Nearly Overconvergent Forms and p-adic L-Functions for Symplectic Groups

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Abstract

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We reformulate Shimura's theory of nearly holomorphic forms for Siegel modular forms using automorphic sheaves over Siegel varieties. This sheaf-theoretic reformulation allows us to define and study basic properties of nearly overconvergent Siegel modular forms as well as their p-adic families. Besides, it finds applications in the construction, via the doubling method, of p-adic partial standard L-functions associated to Siegel cuspidal Hecke eigensystems. We illustrate how the sheaf-theoretic definition of nearly holomorphic forms and Maass–Shimura differential operators helps with the choice of the archimedean sections for the Siegel Eisenstein series on the doubling group $\mathrm{Sp}(4n)_{/\mathbb{Q}}$ and the study of the p-adic properties of their restrictions to $\mathrm{Sp}(2n)_{/\mathbb{Q}} \times \mathrm{Sp}(2n)_{/\mathbb{Q}}$. The selection of archimedean sections, together with p-adic interpolation considerations, then naturally gives the sections at the place p. We compute p-adic zeta integrals corresponding to those sections. Finally, we construct the p-adic standard L-functions associated to ordinary families of Siegel Hecke eigensystems and obtain their interpolation properties.

Contents

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Chapter 1

Introduction

In this work we introduce a sheaf-theoretic definition of Shimura's nearly holomorphic forms and differential operators, and apply it to study Hida and Coleman families of nearly holomorphic Siegel modular forms. Then using the tool of nearly holomorphic forms, we construct the p-adic partial standard L-functions for ordinary families of Hecke eigensystems on symplectic groups by utilizing the doubling method.

The doubling method and the standard L-functions for symplectic groups. One of the main motivations for Shimura to develop his theory of nearly holomorphic forms is to study the algebraicity of special L-values and Klingen Eisenstein series via the doubling method. The doubling method, developed by Garrett [24], Piatetski-Shapiro–Rallis [51] and Shimura [57], provides integral representations for the L-functions and Klingen Eisenstein series associated to a large class of cuspidal automorphic representations of reductive groups. It is well known that by applying the doubling method, one can reduce the study of analytic properties of L-functions, as well as algebraicity and p-adic interpolation of special L-values, to that of the Siegel Eisenstein series on the corresponding doubling group.

Let us first briefly recall the doubling method formula, and see how nearly holomorphic forms and differential operators naturally show up in the consideration of the archimedean place. For simplicity we restrict ourselves to the case $G = \mathrm{Sp}(2n)_{/\mathbb{Z}}$, which is the one we will focus on later.

Let $\pi \subset \mathcal{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A}))$ be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$ and ξ be a primitive Dirichlet character. For a finite place v such that both π_v and ξ_v are unramified, the local L-factor is defined as

$$L_v(s, \pi \times \xi) = (1 - \xi(q_v)q_v^{-s})^{-1} \prod_{i=1}^{n} (1 - \xi(q_v)\alpha_{v,i}q_v^{-s})^{-1}(1 - \xi(q_v)\alpha_{v,i}^{-1}q_v^{-s})^{-1},$$

where q_v is the cardinality of the residue field and $\alpha_{v,1}^{\pm 1}, \cdots, \alpha_{v,n}^{\pm 1}$ are the Satake parameters of π_v. Take S to be a finite set of places of $\mathbb{Q}$ containing the archimedean place and all the finite places where π_v or ξ_v is ramified. Consider the partial standard L-function $L^S(s, \pi \times \xi) = \prod_{v \notin S} L_v(s, \pi \times \xi)$. The Euler product converges absolutely for $\mathrm{Re}\,(s) \gg 0$ and has a meromorphic continuation to the whole complex plane with at most simple poles [43, 51].

Let $H = \mathrm{Sp}(4n)_{/\mathbb{Z}}$ and fix the (holomorphic) embedding $\iota : G \times G \hookrightarrow H$. For a cuspidal automorphic form $\varphi \in \pi$, we denote by φ^ϑ its MVW involution (which belongs to the contragredient representation of π, hence to $\overline{\pi}$ by multiplicity one), i.e. $\varphi^\vartheta(g) = \varphi(\vartheta g \vartheta)$. Let $P_H \subset H$ be the doubling Siegel parabolic. Pick a factorizable section $f(s, \xi)$ from the normalized induction $I_{P_H}(s, \xi) = \mathrm{Ind}_{P_H(\mathbb{A})}^{H(\mathbb{A})}(\xi|\cdot|^s \circ \det)$. Let $E(\cdot, f(s, \xi))$ be the Siegel Eisenstein series, and $E^*(\cdot, f(s, \xi))$ be the normalization of $E(\cdot, f(s, \xi))$ by dividing the factor $d^S(s, \xi)$, which is a product of Dirichlet L-functions $d^S(s, \xi)$ (see §4.2 for precise definitions). Given $\varphi_1, \varphi_2 \in \pi$ with factorizable images under $\pi \cong \bigotimes_v' \pi_v$ and assuming that all data outside S are unramified, the doubling method formula [24, 51, 57] reads

$$\left\langle E^*(\iota(\cdot, \cdot), f(s, \xi)), \overline{\varphi}_1 \otimes \varphi_2^\vartheta \right\rangle = L^S(s + \frac{1}{2}, \pi \times \xi) \cdot \prod_{v \in S} \frac{Z_v(f_v(s, \xi), \overline{\varphi}_{1,v}, \varphi_{2,v})}{\left\langle \overline{\varphi}_{1,v}, \varphi_{2,v} \right\rangle_v} \cdot \langle \overline{\varphi}_1, \varphi_2 \rangle, \qquad (1.1)$$

where $Z_v(\cdot, \cdot, \cdot)$ is the local zeta integral for the doubling method. In this work we use the bi-$\mathbb{C}$-linear Petersson inner product with respect to the Haar measure of $G(\mathbb{A})$ specified in **Notation**. From this formula one sees that the strategy of applying it to attain various results about the standard L-function consists of two main ingredients. One is to verify the desired properties for the Siegel Eisenstein series on H (or rather its restriction to $G \times G$), and the other is to get a good handle of the local zeta integrals appearing on the right hand side of the formula.

Using the doubling method, the meromorphic continuation of the standard L-function and possible location of its poles have been obtained in [43, 51].

If $\pi_\infty \cong \mathcal{D}_{\underline{t}}$, the holomorphic discrete series of weight $\underline{t} = (t_1, \cdots, t_n)$ (so $t_1 \geq \cdots \geq t_n \geq n+1$), the set of critical points of $L^S(s, \pi \times \xi)$ consists of integers $s_0 \in \mathbb{Z}$ such that

$$1 \leq s_0 \leq t_n - n, \text{ and } (-1)^{s_0+n} = \xi(-1),$$

or

$$n + 1 - t_n \leq s_0 \leq 0, \text{ and } (-1)^{n+1-s_0} = \xi(-1).$$

The algebraicity of these critical values, divided by certain automorphic periods (depending on π and s_0, but independent of ξ), can also be proved with the help of the doubling method formula [6, 27, 29, 57, 58].

After establishing the algebraicity, one may also consider the p-adic interpolation of those critical values, up to an explicit factor, as the p-part of ξ varies among all finite order characters of $\mathbb{Z}_p^\times$, the point s_0 varies in the critical set, and moreover the Hecke eigensystem associated to π varies in a p-adic family. When π is fixed with $\underline{t}$ being a scalar weight, the one-variable p-adic L-function is constructed using the doubling method in [6] (another construction is given in [13] using a different integral representation but with the restriction that the rank n be even).

In the above applications of the doubling method, careful analysis at the archimedean place is indispensable. It is in his study of algebraicity of the critical L-values [57,58] that Shimura developed the theory of nearly holomorphic forms to deal with the selection of archimedean sections $f_\infty(s, \xi)$ for the Siegel Eisenstein series on the doubling group. Besides Shimura's nearly holomorphic forms, theories of differential operators are also studied, through different approaches, by Böcherer [5], Ibukiyama [34] and Harris [28], with the goal of deducing algebraicity of critical L-values from various integral representations of automorphic L-functions.

Nearly holomorphic and nearly overconvergent forms. There are three main ingredients in Shimura's theory of nearly holomorphic Siegel modular forms:

(1) the space $N^r_\sigma(\mathbb{H}_n,\Gamma)$ of $W_\sigma(\mathbb{C})$-valued nearly holomorphic forms of level Γ and (non-holomorphy) degree r, together with its algebraic structure, where (σ, W_σ) is an algebraic $\mathrm{GL}(n)$-representation of finite rank and Γ is a congruence subgroup of $\mathrm{Sp}(2n,\mathbb{Z})$;

(2) the Maass–Shimura differential operator $D_{\sigma,\mathbb{H}_n} : N^r_\sigma(\mathbb{H}_n,\Gamma) \to N^{r+1}_{\sigma\otimes\tau}(\mathbb{H}_n,\Gamma)$, where τ is the irreducible finite dimensional representation of $\mathrm{GL}(n)$ of weight $(2,0,\cdots,0)$, i.e. the symmetric square of the standard representation;

(3) a holomorphic projection $N^r_\kappa(\mathbb{H}_n,\Gamma) \to N^0_\kappa(\mathbb{H}_n,\Gamma)$ for a generic weight κ.

The advantage of Shimura's theory of nearly holomorphic forms is that it admits some relatively explicit formulas that are not too complicated so that some explicit calculations can be done. This is crucial for p-adic interpolation purposes (see below for more explanation).

In Part I of this thesis we reformulate Shimura's theory using automorphic sheaves over Siegel varieties, and combine it with the ideas and techniques invented by Andreatta–Iovita–Pilloni in [1] to define and study nearly overconvergent Siegel modular forms and their families, i.e. the Coleman theory for nearly holomorphic Siegel modular forms, generalizing the work of Urban for $\mathrm{GL}(2)_{/\mathbb{Q}}$ [63]. The sheaf-theoretic formulation of nearly holomorphic forms and differential operators also makes it convenient to apply them in the study of congruences among Siegel modular forms, via the theory of p-adic forms and Hida's theory for the ordinary part, which we will see in Part II.

Set $\mathbf{G} = \mathrm{GSp}(2n)_{/\mathbb{Z}}$ with Lie algebra $\mathfrak{g} = \mathrm{Lie}\,\mathbf{G}$, and $Q_\mathbf{G} \subset \mathbf{G}$ to be its standard Siegel parabolic. Suppose that the congruence subgroup $\Gamma \subset \mathbf{G}(\mathbb{Z})$ is neat. Consider the Siegel modular variety $Y = Y_{\mathbf{G},\Gamma}$ defined over $\mathbb{Z}[1/N]$, where N is a positive integer depending on Γ, carrying the universal principally polarized abelian scheme $\mathcal{A} \to Y$ of rank n with Γ-level structure. Take a smooth toroidal compactification $X = X_{\mathbf{G},\Gamma}$ with boundary $C = X - Y$ over which the universal abelian scheme $\mathcal{A} \to Y$ extends to a semi-abelian scheme $\mathcal{G} \to X$. Over X there is the principal $Q_\mathbf{G}$-torsor $T^\times_\mathcal{H} = \underline{\mathrm{Isom}}_X\left(\mathcal{O}^{2n}_X, \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y)^{\mathrm{can}}\right)$, where the isomorphisms are required to respect the Hodge filtration and preserve the symplectic pairing of $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y)^{\mathrm{can}}$ up to similitude. Given a free algebraic representation V of $Q_\mathbf{G}$, one gets an automorphic sheaf $\mathcal{V}$ defined as the contracted product $T^\times_\mathcal{H} \times^{Q_\mathbf{G}} V$.

We show in §2.2 that if one wants to consider automorphic sheaves further equipped with integrable connections, then the right objects to consider are $(\mathfrak{g}, Q_{\mathbf{G}})$-modules. It is the $\mathfrak{g}$-module structure combined with the Gauss–Manin connection on $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y)^{\mathrm{can}}$ that gives rise to the desired connection. Given a finite rank algebraic representation σ of $\mathrm{GL}(n)$, regarded as a representation of $Q_{\mathbf{G}}$ (via the natural map sending $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in Q_{\mathbf{G}}$ to $a \in \mathrm{GL}(n)$), in §2.3 we construct a $(\mathfrak{g}, Q_{\mathbf{G}})$-module V_σ equipped with an increasing filtration V_σ^r for $r \geq 0$ satisfying $\mathfrak{g} \cdot V_\sigma^r \subset V_\sigma^{r+1}$. Composing the connection on $\mathcal{V}_\sigma$ and the Kodaira–Spencer isomorphism, a differential operator D_σ for the automorphic sheaf $\mathcal{V}_\sigma^r$ can be defined as

$$D_\sigma : \mathcal{V}_\sigma^r \xrightarrow{\nabla_\sigma} \mathcal{V}_\sigma^{r+1} \otimes_{\mathcal{O}_X} \Omega^1_X(\log C) \xrightarrow{\mathrm{KS}} \mathcal{V}_{\sigma\otimes\tau}^{r+1}(-1) \longrightarrow \mathcal{V}_{\sigma\otimes\tau}^{r+1}. \tag{1.2}$$

Denote by $X_{G,\Gamma,\mathbb{C}}$ a connected component of the base change of X to the field of complex numbers. The following commutative diagram, established in §2.5, shows that the sheaf-theoretic formulation recovers the first two ingredients in Shimura's original theory.

$$\begin{array}{ccc} H^0(X_{G,\Gamma,\mathbb{C}}, \mathcal{V}_\sigma^r) & \xrightarrow{\sim} & N_\sigma^r(\mathbb{H}_n, \Gamma) \\ \downarrow D_\sigma & & \downarrow D_{\mathbb{H}_n,\sigma} \\ H^0(X_{G,\Gamma,\mathbb{C}}, \mathcal{V}_{\sigma\otimes\tau}^{r+1}) & \xrightarrow{\sim} & N_{\sigma\otimes\tau}^{r+1}(\mathbb{H}_n, \Gamma) \end{array}$$

The construction of holomorphic projections is postponed to §3.7, where it is done in the more general setting of nearly overconvergent families.

For our later application we also show in §2.6 an equivalence, as illustrated by the commutative diagram (2.17), between the action of the sub-Lie algebra $\mathfrak{q}_G^+$ of the complexified Lie algebra $\mathfrak{g}_\mathbb{C}$ (see the definition of $\mathfrak{q}_G^+$, loc. cit) and that of the Maass–Shimura differential operator (hence the sheaf-theoretic differential operator as well).

Our reformulation of Shimura's theory is particularly convenient for combining with the tools developed in [1] to give a Coleman theory for nearly holomorphic Siegel modular forms. Fix an odd prime number p that does not divide N. We consider nearly overconvergent Siegel modular forms of tame principal level N. Let K be a finite extension of $\mathbb{Q}_p$ with evaluation v_p such that $v_p(p) = 1$, and we base change to K the compactified Siegel modular variety associated to $\Gamma = \Gamma(N)$. For simplicity we still denote it by X. Denote by X_{Iw} the finite étale cover of X corresponding to the Iwahori level structure at p. For $v \in v_p(\mathcal{O}_K)_{>0}$ one can define a rigid analytic space $\mathcal{X}_{\mathrm{Iw}}(v)$, which is a subspace of the rigid analytic space $X_{\mathrm{Iw,rig}}$ associated to the variety X_{Iw} and a strict neighborhood of the ordinary locus.

Let $\mathcal{W}$ be the weight space for Siegel modular forms whose $\overline{\mathbb{Q}}_p$-points are $\mathrm{Hom}_{\mathrm{cont}}(T_n(\mathbb{Z}_p), \overline{\mathbb{Q}}_p^{\times})$, where T_n is the maximal torus of $\mathrm{GL}(n)$. Suppose that $\mathcal{U} \subset \mathcal{W}$ is an affinoid subdomain such that all of its $\overline{\mathbb{Q}}_p$-points are w-analytic for some $w \in v_p(\mathcal{O}_K)$ within a range determined by v. In the second half of of Part I, following Andreatta–Iovita–Pilloni's construction, we define a Banach sheaf $\mathcal{V}^{\dagger,r}_{\kappa^{un},w}$ over $\mathcal{X}_{\mathrm{Iw}}(v) \times \mathcal{U}$. It admits differential operators similar to (1.2). In §3.5 the cuspidal part of its global sections $N^{\dagger,r}_{\mathcal{U},w,v,\mathrm{cusp}} := H^0\big((\mathcal{X}_{\mathrm{Iw}}(v) \times \mathcal{U}, \mathcal{V}^{\dagger,r}_{\kappa^{un},w}(-C)\big)$, i.e. the module of cuspidal nearly overconvergent forms of degree r and universal weight, is shown to be a projective $\mathcal{A}(\mathcal{U})$-Banach module. Meanwhile the action of $\mathbb{U}_p$-operators on $N^{\dagger,r}_{\mathcal{U},w,v,\mathrm{cusp}}$ is defined in §3.9, and the operator U_p is shown to be compact. Therefore the Coleman–Riesz–Serre spectral theory applies and gives the slope decomposition of $N^{\dagger,\infty}_{\mathcal{U},w,v,\mathrm{cusp}} := \bigcup_{r\geq 0} N^{\dagger,r}_{\mathcal{U},w,v,\mathrm{cusp}}$ with respect to the operator U_p as in §3.11.

Some other properties regarding nearly overconvergent Siegel modular forms, such as the existence of holomorphic projections, the p-adic splitting of the sheaf $\mathcal{V}^{\dagger,r}_{\kappa,w}$ over the ordinary locus and q-expansion characterizations of families of nearly overconvergent forms, are also discussed.

The problem of reformulating Shimura's theory in the context of automorphic sheaves is also discussed in [15, 50], but their results do not fully recover Shimura's theory or extend to give a theory of p-adic families of nearly overconvergent forms. In the GL(2) case, a construction of the Gauss–Manin connections for sheaves of nearly holomorphic and nearly overconvergent forms is also given in [53], where they consider the action of $\mathbb{G}_m$ (the Levi subgroup of the Siegel parabolic when $n = 1$) instead of that of the Lie algebra $\mathfrak{gl}(2)$.

The p-adic partial standard L-functions for Siegel ordinary families. Fixing an embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}$ and an isomorphism between $\mathbb{C}$ and $\overline{\mathbb{Q}}_p$, the goal of Part II is to interpolate,

p-adically, the critical L-values $L(s_0, \pi \times \xi)$ when the Hecke eigensystem of π varies in an n-variable ordinary family, the p-part of ξ varies among all finite order characters of $\mathbb{Z}_p^\times$, and the point s_0 varies in the right half of the critical set, i.e. $s_0 \in \mathbb{Z}$ with $1 \leq s_0 \leq t_n - n$ and $(-1)^{s_0+n} = \xi(-1)$.

Before stating our result we first define the modified Euler factor at p for p-adic interpolation. We assume the ordinarity condition on π, i.e. there exist $(\mathfrak{a}_1, \cdots, \mathfrak{a}_n) \in (\mathcal{O}_{\overline{\mathbb{Q}}_p}^\times)^n$ and $\varphi \in \pi$ such that the $\mathbb{U}_p$-operator $U_{p,\underline{a}}$ acts on φ by $\prod_{j=1}^n \mathfrak{a}_j^{a_j}$ for all $\underline{a} = (a_1, \cdots, a_n) \in \mathbb{Z}^n$ satisfying $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ (see §4.1.2 for the definition of $U_{p,\underline{a}}$, especially the normalization which depends on the $K_{G,\infty}$-type of the automorphic form it acts on). As discussed in §5.5, for a fixed π, if such an $(\mathfrak{a}_1, \cdots, \mathfrak{a}_n) \in (\mathcal{O}_{\overline{\mathbb{Q}}_p}^\times)^n$ exists, then it must be unique, and the corresponding φ must be an eigenvector for the action of $T_G(\mathbb{Z}_p)$, where T_G is the standard maximal torus of G. Furthermore, the ordinarity condition on π implies that the local factor π_p can be embedded into the principal series $\mathrm{Ind}_{B_G(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}(\theta_1, \cdots, \theta_n)$. Here B_G is the the standard Borel subgroup of G. The character θ_j of $\mathbb{Q}_p^\times$, $1 \leq j \leq n$, is defined as $\theta_j(p) = \alpha_j = p^{-(t_j - j)}\mathfrak{a}_j$ (recall that the archimedean factor π_∞, by assumption, is isomorphic to the holomorphic discrete series of lowest $K_{G,\infty}$-type $\underline{t}$), and $\theta_j|_{\mathbb{Z}_p^\times} = \psi_j$ where $\underline{\psi} = (\psi_1, \cdots, \psi_n)$ is the character of $T_G(\mathbb{Z}_p)$ through which it acts on φ.

Now let ϕ be a Dirichlet character whose conductor is prime to p and χ be a Dirichlet character whose conductor is a power of p. For π, ϕ, χ, define the modified Euler factor at p as

$$\begin{aligned} E_p(s, \pi \times \phi^{-1}\chi^{-1}) = &\frac{\left(1 - \chi^\circ(p) \cdot \phi(p) p^{s-1}\right) \prod_{j=1}^n \left(1 - (\chi\psi_j)^\circ(p) \cdot \phi(p)\alpha_j^{-1} p^{s-1}\right)}{\left(1 - \chi^\circ(p) \cdot \phi(p)^{-1} p^{-s}\right) \prod_{j=1}^n \left(1 - (\chi\psi_j)^\circ(p) \cdot \phi(p)^{-1}\alpha_j p^{-s}\right)} \\ &\times \left(\phi(p) p^{s-1}\right)^{c_\chi} G(\chi) \prod_{j=1}^n \left(\phi(p)\alpha_j^{-1} p^{s-1}\right)^{c_{\chi\psi_j}} G(\chi\psi_j). \end{aligned} \tag{1.3}$$

Here $G(\chi)$ is the Gauss sum of χ. The integer c_χ is defined such that the conductor of χ is p^{c_χ}, and χ° takes the value 0 at p, unless $c_\chi = 0$ in which case $\chi^\circ(p) = 1$. Similarly we define $G(\chi\psi_j)$, $c_{\chi\psi_j}$ and $(\chi\psi_j)^\circ$, $1 \leq j \leq n$. One can check that the factor $E_p(s, \pi \times \phi^{-1}\chi^{-1})$ defined above agrees with Coates' definition in [11, §6] of the modified Euler factor at p for the Weil–Deligne representation associated to π_p twisted by the character $\phi^{-1}\chi^{-1}$. Note that the definition there of the modified Euler factor associated to a Weil–Deligne representation does not depend on the monodromy operator. This is compatible with the definition in (1.3) which does not depend on whether the principal series $\mathrm{Ind}_{B_G(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}(\theta_1,\cdots,\theta_n)$ is irreducible or not.

Fix a positive integer N prime to p. Let F be a finite extension of $\mathbb{Q}_p$ containing all N-th roots of unity. Denote by Γ_{T_n} the p-profinite subgroup of $T_n(\mathbb{Z}_p)$ and set $\Lambda_n=\mathcal{O}_F[[\Gamma_{T_n}]]$. The $\mathcal{O}_F[[T_n(\mathbb{Z}_p)]]$-algebra $\mathbb{T}_{\mathrm{ord}}^N$, consisting of unramified Hecke operators and $\mathbb{U}_p$-operators acting on Hida families of tame principal level N, is finite and torsion free over Λ_n. A point $x\in\mathrm{Spec}(\mathbb{T}_{\mathrm{ord}}^N)(\overline{\mathbb{Q}}_p)$ corresponds to an eigensystem of the unramified Hecke operators and $\mathbb{U}_p$-operators. If that eigensystem comes from an irreducible cuspidal automorphic representation $\pi\in\mathcal{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A}))$, then it completely determines the isomorphism class of π_v for all $v\nmid N$ and we write π_x^N for the isomorphism class of the $G(\mathbb{A}^N)$-representation $\bigotimes_{v\nmid N}'\pi_v$.

Given a point $(\kappa,\underline{\tau})$ inside $\mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times\times T_n(\mathbb{Z}_p),\overline{\mathbb{Q}}_p^\times)$, we say it is arithmetic if it can be written as the product of an algebraic character with a finite order character, and we write its algebraic part (resp. finite part) as $\kappa_{\mathrm{alg}}=k$, $\underline{\tau}_{\mathrm{alg}}=\underline{t}=(t_1,\cdots,t_n)$ (resp. $\kappa_{\mathrm{f}}=\chi$, $\underline{\tau}_{\mathrm{f}}=\underline{\psi}=(\psi_1,\cdots,\psi_n)$). A point is called admissible if it is arithmetic with $t_1\geq\cdots\geq t_n\geq k\geq n+1$. Given a geometrically irreducible component $\mathcal{C}$ of $\mathrm{Spec}(\mathbb{T}_{\mathrm{ord}}^N\otimes_{\mathcal{O}_F}F)$ with function field $F_{\mathcal{C}}$, our result is

Theorem 1.0.1. *For every Dirichlet character ϕ with conductor dividing N such that ϕ^2 is non-trivial, and a pair (β_1,β_2) of positive definite symmetric $n\times n$ matrices with rational entries, there exists a p-adic measure $\mu_{\mathcal{C},\phi,\beta_1,\beta_2}\in\mathcal{M}eas(\mathbb{Z}_p^\times,\Lambda_n)\otimes_{\Lambda_n}F_{\mathcal{C}}$ with the following interpolation properties. Suppose that the weight projection map $\mathrm{Spec}(\mathbb{T}_{\mathrm{ord}}^N)\to\mathrm{Spec}(\mathcal{O}_F[[T_n(\mathbb{Z}_p)]])$ is étale at $x\in\mathcal{C}(\overline{\mathbb{Q}}_p)$. Let $\underline{\tau}\in\mathrm{Hom}_{\mathrm{cont}}(T_n(\mathbb{Z}_p),\overline{\mathbb{Q}}_p^\times)$ be the projection of x into the weight space. If $(\kappa,\underline{\tau})$ is admissible, then the evaluation of $\mu_{\mathcal{C},\phi,\beta_1,\beta_2}$ at κ,x is*

$$\begin{aligned}\left(\int_{\mathbb{Z}_p^\times}\kappa\,d\mu_{\mathcal{C},\phi,\beta_1,\beta_2}\right)(x)&=\phi(-1)^n\mathrm{vol}\left(\widehat{\Gamma}(N)\right)\frac{p^{n^2}(p-1)^n}{\prod_{l=1}^n(p^{2l}-1)}\cdot\frac{\Gamma(k-n)\Gamma_{2n}(k)}{2^{k+n-1}(\pi i)^{2nk+k-n}}\\&\quad\times\frac{Z_\infty(f_{\kappa,\underline{\tau},\infty},v_{\underline{t}}^\vee,v_{\underline{t}})}{\langle v_{\underline{t}}^\vee,v_{\underline{t}}\rangle}\cdot\sum_{\varphi\in\mathfrak{s}_x}\frac{\mathfrak{c}(\varphi,\beta_1)\mathfrak{c}(eW(\varphi),\beta_2)}{\langle\varphi,\overline{\varphi}\rangle}\\&\quad\times E_p(k-n,\pi_x^N\times\phi^{-1}\chi^{-1})\cdot L^{Np\infty}(k-n,\pi_x^N\times\phi^{-1}\chi^{-1}),\end{aligned}\tag{1.4}$$

if $\phi\chi(-1)=(-1)^k$, and otherwise the evaluation is 0. Here

- *For a positive integer m the Gamma function $\Gamma_m(s)$ is defined as $\pi^{\frac{m(m-1)}{4}}\prod_{j=0}^{m-1}\Gamma(s-\frac{j}{2})$.*

- $Z_\infty(f_{\kappa,\underline{\tau},\infty}, v_{\underline{t}}^\vee, v_{\underline{t}})$ *is the archimedean zeta integral for the doubling method with* $v_{\underline{t}}$ *being the highest weight vector inside the lowest* $K_{G,\infty}$*-type of* $\mathcal{D}_{\underline{t}}$*. Our choice of the archimedean section* $f_{\kappa,\underline{\tau},\infty}$ *in* §*4.5 guarantees its nonvanishing. When* $\phi\chi(-1) = (-1)^k$ *the section* $f_{\kappa,\underline{\tau},\infty}$ *depends only on the algebraic part* $(k, \underline{t})$ *of* $(\kappa, \underline{\tau})$*.*

- *The finite set* $\mathfrak{s}_x = \{\varphi_1, \cdots, \varphi_d\}$ *consists of an orthogonal basis of the space spanned by cuspidal holomorphic forms on* $G(\mathbb{A})$ *of weight* $\underline{t}$ *and tame principal level* N *belonging to the Hecke eigenspace parametrized by* x*. Here by being orthogonal we mean the basis satisfies* $\langle \varphi_i, \overline{\varphi_j}\rangle = 0$ *if* $i \neq j$*.*

- $\mathfrak{c}(\cdot, \beta_i)$ *is the* β_i*-th Fourier coefficient for* $i = 1, 2$*. The measure depends on the choice of the indices* β_1, β_2*, and in general there is no canonical choice for them, due to the lack of a canonical nonvanishing Fourier coefficient for Siegel modular eigenforms, which can be regarded as the analogue of the first Fourier coefficient in the case of modular forms.*

- *The operator* $W : \pi \to \pi$ *is defined as*

$$W(\varphi)(g) := \int_{N_G(\mathbb{Z}_p)} \overline{\varphi}^{\vartheta}(gu)\, du,$$

where N_G *is the unipotent radical of* B_G*. Proposition 5.7.2 shows that the ordinary projection* $eW(\varphi)$ *is nonzero if* φ *is ordinary . The operator* W *can be viewed as an analogue of the operator sending a modular form* f *of level* $\Gamma_0(N_f)$ *to* $f^c|\begin{pmatrix} 0 & -1 \\ N_f & 0 \end{pmatrix}$*.*

Remark 1.0.2. The condition $\phi^2 \neq 1$ is used to make sure that the p-adic Dirichlet L-functions which appear in our construction have no poles. Without this condition we can pick a prime number ℓ coprime to p, and get a measure $\mu_{\mathcal{C},\ell,\phi,\beta_1,\beta_2} \in \mathcal{M}eas(\mathbb{Z}_p^\times, \Lambda_n) \otimes_{\Lambda_n} F_{\mathcal{C}}$ with almost the same interpolation properties as described above, with the only difference that we need to add the factor $1 - \chi(\ell)^{-1}\ell^{-k+n}$ on the RHS of (1.4).

From the formula (1.1) it is not difficult to see that a main step of constructing the desired p-adic measure is to pick suitable local sections $f_{\kappa,\underline{\tau},v} \in I_{P_H,v}(k - \frac{2n+1}{2}, \phi^{-1}\chi^{-1})$ for all admissible points $(\kappa, \underline{\tau})$ inside $\mathrm{Hom}(\mathbb{Z}_p^\times \times T(\mathbb{Z}_p), \overline{\mathbb{Q}}_p^\times)$, and to compute the Fourier coefficients of the resulting Siegel Eisenstein series as well as the corresponding local zeta integrals. Away from $Np\infty$ we always set $f_{\kappa,\underline{\tau},v}$ to be the unramified section. The two major criteria for selecting $f_{\kappa,\underline{\tau},v}$ for $v \mid Np\infty$ are the nonvanishing and p-adic interpolation conditions, i.e.

(i) the local zeta integral $Z_v(f_{\kappa,\underline{\tau},v}, \overline{\varphi}_{1,v}, \varphi_{2,v})$ does not vanish identically for $\varphi_1, \varphi_2 \in \mathfrak{s}_x$ if the projection of the point $x \in \mathrm{Spec}(\mathbb{T}_{\mathrm{ord}})(\overline{\mathbb{Q}}_p)$ to the weight space is $\underline{\tau}$, and

(ii) the resulting $E^*(\cdot, f_{\kappa,\underline{\tau}})\big|_{G\times G}$ after a further normalization is algebraic and its q-expansion admits p-adic interpolation.

For $v \mid N$, a very simple choice is the so-called "volume section" (see §4.4).

Regarding the selection for $v = p, \infty$, one observation is that if at p we consider sections supported on the "big cell" then, due to the p-adic interpolation condition on q-expansions, the archimedean section $f_{\kappa,\underline{\tau},\infty}$ almost determines the p-adic section $f_{\kappa,\underline{\tau},p}$ and vice versa. Our strategy is to make choices for the archimedean sections incorporating both representation theory results and p-adic considerations.

As for the selection of the archimedean sections, in order to be able to use Shimura's results [54, 57] on the Fourier coefficients of the Siegel Eisenstein series, we restrict ourselves to a Lie H-submodule $R_{2k,0} \subset I_\infty(k - \frac{2n+1}{2}, \phi^{-1}\chi^{-1})$ so that the algebraicity of $E^*(\cdot, f_{\kappa,\underline{\tau}})$ can be verified by the explicit formulas in [54, 57] and the algebraicity of the action of $\mathfrak{q}_H^+$ as stated in Proposition 2.6.1. Inspired by [28, 29], we deduce from results in [35, 46] that there is a unique piece $\sigma_{k,\underline{t}}$ in the decomposition of $R_{2k,0}|_{K_\infty \times K_\infty}$ that meets the nonvanishing condition. It leads to an Eisenstein series whose restriction to $G \times G$ is holomorphic and algebraic.

The section from $\sigma_{k,\underline{t}}$ is already good enough for many applications, such as showing the algebraicity of critical L-values. However simply taking such a section makes it almost impossible to do the necessary computations for showing that the Fourier coefficients admit p-adic interpolation. This problem can be solved by using nearly holomorphic forms. This is one main motivation for us to study nearly holomorphic forms and their p-adic families. We will take a natural section $f_{\kappa,\underline{\tau},\infty}$ in $R_{2k,0}$ that does not lie exactly inside $\sigma_{k,\underline{t}}$ but has a nontrivial projection into it. Therefore it meets the nonvanishing condition, and the restriction to $G \times G$ of the corresponding Siegel Eisenstein series is not holomorphic, but a nearly holomorphic form with nontrivial holomorphic projection. For such a section some explicit computations on the Fourier coefficients can be done, and will serve as the necessary input for applying the machinery of p-adic automorphic forms to produce a Hida or Coleman family on $G \times G$ from the $E^*\left(\cdot, f_{\kappa,\underline{\tau},\infty}\right)\big|_{G\times G}$'s (see Proposition 4.6.1).

