Columbia University

Department of Economics
Discussion Paper Series

Ambiguity, Measurability and Multiple Priors

Massimiliano Amarante

Discussion Paper No.: 0203-23

Department of Economics
Columbia University
New York, NY 10027

August 2003

Ambiguity, Measurability and Multiple Priors

Massimiliano Amarante*
Columbia University†

Revised: August 2003

Abstract

The paper provides a notion of measurability which is suited for a class of Multiple Prior Models. Those characterized by nonatomic countably additive priors. Preferences generating such representations have been recently axiomatized in [12]. A notable feature of our definition of measurability is that an event is measurable if and only if it is unambiguous in the sense of Ghirardato, Maccheroni and Marinacci [8]. In addition, the paper contains a thorough description of the basic properties of the family of measurable/unambiguous sets, of the measure defined on those and of the dependence of the class of measurable sets on the set of priors. The latter is obtained by means of an application of Lyapunov's convexity theorem.

JEL Classification Number: D81

Keywords: ambiguous events, multiple priors, Lyapunov's convexity theorem

1 Introduction

The Knigthian distinction between Risk and Ambiguity has been the subject of an extensive literature. Among other things, this work has given rise to several definitions of ambiguous events. Notably, those given in Epstein and Zhang [7] and in Ghirardato, Maccheroni and Marinacci [8].While there is no agreement as to which is the "right" definition (see [7] and [9] for through discussions, and [1] for a comparison between the two), there is no doubt that ambiguous events cannot behave like measurable sets. On the contrary, unambiguous ones should certainly do so. This is immediate. The ambiguous nature of an event reveals

*I am grateful to Emel Filiz, Paolo Ghirardato, Fabio Maccheroni, Massimo Marinacci and Marco Scarsini for comments and suggestions.

†Department of Economics, Columbia University, 1019 IAB, 420 W 118th street, NY, NY 10027. Email: ma734@columbia.edu.

into a violation of the independence axiom and, hence, into the nonlinearity of the functional that evaluates the bets. By the same token, such a functional should appear linear when restricted to bets which involve only unambiguous events. As an example, the reader might think of the celebrated three-color urn experiment of Ellsberg [6]. There a decision maker is offered bets whose domain is an urn containing 90 balls. He is told that of those 30 are red (R), while the remaining are either blue (B) or green (G) in an unknown proportion. Ellsberg's decision makers prefer betting on red rather than blue ($R \succ B$) but, at the same time, they prefer betting on the union of blue and green rather that on the union red and green ($R \cup G \prec B \cup G$). In Ellsberg's experiment, we can think of having an algebra of events, the one generated by the collection $\{\emptyset, R, B, G\}$. Let $\mathcal{E}$ be the function that associates each element of the algebra with the amount of money that the decision maker is willing to pay to purchase the bet based on that event. It is easily seen that, for Ellsberg's decision makers, $\mathcal{E}$ satisfies the conditions to be an exterior measure. But, then, B and G cannot be $\mathcal{E}$-measurable sets.

In this paper, we focus on a class of preferences satisfying the α-maxmin criterion. For such a class, we provide a definition of measurable events which has the property that an event is measurable if and only if it is unambiguous in the sense of Ghirardato, Maccheroni and Marinacci [8]. Our definition of measurability is in the spirit of the classical definition (Carathéodory's), and contains that as a special case. We then study the basic properties of the family of measurable/unambiguous sets and of the measure defined on those. In addition, we characterize the dependence of the class of measurable sets on the set of priors by means of an application of Lyapunov's convexity theorem.

The class of preferences we focus on is identified by the properties that all the priors in the representation are (a) countably additive and (b) nonatomic. These preferences have been recently axiomatized in [12]. A number of reasons motivate these restrictions. It is well-known that countable additivity produces remarkable properties both from a mathematical viewpoint and a decision-theoretic one. For the latter, the reader is referred to [2] and [14]. As for the nonatomicity of the measures, this is a property that is often imposed in conventional subjective expected utility theory. It occurs, for instance, in Savage. Here, one of the virtues of the assumption is that, by allowing the use of Lyapunov convexity theorem, it leads to a sharp characterization of the class of measurable sets and, hence, of the class of unambiguous sets in the sense of [8].

The paper proceeds as follows. Section 2 is a brief description of the results contained in [12], which characterize the class of preferences we study. This section serves only to make the paper self-contained. Reader familiar with [12] should certainly skip it. Section 3 contains some preliminary observations about the mathematical structure displayed by the problem we study. In fact, it is this structure that suggests, quite naturally, the definition of measurable set that we give in Section 4. There, we also show that our class of measurable sets is always a λ-system, and we define a natural measure on such a class. The section concludes with some remarks comparing our definition of measurability with the usual (abstract) definition of measurable set of Carathéodory. Section 5 relates our notion of measurable set to that of unambiguous event in the sense of [8]. Section 6 studies the dependence of the class of measurable/unambiguous sets on the set of priors appearing in the representation. Two Appendices complete the exposition. Appendix A collects some basic facts about Standard Spaces. Appendix B contains the proofs omitted in the main text.

