Relative Trace Formula for $SO_2 \times SO_3$ and the Waldspurger Formula

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ABSTRACT

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We provide a new relative trace formula approach to the theorem of Waldspurger on toric periods for GL_2, with possible applications to the global Gross-Prasad conjecture for orthogonal groups.

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Chapter 1

Introduction

In this thesis we present a new perspective on the celebrated formula of Waldspurger for toric periods of automorphic forms on GL_2. This formula applies to cusp forms living on the group GL_2 over a global field F. It relates the period integral of such a form ϕ with respect to a (non-split) maximal torus T to the central critical value of a Rankin-Selberg L-function, and very loosely takes the shape

$$|\mathscr{P}(\phi)|^2 = (*)L(\frac{1}{2}, \pi). \tag{1.1}$$

Here, for $\mathbb{A}$ the ring of adeles of F, we have taken $\pi = \chi \otimes \sigma$ to be an automorphic cuspidal representation of $(T \times GL_2)(\mathbb{A})$ such that χ is compatible with the central character of π, i.e. $\chi|_{Z_{GL_2}(\mathbb{A})}\omega_\sigma = 1$, $\phi = \otimes_v \phi_v \in \sigma$ to be a factorizable vector, and we have defined the period integral as

$$\mathscr{P}(\phi) := \int_{T(F)\backslash T(\mathbb{A})} \phi(t)\chi(t)dt.$$

In the above, $L(\frac{1}{2}, \pi) = L(\frac{1}{2}, \chi \times \sigma)$ is the central value of the Rankin-Selberg L-function attached to π, while the quantity $(*)$ is an explicit constant which contains, for instance, both information on the vanishing of the local spaces of linear functionals $\mathrm{Hom}_{T(F_v)}(\pi_v, \mathbb{C})$ as well as the value at $s = 1$ of an appropriate adjoint L-function.

Waldspurger's formula fits snugly into whole web of conjectural identities (see [Gan *et al.*, 2012; Ichino and Ikeda, 2010; Harris, 2012]), today known as the refined global Gan-Gross-Prasad (GGP) conjectures. To give context, we briefly describe these very general formulas. Loosely, they take the same form as (1.1), and relate a period integral of an automorphic form on $SO_m \times SO_n$ or $U_m \times U_n$,

$m \leq n$ to the central critical value of its associated L-function. Indeed, when $\omega_\sigma = 1$ Waldspurger's formula is exactly the refined Gross-Prasad conjecture for $SO_2 \times SO_3$; a twisted form of (1.1) also appears as the refined GGP conjecture for $U_1 \times U_2$.

Enough on conjecture, on to fact. The original proof of Waldspurger's theorem invoked the methods of the theta correspondence [Waldspurger, 1985]. Since then, there have been many other proofs given. Most notably, in a pair of papers ([Jacquet, 1986; Jacquet, 1987], see also [Jacquet and Chen, 2001]), Jacquet provided an ambitious alternate approach to this theorem. His proof relies on writing down and comparing two relative trace formulas, foregoing the constructive methods of the theta correspondence and instead deducing the formula from a spectral identity.

Both methods have enjoyed some measure of success in attacking the GGP conjectures. The theta correspondence has provided a wealth of results in many low rank cases. For $SO_3 \times SO_4$ it gives, through the beautiful work of Ichino [Ichino, 2008], an essentially complete affirmative answer to the conjecture. It also offers insight into many endoscopic cases for $SO_4 \times SO_5$ [Gan and Ichino, 2011], $SO_2 \times SO_5$, and $SO_3 \times SO_6$ (see [Prasad and Takloo-Bighash, 2011; Liu,]). However, all of these theta correspondence arguments rely, in an essential way, on the many exceptional isomorphisms between low rank classical groups–they thus offer no method of attack towards the Gross-Prasad conjecture for high rank groups. On the other hand, the relative trace formula ideas behind Jacquet's alternate proof of the Waldspurger formula do not inherently have any such restriction. Indeed, following the pioneering work of Jacquet and Rallis [Jacquet and Rallis, 2011], they have been applied to great effect, proving many cases of the Gan-Gross-Prasad conjecture for *unitary groups* $U_n \times U_{n+1}$ for arbitrary n (see [Zhang, 2014b; Zhang, 2014a; Yun, 2011]). A similar trace formula approach, proposed in [Liu, 2014], offers hope towards an eventual proof of the full refined formula ([Liu,]) for arbitrary pairs of unitary groups $U_m \times U_n, m \leq n$ (the even more general case of arbitrary codimension!).

It is thus a pertinent question to ask if there exists a trace formula approach, analogous to the work of Jacquet and Rallis in the unitary case, that offers insight into the Gross-Prasad conjecture for *orthogonal groups* $SO_n \times SO_{n+1}$. The goal of this thesis is to offer some evidence towards an affirmative answer to this question. It is our hope that the methods are robust enough to admit generalization. That is, we hope that in future work we can provide a relative trace formula approach to the global Gross-Prasad conjecture for $SO_n \times SO_{n+1}$. For the present though, we restrict our attention to the case of $SO_2 \times SO_3$, i.e. the original case of Waldspurger, and explain a relative trace formula comparison that appears well suited to the interpretation of the Waldspurger formula as a period relation for orthogonal groups.

For the purposes of this introduction, we state our main result quite informally, saving the precise formulation for later. Let F be a global field of characteristic not equal to 2 with $\mathbb{A} := \mathbb{A}_F$ its ring of adeles. Let $W \hookrightarrow V$ be an embedding of a non-degenerate rank 2 quadratic space over F into a non-degenerate rank 3 quadratic space, and denote by $SO_W \times SO_V$ (or simply $SO_2 \times SO_3$ if the spaces are clear from context) the product of the associated special orthogonal groups. For $\pi = \chi \otimes \sigma$ a cuspidal automorphic representation of $SO_2 \times SO_3$, we recall that the L-function $L(s, \pi) = L(s, \chi \times \sigma)$ is a degree 4 L-function over F, and can be thought of as the Rankin-Selberg L-function of the representation $\Pi = \Pi_\chi \otimes \Pi_\sigma$ of $GL_2 \times GL_2$, where Π_χ and Π_σ are the representations of GL_2 arising as the standard functorial transfer from these orthogonal groups. The main result of this thesis is the following theorem.

Main Theorem. *Let $f = f_{(W,V)}$ be a "good" function on $(SO_W \times SO_V)(\mathbb{A})$. Similarly let $f' \otimes \Phi \otimes \Psi$ be a collection of "good" test data, with f a function on $(GL_2 \times GL_2)(\mathbb{A})$, Φ on $\mathbb{A}^{\oplus 2}$, and Ψ on $\mathbb{A}_E$, where $E = F(\sqrt{d_W})$ is the discriminant algebra corresponding to W. Then there exist trace formula distributions $J(f) = J_{(W,V)}(f_{(W,V)})$ and $I(f' \otimes \Phi \otimes \Psi)$ such that:*

1. *$J(f)$ has a spectral decomposition which encodes the information of the Waldspurger-Gross-Prasad period integral $\mathscr{P}$ as π varies.*

2. *$I(f' \otimes \Phi \otimes \Psi)$ has a spectral decomposition which encodes the information of the L-function $L(s, \Pi)$ as Π varies over all representations of $GL_2 \times GL_2$ that arise as functorial transfers from $SO_{W'} \times SO_{V'}$ for some pair of spaces $W' \hookrightarrow V'$ with the same discriminants as W and V.*

3. *For matching sets of "good" test data*

$$(f_{(W',V')})_{(W',V')} \leftrightarrow \sum_i f'_i \otimes \Phi_i \otimes \Psi_i$$

with (W', V') running over quadratic spaces as in (2)., we have an equality of trace formulas

$$2 \sum_{(W',V')} J_{(W',V')}(f_{(W',V')}) = I(\sum_i f'_i \otimes \Phi_i \otimes \Psi_i). \tag{1.2}$$

A more precise statement of this main result can be found later in Theorem 4.4.1 of Chapter 4. We content ourselves for now with a few short remarks on the above result.

First, we can say a little more about what the word "encodes" means in this context. The study of the period integral $\mathscr{P}|_\pi$ can be reduced to the study of a single distribution on $(SO_V \times SO_W)(\mathbb{A})$, the global spherical character J_π. This is given by

$$J_\pi(f) = \sum_{\phi \in \mathrm{ON}(\pi)} \mathscr{P}(\pi(f)\phi)\overline{\mathscr{P}(\phi)}$$

where the sum runs over an orthonormal basis of the space of π. Then, when we say that the trace formula distribution $J(f)$ "encodes" the data, we mean that formally $J(f)$ decomposes as

$$J(f) = \sum_\pi J_\pi(f)$$

where the sum is over π in the automorphic spectrum of $SO_W \times SO_V$. There is a similar interpretation for the distribution $I(f' \otimes \Phi \otimes \Psi)$, but we postpone discussion of it until later, as the definition of the global spherical character for this trace formula is more involved.

Second, we can explain a little about what we mean when we say that test data $(f_{W',V'})_{(W',V')}$ is "good." The proof of the spectral identity (1.2) follows the same basic idea present in all trace formula comparisons. That is, we decompose $J(f)$ and $I(f' \otimes \Phi \otimes \Psi)$ into geometric sides, i.e we write them as sums of global orbital integrals, and then show by local methods that individual terms in the sum can be identified. We say $(f_{(W',V')})_{(W',V')}$ is "good" if the functions $f_{(W',V')}$ satisfy a technical condition that allows us to essentially work with the "simple" forms of our trace formulas, in the sense of Flicker and Kazhdan [Flicker and Kazhdan, 1988]. We refer to the body of this thesis, and in particular to Propositions 2.5.1 and 3.4.1 of Chapters 2 and 3 for the precise definition of "good," and merely remark that the assumption that test data is "good" bypasses any need to seriously consider either the continuous spectrum or any non-regular semisimple orbital integrals. Of course, it also limits the applicability of our trace formula identity. We hope to overcome this technical problem in future work.

We also remark that the factor 2 on the left hand side of 1.2 appears due to considerations on the "geometric" side of $GL_2 \times GL_2$ trace formula. The sum over i on the right hand side of 1.2 occurs for technical reasons as well; we postpone discussion of it until later.

Finally, let us give an explanation for the presence of the sum over (W', V') in (1.2). To do so, it is helpful to tighten up the discussion of the Waldspurger formula that we gave above. We will deviate slightly from the previous notation we used when presenting the informal (1.1), as we would like to favor the viewpoint that the Waldspurger formula is an $SO_2 \times SO_3$ period relation rather than a GL_2 one.

So let F and $\mathbb{A} = \mathbb{A}_F$ be as above. Let $W \hookrightarrow V$ be an embedding of a non-degenerate 2-dimensional quadratic space W into a non-degenerate 3-dimensional quadratic space V. This gives rise to an obvious embedding $\iota : SO_W \hookrightarrow SO_V$. Consider the reductive group $G = G_{(W,V)} = SO_W \times SO_V$ together with its subgroup $H = \Delta SO_W = \{(h, \iota(h)) : h \in SO_W\}$ consisting of the diagonally embedded copy of SO_W. Let $\pi = \chi \otimes \sigma$ be an automorphic cuspidal representation of G, i.e. let σ be an automorphic cuspidal representation of SO_V and let χ be an automorphic character, i.e. a homomorphism

$$\chi : SO_W(F)\backslash SO_W(\mathbb{A}) \to \mathbb{C}^\times.$$

We consider the Waldspurger-Gross-Prasad period functional

$$\begin{aligned} \mathscr{P} = \mathscr{P}_\pi : \pi = \chi \otimes \sigma &\to \mathbb{C} \\ \varphi = \chi \otimes \phi &\mapsto \int_{[SO_W]} \varphi(h) dh = \int_{[SO_W]} \chi(h)\phi(\iota(h)) dh. \end{aligned}$$

Here $\varphi \in \pi$ and dh is the Tamagawa measure on $SO_W(\mathbb{A})$. We often omit writing ι when it is clear from context. We use the (standard) shorthand that $[M]$, for M a group over F, denotes the automorphic quotient $[M] = M(F)\backslash M(\mathbb{A})$.

It is clear from definition that $\mathscr{P}_\pi$ is an element in the global space of homomorphisms

$$\mathscr{P}_\pi \in \mathrm{Hom}_{H(\mathbb{A})}(\pi, \mathbb{C}).$$

It can thus vanish for representation theoretic reasons, i.e. if any of the local spaces $\mathrm{Hom}_{H_v}(\pi_v, \mathbb{C})$ vanish. To account for this, we proceed as follows: fix a decomposition of the restriction of the (sesquilinear) Petersson inner product on $L^2([G])$ to π

$$(\varphi_1, \varphi_2) = \int_{[G]} \varphi_1(g)\overline{\varphi_2(g)} dg = \prod_v (\varphi_{1,v}, \varphi_{2,v})_v$$

via the factorization $\pi = \otimes_v' \pi_v$. That is, fix invariant Hermitian products $(\cdot, \cdot)_v$ on π_v for all v satisfying the product decomposition above. Fix too a decomposition of the global Tamagawa measure

$$dh = \prod dh_v$$

on $SO_W(\mathbb{A})$ into local measures. We can consider the functional $\alpha_v \in \mathrm{Hom}_{H_v \times H_v}(\pi_v \boxtimes \bar{\pi}_v, \mathbb{C})$ given by, for $\varphi_{i,v} \in \pi_v$,

$$\alpha_v(\varphi_{1,v}, \varphi_{2,v}) = \int_{SO_W(F_v)} (\pi_v(h_v)\varphi_{1,v}, \varphi_{2,v})_v dh_v.$$

When π_v is tempered this converges absolutely. The following remarkable properties hold for the local hom-spaces $\mathrm{Hom}_{H_v}(\pi_v, \mathbb{C})$.

1. The local space of functionals $\mathrm{Hom}_{H_v}(\pi_v, \mathbb{C})$ vanishes if and only if $\alpha_v = 0$ identically.

2. We have

   $$\sum_{(W',V')} \sum_{\pi_v^{(W,V')}} \dim \mathrm{Hom}_{SO_{W'}(F_v)}(\pi_v^{(W,V)}, \mathbb{C}) = 1$$

   where the sum runs over all inner forms (W', V') of (W, V) and all members $\pi_v^{(W',V')}$ of the Vogan L-packet of π.

3. When π_v is unramified and φ_v, ϕ_v are spherical vectors satisfying $(\varphi_v, \phi_v)_v = 1$, then

   $$\begin{aligned}\alpha_v(\varphi_v, \phi_v) &= \mathrm{vol}(SO_W(\mathcal{O}_v))\zeta_{F_v}(2)\frac{L(\frac{1}{2}, \pi_v)}{L(1, \pi_v, \mathrm{Ad})} \\ &= \mathrm{vol}(SO_W(\mathcal{O}_v))\zeta_{F_v}(2)\frac{L(\frac{1}{2}, \pi_v)}{L(1, \omega_v)L(1, \sigma_v, \mathrm{Ad})}\end{aligned}$$

   where $\omega_v = \omega_{E_v/F_v}$ is the quadratic multiplicative character associated to the quadratic extension E_v over F_v, with $E_v = F_v[t]/(t^2 - d_W)$ the discriminant algebra of $W \otimes F_v$. We denote, for future use, this ratio of L-functions

   $$\mathscr{L}_v(\pi_v) = \zeta_{F_v}(2)\frac{L(\frac{1}{2}, \pi_v)}{L(1, \omega_v)L(1, \sigma_v, \mathrm{Ad})}.$$

4. If we define

   $$\alpha_v^\natural(\varphi_v, \phi_v) = \frac{1}{\mathscr{L}_v(\pi_v)}\alpha_v(\varphi_v, \phi_v)$$

   then the period integral factors according to the Waldspurger formula

   $$\frac{\mathscr{P}(\varphi)\mathscr{P}(\phi)}{(\varphi, \phi)} = \frac{1}{4}\zeta_F(2)\frac{L(\frac{1}{2}, \pi)}{L(1, \pi, \mathrm{Ad})}\prod_v \frac{\alpha_v^\natural(\varphi_v, \phi_v)}{(\varphi_v, \phi_v)} \tag{1.3}$$

Again, the L-function $L(s, \pi) = L(s, \chi \times \sigma)$ is the L-function of degree 4 over F defined by the Rankin-Selberg convolution of Π_χ and Π_σ; here Π_χ and Π_σ are the two representations corresponding to the functorial liftings of χ and σ to GL_2. The L-function $L(s, \pi, \mathrm{Ad}) = L(s, \chi, \mathrm{Ad})L(s, \sigma, \mathrm{Ad})$ is the adjoint L-function attached to the representation π. Note that $L(s, \chi, \mathrm{Ad}) = L(s, \omega)$.

The appearance in (2) of the Vogan L-packet gives some philosophical justification for the appearance of the sum in the spectral identity 1.2. In practice, it arises from a consideration of the orbit space appearing in the trace formula.

A rough explanation of our main theorem behind us, let us explain what is not shown in this thesis. While the trace formula identity we describe above is suggestive, it alone does not re-prove Waldspurger's formula. To do so, the following steps still must be carried out:

1. The trace formula distributions $J(f)$ and $I(f' \otimes \Phi \otimes \Psi)$ should be upgraded from the "simple trace formula" identities presented below to full distributions. In practice, this means that in the future we will have to confront the full spectral expansion of both trace formulas. We will also have to deal more seriously with convergence issues on the geometric sides of both trace formulas as well.

2. The simple fundamental lemma of Chapter 3 should be generalized to a fundamental lemma for arbitrary elements of the spherical Hecke algebra, rather than just for the unit elements. This will allow for us to spectrally isolate individual global spherical characters in both trace formulas.

3. Appropriate local spherical characters should be defined, and their basic properties analyzed. In addition, the factorization of the global spherical character into a product of local spherical characters must be explicated, with all constants computed carefully.

There is thus still much to be done. However, we still believe that, as a proof of concept, the simple trace formula comparison of this thesis remains convincing evidence for our approach.

Outline

We proceed with a brief tour of this thesis and a sketch of the proof of Theorem 1. At first glance, the method appears familiar.

We proceed by defining and comparing two relative trace formula distributions. In Chapter 2, we define the $SO_W \times SO_V$ trace formula distribution. We begin by reviewing the notion of pure inner forms of orthogonal spaces, and then define the period integral $\mathscr{P}$ and the associated global spherical character J_π. We then define the trace formula $J(f)$: this is the relative trace formula associated to the quotient space

$$\Delta SO_W \backslash (SO_W \times SO_V)/\Delta SO_W \xrightarrow{\sim} SO_V/^{conj} SO_W$$

where, on the right, the notation denotes that action of SO_W on SO_V by conjugation. $J(f)$ formally expands in two ways, either spectrally as a sum of global spherical characters or geometrically as a sum over orbital integrals, indexed by orbits of $SO_V/^{conj} SO_W$. To better facilitate our understanding of this geometric side, we explicitly parameterize this quotient space. As an interesting feature, this quotient space is *singular,* and moreover the space of "bad"[1] orbits does not intersect the singular set; we thus have to work, in an ad hoc manner, with this singularity.

In Chapter 3 we define the $GL_2 \times GL_2$ trace formula. This is more complicated than on the orthogonal side. To define it we utilize a number of auxillary period integrals. Two of these, the symmetric square and exterior square period $\mathscr{P}_{\mathrm{Sym}^2}$ and $\mathscr{P}_{\wedge^2}$, are used to detect functorial liftings from (relevant) pure inner forms $SO_{W'} \times SO_{V'}$ of $SO_W \times SO_V$. We also transcribe the theory of Rankin-Selberg convolutions on $GL_2 \times GL_2$ and denote by $\mathscr{P}_{RS}$ the usual Rankin-Selberg integral. These ingredients in hand, we define $I(f' \otimes \Phi \otimes \Psi)$. In the notation above, it corresponds roughly to

$$(\Delta PGL_2, E(\cdot, s; \Phi)) \backslash ((GL_2 \times GL_2)/(Z \times Z, \omega \otimes 1))/((PGL_2^{(2)} \times N), \overline{\Theta(\cdot, \Psi)} \otimes \psi^{-1})$$

and involves integrating the usual kernel function next to an Eisenstein series, a degenerate theta function, and an additive character. Miraculously, this distribution decomposes into a sum of factorizable pieces, which we call the "orbital" integrals. They *do not* correspond to the action of a group on a variety in the same sort of sense as the orbits of SO_W acting on SO_V. We find the mysterious appearance of these "orbits" to be the most intriguing aspect of this approach.

In Chapter 4, we embark on the comparison of the two trace formulas. First, we match the regular semisimple orbits of $SO_V/^{conj} SO_W$ to the "regular semisimple orbits" appearing in Chapter 3. This is done using the explicit parametrization of orbits carried out in Chapter 2 and the parametrization of "orbits" appearing in Chapter 3. We then formulate and prove two local identities between the two types of orbital integrals: a smooth transfer conjecture (matching of smooth

[1] i.e. non-regular semisimple

functions) and a fundamental lemma type conjecture. Both are straightforward to prove and essentially follow from a direct computation. Amusingly, the proofs of both of these statements rely on an extremely tight analogy between the asymptotics of Whittaker functions on GL_2 and the asymptotic behavior of certain toy orbital integrals. Finally, we explain how these local identities give rise to an equality of spectral sides of our two trace formulas–our main Theorem 1.2.

Part I

Global Motivations

This first part is largely concerned with two relative trace formula (RTF). One, the RTF on the orthogonal groups $SO_W \times SO_V$, is designed to encode the period side of Waldspurger's formula. The other, which lives on the product of two general linear groups $GL_2 \times GL_2$, is designed to encode the Rankin-Selberg convolution L-function for cuspidal automorphic representations Π_χ and Π_σ which are functorial transfers from orthogonal groups of the appropriate dimensions and discriminants. It encode the L-function side of the Waldspurger formula.

These trace formulas serve as the raison d'être for the local results which form the technical heart of this thesis. We therefore go over them in some detail, outlining their appearance as spectral objects and their decomposition as "'geometric" objects. For $SO_W \times SO_V$ this is standard; for the $GL_2 \times GL_2$ trace formula, this is a little more exotic. We begin with the more familiar case.

Chapter 2

The Relative Trace Formula for $SO_2 \times SO_3$

In this first part, we write down a relative trace formula for $SO_W \times SO_V$ that encodes the data of the Waldspurger-Gross-Prasad period. To make our exposition clear, after some background on orthogonal groups and their pure inner forms, we first review the definition of this period as given in the introduction and introduce a global object, the $SO_W \times SO_V$ global spherical character. We then write down the trace formula distribution and expand it, thus obtaining as usual an equality between a spectral part (a sum of global spherical characters) and a geometric part (a sum over orbital integrals). The precise assumptions needed to work with a simple form of the trace formula are detailed in Section 2.5.

Let us start by reviewing the notation and setup described in the introduction. Throughout, F will be a field, usually either local or global, of characteristic not equal to 2. When F is global we denote its ring of adeles $\mathbb{A} := \mathbb{A}_F$. Let U be an orthogonal space over F of dimension n with quadratic form Q. Recall that the (normalized) discriminant of U is defined as

$$d_U = d_{(U,Q)} = (-1)^r \det Q \in F^\times/(F^\times)^2$$

where $r = \lfloor \frac{n}{2} \rfloor$.

Given an orthogonal space U, let as usual SO_U denote the special orthogonal group attached to U, i.e.

$$SO_U = \{g \in GL_U | {}^t gQg = Q \text{ and } \det g = 1\}$$

This is of course a reductive group over F.

In this paper, we are particularly concerned with orthogonal spaces of dimension 2 and 3. We will reserve the notation W for a quadratic space of dimension 2, and V for one of dimension 3. Recall that in these low rank cases, we can explicitly classify all quadratic spaces in terms of other familiar objects.[1]

Proposition 2.0.1. *All quadratic spaces of dimensions* 2 *and* 3 *over* F *are of the following form:*

1. *If* (W, Q) *is dimension* 2*, then*
$$(W, Q) \cong (E, \varepsilon \, \mathrm{N})$$
where E/F *is a (possibly split) quadratic etale algebra,* N *is the norm map from* E *to* F*, and* $\varepsilon \in F^\times / \mathrm{N}\, E^\times$*. The extension* E *is given by* $E = F[x]/(x^2 - d_W)$*.*

2. *If* (V, Q) *is dimension* 3*, then*
$$(V, Q) \cong (B^{\mathrm{Tr}=0}, -d \, \mathrm{N})$$
where B/F *is a (possibly split) quaternion algebra,* $B^{\mathrm{Tr}=0}$ *is the space of trace zero elements of* B*,* N *is the reduced norm map from* B *to* F*, and* $d \in F^\times/(F^\times)^2$*.* d *is given simply by* $d = d_V$*.*

Remark 2.0.2. The second part of the above proposition is often cited as saying that every 3 dimensional quadratic space of discriminant -1 is of the form $(B^{\mathrm{Tr}=0}, \mathrm{N})$. It is convenient to note that given any 3 dimensional quadratic space we can scale the quadratic form to produce a form with discriminant -1. In practice, this means that when dealing with the group SO_V, there is no loss of generality in assuming that V satisfy $d_V = -1$; since $SO_{(V,Q)} = SO_{(V,\varepsilon Q)}$ for any $\varepsilon \in F^\times$ while $d_{(V,\varepsilon Q)} = \varepsilon^3 d_{(V,W)} = \varepsilon d_{(V,Q)}$, we can always assume $d_V = -1$ and get the same group.

[1] This proposition explains the usual formulation of Waldspurger's formula in terms of toric periods on forms of GL_2 or PGL_2. Since we know that $V = (V, Q) = (B^{\mathrm{Tr}=0}, -d_V \, \mathrm{N})$, we can identify $SO_V \cong PB^\times$ acting by conjugation. If we fix $e \in B^{\mathrm{Tr}=0}$ an anisotropic vector and let $W = (e)^\perp$, then letting $E = F(e) \subset B$, we find, picking an element $w \in W \subset B^{\mathrm{Tr}=0}$, that $(W, Q) \cong (E, \mathrm{N}(w)\,\mathrm{N})$. Thus $SO_W \hookrightarrow SO_V$ becomes $E^\times/F^\times \hookrightarrow PB^\times$. We will revisit this isomorphism a little later, in slightly different language, when we rephrase everything in terms of the groups Spin_V and GSpin_V.

2.1 Pure Inner Forms of $SO_W \times SO_V$

Recall that, given a reductive group G over a field F, a *pure inner form* of G is a group G' which is an inner form of G, together with the data of a lifting of the corresponding class $[G']$ in $H^1(F, G/Z)$ to $H^1(F, G)$. In this section, we identify the *relevant* pure inner forms of the group $G = SO_W \times SO_V$ when the base field F is a local or global field. This description will be used in the next section.

2.1.1 Pure Inner Forms for Special Orthogonal Groups and Relevant Pairs

The following interpretation of Galois cohomology for orthogonal groups over a field F is well known. Let U be a non-degenerate quadratic space. Then

Proposition 2.1.1. *The classes in $H^1(F, O_U)$ correspond bijectively to isomorphism classes of non-degenerate quadratic spaces U' which satisfy $\dim U = \dim U'$. The classes of $H^1(F, SO_U)$ correspond to isomorphism classes of non-degenerate quadratic spaces U' which satisfy $\dim U = \dim U'$ and $d(U) = d(U')$.*

Thus pure inner forms of a special orthogonal group SO_U correspond exactly to isomorphism classes of quadratic spaces U' of the same dimension and discriminant.

Let us proceed to the problem of *relevance*. We define this as follows: let U and T be orthogonal spaces over a field F, with $\dim T = \dim U + 1$. We say that the pair (U, T) is *relevant* if $U \hookrightarrow T$ embeds as the orthogonal complement of an anisotropic vector $e \in T$. The following lemma from [Gross and Prasad, 1994] is well known. It gives a canonical base point for the set of pure inner forms of the group $SO_U \times SO_T$, when $U \hookrightarrow T$ is a relevant pair.

Lemma 2.1.2. *There is a unique quasi-split relevant pair $(U_{q.s.}, T_{q.s.})$ with $U_{q.s.}$ and $T_{q.s.}$ pure inner forms of SO_U and SO_T.*

Proof. See [Gross and Prasad, 1994]. □

The crucial term in this lemma is the word quasi-split. Recall that we say an even dimensional orthogonal space U is *quasi-split* if U contains an isotropic subspace of dimension $\frac{\dim U}{2} - 1$. If it contains an isotropic subspace of maximal dimension $\frac{\dim U}{2}$, then we say the even dimensional U is *split*. If U is odd dimensional, then we say U is *split* if it contains an isotropic subspace of maximal dimension $\lfloor \frac{\dim U}{2} \rfloor$.

Remark 2.1.3. There are many quasi-split pure inner forms of an orthogonal space U. For instance, when U is 2-dimensional, all pure inner forms of U are quasi-split. The condition of relevance, however, singles out a particular pair among the quasi-split pure inner forms of (U, T).