The local zeta integrals at p are calculated in §5.3-5.7, which gives the modified Euler factor $E_p(k-n, \pi_x^N \times \phi^{-1}\chi^{-1})$ in (1.4).

In Chapter 6 we apply Hida theory to produce, from the q-expansion-valued measure $\mu_{\mathcal{E},q\text{-exp}}$, a p-adic measure on $\mathbb{Z}_p^\times$ valued in cuspidal ordinary families of p-adic Siegel modular forms on $G \times G$. Combining it with a p-adic analogue of the Petersson inner product, constructed from the geometrically irreducible component $\mathcal{C}$ of $\mathrm{Spec}(\mathbb{T}_{\mathrm{ord}} \otimes_{\mathcal{O}_F} F)$, the measure in Theorem 1.0.1 is constructed.

For unitary groups there are also works done towards the construction of p-adic L-functions [16–20, 30] and Klingen Eisenstein families [64]. Their results are not yet fully available and play no role in our construction. The computations of the factors at p done in [64] assume restrictive conditions on the conductors of the nebentypes. The general cases, without taking into account the ordinary projection, are treated in [19] with an innovative use of the Godement–Jacquet local functional equation, which bypasses the computation of certain inverse Fourier transforms. It is claimed in [17] that the method for section selections there also works for symplectic groups. We expect the sections chosen by that method (although the expressions seem more complicated) to be no different from ours here because, as we have pointed out, the choice of archimedean sections imposes sections at p via p-adic interpolation considerations, and based on the ideas in [28], the choice of archimedean sections here is quite canonical as explained in the proof of Proposition 4.5.1. The nonvanishing of the archimedean zeta integral is not mentioned in [17], but should follow from the arguments in [29].

Notation. We fix an odd prime p and a positive integer $N \geq 3$ prime to p. We also fix an embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}$ and an isomorphism between $\overline{\mathbb{Q}}_p$ and $\mathbb{C}$. Let v_p be the valuation of $\mathbb{Q}_p$ with $v_p(p) = 1$. It extends uniquely (still denoted as v_p) to all algebraic field extensions of $\mathbb{Q}_p$.

For a Dirichlet character ξ we write ξ° to denote the primitive one associated to it. We denote by C_ξ the conductor of ξ° and by $G(\xi)$ the Gauss sum of ξ°. If the conductor C_ξ is a power of p we define the integer c_ξ such that $C_\xi = p^{c_\xi}$. We also regard ξ as a character of $\mathbb{Q}^\times\backslash\mathbb{A}^\times$ by requiring the local component ξ_v at $v \nmid C_\xi$ to be unramified and to take the value $\xi(q_v)$ at $q_v \in \mathbb{Q}_v^\times$, where q_v is the cardinality of the residue field. Hence if C_ξ is a power of p, then $\xi_p|_{\mathbb{Z}_p^\times} = \xi^{-1}$. The Dirichlet characters we consider in the following will almost always be primitive with only one exception. For a finite order character inside $\mathrm{Hom}(\mathbb{Z}_p^\times, \zeta_{p^\infty})$ regarded as a Dirichlet character we require it to take value 0 at p.

Fix a positive integer n. Let $\mathbf{L}_n$ be the free $\mathbb{Z}$-module of rank $2n$ spanned by the basis $e_1, \cdots, e_n, f_1, \cdots, f_n$. We will always use this basis to write related objects in matrix form. Equip $\mathbf{L}_n$ with the symplectic pairing given by $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Then $e_1, \cdots, e_n$ (resp. $f_1, \cdots, f_n$) span a maximal isotropic subspace L_n (resp. L_n^*) and we have the polarization $\mathbf{L}_n = L_n \oplus L_n^*$. We use G to denote the reductive group $G(\mathbf{L}_n) = \mathrm{Sp}(2n)_{/\mathbb{Z}}$, and and $\mathbf{G}$ to denote the similitude group $\mathrm{GSp}(2n)_{/\mathbb{Z}}$ with the multiplier character $\nu : \mathbf{G} \to \mathbb{G}_m$. In matrix form

$$\mathbf{G} = \left\{ g \in \mathrm{GL}(2n) : {}^{\mathrm{t}}g \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} g = \nu(g) \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \right\},$$

and $G = \mathbf{G}^\circ = \nu^{-1}(1)$. Let $Q_\mathbf{G}$ be the standard Siegel parabolic subgroup of $\mathbf{G}$ preserving L_n, whose Levi subgroup and unipotent radical are $M_\mathbf{G}$ and $U_\mathbf{G}$ respectively. There is the natural morphism $\mathbf{p} : Q_\mathbf{G} \to \mathrm{GL}(n)$ sending $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ to a. Denote by B_n the Borel subgroup of $\mathrm{GL}(n)$ of upper triangular matrices, and by N_n, T_n its unipotent radical and maximal torus respectively. We fix the isomorphism of $\mathbb{G}_m^n$ with T_n which sends $(a_1, \cdots, a_n)$ to $\mathrm{diag}(a_1, \cdots, a_n)$. The inverse image under $\mathbf{p}$ of B_n constitutes the standard Borel subgroup $B_\mathbf{G}$ of $\mathbf{G}$, with unipotent radical $N_\mathbf{G}$ and maximal torus $T_\mathbf{G}$. By intersecting with G, we get $Q_G, M_G, U_G, B_G, N_G, T_G$. Sometimes we also use a superscript $\circ$ to mean trivial similitude, for example $M_\mathbf{G}^\circ = M_G$. The morphism $\mathbf{p}$ identifies M_G with $\mathrm{GL}(n)$, and T_G with T_n.

For an algebra E, let $\mathrm{Rep}_E\, Q_\mathbf{G}$ (resp. $\mathrm{Rep}_{E,f}\, \mathrm{GL}(n)$) stand for the category of algebraic representations of the group $Q_{\mathbf{G}}$ (resp. $\mathrm{GL}(n)$) base changed to E on locally free E-modules (resp. locally free E-modules of finite rank). The morphism $\mathbf{p}$ induces a functor $\mathrm{Rep}_{E,f}\,\mathrm{GL}(n) \to \mathrm{Rep}_E\, Q_{\mathbf{G}}$ and we regard every object in $\mathrm{Rep}_{E,f}\,\mathrm{GL}(n)$ also as a $Q_{\mathbf{G}}$-representation.

Define the congruence subgroup $\Gamma_1(N, p^m) \subset \mathrm{Sp}(2n, \mathbb{Z})$ as

$$\{\gamma \in \mathrm{Sp}(2n,\mathbb{Z}) : \gamma \equiv I_{2n} \mod N, \quad \text{and } \gamma \mod p^m \in N_G(\mathbb{Z}/p^m\mathbb{Z})\},$$

and when $m = 0$ we denote it by $\Gamma(N)$.

Let $\mathfrak{g}$ (resp. $\mathfrak{q}_{\mathbf{G}}$) be the Lie algebra of $\mathbf{G}$ (resp. $Q_{\mathbf{G}}$). We use E_{ij} to denote the matrix with 1 in the (i,j) entry and 0 elsewhere, whose size will be clear from the context. Fix the following basis of $\mathfrak{g}$:

$$\begin{aligned} &\eta_0 = -\sum_{1\le i\le n} E_{i+n,i+n}, && \eta_{ij} = E_{ij} - E_{j+n,i+n}, && 1 \le i,j \le n, \\ &\mu_{ii}^+ = E_{i,i+n}, && \mu_{ii}^- = E_{i+n,i}, && 1 \le i \le n, \\ &\mu_{ij}^+ = E_{i,j+n} + E_{j,i+n}, && \mu_{ij}^- = E_{i+n,j} + E_{j+n,i}, && 1 \le i < j \le n. \end{aligned} \tag{1.5}$$

We put $\mathfrak{g}^\circ$ (resp. $\mathfrak{q}_G$) to be the Lie algebra of G (resp. Q_G).

For a positive integer m and an algebra R, denote by $\mathrm{Sym}(m, R)$ the set of $m \times m$ symmetric matrices with entries in R.

Consider the connected Shimura datum (G, u) with

$$\begin{aligned} u : U(1,\mathbb{R}) &\to G^{\mathrm{ad}}(\mathbb{R}) \\ e^{i\theta} &\mapsto \begin{pmatrix} \cos\theta \cdot I_n & \sin\theta \cdot I_n \\ -\sin\theta \cdot I_n & \cos\theta\cdot I_n \end{pmatrix}. \end{aligned}$$

The group $G(\mathbb{R})$ acts on u by conjugation. The centralizer

$$K_{G,\infty} = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a + bi \in U(n,\mathbb{R}) \right\}$$

is a maximal compact subgroup of $G(\mathbb{R})$, and the conjugacy class of u is $G(\mathbb{R})/K_{G,\infty}$, which is isomorphic to the Siegel upper half space

$$\mathbb{H}_n = \left\{ z \in M_n(\mathbb{C}) : {}^{\mathrm{t}}z = z, \;\; \mathrm{Im}\, z > 0 \right\}.$$

The group $G(\mathbb{R})$ acts on $\mathbb{H}_n$ by $g \cdot z = (az+b)(cz+d)^{-1}$ for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\mathbb{R})$, $z \in \mathbb{H}_n$, and we put $\mu(g,z) = cz+d$.

Fix the standard additive character $\mathbf{e}_{\mathbb{A}} = \bigotimes_v \mathbf{e}_v : \mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}^\times$ with local component $\mathbf{e}_v$ defined as $\mathbf{e}_v(x) = \begin{cases} e^{-2\pi i \{x\}_v} & v \neq \infty \\ e^{2\pi i x} & v = \infty \end{cases}$, where $\{x\}_v$ is the fractional part of x.

For a finite place v we fix the Haar measure on $\mathbb{Q}_v$ (resp. $G(\mathbb{Q}_v)$) with $\mathcal{O}_v$ (resp. $G(\mathcal{O}_v)$) having volume 1. For the archimedean place we take the usual Lebesgue measure for $\mathbb{R}$. For the group $G(\mathbb{R})$ we take the product measure where the one on $K_{G,\infty}$ has total volume 1 and the one on the $\mathbb{H}_n$ is $\det(y)^{-n-1} \prod_{1\le i\le j\le n} dx_{ij}\, dy_{ij}$. The Haar measures on $\mathbb{A}$ and $G(\mathbb{A})$ are obtained by taking products of the local ones. For the unipotent group $N_G(\mathbb{Q}_v)$ (resp. $U_G(\mathbb{Q}_v)$), we always take the Haar measure that gives the open compact subgroup $N_G(\mathcal{O}_v)$ (resp. $U_G(\mathcal{O}_v)$) volume 1 if v is finite, and the Haar measure $d_\infty u(x) = \prod_{1\le i\le j\le n} dx_{ij}$ for the archimedean place, where $u(x) = \begin{pmatrix} I_n & x \\ 0 & I_n \end{pmatrix}$ for $x \in \mathrm{Sym}(n,\mathbb{R})$.

All the above notation related to $G = \mathrm{Sp}(2n)$ is defined similarly for $H = \mathrm{Sp}(4n)$.

Part I

Nearly Overconvergent Siegel Modular Forms

Chapter 2

Nearly holomorphic Siegel modular forms

2.1 Automorphic sheaves over Siegel varieties

In this chapter we work with principal level structure $\Gamma(N)$ with $N \geq 3$ coprime to p. It would be easy to see that after inverting p, everything works also for $\Gamma_1(N, p^m)$. Let $Y = Y_{\mathbf{G},\Gamma(N)}$ be the Siegel modular variety parametrizing principally polarized abelian schemes of rank n with principal level N structure defined over $\mathbb{Z}[1/N]$. Over it there is the universal abelian scheme $\mathbf{p} : \mathcal{A} \to Y$. Take a smooth toroidal compactification X of Y with boundary $C = Y - X$. Then $\mathbf{p} : \mathcal{A} \to Y$ extends to a semi-abelian scheme $\mathbf{p} : \mathcal{G} \to X$. Let $\omega(\mathcal{G}/X)$ be the pullback of $\Omega^1_{\mathcal{G}/X}$ along the unity section of $\mathbf{p}$. According to [44, Porposition 6.9], the locally free sheaf $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y) = R^1\mathbf{p}_*\big(\Omega^\bullet_{\mathcal{A}/Y}\big)$ has a canonical extension $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y)^{\mathrm{can}} \cong \mathcal{H}^1_{log\text{-}\mathrm{dR}}(\mathcal{G}/X)$ which is a locally free subsheaf of $(Y \to X)_*\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y)$. This canonical extension $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y)^{\mathrm{can}}$ is endowed with a symplectic pairing under which $\omega(\mathcal{G}/X)$ is maximally isotropic. The Hodge filtration of $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y)$ also extends to

$$0 \longrightarrow \omega(\mathcal{G}/X) \longrightarrow \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y)^{\mathrm{can}} \longrightarrow \underline{\mathrm{Lie}}({}^{\mathrm{t}}\mathcal{G}/X) \longrightarrow 0$$

where ${}^{\mathrm{t}}\mathcal{G}/X$ is the dual semi-abelian scheme of $\mathcal{G}/X$.

There is a standard way to construct, from a representation in $\mathrm{Rep}_{\mathbb{Z}}\, Q_{\mathbf{G}}$, a quasi-coherent sheaf over X whose global sections are equipped with Hecke actions. The free sheaf $\mathcal{O}_X^{2n}$ can be equipped with a two-step filtration with the first n copies as the subsheaf, and a symplectic pairing using the matrix $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Define the right $Q_{\mathbf{G}}$-torsor over X

$$T_{\mathcal{H}}^{\times} = \underline{\mathrm{Isom}}_X \left(\mathcal{O}_X^{2n}, \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y)^{\mathrm{can}}\right)$$

to be the isomorphisms respecting the filtrations and the symplectic pairings up to similitude. The right $Q_{\mathbf{G}}$-action is given as

$$(b \cdot \phi)\,(v) = (\phi \circ b)\,(v) = \phi(bv).$$

for any open subscheme $U = \mathrm{Spec}(R) \subset X$, $\phi \in T_{\mathcal{H}}^{\times}(U)$, $v \in R^{2n}$ and $b \in Q_{\mathbf{G}}(R)$. With this right $Q_{\mathbf{G}}$-torsor one can define the functor

$$\begin{aligned} \mathcal{E} : \mathrm{Rep}_{\mathbb{Z}}\, Q_{\mathbf{G}} &\longrightarrow QCoh(X) \\ V &\longmapsto T_{\mathcal{H}}^{\times} \times^{Q_{\mathbf{G}}} V \end{aligned}$$

from the category of algebraic representations of $Q_{\mathbf{G}}$ on locally free $\mathbb{Z}$-modules to that of quasi-coherent sheaves over X. Moreover for all $V \in \mathrm{Rep}_{\mathbb{Z}}\, Q_{\mathbf{G}}$ the global sections of the associated quasi-coherent sheaf $\mathcal{E}(V)$ come with a Hecke action constructed via algebraic correspondence (cf. [21, §VII.3]). Such an $\mathcal{E}(V)$ together with the Hecke action on its global sections is often called an automorphic sheaf. Morphisms between algebraic $Q_{\mathbf{G}}$-representations induce Hecke equivariant morphisms between global sections of the corresponding quasi-coherent sheaves. The functor $\mathcal{E}$ is exact and faithful [44, Definition 6.13]. Certainly this functor is not fully faithful (see Example 2.4.5). Let V_{st} be the standard representation of $\mathbf{G}$ restricted to $Q_{\mathbf{G}}$ and W_{st} be the standard representation of $\mathrm{GL}(n)$ regarded as a $Q_{\mathbf{G}}$-representation. Then immediately from the definition we see $\mathcal{E}(V_{\mathrm{st}}) \cong \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y)^{\mathrm{can}}$ and $\mathcal{E}(W_{\mathrm{st}}) \cong \omega(\mathcal{G}/X)$.

The multiplier character $\nu : \mathbf{G} \to \mathbb{G}_m$ can be seen as an algebraic representation of $Q_{\mathbf{G}}$ and we denote its corresponding invertible sheaf over X by $\mathcal{E}(\nu)$. As an invertible sheaf $\mathcal{E}(\nu)$ is isomorphic to the trivial structure sheaf $\mathcal{O}_X$. However the Hecke action differs by a Tate twist. For $V \in \mathrm{Rep}_{\mathbb{Z}}\, Q_{\mathbf{G}}$

we define $\mathcal{E}(V)(i)$ to be $\mathcal{E}(V \otimes \nu^i) = \mathcal{E}(V) \otimes \mathcal{E}(\nu)^i$.

Remark 2.1.1. The Hecke actions are only defined on global sections not on the quasi-coherent sheaves. However in the following for simplicity we say a quasi-coherent sheaf with Hecke actions to mean that Hecke operators act on its global sections, and a Hecke equivariant morphism between quasi-coherent sheaves to mean that the induced map on global sections is Hecke equivariant. Also by an isomorphism between two automorphic sheaves we mean a Hecke equivariant one unless otherwise stated.

2.2 $(\mathfrak{g}, Q_{\mathbf{G}})$-modules and Gauss–Manin connection

Let $\mathfrak{g} = \operatorname{Lie}\mathbf{G}$, $\mathfrak{q}_{\mathbf{G}} = \operatorname{Lie} Q_{\mathbf{G}}$ be the Lie algebras of $\mathbf{G}$ and its Siegel parobolic $Q_{\mathbf{G}}$.

Definition 2.2.1. *Let E be an algebra. A $(\mathfrak{g}, Q_{\mathbf{G}})$-module V over E is an algebraic representation of $Q_{\mathbf{G}}$ and $\mathfrak{g}$ base changed to E on locally free E-modules, such that the action of $\mathfrak{q}_{\mathbf{G}} \subset \mathfrak{g}$ on V is the one induced from that of $Q_{\mathbf{G}}$ and for any $g \in Q_{\mathbf{G}}$, $X \in \mathfrak{g}$ and $v \in V$*

$$g \cdot X \cdot g^{-1} \cdot v = \big(\operatorname{Ad}(g)X\big) \cdot v.$$

We denote the category of $(\mathfrak{g}, Q_{\mathbf{G}})$-modules over E by $\operatorname{Rep}_E(\mathfrak{g}, Q_{\mathbf{G}})$.

It is mentioned on [21, p.223] that $\mathbf{G}(\mathbb{C})$-equivariant quasi-coherent $\mathcal{D}$-modules over the compact dual $\mathbb{D}^\vee = \mathbf{G}(\mathbb{C})/Q_{\mathbf{G}}(\mathbb{C})$ correspond to $(\mathfrak{g}, Q_{\mathbf{G}})$-modules. We show below that for an object $V \in \operatorname{Rep}_{\mathbb{Z}}(\mathfrak{g}, Q_{\mathbf{G}})$, we can equip with its associated automorphic sheaf $\mathcal{E}(V)$ an integrable connection using the $\mathfrak{g}$-module structure on V.

For the locally free sheaf $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y) = R^1\mathbf{p}_*\big(\Omega^\bullet_{\mathcal{A}/Y}\big)$ over Y, a canonical integrable connection called the Gauss–Manin connection can be constructed [40]. We record the following result on the extension of the Gauss–Manin connection.

Theorem 2.2.2. *([44, Porposition 6.9]) The Gauss–Manin connection $\nabla : \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y) \longrightarrow \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y) \otimes \Omega^1_Y$ extends to an integrable connection with log poles along the boundary*

$$\nabla : \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y)^{\mathrm{can}} \longrightarrow \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y)^{\mathrm{can}} \otimes \Omega^1_X(\log C),$$

satisfying Griffith transversality and compatible with the symplectic pairing on $\mathcal{H}^1_{\rm dR}(\mathcal{A}/Y)^{\rm can}$.

The locally free sheaf $\mathcal{E}(V)$ is defined as the contracted product $T^{\times}_{\mathcal{H}} \times^{Q_{\mathbf{G}}} V$. Let $U = \mathrm{Spec}(R)$ be an affine open subscheme of X such that $\mathcal{H}^1_{\rm dR}(\mathcal{A}/Y)^{\rm can}(U)$ is free over R. We identify $T^{\times}_{\mathcal{H}}(U)$, local sections of $T^{\times}_{\mathcal{H}} \to X$ over U, with the set of ordered basis $\alpha = (\alpha_1, \cdots, \alpha_{2n})$ of $\mathcal{H}^1_{\rm dR}(\mathcal{A}/Y)^{\rm can}(U)$, which define isomorphisms between R^{2n} and $\mathcal{H}^1_{\rm dR}(\mathcal{A}/Y)^{\rm can}(U)$ preserving both the Hodge filtration and symplectic paring up to similitude. Then by definition, $\mathcal{E}(V)(U)$ is the set of maps $v : T^{\times}_{\mathcal{H}}(U) \to V \otimes R$ such that $v(\alpha g) = g^{-1} \cdot v(\alpha)$ for all $g \in Q_{\mathbf{G}}(R)$ and $\alpha \in T^{\times}_{\mathcal{H}}(U)$. Now given $D \in T_X(U) = \mathrm{Der}_{\mathbb{Z}[1/N]}(R, R)$, a section of the tangent bundle of X over U, by Theorem 2.2.2 there exists $X(D, \alpha) \in \mathfrak{g}(R)$ (in fact $\mathfrak{g}(\mathrm{Frac}(R))$ with logarithm poles along the boundary if U intersects with the boundary) such that

$$\nabla(D)(\alpha) = \alpha \cdot X(D, \alpha). \tag{2.1}$$

For $v \in \mathcal{E}(V)(U)$ we define the operator $\nabla_{\mathcal{E}(V)}(D)$ acting on it as

$$\big(\nabla_{\mathcal{E}(V)}(D)(v)\big)(\alpha) := Dv(\alpha) + X(D, \alpha) \cdot v(\alpha). \tag{2.2}$$

Here D acts on $v(\alpha) \in V \otimes R$ through the action of $\mathrm{Der}_{\mathbb{Z}[1/N]}(R, R)$ on R, i.e. by coefficients. The action of $X(D, \alpha)$ on $v(\alpha)$ is the action of the Lie algebra $\mathfrak{g}$ on V.

Proposition 2.2.3. *The above defined* $\nabla_{\mathcal{E}(V)}(D)(v)$ *belongs to* $\mathcal{E}(V)(U)$ *and the formula* (2.2) *on local sections patches together to an integrable connection with log poles along the boundary*

$$\nabla_{\mathcal{E}(V)} : \mathcal{E}(V) \longrightarrow \mathcal{E}(V) \otimes \Omega^1_X(\log C).$$

Proof. What we need to show is that for any $g \in Q_{\mathbf{G}}(R)$

$$\big(\nabla_{\mathcal{E}(V)}(D)(v)\big)(\alpha \cdot g) = g^{-1} \cdot \big(\nabla_{\mathcal{E}(V)}(D)(v)\big)(\alpha). \tag{2.3}$$

The Gauss–Manin connection ∇ satisfies that

$$\begin{aligned}\nabla(D)(\alpha\cdot g) &= \nabla(D)(\alpha)\cdot g+\alpha\cdot Dg\\ &= (\alpha\cdot g)\cdot(g^{-1}X(D,\alpha)g+g^{-1}Dg)\\ &= (\alpha\cdot g)\cdot(\mathrm{Ad}(g^{-1})X(D,\alpha)+g^{-1}Dg)\end{aligned}$$

i.e.

$$X(D,\alpha\cdot g)=\mathrm{Ad}(g^{-1})X(D,\alpha)+g^{-1}Dg.$$

We compute the left hand side of (2.3) by definition,

$$\begin{aligned}\mathrm{LHS} &= D\cdot v(\alpha\cdot g)+X(D,\alpha\cdot g)\cdot v(\alpha\cdot g)\\ &= D\big(g^{-1}\cdot v(\alpha)\big)+\big(Ad(g^{-1})X(D,\alpha)+g^{-1}Dg\big)\cdot v(\alpha\cdot g)\\ &= \big((Dg^{-1})g\big)\cdot\big(g^{-1}\cdot v(\alpha)\big)+g^{-1}\cdot\big(Dv(\alpha)\big)+\big(\mathrm{Ad}(g^{-1})X(D,\alpha)+g^{-1}Dg\big)\cdot\big(g^{-1}\cdot v(\alpha)\big)\\ &= -\big(g^{-1}Dg\big)\cdot\big(g^{-1}\cdot v(\alpha)\big)+g^{-1}\cdot\big(Dv(\alpha)\big)+\big(g^{-1}\cdot X(D,\alpha)\cdot g\big)\cdot\big(g^{-1}\cdot v(\alpha)\big)\\ &\quad+\big(g^{-1}Dg\big)\cdot\big(g^{-1}\cdot v(\alpha)\big)\\ &= g^{-1}\cdot\big(Dv(\alpha)+X(D,\alpha)\cdot v(\alpha)\big),\end{aligned}$$

which equals to the right hand side. The compatibility of the action of $\mathfrak{g}$ and $Q_{\mathbf{G}}$ is used for the forth equality. □

Remark 2.2.4. If the $(\mathfrak{g},Q_{\mathbf{G}})$-module V can be constructed from the standard representation V_{st} of $\mathbf{G}$ by taking tensor products, symmetric powers and wedge products, then applying the same operations to $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y)^{\mathrm{can}}=\mathcal{E}(V_{\mathrm{st}})$ we get the locally free sheaf $\mathcal{E}(V)$ attached to V, so the Gauss–Manin connection on $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y)^{\mathrm{can}}$ immediately induces a connection on $\mathcal{E}(V)$. This is the approach adopted in by E. Eischen in [15]. The point of our construction here is that V does not need to be a representation of $\mathbf{G}$. The construction works for all $(\mathfrak{g},Q_{\mathbf{G}})$-modules and therefore can be easily adapted to deal with p-adic analytic weights and the universal weight (see §3.2, 3.4, 3.6). There is another construction for the connection $\nabla_{\mathcal{E}(V)}$ in [60, §3.2] using Grothendieck's sheaves of differentials when V is a finite dimensional $\mathbf{G}$-representation. That approach may be modified to deal with the non-algebraic weight except that there might be some issue with taking duality when infinite dimensional representations are involved.

2.3 The $(\mathfrak{g}, Q_{\mathbf{G}})$-module V_κ

Now in order to use the constructions in §2.1 and §2.2 to formulate Shimura's theory of nearly holomorphic forms in a sheaf-theoretic context, what we need is to define a suitable $(\mathfrak{g}, Q_{\mathbf{G}})$-module for a given algebraic representation of $\mathrm{GL}(n)$.

Let $(\sigma, W_\sigma) \in \mathrm{Rep}_{\mathbb{Z},f}\,\mathrm{GL}(n)$ be an algebraic representation of $\mathrm{GL}(n)$ locally free of finite rank. We define the $(\mathfrak{g}, Q_{\mathbf{G}})$-module V_σ as follows. For any algebra R, set

$$V_\sigma(R) := W_\sigma(R) \otimes_R R[\underline{Y}] = W_\sigma(R) \otimes_R R[Y_{ij}]_{1\le i\le j\le n}$$

where $\underline{Y} = (Y_{ij})_{1\le i,j\le n}$ is the symmetric $n \times n$ matrix with the indeterminate $Y_{ij} = Y_{ji}$ in the (i,j) entry. Elements in $V_\sigma(R)$ can be regarded as polynomials in the $\frac{n(n+1)}{2}$ variables Y_{ij} with coefficients in $W_\sigma(R)$. Define the $Q_{\mathbf{G}}$-action on V_σ by

$$(g \cdot P)(\underline{Y}) = a \cdot P(a^{-1}b + a^{-1}\,\underline{Y}\,d) \tag{2.4}$$

for $g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in Q_{\mathbf{G}}(R)$ and $P(\underline{Y}) \in V_\sigma(R)$. Regarding the action of $\mathfrak{g}$, for $\mathfrak{q}_{\mathbf{G}}$ we simply take the one induced from the $Q_{\mathbf{G}}$-action defined above, and we make $\mathfrak{u}^-_{\mathbf{G}} = \mathrm{Lie}\,U^-_{\mathbf{G}}$, for which $\left\{\mu^-_{ij}\right\}_{1\le i\le j\le n}$ (defined in (2.12)) constitutes a basis, act by the formulas

$$\begin{aligned}
(\mu^-_{ij} \cdot P)(\underline{Y}) &= \sum_{1\le k\le n} (Y_{ki}\eta_{kj} + Y_{kj}\eta_{ki}) \cdot P(\underline{Y}) - \sum_{1\le k\le l\le n} (Y_{ki}Y_{jl} + Y_{kj}Y_{il}) \frac{\partial}{\partial Y_{kl}} P(\underline{Y}), \quad i \ne j, \\
(\mu^-_{ii} \cdot P)(\underline{Y}) &= \sum_{1\le k\le n} Y_{ki}\eta_{ki} \cdot P(\underline{Y}) - \sum_{1\le k\le l\le n} Y_{ki}Y_{il} \frac{\partial}{\partial Y_{kl}} P(\underline{Y}).
\end{aligned} \tag{2.5}$$

It remains to show the compatibility of such defined actions of $Q_{\mathbf{G}}$ and $\mathfrak{u}^-_{\mathbf{G}}$. This can be done by direct computation using the formulas. There is also a more conceptual proof. To describe it we construct a representation of the group

$$I_{\mathbf{G}}(\mathbb{Z}_p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{G}(\mathbb{Z}_p) \middle| \; c \equiv 0 \mod p \right\}.$$

Let $Q^-_{\mathbf{G}}(\mathbb{Z}_p)$ be the subgroup of $I_{\mathbf{G}}(\mathbb{Z}_p)$ whose elements have 0 as the right upper $n \times n$ corner. we make it act on $W_\sigma(\mathbb{Q}_p)$ through its Levi part. Equip $W_\sigma(\mathbb{Q}_p)$ with a p-adic norm by choosing a basis of $W_\sigma(\mathbb{Q}_p)$, and since it is finite dimensional all norms defined in this way are equivalent. We consider the p-adic analytic induction $\mathrm{Ind}_{Q^-_{\mathbf{G}}(\mathbb{Z}_p)}^{I_{\mathbf{G}}(\mathbb{Z}_p)} W_\sigma(\mathbb{Q}_p)$. Thanks to the Iwahori decomposition we know

$$\mathrm{Ind}_{Q^-_{\mathbf{G}}(\mathbb{Z}_p)}^{I_{\mathbf{G}}(\mathbb{Z}_p)} W_\sigma(\mathbb{Q}_p) = W_\sigma(\mathbb{Q}_p) \langle Y_{ij} \rangle_{1 \le i < j \le n} = W_\sigma(\mathbb{Q}_p) \langle \underline{Y} \rangle,$$

with $g \in I_{\mathbf{G}}(\mathbb{Z}_p)$ acting on $P(\underline{Y}) \in W_\sigma(\mathbb{Q}_p) \langle \underline{Y} \rangle$ by

$$(g \cdot P)(\underline{Y}) = (a + \underline{Y}c) \cdot P\left((a + \underline{Y}c)^{-1}(b + \underline{Y}d)\right). \tag{2.6}$$

Here $W_\sigma(\mathbb{Q}_p) \langle \underline{Y} \rangle$ is the space of strictly convergent power series in $\underline{Y}$ (i.e. convergent on the closed unit ball). Then the formulas (2.4) and (2.5) can be deduced from (2.6), and the compatibility of the actions of $Q_{\mathbf{G}}$ and $\mathfrak{u}^-_{\mathbf{G}}$ on V_σ follows.

Remark 2.3.1. One can check that the formulas (2.4)(2.5) actually agree with the formulas (2.11)(2.12) given in [35], so as $\mathfrak{g}(\mathbb{C})$-modules, the module $V_\sigma(\mathbb{C})$ defined here should agree with their module $\mathcal{O}^f\left(\mathbf{G}^\circ(\mathbb{R}), K_{G,\infty}, W_\sigma(\mathbb{C})\right)$.

As a $Q_{\mathbf{G}}$-representation, V_σ comes with an increasing filtration

$$\mathrm{Fil}^r V_\sigma = V_\sigma^r = W_\sigma[\underline{Y}]_{\le r}, \tag{2.7}$$

where the subscript $\le r$ means polynomials in $\underline{Y}$ of total degree less or equal to r. $\mathrm{Fil}^r V_\sigma$ can also be characterized as the sum of generalized η_0-eigenspaces with eigenvalues $\ge -r$ [21, p.230]. The eigenvalues of η_0 are also called F-weights there. Regarding the $\mathrm{GL}(n)$-representation W_σ as a $Q_{\mathbf{G}}$-representation we have $V_\sigma^0 = W_\sigma$. It follows from the definition formulas that

$$\mathfrak{g} \cdot V_\sigma^r \subset V_\sigma^{r+1}. \tag{2.8}$$

Let V_{triv} be the $(\mathfrak{g}, Q_{\mathbf{G}})$-module constructed as above by taking σ to be the trivial representation. Denote by J the $Q_{\mathbf{G}}$-representation V_{triv}^1. We note here the following useful isomorphism of $Q_{\mathbf{G}}$-representations

$$V_\sigma^r \cong V_\sigma^0 \otimes \operatorname{Sym}^r J = W_\sigma \otimes \operatorname{Sym}^r J. \tag{2.9}$$

For a dominant weight $\kappa = (k_1, \cdots, k_2) \in X(T_n)^+$ of $\mathrm{GL}(n)$ with respect to B_n. Set $\kappa' = (-k_n, \cdots, -k_1)$. We define W_κ to be the algebraic $\mathrm{GL}(n)$-representation

$$\left\{f : \mathrm{GL}(n) \to \mathbb{A}^1, \text{ morphism of schemes satisfying } f(gb) = \kappa'(b)f(g) \text{ for all } g \in \mathrm{GL}(n) \text{ and } b \in B_n\right\}$$

with $\mathrm{GL}(n)$ acting by left inverse translation. Putting $\sigma = \kappa$ we get the $(\mathfrak{g}, Q_{\mathbf{G}})$-module V_κ and $Q_{\mathbf{G}}$-representations V_κ^r, $r \geq 0$. Denote by τ the symmetric square of the standard representation of $\mathrm{GL}(n)$. Let $\tau^\vee$ be dual representation of τ. In the following most $\mathrm{GL}(n)$-representations we consider are tensor products of some κ with symmetric powers of τ and $\tau^\vee$.