2 Monotone continuous, countably additive multiple priors

The setting we are going to focus on is defined by three objects. First, a collection, $F(S,\Phi)$, of mappings $S \to \Phi$, which represent the alternatives available to the decision maker. S is called the state space and Φ the prize space. Second, a fixed σ-field, Σ, of subsets of S. Third, a preference relation, $\succsim$, on $F(S,\Phi)$. Let $B(\Sigma)$ denote the set of bounded, Σ-measurable functions. Then, the preference relation $\succsim$ is said to satisfy the α-maxmin criterion if and only if there exists a functional $I : B(\Sigma) \to R$ such that for $a, b \in F(S,\Phi)$

$$a \succsim b \qquad iff \qquad I(u \circ a) \geq I(u \circ b)$$

where $u : \Phi \to R$ is a utility function on the prize space, and for every $h \in B(\Sigma)$, I is defined by

$$I(h) = \alpha \min_{P \in \mathcal{C}} \int_S h dP + (1-\alpha) \max_{P \in \mathcal{C}} \int_S h dP$$

with $\mathcal{C}$ being a set of probability measures on (S,Σ), and $\alpha \in [0,1]$.

Preferences satisfying such a criterion have been recently axiomatized in [8] and in [11]. In [12], it has been shown that whenever such preferences satisfy two additional axioms, namely $\succsim$ is (a) both upward and downward atomless and (b) satisfies the axiom of monotone continuity (see [12]), then all the measures in $\mathcal{C}$ are (i) nonatomic; (ii) countably additive; and (iii) there exists a measure $\nu \in \Delta(S)$ such that all the measures in $\mathcal{C}$ are absolutely continuous with respect to ν.

2.1 A Separable Setting

The representation of [12] delivers a collection of measure spaces $\{(S,\Sigma,P)\}_{P\in\mathcal{C}}$, with each P being nonatomic. In this paper, we are going to assume that Σ is a separable σ-algebra. The assumption is harmless as nonseparable spaces are somewhat beyond the realm of mathematical analysis (see Arveson [3], Chapter 3, for a discussion). Yet, it buys a great deal. In fact, from the viewpoint of the measure theoretic properties, every set equipped with a separable nonatomic measure algebra is a representation (i.e., is measure isomorphic) of the unit interval, $X = \{[0,1], \Lambda, \lambda\}$, equipped with the Lebesgue measure. That is, if

(S,Σ,P) is one such measure space, then, by a theorem of von Neumann (see Appendix A), there exists a measurable (mod 0) onto mapping $\pi: X \to S$ which carries the assigned measure structure on S. The reader unfamiliar with this type of problems might consult Appendix A for more on the topic.

3 Preliminaries

In our setting, the collection $\{(S,\Sigma,P)\}_{P\in\mathcal{C}}$ is naturally interpreted as the set of possible probabilistic scenarios for the decision maker. By using the theorem of von Neumann mentioned above, one can give yet another description of such a set. Since each $P \in \mathcal{C}$ is associated to a mapping $\pi_P : X \to S$, the set of possible probabilistic scenarios can be equivalently described by means of a collection of mappings $\mathcal{F} = \{\pi \in \mathcal{F} \mid \pi : X \to S, \pi \text{ onto and measurable}, X \text{ Standard}\}$, one for each $P \in \mathcal{C}$. Each $\pi \in \mathcal{F}$ induces on (S,Σ) the image law $P_\pi(A) = \pi_*\lambda(A) = \lambda\left(\pi^{-1}(A)\right)$, where $A \in \Sigma$ and λ is the Lebesgue measure on X.

The description in terms of the mappings $\mathcal{F}$, while equivalent to that of Section 2, has the advantage of naturally suggesting the definition of measurable event that we introduce in the next section. Before that, however, two observations are in order. First, as the reader has probably already noticed, for any given $P \in \mathcal{C}$, the corresponding mapping π_P is determined only up to an equivalence class. For, if m is a measure-preserving transformation of the unit interval onto itself, then $\pi_P \circ m$ induces the same image law on S. Hence, the set $\mathcal{F}$ should be properly regarded as a quotient space. We will not stress this again as it is evident that the concepts that we introduce below do not depend on the choice of the representative of the equivalence class. Second, it is probably useful to observe that, with our representation, if Γ is a Borel subset of X, and χ_Γ its indicator function, then the random variable y (defined up to an almost everywhere equality) satisfying $\int_A y dP_{\pi_P} = \int_{\pi_P^{-1}(A)} \chi_\Gamma d\lambda$ is just the conditional probability of Γ given π_P.

4 $\mathcal{F}$-measurable Sets

We have just seen that a decision maker described by a Multiple Prior Model can be equivalently described by a family $\mathcal{F}$ of onto measurable mappings $X \to S$. In this section, we abstract from any reference to the decision maker, and give a notion of measurability for subsets $A \in 2^S$ with respect to the family $\mathcal{F}$. It is worth noticing, however, that, when reinterpreted in a decision-theoretic context, our definition is independent of the form of the functional describing the decision maker as it uses the set of priors only.

We begin by defining the exterior measure of a subset $Y \in 2^S$ with respect to the family $\mathcal{F}$. We, then, use it to define measurability with respect to $\mathcal{F}$, and to study the basic properties of the family of measurable sets. Our definition has a strong resemblance with Carathéodory's usual one, but there are noticeable differences as well. The final part of the section comments on this point.