2.1.2 Relevant Pure Inner forms of $SO_2 \times SO_3$

In any case, let us now turn to the specific case at hand. Recall by Proposition 2.0.1 that over any field F, every two dimensional quadratic space is of the form

$$(W, Q) = (E, \varepsilon\,\mathrm{N})$$

for $\varepsilon \in F^\times/NE^\times$, where $E = F[x]/(x^2 - d_W)$ is the discriminant algebra of W. If we wish to emphasize the dependency on ε, then we write $W = W(\varepsilon)$. Similarly, every three dimensional quadratic space is of the form

$$(V, Q) = (B^{\mathrm{Tr}=0}, -d\,\mathrm{N})$$

where B is a quaternion algebra over F and $d = d_V$. By this, or by the exceptional isomorphism $SO_W \cong U_1$, we compute that

$$H^1(F, SO_W) \cong F^\times/\,\mathrm{N}\,E^\times.$$

So let $W \hookrightarrow V$ be a relevant pair of quadratic spaces of dimensions 2 and 3. Since the set of inner forms of $SO_W \times SO_V$ is the same as that of $SO_{W_0} \times SO_{V_0}$, where $W_0 \hookrightarrow V_0$ is the quasi-split relevant inner form, we can assume that $W \hookrightarrow V$ is quasisplit–note that this is no condition on the two dimensional W, hence really only means that V has an isotropic vector.

Note that, after fixing discriminants d_W and d_V, the relevant quasisplit pure inner form $(W_{q.s.}, V_{q.s.})$ corresponds to $\varepsilon = -\frac{d_W}{d_V}$ (which is the same class, modulo $\mathrm{N}\,E^\times$, as d_V), i.e.

$$W_{q.s.} \cong (E, -\frac{d_W}{d_V}\,\mathrm{N}) = W(-\frac{d_W}{d_V})$$

$$V_{q.s.} \cong (E \oplus Fe, -\frac{d_W}{d_V}\,\mathrm{N} \oplus \left\langle \frac{d_V}{d_W} \right\rangle).$$

Assign, to each pure inner form $W(\varepsilon) = (E, \varepsilon\,\mathrm{N})$ of SO_W a class in $F^\times/\,\mathrm{N}\,E^\times$ by

$$W(\varepsilon) \mapsto -\frac{d_V}{d_W}\varepsilon.$$

This describes a bijection (not necessarily a group homomorphism, although $H^1(F, SO_W)$ does have a natural group structure and the two *are* isomorphic as groups)

$$H^1(F, SO_W) \xrightarrow{\sim} F^\times / \operatorname{N} E^\times.$$

We use this map to parameterize all relevant pure inner forms of $W_{q.s.} \hookrightarrow V_{q.s.}$.

Proposition 2.1.4. *Let F be field, and let $W_{q.s.} \hookrightarrow V_{q.s.}$ be a quasi-split relevant pair of quadratic spaces of discriminants d_W and d_V respectively. Then the map*

$$\{\textit{relevant pure inner forms } W \hookrightarrow V \textit{ of } W_{q.s.} \hookrightarrow V_{q.s.}\} \to F^\times / \operatorname{N} E^\times$$

$$(W \hookrightarrow V) \mapsto -\frac{d_V}{d_W}\varepsilon$$

where $W = W(\varepsilon) = (E, \varepsilon \operatorname{N})$, is a bijection.

Proof. There is very little that needs to be said. Fix discriminants d_W and d_V, and consider the quasi-split relevant pair $W_{q.s.} \hookrightarrow V_{q.s.}$ corresponding to these discriminants. Any pure inner form W of $W_{q.s.}$ gives rise to a relevant pair $W \hookrightarrow V$ by simply setting $V = (W \oplus Fe, Q_W \oplus \left\langle \frac{d_V}{d_W} \right\rangle)$. Thus we have a map $F^\times / \operatorname{N} E^\times \to \{$relevant pure inner forms of $W_{q.s.} \hookrightarrow V_{q.s.}\}$ given by $\epsilon \mapsto (W = W(-\frac{d_W}{d_V}\varepsilon)) \mapsto (W \hookrightarrow V)$. It is clearly injective and surjective. □

Remark 2.1.5. In the future, we will occasionally refer to the relevant pure inner form of a given quasi-split $W_{q.s.} \hookrightarrow V_{q.s.}$ associated to $\varepsilon \in F^\times / \operatorname{N} E^\times$; this is the pair $W_\varepsilon \hookrightarrow V_\varepsilon$, where

$$W_\varepsilon = W(-\frac{d_W}{d_V}\varepsilon).$$

Observe that under this notation, the quasisplit inner form is written as $W_1 \hookrightarrow V_1$ rather than $W_{q.s.} \hookrightarrow V_{q.s.}$ (that this is so becomes apparent if we look at the Clifford algebra; see Section 2.4.1). We hope this does not cause any confusion.

This preliminary material on orthogonal groups behind us, let us now temporarily forget the exceptional isomorphisms of Proposition 2.0.1 and instead move on to defining the period integral.

2.2 The ΔSO_W Period

Now let us assume that F is a global field. Let $G = SO_W \times SO_V$, and let $H = \Delta SO_W = \{(h, \iota(h)) : h \in SO_W\}$ be the diagonally embedded subgroup SO_W as above. . Let $\pi = \chi \otimes \sigma$ be an automorphic cuspidal representation of G, i.e. let χ be an automorphic character $\chi : SO_W(F)\backslash SO_W(\mathbb{A}) \to \mathbb{C}^\times$ and σan automorphic cuspidal representation of SO_V.

We must consider the period integral of Waldspurger-Gross-Prasad applied to forms in π. This is defined simply by

$$\begin{aligned} \mathscr{P} : \pi = \chi \otimes \sigma &\to \mathbb{C} \\ \chi \otimes \phi &\mapsto \int_{[SO_W]} \chi(h)\phi(h)dh. \end{aligned}$$

Here $\phi \in \sigma$ and dh is the Tamagawa measure on $SO_W(\mathbb{A})$. The notation $\phi \in \sigma$ is unambiguous, since by the multiplicity one theorem for $SO_V \cong PB^\times$ there is only one realization of σ in the space of automorphic forms.

The period integral $\mathscr{P}(\phi)$ is absolutely convergent when π is cuspidal. If π is not cuspidal, then the period integral must be regularized appropriately; we will avoid this issue in this paper by dealing with a simple version of the relative trace formula that eliminates any non-cuspidal contribution on the spectral side.

2.2.1 The $SO_W \times SO_V$ Global Spherical Character

Let π be an automorphic cuspidal representation of $G = SO_W \times SO_V$. We define the *global spherical character* by taking for $f = f_1 \otimes f_2 \in C_c^\infty(G(\mathbb{A}))$

$$J_\pi(f) = \sum_{\varphi \in \mathrm{ON}(\pi)} \mathscr{P}(\pi(f)\varphi)\overline{\mathscr{P}(\varphi).}$$

Here the sum runs over a fixed orthonormal basis of $\mathrm{ON}(\pi) \subset \pi \subset L^2([G])$, and the operator $\pi(f)$ is defined by

$$\pi(f)\varphi = \int_{G(\mathbb{A})} f(g)\pi(g)\varphi dg.$$

J_π is visibly a distribution of positive type, in the following sense: if $f = F * F^\vee$, where $F^V(g) = \overline{F(g^{-1})}$ then

$$J_\pi(f) \geq 0.$$

Moreover, the following lemma is easily apparent:

Lemma 2.2.1. *Let $\mathscr{P} = \mathscr{P}|_\pi$ denote the period integral, viewed as an element in* $\mathrm{Hom}_{H(\mathbb{A})}(\pi, \mathbb{C})$. *Then $\mathscr{P} = 0$ if and only if $J_\pi = 0$.*

It is in this sense that we say the distribution J_π encodes the period integral.

2.3 The Trace Formula, Formally

2.3.1 The $SO_W \times SO_V$ Distribution

We can now set up the trace formula identity which will be the main result of this part of the thesis. For the moment, all computations will be formal. In the statement of Theorem 2.5.1 we will be more careful, making an assumption on our test data f to ensure that all integrals and sums are absolutely convergent.

Let $f \in C_c^\infty(G(\mathbb{A}))$ be a test function given by $f = f_W \otimes f_V$ with $f_W \in C_c^\infty(SO_W(\mathbb{A}))$ and $f_V \in C_c^\infty(SO_V(\mathbb{A}))$, let $x = (x_1, x_2), y = (y_1, y_2)$ be elements of $G(\mathbb{A}) = SO_W(\mathbb{A}) \times SO_V(\mathbb{A})$, and consider the usual kernel function

$$K_f(x,y) = \sum_{\delta \in G(F)} f(x^{-1}\delta y) = \sum_{\delta_1 \in SO_W(F)} f_W(x_1^{-1}\delta_1 y_1) \sum_{\delta_2 \in SO_V(F)} f_V(x_2^{-1}\delta_2 y_2).$$

It is easy to see that this kernel function is formally represented by the sum

$$K_f(x,y) = \sum_{\varphi \in \mathrm{ON}(L^2([G]))} (R(f)\varphi)(x)\overline{\varphi(y)}.$$

Here $\mathrm{ON}(L^2([G])$ denotes a fixed orthonormal basis of automorphic forms of the L^2 space of $G(F)\backslash G(\mathbb{A})$ and $R(f)\varphi$ is the right regular action of Hecke operators, namely

$$R(f)\varphi(x) = \int_{G(\mathbb{A})} f(y)\varphi(xy)dy.$$

Our main player in this part is the following:

Definition 2.3.1. The $SO_W \times SO_V$*-distribution* $J(f)$ is defined formally as

$$J(f) = \int_{[SO_W]} \int_{[SO_W]} K_f((h_1,h_1),(h_2,h_2))dh_1dh_2.$$

The trace formula identity we are after follows after expanding this distribution $J(f)$ in two ways: spectrally and geometrically.

The spectral expansion provides the justification for why we would consider $J(f)$ at all. Formally, it works as follows: if we write $L^2([G]) = \bigoplus \pi$, where the "sum" is over all automorphic representations of G, then

$$\begin{aligned} K_f(x,y) &= \sum_{\varphi \in \mathrm{ON}(L^2([G]))} (R(f)\varphi)(x)\overline{\varphi(y)} \\ &= \sum_{\pi} \sum_{\varphi \in \mathrm{ON}(\pi)} (\pi(f)\varphi)(x)\overline{\varphi(y)} \end{aligned}$$

and

$$\begin{aligned} J(f) &= \int_{[SO_W]} \int_{[SO_W]} K_f((h_1, \iota(h_1)), (h_2, \iota(h_2))) dh_1 dh_2 \\ &= \sum_{\pi} \sum_{\varphi \in \mathrm{ON}(\pi)} \mathscr{P}(\pi(f)\varphi)\overline{\mathscr{P}(\varphi)} \\ &= \sum_{\pi} J_\pi(f). \end{aligned}$$

Thus $J(f)$ allows us to understand the sum over all global spherical characters. Of course, to justify this sort of spectral expansion requires some work in general. We avoid this by dealing with a simple form of the trace formula. This is done in Section 2.5.

2.3.2 The Geometric Side

We turn to the geometric expansion of $J(f)$ as a sum of orbital integrals. These can be defined purely locally.

For the moment let F denote a local field. Let $W \hookrightarrow V$ be as above, and let $F \in \mathscr{S}(SO_V(F))$. We hope the appearance of two F's does not cause much confusion!

Definition 2.3.2. The *local $SO_W \times SO_V$ orbital integral* corresponding to an element $\gamma \in SO_V(F)$ is defined formally by

$$\mathrm{Orb}'_{SO_W \times SO_V}(\gamma; F) := \int_{\mathrm{Stab}_\gamma(F) \backslash SO_W(F)} F(h^{-1}\gamma h) dh$$

This integral may not converge for some choices of γ. Note too that $\mathrm{Orb}'_{SO_W \times SO_V}(\gamma; f)$ only depends on the orbit of γ under the conjugation action of $SO_W(F)$. We often suppress the subscript $SO_W \times SO_V$ when discussing this integral.

When the stabilizer Stab_γ of γ in SO_W is finite, we write the unadorned

$$\mathrm{Orb}_{SO_W \times SO_V}(\gamma; F) = \int_{SO_W(F)} F(h^{-1}\gamma h) dh$$

for the naive orbital integral.

We record for later use the following very easy fact.

Lemma 2.3.3. *If γ in SO_V has finite stabilizer and Zariski closed orbit (i.e. is regular semisimple in the sense of Section 2.4.4), then*

$$\mathrm{Orb}(\gamma; f) = \int_{SO_W(F)} F(h^{-1}\gamma h) dh$$

converges absolutely.

Proof. If γ has trivial stabilizer and closed orbit, then the integral above runs over a closed subset of $SO_V(F)$. Since F is Schwartz, this integral converges absolutely. The same holds if γ has a finite stabilizer as well. $\square$

We can also easily define, for F a global field, $\gamma \in SO_V(F)$, and $F = \otimes F_v \in \mathscr{S}(SO_V(\mathbb{A}))$ a factorizable function, the *global orbital integral* by setting

$$\begin{aligned} \mathrm{Orb}'_{SO_W \times SO_V}(\gamma; F) := & \int_{\mathrm{Stab}_\gamma(\mathbb{A})\backslash SO_W(\mathbb{A})} F(h^{-1}\gamma h)dh \\ = & \prod_v \mathrm{Orb}'_{SO_W \times SO_V}(\gamma; F_v). \end{aligned}$$

Now, let F be a local field. For later use, we also need to define a map

$$\begin{aligned} S : \mathscr{S}(SO_W(F) \times SO_V(F)) & \to \mathscr{S}(SO_V(F)) \\ f_W \otimes f_V & \mapsto (g \mapsto \int_{SO_W(F)} f_W(h) f_V(gh)dh). \end{aligned}$$

Lemma 2.3.4. *The map S is surjective.*

Proof. This follows from the Dixmier-Malliavin lemma; see Proposition 4.2.15 later in this thesis. It can also be seen directly in this case of SO_W acting on SO_V–we omit the details. $\square$

We return to discussing the trace formula distribution $J(f)$.

Proposition 2.3.5 ($SO_W \times SO_V$ Trace Formula, Geometric Side)**.** *The distribution J formally decomposes as*

$$J(f) = \sum_{\gamma \in SO_V(F)/^{conj} SO_W(F)} \mathrm{vol}([\mathrm{Stab}_\gamma(F)])\mathrm{Orb}'(\gamma; f)$$

Proof. This is nothing more than the double coset computation

$$\begin{aligned} \Delta SO_W \backslash (SO_W \times SO_V)/\Delta SO_W & \xrightarrow{\sim} SO_V/^{conj} SO_W \\ (h, g) & \mapsto h^{-1}g \end{aligned}$$

We explain. Write

$$\begin{aligned} J(f) & = \int_{[SO_W]} \int_{[SO_W]} K_f((h_1, h_1), (h_2, h_2)) dh_1 dh_2 \\ & = \int_{[SO_W]} \int_{[SO_W]} \sum_{\gamma_1 \in SO_W(F)} f_W(h_1^{-1}\gamma_1 h_2) \sum_{\gamma_2 \in SO_V(F)} f_V(h_1^{-1}\gamma_2 h_2) dh_1 dh_2. \end{aligned}$$

We change variables in the second sum, $\gamma_2 \mapsto \gamma_1\gamma_2$ and unfold in γ_1, which yields

$$J(f) = \int_{[SO_W]} \int_{SO_W(\mathbb{A})} f_W(h_1^{-1}h_2) \sum_{\gamma_2 \in SO_V(F)} f_V(h_1^{-1}\gamma_2 h_2) dh_1 dh_2$$

Now, change variables $h_2 \mapsto h_1 h_2$ to find

$$J(f) = \sum_{\gamma_2 \in SO_V(F)} \int_{[SO_W]} (\int_{SO_W(\mathbb{A})} f_W(h_2) f_V(h_1^{-1}\gamma_2 h_1 h_2) dh_2) dh_1$$

and consider the function on $SO_V(\mathbb{A})$ given by

$$F(g) = S(f_W \otimes f_V).$$

Then we have

$$\begin{aligned} J(f) &= \sum_{\gamma \in SO_V(F)} \int_{[SO_W]} F(h^{-1}\gamma h) dh \\ &= \sum_{\gamma \in SO_V(F)/^{conj} SO_W(F)} \sum_{\delta \in \mathrm{Stab}_\gamma(F) \backslash SO_W(F)} \int_{[SO_W]} F(h^{-1}\delta^{-1}\gamma\delta h) dh \\ &= \sum_{\gamma \in SO_V(F)/^{conj} SO_W(F)} \mathrm{vol}([\mathrm{Stab}_\gamma]) \int_{\mathrm{Stab}_\gamma(\mathbb{A}) \backslash SO_W(\mathbb{A})} F(h^{-1}\gamma h) dh \end{aligned}$$

as desired. □

2.4 Classification of Orbits

The orbits appearing in the geometric side of the trace formula above are given by $SO_W(F)$ acting on $SO_V(F)$ by conjugation. In this section we compute the quotient $SO_V/^{conj}SO_W$ - in both the naive sense of points $SO_V(F)/^{conj}SO_W(F)$ and in the more algebraic sense of GIT quotient $SO_V /\!\!/^{conj} SO_W$ - and identify a good open subset of this quotient, the regular semisimple locus. To do so, we essentially "cheat" and appeal to the exceptional isomorphism $SO_V \cong PB^\times$ and $SO_W \cong E^\times/F^\times$ rather than compute "natively on orthogonal groups." These exceptional isomorphisms are packaged very nicely via well-known machinery from linear algebra, which we now recall.

2.4.1 Covers of Special Orthogonal Groups

Let $C(V)$ be the Clifford algebra of $V = (V, Q)$, i.e.

$$C(V) = TV/I_Q$$

where $TV = \bigoplus_{n\geq 0} V^{\otimes n}$ is the tensor algebra of V and I_Q is the two-sided ideal generated by all elements of the form $v \otimes v - Q(v)$. $C(V)$ is a $\mathbb{Z}/2\mathbb{Z}$-graded algebra, with even part denoted by $C_0(V)$ and odd part denoted by $C_1(V)$; this means

$$C(V) = C_0(V) \oplus C_1(V)$$

with multiplication satisfying

$$C_i(V)C_j(V) \subset C_{i+j}(V)$$

where $i + j$ is considered module 2. We define the algebraic group GSpin_V by

$$\mathrm{GSpin}_V = \{x \in C_0(V)^\times : xVx^{-1} = V\}$$

where the equality of spaces takes place in $C(V)$. It is well known (and easy enough to show) that this fits into an exact sequence

$$1 \to \mathbb{G}_m \to \mathrm{GSpin}_V \xrightarrow{\rho} SO_V \to 1$$

where $\rho(x)v = xvx^{-1}$. The group GSpin_V carries a norm map $\mathrm{N} : \mathrm{GSpin}_V \to \mathbb{G}_m$, and we can define the simply connected group Spin_V as the subgroup of GSpin_V of norm 1; it fits into the exact sequence

$$1 \to \mu_2 \to \mathrm{Spin}_V \to SO_V \to 1.$$

In our setting, where V is a 3-dimensional non-degenerate quadratic space, it follows from simple computation that $C_0(V)$ is a quaternion algebra B over F. It is also easy to see that GSpin_V is nothing more than $C_0(V)^\times$ and that the norm N is nothing more than the reduced norm on B; thus, this gives another explanation of the exceptional isomorphism $SO(V) = PB^\times$ and identifies Spin_V with B^1. It is important to note that these exact sequences are as group schemes, or as Galois modules; they give rise to the exact sequences on F-points

$$1 \to F^\times \to \mathrm{GSpin}_V(F) \to SO_V(F) \to 1$$

by Hilbert's Theorem 90, and

$$1 \to \{\pm 1\} \to \mathrm{Spin}_V(F) \to SO_V(F) \to F^\times/(F^\times)^2 \to 1. \tag{2.1}$$

We can similarly define the Clifford algebra $C(W)$ and its even subalgebra $C_0(W)$. Again, it is easy to see that as W is 2-dimensional and non-degenerate, $C_0(W)$ is isomorphic to the quadratic ï¿œetale algebra E/F defined by $E = F[t]/(t^2 - d_W)$ and that $\mathrm{GSpin}_W = C_0(W)^\times = E^\times$. We have an inclusion

$$C_0(W) \hookrightarrow C_0(V)$$

In fact, we can write $C_0(V)$ in terms of $C_0(W)$. Since $V = W \oplus Fe$ as an orthogonal direct sum, it follows that

$$C(V) = C(W)\hat{\otimes}C(Fe)$$

where $\hat{\otimes}$ is $\mathbb{Z}/2\mathbb{Z}$-graded tensor product (super tensor product). From this, some small computation reveals

$$C_0(V) = C_0(W) + We$$

This follows from that fact that $\dim W = 2$, so $C_1(W) = W$. Note that $w \in W$ anti-commutes with e.

2.4.2 The GIT Quotient $\mathcal{B} = SO_V /\!\!/^{conj} SO_W$

To determine the GIT quotient $SO_V /\!\!/^{conj} SO_W$ we first determine the ring of invariant polynomials on GSpin_V with regards to the conjugation action of SO_W; we then consider which among these are invariant under multiplication by the central $\mathbb{G}_m$.

Lemma 2.4.1. *The GIT quotient $Q' := GSpin_V /\!\!/^{conj} SO_W$ is the subvariety $Q' \subset (\mathrm{Res}_{E/F}\,\mathbb{G}_a)\times\mathbb{G}_a$ defined by $Q' = \{(z, b) : \mathrm{N}\,z - Q(e)b \neq 0\}$.*

Proof. Given $x = z + we \in C_0(V)^\times = \mathrm{GSpin}_V = B^\times$, we can compute hxh^{-1}, with $h \in C_0(W)^\times = \mathrm{GSpin}_W = E^\times$. This gives

$$hxh^{-1} = z + \rho(h)(w)e$$

where $\rho(h)(w)$ denotes the action of $h \in \mathrm{GSpin}_W$ on W. Thus, given $x \in \mathrm{GSpin}_V$, we set $z = z(x) = z$ and $b = b(x) = Q(w)$. It is clear that these give rise to all invariant functions. □

We can similarly conclude:

Lemma 2.4.2. *The GIT quotient $Q'' := Spin_V /\!\!/^{conj} SO_W$ satisfies $Q'' \cong \mathrm{Res}_{E/F}\,\mathbb{G}_a$.*

We can now descend from GSpin_V or Spin_V down to SO_V. We again phrase things in terms of the GIT quotient, i.e. in terms of functions on SO_V which are invariant on SO_W.

Proposition 2.4.3. *The GIT quotient*

$$\mathcal{B} := SO_V /\!\!/^{conj} SO_W$$

is isomorphic to the cone

$$\begin{aligned} \mathcal{B} &\cong \mu_2 \backslash\!\!\backslash (\operatorname{Res}_{E/F} \mathbb{G}_a) \\ &\cong \operatorname{Spec}(F[A,B,C]/(B^2 - AC)). \end{aligned}$$

Proof. By Lemma 2.4.2, the GIT quotient of Spin_V under the conjugation action of SO_W is given by $\operatorname{Res}_{E/F} \mathbb{G}_a$. To descend down to $SO_V /\!\!/^{conj} SO_W$ we need to quotient by μ_2. This quotient is easily identified: on $\operatorname{Res}_{E/F} \mathbb{G}_a$, the functions of "real and imaginary parts"

$$\begin{aligned} a(z) &= \frac{1}{2}(z + \bar{z}) \\ c(z) &= \frac{1}{2}(z - \bar{z}) \end{aligned}$$

independently generate the ring of functions; the ring of functions invariant by μ_2 is spanned by

$$\begin{aligned} A(z) &= a(z)^2 \\ B(z) &= a(z)c(z) \\ C(z) &= c(z)^2 \end{aligned}$$

which obviously satisfy the equation $B^2 = AC$.

When working with points over a field, it is slightly more appealing to work with GSpin_V, since

$$\mathrm{GSpin}_V(F) \twoheadrightarrow SO_V(F)$$

while the same is not true for Spin_V. Let us then also show characterization by descending from GSpin_V. By 2.4.1, the ring of functions on GSpin_V invariant under conjugation by SO_W are given by, if $x = z + ew \in C_0(V)$,

$$\begin{aligned} a(x) &= \frac{1}{2}(z + \bar{z}) \\ c(x) &= \frac{1}{2}(z - \bar{z}) \\ q(x) &= q(w) \\ f(x) &= \frac{1}{\mathrm{N}(x)} \end{aligned}$$

subject to the relation

$$f(x)(a(x)^2 - d_W c(x)^2 + Q(e)q(x)) = 1.$$

We can compute the subring of functions also invariant under the action of $\mathbb{G}_m$ on GSpin_V; it is easy to see that this is the subring generated by

$$\begin{aligned} A(x) &= f(x)a(x)^2 = \frac{a(x)^2}{\mathrm{N}(x)} \\ B(x) &= f(x)a(x)c(x) = \frac{a(x)c(x)}{\mathrm{N}(x)} \\ C(x) &= f(x)c(x)^2 = \frac{c(x)^2}{\mathrm{N}(x)} \\ D(x) &= f(x)q(x) = \frac{Q(w)}{\mathrm{N}(x)} \end{aligned}$$

subject to the relations

$$\begin{aligned} B^2 &= AC \\ A - d_W C + Q(e)D &= 1. \end{aligned}$$

The claim follows. □

2.4.3 The Quotient Sets $SO_V(F)/^{conj}SO_W(F)$, B, and $\mathcal{B}(F)$

Let us now determine the quotient $SO_V(F)/^{conj}SO_W(F)$ in the naive sense. Because of the surjectivity of the projection $\mathrm{GSpin}_V(F) \to SO_V(F)$, which follows from Hilbert's Theorem 90, we can compute this quotient in a similar vein to how we computed the GIT quotient above. That is, it is easy to see that:

Proposition 2.4.4. *Fix a set of representatives $\Sigma \subset F^\times$ of the quotient $F^\times/(F^\times)^2$. The orbits of $SO_W(F)$ acting by conjugation on $SO_V(F)$ are given by the data of both*

1. *An element $\alpha \in \Sigma$.*

2. *A pair, $(b, z) \in F \times (E/\{\pm 1\})$ such that $\frac{\alpha b}{Q(e)} \in Q(W(F))$ and $\mathrm{N}\, z = \alpha(1 - b)$.*

Proof. Let x be an element of $\mathrm{GSpin}_V(F) = C_0(V(F))^\times$; we can write $x = z + we$ where $z \in E$ and $w \in W(F)$. The orbit of $x$under the conjugation action of $SO_W(F)$ is determined entirely by the data of z and $Q(w)$–given a different element $x' = z' + w'e$, x' and x lie in the same $SO_W(F)$-orbit if and only if $z = z'$ and $Q(w) = Q(w')$ by Witt's theorem.

To descend from $\mathrm{GSpin}_V(F)$ to $SO_V(F)$, we can scale x by a constant and to assume $\mathrm{N}\, x = \mathrm{N}\, z + Q(e)Q(w) = \alpha$ lies in our fixed set of representatives Σ; the choice of scaling constant is determined only up to ± 1. Set $b = \frac{Q(e)}{\alpha} Q(w)$. The proposition follows. □

It will be notationally useful to give the sets appearing above a name.

Definition 2.4.5. Let $\alpha \in \Sigma$ and let $W \subset V$ be as above. We denote by $B_{(W,V)}(\alpha)$ the set appearing above in Proposition 2.4.4:

$$B_{(W,V)}(\alpha) := \{(b, z) \in F \times (E/\{\pm 1\}) : \frac{\alpha b}{Q(e)} \in Q(W(F)) \text{ and } \mathrm{N}\, z = \alpha(1 - b)\}.$$

We can use the bijection of Section 1 to identify relevant pure inner forms of a fixed quasi-split pair (W_0, V_0) with elements of $F^\times / \mathrm{N}\, E^\times$; if (W, V) corresponds to $\varepsilon \in F^\times / \mathrm{N}\, E^\times$ then we also write

$$B_\varepsilon(\alpha) := B_{(W,V)}(\alpha).$$

We also denote by $B(\alpha)$

$$\begin{aligned} B(\alpha) &:= \{(b, z) \in F \times (E/\{\pm 1\}) : \mathrm{N}\, z = \alpha(1 - b)\} \\ &= \bigcup_{\varepsilon \in \mathrm{N}\, E^\times / F^\times} B_\varepsilon(\alpha) \end{aligned}$$

and set

$$B_\varepsilon := \coprod_{\alpha \in \Sigma} B_\varepsilon(\alpha)$$

and

$$B := \coprod_{\alpha \in \Sigma} B(\alpha).$$

Note in particular that all unions over $\alpha \in \Sigma$ are taken as disjoint unions, **not** as unions inside of an ambient $F \times (E/\{\pm 1\})$.

Remark 2.4.6. Note that $(\alpha; b, z) \in B(\alpha)$ is in $B_\varepsilon(\alpha)$ if and only if $b \in -\frac{\varepsilon}{\alpha} \mathrm{N}\, E$.

Remark 2.4.7. Note too that Proposition 2.4.4 identifies the naive quotient set as the disjoint union

$$\begin{aligned} SO_V(F)/^{conj} SO_W(F) &= \coprod_\alpha B_{(W,V)}(\alpha) \\ &= B_{(W,V)} \end{aligned}$$

Remark 2.4.8. In Chapter 4 of this thesis, we will explain a comparison between the set of (regular semisimple) orbits above and a set of (regular semisimple) "orbits" introduced in Chapter 3. For this comparison, and other computations in Chapter 4, it is expedient to work with the set B rather than $\mathcal{B}(F)$; these two are "birational," hence agree on a dense open subset (their loci of regular semisimple points), but for the study of orbital integrals we choose to work with B as an ambient space rather than $\mathcal{B}(F)$, as it more closely mimics the coordinate description for "orbits" on the other side of the comparison. Hopefully this explains our insistence on the notation introduced above.