Remark 2.3.2. We can twist V_σ by the i-th power of of the multiplier character ν and denote the resulting $(\mathfrak{g}, Q_{\mathbf{G}})$-module by $V_\sigma(i)$. Such a twist will change the F-weights by $-i$ and corresponds to a Tate twist [21, p.222].

2.4 The sheaf $\mathcal{V}_\kappa^r$ of nearly holomorphic forms

Let κ be a dominant weight of $\mathrm{GL}(n)$. With preparations in previous sections we give the following definitions.

Definition 2.4.1. *The locally free sheaf over X of weight κ, (non-holomorphy) degree r nearly holomorphic forms is defined to be $\mathcal{V}_\kappa^r = \mathcal{E}(V_\kappa^r)$.*

When $r=0$, we also use ω_κ to denote $\mathcal{V}_\kappa^0$ which is the sheaf of weight κ holomorphic forms. More generally for $\sigma \in \operatorname{Rep}_{\mathbb{Z},f} \mathrm{GL}(n)$ we define the locally free sheaves $\mathcal{V}_\sigma = \mathcal{E}(V_\sigma)$, $\mathcal{V}_\sigma^r = \mathcal{E}(V_\sigma^r)$ and denote $\mathcal{V}_\sigma^0$ by ω_σ. The nearly holomorphic forms are defined to be global sections of the sheaf $\mathcal{V}_\kappa^r$.

Definition 2.4.2. *Let R be a $\mathbb{Z}[1/N]$-algebra. The space of nearly holomorphic forms (resp. cuspidal nearly holomorphic forms) over R of weight κ, principal level N and (non-holomorphy) degree r is defined to be $N_\kappa^r(\Gamma(N), R) = H^0\left(X_{/R}, \mathcal{V}_\kappa^r\right)$ (resp. $N_{\kappa,\mathrm{cusp}}^r(\Gamma(N), R) = H^0\left(X_{/R}, \mathcal{V}_\kappa^r(-C)\right)$).*

There is the moduli interpretation à la Katz for nearly holomorphic forms which attaches to each $f \in N_\kappa^r(\Gamma(N), R)$ a functorial rule assigning to every quadruple $(A_{/S}, \lambda, \psi_N, \alpha)$ an element in $V_\kappa^r(S) = W_\kappa(S)[\underline{Y}]_{\leq r}$, where S is an R-algebara, $(A_{/S}, \lambda)$ is a principally polarized dimension n abelian scheme, ψ_N is a principal level N structure and α is a basis of $\mathcal{H}_{\mathrm{dR}}^1(A/S)$ respecting the Hodge filtration and symplectic pairing up to similitude, .

It follows directly from Porposition 2.2.3 and (2.8) that the sheaves $\mathcal{V}_\sigma, \mathcal{V}_\sigma^r$ are equipped with the integrable connections

$$\nabla_\sigma : \mathcal{V}_\sigma \longrightarrow \mathcal{V}_\sigma \otimes \Omega_X^1(\log C)$$

and

$$\nabla_\sigma : \mathcal{V}_\sigma^r \longrightarrow \mathcal{V}_\sigma^{r+1} \otimes \Omega_X^1(\log C). \tag{2.10}$$

The global sections of the differential sheaf Ω_X^1 has a natural Hecke action and the extended Kodaira–Spencer isomorphism [44, Porposition 6.9] says that

$$\Omega_X^1(\log C) \cong \mathrm{Sym}^2(\omega(\mathcal{G}/X))(-1) \cong \omega_\tau(-1)$$

Hecke equivariantly. There is a canonical isomorphism of locally free sheaves $t^+ : \mathcal{V}_{\sigma\otimes\tau}^{r+1}(-1) \to \mathcal{V}_{\sigma\otimes\tau}^{r+1}$ which is not Hecke equivariant but commutes with Hecke actions up to a twist by the multiplier character. Composing ∇_σ with it we get the differential operator

$$D_\sigma : \mathcal{V}_\sigma^r \xrightarrow{\nabla_\sigma} \mathcal{V}_\sigma^{r+1} \otimes \Omega_X^1(\log C) \xrightarrow{\mathrm{KS}} \mathcal{V}_{\sigma\otimes\tau}^{r+1}(-1) \xrightarrow{t^+} \mathcal{V}_{\sigma\otimes\tau}^{r+1}.$$

It commutes with Hecke actions up to a multiplier twist (cf. §3.10, [63, §2.5.2, Porposition 3.3.7]).

Put $\mathcal{J} = \mathcal{E}(J)$ and $\mathcal{J}^\vee$ to be its dual. By (2.9) we have

Proposition 2.4.3. *$\mathcal{V}_\sigma^r \cong \omega_\sigma \otimes \mathrm{Sym}^r \mathcal{J}$ as locally free sheaves over X with Hecke actions.*

Remark 2.4.4. In [62, §4.1.2, 4.3.1] Urban defined a locally free sheaf $\mathcal{J}'$ to be the one making the diagram below commutative with bottom row exact.

$$\begin{array}{ccccccccc} 0 \to & \omega(\mathcal{G}/X) \otimes \underline{\mathrm{Lie}}({}^t\mathcal{G}/X)^\vee & \to & \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y)^{\mathrm{can}} \otimes \underline{\mathrm{Lie}}({}^t\mathcal{G}/X)^\vee & \to & \underline{\mathrm{Lie}}({}^t\mathcal{G}/X) \otimes \underline{\mathrm{Lie}}({}^t\mathcal{G}/X)^\vee & \to 0 \\ & \| & & \uparrow & & \uparrow & \\ 0 \to & \mathrm{Sym}^2\big(\omega(\mathcal{G}/X)\big)(-1) & \to & \mathcal{J}' & \to & \mathcal{O}_X & \to 0 \end{array}$$

After that he defined the sheaf of weight κ degree r nearly holomorphic forms to be $\omega_\kappa \otimes \mathrm{Sym}^r \mathcal{J}'^\vee$. One can show that the sheaf $\mathcal{J}^\vee$ satisfies Urban's condition for defining $\mathcal{J}'$. Hence $\mathcal{J} \cong \mathcal{J}'^\vee$ and our definition of sheaves of nearly holomorphic forms agrees with his.

We end this section with an example showing that the locally free sheaves associated to two non-isomorphic $Q_{\mathbf{G}}$-representations can be isomorphic as locally free sheaves without considering the Hecke actions. It also illustrates that the sheaf $\mathcal{J}$ may have splitting that does not come from the $Q_{\mathbf{G}}$-representation and such a splitting can give rise to holomorphic but non-Hecke equivariant differential operators.

Example 2.4.5. Take $n = 1$, $\mathbf{G} = \mathrm{GL}(2)$ and $\mathbf{G}^\circ = \mathrm{SL}(2)$. We show that the sheaf $\mathcal{J}^\vee = \big(\mathcal{V}^1_{\mathrm{triv}}\big)^\vee$ and the first jet sheaf $\mathcal{P}^1(\mathcal{O}_X)$ are isomorphic in $\mathrm{QCoh}(X)$ but their corresponding $Q_{\mathbf{G}}$-representations are not isomorphic. Let V_1, V_2 be the $Q_{\mathbf{G}}$-representations giving rise to $\mathcal{J}^\vee$, $\mathcal{P}^1(\mathcal{O}_X)$ respectively. Write $Y = Y_{11}$. Then $V_1^\vee = \mathrm{triv} \otimes \mathbb{Z}[Y]$ with basis $\{1, Y\}$, and the action of $Q_{\mathbf{G}^\circ}$ is given by

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \cdot P(Y) = P(a^{-1}b + a^{-2}Y),$$

or in the matrix form

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mapsto \begin{pmatrix} a^{-2} & 0 \\ a^{-1}b & 1 \end{pmatrix}.$$

Clearly V_1 is indecomposable as a $Q_{\mathbf{G}}$-representation. On the other hand by [21, Porposition VI 5.1], $V_2^\vee \cong U_1(\mathfrak{g}^\circ) \otimes_{U(\mathfrak{q}_{\mathbf{G}^\circ})} \mathrm{triv}$ as a $Q_{\mathbf{G}}^\circ$-representation, where $\mathfrak{g}^\circ = \mathrm{Lie}\,\mathbf{G}^\circ = \mathfrak{sl}(2) = \mathrm{Span}\{h, x, y\}$ and $\mathfrak{q}_{\mathbf{G}^\circ} = \mathrm{Span}\{h, x\}$ with $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. As a basis of $V_2^\vee$ we can take $\{y \otimes 1, 1 \otimes 1\}$, and we have $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$ act on them by

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \cdot (y \otimes 1) = a^{-2} y \otimes 1, \qquad \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \cdot (1 \otimes 1) = 1 \otimes 1,$$

or in the matrix form

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mapsto \begin{pmatrix} a^{-2} & 0 \\ 0 & 1 \end{pmatrix}.$$

This is saying that the $Q_{\mathbf{G}}$-action on V_2 splits. Hence V_1 and V_2 are not isomorphic as $Q_{\mathbf{G}}$-representations.

However as coherent sheaves $\mathcal{J}^\vee$ and $\mathcal{P}^1(\mathcal{O}_X)$ are indeed isomorphic, because the nearly holomorphic form E_2 splits $\mathcal{J}^\vee \cong \omega(\mathcal{G}/X)^{\otimes 2} \otimes \mathcal{J}$ as locally free sheaves [63, Remark 2.3.7]. Actually this non-Hecke equaivariant splitting gives rise to Serre's ∂ operator that acts on a modular form f of weight k by

$$\partial f = 12\theta f - kPf,$$

where $\theta = q\frac{d}{dq}$ and P is the holomorphic funciton on the upper half plane defined as $P(q) = 1 - 24\sum_{m \geq 1} \sigma_1(n) q^n$ with $q = e^{2\pi i z}$ (cf. [38, §A1.4]). Serre's ∂ operator is a holomorphic differential operator but not Hecke equivariant.

2.5 Equivalence to Shimura's nearly holomorphic forms and differential operators

First recall Shimura's definition of nearly holomorphic forms and Maass–Shimura differential operators. Let $\mathbb{H}_n = \{z \in \mathbb{C}_n^n : {}^{\mathrm{t}}z = z,\ \mathrm{Im}\, z > 0\}$ be the rank n Siegel upper half space and $\Gamma \subset \mathbf{G}^\circ(\mathbb{Z}) = \mathrm{Sp}(2n, \mathbb{Z})$ be a congruence subgroup. As usual $\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \in \mathbf{G}(\mathbb{R})$ acts on $\mathbb{H}_n$ by $\gamma z = (a_\gamma z + b_\gamma) \cdot (c_\gamma z + d_\gamma)^{-1}$. Put $s(z) = (z - \bar{z})^{-1}$ and $\mu(\gamma, z) = c_\gamma z + d_\gamma$.

For an algebraic representation (σ, W_σ) of $\mathrm{GL}(n)$ free of finite rank, Shimura defines [58, §13.11] the space of $W_\sigma(\mathbb{C})$-valued nearly holomorphic forms of degree r, denoted by $N_\sigma^r(\mathbb{H}_n, \Gamma)$, to be the set consisting of functions $f \in C^\infty(\mathbb{H}_n, W_\sigma(\mathbb{C}))$ satisfying

(i) $f(z)$ can be written as a degree $\leq r$ polynomial in the components of $s(z)$ with coefficients being holomorphic maps from $\mathbb{H}_n$ to $W_\sigma(\mathbb{C})$, and

(ii) f transforms under $\gamma \in \Gamma$ by $f(\gamma z) = \sigma(\mu(\gamma, z)) f(z)$.

When $n = 1$ the function f is also required to satisfy the cusp condition, i.e. for every $\gamma \in \mathrm{SL}(2, \mathbb{Z})$ there exists $a_{in} \in \mathbb{C}$ and $M \in \mathbb{N}$ such that

$$\sigma(\mu(\gamma, z))^{-1} f(\gamma z) = \sum_{i=0}^{r} (\pi \mathrm{Im}\, z)^{-i} \sum_{n=0}^{\infty} a_{in} e^{2\pi i z/M}.$$

The Maass–Shimura differential operator $D_{\mathbb{H}_n,\sigma}$ is defined as [58, §12.9]

$$\begin{aligned} D_{\mathbb{H}_n,\sigma} : N_\sigma^r(\mathbb{H}_n, \Gamma) &\longrightarrow N_{\sigma\otimes\tau}^{r+1}(\mathbb{H}_n, \Gamma) \\ f &\longmapsto \sigma(s)\big(d_z(\sigma(s^{-1}) f)\big). \end{aligned} \tag{2.11}$$

Now we show that $N_\sigma^r(\mathbb{H}_n, \Gamma)$, together with the Maass–Shimura differential operator $D_{\mathbb{H}_n,\sigma}$, is nothing but the global sections over $\Gamma \backslash \mathbb{H}_n$ of the sheaf $\mathcal{V}_\sigma^r$ equipped with the differential operator D_σ defined in the previous sections. Let $Y_{\mathbb{C}}^\circ$ be a connected component of Y base changed to $\mathbb{C}$. Then $Y_{\mathbb{C}}^\circ \cong \Gamma(N) \backslash \mathbb{H}_n$ as complex manifolds and the universal abelian variety $\mathbf{p} : \mathcal{A}_{\mathbb{C}} \to Y_{\mathbb{C}}^\circ$ is isomorphic to $\mathbf{p} : \Gamma(N)\backslash\mathbb{C}^n \times \mathbb{H}_n/\mathbb{Z}^{2n} \to \Gamma(N)\backslash\mathbb{H}_n$. Here $(m_1, m_2) \in \mathbb{Z}^{2n}$ and $\gamma \in \Gamma(N)$ act on $(w, z) \in \mathbb{C}^n \times \mathbb{H}_n$ by

$$(w, z) \cdot (m_1, m_2) = (w + m_1 z + m_2, z),$$
$$\gamma \cdot (w, z) = (w\mu(\gamma, z)^{-1}, \gamma z).$$

Let $q : \mathbb{H}_n \to \Gamma(N)\backslash\mathbb{H}_n$ be the quotient map and $\mathcal{A}_{\mathbb{H}_n} = \mathbb{C}^n \times \mathbb{H}_n/\mathbb{Z}^{2n} \to \mathbb{H}_n$ be the pullback of $A_\mathbb{C}$ via q. For each $z = (z_{ij}) \in \mathbb{H}_n$ the fibre $A_{\mathbb{H}_n,z} \cong \mathbb{C}^n/\Lambda_z$, where Λ_z is the lattice spanned by e_i, the vector with 1 as the i-th entry and 0 elsewhere, $1 \leq i \leq n$, and $z_j = {}^{\mathrm{t}}(z_{1j}, z_{2j}, \cdots, z_{nj})$, $1 \leq j \leq n$. Let $\lambda_{\mathbb{H}_n}$ (resp. $\psi_{N,\mathbb{H}_n}$) be the polarization (principal level N structure) of $A_{\mathbb{H}_n}$ such that its fibre at z is given by the real Riemann form $E_z : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{R}$, defined as $E_z(w_1, w_2) = -\mathrm{Im}\left({}^{\mathrm{t}}w_1(\mathrm{Im}\, z)^{-1}\overline{(iw_2)}\right)$ (resp. $\frac{1}{N}e_1, \cdots, \frac{1}{N}e_2, \frac{1}{N}z_1, \cdots, \frac{1}{N}z_n$). The $\{e_i, z_j\}_{1\leq i,j\leq n}$ form a basis of $H_1(A_{\mathbb{H}_n,z}, \mathbb{Z})$. Over $\mathbb{H}_n$ we have a global basis $(\alpha, \beta) = (\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n)$ for the sheaf $q^*\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}_\mathbb{C}/Y_\mathbb{C}) = \mathcal{H}^1_{\mathrm{dR}}(A_{\mathbb{H}_n}/\mathbb{H}_n)$ defined as

$$\alpha_i\left(\sum_{j=1}^n m_{1,j}z_j + m_{2,j}e_j\right) = m_{2,i}, \qquad \beta_i\left(\sum_{j=1}^n m_{1,j}z_j + m_{2,j}e_j\right) = m_{1,i}.$$

The basis (α, β) is horizontal with respect to the Gauss–Manin connection, i.e.

$$\nabla(\alpha_i) = \nabla(\beta_i) = 0, \quad 1 \leq i \leq n.$$

After base changing to $C^\infty(\mathbb{H}_n, \mathbb{C})$, the Hodge decomposition gives another basis of $\mathcal{H}^1_{\mathrm{dR}}(A_{\mathbb{H}_n}/\mathbb{H}_n) \otimes C^\infty(\mathbb{H}_n, \mathbb{C})$, denoted as $(dw, d\bar{w}) = (dw_1, \cdots, dw_n, d\bar{w}_1, \cdots, d\bar{w}_n)$.

Neither $(dw, d\bar{w})$ nor (α, β) gives rise to an element of $\left(q^*T^\times_\mathcal{H}\right)(\mathbb{H}_n) \otimes C^\infty(\mathbb{H}_n, \mathbb{C})$. The basis $(dw, d\bar{w})$ does not satisfy the pairing condition, while (α, β) is not compatible with the Hodge filtration. Nevertheless (dw, β) (resp. $(dw, -d\bar{w} \cdot s)$) does give an element of $\left(q^*T^\times_\mathcal{H}\right)(\mathbb{H}_n)$ (resp.

$\left(q^*T_{\mathcal{H}}^{\times}\right)(\mathbb{H}_n) \otimes C^\infty(\mathbb{H}_n, \mathbb{C})$), and it is easily checked that

$$(dw, -d\bar{w} \cdot s) = (dw, \beta) \cdot \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}. \tag{2.12}$$

By evaluating global sections of $\mathcal{V}_\sigma^r$ over $Y_{\mathbb{C}}^\circ$ at the test object $\big(A_{\mathbb{H}_n}/\mathbb{H}_n, \lambda_{\mathbb{H}_n}, \psi_{N,\mathbb{H}_n}, (dw, -d\bar{w} \cdot s)\big)$, we define a map

$$\begin{aligned} \phi : H^0(Y_{\mathbb{C}}^\circ, \mathcal{V}_\sigma^r) &\longrightarrow N_\sigma^r(\mathbb{H}_n, \Gamma(N)) \\ f &\longmapsto f\big(A_{\mathbb{H}_n}, \lambda_{\mathbb{H}_n}, \psi_{N,\mathbb{H}_n}, (dw, -d\bar{w} \cdot s)\big)|_{\underline{Y}=0} \end{aligned} \tag{2.13}$$

Proposition 2.5.1. *ϕ is well defined and is an isomorphism.*

Proof. We need to check that the above defined $\phi(f)$ does land inside $N_\sigma^r(\mathbb{H}_n, \Gamma(N))$. First look at the evaluation of f at the test object $\big(A_{\mathbb{H}_n}, \lambda_{\mathbb{H}_n}, \psi_{N,\mathbb{H}_n}, (dw, \beta)\big)$. Since (dw, β) is holomorphic we have

$$f\big(A_{\mathbb{H}_n}, \lambda_{\mathbb{H}_n}, \psi_{N,\mathbb{H}_n}, (dw, \beta)\big) = P_f(\underline{Y}),$$

a polynomial in $\underline{Y}$ of degree $\leq r$ with coefficients being holomorphic maps from $\mathbb{H}_n$ to $W_\sigma(\mathbb{C})$. Combining (2.4) and (2.12) we get

$$\begin{aligned} \phi(f) &= f\big(A_{\mathbb{H}_n}, \lambda_{\mathbb{H}_n}, \psi_{N,\mathbb{H}_n}, (dw, -d\bar{w} \cdot s)\big)|_{\underline{Y}=0} \\ &= f\left(A_{\mathbb{H}_n}, \lambda_{\mathbb{H}_n}, \psi_{N,\mathbb{H}_n}, (dw, \beta) \cdot \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}\right)\Bigg|_{\underline{Y}=0} \\ &= \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \cdot f\big(A_{\mathbb{H}_n}, \lambda_{\mathbb{H}_n}, \psi_{N,\mathbb{H}_n}, (dw, \beta)\big)\Bigg|_{\underline{Y}=0} \\ &= P_f(\underline{Y} + s)|_{\underline{Y}=0} \\ &= P_f(s). \end{aligned}$$

This shows that $\phi(f)$ satisfies condition (i) in the definition of $N_\sigma^r(\mathbb{H}_n, \Gamma(N))$. Under the isomorphism

$$\gamma : A_{\mathbb{H}_n,z} \longrightarrow A_{\mathbb{H}_n,\gamma z}$$
$$w \mapsto w \cdot \mu(\gamma, z)^{-1}$$

for $\gamma \in \Gamma(N)$ we have

$$\gamma^*(dw, -d\bar{w} \cdot s) = (dw, -d\bar{w} \cdot s) \begin{pmatrix} \mu(\gamma, z)^{-1} & 0 \\ 0 & \mu(\gamma, z) \end{pmatrix},$$

from which we see that $\phi(f)$ also has the transformation property required in condition (ii). Finally the bijectivity of ϕ can be seen from the fact that essentially it sends $P_f(\underline{Y})$ to $P_f(s)$ and we can recover one of them from the other. □

We continue to prove the compatibility of D_σ and $D_{\mathbb{H}_n,\sigma}$ under the map ϕ.

Proposition 2.5.2. $D_{\mathbb{H}_n,\sigma} \circ \phi = \phi \circ D_\sigma$

Proof. Let $\langle\ ,\ \rangle$ be the canonical pairing between the sheaf of differentials $\Omega^1_{\mathbb{H}_n}$ and the tangent bundle $T_{\mathbb{H}_n}$. Take $\partial/\partial z_{ij} \in T_{\mathbb{H}_n}$ and $f \in H^0(Y^\circ_{\mathbb{C}}, \mathcal{V}^r_\sigma)$. We show that $\langle D_{\mathbb{H}_n,\sigma} \circ \phi(f), \partial z_{ij}\rangle = \langle \phi \circ D_\sigma(f), \partial/\partial z_{ij}\rangle$. Assume $i \neq j$ (the computation for the case $i = j$ is the same and we omit it), the Gauss–Manin connection acts on (dw, β) as

$$\nabla(\partial/\partial z_{ij})(dw, \beta) = (dw, \beta) \cdot \begin{pmatrix} 0 & 0 \\ E_{ij} + E_{ji} & 0 \end{pmatrix} = (dw, \beta) \cdot \mu^-_{ij}. \tag{2.14}$$

Let $P_f(\underline{Y})$ be as in the above proof. According to the definition of D_σ by (2.2),

$$\begin{aligned}
\langle \phi \circ D_\sigma(f),\, \partial/\partial z_{ij}\rangle &= \langle (D_\sigma f)\big(A_{\mathbb{H}_n}, \lambda_{\mathbb{H}_n}, \psi_{N,\mathbb{H}_n}, (dw,\beta)\big),\, \partial/\partial z_{ij}\rangle\big|_{\underline{Y}=s} \\
&= \frac{\partial}{\partial z_{ij}} P_f(\underline{Y}) + (\mu_{ij}^{-}\cdot P_f)(\underline{Y})\Big|_{\underline{Y}=s} \\
&= \frac{\partial}{\partial z_{ij}} P_f(\underline{Y}) + \sum_{1\le k\le n} (Y_{ki}\eta_{kj} + Y_{kj}\eta_{ki})\cdot P_f(\underline{Y}) - \sum_{1\le k\le l\le n} (Y_{ki}Y_{jl} + Y_{kj}Y_{il}) \frac{\partial}{\partial Y_{kl}} P_f(\underline{Y})\Bigg|_{\underline{Y}=s}. \\
&= \frac{\partial}{\partial z_{ij}} P_f(\underline{Y})\Big|_{\underline{Y}=s} + \sum_{1\le k\le n} (s_{ki}\eta_{kj} + s_{kj}\eta_{ki})\cdot P_f(s) - \sum_{1\le k\le l\le n} (s_{ki}s_{jl} + s_{kj}s_{il}) \frac{\partial}{\partial s_{kl}} P_f(s)
\end{aligned}$$

Using

$$\frac{\partial s_{kl}}{\partial z_{ij}} = -\left(s\left(\frac{\partial}{\partial z_{ij}}(z-\bar{z})\right) s\right)_{kl} = -(s_{ik}s_{jl} + s_{il}s_{jk}),$$

we get

$$\begin{aligned}
\langle \phi \circ D_\sigma(f),\, \partial/\partial z_{ij}\rangle &= \frac{\partial}{\partial z_{ij}} P_f(\underline{Y})\Big|_{\underline{Y}=s} + \sum_{1\le k\le n} (s_{ki}\eta_{kj} + s_{kj}\eta_{ki})\cdot P_f(s) + \frac{\partial s_{kl}}{\partial z_{ij}} \frac{\partial}{\partial s_{kl}} P_f(s) \\
&= \frac{\partial}{\partial z_{ij}} \big(P_f(s)\big) + \sum_{1\le k\le n} (s_{ki}\eta_{kj} + s_{kj}\eta_{ki})\cdot P_f(s) \\
&= \frac{\partial}{\partial z_{ij}} \phi(f) + \sum_{1\le k\le n} (s_{ki}\eta_{kj} + s_{kj}\eta_{ki})\cdot \phi(f). \qquad (2.15)
\end{aligned}$$

On the other hand according to the definition of $D_{\mathbb{H}_n,\sigma}$ (2.11)

$$\begin{aligned}\langle D_{\mathbb{H}_n,\sigma}\circ\phi(f),\,\partial/\partial z_{ij}\rangle &= \langle\sigma(s)\big(d_z(\sigma(s^{-1})\phi(f))\big),\,\partial/\partial z_{ij}\rangle\\ &= \sigma(s)\left(\frac{\partial}{\partial z_{ij}}\big(\sigma(s^{-1})\phi(f)\big)\right)\\ &= \frac{\partial}{\partial z_{ij}}\phi(f)+\sigma\left(s\frac{\partial s^{-1}}{\partial z_{ij}}\right)\phi(f)\\ &= \frac{\partial}{\partial z_{ij}}\phi(f)+\sigma\left(\sum_{k=1}^{n}s_{ki}E_{kj}+s_{kj}E_{ki}\right)\phi(f)\\ &= \frac{\partial}{\partial z_{ij}}\phi(f)+\left(\sum_{k=1}^{n}s_{ki}\eta_{kj}+s_{kj}\eta_{ki}\right)\cdot\phi(f).\end{aligned}$$

Comparing with (2.15), we conclude. □

2.6 Equivalence to the action of $\mathfrak{q}_G^+$

Let $C^\infty(\Gamma\backslash G(\mathbb{R}))$ be the $\mathbb{C}$-vector space of smooth functions on $G(\mathbb{R})$ that are invariant under the left translation by Γ. Let W_σ^* be the dual representation of $W_\sigma(\mathbb{C})$ and $\langle,\rangle : W_\sigma\times W_\sigma^*\to\mathbb{A}^1$ be the canonical pairing. For each $w^*\in W_\sigma^*(\mathbb{C})$ there is the embedding

$$\varphi_G(\cdot,w^*) : C_\sigma^\infty(\mathbb{H}_n,\Gamma)\to C^\infty(\Gamma\backslash G(\mathbb{R})), \tag{2.16}$$

defined as

$$\varphi_G(f,w^*)(g)=\left\langle\sigma\left(\mu(g,i)\right)^{-1}f(g\cdot i),w^*\right\rangle$$

for $f\in C_\sigma^\infty(\mathbb{H}_n,\Gamma)$ and $g\in G(\mathbb{R})$. The maximal compact subgroup $K_{G,\infty}$ acts on $W_\sigma(\mathbb{C})$, $W_\sigma^*(\mathbb{C})$ via the isomorphism $K_{G,\infty}\cong U(n,\mathbb{R})\subset\mathrm{GL}(n,\mathbb{C})$. One can check that for $k\in K_{G,\infty}$ we have

$$\varphi_G(f,w^*)(gk)=\varphi_G(f,{}^{\mathrm{t}}k^{-1}\cdot w^*)(g).$$

Therefore if we put $V_f=\{\varphi_G(f,w^*) : w^*\in W_\sigma^*(\mathbb{C})\}$, then it is a subspace of $C^\infty(\Gamma\backslash G(\mathbb{R}))$ closed under the action of $K_{G,\infty}$ and is isomorphic to $W_\sigma(\mathbb{C})$ as a $K_{G,\infty}$-representation.

The torus $\mathbb{C}^\times$ acts on $G(\mathbb{R})$ by

$$(x+iy)\cdot g = \begin{pmatrix} xI_n & yI_n \\ -yI_n & xI_n \end{pmatrix} g \begin{pmatrix} xI_n & yI_n \\ -yI_n & xI_n \end{pmatrix}^{-1},$$

inducing an action of $\mathbb{C}^\times$ on $\mathfrak{g}_\mathbb{C}$. Let $\mathfrak{g}_\mathbb{C}^{a,b}$ be the subspace of $\mathfrak{g}_\mathbb{C}$ on which $z \in \mathbb{C}^\times$ acts by the scalar $z^{-a}\overline{z}^{-b}$. Then $\mathfrak{g}_\mathbb{C}$ decomposes as $\mathfrak{g}_\mathbb{C}^{-1,1} \oplus \mathfrak{g}_\mathbb{C}^{0,0} \oplus \mathfrak{g}_\mathbb{C}^{1,-1}$. We have $\mathfrak{g}_\mathbb{C}^{0,0} = \mathfrak{k}_{G,\mathbb{C}}$, the complexified Lie algebra of $K_{G,\infty}$. Set $\mathfrak{q}_G^+ = \mathfrak{g}_\mathbb{C}^{-1,1}$ and $\mathfrak{q}_G^- = \mathfrak{g}_\mathbb{C}^{1,-1}$. The aim of this section is to show that the $\mathfrak{q}_G^+$-action on $C^\infty(\Gamma, G(\mathbb{R}))$ translates to the Maass–Shimura differential operators on $C_\sigma^\infty(\mathbb{H}_n,\Gamma)$ under the embedding (2.16). This should be a well known fact but we include a proof here for the convenience of our later application.

Fix a basis $\underline{X} = (X_{ij})_{1\le i,j\le n}$, $X_{ij} = X_{ji}$ of τ with $a \in \mathrm{GL}(n)$ acting on it by ${}^{\mathrm{t}}a\underline{X}a$. We will assume that under the trivialization of $\omega(A_{\mathbb{H}_n}/\mathbb{H}_n)$ by the basis $dw_1,\cdots,dw_n$, the element X_{ij} corresponds to $dw_i dw_j = 2\pi i\cdot dz_{ij}$. Denote by $\underline{X}^* = (X_{ij}^*)_{1\le i,j\le n}$, $\underline{X}_{ij}^* = \underline{X}_{ji}^*$ the basis of τ^* dual to $\underline{X}$.

Let $\mathfrak{c} = \frac{1}{\sqrt{2}}\begin{pmatrix} I_n & iI_n \\ iI_n & I_n \end{pmatrix}$ and $\widehat{\mu}_{ij}^+ = \mathfrak{c}\mu_{ij}\mathfrak{c}^{-1}$. Then $\widehat{\mu}_{ij}^+$, $1\le i\le j\le n$, span $\mathfrak{q}_G^+$.

Proposition 2.6.1. *The diagram*

$$\begin{array}{ccc} C_\sigma^\infty(\mathbb{H}_n,\Gamma) & \xrightarrow{\varphi_G(\cdot,w^*)} & C^\infty(\Gamma\backslash G(\mathbb{R})) \\ {\scriptstyle 4\pi i\cdot D_{\sigma,\mathbb{H}_n}}\downarrow & & \downarrow{\scriptstyle \widehat{\mu}_{ij}^+} \\ C_{\sigma\otimes\tau}^\infty(\mathbb{H}_n,\Gamma) & \xrightarrow{\varphi_G(\cdot,w^*\otimes X_{ij}^*)} & C^\infty(\Gamma\backslash G(\mathbb{R})) \end{array} \tag{2.17}$$

commutes.

Proof. We need to show the identity

$$4\pi i\cdot\varphi_G(D_{\sigma,\mathbb{H}_n}f, w^*\otimes X_{ij}^*)(g) = \widehat{\mu}_{ij}^+\varphi_G(f,w^*)(g) \tag{2.18}$$

for all $f \in C^\infty_\sigma(\mathbb{H}_n, \Gamma)$, $g \in G(\mathbb{R})$ and $1 \le i \le j \le n$. Notice that for all $k \in K_{G,\infty}$ we have

$$\varphi_G(D_{\sigma,\mathbb{H}_n} f, w^* \otimes X^*_{ij})(gk) = \varphi_G(D_{\sigma,\mathbb{H}_n} f, {}^{\mathrm{t}}k^{-1} \cdot (w^* \otimes X^*_{ij}))(g),$$

and

$$\widehat{\mu}^+_{ij}\varphi_G(f, w^*)(gk) = \big(Ad(k)\widehat{\mu}^+_{ij}\big)\varphi_G(f, {}^{\mathrm{t}}k^{-1} \cdot w^*)(g).$$

Thus it is enough to show (2.18) for $g \in Q_G(\mathbb{R})$.