Let $\mathcal{F}$ be as defined above, and let $\pi \in \mathcal{F}$. The next definition introduces a notion of exterior measure for subsets $Y \in 2^S$ relative to a given mapping $\pi \in \mathcal{F}$. Then, the following one defines an exterior measure relative to the whole family $\mathcal{F}$.

Definition 1 *Let $Y \in 2^S$. The exterior measure of Y with respect to $\pi \in \mathcal{F}$ is given by*

$$P_\pi^*(Y) = \lambda^* \left(\pi^{-1}(Y)\right)$$

where λ^* is the usual exterior measure on X.

Observe that since π is onto, then P_π^* is defined on 2^S.

Definition 2 *$\forall Y \in 2^S$, the exterior measure of Y with respect to the family $\mathcal{F}$ is given by*

$$P_{\mathcal{F}}^*(Y) = \sup_{\pi \in \mathcal{F}} P_\pi^*(Y)$$

It is an easy matter to see that (i) the sup always exists, and $P_{\mathcal{F}}^* \leq 1$; (ii) $\forall \pi \in \mathcal{F}$ and any $Y \in 2^S$, $P_\pi^*(Y) \leq P_{\mathcal{F}}^*(Y)$. We also have,

Proposition 3 *$\forall \pi \in \mathcal{F}$, and $\forall A, B \in 2^S$,*

$$P_\pi^*(B) \leq \sup_{h \in \mathcal{F}} \lambda^* \left(h^{-1}(A) \cap \pi^{-1}(B)\right) + \sup_{k \in \mathcal{F}} \lambda^* \left(k^{-1}(A^c) \cap \pi^{-1}(B)\right)$$

where A^c is the complement of A.

Of special interests are those sets for which equality obtains in the above Proposition. In fact, as the Lemma below shows, such sets are characterized by the property that their exterior measure (in the sense of Definition 1) does not depend on the particular elements of $\mathcal{F}$ that one chooses.

Lemma 4 *If $\forall B \in 2^S$, and $\forall \pi \in \mathcal{F}$, $A \in 2^S$ is such that*

$$P_\pi^*(B) = \sup_{h \in \mathcal{F}} \lambda^* \left(h^{-1}(A) \cap \pi^{-1}(B)\right) + \sup_{k \in \mathcal{F}} \lambda^* \left(k^{-1}(A^c) \cap \pi^{-1}(B)\right) \qquad (1)$$

then, $\forall \pi \in \mathcal{F}$

$$P_\pi^*(A) = \sup_{\pi \in \mathcal{F}} P_\pi^*(A) = P_{\mathcal{F}}^*(A)$$

Observe that if condition (1) is satisfied, then A is measurable with respect to the image law of π, for each $\pi \in \mathcal{F}$. This follows at once from the inequalities in the proof of Proposition 3. Conversely, if A is not measurable for some $\pi \in \mathcal{F}$ ($\Longleftrightarrow \exists B \in 2^S$ such that $P_\pi^*(B) < P_\pi^*(A \cap B) + P_\pi^*(A^c \cap B)$), then condition (1) is not satisfied. Motivated by these considerations, we can now give our definition of $\mathcal{F}$-measurable sets.

Definition 5 *If* $\forall B \in 2^S$, *and* $\forall \pi \in \mathcal{F}$,

$$P_\pi^*(B) = \sup_{h \in \mathcal{F}} \lambda^* \left(h^{-1}(A) \cap \pi^{-1}(B)\right) + \sup_{k \in \mathcal{F}} \lambda^* \left(k^{-1}(A^c) \cap \pi^{-1}(B)\right)$$

then, we say that $A \in 2^S$ *is measurable sets with respect to* $\mathcal{F}$ *(or,* $\mathcal{F}$ *-measurable) and we define the measure of* A, $P_{\mathcal{F}}(A)$, *to be*

$$P_{\mathcal{F}}(A) = \sup_{\pi \in \mathcal{F}} P_\pi^*(A) = P_{\mathcal{F}}^*(A)$$

The above is an extension of the usual notion of measurability. This is immediately seen by setting $\mathcal{F} = \{f\}$ (see the discussion of Subsection 2.1, also).

Let $\mathcal{M}_{\mathcal{F}}$ denote the class of $\mathcal{F}$-measurable sets. The next proposition shows that such a class is a λ-system.

Proposition 6 $\mathcal{M}_{\mathcal{F}}$ *is a* λ*-system. In addition,* $A \in \mathcal{M}_{\mathcal{F}} \implies P_{\mathcal{F}}(A^c) = 1 - P_{\mathcal{F}}(A)$. *Moreover, if* $P_{\mathcal{F}}^*(A) = 0$, *then* $A \in \mathcal{M}_{\mathcal{F}}$.

In general, $\mathcal{M}_{\mathcal{F}}$ is not a σ-algebra. Clearly, this depends on the fact that the sups on the RHS of condition (1) need not be attained by the same mapping. Here is an example.

