Setup

An interesting data type: Lie-valued euclidean indexed data.

These data might be:

- phase angles as functions of time or space, for example compass directions.
- 3D orientations of a rigid frame of reference as a function of time or space.
- quaternions as a function of time or space.

This can also be extended to quotients of lie groups which gives us the ability to model points on S2, the unit sphere, as functions of time or space.

Some Examples

1. planning aircraft trajectories
2. animation
3. geophysical patterns such as winds, ocean currents, rock orientations
4. time series modeling
5. multivariate financial time series volatility modeling.

New Methods of Data Representation

- Multiresolution analysis
  - limiting interpolation allows us to estimate a functional form for the data.
- Wavelet decomposition
  - when the data are evolving according to a geodesic, all fine scale wavelet coefficients will be zero. This is because we are using a cubic polynomial interpolation scheme, the data will be predicted along a geodesic.

The multiplicative structure for these data is entirely different than traditional wavelet and multiresolution analysis which assumes real-valued and real vector-valued data.

Applications

We can use the wavelet coefficents to shrink, compress, quantize, and perform various operations and we will still retain the SO(3) data structure, ie. lie-group valued.

- Compression
  - done by discarding wavelet coefficients at certain scales when they can be predicted from previous scales.
- Denoising
  - performed in certain cases using thresholding
- Function estimation
  - carried out through successive levels of interpolation

Background Mathematics

Each Lie group has its associated Lie algebra, related through the exponential map. $so(3)$ is the Lie algebra associated with $SO(3)$ and has a one-to-one relationship with $\mathbb{R}^3$.

Using this fact, and knowledge of the exponential map, we can perform familiar interpolation and refinement operations on $SO(3)$ data.

This is also true for lower dimensions, such as the group of unit circles equipped with complex multiplication, and the group of unit spheres in $\mathbb{R}^3$. [a]

[a] Please see the attached printout for more details!

Interpolation in SO(3)

$$
\begin{array}{ccccc}
\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} & & \begin{pmatrix} \hat{b}_{11} & \hat{b}_{12} & \hat{b}_{13} \\ \hat{b}_{21} & \hat{b}_{22} & \hat{b}_{23} \\ \hat{b}_{31} & \hat{b}_{32} & \hat{b}_{33} \end{pmatrix} & & \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \\
\Big| & & \uparrow & & \Big| \\
Log & & Exp & & Log \\
\downarrow & & \Big| & & \downarrow \\
\begin{pmatrix} 0 & \alpha_1 & -\alpha_2 \\ -\alpha_1 & 0 & \alpha_3 \\ \alpha_2 & -\alpha_3 & 0 \end{pmatrix} & \rightarrow & \begin{pmatrix} 0 & \frac{\alpha_1+\gamma_1}{2} & -\frac{\alpha_2+\gamma_2}{2} \\ -\frac{\alpha_1+\gamma_1}{2} & 0 & \frac{\alpha_3+\gamma_3}{2} \\ \frac{\alpha_2+\gamma_2}{2} & -\frac{\alpha_3+\gamma_3}{2} & 0 \end{pmatrix} & \leftarrow & \begin{pmatrix} 0 & \gamma_1 & -\gamma_2 \\ -\gamma_1 & 0 & \gamma_3 \\ \gamma_2 & -\gamma_3 & 0 \end{pmatrix}
\end{array}
$$

Example of Circular Data

Wind Directions over 71 hours from Black Mountain Australia: [a]