Write elements in $Q_G(\mathbb{R})$ as $g = \begin{pmatrix} a & x\,{}^{\mathrm{t}}a^{-1} \\ 0 & {}^{\mathrm{t}}a^{-1} \end{pmatrix}$ with $a \in \mathrm{GL}(n, \mathbb{R})$ and x an $n \times n$ symmetric matrix with real coefficients. Put $y = a\,{}^{\mathrm{t}}a$ which is positive definite symmetric. Then $z = x + iy$ belongs to $\mathbb{H}_n$, and by definition

$$\begin{aligned}
\varphi_G(D_{\sigma,\mathbb{H}_n} f, w^* \otimes \mu(g,i)^{-1} \cdot X^*_{ij})(g) &= \langle \sigma \otimes \tau_n \left(\mu(g,i)^{-1}\right) \sigma(y)^{-1} d_z\big(\sigma(y) f(z)\big), w^* \otimes \mu(g,i)^{-1} \cdot X^*_{ij} \rangle \\
&= \left\langle \sigma\left(\mu(g,i)\right)^{-1} \sigma(y)^{-1} d_z\big(\sigma(y) f(z)\big), w^* \otimes X^*_{ij} \right\rangle \\
&= \frac{1}{2\pi i} \left\langle \sigma(a)^{-1} \frac{\partial}{\partial z_{ij}}\big(\sigma(y) f(z)\big), w^* \right\rangle.
\end{aligned}$$

Given $\alpha \in \mathrm{GL}(n, \mathbb{C})$ we define $\alpha \cdot \widehat{\mu}^+_{ij}$ to be $\mathfrak{c} \begin{pmatrix} {}^{\mathrm{t}}\alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix} \mu^+_{ij} \begin{pmatrix} {}^{\mathrm{t}}\alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \mathfrak{c}^{-1}$. It is easy to see that under this definition if $\alpha \cdot \widehat{\mu}^+_{ij} = \sum_{1 \le i \le j \le n} c_{ij} \widehat{\mu}^+_{ij}$ with $c_{ij} \in \mathbb{C}$, then $\alpha \cdot X^*_{ij} = \sum_{1 \le i \le j \le n} c_{ij} X^*_{ij}$.

Let ε_{ij}, $1 \le i \le j \le n$ be variables and we write ε to mean the $n \times n$ symmetric matrix whose (i,j) and (j,i) entries are ε_{ij}. Then we have

$$\mu(g,i)^{-1} \cdot \sum_{1 \le i \le j \le n} \varepsilon_{ij} \widehat{\mu}^+_{ij} = -\frac{i}{2} \begin{pmatrix} a^{-1}\varepsilon\,{}^{\mathrm{t}}a^{-1} & 0 \\ 0 & -a^{-1}\varepsilon\,{}^{\mathrm{t}}a^{-1} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & a^{-1}\varepsilon\,{}^{\mathrm{t}}a^{-1} \\ a^{-1}\varepsilon\,{}^{\mathrm{t}}a^{-1} & 0 \end{pmatrix}.$$

Now we compute

$$\big(\mu(g,i)^{-1} \cdot \widehat{\mu}^+_{ij}\big)\varphi_G(f, w^*)(g)$$

$$
\begin{aligned}
&= -\frac{i}{2}\frac{\partial}{\partial \varepsilon_{ij}}\varphi_G(f,w^*)\left(\left(\begin{pmatrix} a & x{}^{\mathrm{t}}a^{-1} \\ 0 & {}^{\mathrm{t}}a^{-1}\end{pmatrix}\exp\begin{pmatrix} a^{-1}\varepsilon{}^{\mathrm{t}}a^{-1} & 0 \\ 0 & -a^{-1}\varepsilon{}^{\mathrm{t}}a^{-1}\end{pmatrix}\right)\right)\Bigg|_{\varepsilon=0} \\
&\quad + \frac{1}{2}\frac{\partial}{\partial \varepsilon_{ij}}\varphi_G(f,w^*)\left(\left(\begin{pmatrix} a & x{}^{\mathrm{t}}a^{-1} \\ 0 & {}^{\mathrm{t}}a^{-1}\end{pmatrix}\exp\begin{pmatrix} 0 & a^{-1}\varepsilon{}^{\mathrm{t}}a^{-1} \\ a^{-1}\varepsilon{}^{\mathrm{t}}a^{-1} & 0\end{pmatrix}\right)\right)\Bigg|_{\varepsilon=0} \\
&= -\frac{i}{2}\frac{\partial}{\partial \varepsilon_{ij}}\left\langle \sigma(a)^{-1}\sigma(y+\varepsilon)f(z+2i\varepsilon),w^*\right\rangle\Big|_{\varepsilon=0} + \frac{1}{2}\frac{\partial}{\partial \varepsilon_{ij}}\left\langle \sigma(a)^{-1}\sigma(y-i\varepsilon)f(z+2\varepsilon),w^*\right\rangle\Big|_{\varepsilon=0} \\
&= -\frac{i}{2}\left\langle \sigma(a)^{-1}\frac{\partial}{\partial y'_{ij}}\sigma(y')f(z),w^*\right\rangle\Bigg|_{y'=y} - i\left\langle \sigma(a)^{-1}\sigma(y)\frac{\partial}{\partial y_{ij}}f(z),w^*\right\rangle \\
&\quad -\frac{i}{2}\left\langle \sigma(a)^{-1}\frac{\partial}{\partial y'_{ij}}\sigma(y')f(z),w^*\right\rangle\Bigg|_{y'=y} + \left\langle \sigma(a)^{-1}\sigma(y)\frac{\partial}{\partial x_{ij}}f(z),w^*\right\rangle \\
&= 2\left\langle \sigma(a)^{-1}\left(\frac{1}{2}\frac{\partial}{\partial x'_{ij}} - \frac{i}{2}\frac{\partial}{\partial y'_{ij}}\right)\sigma(y')f(z),w^*\right\rangle\Bigg|_{z'=z} + 2\left\langle \sigma(a)^{-1}\sigma(y)\left(\frac{1}{2}\frac{\partial}{\partial x_{ij}} - \frac{i}{2}\frac{\partial}{\partial y_{ij}}\right)f(z),w^*\right\rangle \\
&= 2\left\langle \sigma(a)^{-1}\frac{\partial}{\partial z_{ij}}\big(\sigma(y)f(z)\big),w^*\right\rangle .
\end{aligned}
$$

Therefore for a given $g \in Q(\mathbb{R})$ we have the identity

$$
4\pi i\cdot \varphi_G(D_{\sigma,\mathbb{H}_n}f, w^*\otimes \mu(g,i)^{-1}\cdot X^*_{ij})(g) = \big(\mu(g,i)^{-1}\cdot \widehat{\mu}^+_{ij}\big)\varphi_G(f,w^*)(g)
$$

for all $1\le i\le j\le n$, from which (2.18) follows. □

Remark 2.6.2. A similar computation as above shows that for the action of $\mathfrak{q}^-_G$ on $C^\infty(\Gamma\backslash G(\mathbb{R}))$ we have

$$
\Big(\widehat{\mu}^-_{ij}\varphi_G(f,w^*)\Big)_{1\le i,j\le n} = \mu(g,i)^{-1}\left(\varphi_G(\frac{\partial f}{\partial \bar z_{ij}},w^*)\right)_{1\le i,j\le n}{}^{\mathrm{t}}\mu(g,i)^{-1}.
$$

Up to scalars the action of $\mathfrak{q}^-_G$ on $C^\infty(\Gamma\backslash G(\mathbb{R}))$ corresponds to the operator

$$
\begin{aligned}
E_{\sigma,\mathbb{H}_n} : C^\infty_\sigma(\mathbb{H}_n,\Gamma) &\longrightarrow C^\infty_{\sigma\otimes\tau_n^*}(\mathbb{H}_n,\Gamma) \\
f &\longmapsto d_{\bar z}f,
\end{aligned}
$$

which translates to the operator E_σ defined as (3.13) by the map (2.13).

In §4.5 we use operators in $\mathfrak{q}^+_H$ to construct the archimedean sections. The corresponding

Eisenstein series are obtained from the scalar weight holomorphic Eisenstein series by the applying the action of $\mathfrak{q}_H^+$. Propositions 2.6.1 will make it clear how to translate the adelic picture to the geometric picture.

2.7 Polynomial q-expansions

We first define the Mumford objects. Then using the moduli interpretation of $N_\kappa^r(\Gamma(N), R) = H^0\left(X_{/R}, \mathcal{V}_\kappa^r\right)$, we evaluate a nearly holomorphic form at a Mumford object to get its polymonial q-expansion. We also include formulas for the action of differential operators on the polynomial q-expansions.

We have fixed the rank $2n$ lattice $\mathbf{L}_n = L_n \oplus L_n^*$ with a symplectic pairing, where L_n, L_n^* are both maximal isotropic and are dual to each other. Following [21, V.1], Let S_{L_n} be the symmetric quotient of $L_n \times L_n$ and $S_{L_n,\geq 0}$ be the intersection of S_{L_n} with the cone dual to the cone inside $S_{L_n}^* \otimes_{\mathbb{Z}} \mathbb{R}$ consisting of semi-positive definite forms. Take a basis $s_1, \cdots, s_{n(n+1)/2}$ of S_{L_n} lying inside $S_{L_n,\geq 0}$, and set $\mathbb{Z}((S_{L_n,\geq 0})) = \mathbb{Z}[[S_{L_n,\geq 0}]][1/s_1 s_2 \cdots s_{n(n+1)/2}]$. For $\beta \in S_{L_n,\geq 0}$, the corresponding element in $\mathbb{Z}[[S_{L_n,\geq 0}]]$ is sometimes written as q^β.

The natural map $L_n \to S_{L_n} \otimes L_n^*$ defines a period group $L_n \subset L_n^* \otimes \mathbb{G}_{m/\mathbb{Z}((S_{L_n,\geq 0}))}$, principally polarized by the duality between L_n and L_n^*. Mumford's construction [21] gives an abelian variety $A_{/\mathbb{Z}((S_{L_n,\geq 0}))}$ with a canonical polarization λ_{can} and a canonical basis $\omega_{\mathrm{can}} = (\omega_{1,\mathrm{can}}, \cdots, \omega_{n,\mathrm{can}})$ of $\omega(A/\mathbb{Z}((S_{L_n,\geq 0})))$. The exact sequence

$$0 \to L_n^* \otimes \prod_l \varprojlim_m \mu_{l^m} \to \prod_l T_l(A) \to L_n \otimes \widehat{\mathbb{Z}} \to 0,$$

after base changing to $\mathbb{Z}((N^{-1}S_{L_n,\geq 0}))[\zeta_N, 1/N]$, gives rise to a principal level N structure $\psi_{N,\mathrm{can}}$ for $A_{/\mathbb{Z}((S_{L_n,\geq 0}))}$. Let $D_{ij} \in \mathrm{Der}(\mathbb{Z}((S_{L_n,\geq 0})), \mathbb{Z}((S_{L_n,\geq 0})))$ be the element dual to $\omega_{i,\mathrm{can}}\omega_{j,\mathrm{can}}$ and $\delta_{i,\mathrm{can}} = \nabla(D_{ii})\omega_{i,\mathrm{can}}$. For $\beta \in S_{L_n,\geq 0}$ we have $D_{ij}(q^\beta) = (2 - \delta_{ij})\beta_{ij} q^\beta$ with $\delta_{ij} = 0$ if $i \neq j$, and 1 if $i = j$. Then $\delta_{\mathrm{can}} = (\delta_{1,\mathrm{can}}, \cdots, \delta_{n,\mathrm{can}})$ together with ω_{can} forms a basis of $\mathcal{H}^1_{\mathrm{dR}}(A/\mathbb{Z}((S_{L_n,\geq 0})))$ respecting both the Hodge filtration and the symplectic pairing.

Evaluating a nearly holomorphic form $f \in N^r_\kappa(\Gamma(N), R)$ at the test object

$$\mathrm{Mum}_N(q) = (A_{/\mathbb{Z}((N^{-1}S_{L_n,\geq 0}))[\zeta_N, 1/Np]}, \lambda_{\mathrm{can}}, \psi_{N,\mathrm{can}}, \omega_{\mathrm{can}}, \delta_{\mathrm{can}})$$

defines its polynomial q-expansion

$$\begin{aligned} N^r_\kappa(\Gamma(N), R) &\longrightarrow \mathbb{Z}[\zeta_N, 1/N][[N^{-1}S_{L_n,\geq 0}]] \otimes W_\kappa(R)[\underline{Y}]_{\leq r} \\ f &\longmapsto f(q, \underline{Y}) = f(\mathrm{Mum}_N(q)). \end{aligned} \tag{2.19}$$

Putting $\underline{Y} = 0$ in the polynomial q-expansion gives the (p-adic) q-expansion map

$$\varepsilon_{q,p\text{-adic}} : N^r_\kappa(\Gamma(N), R) \to R[\zeta_N, 1/N][[N^{-1}S_{L_n,\geq 0}]]. \tag{2.20}$$

This q-expansion map is injective and can be used to give an integral structure on the space of nearly holomorphic forms. We call it p-adic because it agrees with the one obtained by viewing nearly holomorphic forms as p-adic forms and applying the q-expansion map for p-adic forms to them (see §3.12).

Next we compute formulas of differential operators in terms of polynomial q-expansions. Let $\underline{X} = (X_{ij})_{1\leq i,j\leq n}$ be the symmetric matrix with the indeterminate $X_{ij} = X_{ji}$ as the ij-th and ji-th entries for $1 \leq i \leq j \leq n$. The X_{ij}'s form a basis of the $\mathrm{GL}(n)$-representation τ. An element $a \in \mathrm{GL}(n)$ acts on $\underline{X}$ by $a \cdot \underline{X} = {}^{\mathrm{t}}a\,\underline{X}a$. Let $X^\vee_{ij}$ be the basis of $\tau^\vee$ dual to X_{ij}. Then under the trivialization $(\omega_{\mathrm{can}}, \delta_{\mathrm{can}})$, X_{ij} corresponds to $\omega_{i,\mathrm{can}}\omega_{j,\mathrm{can}}$ and $X^\vee_{ij}$ corresponds to D_{ij}. From the construction of $\mathrm{Mum}_N(q)$ one can see that $\nabla(D_{ij})(\omega_{\mathrm{can}}, \delta_{\mathrm{can}}) = (\omega_{\mathrm{can}}, \delta_{\mathrm{can}})\mu^-_{ij}$, i.e. $X\big(D_{ij}, (\omega_{\mathrm{can}}, \delta_{\mathrm{can}})\big) = \mu^-_{ij}$.

Proposition 2.7.1. *Let $f \in N^r_\kappa(\Gamma(N), R)$ be a nearly holomorphic form with polynomial q-expansion $f(q, \underline{Y}) \in \mathbb{Z}[\zeta_N, 1/N][[N^{-1}S_{L_n,\geq 0}]] \otimes W_\kappa(R)[\underline{Y}]_{\leq r}$. Then*

$$(D_\kappa f)(q, \underline{Y}) = \sum_{1\leq i\leq j\leq n} \Big(D_{ij} f(q, \underline{Y}) + \mu^-_{ij} \cdot f(q, \underline{Y})\Big) \otimes X_{ij}$$

Example 2.7.2. If we apply the above proposition to the $n = 1$ case where $\kappa = k \in \mathbb{N}$, we recover the formula given in [63, Porposition 2.4.1] for D_k (denoted δ_k there). In this case the image of the polynomial q-expansion belongs $R[\zeta_N, 1/N][[q^{1/N}]][\underline{Y}]_{\leq r}$ and $D_{11} = q\frac{d}{dq}$. Write $Y = Y_{11}$. The representations κ and τ are both one-dimensional and we omit writing down their basis.

$$
\begin{aligned}
(D_\kappa f)(q, Y) &= D_{11} f(q, Y) + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \cdot f(q, Y) \\
&= q\frac{d}{dq} f(q, Y) + Y\eta_{11} \cdot f(q, Y) - Y^2 \frac{\partial}{\partial Y} f(q, Y) \\
&= (q\frac{d}{dq} - Y^2 \frac{\partial}{\partial q}) f(q, Y) + kY f(q, Y).
\end{aligned}
$$

Chapter 3

Overconvergent nearly holomorphic forms and their p-adic families

3.1 The weight space

Let p be an odd prime number. The weight space $\mathcal{W}$ is the rigid analytic space defined over $\mathbb{Q}_p$ associated to the noetherian complete algebra $\mathbb{Z}_p[[T_n(\mathbb{Z}_p)]]$. Its $\mathbb{C}_p$-points parametrize continuous characters from $T_n(\mathbb{Z}_p)$ to $\mathbb{C}_p^\times$, i.e. $\mathcal{W}(\mathbb{C}_p) = \mathrm{Hom}_{\mathrm{cont}}(T_n(\mathbb{Z}_p), \mathbb{C}_p^\times)$. For $\kappa \in \mathcal{W}$ we can write it as $\kappa = (\kappa_1, \kappa_2, \cdots, \kappa_n)$ with κ_i being a continuous character of $\mathbb{G}_m(\mathbb{Z}_p) \cong \mathbb{Z}_p^\times$ such that $\kappa(\mathrm{diag}(a_1, \cdots, a_n)) = \prod_{i=1}^n \kappa_i(a_i)$. If we fix a topological generator of $1 + p\mathbb{Z}_p$, say $1+p$, then $\mathcal{W}$ can be identified with the disjoint union of n-dimensional open unit balls indexed by $\widehat{T_n(\mathbb{Z}/p\mathbb{Z})}$, the character group of the torsion part $T_n(\mathbb{Z}/p\mathbb{Z})$ of the group $T_n(\mathbb{Z}_p)$. Explicitly we can write the isomorphism as

$$\begin{aligned} \mathcal{W} &\longrightarrow \widehat{T_n(\mathbb{Z}/p\mathbb{Z})} \times \prod_{i=1}^n \mathcal{B}(1, 1^-) \\ \kappa &\longmapsto (\kappa|_{T_n(\mathbb{Z}/p\mathbb{Z})}, \kappa_1(1+p), \kappa_2(1+p), \cdots, \kappa_n(1+p)). \end{aligned}$$

Here $\mathcal{B}(1, 1^-)$ is the 1-dimensional rigid open unit ball centered at 1. If $\mathcal{U} \subset \mathcal{W}$ is an affinoid subdomain we use $\mathcal{A}(\mathcal{U})$ to denote the affinoid algebra of analytic functions on $\mathcal{U}$ and $\mathcal{A}(\mathcal{U})^\circ$ to denote the subset of $\mathcal{A}(\mathcal{U})$ consisting of power bounded elements.

Given $\kappa = (\kappa_1, \cdots, \kappa_n) \in \mathcal{W}(\mathbb{C}_p)$ we say it is algebraic or classical if $\kappa_i(1+p) = (1+p)^{k_i}$ with k_i being an integer for all i and $k_1 \geq k_2 \geq \cdots \geq k_n$ and it is arithmetic if it can be written as the product of an algebraic weight with a locally constant character.

Let μ_{p-1} be the group of $(p-1)$-th roots of unity. There is a universal character

$$\kappa^{\mathrm{un}} : \mathcal{W} \times T_n(\mathbb{Z}_p) \longrightarrow \mu_{p-1} \times \mathcal{B}(1, 1^-).$$

Take L to be a complete field extension of $\mathbb{Q}_p$ inside $\mathbb{C}_p$. Denote by m_L the maximal ideal of $\mathcal{O}_L$. For each $w \in v(m_L)$ we can define over L the rigid analytic group $\mathcal{T}_{1,w} \cong \prod_{i=1}^n \mathcal{B}(1, p^w)$ with $\mathcal{B}(1, p^w)$ being the 1-dimensional closed ball of radius p^w centered at 1 and the rigid analytic group $\mathcal{T}_w = T(\mathbb{Z}_p)\mathcal{T}_{1,w}$. For any affinoid subdomain $\mathcal{U} \subset \mathcal{W}$ there exists some $w \in v(m_L)$ such that the universal character $\kappa^{\mathrm{un}}|_{\mathcal{U} \times T_n(\mathbb{Z}_p)}$ extends to a map between rigid analytic spaces

$$\kappa^{\mathrm{un}} : \mathcal{U} \times \mathcal{T}_w \longrightarrow \mu_{p-1} \times \mathcal{B}(1, 1^-).$$

For such $\mathcal{U}$ and w we say that the universal character κ^{un} over $\mathcal{U}$ is w-analytic. In order to see the existence of such a w it suffices to look at the case where $\mathcal{U}$ is a closed ball inside the identity connected component $\mathcal{W}^\circ$ of the weight space, i.e. $\mathcal{U} = \mathcal{W}(t)^\circ = \prod_{i=1}^n \mathcal{B}(1, p^t)$ for some $t \in v(m_L)$. Let $Y_1, \cdots, Y_n$ (resp. $S_1, \cdots, S_n$) be the coordinates of $\mathcal{W}(t)^\circ$ (resp. the neighborhood $a \cdot \prod_{i=1}^n \mathcal{B}(1, p^w) = \prod_{i=1}^n \mathcal{B}(a_i, p^w)$ of $a = \mathrm{diag}(a_1, \cdots, a_n) \in T_n(\mathbb{Z}_p)$) with coordinate ring $\mathcal{A}(\mathcal{W}(t)^\circ) = L\langle Y_1, \cdots, Y_n\rangle$ (resp. $L\langle S_1, \cdots, S_n\rangle$). The universal character can be extended to $\mathcal{W}(t)^\circ \times a \cdot \prod_{i+1}^n \mathcal{B}(1, p^w)$ as long as $(1+p^tY_i)^{a_i}(1+p^tY_i)^{\frac{\log(1+p^wS_i)}{\log(1+p)}}$ belongs to $L\langle Y_i, S_i\rangle$ for all $1 \leq i \leq n$. The factor $(1+p^tY_i)^{a_i} = \sum_{j=1}^\infty \begin{pmatrix} a_i \\ j \end{pmatrix} p^{tj}Y_i^j$ is always inside $1 + p^t\mathcal{O}_L\langle Y_i\rangle$, and the factor $(1+p^tY_i)^{\frac{\log(1+p^wS_i)}{\log(1+p)}} = \exp\left(\log(1+p^tY_i) \cdot \frac{\log(1+p^wS_i)}{\log(1+p)}\right)$ lies inside $L\langle Y_i, S_i\rangle$ if we choose w large enough such that the supreme norm of the function $\log(1+X)$ over $\mathcal{B}(0, p^t)$ satisfies $|\log(1+X)|_{\mathcal{B}(o,p^t)} < p^{w - \frac{1}{p-1}}$. If the universal weight κ^{un} is w-analytic over $\mathcal{U}$, then it is obvious that any point $\kappa \in \mathcal{U}(L)$ is a w-analytic weight, i.e. the character $\kappa : T_n(\mathbb{Z}_p) \to \mathbb{C}_p^\times$ extends to an analytic map $\kappa : \mathcal{T}_w \to \mu_{p-1} \times \mathcal{B}(1, 1^-)$.

Let $\mathfrak{T}_{1,w}$ be the formal group defined by

$$\mathfrak{T}_{1,w}(R) = \ker\big(T_n(R) \longrightarrow T_n(R/p^w R)\big) \tag{3.1}$$

for all flat, p-adically complete $\mathcal{O}_L$-algebras R. As a formal scheme $\mathfrak{T}_{1,w}$ is isomorphic to $\mathrm{Spf}(\mathcal{O}_L \langle S_1, \cdots, S_n \rangle)$. The identity component $\mathcal{W}(t)^\circ$ of $\mathcal{W}(t)$ has a natural formal model $\mathfrak{W}(t)^\circ$ isomorphic to $\mathrm{Spf}(\mathcal{O}_L \langle Y_1, \cdots, Y_n \rangle)$. Given an affinoid subdomain $\mathcal{U} \subset \mathcal{W}(t)^\circ$ and an open formal subscheme $\mathfrak{U}$ of an admissible blow-up of $\mathfrak{W}(t)^\circ$ such that $\mathcal{U}$ is the rigid fibre of $\mathfrak{U}$, the above discussion shows that for $w \in v(m_L)$ big enough the formal universal character

$$\kappa^{\mathrm{un}} : \mathfrak{U} \times \mathfrak{T}_{1,w} \longrightarrow \widehat{\mathbb{G}}_m$$

can be defined and it specializes to a formal character $\kappa : \mathfrak{T}_{1,w} \to \widehat{\mathbb{G}}_m$ for each $\kappa \in \mathcal{U}(L)$.

3.2 The analytic $(\mathfrak{g}, \mathcal{Q}_w)$-modules $V_{\kappa,w}$ and $V_{\kappa^{\mathrm{un}},w}$

This section is an analogue of §2.3 in the p-adic analytic and formal setting. Fix the p-adic field L and $w \in v(m_L)$ as in the previous section. Let $\mathfrak{A}_L$ be the category of L-affinoid algebras and $\mathbf{Adm}_{\mathcal{O}_L}$ be the category of admissible $\mathcal{O}_L$-algebras, i.e. the flat $\mathcal{O}_L$-algebras that are quotients of $\mathcal{O}_L \langle X_1, \cdots, X_s \rangle$ for some $s \in \mathbb{N}$. First we define several rigid analytic groups and formal groups. Like the formal torus $\mathfrak{T}^\circ_{1,w}$ we define the formal groups $\mathfrak{M}^\circ_{1,w}$ and $\mathfrak{B}_{1,w}$ over $\mathcal{O}_L$ by

$$\mathfrak{M}^\circ_{1,w}(R) = \ker\big(\mathrm{GL}(n, R) \longrightarrow \mathrm{GL}(n, R/p^w R)\big),$$
$$\mathfrak{B}_{1,w}(R) = \ker\big(B_n(R) \longrightarrow B_n(R/p^w R)\big)$$

for all $R \in \mathbf{Adm}_{\mathcal{O}_L}$. Define $\mathfrak{N}_{1,w}$ to be the unipotent part of $\mathfrak{B}_{1,w}$. Let $\mathrm{GL}(n)_{\mathrm{an}}$, $B_{n,\mathrm{an}}$, $N_{n,\mathrm{an}}$, $T_{n,\mathrm{an}}$, $Q_{\mathbf{G},\mathrm{an}}$, $U_{\mathbf{G},\mathrm{an}}$ be the rigid analytic groups associated to the groups schemes $\mathrm{GL}(n)$, B_n, N_n, T_n, $Q_{\mathbf{G}}$, $U_{\mathbf{G}}$, and $\mathrm{GL}(n)_{\mathrm{rig}}$, $B_{n,\mathrm{rig}}$, $T_{n,\mathrm{rig}}$, $Q_{\mathbf{G},\mathrm{rig}}$ be the generic fibre of the formal completion of $\mathrm{GL}(n)$, B_n, T_n, $Q_{\mathbf{G}}$ along p. The rigid fibre $\mathcal{M}^{\circ}_{1,w}$, $\mathcal{B}_{1,w}$, $\mathcal{T}_{1,w}$ of the formal groups $\mathfrak{M}^{\circ}_{1,w}$, $\mathfrak{B}_{1,w}$, $\mathfrak{T}_{1,w}$ can be naturally regarded as rigid analytic subgroups of $\mathrm{GL}(n)_{\mathrm{rig}}$, $B_{n,\mathrm{rig}}$, $T_{n,\mathrm{rig}}$. Set $I_n(\mathbb{Z}_p) = \{g \in \mathrm{GL}(n,\mathbb{Z}_p) : g \mod p \in B_n(\mathbb{Z}/p\mathbb{Z})\}$ to be the Iwahori subgroup of $\mathrm{GL}(n,\mathbb{Z}_p)$ and $N_n^-(\mathbb{Z}_p)$ to be the unipotent subgroup of $I_n(\mathbb{Z}_p)$ consisting of lower triangular matrices with 1 on the diagonal. $I_n(\mathbb{Z}_p) = N_n^-(\mathbb{Z}_p)B_n(\mathbb{Z}_p)$ is the Iwahori decomposition. We define the rigid analytic subgroup $\mathcal{I}_w$ of $\mathrm{GL}(n)_{\mathrm{rig}}$ by $\mathcal{I}_w = I_n(\mathbb{Z}_p) \cdot \mathcal{M}^{\circ}_{1,w}$. Fixing a set S of representatives in $I_n(\mathbb{Z}_p)$ of $I_n(\mathbb{Z}/p^{[w]}\mathbb{Z})$, the group $\mathcal{I}_w$ can be written as the disjoint union $\coprod_{\gamma\in S} \gamma \cdot \mathcal{M}^{\circ}_{1,w}$. Similarly we define $\mathcal{B}_w = B_n(\mathbb{Z}_p)\cdot\mathcal{B}_{1,w} \subset B_{n,\mathrm{rig}}$. The group $\mathcal{T}_w = T_n(\mathbb{Z}_p)\cdot\mathcal{T}_{1,w} \subset T_{n,\mathrm{rig}}$ is already defined in last section. There is a projection $\pi : Q_{\mathbf{G},\mathrm{an}} \to \mathrm{GL}(n)_{\mathrm{an}}$ sending $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ to a. We define the rigid analytic subgroup $\mathcal{Q}_w \subset Q_{\mathbf{G},an}$ as

$$\mathcal{Q}_w = \pi^{-1}(\mathcal{I}_w) \tag{3.2}$$

Note that $\mathcal{Q}_w$ is not contained inside $Q_{\mathbf{G},\mathrm{rig}}$.

Now take $\kappa = (\kappa_1, \cdots, \kappa_n) \in \mathcal{W}(\mathbb{C}_p)$ to be a w-analytic weight and set $\kappa' = (-\kappa_n, \cdots, -\kappa_1)$ which is also w-analytic. Extend κ' to a character of $\mathcal{B}_w$ through the quotient map $\mathcal{B}_w \to \mathcal{T}_w$. Define the w-analytic left $\mathcal{I}_w$-module $W_{\kappa,w}$ by

$$W_{\kappa,w}(R) = \{f : \mathcal{I}_w(R) \to R : f(xb) = \kappa'(b)f(x), \text{ for all } b \in \mathcal{B}_w(R),\ x \in \mathcal{I}_w(R) \text{ and } f \text{ is analytic}\}$$

for all $R \in \mathfrak{A}_L$, with $\mathcal{I}_w$ acting through the left inverse translation, i.e. $a \in \mathcal{I}_w(R)$ acts on $f \in W_{\kappa,w}(R)$ by

$$(a \cdot f)(x) = f(a^{-1}x).$$

Because of the Iwahori decomposition, $W_{\kappa,w}$ consists of analytic functions on

$$N_n^-(\mathbb{Z}/p^{[w]}\mathbb{Z}) \times \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \mathcal{B}(0,p^w) & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{B}(0,p^w) & \mathcal{B}(0,p^w) & \cdots & 1 \end{pmatrix}.$$

Therefore as a module over R we see $W_{\kappa,w}(R) = \oplus_{N^-(\mathbb{Z}/p^{[w]}\mathbb{Z})} R\langle T_{ij}\rangle_{1\le j<i\le n}$, i.e. $|N_n^-(\mathbb{Z}/p^{[w]}\mathbb{Z})|$ copies of strictly convergent power series in $n(n-1)/2$ variables.

From this description we see that there is a natural formal model of $W_{\kappa,w}$, whose R-points are $\oplus_{N_n^-(\mathbb{Z}/p^{[w]}\mathbb{Z})} R\langle T_{ij}\rangle_{1\le j<i\le n}$ for $R \in \mathbf{Adm}_{\mathcal{O}_L}$, equipped with a functorial action of $I_n(\mathbb{Z}_p)$ and $\mathfrak{M}^\circ_{1,w}$. We denote the formal model still by $W_{\kappa,w}$.

With $W_{\kappa,w}$ we define the w-analytic $(\mathfrak{g}, \mathcal{Q}_w)$-module $V_{\kappa,w}$ in the same way as we define the $(\mathfrak{g}, Q_{\mathbf{G}})$-module V_κ from the algebraic representation W_κ of $\mathrm{GL}(n)$ in §2.3. For all $R \in \mathfrak{A}_L$

$$V_{\kappa,w}(R) = W_{\kappa,w}(R) \otimes_R R[\underline{Y}] = W_{\kappa,w}(R) \otimes_R R[Y_{ij}]_{1\le i\le j\le n}.$$

The action of $g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathcal{Q}_w$ and $\mu^-_{ij} \in \mathfrak{u}^-_{\mathbf{G}}$ on $P(Y) \in V_{\kappa,w}$ is given by the formulas

$$(g \cdot P)(\underline{Y}) = a \cdot P(a^{-1}b + a^{-1}\,\underline{Y}d), \tag{3.3}$$

$$\begin{aligned} (\mu^-_{ij} \cdot P)(\underline{Y}) &= \sum_{1\le k\le n} (Y_{ki}\eta_{kj} + Y_{kj}\eta_{ki}) \cdot P(Y) - \sum_{1\le k\le l\le n} (Y_{ki}Y_{jl} + Y_{kj}Y_{il}) \frac{\partial}{\partial Y_{kl}} P(Y), \quad i \neq j, \\ (\mu^-_{ii} \cdot P)(\underline{Y}) &= \sum_{1\le k\le n} Y_{ki}\eta_{ki} \cdot P(\underline{Y}) - \sum_{1\le k\le l\le n} Y_{ki}Y_{il} \frac{\partial}{\partial Y_{kl}} P(\underline{Y}). \end{aligned} \tag{3.4}$$

The compatibility is checked in the same way as in §2.3 and as $\mathcal{Q}_w$-representations there is the filtration

$$\mathrm{Fil}^r\, V_{\kappa,w}(R) = V^r_{\kappa,w}(R) = W_{\kappa,w}(R) \otimes_R R[\underline{Y}]_{\le r}$$

satisfying $\mathfrak{g} \cdot V^r_{\kappa,w} \subset V^{r+1}_{\kappa,w}$. By definition $V^0_{\kappa,w} = W_{\kappa,w}$ as $\mathcal{Q}_w$-representations if we regard the $\mathcal{I}_w$-representation $W_{\kappa,w}$ as a $\mathcal{Q}_w$-representation via the projection $\mathcal{Q}_w \to \mathcal{I}_w$. For $i \in \mathbb{Z}$ we can twist $V_{\kappa,w}$ by the i-th power of the multiplier ν and get the w-analytic $(\mathfrak{g}, \mathcal{Q}_w)$-module $V_{\kappa,w}(i)$.