Example 7 *Let* $\{a, b, c, d\} \subset \Sigma$ *be a partition of* S. *Let* $\mathcal{F} = \{f, g\}$ *and let* f *and* g *be such that*

$$f(x) \in \begin{matrix} a & if & x \in [0, \frac{3}{8}) \\ b & if & x \in [\frac{3}{8}, \frac{1}{2}) \\ c & if & x \in [\frac{1}{2}, \frac{7}{8}) \\ d & if & x \in [\frac{7}{8}, 1] \end{matrix} \quad ; \quad g(x) \in \begin{matrix} a & if & x \in [0, \frac{1}{8}) \\ b & if & x \in [\frac{1}{8}, \frac{1}{2}) \\ c & if & x \in [\frac{1}{2}, \frac{5}{8}) \\ d & if & x \in [\frac{5}{8}, 1] \end{matrix}$$

It is immediate to verify that both $A = \{a \cup b\}$ *and* $B = \{b \cup c\}$ *are* $\mathcal{F}$*-measurable as* $P_f(A) = P_g(B) = \frac{1}{2}$, *but*

$$P_f(A \cup B) = \frac{7}{8} \neq P_g(A \cup B) = \frac{5}{8}$$

or, which is the same,

$$P_f(A \cap B) = \frac{1}{8} \neq P_g(A \cap B) = \frac{1}{2}$$

Equivalently, in terms of condition (1)

$$1 = P_f(S) < \sup_{h \in \mathcal{F}} \lambda^* \left(h^{-1}(A \cup B) \cap f^{-1}(S)\right) + \sup_{k \in \mathcal{F}} \lambda^* \left(k^{-1}(A \cup B)^c \cap f^{-1}(S)\right) = \frac{5}{4}$$

4.1 Comments

Let $f \in \mathcal{F}$ be given, and $B \in 2^S$. As noticed, any such an f induces a probability law on S. A set $A \in 2^S$ splits B into two parts, $A \cap B$ and $A^c \cap B$. In our definition of measurability, we allow that these two parts not only to be evaluated by means of mappings other than f (i.e., according to different probabilities laws among those induced by $\mathcal{F}$), but also that they be evaluated independently. If these two parts "add up" with respect to the probability law induced by f, and do so for every $f \in \mathcal{F}$, then we term A measurable. In this sense, our definition has the flavor of the usual definition of measurability, which singles out those sets which, so to speak, split every other set additively.

In this regard, however, it might be useful to contrast it with another notion which could emerge, rather naturally, from the setting we have been dealing with. Begin by observing that our $P^*_{\mathcal{F}}$ is an exterior measure on 2^S in the usual (abstract) sense. That is, it satisfies

1. $P^*_{\mathcal{F}}(\emptyset) = 0$
2. $A \subset B \Longrightarrow P^*_{\mathcal{F}}(A) \leq P^*_{\mathcal{F}}(B)$
3. if $\{A_n\}$ is a sequence of disjoint sets, $P^*_{\mathcal{F}}(\cup A_n) \leq \sum P^*_{\mathcal{F}}(A_n)$.

Given this, a strict parallel with the standard measure theory would have demanded to term a set $A \subset S$ measurable if $\forall B \in 2^S$

$$P^*_{\mathcal{F}}(B) = P^*_{\mathcal{F}}(A \cap B) + P^*_{\mathcal{F}}(A^c \cap B) \tag{2}$$

Let $\mathcal{A}_{\mathcal{F}}$ be the class of subsets of S which satisfy equation (2). By the usual argument, $\mathcal{A}_{\mathcal{F}}$ is a σ-algebra.

It can be immediately verified that in the example above $\mathcal{A}_{\mathcal{F}}$ is the trivial algebra $\{\emptyset, S\}$.

5 Unambiguous events

In the previous sections, we saw that a subset $A \subset S$ is $\mathcal{F}$-measurable if and only if $P_f(A) = P_g(A)$ for all $f, g \in \mathcal{F}$ (see Lemma 4). When reinterpreted in a decision-theoretic context this means that a set $A \subset S$ is $\mathcal{F}$-measurable if and only if it is unambiguous in the sense of Ghirardato, Maccheroni and Marinacci [8]. It is worth to record this formally.

Let $\succsim$ be a preference relation on $F(S, \Phi)$ satisfying the assumptions of Section 2, and let $\mathcal{C}$ be the set of priors appearing in that representation. Then, as in Section 3, there exists a family $\mathcal{F} = \{f \in \mathcal{F} \mid f : X \to S,\ f \text{ onto}, X \text{ Standard}\}$ which induces those priors.

Proposition 8 *A set $A \subset S$ is unambiguous in the sense of [8] if and only if it is $\mathcal{F}$-measurable.*

It is also worth recording the following simple corollary (by the observation following Lemma 4, there is no need to distinguish between our concept of measurability and the standard one).

Corollary 9 *If $\succsim$ is Savage (in the countably additive sense), then $T \in \Sigma$ is unambiguous iff it is measurable.*

Just observe that $\succsim$ is Savage if and only if all the elements in $\mathcal{F}$ induce the same reduced measure algebra on X.

6 $\mathcal{F}-$measurable sets and the properties of $P_{\mathcal{F}}$

Let us denote by $\mathcal{C}_{\mathcal{F}}$ the set of measures on S induced by the family $\mathcal{F}$, that is $\mathcal{C}_{\mathcal{F}} = \{P_\pi = \pi_* \lambda \mid \pi \in \mathcal{F}\}$. In this section, we are going to study how the class of $\mathcal{F}$-measurable sets depends on $\mathcal{C}_{\mathcal{F}}$, and hence on the family $\mathcal{F}$. In addition, we are going to study the properties of the measure $P_{\mathcal{F}}$ (Definition 5) which we term the *natural measure* on S (with respect to the family $\mathcal{F}$).