We can relate the naive quotient set of Proposition 2.4.4 to the F-points of the GIT quotient described in Proposition 2.4.3. The following lemma is an easy computation.

Lemma 2.4.9. *There is a surjective map*

$$\coprod_{\varepsilon} SO_{V_\varepsilon}(F)/^{conj}SO_{W_\varepsilon}(F) \twoheadrightarrow \mathcal{B}(F).$$

Non-trivial fibers occur only over the loci $A - dB = 1$ and $A = B = 0$. Moreover, this map factors as

$$\begin{array}{ccc} \coprod_{\varepsilon} SO_{V_\varepsilon}(F)/^{conj}SO_{W_\varepsilon}(F) & \twoheadrightarrow & \mathcal{B}(F) \\ & \searrow\!\!\!\!\searrow & \nearrow\!\!\!\!\nearrow \\ & & B \end{array}$$

where the map to $B = \coprod B(\alpha)$ has non-trivial fibers only on the locus $b = 0$ within each $B(\alpha)$, while the map from B to $\mathcal{B}(F)$ has non-trivial fibers only when $z = 0$.

Thus B acts as an intermediate space between the actual orbit space $\coprod_{\varepsilon} SO_{V_\varepsilon}(F)/^{conj}SO_{W_\varepsilon}(F)$ and the true categorical quotient $\mathcal{B}(F)$. It is extremely convenient for the study of orbital integrals carried out in Chapter 4.

2.4.4 The Regular Semisimple Locus

To deal with these quotient spaces appropriately we must first identify a nice open subset of the space GSpin_V relative to the action of SO_W. The standard approach is to use the regular semisimple locus. We recall the definition below.

Definition 2.4.10. An element $g \in SO_V$ is *strictly regular semisimple* with respect to the conjugation action of SO_W if and only if it has trivial stabilizer in SO_W and Zariski closed orbit. We say an orbit in SO_V is strictly regular semisimple if any (hence all) elements in it are regular semisimple in the above sense. We also say that an element or orbit in GSpin_V is strictly regular semisimple if and only if its image in SO_V is.

We similarly say that $g \in SO_V$ is *regular semisimple* if it has minimal dimensional (hence finite) stabilizer and Zariski closed orbit. We denote by

$$SO_V(F)^{\text{s.r.s.s.}} \text{ and } SO_V(F)^{\text{r.s.s.}}$$

the loci of strict regular semisimple and regular semisimple elements respectively.

(Strict) regular semisimplicity in SO_V is easy to detect.

Proposition 2.4.11. *An element $z + we \in GSpin_V$ is strictly regular semisimple if and only if $Q(w) \neq 0$ and $z \neq 0$. It is regular semisimple if and only if $Q(w) \neq 0$.*

Proof. There are essentially two cases at play here: if $E = F(\sqrt{d_W})$ is a field extension, then $Q(w) = 0$ if and only if $w = 0$. Hence, if $Q(w) = 0$, then the stabilizer of $z + we = z$ is all of SO_W, and the element is not regular. On the other hand, if $E = F \times F$ is a split extension, then $Q(w) = 0$ does not imply that the $w = 0$. However, if we fix an isomorphism

$$\iota : (E, \varepsilon N) = (F \times F, (x, y) \mapsto \varepsilon xy) \xrightarrow{\sim} (W, Q)$$

then $Q(w) = 0$ means that w is either of the form $\iota((x, 0))$ or $\iota((0, y))$. A simple computation then shows that the orbit of of $z + we$ is not Zariski closed. It is also easy to see that if $z = 0$, then $\rho(x)$ has non-trivial stabilizer in SO_W. If we let h_0 be the element of SO_W which acts as -1 on W, then the stabilizer of $\rho(x) = \rho(we) = \{1, h_0\}$.

Conversely, it is clear that an element $z + we$ has non-trivial stabilizer in SO_W if and only if $Q(w) = 0$ or $z = 0$. It is also apparent that if $Q(w) \neq 0$, then the orbit of $z + we$ is Zariski closed. □

Thus, the locus in $\mathcal{B}(F)$ where the map of Lemma 2.4.9 has non trivial fibers is exactly the locus of invariants for those elements which are not strictly regular semisimple. That is, if we set

$$\mathcal{B}(F)^{\text{s.r.s.s.}} = \{(A, B, C) : AB = C^2, A - d_W B \neq 1 \text{ and } (A, B) \neq (0, 0)\}$$

and

$$B^{\text{s.r.s.s.}} = \{(\alpha; b, z) : b \neq 0, z \neq 0\}$$

then we have that

$$\coprod_{\varepsilon} SO_{V_\varepsilon}(F)^{\text{s.r.s.s.}} /^{conj} SO_{W_\varepsilon}(F) \cong B^{\text{s.r.s.s.}} \cong \mathcal{B}(F)^{\text{s.r.s.s.}}$$

are all naturally bijective.

2.4.4.1 A Picture

Let us now try to describe some picture of our set B. Define

$$\widetilde{B}(\alpha) := \{(b, z) \in F \times E : \mathrm{N}\, z = \alpha(1-b)\}$$

and set

$$\widetilde{B} := \coprod_{\alpha} \widetilde{B}(\alpha).$$

There is an obvious map

$$\widetilde{B} \to B$$

defined by sending z to its image in $E/\{\pm 1\}$.

Note that $\widetilde{B}$ is the F-points of a smooth variety: it contains smooth subvarieties $\widetilde{B}(\alpha)$ but as it is defined as a disjoint union, these do not intersect. We are interested in the sequence of maps

$$\widetilde{B} \to B \to \mathcal{B}(F)$$

From this perspective, there are two distinct interesting loci in $\widetilde{B}$; the locus when $z = 0$, and the locus where $b = 0$. Note that:

- The map $\widetilde{B}^{\text{s.r.s.s.}} \to \mathcal{B}(F)^{\text{s.r.s.s.}}$ is a 2 to 1 covering map, and $B^{\text{s.r.s.s.}} \to \mathcal{B}(F)^{\text{s.r.s.s.}}$ is an isomorphism.

- For points $(\alpha; 0, z)$ where $b = 0$, we must have $\alpha \in \mathrm{N}\, E^\times$. For such α, we have that for any neighborhood U of $(\alpha; 0, z)$ in $\widetilde{B}(\alpha)$, $U \cap \widetilde{B}_\varepsilon(\alpha) \neq \emptyset$ for all ε. These points correspond to the non-regular-semisimple locus of $\coprod_\varepsilon SO_{V_\varepsilon}(F)/^{conj} SO_{W_\varepsilon}(F)$.

- Points $(\alpha; 1, 0)$, i.e. those points which map to $0 \in \mathcal{B}(F)$, become singular points of B. Each admits a neighborhood U in $\widetilde{B}(\alpha)$ so that $U \cap \widetilde{B}_\varepsilon(\alpha) = \emptyset$ for all but one ε. For this ε, such a point $(\alpha, 1, 0)$ still corresponds to a unique orbit in $SO_{V_\varepsilon}(F)/^{conj} SO_{W_\varepsilon}(F)$, and, moreover, this orbit is regular semisimple (but not strictly regularly semisimple). These points have $\mathrm{Stab}_\gamma = \mu_2$.

To summarize: if we insist on using $\mathcal{B}(F)$ to parameterize our orbit space, then the points corresponding to $z = 0$ appear bad, since they correspond to the singular point of the categorical quotient. However, these points are still well behaved, with finite stabilizer and closed orbit; it is for this reason that we prefer to work with B rather than $\mathcal{B}(F)$.

The following pictures offer some illustration for $F = \mathbb{R}$ and $E = \mathbb{C}$. In the following, we let $\alpha = \pm 1$ and $\varepsilon = \pm 1$.

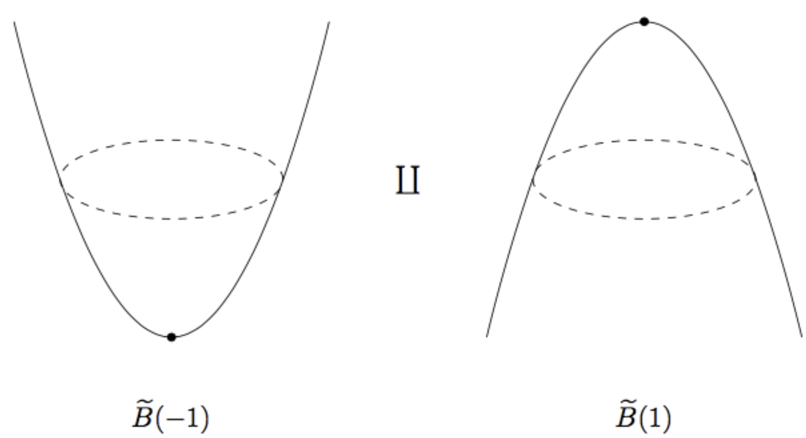

Figure 2.1: The set $\widetilde{B}$ as a disjoint union.

Note that the set $\widetilde{B}$ are the points of a smooth variety, while $\mathcal{B}$ is certainly not a smooth variety. To pass from one to the other, we glue the two points of $\widetilde{B}$ at $z = 0$ together.

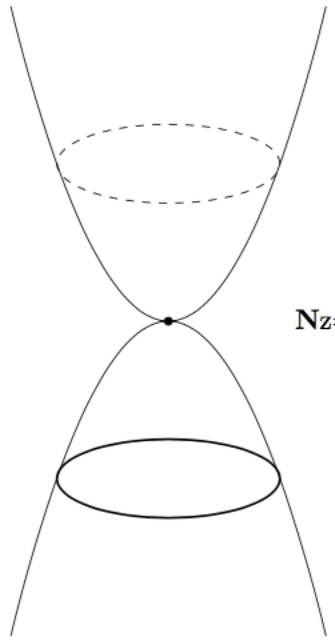

Figure 2.2: Identifying points in $\widetilde{B}$.

The non-regular semisimple locus $D = b = 0$ is marked above. Observe that D does not intersect all of the $\widetilde{B}(\alpha)$; in this example, it is contained entirely in $\widetilde{B}(1)$. Note too that the subset corresponding to $\varepsilon = -1$ is the locus in $\widetilde{B}(1)$ between the point at $\mathrm{N}z = 0, b = 1$ and the circle at $b = 0$; the subset corresponding to $\varepsilon = 1$ is the complement. Thus, near points where $b = 0$, every neighborhood intersects all of the sets $\widetilde{B}_\varepsilon$. In $\widetilde{B}$ the $\varepsilon = 1$ and $\varepsilon = -1$ loci are distinct as $\mathrm{N}z \to 0$; in this intermediate stage, since we have glued the points at $z = 0$ to one another, the two sets are neighbors. Note too that in this intermediate set, a neighborhood of the point at $\mathrm{N}z = 0, b = 1$ intersects all of the pieces $\widetilde{B}(\alpha)$. This is not indicative of what happens in $\widetilde{B}$ upstairs. To pass from this intermediate set to the $\mathbb{R}$ points of the categorical quotient, we have only to quotient out by $z \mapsto -z$. Identifying these points $(\alpha; b, z)$ and $(\alpha; b, -z)$ gives our singular cone.

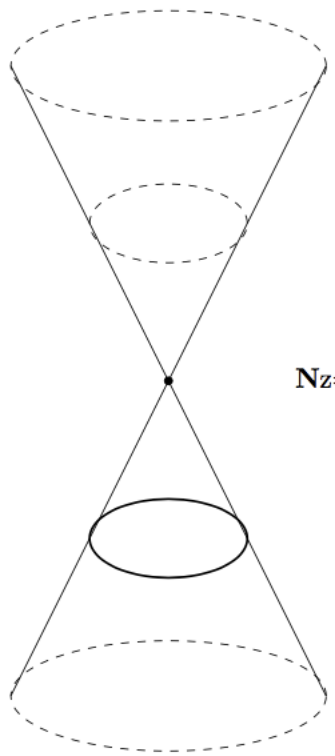

Figure 2.3: The set $\mathcal{B}(\mathbb{R})$.

This completely describes the map $\widetilde{B} \to \mathcal{B}(\mathbb{R})$. This picture of $\widetilde{B}(\mathbb{R})$ is quite helpful to keep in mind when reading Chapter 3 of this thesis–we encourage the reader to do so.

2.4.5 The Quotient $SO_V \mathbin{/\mkern-6mu/}^{conj} O_W$ and the Invariants of Rallis and Schiffman

This subsection is largely disconnected from the remainder of this thesis. Its purpose is to give some relation between the invariant theory of $SO_V/^{conj}SO_W$ outlined above and the relatively well-known theory detailed in the paper [Rallis and Schiffmann, 2007] of Rallis and Schiffman. The uninterested reader should feel free to skip it. Since it is not necessary for the main goals of this thesis, most calculations are omitted or elided.

Definition 2.4.12. Let $U \hookrightarrow T$ be an embedding of non-degenerate quadratic spaces of dimension n and $n+1$. The *Rallis-Schiffman invariants* associated to an element $x \in SO_T$ are the polynomials

$(a_i(x))_{i=1}^n$ and $(b_i(x))_{i=1}^n$. Here

$$\begin{aligned} a_i(x) &= i\text{-th coefficient of the characteristic polynomial of } x \\ &= \operatorname{Tr} \wedge^i x \end{aligned}$$

and

$$b_i(x) = Q(e, x^i e)$$

It is apparent that a_i and b_i are O_U-invariant polynomials on SO_T. In [Rallis and Schiffmann, 2007], Rallis and Schiffman show that they generate the ring of all invariant polynomials. This allows them to realize the GIT quotient $SO_T /\!\!/^{conj} O_U$ as a subset of a $2n$-dimensional affine space. If we let $n = 2$, i.e. we restrict consideration to our case of $SO_V /\!\!/^{conj} O_W$, then we can write this GIT quotient more explicitly.

Lemma 2.4.13. *The Rallis-Schiffman GIT quotient* $SO_V /\!\!/^{conj} O_W$ *is isomorphic to a two dimensional affine space.*

Proof. One only needs to compute the ring of invariant functions corresponding to

$$\mathbb{G}_m \backslash\!\!\backslash \operatorname{GSpin}_V /\!\!/^{conj} O_W$$

The main point is that there is now an additional element, a representative of the nontrivial coset in O_W/SO_W that acts on $W \cong (E, c\mathrm{N})$ by the Galois involution $z \mapsto \bar{z}$. Alternatively, one can compute the Rallis Schiffman invariants a_1, a_2, b_1, b_2 and uncover the appropriate relations between them. □

Carrying out the computations alluded to above also reveals:

Proposition 2.4.14. *The map of GIT quotients* $SO_V /\!\!/^{conj} SO_W \to SO_V /\!\!/^{conj} O_W$ *is given by the inclusion map of rings*

$$F[A, B] \hookrightarrow F[A, B, C]/(C^2 - AB).$$

Notation is as in Proposition 2.4.3.

As for notions of regular semisimplicity, in [Rallis and Schiffmann, 2007], Rallis and Schiffman show that x is O_U-regular semisimple, i.e. has Zariski closed orbit and trivial stabilizer under the conjugation action of O_U on SO_T if and only if the vectors $e, xe, ..., x^n e$ form a basis for T. Hence

Proposition 2.4.15 (Rallis-Schiffman, [Rallis and Schiffmann, 2007]). *An element $g \in SO_V$ is O_W-regular semisimple if and only if e, ge, g^2e form a basis for V. The regular semisimple locus is the complement of the set*

$$\{g \in SO_V : \Delta(g) = 0\}$$

where

$$\Delta(g) = \det((g^{i-1}e, g^{j-1}e)_Q)_{1 \le i,j \le 3})$$

This lets us identify the O_W-regular semisimple locus in GSpin_V more explicitly. We will use the following simple observation repeatedly in the following lemma without further remark: if $z \in C_0(W)$ and $w \in W$, then $zw = w\bar{z}$ inside $C(W)$.

Lemma 2.4.16. *An element $x = z + we \in GSpin_V$, with $z \in C_0(W)$ and $w \in C_1(W) = W$, is O_W-regular semisimple if and only if $Q(w) \neq 0$ and z has non-vanishing real and imaginary parts, i.e. both $z + \bar{z}$ and $z - \bar{z}$ are non-zero.*

Proof. This is a straightforward computation. We include it for completeness. By the above lemma, x is regular semisimple if and only if $e, \rho(x)e, \rho(x)^2e$ is a basis for V; clearly, this occurs if and only if $\rho(x)^{-1}e, e, \rho(x)e$ is also a basis for V.

We expand out the expressions $\rho(x)e = xex^{-1}$ and $\rho(x)^{-1}e = x^{-1}ex$ in the Clifford algebra. Since $x \in C_0(V) = B$ is invertible if and only if $\mathrm{N}(x) = (z + we)(\bar{z} + ew) = \mathrm{N}(z) + Q(e)Q(w)$ is not zero, and since in this case

$$x^{-1} = \frac{1}{\mathrm{N}(x)}(\bar{z} + ew)$$

we write

$$\begin{aligned} xex^{-1} &= \frac{1}{\mathrm{N}(x)}(z + we)e(\bar{z} + ew) \\ &= \frac{1}{\mathrm{N}(x)}(z\bar{z}e + Q(e)zw + Q(e)w\bar{z} - Q(e)Q(w)e) \\ &= \frac{1}{\mathrm{N}(x)}(\mathrm{N}(z) - Q(e)Q(w))e + Q(e)(zw + w\bar{z}) \end{aligned}$$

Similarly

$$x^{-1}ex = \frac{1}{\mathrm{N}(x)}(\mathrm{N}(z) - Q(e)Q(w))e + Q(e)(\bar{z}w + wz)$$

Therefore, the span of the vectors $\{e, \rho(x)e, \rho(x)^{-1}e\}$ is the same as the span of the vectors $\{e, zw+w\bar{z}, \bar{z}w+wz\}$. Noting that $wz = \bar{z}w$, we can further simplify this spanning set to $(e, zw, wz\}$.

To check if $\{e, zw, wz\}$ is linearly independent, we have only to check that $\{zw, wz\}$ is, since these vectors lie in $W = (e)^{\perp}$. To do this, we write down the matrix of inner products

$$Q(w)\begin{pmatrix} \mathrm{N}(z) & \frac{1}{2}(z^2+\bar{z}^2) \\ \frac{1}{2}(z^2+\bar{z}^2) & \mathrm{N}(z) \end{pmatrix}$$

and look at the determinant, which is

$$Q(w)^2(\mathrm{N}(z)^2 - (\frac{z^2+\bar{z}^2}{2})^2)$$

It is easy to compute that, if we write $z = a + b\sqrt{d}$, then the term $\mathrm{N}(z)^2 - (\frac{z^2+\bar{z}^2}{2})^2$ is non-zero if and only if $ab \neq 0$. The lemma follows. □

2.5 A Simple Form of the $SO_W \times SO_V$ Trace Formula

In this section, we again let F be a global field. We make assumptions on our test data f to ensure convergence of the distribution $J(f)$ and to justify the manipulations of the previous sections. The main result is as follows.

Theorem 2.5.1 (The Simple $SO_W \times SO_V$ Trace Formula). *Let the test data f be "good." Then we have an equality*

$$\sum_{\pi} J_\pi(f' \otimes \Phi \otimes \Psi, s) = \sum_{,\gamma \in B^{\mathrm{r.s.s.}}} \mathrm{Orb}(\gamma; f' \otimes \Phi \otimes \Psi)$$

where the sum on the left hand side runs over all automorphic cuspidal representations $\Pi \subset L^2_0([SO_W \times SO_V])$ and the sum on the right hand side runs over all regular semisimple orbits of $SO_V(F)/^{conj}SO_W(F)$. The precise definition of "good" is explained below.

Recall that we have defined the notion of when an element $g \in SO_V$ is regular semisimple. We extend this to the group $SO_W \times SO_V$ by declaring that an element $(h, g) \in SO_W \times SO_V$ is regular semisimple if and only if $h^{-1}g$ (equivalently, gh) is. Then

Definition 2.5.2. A factorizable test function $f = \otimes f_v \in C_c^\infty((SO_W \times SO_V)(\mathbb{A}))$ is "good" if it satisfies the following two conditions:

1. There is a place v_1 so that f_{v_1} is the matrix coefficient of a supercuspidal representation π_{v_1} of $(SO_W \times SO_V)(F_{v_1})$.

2. There is a place $v_2 \neq v_1$ so that the function $f_{v_2} = f_{v_2,W} \otimes f_{v_2,V}$ is supported on $(SO_W \times SO_V)(F_{v_2})^{\text{s.r.s.s.}}$. This implies that the function

$$\begin{aligned} F_{v_2}(g) &= S(f_{v_2,W} \otimes f_{v_2,V})(g) \\ &= \int_{SO_W(F_{v_2})} f_{v_2,W}(h) f_{v_2,V}(h^{-1}g)dh \end{aligned}$$

is supported on $SO_V(F_{v_2})^{\text{r.s.s.}}$.

We will need a lemma.

Lemma 2.5.3. *If a factorizable f satisfies condition (2). above, then the kernel function*

$$K_f(h_1, h_2) = K_f((h_1, h_1), (h_2, h_2))$$

is compactly supported as a function on $[SO_W] \times [SO_W]$. In particular, $J(f)$ converges absolutely and decomposes as

$$J(f) = \sum_{\gamma \in B_{(W,V)}(F)^{\text{r.s.s.}}} \text{Orb}(\gamma; f)$$

Proof. Suppose f is as above. Then we can write, for $G = SO_W \times SO_V$ and $H = \Delta SO_W$,

$$\begin{aligned} K_f(h_1, h_2) = &\sum_{\gamma \in H(F)\backslash G(F)^{\text{s.r.s.s.}}/H(F)} \sum_{(\delta_1,\delta_2) \in H(F)\times H(F)} f(h_1^{-1}\delta_1^{-1}\gamma\delta_2 h_2) \\ &+ \sum_{\gamma \in H(F)\backslash (G(F)^{\text{r.s.s.}} - G(F)^{\text{s.r.s.s.}})/H(F)} \sum_{(\delta_1,\delta_2) \in (\mu_2(F)\backslash H(F)) \times (\mu_2(F)\backslash H(F))} f(h_1^{-1}\delta_1^{-1}\gamma\delta_2 h_2) \end{aligned}$$

where, in the two expressions, the the outer sums are over a set of representatives of strictly regular semisimple orbits

$$\begin{aligned} H(F)\backslash G(F)^{\text{s.r.s.s.}}/H(F) &\xrightarrow{\sim} SO_V(F)^{\text{s.r.s.s.}}/SO_W(F) \\ (h, g) &\mapsto h^{-1}g \end{aligned}$$

and a set of representatives of the regular semisimple but non strictly regular semisimple rational orbits.

We first claim that both of these outer sums only contribute finitely many terms. To see this, consider $\text{Supp}(f) \subset G(\mathbb{A})$. As this is compact, its image under the continuous map $G(\mathbb{A}) \to \mathcal{B}(\mathbb{A})$ is also compact. But $h_1^{-1}\delta_1^{-1}\gamma\delta_2 h_2$ has image in the discrete set $\mathcal{B}(F)$; but for a strictly regular semisimple element of $\mathcal{B}(F)$, there is at most one $H_1(F) \times H_2(F)$ orbit mapping to it. This shows finiteness of the first sum. To see that the second sum is finite, note that the set $H(F)\backslash(G(F)^{\text{r.s.s.}} - G(F)^{\text{s.r.s.s.}})/H(F)$ corresponds to the set $\{(\alpha; 1, 0) \in B(\mathbb{A}) : \alpha \in \Sigma\}$. This is again discrete inside of $\{(\alpha; 1, 0) \in B(\mathbb{A}) : \alpha_v \in \Sigma_v \text{ for all } v\}$ since $F^\times/(F^\times)^2 \hookrightarrow \mathbb{A}^\times/(\mathbb{A}^\times)^2$ is discrete. Moreover, the image of $\text{Supp}(f)$ intersected with this set is again compact.

It remains to show that for fixed γ a regular semisimple element of $G(F)$, the function

$$(h_1, h_2) \mapsto f(h_1^{-1}\gamma h_2)$$

has compact support. But this is clear, since γ regular semisimple implies that

$$\begin{aligned} H(\mathbb{A}) \times H(\mathbb{A}) &\to G(\mathbb{A}) \\ (h_1, h_2) &\mapsto h_1^{-1}\gamma h_2 \end{aligned}$$

induces a surjective map onto a closed subset. □

We can now provide the proof of Theorem 2.5.1.

Proof. We have only to discuss the spectral side of the trace formula, as the geometric side is covered by the lemma above. We appeal to condition (1). in the definition of "good." Since f'_{v_1} is a matrix coefficient of a supercuspidal representation, the operator $R(f)$ acts by 0 on the orthogonal complement to $L^2_0([G])$. Thus, the kernel function can be written as an absolutely convergent sum

$$K_f(x, y) = \sum_{\varphi \in \mathrm{ON}(L^2_0([G]))} R(f)\varphi(x)\overline{\varphi(y)}$$

hence

$$J(f) = \sum_{\pi} J_\pi(f)$$

as desired. □

Chapter 3

The Relative Trace Formula for $GL_2 \times GL_2$

The relative trace formula on the general linear side is more complicated than the one appearing on the special orthogonal side. To set it up, we would like to define a global spherical character on the general linear side and to relate the sum of these global spherical characters to a sum of orbital integrals. However, in order to construct the period functionals that go into defining the $GL_2 \times GL_2$ spherical character we must first introduce some notions and notation.

3.1 Ingredients

Let $G' = GL_2 \times GL_2$, and let $H_1' = \Delta GL_2$, and let $H_2' = GL_2^{(2)} \times ZN$. Here $GL_2^{(2)}(F) = \{g \in GL_2(F) : \det g \text{ is a square}\}$, $Z = Z_{GL_2}$ is the usual center of GL_2, and N is the upper uni-triangular matrices of GL_2. Note that $GL_2^{(2)}$ is not an algebraic group; however, in practice, it should be clear what we mean when we talk of the "R-points" $GL_2^{(2)}(R)$ for a ring R, and so we often abuse notation and often write the expression $GL_2^{(2)}$, pretending it is an honest group over $\mathbb{Z}$.

Let $\Pi = \Pi_1 \otimes \Pi_2$ be an automorphic cuspidal representation of G', i.e. let Π_i be cuspidal representations of GL_2, and denote by ω_1 and ω_2 their (unitary) central characters. We will consider various period integrals of forms in Π along H_1' and H_2' but weighted by various functions (Eisenstein series and theta functions respectively).

When working with GL_2 we find that the notation $[G]$ for the automorphic quotient $[G] :=$

$G(F)\backslash G(\mathbb{A})$ of a group G over F to be somewhat unsatisfactory. Often, it is convenient to additionally quotient out by the center of the adelic points of the group. In this case we write $[PG] := Z_G(\mathbb{A})G(F)\backslash G(\mathbb{A})$.

3.1.1 Eisenstein Series on GL_2 and the Rankin-Selberg Period

To proceed, we must recall the definition of the Eisenstein series on GL_2. Let $\Phi \in \mathscr{S}(\mathbb{A}^2)$ be a Schwartz function on $\mathbb{A}^2$, let ω be a unitary central character of GL_2:

$$\omega : F^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$$

Let

$$P = \left\{ \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \in GL_2 \right\}$$

denote the usual mirabolic subgroup of GL_2, and

$$B = ZP = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in GL_2 \right\}$$

the full parabolic subgroup containing P.

Definition 3.1.1. The *(mirabolic) Eisenstein series* associated to Φ and ω is given, when $\mathrm{Re}(s) \gg 0$, by the following expression

$$E(g) = E(g, s) = E(g, \Phi; s, \omega) := \sum_{\gamma \in B(F)\backslash GL_2(F)} F_s(\gamma g)$$

Here the section F_s is defined by

$$F_s(g) = F(g, \Phi; s, \omega) = \int_{Z(\mathbb{A})} \Phi(e_2^* gz)|\det(gz)|^s \omega(z) dz$$

where $e_2^* = \begin{pmatrix} 0 & 1 \end{pmatrix}$ is the standard row covector. It is clear that $F_s \in \mathrm{Ind}_{B(\mathbb{A})}^{GL_2(\mathbb{A})}(\delta^{s-\frac{1}{2}}(1, \omega^{-1})$, where $\delta\left(\begin{pmatrix} a & * \\ & b \end{pmatrix}\right) = \left|\frac{a}{b}\right|$ is the modular quasicharacter of B, $(1, \omega)\left(\begin{pmatrix} a & * \\ & b \end{pmatrix}\right) := \omega(b)$ and Ind denotes normalized induction.

This Eisenstein series is used crucially in the construction of the global $GL_2 \times GL_2$ Rankin-Selberg L-function, and will appear in our relative trace formula via this application. We record the statement here.