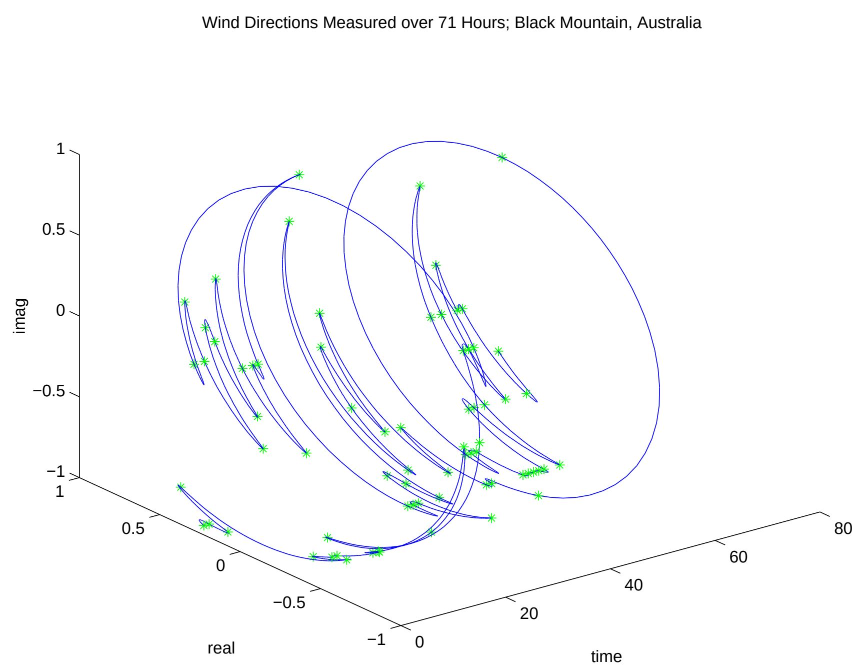


Figure 1: Plot and Interpolation of Wind Direction data

[a] Data obtained from Fisher, N. I. Statistical Analysis of Circular Data 1993

Example of SO(3) Data

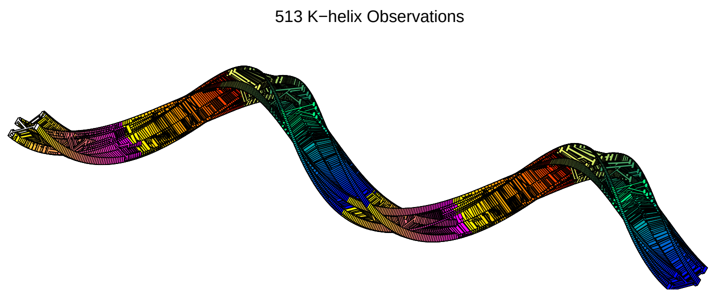


Figure 2: 513 Frames of a Smooth Helix

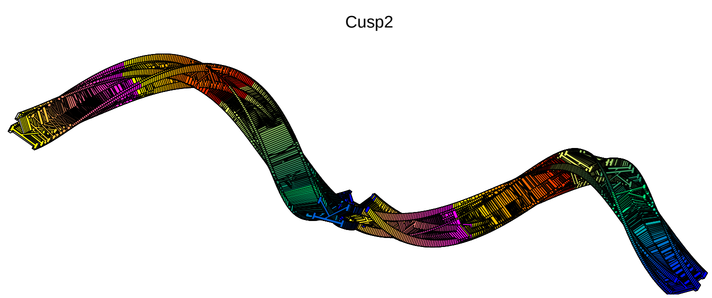
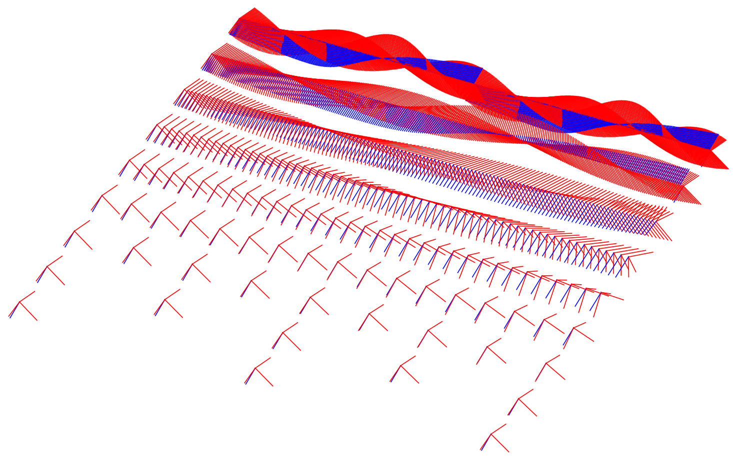


Figure 3: 513 Frames of a Cusped Helix

Wavelet Coefficients for SO(3) Data

Wavelet Coefficients for Smooth Helix



Wavelet Coefficients for Helix with Cusp