From Lemma 4, $P_{\mathcal{F}}$ is simply the restriction of any of the P_f's to the class $\mathcal{M}_{\mathcal{F}}$, the $\mathcal{F}$-measurable subsets of S. Right now, this does not say much. Generally speaking, the only sure thing we know of is that $\emptyset$ and S are in $\mathcal{M}_{\mathcal{F}}$, and, hence, that $P_{\mathcal{F}}$ takes the values 0 and 1. Clearly, if it happens (see below) that $\mathcal{M}_{\mathcal{F}} = \{\emptyset, S\}$, we can say no more.

We say that a set $A \in \mathcal{M}_{\mathcal{F}}$ is non-trivial if its $P_{\mathcal{F}}$-measure is neither 0 nor 1. In this section, we are going to determine conditions under which non-trivial $\mathcal{F}$-measurable sets exist. Then, we will show that, in such a case, the measure $P_{\mathcal{F}}$ is convex-ranged.

Let us begin with a special case. Suppose that each and every measure in $\mathcal{C}_{\mathcal{F}}$ is a convex combination of n measures $\{\mu_1, ..., \mu_n\} \subset \mathcal{C}_{\mathcal{F}}$, that is $\mathcal{C}_{\mathcal{F}}$ is the convex-hull of $\{\mu_1, ..., \mu_n\}$. In such a case, it is a straightforward consequence of Lyapunov's convexity theorem that there exist $\mathcal{F}$-measurable sets of measure β for every $\beta \in [0, 1]$.

To see this, begin by observing that, obviously, if the n measures, $\mu_1, ..., \mu_n$, agree on some set $A \in \Sigma$ so does any other measure in $\mathcal{M}_{\mathcal{F}}$. Now, recall that each measure in $\mathcal{M}_{\mathcal{F}}$ is nonatomic, and, therefore, so are $\mu_1, ..., \mu_n$. By Lyapunov's theorem, the range of the vector-measure $\boldsymbol{\mu} = (\mu_1, ..., \mu_n)$ is a convex subset in R^n. Then, to complete the proof of our assertion, it suffices only to observe that this range contains the n-vectors $(0, ..., 0)$ and $(1, ..., 1)$.

As for the general case, we will proceed under the assumption that all the measures in $C_{\mathcal{F}}$ admit a density with respect to some fixed measure ν. This will allow us to use a simple and elegant construction due to Kingman and Robertson [10]. At any rate, from our viewpoint, the assumption does not entail any loss in the generality of the argument. Ultimately, we are concerned with the properties displayed by the preferences described in Section 2. For those, as shown in [12] (see Section 2 above), the assumption is automatically satisfied.[1]

[1] In fact, the results of this Section could all be established without the assumption about the existence of densities. This can be done by using a paper of Azarnia and Maitland Wright [4], which generalizes Kingman and Robertson's. However, as pointed out in the text, this would not buy anything and would force us to a lengthier and more complicated explanation.

Before stating our results, we need to introduce some notation and terminology, which is mutuated from [10]. For $f \in \mathcal{F}$, let us denote by g_f the density associated to P_f, and let $D_{\mathcal{F}}$ be the set of all such densities. $D_{\mathcal{F}}$ is a subset of $\mathcal{L}^1(S, \Sigma, \nu)$, which has $\mathcal{L}^\infty(S, \Sigma, \nu)$ as dual. Let $D_{\mathcal{F}}^{\perp} = \left\{ \phi \in \mathcal{L}^\infty \mid \int g_f \phi d\nu = 0 \text{ for all } g_f \in D_{\mathcal{F}} \right\}$, and let $D_{\mathcal{F}}^{\perp}(A)$ be the set of all ϕ's vanishing a.e. on the complement of the measurable set A. $D_{\mathcal{F}}$ is said to be *thin* (see [10]) if and only if $D_{\mathcal{F}}^{\perp}(A)$ is different from the zero subspace whenever $\nu(A) > 0$. By letting $[D_{\mathcal{F}}]$ denote the subspace generated by $D_{\mathcal{F}}$, it is clear that $D_{\mathcal{F}}$ is thin if and only if $[D_{\mathcal{F}}]$ is thin. Note, in particular, that any finite subset of $\mathcal{L}^1(S, \Sigma, \nu)$ is thin.

Proposition 10 *(a) If $[D_{\mathcal{F}}]$ is dense in $\mathcal{L}^1(S, \Sigma, \nu)$, then there are no non-trivial $\mathcal{F}$-measurable sets. That is, the natural measure $P_{\mathcal{F}}$ is $\{0, 1\}$ −valued.*

(b) If $[D_{\mathcal{F}}]$ is thin, then there exist $\mathcal{F}$-measurable sets of measure β, for every $\beta \in [0, 1]$. Moreover, $P_{\mathcal{F}}$ is nonatomic.