Recall that J is defined to be the algebraic representation V^1_{triv} of $Q_{\mathbf{G}}$. It restricts to an analytic $\mathcal{Q}_w$-representation and parallel to (2.9) we have

$$V^r_{\kappa,w} \cong V^0_{\kappa,w} \otimes \mathrm{Sym}^r J = W_{\kappa,w} \otimes \mathrm{Sym}^r J$$

as analytic $\mathcal{Q}_w$-representations.

A little more generally, given $(\sigma, W_\sigma) \in \mathrm{Rep}_{\mathbb{Z},f}\,\mathrm{GL}(n)$, an algebraic representation of $\mathrm{GL}(n)$ free of finite rank, the tensor product $W_{\kappa\otimes\sigma,w} = W_{\kappa,w} \otimes W_\sigma$ is an analytic $\mathcal{I}_w$-representation, and we can define the corresponding analytic $(\mathfrak{g}, \mathcal{Q}_w)$-module $V_{\kappa\otimes\sigma,w}$ and $\mathcal{Q}_w$-representation $V^r_{\kappa\otimes\sigma,w}$ for $r \geq 0$.

All of the above constructions carry over to the universal w-analytic weight κ^{un} over an affinoid subdomain $\mathcal{U} \subset \mathcal{W}$.

3.3 The Andreatta–Iovita–Pilloni construction

We briefly recall the constructions in [1, Chapter 3,4,5]. Let ς be the Frobenious endomorphism of $\mathcal{O}_L/p\mathcal{O}_L$. For any finite group scheme H over $\mathcal{O}_L$ we denote by H^D its Cartier dual and ω_H its sheaf of invariant differentials. Given a Barsotti-Tate group G over $\mathcal{O}_L$ of dimension n, the Hasse invariant $\mathrm{Ha}(G) \in \det(\omega_{G[p]^D})^{\otimes p-1}$ is defined to be the determinant of the ς-linear endomorphism on $\omega_{G[p]^D}$ induced by the absolute Frobenius. The Hodge height $\mathrm{Hdg}(G) \in [0,1]$ is defined as the truncated valuation of $\mathrm{Ha}(G)$.

Let $\mathbf{NAdm}_{\mathcal{O}_L}$ be subcategory of $\mathbf{Adm}_{\mathcal{O}_L}$ consisting of those objects that are normal. Fix $R \in \mathbf{NAdm}_{\mathcal{O}_L}$ and suppose that G is a rank n semi-abelian scheme over $S = \mathrm{Spec}(R)$ whose restriction to an open dense subscheme of S is abelian. Take a positive integer $m \in \mathbb{N}_{>0}$ and $v < \frac{1}{2p^{m-1}}$ (resp. $v < \frac{1}{3p^{m-1}}$ if $p = 3$) such that for any $x \in S_{\mathrm{rig}}$ the Hodge height $\mathrm{Hdg}(x) := \mathrm{Hdg}(G_x[p^\infty]) \leq v$. Write R_w to denote R/p^wR. We summarize, in the following theorem, some results about canonical subgroups in families used in [1].

Theorem 3.3.1. *([1, Proposition 4.1.3, Proposition 4.3.1]) There is a finite flat canonical subgroup $H_m \subset G[p^m]$ of level m over S, which, at each point $x \in S_{\mathrm{rig}}$, specializes to the canonical subgroup $H_{m,x} \subset G_x[p^m]$ as constructed in [22, Theorem 6] . Moreover, assuming $H_m^D(R[1/p]) \simeq (\mathbb{Z}/p^m\mathbb{Z})^n$, then there is a free sub-sheaf of R-modules $\mathcal{F} \subset \omega_G$ of rank n containing $p^{\frac{v}{p-1}}\omega_G$, equipped with an isomorphism*

$$\mathrm{HT}_w : H_m^D(R[1/p]) \otimes_{\mathbb{Z}} R_w \xrightarrow{\sim} \mathcal{F} \otimes_R R_w,$$

induced from the Hodge–Tate map on H_m^D, for all $w \in (0, m - v\frac{p^m}{p-1}] \cap v_p(\mathcal{O}_L)$.

Let K be a finite extension of $\mathbb{Q}_p$ with and ϖ be a uniformizer of $\mathcal{O}_K$. Denote by Y the Siegel variety defined over $\mathcal{O}_K$ parametrizing principally polarized abelian schemes of dimension n with principal level N structure. Let X be a smooth toroidal compactification. The universal abelian scheme $\mathcal{A} \to Y$ extends to a semi-abelian scheme $\mathcal{G} \to X$. Set $\mathfrak{X}$ to be the formal scheme obtained by completing X along its special fibre. On the associated rigid analytic space $X_{\mathrm{rig}} = X_{\mathrm{an}}$, we have the Hodge height function $\mathrm{Hdg} : X_{\mathrm{rig}} \to [0,1]$. For $v \in v_p(\mathcal{O}_K)$ we define the open subset $\mathcal{X}(v) = \{x \in X_{\mathrm{rig}} : \mathrm{Hdg}(x) \leq v\}$. Let $\tilde{\mathfrak{X}}(v)$ be the admissible blow-up of $\mathfrak{X}$ along the ideal (Ha, p^v), and $\mathfrak{X}(v)$ be the p-adic completion of the normalization of the largest open formal sub-scheme of $\tilde{\mathfrak{X}}(v)$ where the ideal (Ha, p^v) is generated by Ha. This $\mathfrak{X}(v)$ is a formal model of $\mathcal{X}(v)$. By construction the semi-abelian scheme $\mathcal{G} \to X$ gives rise to semi-ableian schemes over $\mathcal{X}(v)$ and $\mathfrak{X}(v)$, which we still denote by $\mathcal{G}$. For $m \in \mathbb{N}_{>0}$ and $v < \frac{1}{2p^{m-1}}$ (resp. $v < \frac{1}{3p^{m-1}}$ if $p = 3$), there is the level m canonical subgroup $H_m \subset G[p^m]$. Define $\mathcal{X}_1(p^m)(v) = \underline{\mathrm{Isom}}_{\mathcal{X}(v)}((\mathbb{Z}/p^m\mathbb{Z})^n, H_m^D)$ to be the finite étale cover of $\mathcal{X}(v)$ parametrizing the trivializations ψ of the Cartier dual of H_m. The group $\mathrm{GL}(n, \mathbb{Z}/p\mathbb{Z})$ acts on $\mathcal{X}_1(p)(v)$. The quotient $\mathcal{X}_{\mathrm{Iw}}(v) = \mathcal{X}_1(p)(v))/B_n(\mathbb{Z}/p\mathbb{Z})$ by the Borel subgroup is still finite étale over $\mathcal{X}(v)$. As formal models of $\mathcal{X}_1(p^m)(v)$, $\mathcal{X}_{\mathrm{Iw}}(v)$, we take $\mathfrak{X}_1(p^m)(v)$, $\mathfrak{X}_{\mathrm{Iw}}(v)$ to be the normalizations of $\mathfrak{X}(v)$ inside the corresponding rigid spaces. There is the chain of formal schemes

$$\mathfrak{X}_1(p^m)(v) \xrightarrow{\pi_1} \mathfrak{X}_{\mathrm{Iw}}(v) \xrightarrow{\pi_0} \mathfrak{X}(v).$$

Let $\mathfrak{Y}, \mathfrak{Y}(v), \mathfrak{Y}_1(p^m)(v), \mathfrak{Y}_{\mathrm{Iw}}(v)$ be the open formal subschemes of $\mathfrak{X}, \mathfrak{X}(v), \mathfrak{X}_1(p^m)(v), \mathfrak{X}_{\mathrm{Iw}}(v)$ that are the complements of the boundary C. Although $\mathfrak{Y}(v), \mathfrak{Y}_1(p^m)(v), \mathfrak{Y}_{\mathrm{Iw}}(v)$ are not moduli spaces, they admit modular interpretations for $R \in \mathbf{NAdm}$ (cf. [1, Proposition 5.2.1.1]). Let Y_{an} be the analytification of Y with the natural open immersion $Y_{\mathrm{an}} \hookrightarrow X_{\mathrm{an}}$. Set $\mathcal{Y}(v), \mathcal{Y}_1(p^m)(v), \mathcal{Y}_{\mathrm{Iw}}(v)$ to be the fibre products of $\mathcal{X}(v), \mathcal{X}_1(p^m)(v), \mathcal{X}_{\mathrm{Iw}}(v)$ with Y_{an} over X_{an}.

By the construction of $\mathfrak{X}_1(p^m)(v)$, we can apply Theorem 3.3.1 to construct a locally free subsheaf $\mathcal{F} \subset \omega\big(\mathcal{G}/\mathfrak{X}_1(p^m)(v)\big)$ of rank n, equipped with the isomorphism

$$\mathrm{HT}_w \circ \psi : (\mathbb{Z}/p^m\mathbb{Z})^n \otimes_{\mathbb{Z}} \mathcal{O}_{\mathfrak{X}_1(p^m)(v),w} \xrightarrow{\sim} \mathcal{F} \otimes_{\mathcal{O}_K} \mathcal{O}_{K,w} \tag{3.5}$$

for $w \in (0, m - v\frac{p^m}{p-1}] \cap v(\mathcal{O}_K)$.

From now on we assume $w \in (m-1+\frac{v}{p-1}, m - \frac{p^m v}{p-1}] \cap v(\mathcal{O}_K)$, so m is determined by w. Define the $\mathfrak{M}^{\circ}_{1,w}$-torsor $\mathfrak{T}^{\times}_{\mathcal{F},w}(v)$ over $\mathfrak{X}_1(p^m)(v)$ by

$$\mathfrak{T}^{\times}_{\mathcal{F},w}(v) = \underline{\mathrm{Isom}}_{\mathfrak{X}_1(p^m)(v),\psi,w}(\mathcal{O}^n_{\mathfrak{X}_1(p^m)(v)}, \mathcal{F}),$$

where the subscript ψ, w means that we require the isomorphism to be w-compatible with (3.5) as explained below. We always fix the canonical global basis of the n copies of the structure sheaf $\mathcal{O}^n_{\mathfrak{X}_1(p^m)(v)}$ and the canonical basis of the $\mathbb{Z}/p^m\mathbb{Z}$-module $(\mathbb{Z}/p^m\mathbb{Z})^n$. Then locally over $\mathfrak{U} = \mathrm{Spf}(R) \subset \mathfrak{X}_1(p^m)(v)$, an isomorphism α from R^n to $\mathcal{F}(\mathfrak{U})$ corresponds to an ordered basis $\alpha_1, \cdots \alpha_n$ of the free R-module $\mathcal{F}(\mathfrak{U})$ and ψ gives rise to an ordered basis $x_1, \cdots, x_n$ of $H^D_m(R[1/p])$. We say α is w-compatible with (3.5) if $\alpha_i \equiv \mathrm{HT}_w(x_i) \mod p^w R$ for all $1 \le i \le n$. An element $a \in \mathfrak{M}^{\circ}_{1,w}(R)$ acts on α by sending it to $\alpha \circ a$, or equivalently sending the corresponding basis $(\alpha_1, \cdots, \alpha_n)$ to $(\alpha_1, \cdots, \alpha_n) \cdot \alpha$. This action makes $\mathfrak{T}^{\times}_{\mathcal{F},w}(v)$ a principal $\mathfrak{M}^{\circ}_{1,w}$-torsor over $\mathfrak{X}_1(p^m)(v)$.

For a w-analytic weight $\kappa \in \mathcal{W}(K)$ we can form the contracted product and get a locally free formal sheaf

$$\tilde{\mathfrak{w}}^{\dagger}_{\kappa,w} := \mathfrak{T}^{\times}_{\mathcal{F},w}(v) \times^{\mathfrak{M}^{\circ}_{1,w}} W_{\kappa,w}$$

over $\mathfrak{X}_1(p^m)(v)$. In particular this $\tilde{\mathfrak{w}}^{\dagger}_{\kappa,w}$ is a flat formal Banach sheaf in the sense of [1, Definition A.1.1.1]. Therefore we can apply the procedure worked out in [1, A.2.2] to get the associated Banach sheaf $\tilde{\omega}^{\dagger}_{\kappa,w}$ over the rigid analytic fibre $\mathcal{X}_1(p^m)(v)$ (see [1, Definition A.2.1.2] for the definition of a Banach sheaf). For any affinoid subdomain $\mathcal{U} \subset \mathcal{X}_1(p^m)(v)$ and an admissible blow-up $h : \mathfrak{X}' \to \mathfrak{X}_1(p^m)(v)$ such that $\mathcal{U}$ is the rigid fibre of an open formal subscheme $\mathfrak{U}$ of $\mathfrak{X}'$, the local sections of $\tilde{\omega}^{\dagger}_{\kappa,w}$ over $\mathcal{U}$ are

$$\tilde{\omega}^{\dagger}_{\kappa,w}(\mathcal{U}) = h^*\tilde{\mathfrak{w}}^{\dagger}_{\kappa,w}(\mathfrak{U}) \otimes_{\mathcal{O}_K} K,$$

which is naturally equipped with a complete norm (independent of h up to equivalence) with $\tilde{\mathfrak{w}}^{\dagger}_{\kappa,w}(\mathfrak{U})$ being the unit ball.

The group $I_n(\mathbb{Z}/p^m\mathbb{Z})$ acts on $\mathfrak{X}_1(p^m)(v)$ with $\mathfrak{X}_{\mathrm{Iw}}(v)$ as the quotient. Under this action the sheaf $\tilde{\omega}^{\dagger}_{\kappa,w}$ is $I_n(\mathbb{Z}/p^m\mathbb{Z})$-equivariant. In order to see this we need to construct the isomorphism $\varphi_\omega(\bar{\gamma}) : \omega^{\dagger}_{\kappa,w} \to \bar{\gamma}^*\omega^{\dagger}_{\kappa,w}$ for each $\bar{\gamma} \in I_n(\mathbb{Z}/p^m\mathbb{Z})$. Let $\bar{\gamma}^*\mathfrak{T}^{\times}_{\mathcal{F},w}(v)$ be the fibre product

$$\begin{array}{ccc} \bar{\gamma}^*\mathfrak{T}^{\times}_{\mathcal{F},w}(v) & \longrightarrow & \mathfrak{T}^{\times}_{\mathcal{F},w}(v) \\ \downarrow & \square & \downarrow \\ \mathfrak{X}_1(p^m)(v) & \xrightarrow{\bar{\gamma}} & \mathfrak{X}_1(p^m)(v). \end{array}$$

We have $\bar{\gamma}^*\mathfrak{T}^{\times}_{\mathcal{F},w}(v) \cong \underline{\mathrm{Isom}}_{\mathfrak{X}_1(p^m)(v),\psi\circ\bar{\gamma},w}(\mathcal{O}^n_{\mathfrak{X}_1(p^m)(v)}, \bar{\gamma}^*\mathcal{F})$ and $\bar{\gamma}^*\tilde{\mathfrak{w}}^{\dagger}_{\kappa,w} \cong \bar{\gamma}^*\mathfrak{T}^{\times}_{\mathcal{F},w}(v) \times^{\mathfrak{M}^{\circ}_{1,w}} W_{\kappa,w}$. The sheaf $\omega\big(\mathcal{G}/\mathfrak{X}_1(p^m)(v)\big)$ is the pullback of $\omega\big(\mathcal{G}/\mathfrak{X}_{\mathrm{Iw}}(v)\big)$ and hence is naturally $I_n(\mathbb{Z}/p^m\mathbb{Z})$-equivariant. By construction the subsheaf $\mathcal{F} \subset \omega\big(\mathcal{G}/\mathfrak{X}_{\mathrm{Iw}}(v)\big)$ is an $I_n(\mathbb{Z}/p^m\mathbb{Z})$-equivariant subsheaf of $\omega\big(\mathcal{G}/\mathfrak{X}_1(p^m)(v)\big)$ so there is the isomorphism $\varphi_{\mathcal{F}}(\bar{\gamma}) : \mathcal{F} \to \bar{\gamma}^*\mathcal{F}$. Take an open formal subscheme $\mathfrak{U} = \mathrm{Spf}(R) \subset \mathfrak{X}_1(p^m)(v)$ over which the sheaf $\mathcal{F}$ can be trivialized. Local sections of $\tilde{\mathfrak{w}}^{\dagger}_{\kappa,w}$ over $\mathfrak{U}$ are pairs (α, f) with $\alpha \in \mathfrak{T}^{\times}_{\mathcal{F},w}(v)(\mathfrak{U})$ and $f \in W_{\kappa,w}\hat{\otimes}R$ modulo the equivalence relations $(\alpha \circ a, f) \sim (\alpha, a \cdot f)$, $a \in \mathfrak{M}^{\circ}_{1,w}(R)$. Pick a lift $\gamma \in I_n(\mathbb{Z}_p)$ of $\bar{\gamma}$ and define

$$\begin{aligned} \varphi_{\mathfrak{w}}(\bar{\gamma}) : \mathfrak{T}^{\times}_{\mathcal{F},w}(v) \times^{\mathfrak{M}^{\circ}_{1,w}} W_{\kappa,w}(\mathfrak{U}) &\longrightarrow \bar{\gamma}^*\mathfrak{T}^{\times}_{\mathcal{F},w}(v) \times^{\mathfrak{M}^{\circ}_{1,w}} W_{\kappa,w}(\mathfrak{U}) \\ (\alpha, f) &\longmapsto (\varphi_{\mathcal{F}}(\bar{\gamma}) \circ \alpha \circ \gamma, \gamma^{-1} \cdot f). \end{aligned}$$

The map $\varphi_{\mathfrak{w}}(\bar{\gamma})$ is well defined, independent of the choice of the lift γ, and patches to an isomorphism $\varphi_{\mathfrak{w}}(\bar{\gamma}) : \tilde{\mathfrak{w}}^{\dagger}_{\kappa,w} \to \bar{\gamma}^*\tilde{\mathfrak{w}}^{\dagger}_{\kappa,w}$. Inverting p we get $\varphi_\omega(\bar{\gamma}) : \tilde{\omega}^{\dagger}_{\kappa,w} \to \bar{\gamma}^*\tilde{\omega}^{\dagger}_{\kappa,w}$. Since $I_n(\mathbb{Z}/p^m\mathbb{Z})$ is a finite group, the $I_n(\mathbb{Z}/p^m\mathbb{Z})$-invariant of the pushforward $\pi_{1,*}\tilde{\omega}^{\dagger}_{\kappa,w}$ is a Banach sheaf over $\mathcal{X}_{\mathrm{Iw}}(v)$.

Definition 3.3.2. *The Banach sheaf of w-analytic, v-overconvergent, weight κ Siegel modular forms of principal level N is defined as*

$$\omega^{\dagger}_{\kappa,w} := \left(\pi_{1,*}\tilde{\omega}^{\dagger}_{\kappa,w}\right)^{I_n(\mathbb{Z}/p^m\mathbb{Z})}.$$

We also want to associate to the Banach sheaf $\omega^{\dagger}_{\kappa,w}$ a contracted product interpretation, which will bring us some convenience when defining some morphisms. By taking the rigid fibre of the $\mathfrak{M}^{\circ}_{1,w}$-torsor $\mathfrak{T}^{\times}_{\mathcal{F},w}(v)$ over $\mathfrak{X}_1(p^m)(v)$, we get

$$\mathcal{T}^{\times}_{\mathcal{F},w}(v) \xrightarrow{\pi_2} \mathcal{X}_1(p^m)(v) \xrightarrow{\pi_1} \mathcal{X}_{\mathrm{Iw}}(v).$$

The rigid analytic space $\mathcal{T}^{\times}_{\mathcal{F},w}(v)$ is a $\mathcal{M}^{\circ}_{1,w}$-torsor over $\mathcal{X}_1(p^m)(v)$ and the cover $\pi_1 : \mathcal{X}_1(p^m)(v) \to \mathcal{X}_{\mathrm{Iw}}(v)$ is finite étale. The group $I_n(\mathbb{Z}_p)$ acts on $\mathcal{T}^{\times}_{\mathcal{F},w}(v)$ over $\mathcal{X}_{\mathrm{Iw}}(v)$ by sending α to $\varphi_{\mathcal{F}}(\bar{\gamma}) \circ \alpha \circ \gamma$. This $I_n(\mathbb{Z}_p)$-action together with the $\mathcal{M}^{\circ}_{1,w}$-torsor structure on $\mathcal{T}^{\times}_{\mathcal{F},w}(v)$ makes it an $\mathcal{I}_w$-torsor over $\mathcal{X}_{\mathrm{Iw}}(v)$. Let $\mathcal{S}$ be the category whose objects are affinoid subdomains of $\mathcal{X}_{\mathrm{Iw}}(v)$ admitting local sections of the projection $\pi_1 \circ \pi_2$ with inclusions as morphisms. We can define a presheaf on $\mathcal{S}$ by the contracted product

$$\mathcal{T}^{\times}_{\mathcal{F},w}(v) \times^{\mathcal{I}_w} W_{\kappa,w}. \tag{3.6}$$

It is isomorphic to the restriction of the sheaf $\omega^{\dagger}_{\kappa,w}$ to $\mathcal{S}$. We call (3.6) a contracted product interpretation of $\omega^{\dagger}_{\kappa,w}$. Since the objects of $\mathcal{S}$ form a basis of the Grothendieck topology on $\mathcal{X}_1(p^m)(v)$, in order to construct morphisms between the sheaves over $\mathcal{X}_{\mathrm{Iw}}(v)$, it suffices to construct morphisms between their restrictions to $\mathcal{S}$. Therefore the contracted product interpretation $\mathcal{T}^{\times}_{\mathcal{F},w}(v) \times^{\mathcal{I}_w} W_{\kappa,w}$ will be useful in constructing morphisms between sheaves that are related to $\omega^{\dagger}_{\kappa,w}$. By abuse of notation we will write $\omega^{\dagger}_{\kappa,w} = \mathcal{T}^{\times}_{\mathcal{F},w}(v) \times^{\mathcal{I}_w} W_{\kappa,w}$.

Define the $\mathcal{O}_K$ schemes $T_{\omega} = \underline{\mathrm{Hom}}_X\left(\mathcal{O}_X^n, \omega(\mathcal{G}/X)\right)$ and $T^{\times}_{\omega} = \underline{\mathrm{Isom}}_X\left(\mathcal{O}_X^n, \omega(\mathcal{G}/X)\right)$ over X. Let $T_{\omega,\mathrm{an}}$, $T^{\times}_{\omega,an}$ be their rigid analytifications, and $\mathfrak{T}_{\omega}$, $\mathfrak{T}^{\times}_{\omega}$ be their formal completions along the special fibres. Also take $T_{\omega,\mathrm{rig}}$, $T^{\times}_{\omega,\mathrm{rig}}$ to be the rigid fibre of $\mathfrak{T}_{\omega}$, $\mathfrak{T}^{\times}_{\omega}$. Set $\mathcal{T}_{\omega,\mathrm{an}}(v), \mathcal{T}^{\times}_{\omega,\mathrm{an}}(v), \mathcal{T}_{\omega,\mathrm{rig}}(v), \mathcal{T}^{\times}_{\omega,\mathrm{rig}}(v)$ to be the corresponding base changes to $\mathcal{X}_{\mathrm{Iw}}(v)$. Due to the requirement $w \in (m-1+\frac{v}{p-1}, m-$

$v\frac{p^m}{p-1}]\cap v(\mathcal{O}_K)$, the argument of [1, Proposition 5.3.1] shows that there is a natural open immersion

$$\mathcal{T}^{\times}_{\mathcal{F},w}(v)\hookrightarrow \mathcal{T}_{\omega,\mathrm{rig}}(v)\cap\mathcal{T}^{\times}_{\omega,\mathrm{an}}(v).$$

Therefore local sections of the projection $\mathcal{T}^{\times}_{\mathcal{F},w}(v)\to\mathcal{X}_{\mathrm{Iw}}(v)$ correspond to local basis of the sheaf $\omega(\mathcal{G}/\mathcal{X}_{\mathrm{Iw}}(v))$ satisfying w-compatibility conditions defined by the Hodge–Tate map HT_w. Note that $\mathcal{T}^{\times}_{\mathcal{F},w}(v)$ does not lie inside $\mathcal{T}^{\times}_{\omega,\mathrm{rig}}(v)$. When κ is classical this open immersion induces a canonical inclusion of $\omega_\kappa|_{\mathcal{X}_{\mathrm{Iw}}(v)}$ into $\omega^{\dagger}_{\kappa,w}$.

In [1] another two formal schemes are introduced. They are defined as

$$\mathfrak{IW}_w(v)=\mathfrak{T}^{\times}_{\mathcal{F},w}/\mathfrak{B}_{1,w},\qquad\text{and}\qquad \mathfrak{IW}^{+}_w(v)=\mathfrak{T}^{\times}_{\mathcal{F},w}/\mathfrak{N}_{1,w},$$

with maps

$$\mathfrak{IW}^{+}_w(v)\xrightarrow{g}\mathfrak{IW}_w(v)\xrightarrow{\pi_3}\mathfrak{X}_1(p^m)(v)\xrightarrow{\pi_1}\mathfrak{X}_{\mathrm{Iw}}(v).$$

The group $\mathfrak{T}_{1,w}$ acts on $\mathfrak{IW}^{+}_w(v)$ over $\mathfrak{IW}_w(v)$, and so on the pushforward of the structure sheaf $g_*\mathcal{O}_{\mathfrak{IW}^{+}_w(v)}$. Define the invertible sheaf $\mathfrak{L}_\kappa=g_*\mathcal{O}_{\mathfrak{IW}^{+}_w(v)}[\kappa']$ to be the κ'-invariant of the $\mathfrak{T}^{\circ}_{1,w}$-action on $g_*\mathcal{O}_{\mathfrak{IW}^{+}_w(v)}$. Take the rigid fibres $\mathcal{IW}^{+}_w(v)$, $\mathcal{IW}^{+}_w(v)$, $\mathcal{L}_\kappa$. There is a $B_n(\mathbb{Z}_p)$-action on $\mathcal{IW}^{+}_w(v)$ over $\mathcal{X}_{\mathrm{Iw}}(v)$ which, together with κ, makes $\pi_{3,*}\mathcal{L}_\kappa$ a $B_n(\mathbb{Z}/p^m\mathbb{Z})$-equivariant Banach sheaf with respect to the natural $B_n(\mathbb{Z}/p^m\mathbb{Z})$-action on $\mathcal{X}_1(p^m)(v)$ over $\mathcal{X}_{\mathrm{Iw}}(v)$. In [1] the invariant $(\pi_{1,*}\pi_{3,*}\mathcal{L}_\kappa)^{B(\mathbb{Z}/p^m\mathbb{Z})}$ is defined to be the Banach sheaf of w-analytic, v-overconvergent, weight κ Siegel modular forms. It is easy to see that the map $W_{\kappa,w}\to\mathbb{A}^1_K$ by evaluation at identity induces an isomorphism between $(\pi_{1,*}\pi_{3,*}\mathcal{L}_\kappa)^{B(\mathbb{Z}/p^m\mathbb{Z})}$ and the sheaf $\omega^{\dagger}_{\kappa,w}$ in Definition 3.3.2 .

All the above constructions run parallelly for the w-analytic universal weight κ^{un} corresponding to $\mathcal{U}\subset\mathcal{W}$, so that we can define the Banach sheaf $\omega^{\dagger}_{\kappa^{\mathrm{un}},w}$ over $\mathcal{X}_{\mathrm{Iw}}(v)\times\mathcal{U}$ and the flat formal Banach sheaf $\tilde{\mathfrak{w}}^{\dagger}_{\kappa^{\mathrm{un}},w}$ over $\mathfrak{X}_1(p^m)(v)\times\mathfrak{U}$.

3.4 Nearly overconvergent Siegel modular forms

3.4.1 The Banach sheaf $\mathcal{V}^{\dagger,r}_{\kappa,w}$ and its global sections

Recall that in §2.4 we defined the locally free sheaf of finite rank $\mathcal{J}$ over X, and for $\sigma \in \operatorname{Rep}_{\mathbb{Z},f} \operatorname{GL}(n)$ we have $\mathcal{V}^r_\sigma \cong \omega_\sigma \otimes \operatorname{Sym}^r \mathcal{J}$ as locally free sheaves with Hecke actions. Take the rigid analytification of $\mathcal{J}$ and pull it back to $\mathcal{X}_{\mathrm{Iw}}(v)$. We denote the resulting coherent sheaf over $\mathcal{X}_{\mathrm{Iw}}(v)$ still by $\mathcal{J}$. Similarly let $\mathfrak{J}$ be the locally free formal sheaf of finite rank over $\mathfrak{X}_{\mathrm{Iw}}(v)$ obtained by completing $\mathcal{J}$ along the special fibre of X and pulling it back. $\operatorname{Sym}^r \mathcal{J}$ is the rigid fibre of $\operatorname{Sym}^r \mathfrak{J}$. Since $\operatorname{Sym}^r \mathcal{J}$ is locally free of finite rank and $\mathcal{X}_{\mathrm{Iw}}(v)$ is quasi-compact, it can be equipped with a Banach sheaf structure by choosing a cover and local basis. All such structures are equivalent to the one given by the formal model $\operatorname{Sym}^r \mathfrak{J}$. The tensor product of $\operatorname{Sym}^r \mathcal{J}$ with a Banach sheaf is still a Banach sheaf under the tensor product semi-norm. The flatness of $\operatorname{Sym}^r \mathcal{J}$ guarantees that the sheaf conditions of the Banach sheaf are preserved under the operation of tensoring with $\operatorname{Sym}^r \mathcal{J}$. Also the space of local sections of the tensor product sheaf is complete with respect to the tensor product semi-norm (i.e. there is no need to take completed tensor product).

Definition 3.4.1. *The Banach sheaf of w-analytic, v-overconvergent nearly holomorphic forms of principal level N, weight κ (resp. universal weight κ^{un} over $\mathcal{U} \subset \mathcal{W}$) and (non-holomorphy) degree r is defined as*

$$\mathcal{V}^{\dagger,r}_{\kappa,w} := \omega^\dagger_{\kappa,w} \otimes \operatorname{Sym}^r \mathcal{J} \qquad (\textit{resp. } \mathcal{V}^{\dagger,r}_{\kappa^{\mathrm{un}},w} := \omega^\dagger_{\kappa^{\mathrm{un}},w} \otimes \operatorname{Sym}^r \mathcal{J}).$$

The space of global sections of a Banach sheaf over a quasi-compact rigid analytic space can be equipped with a norm by choosing a suitable admissible covering by affinoids. All such norms are equivalent and the space of global sections are complete under these norms.

Definition 3.4.2. *The K-Banach space (resp. $\mathcal{A}(\mathcal{U})$-Banach module) of w-analytic, v-overconvergent nearly holomorphic forms of principal level N, weight κ (resp. universal weight κ^{un} over $\mathcal{U} \subset \mathcal{W}$)*

and (non-holomorphy) degree r and the corresponding cuspidal part are defined as

$$N^{\dagger,r}_{\kappa,w,v} := H^0(\mathcal{X}_{\mathrm{Iw}}(v), \mathcal{V}^{\dagger,r}_{\kappa,w}) \qquad (\textit{resp. } N^{\dagger,r}_{\mathcal{U},w,v} := H^0(\mathcal{X}_{\mathrm{Iw}}(v)\times\mathcal{U}, \mathcal{V}^{\dagger,r}_{\kappa^{\mathrm{un}},w}),$$
$$N^{\dagger,r}_{\kappa,w,v,\mathrm{cusp}} := H^0(\mathcal{X}_{\mathrm{Iw}}(v), \mathcal{V}^{\dagger,r}_{\kappa,w}(-C)) \qquad (\textit{resp. } N^{\dagger,r}_{\mathcal{U},w,v,\mathrm{cusp}} := H^0(\mathcal{X}_{\mathrm{Iw}}(v)\times\mathcal{U}, \mathcal{V}^{\dagger,r}_{\kappa^{\mathrm{un}},w}(-C)).$$

Following [63] we also call overconvergent nearly holomorphic forms nearly overconvergent forms.

For later use we also define a locally free formal Banach sheaf $\tilde{\mathfrak{V}}^{\dagger,r}_{\kappa,w}$ over $\mathfrak{X}_1(p^m)(v)$ by the tensor product $\tilde{\mathfrak{w}}^{\dagger}_{\kappa,w}\otimes\mathrm{Sym}^r\,\mathfrak{J}$. Let $\tilde{\mathcal{V}}^{\dagger,r}_{\kappa,w}$ be its rigid fibre which is an $I_n(\mathbb{Z}/p^m\mathbb{Z})$-equivariant Banach sheaf. Then we have $\mathcal{V}^{\dagger,r}_{\kappa,w} = \left(\pi_{1,*}\tilde{\mathcal{V}}^{\dagger,r}_{\kappa,w}\right)^{I_n(\mathbb{Z}/p^m\mathbb{Z})}$.

3.4.2 The $\mathcal{Q}_w$-torsor $\mathcal{T}^{\times}_{\mathcal{H},w}(v)$ and contracted product interpretation of $\mathcal{V}^{\dagger,r}_{\kappa,w}$

The definition of the Banach sheaf $\mathcal{V}^{\dagger,r}_{\kappa,w}$ as $\omega^{\dagger}_{\kappa,w}\otimes\mathrm{Sym}^r\,\mathcal{J}$ is already convenient for constructing unramified Hecke operators and $\mathbb{U}_p$-operators. However, for the construction of differential operators and holomorphic projections, it is preferable to have a contracted product interpretation involving a $\mathcal{Q}_w$-torsor and the $\mathcal{Q}_w$-submodule $V^r_{\kappa,w}$ of the $(\mathfrak{g},\mathcal{Q}_w)$-module $V_{\kappa,w}$ defined in §3.2.