Theorem 3.1.2 (Jacquet, Piatetski-Shapiro, Shalika, Rankin, Selberg)**.** *Let Π_1 and Π_2 be irreducible automorphic cuspidal representations of $GL_2(\mathbb{A})$, with central characters ω_1 and ω_2. Let $\varphi_i \in \Pi_i$ be forms, and let $\Phi \in \mathscr{S}(\mathbb{A}^2)$ be a Schwartz function. Then the integral*

$$\begin{aligned} I(s;\varphi_1,\varphi_2,\Phi) : &= \int_{Z(\mathbb{A})GL_2(F)\backslash GL_2(\mathbb{A})} \varphi_1(g)\varphi_2(g)E(g,\Phi;s,\omega_1\omega_2)dg \\ &= \int_{[PGL_2]} \varphi_1(g)\varphi_2(g)E(g,s)dg \end{aligned}$$

represents the Rankin-Selberg L-function $L(s,\Pi_1\times\Pi_2)$. That is, for factorizable forms $\varphi_i = \otimes_v' \varphi_{i,v}$ and test data $\Phi = \otimes_v' \Phi_v$ the integral unfolds to an Euler product

$$I(s;\varphi_1,\varphi_2,\Phi) = \int_{N(\mathbb{A})\backslash GL_2(\mathbb{A})} W_{\varphi_1}(g)W_{\varphi_2}(g)\Phi(e_n g)|\det g|^s dg$$

where

$$W_{\varphi_i}(g) = \int_{[N]} \varphi_i(ng)\psi^{-1}(n)dn$$

is the ψ-th Fourier coefficient. Moreover, the local integrals

$$I_v(s;\varphi_{1,v},\varphi_{2,v},\Phi_v) = \int_{N(F_v)\backslash GL_2(F_v)} W_{\varphi_{1,v}}(g)W_{\varphi_{2,v}}(g)\Phi_v(e_n g)|\det g|^s dg$$

are exactly equal to the local factors $L_v(s,\Pi_{1,v}\times\Pi_{2,v})$ for unramified places and unramified data.

We can quickly remark that ramified places can be controlled by non-vanishing results for the local integrals, and thus this integral gives a completely robust understanding of the analytic properties of $L(s,\Pi_1\times\Pi_2)$. The functional equation follows from the functional equation of the Eisenstein series.

3.1.1.1 Fourier Coefficients of $E(g,s)$

The Fourier expansion of the GL_2-Eisenstein series will be used later. The relevant and extremely well-known computation follows.

Lemma 3.1.3. *(Fourier Coefficients of the Eisenstein Series) Let $\Phi = \otimes\Phi_v$ be factorizable Schwartz function. The Eisenstein series*

$$E(g) = E(g,\Phi;s,\omega) = \sum_{\gamma\in B(F)\backslash GL_2(F)} F(\gamma g,\Phi;s,\omega)$$

has α-th Fourier coefficient given by

$$\begin{aligned} E^{\alpha}(g) &= \int_{F\backslash\mathbb{A}} E(n(x)g)\psi(-\alpha x)dx \\ &= \begin{cases} \prod_v \int_{F_v} F_{v,s}(w_0 n(x)a(\alpha)g)\psi(-x)dx & \text{if } \alpha \neq 0 \\ \prod_v F_{v,s}(g) + \prod_v \int_{F_v} F_{v,s}(w_0 n(x)g) & \text{if } \alpha = 0 \end{cases} \end{aligned}$$

where $w_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, *and, for* $\alpha \in F$, $a(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$.

Proof. Recall the Eisenstein series is given by

$$E(g) = \sum_{\gamma \in B(F)\backslash GL_2(F)} F_s(\gamma g)$$

Taking a set of representatives of $B(F)\backslash GL_2(F)$ to be the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (the small Bruhat cell) and matrices of the form $wn(t)$ (the large cell), where $w_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $n(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, we find

$$E^{\alpha}(g) = \int_{F\backslash\mathbb{A}} \left(F_s(n(x)g) + \sum_{t \in F} F_s(w_0 n(t) n(x) g) \right) \psi(-\alpha x)dx$$

Note that

$$\int_{F\backslash\mathbb{A}} F_s(n(x)g)\psi(-\alpha x)dx = F_s(g) \int_{F\backslash\mathbb{A}} \psi(-\alpha x)dx = 0$$

unless $\alpha = 0$.

So suppose first that $\alpha \neq 0$. Then we unfold to find

$$\begin{aligned} E^{\alpha}(g) &= \int_{F\backslash\mathbb{A}} \sum_{t \in F} F_s(w_0 n(x+t)g)\psi(-\alpha x)dx \\ &= \int_{\mathbb{A}} F_s(w_0 n(x)g)\psi(-\alpha x)dx \\ &= \int_{\mathbb{A}} F_s(\begin{pmatrix} 1 & 0 \\ 0 & \alpha^{-1} \end{pmatrix} w_0 n(x) \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g)\psi(-x)dx \\ &= \int_{\mathbb{A}} F_s(w_0 n(x) \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g)\psi(-x)dx \end{aligned}$$

which is factorizable.

If $\alpha = 0$, the computation is similar. Proceeding as above, we get

$$E^0(g) = F_s(g) + \int_{\mathbb{A}} F_s(w_0 n(x) g) dx$$

□

3.1.2 Theta Functions and the Symmetric Square Period

As above, let W denote a two dimensional quadratic space over F. We are interested in the functorial transfer of automorphic forms from SO_W to GL_2; our goal in the section is to explain a characterization of the joint image of this transfer map as W varies. Problems of functorial transfer are often formulated in the terminology of L-groups–let us quickly survey our situation using such language. Given the group SO_W, we can associate its L-group. This is the semidirect product of its connected component $({}^L SO_W)^0 = SO_{(\mathbb{C}^2, w)}$, where $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, with the Galois group $\mathrm{Gal}(\bar{F}/F)$, where $\mathrm{Gal}(\bar{F}/F)$ acts by outer automorphisms on $SO_{(\mathbb{C}^2,w)}$. There is an L-homomorphism, the "standard representation," [1]

$$\mathrm{St}: {}^L SO_W = SO_{(\mathbb{C}^2,w)} \rtimes \mathrm{Gal}(\bar{F}/F) \to GL_2(\mathbb{C}) \times \mathrm{Gal}(\bar{F}/F) = {}^L GL_2$$

which sends $g\sigma$ in $SO_{(\mathbb{C}^2,w)} \rtimes \mathrm{Gal}(\bar{F}/F)$, where $g \in SO_{(\mathbb{C}^2,w)}$ and $\sigma \in \mathrm{Gal}(\bar{F}/F)$, to $(gi(\sigma).\sigma)$ in $GL_2(\mathbb{C}) \times \mathrm{Gal}(\bar{F}/F)$.

Here i is the obvious map

$$i : \mathrm{Gal}(\bar{F}/F) \to \mathrm{Gal}(E/F) \to \{1, w\} \subset SO_{(\mathbb{C}^2,w)}$$

where $E = F(\sqrt{d(W)})$. In particular, $i(\sigma) = 1$ if $d(W) = 1$. Note too that $w \in SO_{(\mathbb{C}^2,w)}$, so that $\mathrm{St}(g\sigma) \in SO_{(\mathbb{C}^2,w)}$ for all $g\sigma \in {}^L SO_W$.

Summarizing, the image of St in $GL_2(\mathbb{C})$ is given (up to conjugation) by

$$\mathrm{St}({}^L SO_W) = \begin{cases} O_2(\mathbb{C}) & \text{if } d(W) \notin (F^\times)^2 \\ SO_2(\mathbb{C}) & \text{if } d(W) \in (F^\times)^2 \end{cases}$$

[1] Actually, when $d(W) \notin (F^\times)^2$, there are actually two distinct L-homomorphisms that deserve the name of "standard representation." However, following Arthur [Arthur, 2013], we fix one such map explicitly in our description above.

Following standard practice, we will occasionally confuse ${}^L SO_W$ with its image $\mathrm{St}({}^L SO_W)$. This allows us to summarize the situation with the imprecise pronouncement that the L-group of SO_W is either $SO_2(\mathbb{C})$ or $O_2(\mathbb{C})$ according to whether W is split or not. In any case we expect a transfer map of automorphic forms on SO_W to GL_2 corresponding to the L-homomorphism St. This transfer map can be constructed explicitly using the theta lift from SO_W to the double cover of SL_2–however, we shy away from using this construction in our paper. This for the following reason. The concurrence of the theta lift and the functorial transfer from SO_2 to GL_2 depends on the accidental isomorphism $Sp_2 = SL_2$. Since this paper is an attempt to reprove an old theorem while avoiding the use of accidental isomorphisms, we believe it would be somewhat self-defeating to rely on this characterization of functorial lifts.

Instead, we can characterize those forms which are lifts from SO_W by examining the poles of an auxiliary L-function. Since, for any 2 dimensional quadratic space W, ${}^L SO_W \subset O_2(\mathbb{C})$, it quickly follows that a representation Π of GL_2 should be a functorial lift from *some* special orthogonal group of *some* two dimensional quadratic space if and only if the symmetric square L function $L(s, \Pi, \mathrm{Sym}^2)$ has a pole at $s = 1$. (This is now a theorem, for all quasi-split even special orthogonal groups, due to work of Arthur [Arthur, 2013].) The work of Gelbart and Jacquet, Bump and Ginzburg, and Takeda ([Gelbart and Jacquet, 1978; Takeda, 2014; Bump and Ginzburg, 1992]) provide an integral representation of this L-function–the integral in question varies slightly from author to author. We follow Takeda's exposition, which corrects the (undefined!) integral of Bump-Ginzburg [Takeda, 2014]. Using this integral representation to compute the residue of $L(s, \Pi, \mathrm{Sym}^2)$ at $s = 1$ provides the following theorem:

Theorem 3.1.4 (Takeda, [Takeda, 2014]). *Let Π be an irreducible cuspidal automorphic representation of $GL_2(\mathbb{A})$ with central character ω. Then $L(s, \Pi, \mathrm{Sym}^2)$ has a pole at $s = 1$ if and only if $\omega^2 = 1$, $\omega = (\cdot, d)_{\mathbb{A}}$, and the linear functional*

$$\begin{aligned} \mathscr{P}_{\mathrm{Sym}^2} : \Pi \otimes \mathbf{r}^{\psi} \otimes \mathbf{r}^{\psi_{-d}} &\to \mathbb{C} \\ \varphi \otimes \Psi_1 \otimes \Psi_2 &\mapsto \int_{Z_{GL_2}(\mathbb{A}) GL_2^{(2)}(F) \backslash GL_2^{(2)}(\mathbb{A})} \varphi(h) \Theta_1(\kappa(h), \mathbf{r}^{\psi}, \Psi_1) \Theta_2(\kappa(h), \mathbf{r}^{\psi_{-d}}, \Psi_2) dh \end{aligned}$$

is not identically zero.

In the above, $\varphi \in \Pi$, Θ_1 and Θ_2 are theta functions on the metaplectic double cover of $GL_2^{(2)}(\mathbb{A})$ corresponding to Schwartz functions Ψ_1, Ψ_2 on $\mathbb{A}$. The representations $\mathbf{r}_1^{\psi}$ and $\mathbf{r}_1^{\psi_{-d}}$ are Weil repre-

sentations associated to the additive characters ψ and ψ_{-d}. κ is the canonical set-theoretic section $GL_2(\mathbb{A}) \to \widetilde{GL_2(\mathbb{A})}$.

Remark 3.1.5. Neither the above theorem, nor its generalization to GL_n, appear explicitly in Takeda's paper [Takeda, 2014] which gives the integral representation of the symmetric square L-function. However, it is easy to deduce from the results of his paper. We omit a discussion of this derivation, as it is somewhat tedious. Instead, we merely remark that the theorem follows from a computation of the residue of a half-integral weight Eisenstein series on $\widetilde{GL_2(\mathbb{A})}$.

The theorem suggests the following terminology, which we adopt.

Definition 3.1.6. Call $\mathscr{P}_{\mathrm{Sym}^2}$ the symmetric-square period. If $\mathscr{P}_{\mathrm{Sym}^2} \neq 0$, we say that Π is $\mathscr{P}_{\mathrm{Sym}^2}$-distinguished or symmetric-square-distinguished.

This linear functional will play an important role in our trace formula. Therefore, let us provide a quick survey of the definitions of the representations and functions appearing above.

3.1.2.1 The Weil Representations

There is a well-known and canonically defined double cover $\widetilde{GL_2(\mathbb{A})}. \to GL_2(\mathbb{A})$. This cover comes with two important set-theoretic sections, which we denote κ and $\mathbf{s}$, from $GL_2(\mathbb{A}) \to \widetilde{GL_2(\mathbb{A})}$. Let $GL_2^{(2)}(\mathbb{A})$ denote the subgroup of $GL_2(\mathbb{A})$ consisting of matrices with square determinant, and $\widetilde{GL_2^{(2)}}(\mathbb{A})$ its inverse image. There is a Weil representation $\mathbf{r}^\psi$ of $\widetilde{GL_2^{(2)}}(\mathbb{A})$. This depends on the choice of a unitary additive character $\psi : F\backslash\mathbb{A} \to \mathbb{C}^\times$. The action is on the space of even Schwartz functions on $\mathbb{A}$, $\mathscr{S}^+(\mathbb{A})$, and is given by the formulas

$$\mathbf{r}^\psi(\mathbf{s}(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}))f(x) = \gamma(\psi, \cdot^2)\hat{f}(x)$$

$$\mathbf{r}^\psi(\mathbf{s}(\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}))f(x) = \psi(bx^2)f(x)$$

$$\mathbf{r}^\psi(\mathbf{s}(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix})f(x) = |a|^{\frac{1}{2}}\mu_\psi(a)f(ax)$$

$$\mathbf{r}^\psi(\mathbf{s}(\begin{pmatrix} 1 & 0 \\ 0 & a^2 \end{pmatrix})f(x) = |a|^{-\frac{1}{2}}f(a^{-1}x)$$

$$\mathbf{r}^{\psi}(\xi)f(x) = \xi f(x)$$

Here ξ lies in the central $\{\pm 1\}$ used to define the double cover. In the above formulae, $\hat{f}(x) = \int_F f(y)\psi(2xy)dy$ is the Fourier transform with dy a self-dual measure on F, and $\gamma(\psi, \cdot^2)$ is the Weil index of the character of second degree $x \mapsto \psi(x^2)$, while $\mu_\psi(a) = \gamma(\psi_a, \cdot^2)/\gamma(\psi, \cdot^2)$.

Consider the two dimensional quadratic space (E, N). We can, in a similar manner to what we did above, define the representation $\mathbf{R}^{\psi}$ of $GL^{(2)}(2, \mathbb{A})$ acting on $\mathscr{S}^+(\mathbb{A}_E)$ by the formulas

$$\mathbf{R}^{\psi}(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})f(z) = \gamma(\psi, \mathrm{N})\hat{f}(z)$$

$$\mathbf{R}^{\psi}(\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix})f(z) = \psi(b\,\mathrm{N}\,z)f(z)$$

$$\mathbf{R}^{\psi}(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix})f(z) = |a|\omega(a)f(az)$$

$$\mathbf{R}^{\psi}(\begin{pmatrix} 1 & 0 \\ 0 & a^2 \end{pmatrix})f(z) = |a|^{-1}f(a^{-1}z)$$

The two constructions are related by the identity

$$\mathbf{R}^{\psi} \cong \mathbf{r}^{\psi}\hat{\otimes}\mathbf{r}^{\psi_{-d}}$$

Here, $\hat{f}(z) = \int_{\mathbb{A}_E} f(z')\psi(\mathrm{N}(z+z') - \mathrm{N}(z) - \mathrm{N}(z'))dz'$ is the Fourier transform, with dz' normalized so as to be self-dual. Note that $\mathbf{r}^{\psi}\hat{\otimes}\mathbf{r}^{\psi_{-d}} \cong \mathbf{R}^{\psi}$ is a non-genuine representations of $\widetilde{GL_2^{(2)}}(\mathbb{A})$, i.e. it factors through $\widetilde{GL_2^{(2)}}(\mathbb{A}) \to GL_2^{(2)}(\mathbb{A})$.

The representations $\mathbf{r}^{\psi}$ and $\mathbf{R}^{\psi}$ both have automorphic realizations. We can embed $\mathbf{r}^{\psi}$ into the space of automorphic forms on $\widetilde{GL_2^{(2)}}(\mathbb{A})$ by considering the theta function associated to a function $\Psi \in \mathscr{S}^+(\mathbb{A})$

$$\Theta(\tilde{h}, \mathbf{r}^{\psi}, \Psi) = \sum_{\xi \in F} (\mathbf{r}^{\psi}(\tilde{h})\Psi)(\xi)$$

Similarly we can associate to $\Psi \in \mathscr{S}^+(\mathbb{A}_E)$ the theta function

$$\Theta(h, \mathbf{R}^{\psi}, \Psi) = \sum_{\zeta \in E} (\mathbf{R}^{\psi}(h)\Psi)(\zeta).$$

When it does not cause confusion, we suppress the symbols $\mathbf{r}^\psi$ and $\mathbf{R}^\psi$ when talking about the appropriate theta functions.

Let us return to the context of Theorem 3.1.4. Given two Ψ_1, Ψ_2, we can identify $\Psi_1 \otimes \Psi_2 \in \mathscr{S}^+(\mathbb{A}) \otimes \mathscr{S}^+(\mathbb{A})$ with $\Psi \in \mathscr{S}^+(\mathbb{A}_E)$ by $\Psi_1(\xi_1)\Psi_2(\xi_2) = \Psi(\xi_1 + \xi_2\sqrt{d})$. We can thus rewrite

$$\Theta_1(\widetilde{h}, \mathbf{r}^\psi, \Psi_1)\Theta_2(\tilde{h}, \mathbf{r}^\psi, \Psi_2) = \Theta(h, \mathbf{R}^\psi, \Psi)$$

and hence we can rewrite Theorem 3.1.4 in the following manner

Corollary 3.1.7 (Takeda, [Takeda, 2014])**.** *Let Π be an irreducible cuspidal automorphic representation of $GL_2(\mathbb{A})$ with central character ω. Then $L(s, \Pi, \mathrm{Sym}^2)$ has a pole at $s = 1$ if and only if $\omega^2 = 1$, i.e. $\omega = (\cdot, d)_{\mathbb{A}}$ for $d \in F^\times/(F^\times)^2$, and, if we set $E = F[x]/(x^2 - d)$ to be the quadratic etale algebra associated to d, then there exists $\varphi \in \Pi$ and $\Psi \in \mathscr{S}(\mathbb{A}_E)$ so that*

$$\begin{array}{rcl} \mathscr{P}_{\mathrm{Sym}^2} : \Pi \otimes \mathbf{R}^\psi & \to & \mathbb{C} \\ \varphi \otimes \Psi & \mapsto & \displaystyle\int_{Z_{GL(2)}(\mathbb{A})GL_2^{(2)}(F)\backslash GL_2^{(2)}(\mathbb{A})} \varphi(h)\Theta(h, \mathbf{R}^\psi, \Psi) \end{array}$$

is not zero.

Of course, this corollary is nothing more than a direct restatement of 3.1.4, with a few expressions renamed.

3.1.2.2 Fourier Coefficients of $\Theta(h, \mathbf{R}^\psi, \Psi)$

For later use, we include the following simple lemma.

Lemma 3.1.8. *Let $\Theta(h) = \Theta(h, \mathbf{R}^\psi, \Psi)$ be as above. Then the α-th Fourier coefficient is given by*

$$\Theta^\alpha(h) = \sum_{\substack{\zeta \in E \\ \mathrm{N}\,\zeta = \alpha}} (\mathbf{R}^\psi(h)\Psi)(\zeta)$$

Proof. We compute

$$\begin{aligned} \Theta^\alpha(h) &= \int_{F\backslash\mathbb{A}} \Theta(n(x)h)\psi(-\alpha x)dx \\ &= \int_{F\backslash\mathbb{A}} \sum_{\zeta\in E} (\mathbf{R}^\psi(n(x)h)\Psi)(\zeta)\psi(-\alpha x)dx \\ &= \sum_{\zeta\in E} (\mathbf{R}^\psi(h)\Psi)(\zeta) \int_{F\backslash\mathbb{A}} \psi(\mathrm{N}(\zeta))\psi(-\alpha x)dx \end{aligned}$$

This last integral is 0 unless $\mathrm{N}(\zeta) = \alpha$. By our normalization of measures, it is 1 when $\mathrm{N}(\zeta) = \alpha$. □

3.1.3 The Exterior Square Period

Now, let V denote a three dimensional quadratic space over F. We are interested in the functorial transfer of automorphic forms from SO_V to GL_2. This is substantially simpler than the SO_2 case. Recall that the L-group of SO_V can be identified with

$$^L SO_V = Sp_2(\mathbb{C}) \times \mathrm{Gal}(\bar{F}/F) = SL_2(\mathbb{C}) \times \mathrm{Gal}(\bar{F}/F)$$

The morphism of L-groups

$$\mathrm{St} : {}^L SO_V \hookrightarrow {}^L GL_2 = GL_2(\mathbb{C}) \times \mathrm{Gal}(\bar{F}/F)$$

suggests a transfer map on automorphic representations; since in our case $\dim V = 3$, $SO(V) \cong PB^\times$, and this is explicitly given by the Jacquet-Langlands lift from $PB^\times$ to PGL_2. We would like to characterize the joint image of this transfer map as V varies over all 3-dimensional quadratic spaces.

This is easy. We can again examine the poles of an auxiliary L-function. Since, for any 3 dimensional quadratic space V, ${}^L SO_V$ is essentially $Sp_2(\mathbb{C}) = SL_2(\mathbb{C})$, it quickly follows that a representation Π of GL_2 should be a functorial lift from *some* special orthogonal group of *some* three dimensional quadratic space if and only if the exterior square L function $L(s, \Pi, \wedge^2)$ has a pole at $s = 1$. In this low rank case, $\wedge^2 = \det$ and hence $L(s, \Pi, \wedge^2) = L(s, \omega)$ where $\omega = \omega_\Pi$ is the central character of Π. That is, Π is a lift from some SO_V if and only if it has trivial central character. Since we would also like to consider Π which are generic, we make the following definition. Let ψ be a non-trivial global additive character.

Definition 3.1.9. The *exterior square period* $\mathscr{P}_{\wedge^2} = \mathscr{P}^\psi_{\wedge^2}$ is defined by

$$\begin{aligned} \mathscr{P}_{\wedge^2} \quad : \Pi \to \quad & \mathbb{C} \\ \varphi \qquad \mapsto \quad & \int_{[Z]} \int_{[N]} \varphi(zn)\psi(n) dh dz \end{aligned}$$

Here, as usual, $Z = Z_{GL_2}$ is the subgroup of diagonal matrices and N is the subgroup of upper uni-triangular matrices.

We have essentially shown

Lemma 3.1.10. *Let notation be as above. Then $L(s, \Pi, \wedge^2)$ has a pole at $s = 1$ if and only if the linear functional $\mathscr{P}_{\wedge^2}$ is not identically zero.*

This, or rather its analogous statement in higher rank, has also been shown for GL_n, $n \geq 2$. See [Jacquet and Shalika, 1990].

3.1.4 The $GL_2 \times GL_2$ Global Spherical Character

We have finally assembled all of the pieces necessary to introduce the main character on the spectral side of our trace formula. Let $\Pi = \Pi_1 \otimes \Pi_2$ be an automorphic cuspidal representation of $G' = GL_2 \times GL_2$. Let ω be an automorphic quadratic character of $\mathbb{A}^\times$ and let E be the associated quadratic extension of F. We suppose that Π_1 has central character ω. We define the *global spherical character* by taking for $f' = f_1' \otimes f_2' \in C_c^\infty(G'(\mathbb{A}))$, $\Phi \in \mathscr{S}(\mathbb{A}^{\oplus 2})$, and $\Psi \in \mathscr{S}^+(\mathbb{A}_E)$

$$I_\Pi(s; f' \otimes \Phi \otimes \Psi) = \sum_{\varphi = \varphi_1 \otimes \varphi_2 \in \mathrm{ON}(\Pi)} \mathscr{P}_{RS}(\Pi(f')\varphi)\overline{\mathscr{P}^{\psi^{-1}}_{\mathrm{Sym}^2}(\varphi_1)\mathscr{P}^{\psi}_{\wedge^2}(\phi_2)}.$$

Here

$$\begin{aligned} \mathscr{P}_{\mathrm{RS}}(\varphi_1 \otimes \varphi_2) &= \int_{[PGL_2]} \varphi_1(g)\varphi_2(g)E(g,s) \\ &= \int_{Z(\mathbb{A})GL_2(F)\backslash GL_2(\mathbb{A})} \varphi_1(g)\varphi_2(g)E(g,\Phi;s,\omega)dg \end{aligned}$$

is the Rankin-Selberg integral. The sum runs over a fixed orthonormal basis of $\mathrm{ON}(\Pi) \subset \Pi \subset L^2([G'])$, and the operator $\Pi(f)$ is defined by

$$\Pi(f)\varphi = \int_{G(\mathbb{A})} f(g)\Pi(g)\varphi dg.$$

We can make some simple remarks:

- The distribution I_Π vanishes for all $f' \otimes \Phi \otimes \Psi$ unless there exist a two dimensional quadratic space W and three dimensional quadratic space V such that
  - the discriminant d_W satisfies $\omega = (\cdot, d_W)$
  - $\Pi = \Pi_1 \otimes \Pi_2$ has Π_1 a functorial lift from SO_W and Π_2 a functorial lift from SO_V.
- The distribution I_Π vanishes for all $f' \otimes \Phi \otimes \Psi$ if $L(\frac{1}{2}, \Pi) = 0$.

It is in this sense that we say in the introduction that I_Π encodes the data of the L-value $L(\frac{1}{2}, \pi)$ for πa representation of $SO_W \times SO_V$.

3.2 The Trace Formula, Formally

3.2.1 The $GL_2 \times GL_2$ Distribution

Let $f' \in \mathscr{S}(G'(\mathbb{A}))$ be a Schwartz function given by $f' = f_1' \otimes f_2'$ with $f_1', f_2' \in \mathscr{S}(GL_2(\mathbb{A}))$, let $x = (x_1, x_2), y = (y_1, y_2) \in G'(\mathbb{A})$, and define the kernel function

$$K_{f'}(x,y) = \sum_{\gamma \in G'(F)} f'(x^{-1}\gamma y) = \sum_{\gamma_1 \in GL_2(F)} f_1'(x_1^{-1}\gamma_1 y_1) \sum_{\gamma_2 \in GL_2(F)} f_2'(x_2^{-1}\gamma_2 y_2)$$

It is easy to see that this kernel function is formally represented by the sum

$$K_{f'}(x,y) = \sum_{\phi \in \mathrm{ON}(L^2([G']))} (R(f')\phi)(x)\overline{\phi(y)}$$

Here $\mathrm{ON}(L^2([G'])$ denotes a fixed orthonormal basis of automorphic forms of the L^2 space of $G'(F)\backslash G'(\mathbb{A})$ and $R(f')\phi$ is the right regular action representation, namely

$$R(f')\phi(x) = \int_{G'(\mathbb{A})} f'(y)\phi(xy)dy$$

Rather than sum over all ϕ in an orthonormal basis of the full L^2 space, it is often helpful to sum only over forms with given central character. That is, let

$$\eta = \eta_1 \otimes \eta_2 : [Z_{GL_2} \times Z_{GL_2}] = F^\times\backslash\mathbb{A}^\times \times F^\times\backslash\mathbb{A}^\times \to \mathbb{C}^1$$

be an automorphic unitary character of the center of G'. Then we define

$$K_{f',\eta}(x,y) = \int_{[Z\times Z]} K_{f'}(x,zy)\eta(z)dz$$

Let $L^2([G'],\eta)$ denote the L^2 completion of the space of square integrable automorphic forms on G' with central character η. Formally we have

$$K_{f',\eta}(x,y) = \sum_{\phi \in \mathrm{ON}(L^2[G'],\eta)} (R(f')\phi)(x)\overline{\phi(y)}$$

Now let $\eta = \omega \otimes 1$, where ω is a quadratic character of $F^\times\backslash\mathbb{A}^\times$. Our main object of interest is the following.

Definition 3.2.1. The *$GL_2 \times GL_2$-distribution* $I^\psi(s; f' \otimes \Phi \otimes \Psi) = I(s; f' \otimes \Phi \otimes \Psi)$ is defined formally as

$$I(s; f' \otimes \Phi \otimes \Psi) = \int_{[N]} \int_{[PGL_2^{(2)}]} \int_{[PGL_2]} K_{f',\omega\otimes 1}((g,g),(h,n))E(g,s)\overline{\Theta(h)}\psi(n^{-1})dgdhdn$$

Here $\Psi \in \mathscr{S}^+(\mathbb{A}_E)$ is used to defined $\Theta(h) = \Theta(h, \mathbf{R}^{\psi^{-1}}, \Psi)$, $\omega(\cdot) = (\cdot, d)_{\mathbb{A}} = \omega_{E/F}(\cdot)$ is the quadratic idele class character corresponding to the quadratic extension $E = F[\sqrt{d}]$, and $\Phi \in \mathscr{S}(\mathbb{A})$ is a test function used to define the Eisenstein series $E(g, s) = E(g, \Phi; s, \omega)$ appearing in the Rankin-Selberg period.

Remark 3.2.2. Note that we use the theta function corresponding to $\mathbf{R}^{\psi^{-1}}$. This is merely to slightly simplify some expressions later on.

Remark 3.2.3. Note that we are "ignoring" one integration over $[Z]$ that should be appearing in the the exterior square period; this is since this integration is hidden in the definition of $K_{f,\omega\otimes 1}$.