Proof. We follow Kingman and Robertson ([10]). Let $K = \{\psi \in L^\infty([0, 1]) : 0 \le \psi(x) \le 1 \quad a.e.\}$. K is compact and convex in the weak*-topology $\sigma(\mathcal{L}^1, \mathcal{L}^\infty)$. The set $K_0 \subset K$ of all extremal points of K consists of the indicator functions of all measurable sets. Define a mapping $u : \mathcal{L}^\infty \to R^{\mathcal{F}}$ (the latter equipped with the product topology) by

$$u(\psi) = \left(\int \psi g_f d\nu \right)_{f \in \mathcal{F}} = \left(\int \psi dP_f \right)_{f \in \mathcal{F}}$$

This mapping is obviously linear and continuous, and

$$Ker\ u = u^{-1}(0) = \left\{ \psi \in \mathcal{L}^\infty([0, 1]) : \int \psi g_f d\nu = 0, \forall f \right\} = D_{\mathcal{F}}^{\perp}$$

The range of the vector measure $\left(\int dP_f \right)_{f \in \mathcal{F}}$ is $u(K_0)$. Hence, the problem of existence of non-trivial $\mathcal{F}$-measurable sets can be set in this way. Let $\mathbf{1} \in R^{\mathcal{F}}$ be the vector whose coordinates are all one, and let $\beta \in [0, 1]$. Then, $\beta\mathbf{1} \in R^{\mathcal{F}}$ and there exists an $\mathcal{F}$-measurable set A of measure β if $\chi_A \in u^{-1}(\beta\mathbf{1})$, where χ_A is the indicator function of the set A. We have two cases.

(i) $D_{\mathcal{F}}$ spans a dense subspace of $\mathcal{L}^1$. This means that $D_{\mathcal{F}}^{\perp} = \{0\}$ (i.e., $D_{\mathcal{F}}$ is not thin), and it is equivalent to the fact that u is one-to-one. Since, we always have $u(\beta\chi_S) = \beta\mathbf{1}$, a set A is $\mathcal{F}$-measurable if and only if its indicator function is of the form $\beta\chi_S$, which can occur only if $\beta = 0$ or $\beta = 1$. That is, the only $\mathcal{F}$-measurable sets have either measure 0 or 1.

(ii) $D_{\mathcal{F}}$ is thin. In such a case, we have $K \subset K_0 + D_{\mathcal{F}}^{\perp}$ (see [10], p.348). Hence, $u(K_0) = u(K)$, which is immediately seen to be compact and convex as u is linear and continuous. Hence, the first part of the statement (b).

To see that $P_{\mathcal{F}}$ is non-atomic, proceed as follows. Pick $\beta \in (0, 1)$. By the first part of (b), there exists $A \in \Sigma$ such that $P_f(A) = P_h(A) = \beta$, for any $f, h \in \mathcal{F}$. For any $f \in \mathcal{F}$, and for any $B \in \Sigma$ such that $B \subset A$, define

$$P_f \mid_A (B) = \frac{P_f(B)}{P_f(A)}$$

Since, $D_{\mathcal{F}}(A)$ is thin in $\mathcal{L}^1(A, \Sigma\mid_A, \nu\mid_A)$, (again, by the first part of (b)) for any $\gamma \in [0,1]$, there exists a $B \in \Sigma\mid_A$ such that for any$f, h \in \mathcal{F}$

$$P_f\mid_A(B) = P_h\mid_A(B) = \gamma$$

Hence, for any $f \in \mathcal{F}$, $P_f(B) = \gamma P_f(A)$. Since, $A \in \mathcal{M}_{\mathcal{F}}$, it follows at once that $B \in \mathcal{M}_{\mathcal{F}}$ and

$$P_{\mathcal{F}}(B) = \gamma P_{\mathcal{F}}(A)$$

■

We conclude the section by exhibiting a case where no nontrivial $\mathcal{F}$-measurable sets exist.

Example 11 *Recall that by means of the von Neumann's theorem mentioned above, we can refer, without loss, to* $[0,1]$ *equipped with the Lebesgue* σ*-algebra. Consider the system of functions* $\{\phi_i\}$ *defined as follows.* ϕ_1 *is identically* 1 *on* $[0,1]$*;* ϕ_2 *is* 1 *on* $[0,\frac{1}{2})$ *and* -1 *on* $[\frac{1}{2},1]$*;* ϕ_3 *is* 1 *on* $[0,\frac{1}{4}]$*,* -1 *on* $[\frac{1}{4},\frac{1}{2})$*, and* 0 *on* $[\frac{1}{2},1]$*;* ϕ_4 *is* 0 *on* $[0,\frac{1}{2})$*,* 1 *on* $[\frac{1}{2},\frac{3}{4})$ *and* -1 *on* $[\frac{3}{4},1]$*, etc.. In general, for positive integers* r *and* $k \leq 2^{r-1}$*,* $\phi_{2^{r-1}+k}$ *is* 1 *on* I^r_{2k-1}*,* -1 *on* I^r_{2k}*, and* 0 *otherwise, where* $[0,1]$ *is partitioned into* 2^r *intervals* $\left\{I^r_j : 1 \leq j \leq 2^r\right\}$ *of equal length. The system* $\{\phi_i\}$ *just described is known as the Haar system. It easily seen to be a (Schauder) basis for the space* $\mathcal{L}^1([0,1])$*. In particular, since, except for the endpoints, the linear hull of* $\{\phi_i\}$ *contains all characteristic functions of the intervals* $\left\{I^r_j\right\}$*,* $\{\phi_i\}$ *spans a dense subspace of* $\mathcal{L}^1([0,1])$*. Define a new system by*

$$\begin{aligned} f_1 &= \phi_1 \\ f_{2^{r-1}+k} &= \frac{1}{2}\phi_{2^{r-1}+k} + 1 \end{aligned}$$

Obviously, the linear hull of $\{f_i\}$ *coincides with that of* $\{\phi_i\}$*. Finally, define a system of measures* $\{P_i\}$ *by* $P_i(A) = \int f_i d\lambda$*, for* $A \in \Lambda$ *and* λ *the Lebesgue measure.*

References

[1] Amarante, M. (2003), "Ambiguous Events", Columbia University, mimeo.