The $\mathcal{O}_K$-scheme $T^{\times}_{\mathcal{H}} = \underline{\mathrm{Isom}}_X\left(\mathcal{O}^{2n}_X, \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y)^{\mathrm{can}}\right)$ is defined as in §2.1. Let $T^{\times}_{\mathcal{H},\mathrm{an}}$ be its analytification and $\mathcal{T}^{\times}_{\mathcal{H},\mathrm{an}}(v)$ be the base change to $\mathcal{X}_{\mathrm{Iw}}(v)$. There is a natural projection

$$\mathcal{T}^{\times}_{\mathcal{H},\mathrm{an}}(v) \longrightarrow \mathcal{T}^{\times}_{\omega,\mathrm{an}}(v).$$

We define the $\mathcal{Q}_w$-torsor $\mathcal{T}^{\times}_{\mathcal{H},w}$ over $\mathcal{X}_{\mathrm{Iw}}(v)$ as

$$\mathcal{T}^{\times}_{\mathcal{H},w}(v) := \mathcal{T}^{\times}_{\mathcal{H},\mathrm{an}}(v)\times_{\mathcal{T}^{\times}_{\omega,\mathrm{an}}(v)}\mathcal{T}^{\times}_{\mathcal{F},w}(v).$$

It is not difficult to see that the Banach sheaf $\mathcal{V}^{\dagger,r}_{\kappa,w}$ admits the following contracted product interpretation

$$\mathcal{V}^{\dagger,r}_{\kappa,w} = \mathcal{T}^{\times}_{\mathcal{H},w}(v)\times^{\mathcal{Q}_w} V^r_{\kappa,w}.$$

3.4.3 Summary

We record below several interpretations of the Banach sheaf $\mathcal{V}^{\dagger,r}_{\kappa,w}$ over $\mathcal{X}_{\mathrm{Iw}}(v)$ and its global sections, which we will use later for convenience according to different purposes.

(i) $\mathcal{V}^{\dagger,r}_{\kappa,w} = \omega^{\dagger}_{\kappa,w} \otimes \mathrm{Sym}^r \mathcal{J}$,

(ii) $\mathcal{V}^{\dagger,r}_{\kappa,w} = (\pi_{1,*}\pi_{3,*}\mathcal{L}_\kappa)^{B_n(\mathbb{Z}/p^m\mathbb{Z})} \otimes \mathrm{Sym}^r \mathcal{J}$, and for global sections

$$N^{\dagger,r}_{\kappa,w,v} = H^0\big(\mathcal{IW}_w(v), \mathcal{L}_\kappa \otimes (\pi_1 \circ \pi_3)^* \mathrm{Sym}^r \mathcal{J}\big)^{B_n(\mathbb{Z}/p^m\mathbb{Z})},$$

(iii) $\mathcal{V}^{\dagger,r}_{\kappa,w} = (\pi_{1,*}\tilde{\mathcal{V}}^{\dagger,r}_{\kappa,w})^{I(\mathbb{Z}/p^m\mathbb{Z})}$, and for global sections

$$\begin{aligned} N^{\dagger,r}_{\kappa,w,v} &= H^0(\mathcal{X}_1(p^m)(v), \tilde{\mathcal{V}}^{\dagger,r}_{\kappa,w})^{I(\mathbb{Z}/p^m\mathbb{Z})} \\ &= \big(H^0(\mathfrak{X}_1(p^m)(v), \tilde{\mathfrak{w}}^{\dagger}_{\kappa,w} \otimes \pi_1^* \mathrm{Sym}^r \mathfrak{J})[1/p]\big)^{I(\mathbb{Z}/p^m\mathbb{Z})}, \end{aligned}$$

(iv) $\mathcal{V}^{\dagger,r}_{\kappa,w} = \mathcal{T}^{\times}_{\mathcal{H},w}(v) \times^{\mathcal{Q}_w} V^r_{\kappa,w}$.

It is easy to see that in all the above constructions we can replace κ by the w-analytic universal weight κ^{un} corresponding to $\mathcal{U} \subset \mathcal{W}$, and consider the Banach sheaf $\mathcal{V}^{\dagger,r}_{\kappa^{\mathrm{un}},w}$ over $\mathcal{X}_{\mathrm{Iw}}(v) \times \mathcal{U}$ as well as the $\mathcal{A}(\mathcal{U})$-Banach module $N^{\dagger,r}_{\mathcal{U},w,v} := H^0(\mathcal{X}_{\mathrm{Iw}}(v) \times \mathcal{U}, \mathcal{V}^{\dagger,r}_{\kappa^{\mathrm{un}},w})$.

In the following we need also to consider the Banach sheaf $\mathcal{V}^{\dagger,r}_{\kappa\otimes\sigma,w} := \omega^{\dagger}_{\kappa,w} \otimes \omega_\sigma \otimes \mathrm{Sym}^r \mathcal{J}$ and its global sections $N^{\dagger,r}_{\kappa\otimes\sigma,w,v}$ for some $(\sigma, W_\sigma) \in \mathrm{Rep}_{\mathbb{Z},f}\,\mathrm{GL}(n)$. Here ω_σ is the base change to $\mathcal{X}_{\mathrm{Iw}}(v)$ of the analytification of the automorphic sheaf $\mathcal{E}(W_\sigma)$. From $\mathcal{E}(W_\sigma)$ one also gets the locally free formal sheaf of finite rank $\mathfrak{w}_\sigma$ over $\mathfrak{X}_{\mathrm{Iw}}(v)$ whose rigid fibre is ω_σ. When working with $\mathcal{V}^{\dagger,r}_{\kappa\otimes\sigma,w}$ and $N^{\dagger,r}_{\kappa\otimes\sigma,w,v}$, we can replace $\mathrm{Sym}^r \mathcal{J}, \mathrm{Sym}^r \mathfrak{J}$ in (ii)(iii) by $\omega_\sigma \otimes \mathrm{Sym}^r \mathcal{J}$ and $\mathfrak{w}_\sigma \otimes \mathrm{Sym}^r \mathfrak{J}$ and $V^r_{\kappa,w}$ in (iv) by $V^r_{\kappa\otimes\sigma,w}$.

3.5 The Banach $\mathcal{A}(\mathcal{U})$-module $N^{\dagger,r}_{\mathcal{U},w,v,\mathrm{cusp}}$ is projective

The goal of this section is to prove the proposition below following the arguments in [1, §8].

Proposition 3.5.1. *$N^{\dagger,r}_{\mathcal{U},w,v,\mathrm{cusp}}$ is a projective Banach $\mathcal{A}(\mathcal{U})$-module. For every $\kappa \in \mathcal{U}$ with the corresponding maximal ideal $\mathfrak{m}_\kappa \subset \mathcal{A}(\mathcal{U})$ we have $N^{\dagger,r}_{\mathcal{U},w,v,\mathrm{cusp}} \otimes \mathcal{A}(\mathcal{U})/\mathfrak{m}_\kappa\mathcal{A}(\mathcal{U}) \xrightarrow{\sim} N^{\dagger,r}_{\kappa,w,v,\mathrm{cusp}}$.*

Proof. We use the interpretation (iii) in §3.4.3 and the same proof works if we replace κ^{un} by $\kappa^{\mathrm{un}}\otimes\sigma$ with $\sigma \in \mathrm{Rep}_{\mathbb{Z},f}\,\mathrm{GL}(n)$. Our case differs very little from that in [1, §8]. Instead of repeating the whole proof here, we just point out the main ingredients there and explain that their arguments for the formal Banach sheaf $\tilde{\mathfrak{w}}^{\dagger}_{\kappa^{\mathrm{un}},w}(-D)$ are applicable to $\tilde{\mathfrak{V}}^{\dagger}_{\kappa^{\mathrm{un}},w}(-D) = \tilde{\mathfrak{w}}^{\dagger}_{\kappa^{\mathrm{un}},w} \otimes \pi_1^* \mathrm{Sym}^r \mathfrak{J}(-D)$. Below for simplicity we write $\pi_1^* \mathrm{Sym}^r \mathfrak{J}$ as $\mathrm{Sym}^r \mathfrak{J}$.

We use the notation in [1, §8.2]. Let $X^\star$ be the minimal compactification of Y. There is a proper morphism $X \to X^\star$. Like $\mathfrak{X}(v)$ one can define $\mathfrak{X}^\star(v)$ to be the p-adic completion of the normalization of the largest open formal subscheme of the blow-up of $\mathfrak{X}^\star$ along the ideal (Ha, p^v) where it is generated by Ha. We have the projection $\eta : \mathfrak{X}_1(p^m)(v) \to \mathfrak{X}^\star(v)$. We may assume that $\mathcal{U}$ lies inside the identity component $\mathcal{W}^\circ$ and take $\mathfrak{U}$ to be the open formal subscheme of an admissible blow-up of $\mathfrak{W}^\circ$ whose rigid fibre is $\mathcal{U}$. We use the subscript l to mean reduction modulo ϖ^l. [1, Corollary 8.1.6.2] shows that $\tilde{\mathfrak{V}}^{\dagger,r}_{\kappa^{\mathrm{un}},w}$ is a small formal Banach sheaf over $\mathfrak{X}_1(p^m)(v)$ with $\mathrm{Sym}^r \mathfrak{J}_1$ as the required coherent sheaf in the definition of small formal Banach sheaves (cf. [1, Definition A.1.2.1]).

First we claim that the proposition follows from the following base change property for $\tilde{\mathfrak{V}}^{\dagger}_{\kappa^{\mathrm{un}},w}(-C)$. For all $l \in \mathbb{N}$, considering the the diagram

$$\begin{array}{ccc} X_1(p^m)(v)_l \times \mathfrak{U}_l & \xrightarrow{i} & X_1(p^m)(v)_{l+1} \times \mathfrak{U}_{l+1} \\ \downarrow{\scriptstyle \eta_l \times 1} & & \downarrow{\scriptstyle \eta_{l+1} \times 1} \\ X^\star(v)_l \times \mathfrak{U}_l & \xrightarrow{i'} & X^\star(v)_{l+1} \times \mathfrak{U}_{l+1} \end{array},$$

the base change property for $\tilde{\mathfrak{V}}^{\dagger}_{\kappa^{\mathrm{un}},w}(-C)$ is

$$i'^*(\eta_{l+1} \times 1)_* \tilde{\mathfrak{V}}^{\dagger}_{\kappa^{\mathrm{un}},w,l+1}(-C) = (\eta_l \times 1)_* \tilde{\mathfrak{V}}^{\dagger}_{\kappa^{\mathrm{un}},w,l}(-C). \tag{3.7}$$

Once this base change property is proved, we deduce that $(\eta \times 1)_* \tilde{\mathfrak{V}}^{\dagger}_{\kappa^{\mathrm{un}},w}(-C)$ is a small formal Banach sheaf with $(\eta \times 1)_* \mathrm{Sym}^r \mathfrak{J}_1$ as the required coherent sheaf. Then applying [1, Theorem A.1.2.2] and the arguments in [1, Corollary 8.2.3.1, 8.2.3.2], we conclude that the module $H^0\big(\mathcal{X}_1(p^m)(v)\times\mathcal{U},\tilde{\mathcal{V}}^{\dagger}_{\kappa^{\mathrm{un}},w}(-C)\big)$ is a projective Banach $\mathcal{A}(\mathcal{U})$-module and the map

$$H^0\big(\mathcal{X}_1(p^m)(v)\times\mathcal{U},\tilde{\mathcal{V}}^{\dagger}_{\kappa^{\mathrm{un}},w}(-C)\big)\otimes\mathcal{A}(\mathcal{U})/\mathfrak{m}_\kappa\mathcal{A}(\mathcal{U})\longrightarrow H^0\big(\mathcal{X}_1(p^m)(v),\tilde{\mathcal{V}}^{\dagger}_{\kappa,w}(-C)\big)$$

is an isomorphism. The statement of the proposition follows by taking the invariant of the finite group $I_n(\mathbb{Z}/p^m\mathbb{Z})$.

We are left to show the base change property (3.7). Let $V'\subset V=\mathbb{Z}^{\oplus 2n}$ be an isotropic direct factor of rank r' and $Y_{V'}$ be the V'-stratum of $X^\star$ with universal abelian scheme $A_{V'}\to Y_{V'}$. We start by recalling the description of the localization of the projection from the toroidal compactification to the minimal compactification at a point belonging to the stratum $Y_{V'}$ of $X^\star$ given in [1, §8.2]. There are the abelian scheme $\mathcal{B}_{V'}\to Y_{V'}$ parametrizing the extensions of $A_{V'}$ by $V'\otimes\mathbb{G}_m$ and an isogeny $\mathcal{B}_{V'}\to A_{V'}^{r'}$ of degree a power of N. Over $\mathcal{B}_{V'}$ lies $\mathcal{M}_{V'}$ which is a torsor under the torus with character group $S^\vee_{V'}$, isogeneous to $\mathrm{Hom}(\mathrm{Sym}^2 V/V'^\perp,\mathbb{G}_m)$. Set $\mathcal{M}_{V'}\to\mathcal{M}_{V',\mathcal{S}}$ to be torus embedding associated to a polyhedral decomposition $\mathcal{S}$ of the cone $C(V/V'^\perp)$ of symmetric semi-definite bilinear forms on $V/V'^\perp$. In the same manner as in §3.3 one defines $\mathfrak{Y}_{V'}(v),\mathfrak{Y}_1(p^m)_{V'}(v),\mathfrak{B}_{V'}(v),\mathfrak{M}_{V',\mathcal{S}}(v)$. Put $\mathfrak{B}_1(p^m)_{V'}(v)=\mathfrak{B}_{V'}(v)\times_{\mathfrak{A}^r_{V'}}(\mathfrak{A}_{V'}/H_{m,V'})^r$ and $\mathfrak{M}_1(p^m)_{V',\mathcal{S}}(v)=\mathfrak{M}_{V',\mathcal{S}}(v)\times_{\mathfrak{B}_{V'}(v)}\mathfrak{B}_1(p^m)_{V'}(v)$. Take a geometric point $\bar{x}\in X^\star(v)_l$ and consider the projection $X_1(p^m)(v)_l\to X^\star(v)_l$ localized at $\bar{x}$. The completion $\widehat{X_1(p^m)(v)}_{l,\bar{x}}$ is isomorphic to a disjoint union of spaces $\mathcal{M}_1\widehat{(p^m)_{V',\mathcal{S}}(v)}_{l,\bar{y}}/\Gamma_1(p^m)_{V'}$ with some geometric point $\bar{y}\in Y_1(p^m)_{V'}(v)_l$. The spaces fit into the diagram

$$\begin{array}{ccccc}\mathcal{M}_1\widehat{(p^m)_{V',\mathcal{S}}(v)}_{l,\bar{y}} & \xrightarrow{h_2} & \mathcal{M}_1\widehat{(p^m)_{V',\mathcal{S}}(v)}_{l,\bar{y}}/\Gamma_1(p^m)_{V'} & \longrightarrow & \widehat{X_1(p^m)(v)}_{l,\bar{x}}\ . \\ \downarrow{\scriptstyle h_1} & & \downarrow & & \\ \mathcal{B}_1\widehat{(p^m)_{V'}(v)}_{l,\bar{y}} & \longrightarrow & Y_1\widehat{(p^m)_{V'}(v)}_{l,\bar{y}} & & \end{array} \tag{3.8}$$

Because of the exact sequence

$$0\to\tilde{\mathfrak{w}}^{\dagger}_{\kappa,w,1}\otimes\mathrm{Sym}^r\mathfrak{J}_1(-C)\overset{\varpi^{l-1}}{\to}\tilde{\mathfrak{w}}^{\dagger}_{\kappa,w,l}\otimes\mathrm{Sym}^r\mathfrak{J}_l(-C)\to\tilde{\mathfrak{w}}^{\dagger}_{\kappa,w,l-1}\otimes\mathrm{Sym}^r\mathfrak{J}_{l-1}(-C)\to 0,$$

the base change property for $\tilde{\mathfrak{V}}^{\dagger}_{\kappa^{\mathrm{un}},w}(-C)$ will follow from the vanishing result

$$H^1\big(\mathcal{M}_1(\widehat{p^m)_{V'}},\mathcal{S}(v)_{1,\bar{y}}/\Gamma_1(p^m)_{V'}, \tilde{\mathfrak{w}}^{\dagger}_{\kappa,w,1}\otimes \mathrm{Sym}^r\,\mathfrak{J}_1(-C)\big)=0 \tag{3.9}$$

for all $\kappa\in\mathcal{U}$. The coherent $\mathrm{Sym}^r\,\mathfrak{J}$ has a filtration with graded pieces being automorphic sheaves attached to algebraic $\mathrm{GL}(n)$-representations that are free of finite rank, and the sheaf $\tilde{\mathfrak{w}}^{\dagger}_{\kappa,w,1}$ is an inductive limit of iterated extensions of the trivial sheaf [1, Corollary 8.1.6.2]. Therefore (3.9) is a corollary of the general vanishing result: for all $\sigma\in\mathrm{Rep}_{\mathbb{Z},f}\,\mathrm{GL}(n)$ and $i>0$,

$$H^i\big(\mathcal{M}_1(\widehat{p^m)_{V'}},\mathcal{S}(v)_{1,\bar{y}}/\Gamma_1(p^m)_{V'}, \mathfrak{w}_{\sigma,1}(-C)\big)=0 \tag{3.10}$$

where $\mathfrak{w}_{\sigma,1}$ is the pullback to $\mathcal{M}_1(\widehat{p^m)_{V'}},\mathcal{S}(v)_{1,\bar{y}}/\Gamma_1(p^m)_{V'}$ of the automorphic sheaf ω_σ on X. The proof of (3.10) is an adaption of [45, §8.2] in the situation (3.8). It is enough to show

$$H^i\Big(\Gamma_1(p^m)_{V'}, H^j\big(\mathcal{M}_1(\widehat{p^m)_{V'}},\mathcal{S}(v)_{1,\bar{y}}, h_2^*\mathfrak{w}_{\sigma,1}(-C)\big)\Big)=0 \quad \text{if} \quad i+j>0.$$

Over $\mathcal{B}_{V'}$ there is the universal semi-abelian scheme

$$0\longrightarrow V'\otimes\mathbb{G}_m\longrightarrow G_{V'}\longrightarrow A_{V'}\longrightarrow 0,$$

so using the $\mathrm{GL}(n)$-torsor $\underline{\mathrm{Isom}}_{\mathcal{B}_{V'}}\big(\mathcal{O}^n_{\mathcal{B}_{V'}},\omega(G_{V'}/\mathcal{B}_{V'})\big)$ one constructs a locally free sheaf of finite rank $\underline{\omega}_\sigma$ over $\mathcal{B}_{V'}$. Its pullback $\underline{\mathfrak{w}}_{\sigma,1}$ to $\mathcal{B}_1(\widehat{p^m)_{V'}}(v)_{l,\bar{y}}$ satisfies

$$h_1^*\underline{\mathfrak{w}}_{\sigma,1}=h_2^*\mathfrak{w}_{\sigma,1}.$$

The action of $\Gamma_1(p^m)_{V'}$ on $S_{V'}$ factors through a quotient $\Gamma_1'(p^m)_{V'}$ whose action on the set $\big\{\lambda\in S_{V'}\cap C(V/V'^{\perp})^\vee:\lambda>0\big\}$ is free. Take S_0 to be a set of representatives of the orbits. Applying [45, Lemma 8.2.3.12], [21, Theorem V.2.7] we get

$$\begin{aligned}
&H^i\Big(\Gamma_1(p^m)_{V'}, H^j\big(\mathcal{M}_1\widehat{(p^m)_{V'}},_{\mathcal{S}}(v)_{1,\bar{y}}, h_2^*\mathfrak{w}_{\sigma,1}(-C)\big)\Big)\\
&= H^i\Big(\Gamma_1(p^m)_{V'}, \prod_{\lambda\in S_{V'}\cap C(V/V'^{\perp})^\vee,\lambda>0} H^j\big(\mathcal{B}_1\widehat{(p^m)_{V'}}(v)_{l,\bar{y}}, \mathcal{L}(\lambda)\otimes\underline{\mathfrak{w}}_{\sigma,1}\big)\Big)\\
&= \begin{cases} \prod_{\lambda\in S_0} H^j\big(\mathcal{B}_1\widehat{(p^m)_{V'}}(v)_{l,\bar{y}}, \mathcal{L}(\lambda)\otimes\underline{\mathfrak{w}}_{\sigma,1}\big) & i=0\\ 0 & i>0\end{cases}.
\end{aligned}$$

Here $\mathcal{L}(\lambda)$ is an ample invertible sheaf over the abelian scheme $\mathfrak{B}_1(p^m)_{V'}(v)$ for $\lambda\in S_0$ [21, p. 143]. We reduce to show

$$H^j\big(\mathcal{B}_1\widehat{(p^m)_{V'}}(v)_{l,\bar{y}}, \mathcal{L}(\lambda)\otimes\underline{\mathfrak{w}}_{\sigma,1}\big)=0 \quad\text{if}\quad j>0. \tag{3.11}$$

One observation is that, over $\mathcal{B}_1\widehat{(p^m)_{V'}}(v)_{l,\bar{y}}$, the sheaf of invariant differentials of the torus part and the quotient abelian part of the semi-abelian scheme $G_{V'}$ can be trivialized. Hence the sheaf $\underline{\omega}_\sigma$ can be constructed using a torsor of a unipotent subgroup $N_{V'}\subset \mathrm{GL}(n)$ with the $N_{V'}$-representation $\sigma|_{N_{V'}}$. Then [45, Lemma 8.2.4.16] says that $\sigma|_{N_{V'}}$ admits a filtration with $N_{V'}$ acting trivially on each graded piece. Thus $\underline{\omega}_\sigma$ is an iterated extension of the trivial sheaf, and (3.11) follows from the vanishing results for $H^j\big(\mathcal{B}_1\widehat{(p^m)_{V'}}(v)_{l,\bar{y}}, \mathcal{L}(\lambda)\big)$, $j>0$ [49, §III.16]. $\square$

3.6 The differential operators

Let $\Omega^1_{\mathcal{X}_{\mathrm{Iw}}(v)}$ be the sheaf of differentials on $\mathcal{X}_{\mathrm{Iw}}(v)$ defined as in [23, Ex. 4.4.1]. Over $\mathcal{X}_{\mathrm{Iw}}(v)$ we have the integrable Gauss–Manin connection

$$\nabla : \mathcal{H}^1_{\mathrm{dR}}(\mathcal{G}/\mathcal{X}_{\mathrm{Iw}}(v))^{\mathrm{can}} \to \mathcal{H}^1_{\mathrm{dR}}(\mathcal{G}/\mathcal{X}_{\mathrm{Iw}}(v))^{\mathrm{can}}\otimes\Omega^1_{\mathcal{X}_{\mathrm{Iw}}(v)}(\log C).$$

For a w-analytic weight $\kappa\in\mathcal{W}(\mathbb{C}_p)$ and $\sigma\in\mathrm{Rep}_{\mathbb{Z},f}\,\mathrm{GL}(n)$, we defined in §3.2 the $(\mathfrak{g},\mathcal{Q}_w)$-module $V_{\kappa\otimes\sigma,w}$. The Banach sheaf $\mathcal{V}^{\dagger,r}_{\kappa\otimes\sigma,w}=\omega^\dagger_{\kappa,w}\otimes\omega_\sigma\otimes\mathrm{Sym}^r\mathcal{J}$ on $\mathcal{X}_{\mathrm{Iw}}(v)$ has the contracted product interpretation $\mathcal{T}^\times_{\mathcal{H},w}(v)\times^{\mathcal{Q}_w}V_{\kappa\otimes\sigma,w}$. Using this contracted product interpretation and the construction in §2.2, we obtain a connection

$$\nabla_{\kappa\otimes\sigma,w} : \mathcal{V}^{\dagger,r}_{\kappa\otimes\sigma,w} \longrightarrow \mathcal{V}^{\dagger,r+1}_{\kappa\otimes\sigma,w} \otimes \Omega_{\mathcal{X}_{\mathrm{Iw}(v)}}(\log C) \cong \mathcal{V}^{\dagger,r+1}_{\kappa\otimes\sigma\otimes\tau,w}(-1).$$

Recall that τ is the symmetric square of the standard representation of $\mathrm{GL}(n)$. Composing it with $t^+ : \mathcal{V}^{\dagger,r+1}_{\kappa\otimes\sigma\otimes\tau,w}(-1) \to \mathcal{V}^{r+1}_{\kappa\otimes\sigma\otimes\tau,w}$ we get the following differential operator which can be thought of as an p-adic analytic version of the Maass–Shimura differential operators

$$D_{\kappa\otimes\sigma,w} : \mathcal{V}^{\dagger,r}_{\kappa\otimes\sigma,w} \longrightarrow \mathcal{V}^{\dagger,r+1}_{\kappa\otimes\sigma\otimes\tau,w}.$$

Besides, there is the Shimura's E-operator [58, §12.9], whose construction relies only on the fact that we have the morphism of $\mathcal{Q}_w$-representations

$$V^{r}_{\kappa\otimes\sigma,w}/V^{0}_{\kappa\otimes\sigma,w} \longrightarrow V^{r-1}_{\kappa\otimes\sigma,w} \otimes V^{0}_{\tau^\vee}(1) = V^{r-1}_{\kappa\otimes\sigma\otimes\tau^\vee,w}(1).$$

To be explicit, let $\underline{Z} = (Z_{ij})_{1\leq i,j\leq n}$ be the basis of $\tau^\vee$ with $a \in \mathrm{GL}(n)$ acting on $\underline{Z}$ by $a^{-1}\,\underline{Z}{}^{\mathrm{t}}a^{-1}$. Then the morphism is given by $\sum\limits_{1\leq i\leq j\leq n} Z_{ij}\frac{\partial}{\partial Y_{ij}}$. The r-th iteration divided by $r!$ is an isomorphism

$$\frac{1}{r!}\Big(\sum_{1\leq i\leq j\leq n} Z_{ij}\frac{\partial}{\partial Y_{ij}}\Big)^r : V^{r}_{\kappa\otimes\sigma,w}/V^{r-1}_{\kappa\otimes\sigma,w} \xrightarrow{\sim} V^{0}_{\kappa\otimes\sigma\otimes\mathrm{Sym}^r\,\tau^\vee,w}(r). \tag{3.12}$$

We write the induced operator on the Banach sheaves as

$$\varepsilon_{\kappa\otimes\sigma,w} : \mathcal{V}^{\dagger,r}_{\kappa\otimes\sigma,w} \longrightarrow \mathcal{V}^{\dagger,r-1}_{\kappa\otimes\sigma\otimes\tau^\vee,w}(1),$$

and its composition with $t^- : \mathcal{V}^{\dagger,r-1}_{\kappa\otimes\sigma\otimes\tau^\vee,w}(1) \to \mathcal{V}^{\dagger,r-1}_{\kappa\otimes\sigma\otimes\tau^\vee,w}$ as

$$E_{\kappa\otimes\sigma,w} : \mathcal{V}^{\dagger,r}_{\kappa\otimes\sigma,w} \longrightarrow \mathcal{V}^{\dagger,r-1}_{\kappa\otimes\sigma\otimes\tau^\vee,w}(1) \xrightarrow{t^-} \mathcal{V}^{\dagger,r-1}_{\kappa\otimes\sigma\otimes\tau^\vee,w}. \tag{3.13}$$

We can also iterate the operators and obtain

$$D^e_{\kappa\otimes\sigma,w} : \mathcal{V}^{\dagger,r}_{\sigma,w} \longrightarrow \mathcal{V}^{\dagger,r+e}_{\kappa\otimes\sigma\otimes\mathrm{Sym}^e\,\tau,w},$$
$$E^e_{\kappa\otimes\sigma,w} : \mathcal{V}^{\dagger,r}_{\sigma,w} \longrightarrow \mathcal{V}^{\dagger,r-e}_{\kappa\otimes\sigma\otimes\mathrm{Sym}^e\,\tau^\vee,w},$$

for $e \in \mathbb{N}$. A section of the sheaf $\mathcal{V}^{\dagger,r}_{\kappa\otimes\sigma,w}$ lies inside $\mathcal{V}^{\dagger,r'}_{\kappa\otimes\sigma,w}$ for $0 \leq r' < r$ if and only is it is annihilated by $E^{r'+1}_{\kappa\otimes\sigma,w}$.

3.7 The holomorphic projection

Besides the definition of the space of nearly holomorphic forms, its algebraic structure and the Maass–Shimura differential operators, another main ingredient in Shimura's theory of nearly holomorphic forms is the holomorphic projection. Shimura's construction [58, Proposition 14.2] can be adapted to our p-adic analytic context.

Define the functions $\mathrm{Log}_1, \cdots, \mathrm{Log}_n$ on the weight space $\mathcal{W}$ by

$$\mathrm{Log}_i(\kappa) := \frac{\log_p(\kappa_i(1+p)^t)}{\log_p((1+p)^t)},$$

for $\kappa = (\kappa_1, \cdots, \kappa_n) \in \mathcal{W}$ and some $t \in \mathbb{N}$ sufficiently large. Let $K(\mathrm{Log}_1, \cdots, \mathrm{Log}_n)$ be the the fraction field of $K[\mathrm{Log}_1, \cdots, \mathrm{Log}_n]$. For an affinoid subdomain $\mathcal{U} \subset \mathcal{W}$ such that $\kappa^{\mathrm{un}}|_{\mathcal{U}}$ is w-analytic, we prove in this section the following proposition.

Proposition 3.7.1. *There is an $\mathcal{A}(\mathcal{U})$-linear continuous map*

$$\mathscr{A} : N^{\dagger,r}_{\mathcal{U},w,v} \longrightarrow N^{\dagger,0}_{\mathcal{U},w,v} \otimes_K K(\mathrm{Log}_1, \cdots, \mathrm{Log}_n)$$

whose restriction to $N^{\dagger,0}_{\mathcal{U},w,v}$ is the identity.

In order to simplify notation for the rest of this section we omit all the subscripts from the differential operators and E-operators as well as the subscript w from $\mathcal{V}^{\dagger,r}_{\kappa^{\mathrm{un}}\otimes\mathrm{Sym}^e\,\tau\otimes\mathrm{Sym}^{e'}\,\tau^\vee,w}$.

Suppose $\mathrm{Spm}(R) \subset \mathcal{X}_{\mathrm{Iw}}(v)$ is an affinoid subdomain such that there exists a section $\alpha \in$

$\mathcal{T}^\times_{\mathcal{H},w}(v)(R)$, we regard α as a basis $(\alpha_1,\cdots,\alpha_{2n})$ of $\mathcal{H}^1_{\mathrm{dR}}(A/R)$ satisfying certain conditions. Given $D\in\mathrm{Der}_K(R,R)$ in order to decide the action of $\nabla(D)$ on sections of $\mathcal{V}^{\dagger,r}_{\kappa^{\mathrm{un}}\otimes\sigma}$ over $\mathrm{Spm}(R)$, we need to consider the element $X(D,\alpha)\in\mathfrak{g}\otimes R$ defined by

$$\nabla(D)\alpha=\alpha\cdot X(D,\alpha).$$

Let $\overline{X(D,\alpha)}$ denote the image of $X(D,\alpha)$ inside the quotient $\mathfrak{g}/\mathfrak{q}_{\mathbf{G}}\cong\mathfrak{u}^-_{\mathbf{G}}$. The Levi subgroup $M_{\mathbf{G}}$ acts on $\mathfrak{u}^-_{\mathbf{G}}$ by conjugation. Hence $a\in\mathrm{GL}(n,R)$ acts on $\overline{X(D,\alpha)}$ by sending it to ${}^{\mathrm{t}}a^{-1}\overline{X(D,\alpha)}a^{-1}$. This $\mathrm{GL}(n)$-action is isomorphic to $\tau^\vee$. Given $\alpha\in\mathcal{T}^\times_{\mathcal{H},w}(v)(R)$ and a basis $\{e_i\}_{1\le i\le n(n+1)/2}$ of the $\mathrm{GL}(n)$-representation τ, the dual basis $\{e_i^\vee\}_{1\le i\le n(n+1)/2}$ gives rise to a basis $\left\{D_{e_i^\vee,\alpha}\right\}_{1\le i\le n(n+1)/2}$ of the tangent space $\mathrm{Der}_K(R,R)$. One can check by definition that the element $\overline{X(D_{e_i^\vee,\alpha},\alpha)}$ inside $\mathfrak{u}^-_{\mathbf{G}}\otimes R$ is independent of the choice of $\alpha\in\mathcal{T}^\times_{\mathcal{H},w}(v)(R)$, and we abbreviate it as $\overline{X(e_i^\vee)}$.

Lemma 3.7.2. $\left\{\overline{X(D_{e_i^\vee})}\right\}_{1\le i\le n(n+1)/2}$ *form a basis of* $\mathfrak{u}^-_{\mathbf{G}}\otimes R\cong\tau^\vee(R)$*, which is dual to the basis* $\{e_i\}_{1\le i\le n(n+1)/2}$.