As before, the distribution $I^{\psi}(s; f' \otimes \Phi \otimes \Psi)$ is engineered to decompose formally as a sum of global spherical characters $I_{\Pi}(s; f' \otimes \Phi \otimes \Psi)$. It works as follows: if we write $L^2([G'], \omega \otimes 1) = \bigoplus \Pi$, where the "sum" is over all automorphic representations Π of $G' = GL_2 \times GL_2$ with central characters given by $\omega \otimes 1$, then

$$\begin{aligned} K_{f',\omega\otimes 1}(x, y) &= \sum_{\varphi \in \mathrm{ON}(L^2([G'],\omega\otimes 1))} (R(f')\varphi)(x)\overline{\varphi(y)} \\ &= \sum_{\Pi} \sum_{\varphi \in \mathrm{ON}(\Pi)} (\Pi(f')\varphi)(x)\overline{\varphi(y)} \end{aligned}$$

and thus

$$\begin{aligned} I(s; f' \otimes \Phi \otimes \Psi) &= \int_{[N]} \int_{[PGL_2^{(2)}]} \int_{[PGL_2]} K_{f',\omega\otimes 1}((g, g), (h, n)) E(g, s)\overline{\Theta(h)}\psi(n^{-1}) dg dh dn \\ &= \sum_{\Pi} \sum_{\varphi \in \mathrm{ON}(\Pi)} \mathscr{P}_{RS}(\Pi(f')\varphi)\overline{\mathscr{P}^{\psi^{-1}}_{\mathrm{Sym}^2}(\varphi_1)\mathscr{P}^{\psi}_{\wedge^2}(\phi_2)}. \\ &= \sum_{\Pi} I_{\Pi}(s; f' \otimes \Phi \otimes \Psi). \end{aligned}$$

Thus I formally decomposes as a sum over all global spherical characters. Justifying this sort of spectral expansion requires some work in general. We avoid this by dealing with a simple form of the trace formula. This is done in Section 3.4.

3.2.2 The "Geometric" Side

We can expand the distribution $I(f, \Phi, \Psi, s)$ as a sum over "orbital integrals". Our use of quotation marks merits some discussion: the "orbital integrals" appearing in our expansion of the distribution do *not* come from the action of a group acting on a space, hence are not integrals over orbits in the usual sense. Instead, they manifest in an unfolding argument through the Fourier coefficients of various global objects.[2] The following definition of orbital integral may then look strange to the reader accustomed to the usual relative trace formula scenario. To avoid cluttering the page with notation we have also opted to omit writing the subscript v for nearly all objects in the following definition, even though these are all purely local. We hope this does not lead to confusion.

So for the moment let F denote a local field. Fix $d \in F^\times$ and denote by $E = F[x]/(x^2 - d)$ the quadratic etale algebra over F corresponding to d. Let $f' = f_1' \otimes f_2'$ with $f_i' \in C_c^\infty(GL_2(F))$‘, let $\Phi \in \mathscr{S}(F^2)$, and let $\Psi \in \mathscr{S}^+(E)$ be an even Schwartz function on E, and let $\omega = \omega_{E/F}$. Fix a non-trivial additive character ψ of F.

Definition 3.2.4. Let $\alpha \in F^\times$ and set

$$B'(\alpha) = \{\gamma' = (b', z') \in F \times (E/\{\pm 1\}) \ : \ \mathrm{N}\, z' = \alpha(1 - b')\}$$

and

$$B'(\alpha)^{\text{r.s.s.}} = \{\gamma' = (b', z') \in F \times (E/\{\pm 1\}) : \mathrm{N}\, z' = \alpha(1 - b'), b' \neq 0\}.$$

We define the *non-normalized local $GL_2 \times GL_2$-orbital integral* corresponding to $\alpha \in F^\times$ and $\gamma' \in B'(\alpha)^{\text{r.s.s.}}$ to be

$$\begin{aligned} \mathscr{O}^{\psi}{}_{GL_2 \times GL_2}(\alpha, \gamma'; s; f' \otimes \Phi \otimes \Psi) \ &:= \int f_1'(g^{-1}a(\alpha)h) f_2'(g^{-1}) W_{F_s}^{\psi^{-1}}(a(b')g)\overline{(\mathbf{R}^{\psi^{-1}}(h)\Psi)(z')}\, dh dg \\ &= \int f_1'(g^{-1}h) f_2'(g^{-1}a(\alpha^{-1})) W_{F_s}^{\psi^{-1}}(a(b'\alpha)g)\overline{\mathbf{R}^{\psi^{-1}}(h)\Psi(z')}\, dh dg \end{aligned}$$

where the integration runs over $g \in GL_2(F)$ and $h \in GL_2^{(2)}(F)$. Here $W_{F_s}^{\psi^{-1}}(g)$ denotes the Whittaker integral

$$W_{F_s}^{\psi^{-1}}(g) = \int_F F_s(w_0 n(x) g)\psi^{-1}(x)\, dx$$

where

$$\begin{aligned} F_s(g) \ &= \int_{Z_{GL_2}(F)} \Phi(e_2^* g z) |\det(gz)|^s \omega(z)\, dz \\ &= |\det g|^s \int_{\mathbb{A}^\times} \Phi(a e_2^* g) |a|^{2s} \omega(a)\, d^\times a. \end{aligned}$$

[2]Perhaps a better name for these objects would be *unfolded integrals*.

We have

$$F_s \in I(|\cdot|^{s-\frac{1}{2}}, |\cdot|^{\frac{1}{2}-s}\omega^{-1}) := \mathrm{Ind}_{B(F)}^{GL_2(F)}(|\cdot|^{s-\frac{1}{2}}.|\cdot|^{\frac{1}{2}-s}\omega^{-1}).$$

When b' is non-zero element of a global field, the term $W_{F_s}^{\psi^{-1}}(a(b')g)$ appears as the local component of the b'-th Fourier coefficient of the Eisenstein series corresponding to F_s.

Remark 3.2.5. While we are working entirely formally in this section, ignoring most issues of convergence, it is still useful to say a word about the integral defining F_s. A priori, it only converges for $\mathrm{Re}(s) > \frac{1}{2}$. In the future, we would like to talk about the section $F_{\frac{1}{2}}$. In order to do so, we need to make sense of this integral by analytic continuation. However, note that as long as ω is non-trivial, then there is no pole of this integral at $s = \frac{1}{2}$ by Tate's thesis, thus there is no real issue in defining $F_{\frac{1}{2}}$ to be the value of this analytic continuation at $s = \frac{1}{2}$.

To keep expressions short, in this section we will often omit the subscript $GL_2 \times GL_2$ in our notation for the orbital integral.

The non-normalized orbital integral above has a few distasteful characteristics. First, we would prefer that the orbital integral not depend on the precise value of $\alpha \in F^\times$, but only on the class of α in $F^\times/(F^\times)^2$; of course, the set $B'(\alpha)$ as a subset of $F \times (E/\pm 1)$ depends on α, not just the class of αmodulo squares. Thus, to resolve this issue, we make the following easy definition. We can relate the two sets $B'(\alpha)^{\text{r.s.s.}}$ and $B'(\alpha t^2)^{\text{r.s.s.}}$ for any $t \in F^\times$:

Definition 3.2.6. Given any $t \in F^\times$, denote by $\mathbf{t}$ the bijection

$$\begin{aligned} \mathbf{t} : B'(\alpha) &\to B'(\alpha t^2) \\ \gamma' = (b, z) &\mapsto \mathbf{t}(\gamma') = (b, tz). \end{aligned}$$

This allows us to identify $B'(\alpha_1)$ and $B'(\alpha_2)$ when α_1 and α_2 lie in the same class modulo $(F^\times)^2$. Under this identification $\mathbf{t}$ it is easy to see that a slight modification of $\mathscr{O}^\psi(f \otimes \Phi \otimes \Psi; \alpha, \gamma')$ is invariant by $\alpha \mapsto \alpha t^2$.

Lemma 3.2.7. *Let $t \in F^\times$. Then*

$$|\alpha t^2|^{\frac{1}{2}} \mathscr{O}^\psi(\alpha t^2, \mathbf{t}(\gamma'); s; f' \otimes \Phi \otimes \Psi) = |\alpha|^{\frac{1}{2}} \mathscr{O}^\psi(\alpha, \gamma'; s; f' \otimes \Phi \otimes \Psi)$$

Proof. We simply compute

$$|\alpha t^2|^{\frac{1}{2}} \mathscr{O}^\psi(\alpha t^2, \mathbf{t}(\gamma'); s; f' \otimes \Phi \otimes \Psi)).$$

This, by definition is

$$|\alpha t^2|^{\frac{1}{2}} \int f_1'(g^{-1} \begin{pmatrix} \alpha t^2 & \\ & 1 \end{pmatrix} h) f_2'(g^{-1}) W_{F_s}^{\psi^{-1}}(a(b')g) \overline{(\mathbf{R}^{\psi^{-1}}(h)\Psi)(tz')} dhdg$$

which, changing variables, gives

$$|\alpha t^2|^{\frac{1}{2}} \int f_1'(g^{-1}a(\alpha)h) f_2'(g^{-1}) W_{F_s}^{\psi^{-1}}(a(b')g) \overline{(\mathbf{R}^{\psi^{-1}}(\begin{pmatrix} t^{-2} & \\ & 1 \end{pmatrix} h)\Psi)(tz')} dhdg.$$

This simplifies into

$$|\alpha|^{\frac{1}{2}} \int f_1'(g^{-1}a(\alpha)h) f_2'(g^{-1}) W_{F_s}^{\psi^{-1}}(a(b')g) \overline{(\mathbf{R}^{\psi^{-1}}(h)\Psi)(z')} dhdg$$

which is

$$|\alpha|^{\frac{1}{2}} \mathscr{O}^{\psi}(\alpha, \gamma'; s; f' \otimes \Phi \otimes \Psi).$$

□

Remark 3.2.8. The lemma says that if we identify $\gamma' \in B'(\alpha)$ with $\mathbf{t}(\gamma') \in B'(\alpha t^2)$, then the function $|\alpha|^{\frac{1}{2}} \mathscr{O}^{\psi}(\alpha, \gamma'; s; f' \otimes \Phi \otimes \Psi)$ depends only on the class of α in $F^\times/(F^\times)^2$.

Another unpleasant characteristic of the local orbital integral is its dependency on a choice of additive character ψ. We isolate this dependency in the following lemma.

Lemma 3.2.9. *Let $a \in F^\times$. The non-normalized orbital integral satisfies*

$$\mathscr{O}^{\psi_a}(\alpha, \gamma'; s; f' \otimes \Phi \otimes \Psi) = |a|^{s-1}\omega(a) \mathscr{O}^{\psi}(\alpha, \gamma'; s; (R(a)f') \otimes \Phi \otimes \Psi)$$

Proof. Recall that

$$\begin{aligned} W_{F_s}^{\psi_a^{-1}}(a(b')g) &= \int_F F_s(w_0 n(x) a(b')g)\psi^{-1}(ax)dx \\ &= \frac{1}{|a|}\int_F F_s(w_0 \begin{pmatrix} a^{-1} & \\ & 1 \end{pmatrix} n(x) \begin{pmatrix} a & \\ & 1 \end{pmatrix} a(b')g)\psi^{-1}(x)dx \\ &= |a|^{s-1}\omega(a)\int_F F_s(w_0 n(x) a(ab')g)\psi^{-1}(x)dx \\ &= |a|^{s-1}\omega(a) W_{F_s}^{\psi^{-1}}(a(ab')g) \end{aligned}$$

and

$$\mathbf{R}^{\psi_a}(h) = \mathbf{R}^{\psi}(\begin{pmatrix} a & \\ & 1 \end{pmatrix} h \begin{pmatrix} a^{-1} & \\ & 1 \end{pmatrix})$$

hence, by definition,

$$\mathscr{O}^{\psi_a}(\alpha, \gamma'; s; f' \otimes \Phi \otimes \Psi) = \int f_1'(g^{-1}a(\alpha)h) f_2'(g^{-1}) W_{F_s}^{\psi_a^{-1}}(a(b')g)\overline{(\mathbf{R}^{\psi_a^{-1}}(h)\Psi)(z')}dhdg$$

We can rewrite this as

$$|a|^{s-1}\omega(a) \int f_1'(g^{-1}a(\alpha)h) f_2'(g^{-1}) W_{F_s}^{\psi^{-1}}(a(b'a)g)\overline{(\mathbf{R}^{\psi^{-1}}(\begin{pmatrix} a & \\ & 1 \end{pmatrix} h \begin{pmatrix} a^{-1} & \\ & 1 \end{pmatrix})\Psi)(z')}dhdg$$

which further simplifies to

$$|a|^{s-1}\omega(a) \int f_1'(g^{-1}a(\alpha)h \begin{pmatrix} a & \\ & 1 \end{pmatrix}) f_2'(g^{-1} \begin{pmatrix} a & \\ & 1 \end{pmatrix}) W_{F_s}^{\psi^{-1}}(a(b')g)\overline{(\mathbf{R}^{\psi^{-1}}(h)\Psi)(z')}dhdg$$

which is exactly

$$|a|^{s-1}\omega(a)\mathscr{O}^{\psi}(\alpha, \gamma'; s; (R(a)f') \otimes \Phi \otimes \Psi)$$

as desired. □

We are in particular interested in the value of the orbital integral when $s = \frac{1}{2}$, in which case this gives

$$|a|^{\frac{1}{2}}\omega(a)\mathscr{O}^{\psi_a}(\alpha, \gamma'; \frac{1}{2}; f' \otimes \Phi \otimes \Psi) = \mathscr{O}^{\psi}(\alpha, \gamma'; \frac{1}{2}; (R(a)f') \otimes \Phi \otimes \Psi).$$

The presence of the action by $R(a)$ may appear slightly concerning. However, note that if we define the *global orbital integral*

$$\mathscr{O}^{\psi}(\alpha, \gamma', s; f' \otimes \Phi \otimes \Psi) = \prod_v \mathscr{O}_v^{\psi_v}(\alpha, \gamma', s; f_v' \otimes \Phi_v \otimes \Psi_v)$$

then if $\psi = \otimes_v \psi_v$ is a global additive character, then changing to a different additive character $\psi_a(\cdot) = \psi(a\cdot)$ with a a global element only changes finitely many of the local orbital integrals (since f' must be K_v right invariant for almost all v).

This discussion behind us, we define the *normalized* $GL_2 \times GL_2$ *orbital integral* by:

Definition 3.2.10. Let

$$\mathrm{Orb}^{\psi}_{GL_2\times GL_2}(\alpha, \gamma'; s; f' \otimes \Phi \otimes \Psi) := |\alpha|^{\frac{1}{2}}|b'|^{-\frac{1}{2}}\mathscr{O}^{\psi}{}_{GL_2\times GL_2}(\alpha, \gamma'; s; f' \otimes \Phi \otimes \Psi).$$

The factor $|\alpha|^{\frac{1}{2}}$ has been discussed above. The factor $|b'|^{-\frac{1}{2}}$ appears due to considerations occurring in Chapter 4. It is a "transfer factor," allowing us to relate the local orbital integrals of this section to the local orbital integrals defined in Chapter 2. Note too that because of the product formula, the global orbital integral satisfies

$$\prod_v \mathrm{Orb}_v^{\psi_v}(\alpha, \gamma'; s; f'_v \otimes \Phi_v \otimes \Psi_v) = \prod_v \mathcal{O}_v^{\psi_v}(\alpha, \gamma'; s; f'_v \otimes \Phi_v \otimes \Psi_v).$$

Following these definitions, we return to the trace formula distribution. As before, let F now denote our global field of characteristic not 2. Let $d \in F^\times$, $E = F[t]/(t^2 - d)$ the corresponding quadratic extension, and fix, once and for all, a set of representatives Σ of $F^\times/(F^\times)^2$. Let

$$M : I(|\cdot|^{s-\frac{1}{2}}, |\cdot|^{\frac{1}{2}-s}\omega) \to I(|\cdot|^{\frac{1}{2}-s}\omega, |\cdot|^{s-\frac{1}{2}})$$

be the intertwining operator defined formally by

$$M(F_s)(g) = \int_{\mathbb{A}} F_s(w_0 n(x) g) dx.$$

Proposition 3.2.11 ($GL_2 \times GL_2$ Trace Formula, Geometric Side)**.** *Let f', Φ, Ψ be factorizable test data. The distribution I decomposes formally*

$$\begin{aligned} I^\psi(s; f' \otimes \Phi \otimes \Psi) \quad = \quad & 2\sum_\alpha \sum_{\gamma' \in B'(\alpha)^{\text{r.s.s.}}} \mathrm{Orb}^\psi(\alpha, \gamma'; s; f' \otimes \Phi \otimes \Psi) \\ & + \sum_\alpha \sum_{\mathrm{N}\,\zeta = \alpha} \int f'_1(g^{-1} a(\alpha) h) f'_2(g^{-1}) F_s(g) \overline{(\mathbf{R}^{\psi^{-1}}(h)\Psi)(\zeta)} \\ & + \sum_\alpha \sum_{\mathrm{N}\,\zeta = \alpha} \int f'_1(g^{-1} a(\alpha) h) f'_2(g^{-1}) M(F_s)(g) \overline{(\mathbf{R}^{\psi^{-1}}(h)\Psi)(\zeta)} \end{aligned}$$

Proof. We expand out the distribution $I(f' \otimes \Phi \otimes \Psi, s)$. Using the definition of $K_{f',\omega\otimes 1}$, we find that

$$\begin{aligned} I(s; f' \otimes \Phi \otimes \Psi) \quad &= \quad \int K_{f',\omega\otimes 1}((g,g),(h,n)) E(g,s) \overline{\Theta(h)} \psi(n^{-1}) \\ &= \quad \int \sum_{\gamma_1 \in GL_2(F)} f'_1(g^{-1}\gamma_1 z_1 h) \sum_{\gamma_2 \in GL_2(F)} f'_2(g^{-1}\gamma_2 z_2 n) \omega(z_1) E(g,s) \overline{\Theta(h)} \psi(n^{-1}) \end{aligned}$$

Here the integration is done over $z_1, z_2 \in [Z]$, $g \in [PGL_2]$, $h \in [PGL_2^{(2)}]$, and $n \in [N]$. So as not to clutter notation, during this proof we will almost always omit the symbols $dg, dh, dn, ...$ in our integrals.

First, we change variables $z_1 \mapsto z_2 z_1$. This gives

$$\begin{aligned} I(s; f' \otimes \Phi \otimes \Psi) &= \int \sum_{\gamma_1 \in GL_2(F)} f_1'(g^{-1} z_2 \gamma_1 z_1 h) \sum_{\gamma_2 \in GL_2(F)} f_2'(g^{-1} z_2 \gamma_2 n) \omega(z_2) E(g,s) \overline{\omega(z_1) \Theta(h)} \psi(n^{-1}) \\ &= \int \sum_{\gamma_1 \in GL_2(F)} f_1'(g^{-1} z_2 \gamma_1 z_1 h) \sum_{\gamma_2 \in GL_2(F)} f_2'(g^{-1} z_2 \gamma_2 n) E(z_2^{-1} g, s) \overline{\Theta(h)} \psi(n^{-1}) \end{aligned}$$

Collapsing the integrations over z_1 and z_2 into the integrations over g and h, we can rewrite

$$I(s; f' \otimes \Phi \otimes \Psi) = \int \sum_{\gamma_1 \in GL_2(F)} f_1'(g^{-1} \gamma_1 h) \sum_{\gamma_2 \in GL_2(F)} f_2'(g^{-1} \gamma_2 n) E(g,s) \overline{\Theta(h)} \psi(n^{-1})$$

where now $g \in [GL_2]$, $h \in [GL_2^{(2)}]$, and $n \in [N]$.

Now, we change variables in the first sum, $\gamma_1 \mapsto \gamma_2 \gamma_1$ and unfold the summation in γ_2 to find

$$I(s; f' \otimes \Phi \otimes \Psi) = \int \sum_{\gamma_1 \in GL_2(F)} f_1'(g^{-1} \gamma_1 h) f_2'(g^{-1} n) E(g,s) \overline{\Theta(h)} \psi(n^{-1})$$

where $g \in GL_2(\mathbb{A})$, $h \in [GL_2^{(2)}]$, and $n \in [N]$. We can write every element $\gamma_1 \in GL_2(F)$ uniquely as $\gamma_1 = a(\alpha)\delta$ where δ runs over $GL_2^{(2)}(F)$ and $a(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$ with α running over Σ, a fixed set of representatives of $F^\times/(F^\times)^2$. We unfold in δ and write

$$I(s; f' \otimes \Phi \otimes \Psi) = \sum_{\alpha \in \Sigma} \int f_1'(g^{-1} a(\alpha) h) f_2'(g^{-1} n) E(g,s) \overline{\Theta(h)} \psi(n^{-1})$$

where now $g \in GL_2(\mathbb{A})$, $h \in GL_2^{(2)}(\mathbb{A})$, and $n \in [N]$.

Change variables $g \mapsto ng$ to find

$$I(s; f' \otimes \Phi \otimes \Psi) = \sum_{\alpha \in \Sigma} \int f_2'(g^{-1}) f_1'(g^{-1} n^{-1} a(\alpha) h) E(ng,s) \overline{\Theta(h)} \psi(n^{-1})$$

We would like to change variables in h to move the n variable out of f_1'. To accomplish this, we write $n = n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, $x \in F\backslash\mathbb{A}$ and conjugate n^{-1} by $a(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$ to find

$$I(s; f' \otimes \Phi \otimes \Psi) = \sum_{\alpha \in \Sigma} \int f_2'(g^{-1}) f_1'(g^{-1} a(\alpha) n(-\alpha^{-1} x) h) E(n(x) g, s) \overline{\Theta(h)} \psi(-x)$$

Now change variables $x \mapsto \alpha x$, noting that as $\alpha \in F^\times$ the measure on $[N]$ remains unchanged, and then change variables $h \mapsto n(x)h$. This gives

$$\begin{aligned} I(s; f' \otimes \Phi \otimes \Psi) &= \sum_{\alpha \in \Sigma} \int f_2'(g^{-1}) f_1'(g^{-1} a(\alpha) n(-x) h) E(n(\alpha x) g, s) \overline{\Theta(h)} \psi(-\alpha x) \\ &= \sum_{\alpha \in \Sigma} \int f_2'(g^{-1}) f_1'(g^{-1} a(\alpha) h) E(n(\alpha x) g, s) \overline{\Theta(n(x) h)} \psi(-\alpha x) \end{aligned}$$

The integration is running over $g \in GL_2(\mathbb{A})$, $h \in GL_2^{(2)}(\mathbb{A})$, and $x \in F \backslash \mathbb{A}$. Note that the integration in x can now pass through f_1' and f_2'.

We are led to consider, for each α in our fixed set of representatives of $F^\times / \mathrm{N}\, E^\times$, the inner integral

$$\begin{aligned} \mathrm{FC}(\alpha) &= \mathrm{FC}(\alpha, g, h; \Phi, \Psi, s) \\ &:= \int_{F \backslash \mathbb{A}} E(n(\alpha x) g, \Phi; s, \omega) \overline{\Theta(n(x) h, \mathbf{R}^{\psi^{-1}}, \Psi)} \psi(-\alpha x) dx \end{aligned}$$

We use the Fourier expansions

$$E(n(x)g) = \sum_{\beta' \in F} E^{\beta'}(g) \psi(\beta' x)$$

$$\Theta(n(x)h) = \sum_{\beta'' \in F} \Theta^{\beta''}(h) \psi(\beta'' x)$$

to determine that

$$\begin{aligned} \mathrm{FC}(\alpha) &= \int_{F \backslash \mathbb{A}} \left(\sum_{\beta' \in F} E^{\beta'}(g) \psi(\alpha \beta' x) \right) \left(\sum_{\beta'' \in F} \overline{\Theta^{\beta''}(h)} \psi(-\beta'' x) \right) \psi(-\alpha x) dx \\ &= \sum_{\substack{\beta', \beta'' \in F \\ -\beta'' = \alpha(1 - \beta')}} E^{\beta'}(g) \overline{\Theta^{\beta''}(h)} \end{aligned}$$

That is, this inner integral is a sum of products of Fourier coefficients of the Eisenstein series and theta functions. Using Lemma 3.1.8 (while keeping in mind that we are using the theta function associated to $\mathbf{R}^{\psi^{-1}}$ instead of $\mathbf{R}^{\psi}$) we find that

$$\mathrm{FC}(\alpha) = \sum_{\substack{\beta' \in F, \zeta' \in E \\ \mathrm{N}\, \zeta' = \alpha(1 - \beta')}} E^{\beta'}(g) \overline{(\mathbf{R}^{\psi^{-1}}(h) \Psi)(\zeta')}$$

Writing this expression back into the decomposition of the distribution I gives.

$$I(s; f' \otimes \Phi \otimes \Psi) = \sum_{\alpha \in \Sigma} \sum_{\substack{\beta' \in F, \zeta' \in E \\ \mathrm{N}\,\zeta' = \alpha(1-\beta')}} \int f_2'(g^{-1}) f_1'(g^{-1} a(\alpha) h) E^{\beta'}(g) \overline{(\mathbf{R}^{\psi^{-1}}(h)\Psi)(\zeta')}$$

where $g \in GL_2(\mathbb{A})$ and $h \in GL_2^{(2)}(\mathbb{A})$. Applying Lemma 3.1.3 and noting that $(\mathbf{R}^{\psi^{-1}}\Psi)(\zeta') = (\mathbf{R}^{\psi^{-1}}\Psi)(-\zeta')$ concludes the calculation. □

3.3 Classification of "Orbits"

Following the introduction of the sets $B'(\alpha)$, $B'(\alpha)^{\text{r.s.s.}}$, and $B'^{\text{r.s.s.}} = \coprod_{\alpha \in \Sigma} B'(\alpha)^{\text{r.s.s.}}$ appearing on the geometric side of the trace formula, we make the following definition.

Definition 3.3.1. Let F be either a local or global field. The *regular semisimple* $GL_2 \times GL_2$ *"orbits"* consist of the set

$$B'^{\text{r.s.s.}} = \coprod_{\alpha \in \Sigma} B'(\alpha)^{\text{r.s.s.}}$$

where

$$B'(\alpha) = \{(b', z') \in F \times E/\{\pm 1\} : \mathrm{N}\, z' = \alpha(1 - b'), b' \neq 0\}$$

We similarly define the *non-regular semisimple "orbits"* to be

$$D' = D_1' \coprod D_2'$$

where each D_i'

$$D_i' = \coprod_{\alpha} \{z' \in E : \mathrm{N}\, z = \alpha\}.$$

These correspond to the non-regular semisimple terms appearing in Proposition 3.2.11.

We are unfortunately unable to give a less ad hoc definition or construction of these "orbits," and so cannot relate our notion of regular semisimple above to a true notion of regular semisimplicity (in terms of a group action on a variety). We find this feature of the "orbits" extremely mysterious!

3.4 A Simple Form of the $GL_2 \times GL_2$ Trace Formula

In this section, we make assumptions on our test data $f' \otimes \Phi \otimes \Psi$ to ensure convergence of the distribution $I(s; f' \otimes \Phi \otimes \Psi)$ and to justify the manipulations of Proposition 3.2.11. The main result is as follows.

Theorem 3.4.1 (The Simple $GL_2 \times GL_2$ Trace Formula)**.** *Let the test data $f' \otimes \Phi \otimes \Psi$ be "good." Then we have an equality*

$$\sum_{\Pi} I_{\Pi}^{\psi}(s; f' \otimes \Phi \otimes \Psi) = 2 \sum_{\substack{\alpha \in \Sigma \\ '\gamma' \in B'(\alpha)^{\text{r.s.s.}}}} Orb^{\psi}(\alpha, \gamma'; s; f' \otimes \Phi \otimes \Psi)$$

where the sum on the left hand side runs over all automorphic cuspidal representations $\Pi \subset L_0^2([GL_2 \times GL_2])$ and the sum on the right hand side runs over all regular semisimple "orbits."

We must describe the condition of test data being "good."

Definition 3.4.2. A pure tensor of test data $f' \otimes \Phi \otimes \Psi$ is "good" if it satisfies the following conditions:

1. For one place v_1, the function f'_{v_1} is essentially the matrix coefficient of a supercuspidal representation. For us, this means that $\int_{Z_{G'}(F_{v_1})} f'_{v_1}((z_1, z_2)(g_1, g_2))\omega(z_1)dz_1dz_2$ is the matrix coefficient of a supercuspidal representation of $GL_2 \times GL_2$.

2. For a non-archimedean place $v_2 \neq v_1$, $f' \otimes \Phi \otimes \Psi$ satisfies

$$\int f'_{2,v_2}(g^{-1})f'_{1,v_2}(g^{-1}a(\alpha)h)F_{v_2,s}(g)\overline{\mathbf{R}^{\psi_{v_2}}(h)\Psi_{v_2}(z')}dgdh = 0$$

   for all choices $\alpha \in F_{v_2}^{\times}/(F_{v_2}^{\times})^2$ and $z' \in E_{v_2}$, where the integral is over $g \in GL_2(F_{v_2})$ and $h \in GL_2^{(2)}(F_{v_2})$.

3. At the place v_2, $f' \otimes \Phi \otimes \Psi$ satisfies

$$\int f'_{2,v_2}(g^{-1})f'_{1,v_2}(g^{-1}a(\alpha)h)M(F_{v_2,s})(g)\overline{\mathbf{R}^{\psi_{v_2}}(h)\Psi_{v_2}(z')}dgdh = 0$$

   for all $\alpha \in F_{v_2}^{\times}/(F_{v_2}^{\times})^2$ and $z' \in E_{v_2}$.

