ellipsoid, rotated from the x-axis by the angle β given above. If the rotation by β is carried out on the equation representation, the result is:

$$(x^2/a^2)[\cos^2\beta + (a^2/b^2)(\tan^2\alpha\cos^2\beta - 2\tan\alpha\cos\beta\sin\beta + \sin^2\beta)]$$

$$+ (y^2/b^2)[\cos^2\beta + 2\tan\alpha\cos\beta\sin\beta + \tan^2\alpha\sin^2\beta + (b^2/a^2)\sin^2\beta]$$

$$+ (z^2/c^2)$$

$$= 1$$

The eccentricity is different from that of the original right ellipsoid, as we would expect.

So the "oblique" figure can also be represented as a Right non-Linear SHGC with the Axis in the x-y plane at angle β. Thus "being a right ellipsoid" is a non-Linear property of SHGCs which is invariant under skew transformations. To put it another way, there is no such thing as an oblique ellipsoid.

It is interesting that while the z-axis representation depends on being able to take advantage of the freedom to orient the Axis anywhere in three dimensions, the "angle β" representation also works as a counter-example to the two-dimensional analog of the Slant Theorem.

So there are some Non-linear Oblique SHGCs which are equivalent to Right SHGCs, and therefore the "only if" part of the Slant Theorem does not hold without the "same axis" condition.

Families of descriptions with different axes

Define an H-axis for a shape as a line which is the Axis for some SHGC description for that shape. An RH-axis is an Axis for some Right SHGC description. Shafer in his Pivot Theorem ([1], p. 105 and Appendix F) has described families of H-axes which all use the same cross-sections, which exist only for Linear SHGCs. Other classes of shapes that have multiple H-axes:

1. There is an H-axis lying in the x-y plane, and the equation representation for the shape can be written in the form

$$f(x,y) = g(z)$$

and f satisfies

$$f(kx,ky) = h(k)f(x,y) \text{ for some function } h.$$

Then the z-axis is an RH-axis. For example:

$$(x/a)^4 + (y/b)^4 + (z/c)^4 = 1$$

2. The cross-section of a Right SHGC itself has multiple H-axes in its plane. For example, a square has four H-axes, a regular pentagon five, and a circle infinity. Any radius function can be used. These also satisfy the property of the previous type above. Included here could be spheres, cylinders, right prisms with polygonal bases, tetrahedra, octahedra.

3. Various elongations and skews of the first two types. Included here would be oblique prisms with polygonal bases and irregular tetrahedrons. The ellipsoid can be seen as an elongated sphere.

REFERENCES

[1] S. A. Shafer, *Shadow Geometry and Occluding Contours of Generalized Cylinders.* PhD dissertation, Carnegie-Mellon University, May 1983.