Proof. The statement is an equality statement and does not depend on the choice of $\{e_i\}_{1\le i\le n(n+1)/2}$. Hence it suffices to prove it for the Siegel variety Y, and we can further reduce to the Siegel upper half space $\mathbb{H}_n$ and take α to be the holomorphic basis (dw,β) of $\mathcal{H}^1_{\mathrm{dR}}(A_{\mathbb{H}_n}/\mathbb{H}_n)$ constructed in §2.5. Denote by KS the Kodaira–Spencer map. Explicit computation using (2.14) shows that

$$\mathrm{KS}(dw_i dw_j)=2\pi i\cdot dz_{ij}\quad 1\le i\le j\le n. \tag{3.14}$$

Put $\underline{X}=(X_{ij})$ as in §2.7. Then $(X_{ij})_{1\le i\le j\le n}$ can be regarded as a basis spanning the representation τ. It is dual to the basis μ^-_{ij} of $\mathfrak{u}^-_{\mathbf{G}}$. (3.14) shows that dz_{ij} corresponds to X_{ij} under the basis (dw,β) so $\partial/\partial z_{ij}=D_{X^\vee_{ij},(dw,\beta)}$. By (2.14) we have $\overline{X(X^\vee_{ij})}=\mu^-_{ij}$ and the statement is proved. □

The morphism $\tau\otimes\tau^\vee\to\mathrm{triv}$ of $\mathrm{GL}(n)$-representations induces the contraction operator

$$\theta^e:\mathcal{V}^{\dagger,r}_{\kappa^{\mathrm{un}}\otimes\mathrm{Sym}^e\tau\otimes\mathrm{Sym}^e\tau^\vee}\longrightarrow\mathcal{V}^{\dagger,r}_{\kappa^{\mathrm{un}}}.$$

Lemma 3.7.3. *The composition*

$$E^e\theta^e D^e : \mathcal{V}^{\dagger,0}_{\kappa^{\mathrm{un}}\otimes \mathrm{Sym}^e\,\tau^\vee} \xrightarrow{D^e} \mathcal{V}^{\dagger,e}_{\kappa^{\mathrm{un}}\otimes \mathrm{Sym}^e\,\tau^\vee\otimes \mathrm{Sym}^e\,\tau} \xrightarrow{\theta^e} \mathcal{V}^{\dagger,e}_{\kappa^{\mathrm{un}}} \xrightarrow{E^e} \mathcal{V}^{\dagger,0}_{\kappa^{\mathrm{un}}\otimes \mathrm{Sym}^e\,\tau^\vee}$$

is an $\mathcal{O}_{\mathcal{X}_{\mathrm{Iw}(v)}\times\mathcal{U}}$-linear morphism of Banach sheaves over $\mathcal{X}_{\mathrm{Iw}(v)}\times\mathcal{U}$, induced by an endomorphism of the $\mathcal{Q}_w$-representation $V^0_{\kappa^{\mathrm{un}}\otimes\mathrm{Sym}^e\,\tau^\vee}$.

Proof. There exists a contraction map $\tilde{\theta}^e : \mathcal{V}^{\dagger,0}_{\kappa^{\mathrm{un}}\otimes \mathrm{Sym}^e\,\tau^\vee\otimes \mathrm{Sym}^e\,\tau\otimes\mathrm{Sym}^e\,\tau^\vee} \to \mathcal{V}^{\dagger,0}_{\kappa^{\mathrm{un}}\otimes \mathrm{Sym}^e\,\tau^\vee}$ induced from a morphism of the corresponding representations such that $E^e\theta^e D^e = \tilde{\theta}^e E^e D^e$. Therefore it is enough to show that the map $E^eD^e : \mathcal{V}^{\dagger,0}_{\kappa^{\mathrm{un}}\otimes \mathrm{Sym}^e\,\tau^\vee} \longrightarrow \mathcal{V}^{\dagger,0}_{\kappa^{\mathrm{un}}\otimes \mathrm{Sym}^e\,\tau^\vee\otimes \mathrm{Sym}^e\,\tau\otimes\mathrm{Sym}^e\,\tau^\vee}$ is induced from a morphism of $\mathcal{I}_w$-representations. Still take $\underline{X} = (X_{ij})$ as a basis of τ and write $V_{\kappa^{\mathrm{un}},w} = W_{\kappa^{\mathrm{un}},w}[\underline{Y}]$ with $\underline{Y} = (Y_{ij})_{1\le i\le j\le n}$ as in §3.2. Locally over $\mathrm{Spm}(R)\subset\mathcal{X}_{\mathrm{Iw}}(v)$, we fix a section $\alpha\in\mathcal{T}^\times_{\mathcal{H},w}(v)(R)$ and let $D_{X^\vee_{ij},\alpha}$ be the basis of $\mathrm{Der}_K(R,R)$ associated to $X^\vee_{ij}$ and α. With these choices of local coordinates the map E^eD^e can be written as

$$E^eD^e : \mathcal{T}^\times_{\mathcal{H},w}(v)(R)\times^{\mathcal{Q}_w(R)} V^0_{\kappa^{\mathrm{un}}\otimes \mathrm{Sym}^e\,\tau^\vee}(R) \longrightarrow \mathcal{T}^\times_{\mathcal{H},w}(v)(R)\times^{\mathcal{Q}_w(R)} V^0_{\kappa^{\mathrm{un}}\otimes \mathrm{Sym}^e\,\tau^\vee\otimes \mathrm{Sym}^e\,\tau\otimes\mathrm{Sym}^e\,\tau^\vee}(R)$$
$$(\alpha,u)\longmapsto(\alpha, P_{\alpha,u,e}(\underline{X},\underline{Y})),$$

with $P_{\alpha,u,e}(\underline{X},\underline{Y})$ being a homogenous polynomial of degree e in $\underline{X}$ and degree e in $\underline{Y}$ whose coefficients lie in $V^0_{\kappa^{\mathrm{un}}\otimes\mathrm{Sym}^e\,\tau^\vee}(R)$. The claim that E^eD^e is induced from a morphism of $\mathcal{I}_w$-representations is equivalent to the equality

$$a\cdot\big(P_{\alpha,u,e}(\underline{X},\underline{Y})\big) = P_{\alpha\cdot a,u,e}(\underline{X},\underline{Y}), \tag{3.15}$$

for all $a\in\mathcal{I}_w(R)$ and $u\in V^0_{\kappa^{\mathrm{un}}\otimes\mathrm{Sym}^e\,\tau^\vee}(R)$. Note that by (2.2) the operator E^e annihilates all terms in $D^e\left((\alpha,u)\right)$ involving derivations of the base ring R or the action of $\mathfrak{q}_{\mathbf{G}}\subset\mathfrak{g}$, because they do not increase the degree in $\underline{Y}$. we get

$$P_{\alpha,u,e}(\underline{X},\underline{Y}) = \sum_{1\le i\le j\le n}\Big(\overline{X(X^\vee_{ij})}\cdot P_{\alpha,u,e-1}(\underline{X},\underline{Y})\Big)X_{ij},$$

where $\overline{X(X_{ij}^\vee)}$ is regarded as an element of $\mathfrak{u}^-$ through $\mathfrak{u}^- \cong \mathfrak{g}/\mathfrak{q}$. We show (3.15) by induction. The $e = 0$ case is true by definition of the contracted product. Assuming it is true for $e-1$, then

$$\begin{aligned} a \cdot P_{\alpha,u,e}(\underline{X}, \underline{Y}) &= a \cdot \sum_{1\le i\le j\le n} \left(\overline{X(X_{ij}^\vee)} \cdot P_{\alpha,u,e-1}(\underline{X}, \underline{Y})\right) X_{ij} \\ &= \sum_{1\le i\le j\le n} \left(\left({}^{\mathrm{t}}a^{-1}\overline{X(X_{ij}^\vee)}a^{-1}\right) \cdot \left(a \cdot P_{\alpha,u,e-1}(\underline{X}, \underline{Y})\right)\right) \left(a \cdot X_{ij}\right) \\ &= \sum_{1\le i\le j\le n} \left(\overline{X(X_{ij}^\vee)} \cdot \left(a \cdot P_{\alpha,u,e-1}(\underline{X}, \underline{Y})\right)\right) X_{ij} \\ &= \sum_{1\le i\le j\le n} \left(\overline{X(X_{ij}^\vee)} \cdot P_{\alpha\cdot a,u,e-1}(\underline{X}, \underline{Y})\right) X_{ij} \\ &= P_{\alpha\cdot a,u,e}(\underline{X}, \underline{Y}). \end{aligned}$$

The second equality uses the compatibility of the action of $\mathfrak{g}$ and $\mathcal{I}_w$, and the third equality follows from Lemma 3.7.2. ☐

Denote by $\varphi(\kappa^{\mathrm{un}}, e)$ the endomorphism of $W_{\kappa^{\mathrm{un}}\otimes \mathrm{Sym}^e \tau^\vee} = V^0_{\kappa^{\mathrm{un}}\otimes \mathrm{Sym}^e \tau^\vee}$ giving rise to $E^e\theta^e D^e$.

Lemma 3.7.4. *There exists an element $\widetilde{\varphi} \in \mathrm{End}(W_{\kappa^{\mathrm{un}}\otimes \mathrm{Sym}^e \tau^\vee, w})$ and a nonzero $\eta \in K[\mathrm{Log}_1, \cdots, \mathrm{Log}_n]$ such that $\widetilde{\varphi} \circ \varphi(\kappa^{\mathrm{un}}, e) = \varphi(\kappa^{\mathrm{un}}, e) \circ \widetilde{\varphi} = \eta$.*

Proof. As an $\mathcal{A}(\mathcal{U})$-Banach module, we have $W_{\kappa^{\mathrm{un}},w} \cong \oplus_{N^-(\mathbb{Z}/p^{[w]}\mathbb{Z})}\mathcal{A}(\mathcal{U})\langle \underline{T}\rangle$, the direct sum of $|N^-(\mathbb{Z}/p^{[w]}\mathbb{Z})|$ copies of strictly convergent power series in $\underline{T}$ with $\underline{T} = (T_{ij})_{1\le i<j\le n}$. Let $W^0 = \mathcal{A}(\mathcal{U})[\underline{T}]$ be the polynomial part of one copy. Fix a basis $\underline{Z} = (Z_{ij})_{1\le i,j\le n}$, $Z_{ij} = Z_{ji}$ of $\tau^\vee$ with $a \in \mathrm{GL}(n)$ acting by $a \cdot \underline{Z} = {}^{\mathrm{t}}a^{-1}\,\underline{Z}a^{-1}$. Then $W_{\kappa^{\mathrm{un}}\otimes \mathrm{Sym}^e \tau^\vee, w} \cong \oplus_{N^-(\mathbb{Z}/p^{[w]}\mathbb{Z})}\mathcal{A}(\mathcal{U})[\underline{Z}]_e\langle \underline{T}\rangle$ where the subscript e means homogenous polynomials of degree e. Like W^0, set $W^0_e = W^0 \otimes \mathrm{Sym}^e \tau^\vee = \mathcal{A}(\mathcal{U})[\underline{Z}]_e[\underline{T}]$. Both W^0 and W^0_e are closed under the action of $\mathfrak{gl}(n)$. Only Lie algebra action is involved in the differential operators, so $\varphi(\kappa^{\mathrm{un}}, e)$ restricts to an endomorphism of the $\mathfrak{gl}(n)$-module W^0_e. We can write W^0_e as a direct sum of its weight spaces $W^0_e = \oplus_\lambda W^0_{e,\lambda}$ and each $W^0_{e,\lambda}$ is free of finite rank generated by some monomials of the form $\prod_{1\le i<j\le n} T_{ij}^{s_{ij}} \cdot \prod_{1\le k\le l\le n} Z_{kl}^{t_{kl}}$, $s_{ij}, t_{kl} \ge 0$, $\sum t_{kl} = e$. The endomorphism $\varphi(\kappa^{\mathrm{un}}, e)$ restricts to an $\mathcal{A}(\mathcal{U})$-linear map $\varphi_\lambda : W^0_{e,\lambda} \to W^0_{e,\lambda}$ for each λ and the corresponding matrix, with respect to the basis consisting of monomials, has entries in

$\mathcal{O}_K[\mathrm{Log}_1, \cdots, \mathrm{Log}_n]$. The first claim is that the determinant of φ_λ in non-zero. For $\kappa \in \mathcal{U}$ write $\varphi_{\lambda,\kappa}$ to denote the specialization of φ_λ at κ. Fix an arbitrary $\kappa = (\kappa_1, \cdots, \kappa_n) \in \mathcal{U}$ and consider $\kappa + k = (\kappa_1 + k, \cdots, \kappa_n + k)$ with k varying in $\mathbb{N}$. Set $Q(k)$ to be the determinant of $\varphi_{\lambda,\kappa+k}$. It is a polynomial in k and is non-zero as observed in [58, (14.3)]. Hence the determinant of φ_λ cannot be zero. Then in order to show the existence of the $\tilde{\varphi}$, it suffices to show that there exists $\eta \in \mathcal{O}_K[\mathrm{Log}_1, \cdots, \mathrm{Log}_n]$ such that the minimal polynomial P_λ of φ_λ divides η in $\mathcal{O}_K[\mathrm{Log}_1, \cdots, \mathrm{Log}_n]$ for all λ. Let L be the algebraic closure of the field $K(\mathrm{Log}_1, \cdots, \mathrm{Log}_n)$. For a generic $\kappa \in \mathcal{U}$, the specialization W^0_κ of W^0 at κ is isomorphic to the irreducible Verma module with highest weight κ. According to [3, Lemma 5], for a generic κ, the $\mathfrak{gl}(n)$-module $W^0_{e,\kappa} = W^0_\kappa \otimes \mathrm{Sym}^e \tau^\vee$ admits a Jordan-Hölder series with irreducible Verma modules as graded pieces and the length is finite, independent of κ. Let l be this length. It follows that the subset of L, consisting all the eigenvalues of φ_λ for all λ, is finite, and also for each vector $u \in W^0_{e,\lambda,\kappa}$ with κ generic, the dimension of the space $\mathrm{Span}\{\varphi^m_{\lambda,\kappa}(u) : m \in \mathbb{N}\}$ is bounded by l. Therefore as λ varies the degree of the minimal polynomial P_λ is uniformly bounded and all the roots are contained in a finite set. This implies the existence of the desired $\eta \in \mathcal{O}_K[\mathrm{Log}_1, \cdots, \mathrm{Log}_n]$. □

Proof of Proposition 3.7.1. Let $\tilde{\varphi}, \eta$ be as in the previous lemma for $e = r$. Then $\eta^{-1}\tilde{\varphi}$ induces the morphism

$$\Phi_r : \mathcal{V}^{\dagger,0}_{\kappa^{\mathrm{un}}\otimes \mathrm{Sym}^r \tau^\vee, w} \longrightarrow \mathcal{V}^{\dagger,0}_{\kappa^{\mathrm{un}}\otimes \mathrm{Sym}^r \tau^\vee, w} \otimes_K K(\mathrm{Log}_1, \cdots, \mathrm{Log}_n),$$

which is the inverse of $E^r\theta^r D^r$. Set $\mathscr{A}_r = 1 - \theta^r D^r \Phi_r E^r$. Then

$$E^r \mathscr{A}_r = E^r(1 - \theta^r D^r \Phi_r E^r) = E^r - (E^r\theta^r D^r \Phi_r)E^r = E^r - E^r = 0,$$

showing that $\mathscr{A}_r$ sends $N^{\dagger,r}_{\mathcal{U},w,v}$ into $N^{\dagger,r-1}_{\mathcal{U},w,v} \otimes_K K(\mathrm{Log}_1, \cdots, \mathrm{Log}_n)$. Meanwhile $\mathscr{A}_r$ is identity on $N^{\dagger,r-1}_{\mathcal{U},w,v}$ because E^r annihilates $N^{\dagger,r-1}_{\mathcal{U},w,v}$. By induction we obtain the desired $\mathscr{A} = \mathscr{A}_1 \circ \mathscr{A}_2 \circ \cdots \circ \mathscr{A}_r$. □

Corollary 3.7.5. *There exists a nonzero $\eta \in K[\mathrm{Log}_1, \cdots, \mathrm{Log}_n]$ such that each $F \in N^{\dagger,r}_{\mathcal{U},w,v}$ can be written as*

$$\eta F = F_0 + \theta D F_1 + \cdots + \theta^r D^r F_r$$

with $F_i \in N^{\dagger,0}_{\mathcal{U}\otimes \mathrm{Sym}^i \tau^\vee, w, v}$.

3.8 Unramified Hecke operators

Let ℓ be a prime integer with $(\ell, Np) = 1$. For $\gamma_\ell \in \mathrm{GSp}(2n, \mathbb{Z}_\ell)\backslash \mathrm{GSp}(2n, \mathbb{Q}_\ell)/\mathrm{GSp}(2n, \mathbb{Z}_\ell)$, the action of the Hecke operator T_{γ_ℓ} on $N^{\dagger,r}_{\kappa,w,v}$ can be defined in the standard way using algebraic correspondence of ℓ-(quasi-)isogenies of type γ_ℓ. Let $Y_{\mathrm{Iw},K}$ be the moduli scheme over K parametrizing principally polarized abelian schemes (A, λ) with a principal level N structure and a self-dual full flag $\mathrm{Fil}_\bullet A[p]$. Define $C_{\gamma_\ell} \subset Y_{\mathrm{Iw},K} \times Y_{\mathrm{Iw},K}$ to be the moduli space, whose R-points $C_{\gamma_\ell}(R)$ for any K-algebra R consists of (quasi-)isogenies

$$\pi : (A_1, \lambda_1, \psi_{N,1}, \mathrm{Fil}_\bullet A_1[p]) \to (A_2, \lambda_2, \psi_{N,2}, \mathrm{Fil}_\bullet A_2[p])$$

of type γ_ℓ with degree being a power of ℓ. Here for $i = 1, 2$, $\lambda_i, \psi_{N,i}$ and $\mathrm{Fil}_\bullet A_i[p]$ need to satisfy $\pi^*\lambda_2 = \nu(\gamma_\ell)\lambda_1$, $\pi \circ \psi_{N,1} = \psi_{N,2}$, $\pi \circ \mathrm{Fil}_\bullet A_1[p] = \mathrm{Fil}_\bullet A_2[p]$. Being of type γ_ℓ means that under certain $\mathbb{Z}_\ell$-basis of the Tate modules $T_\ell(A_i)$, the matrix of the morphism induced by π on Tate modules is γ_ℓ. Denote by p_1 (resp. p_2) the projection of C_{γ_ℓ} to the first (resp. second) factor. Put $\mathcal{C}_{\gamma_\ell}(v) = C_{\gamma_\ell,\mathrm{an}} \times_{p_1, Y_{\mathrm{Iw},K,\mathrm{an}}} \mathcal{Y}_{\mathrm{Iw}}(v)$. Then we have the picture

$$\begin{array}{ccc} & \mathcal{C}_{\gamma_\ell}(v) & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ \mathcal{Y}_{\mathrm{Iw}}(v) & & \mathcal{Y}_{\mathrm{Iw}}(v). \end{array} \tag{3.16}$$

Write $p_i^*\mathcal{T}^\times_{\mathcal{H},w}(v) = \mathcal{C}_{\gamma_\ell}(v) \times_{p_i, \mathcal{Y}_{\mathrm{Iw}}(v)} \mathcal{T}^\times_{\mathcal{H},w}(v)$. Due to the functoriality of the Hodge–Tate map and the canonical subgroups, the (quasi-)isogeny π induces an isomorphism $\pi^* : p_2^*\mathcal{T}^\times_{\mathcal{H},w}(v) \to p_1^*\mathcal{T}^\times_{\mathcal{H},w}(v)$ (cf. [1, Lemma 6.1.1]). Applying π^* to the first factor of the contracted product $p_i^*\mathcal{T}^\times_{\mathcal{H},w}(v) \times^{\mathcal{Q}_w} V^r_{\kappa,w}$ we obtain

$$\pi^* : p_2^*\mathcal{V}^{\dagger,r}_{\kappa,w} \xrightarrow{\sim} p_1^*\mathcal{V}^{\dagger,r}_{\kappa,w}.$$

The Hecke operator T_{γ_ℓ} is defined as the composition

$$H^0(\mathcal{Y}_{\mathrm{Iw}}(v), \mathcal{V}^{\dagger,r}_{\kappa,w}) \xrightarrow{p_2*} H^0(\mathcal{C}_{\gamma_\ell}(v), p_2^*\mathcal{V}^{\dagger,r}_{\kappa,w}) \xrightarrow{\pi^*} H^0(\mathcal{C}_{\gamma_\ell}(v), p_1^*\mathcal{V}^{\dagger,r}_{\kappa,w}) \xrightarrow{\mathrm{Tr}\, p_1} H^0(\mathcal{Y}_{\mathrm{Iw}}(v), \mathcal{V}^{\dagger,r}_{\kappa,w}).$$

Such defined T_{γ_ℓ} maps bounded functions to bounded functions so it defines an action on $N^{\dagger,r}_{\kappa,w,v}$ by the discussion of [1, §5.5]. Its action also preserves the cuspidal part (see Remark 3.9.5).

3.9 The $\mathbb{U}_p$-operators

Let $T^+_{\mathbf{G}} = \{\mathrm{diag}(p^{a_1}, \cdots, p^{a_n}, p^{a_0-a_1}, \cdots, p^{a_0-a_n}) \in T_{\mathbf{G}}(\mathbb{Q}) : a_1 \leq \cdots \leq a_n, a_0 \geq 2a_n\}$. Set

$$\gamma_{p,i} = \begin{pmatrix} I_i & 0 & 0 & 0 \\ 0 & pI_{n-i} & 0 & 0 \\ 0 & 0 & p^2 I_i & 0 \\ 0 & 0 & 0 & pI_{n-i} \end{pmatrix}, \quad 1 \leq i \leq n-1, \quad \text{and} \quad \gamma_{p,n} = \begin{pmatrix} I_n & 0 \\ 0 & pI_n \end{pmatrix}. \tag{3.17}$$

We want to attach a Hecke operator to each element of $T^+_{\mathbf{G}}$. All such operators will be called $\mathbb{U}_p$-operators. An element $\gamma_p \in T^+_{\mathbf{G}}$ can be uniquely written as $\gamma_p = p^{s_0}\prod_{j=1}^n \gamma_{p,j}^{s_j}$ with $s_0 \in \mathbb{Z}$, $s_1 \cdots, s_n \in \mathbb{N}$. We make the scalar p act on $Y_{\mathrm{Iw},K}$ by sending $(A, \lambda, \psi_N, \mathrm{Fil}_\bullet A[p])$ to $(A, \lambda, \psi_N \circ p, \mathrm{Fil}_\bullet A[p])$. This action is invertible and induces a map on the global sections of the sheaf $\mathcal{V}^{\dagger,r}_{\kappa,w}$, which we take as the Hecke operator corresponding to $p \in T^+_{\mathbf{G}}$ and denote by $\langle p \rangle$. We define the Hecke operator attached to p^{s_0} as $\langle p \rangle^{s_0}$ for all $s_0 \in \mathbb{Z}$. It remains to define the operators $U_{p,i}$ associated to $\gamma_{p,i}$ for $1 \leq i \leq n$.

3.9.1 The operator $U_{p,n}$

Let $C_n \subset Y_{\mathrm{Iw},K} \times Y_{\mathrm{Iw},K}$ be the moduli space parametrizing the quintuples $(A, \lambda, \psi_N, \mathrm{Fil}_\bullet A[p], L)$, with $(A, \lambda, \psi_N, \mathrm{Fil}_\bullet A[p])$ being the moduli problem defining $Y_{\mathrm{Iw},K}$ and $L \subset A[p]$ satisfying $L \oplus \mathrm{Fil}_n A[p] = A[p]$. Denote by $\pi : A \to A/L$ the universal isogeny. There are two projections p_1, p_2 from C_n to $Y_{\mathrm{Iw},K}$. The first one is by forgetting L, and the other sends $(A, \lambda, \psi_N, \mathrm{Fil}_\bullet A[p], L)$ to $(A/L, \lambda', \pi \circ \psi_N, \mathrm{Fil}_\bullet A/L[p])$, with λ' defined by $\pi^*\lambda' = p\lambda$ and $\mathrm{Fil}_i A/L[p] = \pi \circ \mathrm{Fil}_i A[p]$,

$1 \leq i \leq n$. Consider $\mathcal{C}_n(v) = C_{n,\mathrm{an}} \times_{p_1, Y_{\mathrm{Iw},K}} \mathcal{Y}_{\mathrm{Iw}}(v) \subset \mathcal{Y}_{\mathrm{Iw}}(v) \times \mathcal{Y}_{\mathrm{Iw}}(v)$, which parametrizes $(A, \lambda, \psi_N, \mathrm{Fil}_\bullet A[p], L)$ with $\mathrm{Hdg}(A[p^\infty]) \leq v$ and $\mathrm{Fil}_n A[p] = H_1$, the level 1 canonical subgroup. According to [22, Theorem 8], there is the diagram

$$\begin{array}{ccc} & \mathcal{C}_n(v) & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ \mathcal{Y}_{\mathrm{Iw}}(v) & & \mathcal{Y}_{\mathrm{Iw}}(p)(\frac{v}{p}). \end{array} \tag{3.18}$$

The universal isogeny π induces an isomorphism $\pi^* : p_2^* \mathcal{T}^\times_{\mathcal{H},w}(\frac{v}{p}) \to p_1^* \mathcal{T}^\times_{\mathcal{H},w}(v)$ (cf. [1, Lemma 6.2.1.2]) that gives rise to $\pi^* : p_2^* \mathcal{V}^{\dagger,r}_{\kappa,w} \xrightarrow{\sim} p_1^* \mathcal{V}^{\dagger,r}_{\kappa,w}$. The operator $U_{p,n}$ is defined as the composition

$$\begin{aligned} H^0(\mathcal{Y}_{\mathrm{Iw}}(p)(\frac{v}{p}), \mathcal{V}^{\dagger,r}_{\kappa,w}) &\xrightarrow{p_2^*} H^0(\mathcal{C}_n(v), p_2^* \mathcal{V}^{\dagger,r}_{\kappa,w}) \xrightarrow{\pi^*} H^0(\mathcal{C}_n(v), p_1^* \mathcal{V}^{\dagger,r}_{\kappa,w}) \\ &\xrightarrow{p^{-n(n+1)/2} \operatorname{Tr} p_1} H^0(\mathcal{Y}_{\mathrm{Iw}}(v), \mathcal{V}^{\dagger,r}_{\kappa,w}). \end{aligned} \tag{3.19}$$

See §3.9.5 for the normalizer $p^{-n(n+1)/2}$.

3.9.2 The operators $U_{p,i}$, $1 \leq i \leq n-1$

First for $\underline{w} = (w_{jk})_{1 \leq k < j \leq n}$ satisfying

(i) $w_{jk} = w$ or $w - 1$ for some w as before,

(ii) $w_{j+1,k} \geq w_{j,k}$, and $w_{j,k-1} \geq w_{jk}$,

we introduce the $\underline{w}$-analyticity, generalizing the w-analyticity for a scalar w. Recall $N_n^-(\mathbb{Z}_p) \subset I_n(\mathbb{Z}_p)$ is the subgroup of lower triangular elements with 1 as diagonal entries. Let $\mathcal{N}^-_{\underline{w}}$ be the rigid analytic group

$$N_n^-(\mathbb{Z}_p) \cdot \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \mathcal{B}(0, p^{w_{21}}) & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{B}(0, p^{w_{n1}}) & \mathcal{B}(0, p^{w_{n2}}) & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ p\mathbb{Z}_p + \mathcal{B}(0, p^{w_{21}}) & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p\mathbb{Z}_p + \mathcal{B}(0, p^{w_{n1}}) & p\mathbb{Z}_p + \mathcal{B}(0, p^{w_{n2}}) & \cdots & 1 \end{pmatrix}.$$

Then $\mathcal{I}'_{\underline{w}} = \mathcal{N}^-_{\underline{w}} \mathcal{T}_{w-1} N_{n,an}$ is a rigid analytic space with the group $\mathcal{T}_{w-1} N_{n,\mathrm{an}}$ acting by the right multiplication. Due to the requirement (i)(ii) on $\underline{w}$, the space $\mathcal{I}'_{\underline{w}}$ is also stable under the left multiplication by the group $\mathcal{I}_w$. Like in §3.2 we define the $\mathcal{I}_w$-module $W_{\kappa,\underline{w}}$ by

$$W_{\kappa,\underline{w}}(R) = \left\{ \begin{array}{l} f : \mathcal{I}'_{\underline{w}}(R) \to R, \;\; f|_{\mathcal{N}^-_{\underline{w}}} \text{ is analytic and } f(xtn) = \kappa'(t) f(x) \\ \text{for all } x \in \mathcal{I}'_{\underline{w}}(R), t \in \mathcal{T}_{w-1}(R), n \in N_{n,an}(R) \end{array} \right\}$$

for all $R \in \mathfrak{A}_L$. The group $\mathcal{I}_w$ acts on it through the left inverse translation. We write $\underline{w}' \leq \underline{w}$ if $w'_{jk} \leq w_{jk}$ for all $1 \leq k < j \leq n$. For $\underline{w}' \leq \underline{w}$ the module $W_{\kappa,\underline{w}'}$ is contained in $W_{\kappa,\underline{w}}$, and elements in $W_{\kappa,\underline{w}'}$ satisfy stronger analyticity condition. By the same formulas as (3.3), (3.4) we define $V_{\kappa,\underline{w}}$. The contracted products $\mathcal{T}^\times_{\mathcal{F},w}(v) \times^{\mathcal{I}_w} W_{\kappa,\underline{w}}$ and $\mathcal{T}^\times_{\mathcal{H},w}(v) \times^{\mathcal{Q}_w} V_{\kappa,\underline{w}}$ define sub-Banach sheaves $\omega^\dagger_{\kappa,\underline{w}}$ of $\omega^\dagger_{\kappa,w}$, and $\mathcal{V}^{\dagger,r}_{\kappa,\underline{w}}$ of $\mathcal{V}^{\dagger,r}_{\kappa,w}$.

Next we extend the action of $\mathcal{I}_w$ on $W_{\kappa,\underline{w}}$ to $\Delta^-_{I,w} = \mathcal{I}_w T^-_n \mathcal{I}_w$, where $T^-_n = \{\mathrm{diag}(p^{b_1}, \cdots, p^{b_n}) \in \mathrm{GL}(n, \mathbb{Q}) : b_1 \geq \cdots \geq b_n\}$. With this extension the $\mathcal{Q}_w$-action on $V^r_{\kappa,\underline{w}}$ extends to $\Delta^-_{Q,w} = \mathcal{Q}_w T^-_{\mathbf{G}} \mathcal{Q}_w$ where $T^-_{\mathbf{G}} = \{\mathrm{diag}(p^{b_1}, \cdots, p^{b_n}, p^{b_0-b_1}, \cdots, p^{b_0-b_n}) \in T_{\mathbf{G}}(\mathbb{Q}) : b_1 \geq \cdots \geq b_n, b_0 \geq 2b_1\}$. Given $h = h' t_h h''$ with $h', h'' \in \mathcal{I}_w$ and $t_h \in T^-_{\mathbf{G}}$, we make it act on $f \in W_{\kappa,\underline{w}}$ by

$$(f \cdot h)(x) = f(h^{-1} x t_h). \tag{3.20}$$

It can be checked that this is a well defined action and has norm less or equal to 1 with respect to the supreme norm on $W_{\kappa,\underline{w}}$. If $t_h = \mathrm{diag}(p^{b_1}, \cdots, p^{b_n})$, then h sends $W_{\kappa,\underline{w}}$ into $W_{\kappa,\underline{w}'}$, with $w'_{jk} = \max_{k \leq t < s \leq j} \{w_{st} + b_s - b_t, w_{jk} - 1\} \leq w_{jk}$, increasing the analyticity.

Now fix $1 \leq i \leq n-1$ and consider the moduli scheme C_i over K parametrizing $(A, \lambda, \psi_N, \mathrm{Fil}_\bullet\, A[p], L)$, where $(A, \lambda, \psi_N, \mathrm{Fil}_\bullet\, A[p])$ is the moduli problem defining Y_{Iw}, and $L \subset A[p^2]$ is a Lagrangian subgroup such that $L[p] \oplus \mathrm{Fil}_i\, A[p] = A[p]$. Denote by $\pi : A \to A/L$ the universal isogeny. Define the projection $p_1 : C_i \to Y_{\mathrm{Iw},K}$ by forgetting L, and $p_2 : C_i \to Y_{\mathrm{Iw},K}$ by sending $(A, \lambda, \psi_N, \mathrm{Fil}_\bullet\, A[p], L)$ to $(A/L, \lambda', \pi \circ \psi_N, \mathrm{Fil}_\bullet\, A/L[p])$. Here the polarization λ' is defined by $\pi^* \lambda' = p^2 \lambda$ and $\mathrm{Fil}_\bullet\, A/L[p]$

is defined as

$$\begin{aligned}
\operatorname{Fil}_j A/L[p] &= \pi\big(\operatorname{Fil}_j A[p]\big), && 1 \le j \le i,\\
\operatorname{Fil}_j A/L[p] &= \pi\big(\operatorname{Fil}_j A[p] + p^{-1}(\operatorname{Fil}_j A[p] \cap L)\big), && i < j \le n,\\
\operatorname{Fil}_j A/L[p] &= \big(\operatorname{Fil}_{2n-j} A/L[p]\big)^{\perp}, && n+1 \le j \le 2n.
\end{aligned}$$

For example if $x_1, \cdots, x_n, x_{n+1}, \cdots, x_{2n}$ is a basis of $A[p^2]$ compatible with $\operatorname{Fil}_\bullet A[p]$ and the Weil pairing, then L can be taken to be $\langle px_{i+1}, \cdots, px_n, px_{n+1}, \cdots, px_{2n-i}, x_{2n-i+1}, \cdots, x_{2n}\rangle$ and correspondingly $\operatorname{Fil}_\bullet A/L[p]$ is

$$\langle p\bar{x}_1\rangle \subset \cdots \subset \langle p\bar{x}_1, \cdots, p\bar{x}_i\rangle \subset \langle p\bar{x}_1, \cdots, p\bar{x}_i, \bar{x}_{i+1}\rangle \subset \cdots \subset \langle p\bar{x}_1, \cdots, p\bar{x}_i, \bar{x}_{i+1}, \cdots, \bar{x}_n\rangle \subset \cdots$$

where $\bar{x}_j$ stands for $x_j \mod L$.