4. The sum

$$\sum_{\substack{\alpha \in \Sigma \\ \beta', \beta'' \in F \\ -\beta'' = \alpha(1-\beta')}} \int |f_2(g^{-1})f_1(g^{-1}a(\alpha)h)E^{\beta'}(g)\overline{\Theta^{\beta''}(h)}|dgdh$$

   converges. Here the integral runs over $g \in GL_2(\mathbb{A})$ and $h \in GL_2^{(2)}(\mathbb{A})$.

We say that a mixed tensor $\sum_i f'_i \otimes \Phi_i \otimes \Psi_i$ is "good" if all of the $ef'_i \otimes \Phi_i \otimes \Psi_i$ are.

Remark 3.4.3. The restrictive conditions in the definition of "good" arise because of one reason: since we have no notion of regular semisimple elements of the group, we cannot ask that test data on the groups is regular semisimply supported at a place. Instead, we have to manually ensure that the geometric side of our trace formula converges and consists only of regular semisimple orbital integrals, hence the awkward conditions (2)., (3)., and (4).

Better said, condition (4). ensures that the geometric side of our trace formula converges. (2). and (3). directly ask that for one place v_2 the two types of non regular semisimple orbital integrals vanish, hence that the trace formula includes only regular semisimple orbital integrals. Condition (1). is the usual simple trace formula restriction that forces the spectral side to be entirely cuspidal.

Remark 3.4.4. It is an interesting question to ask if there are better restrictions we can choose to produce a simple trace formula. Condition (4)., in particular, does not appear to be local or checkable in any real sense. However, the reader should also keep in mind that, if one carries out a true trace formula analysis and regularizes both the spectral and geometric sides above, these simple trace formula conditions should be completely removable–thus, these assumptions should be thought of as a something of a placeholder until a more complete analysis of the $GL_2 \times GL_2$ trace formula is carried out.

We require the following lemma.

Lemma 3.4.5. *The integrand*

$$K_{f',\omega\otimes 1}((g,g),(h,n)E(g,s)\overline{\Theta(h)}\psi(n)^{-1}$$

appearing in the definition of $I(s; f'\otimes\Phi\otimes\Psi)$ *is compactly supported on* $GL_2(\mathbb{A})\times GL_2^{(2)}(\mathbb{A})\times N(\mathbb{A})$ *modulo* $Z(\mathbb{A})GL_2(F)\times Z(\mathbb{A})GL_2^{(2)}(F)\times N(F)$. *Moreover, if* $f'\otimes\Phi\otimes\Psi$ *satisfies (2).,(3)., and (4). above, then*

$$I(s; f'\otimes\Phi\otimes\Psi) = 2\sum_{\substack{\alpha\in\Sigma\\ ,\gamma'\in B'(\alpha)^{\text{r.s.s.}}}} \text{Orb}^{\psi}(\alpha,\gamma'; s; f'\otimes\Phi\otimes\Psi)$$

where the sum on the right hand side is finite.

Proof. Note that we can expand out

$$K_{f',\omega\otimes 1}((g,g),(h,n)E(g,s)\overline{\Theta(h)}\psi(n)^{-1}$$

as

$$\sum_{\alpha\in\Sigma}\sum_{\substack{\gamma\in GL_2(F)\\ \delta\in GL_2^{(2)}(F)}}\int_{[Z\times Z]} f_1(g^{-1}\gamma a(\alpha)\delta z_1z_2h)f_2(g^{-1}\gamma z_2 n)\omega(z_1z_2)E(g,s)\overline{\Theta(h)}\psi(n)^{-1}dz_1dz_2.$$

Observe that, as the image of $\det(\mathrm{Supp}(f_i))$ in $\mathbb{A}^\times/(\mathbb{A}^\times)^2$ is a compact set and since $F^\times/(F^\times)\hookrightarrow \mathbb{A}^\times/(\mathbb{A}^\times)^2$ discretely, the sum over $\alpha\in\Sigma$ is always finite. Thus, we are always justified in our computation of $I(s;f'\otimes\Phi\otimes\Psi)$ in writing

$$I(s;f'\otimes\Phi\otimes\Psi)=\sum_{\alpha\in\Sigma}\int\sum_{\substack{\gamma\in GL_2(F)\\ \delta\in GL_2^{(2)}(F)}} f_1(g^{-1}\gamma a(\alpha)\delta h)f_2(g^{-1}\gamma n)E(g,s)\overline{\Theta(h)}$$

and unfolding. Condition (4). ensures that we are also allowed to continue the formal calculation above and interchange the sum over Fourier coefficients with the outer integral; then, conditions (2). and (3). guarantee that only regular semisimple orbits will contribute. The finiteness of the sum follows, as conditions (2). and (3). force the local orbital integral, as a function on the regular semisimple locus $B'(F_{v_2})^{\text{r.s.s.}}$, to be compactly supported. This uses that the germs of the regular semisimple orbital integral are controlled by the non-regular semisimple orbital integrals–this can be seen directly from the constructions in Chapter 4. □

As for the spectral side, we have the following lemma.

Lemma 3.4.6. *If $f'\otimes\Phi\otimes\Psi$ satisfies (1)., then we have an equality*

$$I(s;f'\otimes\Phi\otimes\Psi)=\sum_{\Pi} I_\Pi(s;f'\otimes\Phi\otimes\Psi).$$

Proof. By condition (1). above, the kernel function $K_{f',\omega\otimes 1}(x,y)$ is equal to $K^0_{f',\omega\otimes 1}(x,y)$, its projection onto the cuspidal component. But this kernel function $K^0_{f',\omega,1}(x,y)$ is L^2. We claim it is also of rapid decay in both variables. To see this we may assume, since any f' is the finite linear combination of convolutions (by the Dixmier-Malliavin lemma–see Proposition 4.2.15), that f' is the triple convolution of smooth functions $f'=l'*m'*n'$. But then $K^0_{f,\omega\otimes 1}(x,y)=l'*K^0_{m',\omega\otimes 1}(x,y)*n'$ where the two convolutions are in the x and y variables respectively. Since $K^0_{f,\omega\otimes 1}$ is then, for each variable, the convolution of a smooth function with an L^2 function, it follows that $K^0_{f,\omega\otimes 1}$ is of rapid decay. See [Jacquet and Zagier, 1987].

Thus, the integral

$$I(s; f' \otimes \Phi \otimes \Psi) = \int K_{f',\omega\otimes 1}((g,g),(h,n))E(g,s)\overline{\Theta(h)}\psi(n^{-1})dndhdg$$

converges absolutely. By the spectral expansion of $K^0_{f',\omega\otimes 1}$, we conclude the lemma. □

We can now prove Theorem 3.4.1.

Proof. We simply apply the above Lemma, and note once again that condition (1). of "good" test data ensures that we have an absolutely convergent sum

$$I(\frac{1}{2}, f' \otimes \Phi \otimes \Psi) = \sum_{\Pi} I_{\Pi}(\frac{1}{2}, f' \otimes \Phi \otimes \Psi)$$

where Π only runs over the cuspidal spectrum. □

Part II

Local Considerations

This part is the technical heart and soul of this thesis. In it, we begin a comparison of the geometric sides of the two trace formulas. This relies on two local results: the so-called "smooth transfer conjecture" and a so-called fundamental lemma. To prove these, we embark on analysis of the local regular semisimple orbital integrals. These distributions, which appeared earlier as factors of the global orbital integrals we encountered in Part 1, can be studied through their asymptotics and can be explicitly computed for unramified data. We carry out these computations. Finally, this part concludes with a weak application of these results, which merely states that the two trace formulas, in "simple forms" can be compared.

Chapter 4

Comparison

The setup of the trace formulas of parts 1 and 2 behind us, we now embark on a comparison of the two. As in most trace formula comparisons, this proceeds predictably. First, we write down a matching between orbits, identifying the regular semisimple loci of the two quotient spaces $B^{\text{r.s.s.}}$ and ${B'}^{\text{r.s.s.}}$. We then propose and verify statements of smooth transfer and fundamental lemma type; these are Theorems 4.2.3 and Theorem 4.3.1. Finally, in Section 4.4, we provide the (now easy) proof of our main theorem, alluded to in the introduction.

4.1 Matching of Orbits

In the notation we have used, the matching of orbits seems to be an afterthought. We spell out the bijection here, although the reader has likely already noted the similarities between the two orbit spaces appearing in Chapters 2 and 3 and the notation we have used to describe them.

Let $V = W \oplus Fe$ be a three dimensional quadratic space as above, and, as usual, denote by $E = F(\sqrt{d_W})$ the discriminant algebra of W. Recall that, by Proposition 2.4.3, a regular semisimple orbit of $SO_V(F)/^{conj}SO_W(F)$ is given by

- An element $\alpha \in \Sigma$, our fixed set of representatives of $F^\times/(F^\times)^2$.
- A pair $(b, z) \in F \times (E/\{\pm 1\})$ satisfying:

  1. $\mathrm{N}\, z = \alpha(1 - b)$
  2. $b \neq 0$

3. $\alpha b \frac{d_W}{d_V} \in Q(W(F))$.

Of these last 3 conditions, 2. asks that $(\alpha; b, z)$ be *regular semisimple*, while 3. specifies that the triple $(\alpha; b, z)$ corresponds to an orbit of SO_W on SO_V rather than for a pure inner form.

Note that $(\alpha; b, z)$ corresponds to a relevant pair $W_\varepsilon \hookrightarrow V_\varepsilon$ if and only if $b \in \frac{d_V \varepsilon}{d_W \alpha} \operatorname{N} E^\times$.

We can therefore write

$$B^{\text{r.s.s.}} = \coprod_{\varepsilon} SO_{V_\varepsilon}(F)^{\text{r.s.s.}} /^{conj} SO_{W_\varepsilon}(F)$$

and, as described in Section 2.4, $B^{\text{r.s.s.}}$ can be identified with the collection of triples $(\alpha; b, z)$ with $\alpha \in \Sigma$, $(b, z) \in F \times (E/\{\pm 1\})$ satisfying $\operatorname{N} z = \alpha(1 - b)$, and $b \neq 0$.

On the other hand, the regular semisimple orbits appearing on the general linear side are parametrized by the set

$$B'^{\text{r.s.s.}}$$

where an element of $B'^{\text{r.s.s.}}$ consists of a triple $(\alpha'; b', z')$ with $\alpha' \in \Sigma$, $(b', z') \in F \times (E/\{\pm 1\})$ satisfying $\operatorname{N} z' = \alpha(1 - b')$, and $b' \neq 0$. We thus have an obvious bijective map

$$B^{\text{r.s.s.}} \to B'^{\text{r.s.s.}}$$

given by assigning to an element of $(\alpha; b, z) \in B^{\text{r.s.s.}}$ the element $(\alpha; b, z) \in B'^{\text{r.s.s.}}$.

Definition 4.1.1. We say that two regular semisimple orbits $(\alpha; b, z)$ and (α', b', z') match if $\alpha = \alpha' \in \Sigma$, $b = b'$, and $z = z'$. We write

$$(\alpha; b, z) \leftrightarrow (\alpha'; b', z').$$

We will also occasionally use the notation γ or (α, γ) for an element of $B^{\text{r.s.s.}}$; in this case, we write

$$\gamma \leftrightarrow (\alpha'; b', z')$$

to indicate that the two orbits match.

4.2 Matching of Smooth Functions

Let F be a local field of characteristic not equal to 2. Recall that, given two discriminants d_W and d_V, isomorphism classes of relevant pairs (W, V) of quadratic spaces over F with discriminants equal to d_W and d_V are indexed by the group $F^\times/\mathrm{N}\,E^\times$, where $E = F(\sqrt{d_W})$ is the quadratic ï¿œetale algebra associated to d_W. In particular, the set of pure inner forms has order 2 if the E/F is field extension, and is a singleton set if $E = F \times F$ is split. Given discriminants d_W and d_V and norm class $\varepsilon \in F^\times/\mathrm{N}\,E^\times$ we write, following the notation described just after Proposition 2.1.4, $(W_\varepsilon, V_\varepsilon)$ for the associated relevant pair, letting, as in Section 1, $\varepsilon = 1$ correspond to the quasi-split relevant pair.

Definition 4.2.1. Let $(F_\varepsilon)_{\varepsilon\in F^\times/\mathrm{N}\,E^\times}$, $F_\varepsilon \in \mathscr{S}(SO_{V_\varepsilon}(F))$ be a tuple of Schwartz functions, indexed by elements $\varepsilon \in F^\times/\mathrm{N}\,E^\times$, i.e. by pairs $(W_\varepsilon, V_\varepsilon)$ of relevant quadratic spaces of dimensions 2 and 3 with given discriminants d_W and d_V. We say that $(F_\varepsilon)_\varepsilon$ *matches* the pure tensor $f' \otimes \Phi \otimes \Psi$, where $f' \in \mathscr{S}(GL_2(F) \times GL_2(F), \Phi \in \mathscr{S}(F^2)$, and $\Psi \in \mathscr{S}^+(E)$, if, given any matching regular semisimple $\gamma \leftrightarrow \gamma'$, we have

$$\sum_{\varepsilon\in F^\times/\mathrm{N}\,E^\times} \mathrm{Orb}_{SO_{W_\varepsilon}\times SO_{V_\varepsilon}}(\gamma, F_\varepsilon) = \mathrm{Orb}_{GL_2\times GL_2}(\gamma', f' \otimes \Phi \otimes \Psi).$$

Note that the sum on the left hand side is, if we look at a given regular semisimple γ, a single term.

We can similarly say that, given a collection $(F_\varepsilon)_\varepsilon$ and a finite number of pure tensors $f'_i\otimes\Phi_i\otimes\Psi_i$, that $(F_\varepsilon)_\varepsilon$ matches the mixed tensor $\sum_i f'_i \otimes \Phi_i \otimes \Psi_i$ if and only if

$$\sum_{\varepsilon\in F^\times/\mathrm{N}\,E^\times} \mathrm{Orb}_{SO_{W_\varepsilon}\times SO_{V_\varepsilon}}(\gamma, F_\varepsilon) = \sum_i \mathrm{Orb}_{GL_2\times GL_2}(\gamma', f'_i \otimes \Phi_i \otimes \Psi_i)$$

for matching regular semisimple $\gamma \leftrightarrow \gamma'$.

Given this definition, it is natural to ask if there are many pairs of matching functions. We formulate this as

Conjecture 4.2.2 (Strong Smooth Transfer)**.** *Let $(F_\varepsilon)_\varepsilon$ be any tuple of Schwartz functions as above. Then there exists a pure tensor $f' \otimes \Phi \otimes \Psi$ which matches $(F_\varepsilon)_\varepsilon$. Conversely, given a pure tensor $f' \otimes \Phi \otimes \Psi$ as above, there exists a tuple $(F_\varepsilon)_\varepsilon$ which matches $f' \otimes \Phi \otimes \Psi$.*

Unfortunately, we do not currently know how to prove this conjecture as stated. However, we can show the following weaker form. The presence of mixed tensors in the theorem accounts for the unfortunate sum occurring on the $GL_2 \times GL_2$ side of our main theorem 4.4.1.

Theorem 4.2.3 (Weak Smooth Transfer)**.** *Let $(F_\varepsilon)_\varepsilon$ be any tuple of Schwartz functions as above. Then there exists a mixed tensor $\sum_i f_i' \otimes \Phi_i \otimes \Psi_i$ which matches $(F_\varepsilon)_\varepsilon$. Conversely, given a mixed tensor $\sum_i f_i' \otimes \Phi_i \otimes \Psi_i$ as above, there exists a tuple $(F_\varepsilon)_\varepsilon$ which matches $\sum_i f_i' \otimes \Phi_i \otimes \Psi_i$.*

The proof of Theorem 4.2.3 encompasses the remainder of this section. The proof is direct–we merely identify those functions on c which occur as (finite linear combinations of) orbital integrals, viewed as functions on the base, for both the orthogonal side and the general linear side of our comparison.

However, in order to talk about these spaces of orbital integrals it behooves us to take a moment and describe the ambient space of functions in which they lie. We use the spaces $C^\infty(B^{\text{r.s.s.}})$ and $C^\infty(B'^{\text{r.s.s.}})$; since $B^{\text{r.s.s.}}$ (resp. $B'^{\text{r.s.s.}}$) is not smooth manifold and has a singularity at $z = 0$, these spaces of smooth functions are defined in an ad hoc manner below.

Definition 4.2.4. Let $\widetilde{B}^{\text{r.s.s.}}$ be given by

$$\widetilde{B}^{\text{r.s.s.}} := \coprod_\alpha \widetilde{B}(\alpha)^{\text{r.s.s.}}$$

where

$$\widetilde{B}(\alpha)^{\text{r.s.s.}} := \{(b, z) \in F \times E : \mathrm{N}\, z = \alpha(1-b), b \neq 0\}.$$

We similarly define $\widetilde{B}'^{\text{r.s.s.}}$ by the same formulas. We identify

$$C^\infty(B(\alpha)^{\text{r.s.s.}}) := \{J \in C^\infty(\widetilde{B}(\alpha)^{\text{r.s.s.}}) : J(z, b) = J(-z, b)\}$$

and set

$$C^\infty(B^{\text{r.s.s.}}) = \bigoplus_{\alpha\in\Sigma} C^\infty(B(\alpha)^{\text{r.s.s.}})$$

and similarly for B'. When talking about functions $J = (J_\alpha)_\alpha$ in $C^\infty(B^{\text{r.s.s.}})$ we write the evaluation J_α a point $(b, z) \in B(\alpha)^{\text{r.s.s.}}$ as $J(\alpha; b, z)$. We hope this does not cause much confusion.

We also define $\mathscr{S}(B^{\text{r.s.s.}})$ and $\mathscr{S}(B'^{\text{r.s.s.}})$ in a similar way, as z-even Schwartz functions on Nash manifolds. Note in particular that functions in $\mathscr{S}(B^{\text{r.s.s.}})$, when viewed as functions on $\mathcal{B}(F)$, vanish along

$$D := \{(\alpha; b, z) \in B : b = 0\}.$$

We will also need to define the space $\mathscr{S}(B)$ and $\mathscr{S}(B')$. We do so by first setting $\mathscr{S}(\widetilde{B}(\alpha))$ to be the space generated by restrictions of functions in $\mathscr{S}(F \times E)$ to $\widetilde{B}(\alpha)$, letting $\mathscr{S}(\widetilde{B}) = \bigoplus \mathscr{S}(\widetilde{B}(\alpha))$, then we letting $\mathscr{S}(B)$ be the space of z-even functions in $\mathscr{S}(\widetilde{B})$. The same construction defines $\mathscr{S}(B')$.

We can now define the spaces of orbital integrals alluded to before.

Definition 4.2.5. Let Ω denote the space of functions on $B^{\text{r.s.s.}}$ which occur as orbital integrals, i.e. let

$$\Omega := \left\{ J(\alpha; b, z) \in C^\infty(B^{\text{r.s.s.}}) : J(\alpha; b, z) = \sum_{\varepsilon \in F^\times / \operatorname{N} E^\times} \operatorname{Orb}_{SO_{W_\varepsilon} \times SO_{V_\varepsilon}}(\alpha; b, z; F_\varepsilon) \right\}$$

for some $(F_\varepsilon)_\varepsilon \in \prod_{\varepsilon \in \in F^\times / \operatorname{N} E^\times} \mathscr{S}(SO_{V_\varepsilon}(F))$. Similarly, denote by

$$\Omega' := \operatorname{Span} \left\{ I(\alpha; b', z') \in C^\infty(B'^{\text{r.s.s.}}) : I(\alpha; b', z') = \operatorname{Orb}_{GL_2 \times GL_2}(\alpha; b', z'; f' \otimes \Phi \otimes \Psi) \right\}$$

for some $f' \otimes \Phi \otimes \Psi$. Here Span means in the algebraic sense of finite linear combinations. Note that it is unclear, if we omit the term "Span" in the definition of Ω', that the corresponding space of functions is even a vector space (while the analogous statement for Ω) is clear.

We can then restate Theorem 4.2.3 simply: if we identify the spaces $B^{\text{r.s.s.}} \cong B'^{\text{r.s.s.}}$ using the matching of regular semisimple orbits described in Section 8, then $\Omega = \Omega'$.

The argument is spread out throughout the next few subsections. In Section 4.2.1 we identify Ω as a subset of $C^\infty(B^{\text{r.s.s.}})$; in Section 4.2.2 we do the same thing for Ω' in $C^\infty(B'^{\text{r.s.s.}})$. That the orbit matching identifies these two spaces will then be readily apparent.

4.2.1 The Space Ω of Special Orthogonal Orbital Integrals

In this subsection, we examine the nature of those functions living in Ω, the space of special orthogonal orbital integrals. The main result is the following.

Proposition 4.2.6. *Let $J = J(\alpha; b, z) \in C^\infty(B^{\text{r.s.s.}})$. Then $J \in \Omega$ if and only if there exist Schwartz functions $\phi_1, \phi_2 \in \mathscr{S}(B)$ so that*

- *If E/F is a field extension, then*

$$J(\alpha; b, z) = \phi_1(\alpha; b, z) + \omega(\alpha b)\phi_2(\alpha; b, z).$$

- *If $E = F \times F$ is a split extension, then*

$$J(\alpha; b, z) = \phi_1(\alpha; b, z) + \log(|\alpha b|)\phi_2(\alpha; b, z).$$

As usual, $\omega = \omega_{E/F}$ is the quadratic character of $F^\times$ associated to the extension E/F.

Moreover, in both cases, the functions $\phi_i(\alpha; 0, z)$ can be related to non-regular semisimple orbital integrals.

This proposition characterizes the orbital integrals in terms of their growth or oscillatory behavior as $b \to 0$. Thus, in practice, in order to prove this proposition we can essentially restrict attention to a neighborhood U of $D = \{b = 0\}$, the non-regular semisimple locus. Note that it satisfies

$$D \subset \bigcup_{\alpha \in \mathrm{N}\, E^\times \cap \Sigma} B(\alpha),$$

that is, it corresponds only to certain α. Therefore, we can always assume that the neighborhood U of D satisfies

$$U \subset \bigcup_{\alpha \in \mathrm{N}\, E^\times \cap \Sigma} B(\alpha)$$

and that U does not contain any points with $z = 0$. We use ϕ_i^U to denote smooth functions on U.

As one may expect, the proof of4.2.6 breaks into two cases: when E/F is a field extension, and when $E = F \times F$ is a split extension. We correspondingly break the discussion of these cases up below. The relation of $\phi_i(\alpha; 0, z)$, $\alpha \in \mathrm{N}\, E^\times/(F^\times)^2$ to non-regular-semisimple orbital integrals is also explicated below.

Before moving into a case-by-case analysis, though, let us quickly note the following result.

Lemma 4.2.7. *Let $(F_\varepsilon)_\varepsilon \in \prod_{\varepsilon \in F^\times / \mathrm{N}\, E^\times} \mathscr{S}(SO_{V_\epsilon}(F)^{\text{s.r.s.s.}})$ be a tuple of Schwartz functions supported on the strictly regular semisimple locus. Then*

$$\sum_{\varepsilon \in F^\times / \mathrm{N}\, E^\times} \mathrm{Orb}_{SO_{W_\varepsilon} \times SO_{V_\varepsilon}}(\alpha; b, z; F_\varepsilon) \in \mathscr{S}(B^{\text{s.r.s.s.}})$$

Conversely, every Schwartz function on $B^{\text{s.r.s.s.}}$ can be obtained in this way.

Proof. There is a G-isomorphism

$$\bigcup_\varepsilon \bigcup_\alpha B_\varepsilon(\alpha)^{\text{s.r.s.s.}} \times SO_{W_\epsilon}(F) \xrightarrow{\sim} \bigcup_\varepsilon SO_{V_\varepsilon}(F)^{\text{s.r.s.s.}}$$

and it is easy to see that it induces a surjective map

$$\mathscr{S}(\bigcup_{\varepsilon}\bigcup_{\alpha} B_{\varepsilon}(\alpha)^{\text{s.r.s.s.}} \times SO_{W_{\epsilon}}(F)) \to \mathscr{S}(\bigcup_{\varepsilon}\bigcup_{\alpha} B_{\varepsilon}(\alpha)^{\text{s.r.s.s.}}).$$

□

Thus, we always have $\mathscr{S}(B^{\text{s.r.s.s.}}) \subset \Omega$. In fact, we always have $\mathscr{S}(B^{\text{r.s.s.}}) \subset \Omega$ as well. This follows from looking at behavior near the locus of regular semisimple but non-strictly regular semisimple points. Since this set consists merely of the finite collection of points $(\alpha; 1, 0)$, and since these orbits are Zariski dense but have stabilizer ± 1, it is easy to see that orbital integrals always remain smooth near these points. We omit the details.

4.2.1.1 E is a Field Extension of F

So now let us assume that E/F is a genuine extension of fields. Note that then $SO_W(F) \cong E^\times/F^\times \cong U_1(F)$ is a compact group. The special orthogonal orbital integrals have the following behavior as as $b \to 0$.

Lemma 4.2.8. *Let E/F be a genuine quadratic extension of local fields. Let $U_1(F)$ act on E by multiplication, and let $f \in \mathscr{S}(E)$ be a Schwartz function. Consider the space V of functions G on $\mathrm{N}\, E^\times$ consisting of orbital integrals*

$$G(x) = \int_{U_1(F)} f(uz)du$$

where $x \in \mathrm{N}\, E^\times$ and $\mathrm{N}\, z = x$. Then V is the space of all functions G on $\mathrm{N}\, E^\times$ such that there is a neighborhood $U \subset F$ of 0 and a smooth function $\phi(x)$ on $U \cap \mathrm{N}\, E^\times$ so that

$$\lim_{\substack{x \in \mathrm{N}\, E^\times \\ x \to 0}} \phi(x) \ \textit{exists}$$

and with $G - \phi \in \mathscr{S}(\mathrm{N}\, E^\times)$.

Proof. In the non-Archimedean case, this is apparent. If $F = \mathbb{R}$ (and $E = \mathbb{C}$) then the orbital integral reduces to the asymptotics, as $r \to 0$, of integrals of the form

$$\int_0^{2\pi} f(re^{i\vartheta})d\theta$$

for f a Schwartz function on $\mathbb{C}$. The conclusion follows immediately. □

This quickly implies the following lemma.

Lemma 4.2.9. *Let $F_\varepsilon \in \mathscr{S}(SO_{V_\varepsilon}(F))$. Then there exists a neighborhood U of $D = \{b = 0\}$, the non-regular semisimple locus of B, and a smooth function $\phi^{U,\varepsilon}(\alpha; b, z)$ on $(U \backslash D) \cap B_\varepsilon$ such that the limit*

$$\lim_{\substack{b \in \frac{d_V}{d_W \alpha} \varepsilon \operatorname{N} E^\times \\ b \to 0}} \phi^{U,\varepsilon}(\alpha; b, z) = \operatorname{vol}(SO_{W_\varepsilon}(F)) F_\varepsilon(\alpha; 0, z)$$

hence in particular this limit exists and is smooth on D. Moreover

$$\operatorname{Orb}_{SO_{W_\varepsilon} \times SO_{V_\varepsilon}}(\alpha; b, z; F_\varepsilon)|_{U \cap B_\varepsilon^{\text{r.s.s.}}} - \phi^{U,\varepsilon}(\alpha; b, z)$$

is the restriction of a Schwartz function on $B_\varepsilon^{\text{r.s.s.}}$ to $U \cap B_\varepsilon^{\text{r.s.s.}}$.

If a smooth function $J(\alpha; b, z)$ on $B^{\text{r.s.s.}}$ is supported only on $B_\varepsilon^{\text{r.s.s.}} \subset B^{\text{r.s.s.}}$ and has the limit

$$J(\alpha; 0, z) := \lim_{\substack{b \in \frac{d_V}{d_W \alpha} \varepsilon \operatorname{N} E^\times \\ b \to 0}} J(\alpha; b, z)$$

exist with $J(\alpha; 0, z)$ smooth on D, then $J(\alpha; b, z)$ is the restriction of an orbital integral of some function $F_\varepsilon \in \mathscr{S}(SO_{V_\varepsilon}(F))$ to an open subset of the form $U \cap B_\varepsilon^{\text{r.s.s.}}$.

Proof. This comes down to noting that that $SO_{W_\varepsilon}(F)$ action is, on the w part of the $z + we$ decomposition, nothing more than the action of $U_1(F)$ acting on E by multiplication. ∎

Note that given $\phi^{U,\varepsilon}$ as above, we can extend it to a smooth function on all of U, which we write as $\tilde{\phi}^{U,\varepsilon}$. We can write, recalling that $(\alpha; b, z) \in B_\varepsilon^{\text{r.s.s.}}$ if and only if $b \in -\frac{\varepsilon}{\alpha} \operatorname{N} E^\times$,

$$\mathbb{1}_{B_\varepsilon}(\alpha; b, z) = \frac{1}{2}(1 + \omega(-\varepsilon)^{-1}\omega(\alpha b)).$$

Therefore, we can write

$$\phi^{U,\varepsilon}(\alpha; b, z) = \frac{1}{2}(1 + \omega(-\varepsilon)\omega(\alpha b))\tilde{\phi}^{U,\varepsilon}(\alpha; b, z).$$

Applying this to each individual term in a pair of functions $(F_\varepsilon)_{\varepsilon \in F^\times / \operatorname{N} E^\times}$ shows the following proposition. We write the pair as $(F_{\varepsilon_0}, F_{\epsilon_1})$ where $\varepsilon_0 = 1$ and ε_1 denotes the non-trivial element of $F^\times / \operatorname{N} E^\times$.