[2] Arrow, K. (1970), *Essays in the Theory of Risk-Bearing* ,Amsterdam, North Holland.

[3] Arveson, W. (1976), *An Invitation to C*-Algebras*, Springer.

[4] Azarnia N. and J.D. Maitland Wright (1982), "On the Lyapunov-Knowles Theorem", *The Quarterly Journal of Mathematics*, 33, 257-61.

[5] Billingsley, P. (1965), *Ergodic Theory and Information*, Wiley, NY.

[6] Ellsberg, D. (1961), "Risk, Ambiguity and the Savage Axioms", *QJE* 75, 643-69.

[7] Epstein, L.G. and J. Zhang (2000), "Subjective Probabilities on Subjectively Unambiguous Events", *Econometrica* 69, 265-306.

[8] Ghirardato, P., F. Maccheroni and M. Marinacci (2002), "Ambiguity from the differential viewpoint", ICER WP 17.

[9] Ghirardato, P. and M. Marinacci (2002), "Ambiguity made precise: A comparative foundation" *JET* 102, 251-89.

[10] Kingman J.F.C. and A.P. Robertson (1968), "On a Theorem of Lyapunov", *Journal of the London Mathematical Society*, 43, 347-51.

[11] Kopylov, I. (2001), "Procedural rationality in the multiple prior model", University of Rochester, mimeo.

[12] Marinacci, M., F. Maccheroni, A. Chateauneuf and J-M. Tallon (2003), "Monotone Continuous Multiple Priors", University of Turin, mimeo.

[13] Royden H.L. (1988), *Real Analysis*, Macmillan.

[14] Stinchcombe, M. (1997), "Countably Additive Subjective Probabilities", *RES* 64, 125-46.

APPENDIX A. STANDARD SPACES

A Polish space, (X, τ), is a separable, completely metrizable topological space. Given the topology τ on X, the Borel σ–field is the one generated by the closed sets.

A Standard space is a Polish space stripped down to its Borel structure. A Standard Borel space along with a nonatomic measure is a called a Standard Lebesgue space.

In the main text, we used the following two theorems in several occasions

Theorem 12 ***(Carathéodory**, see [13] p. 399)* *If (Σ, P) is a normalized, separable, nonatomic measure algebra, then there is an isomorphism of (Σ, P) onto the measure algebra of the unit interval.*

Theorem 13 *(**von Neumann**, see [5] p. 69) If (Y, Γ, Q) and (S, Σ, P) are Lebesgue spaces, and Φ is a homomorphism of their measure algebras, then Φ arises from a point homomorphism mod 0 (i.e., a measurable mapping $\varphi : Y \setminus Y_0 \to S$, where Y_0 has Q-measure 0).*

APPENDIX B. Omitted Proofs

Proof of Proposition 3. $\forall B \in 2^S$, and any $\pi \in \mathcal{F}$,

$$\begin{aligned}
P^*_\pi(B) &\leq P^*_\pi(A \cap B) + P^*_\pi(A^c \cap B) \\
&= \lambda^*\left(\pi^{-1}(A) \cap \pi^{-1}(B)\right) + \lambda^*\left(\pi^{-1}(A^c) \cap \pi^{-1}(B)\right) \\
&\leq \sup_{h \in \mathcal{F}} \lambda^*\left(h^{-1}(A) \cap \pi^{-1}(B)\right) + \sup_{k \in \mathcal{F}} \lambda^*\left(k^{-1}(A^c) \cap \pi^{-1}(B)\right)
\end{aligned}$$

■

Proof of Lemma 4. Suppose that (1) is satisfied, but $\exists \gamma \in \mathcal{F}$ such that $P^*_\gamma(A) > P^*_\pi(A)$. We are going to show that $\exists B \in 2^S$ such that (1) is violated for that B.

Set $B = S$. Then,

$$\begin{aligned}
P^*_\pi(S) &= \lambda^*\left(\pi^{-1}(A) \cap \pi^{-1}(S)\right) + \lambda^*\left(\pi^{-1}(A^c) \cap \pi^{-1}(S)\right) \\
&= P^*_\pi(A) + \lambda^*\left(\pi^{-1}(A^c) \cap \pi^{-1}(S)\right) \\
&< P^*_\gamma(A) + \lambda^*\left(\pi^{-1}(A^c) \cap \pi^{-1}(S)\right) \\
&= \lambda^*\left(\gamma^{-1}(A) \cap \pi^{-1}(S)\right) + \lambda^*\left(\pi^{-1}(A^c) \cap \pi^{-1}(S)\right) \\
&\leq \sup_{h \in \mathcal{F}} \lambda^*\left(h^{-1}(A) \cap \pi^{-1}(S)\right) + \sup_{k \in \mathcal{F}} \lambda^*\left(k^{-1}(A^c) \cap \pi^{-1}(S)\right)
\end{aligned}$$

■

Proof of Proposition 6. All the listed properties are obvious. It is immediate to check that $\emptyset, S \in \mathcal{M}_\mathcal{F}$, and that $A \in \mathcal{M}_\mathcal{F} \Longrightarrow A^c \in \mathcal{M}_\mathcal{F}$ because of the symmetry of condition (1). By replacing B with S in condition (1), one has that if $A \in \mathcal{M}_\mathcal{F}$, then $P_\mathcal{F}(A^c) = 1 - P_\mathcal{F}(A)$.