Proposition 4.2.10. *If E/F is a field extension, then Proposition 4.3.2 is true. Moreover, in this case, we can write*

$$\begin{aligned}\phi_1^U(\alpha; b, z) &= \frac{1}{2}(\tilde{\phi}^{U,\varepsilon_0}(\alpha; b, z) + \tilde{\phi}^{U,\varepsilon_1}(\alpha; b, z)) \\ \phi_2^U(\alpha; b, z) &= \frac{\omega(-1)}{2}(\tilde{\phi}^{U,\varepsilon_0}(\alpha; b, z) - \tilde{\phi}^{U,\varepsilon_1}(\alpha; b, z)).\end{aligned}$$

Moreover, we have

$$\begin{aligned}\phi_1^U(\alpha; 0, z) &= \frac{\mathrm{vol}(SO_W(F))}{2}(F_{\varepsilon_0}(\alpha; 0, z) + F_{\varepsilon_1}(\alpha; 0, z)) \\ \phi_2^U(\alpha; 0, z) &= \frac{\mathrm{vol}(SO_W(F))}{2}\omega(-1)(F_{\varepsilon_0}(\alpha; 0, z) - F_{\varepsilon_1}(\alpha; 0, z))\end{aligned}$$

Thus, the $\phi_i^U(\alpha; 0, z)$ are related to non-regular semisimple orbital integrals.

Proof. The behavior of the orbital integral as $b \to 0$ is considered above. It remains to check that the orbital integral $\sum_\varepsilon \mathrm{Orb}_{SO_{W_\varepsilon} \times SO_{V_\varepsilon}}(\alpha; b, z; F_\varepsilon)$ is smooth as $z \to 0$, but this is apparent since the points $(\alpha; \alpha, 0)$ correspond to orbits which a stabilizer only of order 2. $\square$

4.2.1.2 $E = F \times F$ is a Split Extension

If $E = F \times F$ is a split quadratic algebra over F, then the computation of the the space of orbital integrals is slightly more involved. Now there is only one possible relevant pair $W \hookrightarrow V$, which we denote by $W_{spl} \hookrightarrow V_{spl}$. We give the group $\mathrm{GSpin}_{V_{spl}}$ explicit coordinates $z + we = (r, s) + \iota(a, b)e$ by fixing an isomorphism

$$\iota : (E = F \times F, \mathrm{N} : (a, b) \mapsto ab) \to (W^{spl}, Q^{spl}).$$

However, the group $SO_{W_{spl}}(F) \cong F^\times$ is now not longer compact, which makes the analysis of the orbital integrals slightly more delicate. We will require the following lemma, which is the split analogue of Lemma 4.2.8. The proof of this lemma may be found in [Sakellaridis, 2013] as part of Sakellaridis' analysis of his "baby case." One can also prove this lemma more directly by examining the behavior of a certain Tate integral–we omit this calculation.

Lemma 4.2.11. *Let $f \in \mathscr{S}(F \times F)$ be a Schwartz function. Then there exist two Schwartz functions $\phi_1, \phi_2 \in \mathscr{S}(F)$ such that, on the locus where $a = xy$ is non-zero,*

$$\int_{F^\times} f(tx, t^{-1}y) d^\times t = \phi_1(a) + \phi_2(a) \log |a|. \tag{4.1}$$

where $\phi_2(0)$ is a constant multiple of $f(0,0)$ and $\phi_2(0)$ is a constant multiple of the limit, as $s \to 0$ of the difference of the one dimensional Tate integrals

$$\zeta(s, \Phi|_{y=0}) = \int_{F^\times} f(x,0)|x|^s d^\times x$$

and

$$\zeta(s, \Phi|_{x=0}) = \int_{F^\times} f(0,y)|y|^s d^\times y.$$

Moreover, given any $\phi_1, \phi_2 \in \mathscr{S}(F)$, there exists a Schwartz function $f \in \mathscr{S}(F^{\oplus 2})$ such that the equality 4.1 holds.

This lemma allows us to immediately deduce the following proposition.

Proposition 4.2.12. *Let $F \in \mathscr{S}(SO_{V_{spl}}(F))$. Then there exists smooth functions $\phi_1^U(\alpha; b, z)$ and $\phi_2^U(\alpha; b, z)$ on an open neighborhood U of D in B such that*

$$\mathrm{Orb}_{SO_{W_{spl}} \times SO_{V_{spl}}}(\alpha; b, z; F) - \left(\phi_1^U(\alpha; b, z) + \phi_2^U(\alpha; b, z) \log|\alpha b|\right)$$

is the restriction of a Schwartz function on $B^{\mathrm{r.s.s.}}$ to $U \cap B^{\mathrm{r.s.s.}}$.

Moreover if a smooth function $J(\alpha; b, z)$ is of this form, then $J(\alpha; b, z)$ lies in Ω.

Proof. The $SO_{W_{spl}}$ action on the w part of the $z + we$ decomposition is nothing more than the action of $F^\times$ described in 4.2.11. Therefore, as $b \to 0$, the behavior of the orbital integral becomes exactly that behavior described above. That the orbital integral remains smooth as $z \to 0$ follows for the same reasons as in 4.2.10. □

This concludes the proof of Proposition 4.2.6.

4.2.2 The Space Ω' of General Linear Orbital Integrals

In this subsection we determine the possible singularities of the $GL_2 \times GL_2$ orbital integrals, i.e. we write down the space Ω' completely. Recall that we have defined

$$\mathrm{Orb}^{\psi}_{GL_2 \times GL_2}(\alpha, b', z'; f' \otimes \Phi \otimes \Psi) = |\frac{\alpha}{b'}|^{\frac{1}{2}} \mathscr{O}^{\psi}{}_{GL_2 \times GL_2}(\alpha, b', z'; f' \otimes \Phi \otimes \Psi)$$

where

$$\begin{aligned} \mathscr{O}^{\psi}(\alpha, b', z'; f' \otimes \Phi \otimes \Psi) &= \int f_1'(g^{-1}a(\alpha)h) f_2'(g^{-1}) W^{\psi^{-1}}_{F_{\frac{1}{2}}}(a(b')g) \overline{\mathbf{R}^{\psi^{-1}}(h)\Psi(z)} dg dh \\ &= \int f_1'(g^{-1}h) f_2'(g^{-1}a(\alpha^{-1})) W^{\psi^{-1}}_{F_{\frac{1}{2}}}(a(\alpha b')g) \overline{\mathbf{R}^{\psi^{-1}}(h)\Psi(z)} dg dh. \end{aligned}$$

First, some preliminaries.

4.2.2.1 Asymptotics of Whittaker Functions

As we have discussed, the Whittaker term in the above expression is defined by considering the smooth principal series, defined, for $s \in \mathbb{C}$, by

$$I(|\cdot|^{s-\frac{1}{2}}, |\cdot|^{\frac{1}{2}-s}\omega^{-1}) := \mathrm{Ind}_{B(F)}^{GL_2(F)}(|\cdot|^{s-\frac{1}{2}}.|\cdot|^{\frac{1}{2}-s}\omega^{-1}).$$

When F is non-archimedean, we need not concern ourselves with topologizing $I(|\cdot|^{s-\frac{1}{2}}, |\cdot|^{\frac{1}{2}-s}\omega^{-1})$, which is defined as the usual induction space. When $F = \mathbb{R}$ or $\mathbb{C}$, then when we write $I(|\cdot|^{s-\frac{1}{2}}, |\cdot|^{\frac{1}{2}-s}\omega^{-1})$ we mean the "smooth induction", i.e. the space of smooth moderate growth sections of the appropriate line bundle on $B(F)\backslash GL_2(F)$. It is a Frechet space.

We can construct such smooth sections $F_s = F(\cdot, \Phi; s, \omega) \in I(|\cdot|^{s-\frac{1}{2}}, |\cdot|^{\frac{1}{2}-s}\omega^{-1})$ from the data of $\Phi \in \mathscr{S}(F^{\oplus 2})$ by setting

$$F_s(g) = \int_{Z_{GL_2}(F)} \Phi(e_2^* gz)|\det(gz)|^s \omega(z) dz.$$

We consider

$$W_{F_s}^{\psi^{-1}}(g) = \Lambda_0(g.F_s)$$

where

$$\Lambda_0(F_s) = \int_F F_s(w_0 n(x))\psi^{-1}(x)dx$$

is the usual Whittaker functional for principal series. Note that as F_s is clearly a smooth section in $I(|\cdot|^{s-\frac{1}{2}}, |\cdot|^{\frac{1}{2}-s}\omega^{-1})$, $W_{F_s}^{\psi^{-1}}$ is also a smooth vector in the space of the Whittaker model of $I(|\cdot|^{s-\frac{1}{2}}, |\cdot|^{\frac{1}{2}-s}\omega^{-1})$.

Now, calling $W = W_{F_{\frac{1}{2}}}^{\psi^{-1}}$, we are interested in

$$W_{F_{\frac{1}{2}}}^{\psi^{-1}}(a(t)g) = g.W(a(t)).$$

If we fix g and view this as a function of t, then the problem of describing the behavior as $t \to 0$ of $W_{F_{\frac{1}{2}}}^{\psi}(a(t)g)$ becomes that of describing the (smooth) Kirillov model $\mathscr{K}(1,\omega)$ of $I(1,\omega)$. Better said, we have the following identifications between spaces:

$$\begin{array}{ccccc} I(1,\omega) & \tilde{\to} & \mathscr{W}(1,\omega) & \tilde{\to} & \mathscr{K}(1,\omega) \\ F & \mapsto & (W : g \mapsto \lambda_0(\pi(g)F) & \to & (t \mapsto W(a(t))) \end{array}$$

and we can apply the following well-known result.

Lemma 4.2.13 (Asymptotics of Whittaker Functions)**.** *Suppose that* $\chi_1\chi_2^{-1} \neq |\cdot|^{\pm 1}$*, so that the principal series representation* $I(\chi_1, \chi_2)$ *is irreducible. Then the Kirillov model* $\mathscr{K}(\chi_1, \chi_2) \subset C^\infty(F^\times)$ *consists of all functions of the formao*

$$\phi(t) = \begin{cases} |t|^{\frac{1}{2}}(\phi_1(t)\chi_1(t) + \phi_2(t)\chi_2(t)) & \text{if } \chi_1 \neq \chi_2 \\ |t|^{\frac{1}{2}}\chi(t)(\phi_1(t) + \phi_2(t)\log|t|) & \text{if } \chi_1 = \chi_2 = \chi \end{cases}$$

where $\phi_i(t)$ *are Schwartz functions on* F*.*

Proof. When F is non-Archimedean, this follows almost directly from the calculation of the Jacquet module of a principal series of GL_2–see [Bump, 1997] for details. When $F = \mathbb{R}$ or $\mathbb{C}$, this is slightly more delicate. We refer the reader to [Wallach, 1992], Chapter 15, for a discussion of this in the generality of an arbitrary reductive group G. For a more down-to-earth description of this Archimedean case for GL_2, we also refer to [Jacquet, 2004]. □

Specializing to $\chi_1 = 1, \chi_2 = \omega$ with $\omega = \omega_{E/F}$, this gives

Lemma 4.2.14. *The functions* $b' \mapsto W^{\psi^{-1}}_{F_{\frac{1}{2}}}(a(\alpha b'))$ *appearing in the* $GL_2 \times GL_2$ *orbital integral are exactly those functions of* b' *which can be written as*

$$\phi(x) = \begin{cases} |\alpha b'|^{\frac{1}{2}}(\phi_1(\alpha b') + \phi_2(\alpha b')\omega(\alpha b')) & \text{if } E/F \text{ is a field extension} \\ |\alpha b'|^{\frac{1}{2}}(\phi_1(\alpha b') + \phi_2(\alpha b')\log|\alpha b'|) & \text{if } E = F \times F \text{ is split.} \end{cases}$$

Before proceeding further, we must mention a tool that will come up repeatedly in our discussion.

4.2.2.2 The Dixmier-Malliavin Lemma

We record the statement of this extremely useful fact here.

Proposition 4.2.15 (The Dixmier-Malliavin Lemma, [Dixmier and Malliavin, 1978])**.** *Let* G *be a Lie group acting continuously on a Frechet space* V*. Then every element* $v \in V^\infty$*, the smooth vectors in* V *can be written as a finite sum*

$$v = \sum \pi(f_i)v_i$$

with $v_i \in V$ *and* $f_i \in C_c^\infty(G)$*. Better said*

$$C_c^\infty(G)V = V^\infty.$$

Of course, the analogous statement for a group over a non-archimedean field is obvious.

The Dixmier-Malliavin theorem has an often cited corollary:

Corollary 4.2.16. *Let G be a Lie group. Every element $f \in \mathscr{S}(G)$ can be written as a finite linear combination of convolutions*

$$f = \sum_i f_i * g_i$$

where $f_i \in C_c^\infty(G)$ and $g_i \in \mathscr{S}(G)$.

4.2.2.3 The Space Ω'

Let us now begin our examination of the space of orbital integrals Ω'. We write $K(t)$ for the function $W_{F_{\frac{1}{2}}}^{\psi^{-1}}(a(t))$ and denote the right regular action of $GL_2(F)$ given by $g : W_{F_{\frac{1}{2}}}^{\psi^{-1}}(\cdot) \to W_{F_{\frac{1}{2}}}^{\psi^{-1}}(\cdot g)$ by $g : K \mapsto g.K$. We then write

$$\mathscr{O}^\psi(\alpha, b', z'; f' \otimes \Phi \otimes \Psi) = \int f_1'(g^{-1}h) f_2'(g^{-1}a(\alpha^{-1})) W_{F_{\frac{1}{2}}}^{\psi^{-1}}(a(\alpha b')g)\overline{\mathbf{R}^{\psi^{-1}}(h)\Psi(z)}dgdh.$$

We break the integration in g up into a sum over right cosets, writing $GL_2(F) = \coprod_{\alpha' \in \Sigma} GL_2^{(2)}(F)a(\alpha')$. This gives

$$\sum_{\alpha' \in \Sigma} \int f_1'(a(\alpha'^{-1})g^{-1}h) f_2'(a(\alpha'^{-1})g^{-1}a(\alpha^{-1}))(ga(\alpha')).K(\alpha b')\overline{\mathbf{R}^{\psi^{-1}}(h)\Psi(z)}dgdh$$

where now the integration in g runs merely over $GL_2^{(2)}(F)$. Changing variables $h \mapsto gh$ provides

$$\sum_{\alpha' \in \Sigma} \int_{GL_2^{(2)}(F)} f_2'(a(\alpha'^{-1})g^{-1}a(\alpha^{-1}))((ga(\alpha')).K)(\alpha b')\overline{\mathbf{R}^{\psi^{-1}}(g)\left(\int_{GL_2^{(2)}(F)} \overline{f_1'(a(\alpha')h)}\mathbf{R}^{\psi^{-1}}(h)\Psi(z)dh\right)}dg.$$

We can view each expression

$$f_1'(a(\alpha'^{-1})h) = {}^{\alpha'} f_1'(h)$$

and

$$f_2'(a(\alpha')g^{-1}a(\alpha^{-1})) = {}^{\alpha'} f_2'(g^{-1}a(\alpha^{-1}))$$

as a collection of independent functions on $GL_2^{(2)}(F)$, indexed by α'. We also write

$$a(\alpha').K =^{\alpha'} K.$$

This gives

$$\sum_{\alpha'\in\Sigma}\int_{GL_2^{(2)}(F)} {}^{\alpha'}f_2'(g^{-1}a(\alpha^{-1}))(g.{}^{\alpha'}K)(\alpha b')\overline{\mathbf{R}^{\psi^{-1}}(g)}\overline{\left(\int_{GL_2^{(2)}(F)}\overline{{}^{\alpha'}f_1'(h)}\mathbf{R}^{\psi^{-1}}(h)\Psi(z)dh\right)}dg$$

where, after the introduction of α', all integrations over g are merely over $GL_2^{(2)}(F)$, hence the change in variables $h \mapsto gh$ is justified.

So let us examine each term in the sum. Each looks like

$$\int_{GL_2^{(2)}(F)}({}^{\alpha'}f_2'(g^{-1}a(\alpha^{-1})(g.{}^{\alpha'}K)(\alpha b')\overline{\mathbf{R}^{\psi^{-1}}(g)(\mathbf{R}^{\psi^{-1}}(\overline{{}^{\alpha'}f_1'})\Psi)(z)dh}.$$

It is clear that $\mathbf{R}^{\psi^{-1}}(\overline{{}^{\alpha'}f_1'})\Psi \in \mathscr{S}^+(E)$. We can apply the Dixmier-Malliavin lemma (and its trivial analogue for F non-archimedean) to find a finite collection of $f_{1,i}'$ and Ψ_i so that, given any desired collection of $(\Psi_\alpha')_{\alpha\in\Sigma}$ of even Schwartz functions on E,

$$\sum_i(\mathbf{R}^{\psi^{-1}}(\overline{{}^{\alpha'}f_{1,i}'})\Psi_i) = \Psi_\alpha'$$

for all α.

We have only to consider integrals of the form

$$\int_{GL_2^{(2)}(F)} f_2'(g^{-1}a(\alpha^{-1}))(g.K)(\alpha b')\overline{\mathbf{R}^{\psi^{-1}}(g)\Psi'(z)}dg.$$

This integral involves the diagonal action of $g \in GL_2^{(2)}(F)$ on a tensor product representation V. We can say more about V: note that as both $\mathscr{K}(1,\omega)$ and $\mathscr{S}^+(E)$ are Casselman-Wallach representations (smooth admissible moderate growth Frechet representations), hence nuclear Frechet spaces in the archimedean case, and admissible representations in the non-archimedean case, we can talk about the representation on the completed tensor product

$$V = \mathscr{K}(1,\omega)\hat{\otimes}\mathscr{S}^+(E).$$

V is again a Frechet space, and we identify it with the space of smooth functions on $F^\times \times E$ which can be written as

$$\begin{cases} |t|^{\frac{1}{2}}(\phi_1(t,z)+\phi_2(t,z)\omega(t)) & \text{if } E/F \text{ is a field extension} \\ |t|^{\frac{1}{2}}(\phi_1(t,z)+\phi_2(t,z)\log|t|) & \text{if } E = F\times F \end{cases}$$

for $\phi_i \in \mathscr{S}(F\times E)$. In any case, we can now apply the Dixmier Malliavin theorem a final time to show that, given any finite collection ϕ_α of functions in V, we can find a mixed tensor $\sum_i f'_{1,i}\otimes f'_{2,i}\otimes\Phi\otimes\Psi$ so that

$$\sum_i \mathscr{O}^\psi(\alpha, b', z'; f'_i \otimes \Phi_i \otimes \Psi_i) = \phi_\alpha(\alpha b', z').$$

We have shown the following proposition.

Proposition 4.2.17. *The space of functions $\Omega' \subset C^\infty(B'^{\text{r.s.s.}})$ consists of those functions I for which there exist Schwartz functions $\phi_1, \phi_2 \in \mathscr{S}(B')$ so that*

- *If E/F is a field extension, then*

$$I(\alpha; b, z) = \phi_1(\alpha; b, z) + \omega(\alpha b)\phi_2(\alpha; b, z).$$

- *If $E = F \times F$ is a split extension, then*

$$I(\alpha; b, z) = \phi_1(\alpha; b, z) + \log(|\alpha b|)\phi_2(\alpha; b, z).$$

This concludes the proof of Theorem 4.2.3.

4.3 The Fundamental Lemma

Let F be a non-archimedean local field of characteristic not equal to 2. Fix two discriminants d_W and d_V in $F^\times/(F^\times)$, and fix two lifts of d_W and d_V to $F^\times$. We will denote these lifts again by the same symbols d_W and d_V. In this section, we will consider two cases: when $E = F(\sqrt{d_W})$ is either an unramified quadratic extension of F or a split extension, i.e. $E \cong F \times F$. For later simplicity of notation, we assume that our representative of d_W satisfies $\operatorname{val}_\varpi d_W = 0$. We also assume, to declutter the notation a little, that $\operatorname{val}_\varpi d_V = 0$ as well–while this may appear restrictive, in application d_V is an element of our global field, hence for almost all places our assumption is satisfied. In any case, the computations of this section do not rely in any significant way on this assumption.

Suppose first that E/F is an unramified quadratic extension. There are two isomorphism classes of pairs of relevant quadratic spaces of discriminant d_W and d_V corresponding to $\varepsilon = 1$ or $\varepsilon = \varpi$;

we denote them (in this section only) by

$$\begin{aligned} V_0 &= W_0 \oplus F e_0 \\ V_1 &= W_1 \oplus F e_1. \end{aligned}$$

They are equipped with quadratic forms Q_0 and Q_1 respectively. The first space V_0 is split over F, i.e. contains an isotropic vector[1], and is found by setting

$$(W_0, Q_0|_{W_0}) \cong (E, \mathrm{N})$$

while we also set

$$(W_1, Q_1|_{W_1}) \cong (E, \varpi \mathrm{N}).$$

For notational use, let us denote these isomorphisms by ι, i.e. whenever we wish to regard an element $z \in E$ as a vector in W_i, we write it as $\iota(z)$. For both spaces, we set

$$Q_i(e_i) = \frac{d_V}{d_W}.$$

It is psychologically helpful to recall that the orthogonal groups associated to these spaces can be thought of as

$$\begin{aligned} SO_{V_0} &\cong PGL_2 \\ SO_{W_0} &\cong \mathrm{Res}_{E/F}\, \mathbb{G}_m/\mathbb{G}_m \end{aligned}$$

and

$$\begin{aligned} SO_{V_1} &\cong PB^\times \\ SO_{W_1} &\cong \mathrm{Res}_{E/F}\, \mathbb{G}_m/\mathbb{G}_m \end{aligned}$$

where B/F is the unique non-split quaternion algebra over F. Note that $SO_{V_1}(F)$ is compact, while $SO_{V_0}(F)$ is not. Fixing a lattice $\mathscr{L}_0 \subset V_0$ allows us to define a maximal compact subgroup $K_0 = SO(\mathscr{L}_0)$ of $SO_{V_0}(F)$; we take

$$\mathscr{L}_0 = \iota(\mathscr{O}_E) + \mathscr{O} e_0$$

[1] To see that V_0 has an isotropic, note that by assumption $-\frac{d_V}{d_W} \in \mathrm{N}\, E^\times$.

with form Q_0 as described above.

Now suppose that E/F is split, i.e. $E = F \times F$. In this case, the situation degenerates and there is only one isomorphism class among pairs of relevant quadratic spaces with discriminants d_W and d_V; we denote it by

$$V^{spl} = W^{spl} \oplus Fe^{spl}$$

and its quadratic form by Q^{spl}. We have

$$(W^{spl}, Q^{spl}|_{W^{spl}}) \cong (E, \mathrm{N}) \cong (F \times F, (x,y) \mapsto xy)$$

and

$$Q^{spl}(e^{spl}) = \frac{d_V}{d_W}.$$

Note we have

$$\begin{aligned} SO_{V^{spl}} &\cong PGL_2 \\ SO_{W^{spl}} &\cong \mathbb{G}_m. \end{aligned}$$

Again, we can define a maximal compact subgroup $K^{spl} = SO(\mathscr{L}^{spl})$ where we define the lattice

$$\mathscr{L}^{spl} = \iota(\mathscr{O}_{F\times F}) + \mathscr{O}e^{spl}$$

with form Q^{spl} as above.

The matching of smooth functions described in the previous section assumes a very concrete appearance in this unramified situation.

Theorem 4.3.1 (The Fundamental Lemma). *Let notation be as above, with F a non-archimedean local field of residue characteristic not equal to 2. We write $\mathbb{1}_{GL_2(\mathscr{O})\times GL_2(\mathscr{O})} \otimes \mathbb{1}_{\mathscr{O}^{\oplus 2}} \otimes \mathbb{1}_{\mathscr{O}_E}$ simply as $\mathbb{1}$. Then:*

1. *If E/F is an unramified quadratic extension, then the functions $\mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}$ and $(\mathbb{1}_{K_0}, 0)$ match, in the sense that*

$$\mathrm{Orb}^{\psi}(\alpha, \gamma'; \mathbb{1}) = \begin{cases} \mathrm{Orb}(\alpha, \gamma; \mathbb{1}_{K_0}) & \textit{if } (\alpha, \gamma') \leftrightarrow (\alpha, \gamma) \textit{ for } (\alpha, \gamma) \in SO_{V_0}(F) \\ 0 & \textit{else} \end{cases}$$

2. *If E/F is a split extension, then the functions $\mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}$ and $\mathbb{1}_{K^{spl}}$ match, in the sense that*

$$\mathrm{Orb}^{\psi}(\alpha, \gamma'; \mathbb{1}) = \mathrm{Orb}(\alpha, \gamma; \mathbb{1}_{K^{spl}})$$

for matching regular semisimple elements $\gamma \leftrightarrow \gamma'$.

The proof of this theorem occupies the remainder of the section. The argument is straightforward: we simply compute the orbital integrals on both sides of the equality explicitly. This is broken up into the next few subsections.

4.3.1 The Special Orthogonal Orbital Integral

In this subsection, we evaluate the local special orthogonal orbital integral for unramified test data when $E = F(\sqrt{d_W})$ is split or unramified over F. The result is the following:

Proposition 4.3.2. *Let F be a non-archimedean local field of residue characteristic not equal to 2. Then:*

1. *If E/F is an unramified quadratic extension, and $\gamma = z + we_0$ is regular semisimple, then*

$$\mathrm{Orb}_{SO_{W_0} \times SO_{V_0}}(\gamma, \mathbb{1}_{K_0}) = \begin{cases} 1 + q^{-1} & \textit{if } \mathrm{val}_{\varpi}\, \alpha \textit{ is even and } |\,\mathrm{N}(z)|, \frac{|Q_0(e_0)Q_0(w)|}{|\alpha|} \leq 1 \\ 0 & \textit{else} \end{cases}$$

2. *If $E = F \times F$ is split over F, and $\gamma = z + we^{spl} = (r, s) + \iota(a, b)e^{spl}$ is regular semisimple, then*

$$\mathrm{Orb}_{SO_{W^{spl}} \times SO_{V^{spl}}}(\gamma, \mathbb{1}_{K^{spl}}) = \begin{cases} (1 - q^{-1})(\mathrm{val}_{\varpi} \frac{ab}{\alpha} + 1) & \textit{if } \mathrm{val}_{\varpi}\, \alpha \textit{ is even and } |r|, |s|, \frac{|Q^{spl}(e^{spl})ab|}{|\alpha|} \leq 1 \\ 0 & \textit{else} \end{cases}.$$

We proceed case by case.

4.3.1.1 The Case of E/F Unramified

To compute the orbital integral

$$\mathrm{Orb}_{SO_{W_0} \times SO_{V_0}}(\gamma, \mathbb{1}_{K_0}) = \int_{SO_{W_0}(F)} \mathbb{1}_{K_0}(h^{-1}\gamma h)dh$$

for $\gamma \in SO_{V_0}(F)$ a regular semisimple (with respect to the action of SO_{W_0}) element, we pull back to the group GSpin_{V_0} and work with the familiar coordinates $x = z + we_0$. Denote by $\rho : \mathrm{GSpin}_{V_0} \to SO_{V_0}$ the usual projection map, and consider the group $\rho^{-1}(K_0) \subset \mathrm{GSpin}_{V_0}(F)$. In the following computation, we will again repeatedly appeal to the observation made before Lemma 2.4.16: that in $C(W)$, $zw = w\bar{z}$.

Lemma 4.3.3. *The group $\rho^{-1}(K_0)$ is given by*

$$\rho^{-1}(K_0) = \{x = z + we_0 \in C_0(V_0)^{\times} : \left|\frac{Q_0(w)}{\mathrm{N}(x)}\right| \leq 1\}.$$

Proof. This is an easy computation. For this proof, let us call

$$\widetilde{K_0} = \{x = z + we_0 : \left|\frac{Q_0(w)}{\mathrm{N}(x)}\right| \leq 1\}.$$

We wish to show that $\rho^{-1}(K_0) = \widetilde{K_0}$. To see that $\rho^{-1}(K_0) = \rho^{-1}(SO(\mathscr{L}_0)) \subseteq \widetilde{K_0}$, we note that

$$\begin{aligned}\rho(x)(e_0) &= xe_0x^{-1} = \frac{1}{\mathrm{N}(x)}(\mathrm{N}(z) - Q_0(e_0)Q_0(w))e_0 + Q_0(e_0)(zw + w\bar{z}) \\ &= (1 - 2\frac{Q_0(e_0)Q_0(w)}{\mathrm{N}(x)})e_0 + 2\frac{Q_0(e_0)}{\mathrm{N}(x)}zw\end{aligned}$$

and observe that, by looking at the e_0-component of the above expression, $\rho(x)(e_0) \in \mathscr{L}_0$ only if $\left|\frac{Q_0(w)}{\mathrm{N}(x)}\right| \leq 1$.