To complete the proof that $\mathcal{M}_\mathcal{F}$ is a λ-system, let $\{E_i\} \subset \mathcal{M}_\mathcal{F}$ be a sequence of disjoint sets.

Begin by observing that if $A, B \subset S$ and Γ is any subset of X, from the property of the exterior measure, λ^*, on X, it follows that for any $h \in \mathcal{F}$ one has

$$\lambda^*\left(h^{-1}(A) \cap \Gamma\right) + \lambda^*\left(h^{-1}(B) \cap \Gamma\right) \geq \lambda^*\left(h^{-1}(A \cup B) \cap \Gamma\right) + \lambda^*\left(h^{-1}(A \cap B) \cap \Gamma\right)$$

Now, let $E_1, E_2 \in \mathcal{M}_\mathcal{F}$, $E_1 \cap E_2 = \emptyset$. For any $\pi \in \mathcal{F}$ and any $B \in 2^S$, we have

$$\begin{aligned}
&\sup_{h \in \mathcal{F}} \lambda^*\left(h^{-1}(E_1^c \cap E_2^c) \cap \pi^{-1}(B)\right) + \sup_{k \in \mathcal{F}} \lambda^*\left(k^{-1}(E_1 \cup E_2) \cap \pi^{-1}(B)\right) \\
\leq\; &\sup_{h \in \mathcal{F}} \left[\lambda^*\left(h^{-1}(E_1^c) \cap \pi^{-1}(B)\right) + \lambda^*\left(h^{-1}(E_2^c) \cap \pi^{-1}(B)\right) - \lambda^*\left(h^{-1}(E_1^c \cup E_2^c) \cap \pi^{-1}(B)\right)\right] \\
&+ \sup_{k \in \mathcal{F}} \left[\lambda^*\left(k^{-1}(E_1) \cap \pi^{-1}(B)\right) + \lambda^*\left(k^{-1}(E_2) \cap \pi^{-1}(B)\right) - \lambda^*\left(k^{-1}(E_1 \cap E_2) \cap \pi^{-1}(B)\right)\right] \\
\leq\; &\sup_{h \in \mathcal{F}} \lambda^*\left(h^{-1}(E_1^c) \cap \pi^{-1}(B)\right) + \sup_{h \in \mathcal{F}} \lambda^*\left(h^{-1}(E_2^c) \cap \pi^{-1}(B)\right) - P^*_\pi(B) \\
&+ \sup_{k \in \mathcal{F}} \lambda^*\left(k^{-1}(E_1) \cap \pi^{-1}(B)\right) + \sup_{k \in \mathcal{F}} \lambda^*\left(k^{-1}(E_2) \cap \pi^{-1}(B)\right) \\
\leq\; &P^*_\pi(B)
\end{aligned}$$

where we have used the usual properties of λ^* and the measurability of E_1 and E_2. Then, by Proposition 3, $E_1^c \cap E_2^c$ is measurable and so is $E_1 \cup E_2$. Inductively, any finite union of the E_i's is measurable. Finally, let $E = \overset{\infty}{\cup} E_i$, $F_n = \overset{n}{\cup} E_i$. Then, $F_n \in \mathcal{M}_{\mathcal{F}}$, $F_n^c \supset E^c$, and for any $\pi \in \mathcal{F}$ and any $B \in 2^S$,

$$\begin{aligned} P_\pi^*(B) &= \sup_{h \in \mathcal{F}} \lambda^* \left(h^{-1}(F_n) \cap \pi^{-1}(B) \right) + \sup_{k \in \mathcal{F}} \lambda^* \left(k^{-1}(F_n^c) \cap \pi^{-1}(B) \right) \\ &\geq \sup_{h \in \mathcal{F}} \lambda^* \left(h^{-1}(F_n) \cap \pi^{-1}(B) \right) + \sup_{k \in \mathcal{F}} \lambda^* \left(k^{-1}(E^c) \cap \pi^{-1}(B) \right) \end{aligned}$$

Since the inequality holds for any n, and the LHS is independent of n, the measurability of E follows.

Finally, it is obvious that if $P_{\mathcal{F}}^*(A) = 0$, then for any $\pi \in \mathcal{F}$ and any $B \in 2^S$,

$$P_\pi^*(B) \geq \sup_{h \in \mathcal{F}} \lambda^* \left(h^{-1}(A) \cap \pi^{-1}(B) \right) + \sup_{k \in \mathcal{F}} \lambda^* \left(k^{-1}(A^c) \cap \pi^{-1}(B) \right)$$

■