Conversely, we have $\widetilde{K_0} \subseteq \rho^{-1}(K_0)$. To see this, let $x \in \tilde{K}_0$. By the computation above, $\rho(x)(e_0) \in \mathscr{L}_0$ since $\left|Q_0(2\frac{Q_0(e_0)}{\mathrm{N}(x)}zw)\right| = \left|Q_0(\frac{zw}{\mathrm{N}(x)})\right| \left|\frac{\mathrm{N}(z)Q_0(w)}{\mathrm{N}(x)^2}\right| \leq 1$. We can also compute, for $w' \in \mathscr{L}_0 \cap W_0$, that

$$\begin{aligned}\rho(x)(w') &= xw'x^{-1} = \frac{1}{\mathrm{N}(x)}(-(w'wz + wzw')e_0 + (z^2w' - Q_0(e_0)ww'w)) \\ &= \frac{1}{\mathrm{N}(x)}(-2(wz, w')_{Q_0}e_0 + (z^2w' - Q_0(e_0)ww'w)).\end{aligned}$$

Note that both $z^2w', ww'w \in W$. Observe that $\left|Q_0(\frac{wz}{\mathrm{N}(x)})\right| = \left|\frac{\mathrm{N}(z)Q_0(w)}{\mathrm{N}(x)^2}\right|$, so that the vector $\frac{1}{\mathrm{N}(x)}wz \in \mathscr{L}_0 \cap W_0$. Since E/F is unramified, $\mathscr{L}_0 \cap W_0 \cong (\mathscr{O}_E, \mathrm{N}) \subset (E, \mathrm{N}) \cong W_0$ is a self-dual lattice, and thus $-2(\frac{wz}{\mathrm{N}(x)}, w')_{Q_0} \in \mathscr{O}$. It is also easy to see that $\left|Q_0(\frac{z^2w'}{\mathrm{N}(x)})\right|, \left|Q_0(\frac{Q_0(e_0)ww'w}{\mathrm{N}(x)})\right| \leq 1$. Thus, $x \in \tilde{K}_0$ must preserve $\mathscr{L}_0$. □

This preliminary lemma behind us, we can now write

$$\begin{aligned}\mathrm{Orb}_{SO_{W_0}\times SO_{V_0}}(\gamma,\mathbb{1}_{K_0}) &= \int_{SO_{W_0}(F)} \mathbb{1}_{K_0}(h^{-1}\gamma h)dh \\ &= \int_{F^\times\backslash E^\times} \mathbb{1}_{\rho^{-1}(K_0)}(u^{-1}\gamma u)du\end{aligned}$$

where here we confuse $\gamma \in SO_{V_0}(F)$ with a choice of lifting to $\mathrm{GSpin}_{V_0}(F)$. We ask that the lifting satisfy $\mathrm{N}(\gamma) \in \Sigma$, our fixed set of representatives of $F^\times/(F^\times)^2$; as before, we call $\mathrm{N}(\gamma) = \alpha \in \Sigma$. Note that if $\mathrm{val}_\varpi \alpha$ is odd, then $\gamma \notin \rho^{-1}(K_0)$; thus the orbital integral vanishes for trivial reasons in this case, and we can assume that $\mathrm{val}_\varpi \alpha = 0$.

We can thus write $\gamma = z + we$, and recall that regular semisimple means that $Q(w) \neq 0$. We have

$$\begin{aligned}\mathrm{Orb}_{SO_{W_0}\times SO_{V_0}}(\gamma,\mathbb{1}_{K_0}) &= \int_{F^\times\backslash E^\times} \mathbb{1}_{\rho^{-1}(K_0)}(z+\rho(u^{-1})(w)e_0)du \\ &= \begin{cases}\mathrm{vol}(SO_{W_0}(F)) & \text{if } \mathrm{val}_\varpi \mathrm{N}(\gamma) = 0 \text{ and} \frac{|\mathrm{N}(z)|}{|\mathrm{N}(\gamma)|}, \frac{|Q_0(w)|}{|\mathrm{N}(\gamma)|} \leq 1 \\ 0 & \text{else}\end{cases}. \\ &= \begin{cases}1+q^{-1} & \text{if } \mathrm{val}_\varpi \alpha = 0 \text{ and } |\mathrm{N}(z)|, \frac{|Q_0(e_0)Q_0(w)|}{|\alpha|} \leq 1 \\ 0 & \text{else}\end{cases}.\end{aligned}$$

The factor $1+q^{-1}$ appears as the local Tamagawa measure (which we have been using throughout!) for $SO_{W_0}(F) \cong U_1(F)$ gives $SO_{W_0}(F)$ measure $1+q^{-1}$.

4.3.1.2 The Case of E/F Split

Again, we wish to compute the orbital integral

$$\mathrm{Orb}_{SO_{W^{spl}}\times SO_{V^{spl}}}(\gamma,\mathbb{1}_{K^{spl}}) = \int_{SO_{W^{spl}}(F)} \mathbb{1}_{K^{spl}}(h^{-1}\gamma h)dh$$

for $\gamma \in SO_{V^{spl}}(F)$ a regular semisimple (with respect to the action of $SO_{W^{spl}}$) element. Recall that we have fixed an isomorphism

$$\iota : (\mathscr{O}_E = \mathscr{O}\times\mathscr{O}, \mathrm{N} : (a,b)\mapsto ab) \to (\mathscr{L}^{spl}\cap W^{spl}, Q^{spl})$$

which induces an isomorphism of quadratic spaces over F

$$\iota : (E = F\times F, \mathrm{N} : (a,b)\mapsto ab) \to (W^{spl}, Q^{spl}).$$

Given this, let us be overly explicit about the nature of the group $\mathrm{GSpin}_{W^{spl}}$. Given an element (r,s) of $\mathbb{G}_m \times \mathbb{G}_m = \mathbb{G}_m^2$, define the element $z(r,s) \in C_0(W)$ by letting $z(r,s)$ satisfy

$$z(r,s)\iota(a,b) = \iota(ra, sb)$$

in the Clifford algebra $C(W)$.[2] Similarly, given an element t of $\mathbb{G}_m$, let it act on W by associating to t an element $\zeta(t) \in SO_{W^{spl}}$ which acts by $\zeta(t)(\iota(a,b)) = (ta, t^{-1}b)$. Then we have the following isomorphism of exact sequences:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{G}_m & \xrightarrow{u\mapsto(u,u)} & \mathbb{G}_m^2 & \xrightarrow{(r,s)\mapsto\frac{r}{s}} & \mathbb{G}_m & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle z} & & \downarrow{\scriptstyle \zeta} & & \\
1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \mathrm{GSpin}_{W^{spl}} & \xrightarrow[\rho]{} & SO_{W^{spl}} & \longrightarrow & 1
\end{array}$$

In this way, we fix an isomorphism between $\mathrm{GSpin}_{W^{spl}}$ and $\mathbb{G}_m^2$ that trivializes all calculations in the Clifford algebra.

As before, we work on the central extension $\mathrm{GSpin}_{V^{spl}} = C_0(V^{spl})^\times \cong GL_2$ and with coordinates $x = (r,s) + \iota(a,b)e^{spl}$, where $(r,s),(a,b) \in F \times F = E$. Denote by $\rho : \mathrm{GSpin}_{V^{spl}} \to SO_{V^{spl}}$ the projection map, and consider $\rho^{-1}(K^{spl}) \subset \mathrm{GSpin}_{V_0}(F)$.

Lemma 4.3.4. *The group $\rho^{-1}(K^{spl})$ is given by*

$$\rho^{-1}(K^{spl}) = \{(r,s) + \iota(a,b)e^{spl} \in C_0(V^{spl})^\times : \left|\frac{r^2}{\mathrm{N}(x)}\right|, \left|\frac{s^2}{\mathrm{N}(x)}\right|, \left|\frac{a^2}{\mathrm{N}(x)}\right|, \left|\frac{b^2}{\mathrm{N}(x)}\right| \leq 1\}.$$

Proof. As in Lemma 4.3.3, this is routine computation. Call

$$\widetilde{K^{spl}} = \{z + we^{spl} = (r,s) + \iota((a,b))e^{spl} : \left|\frac{r^2}{\mathrm{N}(x)}\right|, \left|\frac{s^2}{\mathrm{N}(x)}\right|, \left|\frac{a^2}{\mathrm{N}(x)}\right|, \left|\frac{b^2}{\mathrm{N}(x)}\right| \leq 1\}$$

We wish to show that $\rho^{-1}(K^{spl}) = \widetilde{K^{spl}}$. To see one inclusion $\rho^{-1}(K^{spl}) \subseteq \widetilde{K^{spl}}$, note, as in Lemma 4.3.3, that if $x = (r,s) + \iota(a,b) \in \rho^{-1}(K^{spl})$ and $w' = \iota(a',b')$, we can compute that

$$\begin{aligned}
\rho(x)(w') &= \frac{1}{\mathrm{N}(x)}(-(w'\bar{z}w + wzw')e^{spl} + (z^2w' - Q^{spl}(e^{spl})ww'w)) \\
&= \frac{1}{\mathrm{N}(x)}(-2(\bar{z}w, w')_{Q^{spl}}e^{spl} + (z^2w' - Q^{spl}(e^{spl})ww'w)) \\
&= \frac{1}{\mathrm{N}(x)}(-(sab' + rba')e^{spl} + (\iota(r^2a' - Q^{spl}(e^{spl})a^2b', s^2b' - Q^{spl}(e^{spl})b^2a').
\end{aligned}$$

[2] When working in $C_0(W^{spl})$ in the future, we will omit writing the z, instead confusing the element $z(r,s) \in C_0(W^{spl})$ with the abstract element $(r,s) \in F \times F$.

Taking $a' = 1, b' = 0$ gives

$$\rho(x)(\iota(1,0)) = \frac{1}{\mathrm{N}(x)}(-rbe^{spl} + \iota(r^2, -Q^{spl}(e^{spl})b^2))$$

while taking $a' = 0, b' = 1$ gives

$$\rho(x)(\iota(0,1)) = \frac{1}{\mathrm{N}(x)}(-sae^{spl} + \iota(-Q^{spl}(e^{spl})a^2, s^2).$$

We conclude that if $x \in \rho^{-1}(K^{spl})$, then $\left|\frac{r^2}{\mathrm{N}(x)}\right|, \left|\frac{s^2}{\mathrm{N}(x)}\right|, \left|\frac{a^2}{\mathrm{N}(x)}\right|, \left|\frac{b^2}{\mathrm{N}(x)}\right| \leq 1$.

Conversely, we have $\widetilde{K^{spl}} \subset \rho^{-1}(K^{spl})$. If $x \in \widetilde{K^{spl}}$, then it is apparent by the formulas above that $\rho(x)(\iota(1,0)), \rho(x)(\iota(0,1)) \in \mathscr{L}^{spl}$. A similar computation reveals

$$\rho(x)(e^{spl}) = (1 - 2\frac{Q^{spl}(e^{spl})ab}{\mathrm{N}(x)})e_0 + 2\frac{Q^{spl}(e^{spl})}{\mathrm{N}(x)}\iota(ra, sb),$$

hence $\rho(x)(e^{spl})$ also lies in $\mathscr{L}^{spl}$. □

This description in hand, let us compute the orbital integral directly. Again, we confuse a regular semisimple $\gamma \in SO_{V^{spl}}(F)$ with a choice of lifting to $\mathrm{GSpin}_{V^{spl}}(F)$, and, as before, we ask that this lifting satisfy $\mathrm{N}(\gamma) \in \Sigma$, where Σ denotes our fixed set of representatives of $F^\times/(F^\times)^2$. We call $\mathrm{N}(\gamma) = \alpha$. As before, if $\mathrm{val}_\varpi \alpha$ is odd, then $\gamma \notin \rho^{-1}(K^{spl})$; thus the orbital integral vanishes for trivial reasons in this case, hence we can assume that $\mathrm{val}_\varpi \alpha = 0$. We write for $\gamma = (r,s) + \iota(a,b)e^{spl}$ a regular semisimple element of $\mathrm{GSpin}_{V^{spl}}(F)$

$$\begin{aligned}\mathrm{Orb}_{SO_{W^{spl}} \times SO_{V^{spl}}}(\gamma, \mathbb{1}_{K^{spl}}) &= \int_{SO_{W^{spl}}(F)} \mathbb{1}_{K^{spl}}(h^{-1}\gamma h)dh \\ &= \int_{F^\times} \mathbb{1}_{\rho^{-1}(K^{spl})}((r,s) - \iota(t^{-1}a, tb)e^{spl})d^\times t\end{aligned}$$

where here we break from our earlier tradition and write the multiplicative Haar measure on $F^\times$ as $d^\times t = \frac{d^+t}{|t|}$ where d^+t is the usual additive Haar measure on F giving $\mathscr{O}$ volume 1 (this deviates from other notation in this paper, where we have always used the unadorned symbols $dg, dh, dt, \ldots$to denote the Haar measure on the group over which we were integrating).

Clearly, if $|\left|\frac{ab}{\mathrm{N}(\gamma)}\right| > 1$ or if one of $\left|\frac{r^2}{\mathrm{N}(\gamma)}\right|, \left|\frac{s^2}{\mathrm{N}(\gamma)}\right| > 1$ then this integral is zero. Similarly, if $\mathrm{val}_\varpi \mathrm{N}(\gamma) = \mathrm{val}_\varpi \alpha$ is odd then the integral vanishes as well, so we can assume that $\mathrm{val}_\varpi \alpha$ Denote by $l = \frac{1}{2}\mathrm{val}_\varpi \frac{a^2}{\mathrm{N}(\gamma)}$ and $k = \frac{1}{2}\mathrm{val}_\varpi \frac{b^2}{\mathrm{N}(\gamma)}$, and $l + k = \mathrm{val}_\varpi \frac{ab}{\mathrm{N}(\gamma)}$. Observe that whenever the orbital integral is non-zero, these are all integers, as $\mathrm{N}(\gamma)$ has even valuation. Then,

$$\begin{aligned}
\mathrm{Orb}_{SO_{W^{spl}}\times SO_{V^{spl}}}(\gamma, \mathbb{1}_{K^{spl}}) &= \int_{F^\times} \mathbb{1}_{\rho^{-1}(K^{spl})}((r,s) - \iota(t^{-1}a, tb)e^{spl})d^\times t \\
&= \begin{cases} \int_{\{t\in F: -k\le \mathrm{val}_\varpi t \le l\}} d^\times t & \text{if } \mathrm{val}_\varpi \mathrm{N}(\gamma) = 0 \text{ and } \frac{|ab|}{|\mathrm{N}(\gamma)|}, \frac{|r^2|}{|\mathrm{N}(\gamma)|} \frac{|s^2|}{|\mathrm{N}(\gamma)|} \le 1 \\ 0 & \text{else} \end{cases} \\
&= \begin{cases} \sum_{i=-k}^{l} \int_{\varpi^i \mathscr{O}^\times} d^\times t & \text{if } \mathrm{val}_\varpi \alpha \text{ is even and } \frac{|ab|}{|\alpha|}, |r^2|, |s^2| \le 1 \\ 0 & \text{else} \end{cases} \\
&= \begin{cases} (1-q^{-1})(\mathrm{val}_\varpi \frac{ab}{\alpha} + 1) & \text{if } \mathrm{val}_\varpi \alpha \text{ is even and } \frac{|Q^{spl}(e^{spl})ab|}{|\alpha|}, |r|, |s| \le 1 \\ 0 & \text{else} \end{cases}.
\end{aligned}$$

The factor $1 - q^{-1}$ appears as the volume of $\mathscr{O}^\times$for our local Tamagawa measure $d^\times t = \frac{d^+t}{|t|}$. This concludes the proof of Proposition 4.3.2.

4.3.2 The General Linear Orbital Integral

Let notation be as before, with F a non-archimedean local field of residue characteristic not equal to 2, $E = F(\sqrt{d_W})$ over F a quadratic ï¿œtale algebra which is either unramified or split, and $\omega = \omega_{E/F}$ the quadratic character determined by E. Let ψ be an additive character of F with conductor $\mathscr{O}$.

We must consider the non-normalized unramified $GL_2 \times GL_2$ orbital integral. In the notation of Chapter 3, this is

$$\mathscr{O}^\psi{}_{GL_2\times GL_2}(\mathbb{1}_{GL_2(\mathscr{O})\times GL_2(\mathscr{O})} \otimes \mathbb{1}_{\mathscr{O}^{\oplus 2}} \otimes \mathbb{1}_{\mathscr{O}_E}, s; \alpha, \gamma').$$

In this subsection we shorten this, to save ink, to

$$\begin{aligned}
\mathscr{O}^\psi(s;\gamma') &:= \mathscr{O}^\psi{}_{GL_2\times GL_2}(\mathbb{1}_{GL_2(\mathscr{O})\times GL_2(\mathscr{O})} \otimes \mathbb{1}_{\mathscr{O}^{\oplus 2}} \otimes \mathbb{1}_{\mathscr{O}_E}, s; \alpha, b', z') \\
&= \int \mathbb{1}_{GL_2(\mathscr{O})}(g^{-1}a(\alpha)h)\mathbb{1}_{GL_2(\mathscr{O})}(g^{-1})W^{\psi^{-1}}_{F_s^\circ}(a(b')g)\overline{(\mathbf{R}^{\psi^{-1}}(h)\mathbb{1}_{\mathscr{O}_E})(z')}dh.
\end{aligned}$$

Recall that here $\alpha \in \Sigma$ for Σ our fixed set of representatives of $F^\times/(F^\times)^2$, and $\gamma' = (\alpha; z', b'), (z', b') \in B(\alpha)$ is a regular semisimple tuple satisfying $\mathrm{N}(z') = \alpha(1-b')$. The integration above runs over $g \in GL_2(F)$, $h \in GL_2^{(2)}(F)$. The integral above simplifies to

$$\mathscr{O}^\psi(s;\gamma') = \int \mathbb{1}_{GL_2(\mathscr{O})}(a(\alpha)h)W^{\psi^{-1}}_{F_s^\circ}(a(b')g)\overline{\mathbf{R}^{\psi^{-1}}(h)\mathbb{1}_{\mathscr{O}_E}(z')}$$

where now $g \in GL_2(\mathscr{O})$ and $h \in GL_2^{(2)}(F)$.

Note the following: if $\mathrm{val}_\varpi(\alpha)$ is odd, then since $h \in GL_2^{(2)}(F)$ has $\mathrm{val}_\varpi \det h$ even, $a(\alpha)h$ can never be in $GL_2(\mathscr{O})$. Thus, in this case, the integral vanishes; otherwise, we can choose a representative of $\alpha \in F^\times/(F^\times)^2$ lying in $\mathscr{O}^\times$, and write

$$\mathscr{O}^\psi(s;\gamma') = \begin{cases} \int W_{F_s^\circ}^{\psi^{-1}}(a(b')g)\overline{\mathbf{R}^{\psi^{-1}}(h)\mathbb{1}_{\mathscr{O}_E}(z')} & \text{if } \mathrm{val}_\varpi \alpha \text{ is even} \\ 0 & \text{else} \end{cases}$$

where $g \in GL_2(\mathscr{O})$ and $h \in GL_2^{(2)}(\mathscr{O})$. It is easy to see (via the formulae defining the Weil representation) that

$$\mathbf{R}^{\psi^{-1}}(h)\mathbb{1}_{\mathscr{O}_E} = \mathbb{1}_{\mathscr{O}_E}$$

if $h \in GL_2^{(2)}(\mathscr{O})$. Recall too that $W_{F_s^\circ}^{\psi^{-1}}(a(b')g)$, which is defined by

$$W_{F_s^\circ}^{\psi^{-1}}(g) = \int_F F_s^\circ(w_0 n(x)g))\psi(-x)dx$$

where

$$F_s^\circ(g) = \int_{Z(F)} \mathbb{1}_{\mathscr{O}^{\oplus 2}}(e_2^* gz)\,|\det gz|^s\,\omega(z)dz$$

is the spherical vector in $\mathrm{Ind}_{B(F)}^{GL_2(F)}(\delta^{s-\frac{1}{2}} \otimes (1,\omega^{-1})) = \mathrm{Ind}_{B(F)}^{GL_2(F)}(|\cdot|^{s-\frac{1}{2}}, |\cdot|^{\frac{1}{2}-s}\omega^{-1)}$ satisfying $F_s^o(bk) = \delta^s(b)\omega^{-1}(b_{22})$ can also be easily computed. That is, observe that when $g \in GL_2(\mathscr{O})$, we have $W_{F_s^\circ}^{\psi^{-1}}(a(b')g) = W_{F_s^\circ}^{\psi^{-1}}(a(b'))$. We can thus appeal to the following well known result:

Proposition 4.3.5 (The Casselman-Shalika-Shintani Formula)**.** *Let χ_1, χ_2 be unramified characters, call $\alpha_i = \chi_i(\varpi)$, and let $f^\circ \in \mathrm{Ind}_{B(F)}^{GL_2(F)}(\chi_1,\chi_2)$ be the unique spherical vector satisfying $f^\circ(1) = 1$. Define the spherical Whittaker function associated to f° by setting*

$$W^\circ(g) = \int_F f^\circ(w_0 n(x)g)\psi(-x)dx.$$

A priori, this is only absolutely convergent for dominant (χ_1,χ_2), i.e. when $|\alpha_1|/|\alpha_2| \leq 1$, but can be be analytically continued to all (χ_1,χ_2). Then

$$W^\circ(a(b')) = \begin{cases} (1-q^{-1}\alpha_1\alpha_2^{-1})q^{-\frac{m}{2}}\frac{\alpha_1^{m+1}-\alpha_2^{m+1}}{\alpha_1-\alpha_2} & \text{if } b' \in \mathscr{O},\ m = \mathrm{val}_\varpi b' \\ 0 & \text{else} \end{cases}.$$

There are two cases to consider.

First, suppose E/F is an unramified quadratic extension. Then $\omega(\varpi) = -1$ and the Casselman-Shalika formula gives, for $g \in GL_2(\mathscr{O})$

$$\begin{aligned} W^{\psi^{-1}}_{F_s^\circ}(a(b')g) &= W^{\psi^{-1}}_{F_s^\circ}(a(b')) \\ &= \begin{cases} q^{-\frac{m}{2}}(1+q^{2s-2})\frac{q^{(m+1)(\frac{1}{2}-s)}-(-1)^{m+1}q^{(m+1)(s-\frac{1}{2})}}{q^{\frac{1}{2}-s}+q^{s-\frac{1}{2}}} & \text{if } b' \in \mathscr{O},\ m = \operatorname{val}_\varpi b' \\ 0 & \text{else} \end{cases}. \end{aligned}$$

Specializing to $s = \frac{1}{2}$, we find

$$\begin{aligned} W^{\psi^{-1}}_{F_s^\circ}(a(b')g) &= \begin{cases} q^{-\frac{m}{2}}(1+q^{-1})\frac{1-(-1)^{m+1}1}{2} & \text{if } b' \in \mathscr{O},\ m = \operatorname{val}_\varpi b' \\ 0 & \text{else} \end{cases} \\ &= \begin{cases} |b'|^{\frac{1}{2}}(1+q^{-1}) & \text{if } b' \in \mathscr{O},\ m = \operatorname{val}_\varpi b' \text{ even} \\ 0 & \text{else} \end{cases} \end{aligned}$$

Summarizing, if E/F is unramified, then

$$\mathscr{O}^\psi(\frac{1}{2};\alpha,\gamma') = \begin{cases} |b'|^{\frac{1}{2}}(1+q^{-1}) & \text{if } b \in \mathscr{O}, z' \in \mathscr{O}_E,\ m = \operatorname{val}_\varpi b' \text{ even} \\ 0 & \text{else} \end{cases}.$$

Now, suppose that $E = F \times F$ is a split extension Then $\omega(\varpi) = 1$ and the Casselman-Shalika formula says that if $g \in GL_2(\mathscr{O})$

$$\begin{aligned} W^{\psi^{-1}}_{F_s^\circ}(a(b')g) &= W^{\psi^{-1}}_{F_s^\circ}(a(b')) \\ &= \begin{cases} q^{-\frac{m}{2}}(1-q^{2s-2})(q^{m(\frac{1}{2}-s)}+q^{(m-2)(\frac{1}{2}-s)}+\ldots+q^{-m(\frac{1}{2}-s)}) & \text{if } b' \in \mathscr{O},\ m = \operatorname{val}_\varpi b' \\ 0 & \text{else} \end{cases}. \end{aligned}$$

Specializing to $s = \frac{1}{2}$, we find

$$\begin{aligned} W^{\psi^{-1}}_{F_s^\circ}(a(b')g) &= \begin{cases} q^{-\frac{m}{2}}(1-q^{-1})(m+1) & \text{if } b' \in \mathscr{O},\ m = \operatorname{val}_\varpi b' \\ 0 & \text{else} \end{cases} \\ &= \begin{cases} |b'|^{\frac{1}{2}}(1-q^{-1})(m+1) & \text{if } b' \in \mathscr{O},\ m = \operatorname{val}_\varpi b' \\ 0 & \text{else} \end{cases} \end{aligned}$$

Summarizing, if E/F is split, then

$$\mathscr{O}^{\psi}(\frac{1}{2};\gamma') = \begin{cases} |b'|^{\frac{1}{2}}(1-q^{-1})(m+1) & \text{if } b \in \mathscr{O}, z' = (r', s') \in \mathscr{O}^{\oplus 2},\ m = \mathrm{val}_{\varpi} b' \text{ even} \\ 0 & \text{else} \end{cases}.$$

In total, we have shown the following proposition.

Proposition 4.3.6. *Let F be a non-archimedean local field of residue characteristic not equal to 2. Let $\gamma' = (\alpha; z', b')$ be a regular semisimple "orbit". Then:*

1. *If E/F is an unramified quadratic extension,*

$$\mathscr{O}^{\psi}(\mathbb{1}_{GL_2(\mathscr{O})\times GL_2(\mathscr{O})}\otimes\mathbb{1}_{\mathscr{O}^{\oplus 2}}\otimes\mathbb{1}_{\mathscr{O}_E}, \frac{1}{2}; \alpha, b', z') = \begin{cases} |b'|^{\frac{1}{2}}(1+q^{-1}) & \textit{if } b \in \mathscr{O}, z' \in \mathscr{O}_E, m = \mathrm{val}_{\varpi} b' \textit{ is even} \\ 0 & \textit{else} \end{cases}$$

2. *If $E = F \times F$ is split over F,*

$$\mathscr{O}^{\psi}(\mathbb{1}_{GL_2(\mathscr{O})\times GL_2(\mathscr{O})}\otimes\mathbb{1}_{\mathscr{O}^{\oplus 2}}\otimes\mathbb{1}_{\mathscr{O}_E}, \frac{1}{2}; \alpha, b', z') = \begin{cases} |b'|^{\frac{1}{2}}(1-q^{-1})(m+1) & \substack{\textit{if } b\in\mathscr{O}, z'=(r',s')\in\mathscr{O}^{\oplus 2} \\ \mathrm{val}_{\varpi}\alpha \textit{ is even}, m=\mathrm{val}_{\varpi} b'} \\ 0 & \textit{else} \end{cases}.$$

Now, simply recall that if E/F is unramified, a regular semisimple "orbit" $(\alpha; z', b')$ matches an element (orbit) in $SO_{V_0}(F)/^{conj}SO_{W_0}(F)$ if and only if $\mathrm{val}_{\varpi} b' = m$ is even. This concludes the proof of Theorem 4.3.1.

4.4 The Spectral Identity

In this short section, we apply the results above to deduce our Main Theorem 1, a spectral identity between two relative trace formulas. There is essentially nothing left to prove.

We now state our theorem more precisely than in the introduction. As before, let $G = G_{\varepsilon} = SO_{W_{\varepsilon}} \times SO_{V_{\varepsilon}}$ and let $G' = GL_2 \times GL_2$. All sums over i in the mixed tensors on the $GL_2 \times GL_2$ side are finite.

Theorem 4.4.1 (The Spectral Identity)**.** *Suppose that $(f_{\varepsilon})_{\varepsilon}$ and $\sum f'_i \otimes \Phi_i \otimes \Psi_i$ are a matching set of "good" test data. Then we have the equality*

$$2\sum_{\varepsilon} J(f_{\varepsilon}) = \sum_i I(f'_i \otimes \Phi_i \otimes \Psi_i)$$

which, upon expanding the spectral side of the trace formula, gives

$$2\sum_{\varepsilon}\sum_{\pi\subset L_0^2([G_\varepsilon])} J_\pi(f_\varepsilon) = \sum_i \sum_{\Pi\subset L_0^2([G'])} I_\Pi(f_i'\otimes\Phi_i\otimes\Psi_i).$$

Proof. The proof is now easy. By Proposition 2.5.1 and Proposition 3.4.1, if the test data $(f_\varepsilon)_\varepsilon$ and $\sum f_i'\otimes\Phi_i\otimes\Psi_i$ are good, then the trace formulas decompose into an equality of a spectral side, where only the cuspidal spectra contribute, and a geometric side, where only regular semisimple orbital integrals contribute.

By the assumption of matching, the geometric sides of the two trace formulas are equal. This concludes the proof. □

It is not a priori clear that the assumption that $(f_\varepsilon)_\varepsilon \leftrightarrow \sum f_i'\otimes\Phi_i\otimes\Psi_i$ are matching sets of "good" data does not force both sides of the spectral identity to vanish, i.e. that the theorem above always reduces to $0 = 0$. This issue, of course, would disappear if there were more refined versions of these two trace formulas to compare (as then there would be no condition on "good"). We leave this work to the future.

Part III

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