Set $\mathcal{C}_i(v) = C_{i,\mathrm{an}} \times_{p_1, Y_{\mathrm{Iw,an}}} \mathcal{Y}_{\mathrm{Iw}}(v)$. In order to form a diagram analogous to (3.16), (3.18) we need

Proposition 3.9.1. *([1, Proposition 6.2.2.1]) If* $\operatorname{Hdg}(A[p^\infty]) < \frac{p-2}{p(2p-2)}$ *and* $\operatorname{Fil}_n A[p]$ *is the canonical subgroup of level* 1*, then* $\operatorname{Hdg}(A[p^\infty]/L) \le \operatorname{Hdg}(A[p^\infty])$ *and the* $\operatorname{Fil}_n A/L[p]$ *defined above is the canonical subgroup of level* 1 *of* A/L*.*

Then we have the diagram

$$\begin{array}{ccccc}
 & & \mathcal{C}_i(v) & & \\
 & \swarrow^{p_1} & & \searrow^{p_2} & \\
\mathcal{Y}_{\mathrm{Iw}}(v) & & & & \mathcal{Y}_{\mathrm{Iw}}(v).
\end{array} \tag{3.21}$$

Now the pullback $\pi^* : p_2^*\mathcal{T}^{\times}_{\mathcal{H},\mathrm{an}}(v) \xrightarrow{\sim} p_1^*\mathcal{T}^{\times}_{\mathcal{H},\mathrm{an}}(v)$ does not send $p_2^*\mathcal{T}^{\times}_{\mathcal{H},w}(v)$ into $p_1^*\mathcal{T}^{\times}_{\mathcal{H},w}(v)$, but to

$$p_1^*\mathcal{T}^{\times}_{\mathcal{H},w}(v) \circ \begin{pmatrix} pI_{n-i} & 0 & 0 & 0 \\ 0 & I_i & 0 & 0 \\ 0 & 0 & pI_{n-i} & 0 \\ 0 & 0 & 0 & p^2 I_i \end{pmatrix} \mathcal{Q}_w \subset p_1^*\mathcal{T}^{\times}_{\mathcal{H},w}(v) \circ \Delta^-_{Q,w}.$$

Given local section (α, u) of the contracted product $p_2^*\mathcal{T}^{\times}_{\mathcal{H},w}(v) \times^{\mathcal{Q}_w} V^r_{\kappa,\underline{w}}$, there is a $\gamma_\alpha \in \Delta^-_{Q,w}$ such that $(\pi^*\alpha)\circ\gamma_\alpha^{-1}$ lies inside $p_1^*\mathcal{T}^{\times}_{\mathcal{H},w}(v)$, and we can define

$$\begin{aligned} \widetilde{\pi}^* : p_2^*\mathcal{T}^{\times}_{\mathcal{H},w}(v) \times^{\mathcal{Q}_w} V^r_{\kappa,\underline{w}} &\longrightarrow p_1^*\mathcal{T}^{\times}_{\mathcal{H},w}(v) \times^{\mathcal{Q}_w} V^r_{\kappa,\underline{w}'} \\ (\alpha, u) &\longmapsto \big((\pi^*\alpha)\circ\gamma_\alpha^{-1}, \gamma_\alpha \cdot u\big), \end{aligned} \tag{3.22}$$

with $w'_{jk} = \begin{cases} \max\{w_{jk}-1, w-1\}, & \text{if } 1 \le k \le n-i < j \le n, \\ w_{jk}, & \text{otherwise} \end{cases}$. It is easy to see that the right hand side of (3.22) does not depend on the choice of γ_α and $\widetilde{\pi}^*$ is well defined.

The operator $U_{p,i}$ is defined as the composition

$$\begin{aligned} H^0(\mathcal{Y}_{\mathrm{Iw}}(v), \mathcal{V}^{\dagger,r}_{\kappa,\underline{w}}) &\xrightarrow{p_{2*}} H^0(\mathcal{C}_i(v), p_2^*\mathcal{V}^{\dagger,r}_{\kappa,\underline{w}}) \xrightarrow{\widetilde{\pi}^*} H^0(\mathcal{C}_i(v), p_1^*\mathcal{V}^{\dagger,r}_{\kappa,\underline{w}'}) \\ &\xrightarrow{p^{-i(n+1)}\mathrm{Tr}\, p_1} H^0(\mathcal{Y}_{\mathrm{Iw}}(v), \mathcal{V}^{\dagger,r}_{\kappa,\underline{w}'}). \end{aligned} \tag{3.23}$$

The normalizer $p^{-i(n+1)}$ is justified in §3.9.5.

3.9.3 A compact operator U_p

From (3.19), (3.23) we see that the composition $U_{p,n}\circ U_{p,n-1}\circ\cdots\circ U_{p,1}$ maps $N^{\dagger,r}_{\kappa,w,v}$ continuously into $N^{\dagger,r}_{\kappa,w-1,pv}$. Let $\mathrm{res} : N^{\dagger,r}_{\kappa,w-1,pv} \to N^{\dagger,r}_{\kappa,w,v}$ be the natural restriction map. Define the operator U_p as

$$U_p = \mathrm{res}\circ U_{p,n}\circ U_{p,n-1}\circ\cdots\circ U_{p,1} : N^{\dagger,r}_{\kappa,w,v} \longrightarrow N^{\dagger,r}_{\kappa,w,v}.$$

In the following we show that the map res : $N^{\dagger,r}_{\kappa,w-1,pv} \to N^{\dagger,r}_{\kappa,w,v}$ is a compact morphism between two K-Banach modules. To this end it will be convenient to use the interpretation (ii) of $N^{\dagger,r}_{\kappa,w,v}$ in §3.4.3, i.e.

$$N^{\dagger,r}_{\kappa,w,v} = H^0(\mathcal{IW}_w(v), \mathcal{L}_\kappa \otimes (\pi_1 \circ \pi_3)^* \operatorname{Sym}^r \mathcal{J})^{B_n(\mathbb{Z}/p^m\mathbb{Z})}.$$

Since the group $B_n(\mathbb{Z}/p^m\mathbb{Z})$ is finite, there is a continuous projection from $H^0(\mathcal{IW}_w(v), \mathcal{L}_\kappa \otimes (\pi_1 \circ \pi_3)^* \operatorname{Sym}^r \mathcal{J})$ to its $B_n(\mathbb{Z}/p^m\mathbb{Z})$-invariant part. Thus it is enough to show the compactness of the restriction

$$H^0(\mathcal{IW}_{w-1}(pv), \mathcal{L}_\kappa \otimes (\pi_1 \circ \pi_3)^* \operatorname{Sym}^r \mathcal{J}) \longrightarrow H^0(\mathcal{IW}_w(v), \mathcal{L}_\kappa \otimes (\pi_1 \circ \pi_3)^* \operatorname{Sym}^r \mathcal{J}).$$

Note that the sheaf $\mathcal{L}_\kappa \otimes (\pi_1 \circ \pi_3)^* \operatorname{Sym}^r \mathcal{J}$ is coherent. Applying [41, Proposition 2.4.1] we reduce to prove that $\mathcal{IW}_w(v)$ is relatively compact inside $\mathcal{IW}_{w-1}(pv)$ (relative to $\operatorname{Spm}(K)$).

According to [41, Definition 2.1.1], given a quasi-compact rigid analytic space $\mathcal{Z}$ and $\mathcal{V} \subset \mathcal{Z}$, an admissible open quasi-compact subset, $\mathcal{V}$ is called relatively compact inside $\mathcal{Z}$ (relative to $\operatorname{Spm}(K)$), written as $\mathcal{V} \Subset \mathcal{Z}$, if there exists a formal model $\mathfrak{Z}$ of $\mathcal{Z}$ together with an open sub-formal scheme $\mathfrak{V} \subset \mathfrak{Z}$ with rigid fibre $\mathfrak{V}_{\text{rig}} = \mathcal{V}$, such that the closure $\overline{\mathfrak{V}_0}$ of the reduction $\mathfrak{V}_0$ inside $\mathfrak{Z}_0$ is proper (over $\operatorname{Spec}(k)$, $k = \mathcal{O}_K/\varpi$).

Lemma 3.9.2. *$\mathcal{X}_1(p^m)(v)$ is relatively compact inside $\mathcal{X}_1(p^m)(pv)$.*

Proof. First note that $\mathcal{X}(pv) \Subset \mathcal{X}$ since X is proper. Then using [41, Proposition 2.3.1] we get $\mathcal{X}(v) \Subset \mathcal{X}(pv)$. Both of the projections $\mathcal{X}_1(p^m)(v) \to \mathcal{X}(v)$ and $\mathcal{X}_1(p^m)(pv) \to \mathcal{X}(pv)$ are finite. The statement follows from [41, Lemma 2.1.8]. □

Proposition 3.9.3. *$\mathcal{IW}_w(v)$ is relatively compact inside $\mathcal{IW}_{w-1}(pv)$.*

Proof. By construction we have the formal model $f : \mathfrak{IW}_{w-1}(pv) \to \mathfrak{X}_1(p^m)(pv)$. By the previous lemma we can take an admissible formal blow-up $\mathfrak{X}_1(p^m)(pv)' \to \mathfrak{X}_1(p^m)(pv)$ with an open formal subscheme $\mathfrak{X}_1(p^m)(v)' \subset \mathfrak{X}_1(p^m)(pv)'$, such that $\mathfrak{X}_1(p^m)(v)'_{\text{rig}} = \mathcal{X}_1(p^m)(v)$ and the closure

$\overline{\mathfrak{X}_1(p^m)(v)'_0}$ inside $\mathfrak{X}_1(p^m)(pv)'_0$ is proper. Base changing f via the blow-up we get

$$\begin{array}{ccc}
\mathfrak{IW}_{w-1}(v)' & \hookrightarrow & \mathfrak{IW}_{w-1}(pv)' \\
\downarrow & \square & \downarrow \\
\mathfrak{X}_1(p^m)(v)' & \hookrightarrow & \mathfrak{X}_1(p^m)(pv)'
\end{array}.$$

There is an open covering of $\mathfrak{X}_1(p^m)(pv)'$ by affine open subschemes such that over each member $\mathrm{Spf}(R)$ of it, $\mathfrak{IW}_{w-1}(pv)' \times_{\mathfrak{X}_1(p^m)(pv)'} \mathrm{Spf}(R)$ is isomorphic to

$$\begin{aligned}
\mathrm{Spf}(R) \times \begin{pmatrix} 1 & 0 & \cdots & 0 \\ p^{w-1}\mathfrak{B}(0,1) & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p^{w-1}\mathfrak{B}(0,1) & p^{w-1}\mathfrak{B}(0,1) & \cdots & 1 \end{pmatrix} &\cong \mathrm{Spf}(R) \times \mathfrak{B}(0,1)^{n(n-1)/2} \\
&\cong \mathrm{Spf}\left(R\left\langle T_{ij}\right\rangle_{1\le j<i\le n}\right).
\end{aligned}$$

Over $\mathfrak{IW}_{w-1}(pv)' \times_{\mathfrak{X}_1(p^m)(pv)'} \mathrm{Spf}(R)$ one can define the ideal sheaf attached to the ideal generated by p and T_{ij}, $1 \le j < i \le n$, which is independent of the choice of the coordinate T_{ij}. Such locally defined ideal sheaves glue together to an ideal sheaf $\mathscr{I}$ over $\mathfrak{IW}_{w-1}(pv)'$. Let $\mathfrak{IW}_{w-1}(pv)''$ be the blow-up of $\mathfrak{IW}_{w-1}(pv)'$ along $\mathscr{I}$, and take $\mathfrak{IW}_w(pv)''$ to be its open sub-formal scheme where the ideal sheaf $\mathscr{I}$ is generated by p. From the local description of $\mathscr{I}$, we know that the closure $\overline{\mathfrak{IW}_w(pv)''_0}$ of $\mathfrak{IW}_w(pv)''_0$ inside $\mathfrak{IW}_{w-1}(pv)''_0$ is proper over the base $\mathfrak{X}_1(p^m)(pv)'_0$. Take $\mathfrak{IW}_w(v)''$ to be the inverse image of $\mathfrak{X}_1(p^m)(v)'$ under the projection $\mathfrak{IW}_w(pv)'' \to \mathfrak{X}_1(p^m)(pv)'$. Then $\mathfrak{IW}_w(v)''$ is an open sub-formal scheme of $\mathfrak{IW}_{w-1}(pv)''$ with rigid fibre equal to $\mathcal{IW}_w(v)$. Now we have the picture

$$\begin{array}{ccccccc}
\mathfrak{IW}_w(v)''_0 & \hookrightarrow & \mathfrak{Z}_0 & \hookrightarrow & \overline{\mathfrak{IW}_w(pv)''_0} & \hookrightarrow & \mathfrak{IW}_{w-1}(pv)''_0 \\
\downarrow & & \downarrow g & \square & \downarrow h & \swarrow & \\
\mathfrak{X}_1(p^m)(v)'_0 & \hookrightarrow & \overline{\mathfrak{X}_1(p^m)(v)'_0} & \hookrightarrow & \mathfrak{X}_1(p^m)(pv)'_0 & &
\end{array}$$

with the vertical map h being proper. Due to the properness of the scheme $\overline{\mathfrak{X}_1(p^m)(v)'_0}$ and the map g (implied by that of h), the scheme $\mathfrak{Z}_0$ is proper. Then the closure of $\mathfrak{IW}_w(v)''_0$ inside $\mathfrak{IW}_{w-1}(pv)''_0$

must be proper since it is contained in $\mathfrak{Z}$. □

All the arguments apply to the universal weight case by working relatively over $\mathcal{U} \subset \mathcal{W}$ as well as the cuspidal case by replacing $\mathrm{Sym}^r \mathcal{J}$ with $\mathrm{Sym}^r \mathcal{J}(-C)$. We record the following corollary.

Corollary 3.9.4. *The operators $U_p : N^{\dagger,r}_{\kappa,w,v} \to N^{\dagger,r}_{\kappa,w,v}$ and $U_p : N^{\dagger,r}_{\kappa,w,v,\mathrm{cusp}} \to N^{\dagger,r}_{\kappa,w,v,\mathrm{cusp}}$ (resp. $U_p : N^{\dagger,r}_{\mathcal{U},w,v} \to N^{\dagger,r}_{\mathcal{U},w,v}$ and $U_p : N^{\dagger,r}_{\mathcal{U},w,v,\mathrm{cusp}} \to N^{\dagger,r}_{\mathcal{U},w,v,\mathrm{cusp}}$) are compact operators of K-Banach spaces (resp. $\mathcal{A}(\mathcal{U})$-Banach modules).*

3.9.4 Tensoring with $\tau, \tau^\vee$

We consider the algebraic $\mathrm{GL}(n)$-representations $(\sigma_{\mathrm{alg}}, W_{\sigma_{\mathrm{alg}}})$ that are obtained by taking tensor products of symmetric powers $\mathrm{Sym}^{e_1} \tau_{\mathrm{alg}}$ and $\mathrm{Sym}^{e_2} \tau^\vee_{\mathrm{alg}}$ with $e_1, e_2 \in \mathbb{N}$. Here we add the subscript to indicate that the action of $\Delta^-_{I,w}$ is the one given by the algebraic action of $\mathrm{GL}(n)$. The notation $\sigma, \tau, \tau^\vee$ will be saved for the $\Delta^-_{I,w}$-modules which are obtained from the algebraic ones by a renormalization explained below.

First we define two characters χ_1, χ_2 on the semi-group $\Delta^-_{I,w}$. Given $h = h' t_h h''$ with $h', h'' \in \mathcal{I}_w$ and $t_h = \mathrm{diag}(p^{b_1}, \cdots, p^{b_n}) \in T^-_n$, put

$$\chi_1(h) = p^{-2b_n}, \qquad \chi_2(h) = p^{2b_1}.$$

We define the $\Delta^-_{I,w}$-modules $\tau, \tau^\vee$ as

$$\tau := \tau_{\mathrm{alg}} \otimes \chi_1, \qquad \tau^\vee := \tau^\vee_{\mathrm{alg}} \otimes \chi_2.$$

Then by taking tensor products of $\tau, \tau^\vee$, we associate to each σ_{alg} the renormalized $\Delta^-_{I,w}$-module σ. The reason we consider this renormalization of σ_{alg} is that it makes the action of $\Delta^-_{Q,w}$ on V^r_σ integral.

Then all the $\mathbb{U}_p$-operators can be constructed for $N^{\dagger,r}_{\kappa\otimes\sigma,w,v}$ in exactly the same way as when σ is trivial, and Corollary 3.9.4 holds for the action of U_p on $N^{\dagger,r}_{\kappa\otimes\sigma,w,v}$ and $N^{\dagger,r}_{\kappa\otimes\sigma,w,v,\mathrm{cusp}}$. There is no need to distinguish σ and σ_{alg} when constructing of $\mathcal{V}^{\dagger,r}_{\kappa\otimes\sigma,w}$. Their difference only concerns the action of $\mathbb{U}_p$-operators.

3.9.5 The normalizations of the $\mathbb{U}_p$-operators

We show in this section that by our choice of the normalizations of the $\mathbb{U}_p$-operators, all the eigenvalues of the compactor operator U_p acting on $N^{\dagger,r}_{\kappa\otimes\sigma,w,v}$ are p-adically integral for all w-analytic κ. Since $\mathcal{V}^r_{\kappa\otimes\sigma,w}$ has a filtration with $\mathcal{V}^0_{\kappa\otimes\sigma\otimes\mathrm{Sym}^e\,\tau^\vee,w}$ as graded pieces, it is enough to consider the case $r=0$.

For a positive integer $l\in\mathbb{N}$, let Y_l be the Siegel variety modulo p^l and $Y_l[1/\mathrm{Ha}]$ be the ordinary locus. Denote by $S(p^m)_l$ the finite étale cover of $Y_l[1/\mathrm{Ha}]$, parametrizing the quintuples $(A,\lambda,\psi_N,\mathrm{Fil}_\bullet\,{}^{\mathrm{t}}A[p^m]^{\text{ét}},(\phi_j)_{1\le j\le n})$, where (A,λ,ψ_N) is a principally polarized ordinary abelian scheme of rank n with principal level N structure defined over an $\mathcal{O}_K/p^l$-algebra, and $\mathrm{Fil}_\bullet\,{}^{\mathrm{t}}A[p^m]^{\text{ét}}$ is a complete flag of the free $\mathbb{Z}/p^m\mathbb{Z}$-module ${}^{\mathrm{t}}A[p^m]^{\text{ét}}$ with trivializations of graded pieces $\phi_j:\mathbb{Z}/p^m\mathbb{Z}\simeq\mathrm{Fil}_j/\mathrm{Fil}_{j+1}\,{}^{\mathrm{t}}A[p^m]^{\text{ét}}$. Put $\mathfrak{S}(p^\infty)=\varinjlim_l\varprojlim_m S(p^m)_l$. The Hodge–Tate map gives rise to the embedding

$$\begin{array}{ccc} \mathfrak{S}(p^\infty) & \hookrightarrow & \mathfrak{IW}^+_w(v)\ , \\ & \searrow & \downarrow \\ & & \mathfrak{X}_{\mathrm{Iw}}(v) \end{array} \tag{3.24}$$

which induces an injective map

$$\mathrm{res}: N^{\dagger,r}_{\kappa\otimes\sigma,w,v}\to H^0(\mathfrak{S}(p^\infty),\mathfrak{V}^r_\sigma)[1/p], \tag{3.25}$$

where $\mathfrak{V}^r_\sigma$ is the pullback to $\mathfrak{S}(p^\infty)$ of the locally free sheaf $\mathcal{V}^r_\sigma$ of finite rank over X. In the following we define $\mathbb{U}_p$-operators acting on $H^0(\mathfrak{S}(p^\infty),\mathfrak{V}^r_\sigma)$ such that res is $\mathbb{U}_p$-equivariant. Then the integrality of the U_p-eigenvalues on $N^{\dagger,r}_{\kappa\otimes\sigma,w,v}$ follows. We deal with the case of the operator $U_{p,i}$ for $1\le i\le n-1$. Other cases are basically the same.

First we construct the correspondence analogous to (3.21)

$$\begin{array}{ccc} & C_{i,m,l}(0) & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ S(p^m)_l & & S(p^{m-1})_l \end{array},$$

where $C_{i,m,l}(0)$ parametrizes the sextuples $(A, \lambda, \psi_N, \mathrm{Fil}_\bullet\, {}^{\mathrm{t}}A[p^m]^{\text{ét}}, (\phi_j)_{1\le j\le n}, L)$ whose first five components form the quintuple defining $S(p^m)_l$. The flag $\mathrm{Fil}_\bullet\, {}^{\mathrm{t}}A[p^m]^{\text{ét}}$ gives a self-dual flag of $\mathrm{Fil}_\bullet\, A[p]$ and $L \subset A[p^2]$ is the one used in defining C_i. The projection p_1 is forgetting L. The universal isogeny $\pi : A \to A' = A/L$ induces a map ${}^{\mathrm{t}}\pi : {}^{\mathrm{t}}A'[p^m]^{\text{ét}} \to {}^{\mathrm{t}}A[p^m]^{\text{ét}}$ and a well-defined map $p\cdot {}^{\mathrm{t}}\pi^{-1} : {}^{\mathrm{t}}A[p^m]^{\text{ét}} \to {}^{\mathrm{t}}A'[p^m]^{\text{ét}}$. Set $\mathrm{Fil}_j\, {}^{\mathrm{t}}A'[p^{m-1}]^{\text{ét}} = p\cdot {}^{\mathrm{t}}\pi^{-1}(\mathrm{Fil}_j\, {}^{\mathrm{t}}A[p^m]^{\text{ét}}) \cap {}^{\mathrm{t}}A'[p^{m-1}]^{\text{ét}}$ and $\phi'_j = \begin{cases} p^2 \cdot {}^{\mathrm{t}}\pi^{-1} \circ \phi_j, & \text{if } 1 \le j \le n-i, \\ p \cdot {}^{\mathrm{t}}\pi^{-1} \circ \phi_j, & \text{if } n-i+1 \le j \le n. \end{cases}$ The projection p_2 sends $(A, \lambda, \psi_N, \mathrm{Fil}_\bullet\, {}^{\mathrm{t}}A[p^m]^{\text{ét}}, (\phi_j)_{1\le j\le n}, L)$ to $(A', \lambda', \pi\circ\psi_N, \mathrm{Fil}_\bullet\, {}^{\mathrm{t}}A'[p^{m-1}]^{\text{ét}}, (\phi'_j)_{1\le j\le n})$. Taking the inverse limit with respect to m followed by the direct limit with respect to l, we get $\mathfrak{C}_{i,\infty}(0) = \varinjlim_l \varprojlim_m C_{i,m,l}(0)$ and the correspondence

$$\begin{array}{ccc} & \mathfrak{C}_{i,\infty}(0) & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ \mathfrak{S}(p^\infty) & & \mathfrak{S}(p^\infty). \end{array}$$

By our choice of the normalization of the $\Delta^-_{I,w}$-action on V^r_σ in the previous section, the group $I_n(\mathbb{Z}_p)T^-_n I_n(\mathbb{Z}_p)$ acts on it integrally. This guarantees that the map $\widetilde{\pi}^* : p_2^*\mathfrak{V}_\sigma \to p_1^*\mathfrak{V}_\sigma$ can be defined in a manner similar to (3.22). Once we have checked that $\mathrm{Im}\,(\mathrm{Tr}\, p_1) \subset p^{i(n+1)} H^0(\mathfrak{S}(p^\infty), \mathfrak{V}^r_\sigma)$, we can define the operator $U_{p,i}$ as

$$H^0(\mathfrak{S}(p^\infty), \mathfrak{V}^r_\sigma) \xrightarrow{p_2^*} H^0(\mathfrak{C}_{i,\infty}(0), p_2^*\mathfrak{V}^r_\sigma) \xrightarrow{\widetilde{\pi}^*} H^0(\mathfrak{C}_{i,\infty}(0), p_1^*\mathfrak{V}^r_\sigma) \xrightarrow{p^{-i(n+1)}\mathrm{Tr}\, p_1} H^0(\mathfrak{S}(p^\infty), \mathfrak{V}^r_\sigma).$$

It is not difficult to see that with such defined $\mathbb{U}_p$-operators on $H^0(\mathfrak{S}(p^\infty), \mathfrak{w}_\sigma)$, the map res is $\mathbb{U}_p$-equivariant.

In the rest of this section we show the inclusion

$$\mathrm{Im}\,(\mathrm{Tr}\,p_1) \subset p^{i(n+1)} H^0(\mathfrak{S}(p^\infty), \mathfrak{V}_\sigma^r).$$

Essentially this containment reflects the fact that the projection p_1 is ramified and $p^{i(n+1)}$ is its pure inseparability degree. Thanks to the projection formula we have

$$p_{1,*}p_1^*\mathfrak{V}_\sigma^r = p_{1,*}\mathcal{O}_{\mathfrak{C}_{i,\infty}(0)} \otimes \mathfrak{V}_\sigma^r.$$

Therefore it suffices to show

$$\mathrm{Tr}\,p_1(p_{1,*}\mathcal{O}_{\mathfrak{C}_{i,\infty}(0)}) \subset p^{i(n+1)}\mathcal{O}_{\mathfrak{S}(p^\infty)}. \tag{3.26}$$

Let $S(p^\infty)_0$ be the reduction of $\mathfrak{S}(p^\infty)$ and take $y_0 \in S(p^\infty)_0$, $y_0' \in p_2(p_1^{-1}(y_0))$. We show (3.26) in the formal neighborhoods $\widehat{\mathfrak{S}(p^\infty)}_{y_0}, \widehat{\mathfrak{C}_{i,\infty}(0)}_{(y_0,y_0')}$. We explicate the projection p_1 using the Serre–Tate coordinates [33, §8.2, 8.3]. The formal neighborhood $\widehat{\mathfrak{S}(p^\infty)}_{y_0}$ is isomorphic to $\mathrm{Hom}_{\mathrm{sym}}(T_pA_{y_0}^{\text{ét}} \times T_p{}^{\mathrm{t}}A_{y_0}^{\text{ét}}, \widehat{\mathbb{G}}_m)$. A point $z \in \widehat{\mathfrak{S}(p^\infty)}_{y_0}$ corresponds to a bilinear map $q : T_pA_{y_0}^{\text{ét}} \times T_p{}^{\mathrm{t}}A_{y_0}^{\text{ét}} \to \widehat{\mathbb{G}}_m$ that is symmetric if we identify ${}^{\mathrm{t}}A_{y_0}^{\text{ét}}$ with $A_{y_0}^{\text{ét}}$ via the polarization. Given any basis $x_1, \cdots, x_n$ of $T_pA_{y_0}^{\text{ét}}$, let ${}^{\mathrm{t}}x_1, \cdots, {}^{\mathrm{t}}x_n$ be its image under the polarization, which is a basis of ${}^{\mathrm{t}}A_{y_0}^{\text{ét}}$. Write $q(x_i, {}^{\mathrm{t}}x_j) = 1 + T_{jk}$, $1 \le j, k \le n$. We know that $T_{jk} = T_{kj}$. The $\{T_{jk}\}_{1\le j\le k\le n}$ is a Serre–Tate coordinate of $\widehat{\mathfrak{S}(p^\infty)}_{y_0}$. Similarly for $\widehat{\mathfrak{S}(p^\infty)}_{y_0'}$ with a given basis $x_1', \cdots, x_n'$ of $T_pA_{y_0'}^{\text{ét}}$ we get a corresponding Serre–Tate coordinate $\left\{T_{jk}'\right\}_{1\le j\le k\le n}$. The isogeny $\pi : A_{y_0} \to A_{y_0'}$ induces a map on the Tate modules. Now fix basis $x_1, \cdots, x_n$ and $x_1', \cdots, x_n'$ of $T_pA_{y_0}^{\text{ét}}$ and $T_pA_{y_0'}^{\text{ét}}$, such that with respect to them the matrix for the map $\pi : T_pA_{y_0}^{\text{ét}} \to T_pA_{y_0'}^{\text{ét}}$ is given by $\begin{pmatrix} pI_{n-i} & 0 \\ 0 & p^2I_i \end{pmatrix}$. Then under the basis ${}^{\mathrm{t}}x_1, \cdots, {}^{\mathrm{t}}x_n$ and ${}^{\mathrm{t}}x_1', \cdots, {}^{\mathrm{t}}x_n'$ of $T_p{}^{\mathrm{t}}A_{y_0}^{\text{ét}}$ and $T_p{}^{\mathrm{t}}A_{y_0'}^{\text{ét}}$, obtained from $x_1, \cdots, x_n$ and $x_1', \cdots, x_n'$ by the polarization, the matrix for ${}^{\mathrm{t}}\pi : T_p{}^{\mathrm{t}}A_{y_0'}^{\text{ét}} \to T_p{}^{\mathrm{t}}A_{y_0}^{\text{ét}}$ is given by $\begin{pmatrix} pI_{n-i} & 0 \\ 0 & I_i \end{pmatrix}$. For each $(z, z') \in \widehat{\mathfrak{C}_{i,\infty}(0)}_{(y_0,y_0')} \subset \widehat{\mathfrak{S}(p^\infty)}_{y_0} \times \widehat{\mathfrak{S}(p^\infty)}_{y_0'}$, let $q : T_pA_{y_0}^{\text{ét}} \times T_p{}^{\mathrm{t}}A_{y_0}^{\text{ét}} \to \widehat{\mathbb{G}}_m$

(resp. $q' : T_p A^{\text{ét}}_{y_0'} \times T_p {}^{\mathrm{t}}A^{\text{ét}}_{y_0'} \to \widehat{\mathbb{G}}_m$) be the corresponding bilinear map for z (resp. z'). We have $q\big(x_j, {}^{\mathrm{t}}\pi(x_k')\big) = q'\big(\pi(x_j), x_k'\big)$. Translating to the coordinates T_{jk} and T'_{jk}, we see that T'_{jk} can be taken to be the local coordinates of $\widehat{\mathfrak{C}_{i,\infty}(0)}_{(y_0,y_0')}$, and the projection $p_1 : \widehat{\mathfrak{C}_{i,\infty}(0)}_{(y_0,y_0')} \to \widehat{\mathfrak{S}(p^\infty)}_{y_0}$ is given by

$$
\begin{aligned}
\mathcal{O}_K[[T_{jk}]] &\longrightarrow \mathcal{O}_K[[T'_{jk}]] \\
T_{jk} &\longmapsto T'_{jk} && \text{if } 1 \le j \le k \le n-i, \\
T_{jk} &\longmapsto (T'_{jk}+1)^p - 1 && \text{if } 1 \le j \le n-i < k \le n, \\
T_{jk} &\longmapsto (T'_{jk}+1)^{p^2} - 1 && \text{if } n-i+1 \le j \le k \le n.
\end{aligned}
$$

An easy computation shows that the pure inseparability degree of p_1 is $p^{i(n+1)}$ and $\mathrm{Im}\,(\mathrm{Tr}\, p_1) \subset p^{i(n+1)}\mathcal{O}_K[[T_{jk}]]$.

Before ending this section we include the following remark concerning the Hecke actions preserving the cuspidality.

Remark 3.9.5. The injection (3.25) is equivariant under the action of both unramified Hecke operators and $\mathbb{U}_p$-operators. It is also easy to check that

$$N^{\dagger,r}_{\kappa\otimes\sigma,w,v,\mathrm{cusp}} = N^{\dagger,r}_{\kappa\otimes\sigma,w,v} \cap H^0(\mathfrak{S}(p^\infty), \mathfrak{V}^r_\sigma(-C))[1/p].$$

Hence it is enough to notice that the space $H^0(\mathfrak{S}(p^\infty), \mathfrak{V}^r_\sigma(-C))$ is preserved under those operators. This follows from the fact that classical cuspidal nearly homomorphic forms are stable under Hecke actions, and that the classical cuspidal nearly homomorphic forms are dense inside $H^0(\mathfrak{S}(p^\infty), \mathfrak{V}^r_\rho(-C))$.

3.10 Interchanging the Hecke and differential operators

Let σ be as in §3.9.4. In this section we discuss the commutator of the $\mathbb{U}_p$-operators and unramified Hecke operators, with the operators $D_{\kappa\otimes\sigma,w}$ and $E_{\kappa\otimes\sigma,w}$ acting on $N^{\dagger,r}_{\kappa\otimes\sigma,w,v}$. Recall that the operators $D_{\kappa\otimes\sigma,w}$ and $E_{\kappa\otimes\sigma,w}$ are defined as the compositions

$$D_{\kappa\otimes\sigma,w} : \mathcal{V}^r_{\kappa\otimes\sigma,w,v} \xrightarrow{\nabla_{\kappa\otimes\sigma,w}} \mathcal{V}^r_{\kappa\otimes\sigma\otimes\tau_{\mathrm{alg}},w,v}(-1) \xrightarrow{t^+} \mathcal{V}^r_{\kappa\otimes\sigma\otimes\tau,w,v},$$
$$E_{\kappa\otimes\sigma,w} : \mathcal{V}^r_{\kappa\otimes\sigma,w,v} \xrightarrow{\varepsilon_{\kappa\otimes\sigma,w}} \mathcal{V}^r_{\kappa\otimes\sigma\otimes\tau^\vee_{\mathrm{alg}},w,v}(1) \xrightarrow{t^-} \mathcal{V}^r_{\kappa\otimes\sigma\otimes\tau^\vee,w,v}.$$

We first show that the $\mathbb{U}_p$-operators and unramified Hecke operators commute with the connection $\nabla_{\kappa\otimes\sigma,w}$ and the operator $\varepsilon_{\kappa\otimes\sigma,w}$, and then see how interchanging the order of the $\mathbb{U}_p$-operators and the maps t^+, t^- leads to a certain power of p.

Lemma 3.10.1. *The $\mathbb{U}_p$-operators and unramified Hecke operators commute with the connection $\nabla_{\kappa\otimes\sigma,w}$ and the operator $\varepsilon_{\kappa\otimes\sigma,w}$.*

Proof. The $Q_{\mathbf{G}}$-representation J admits a filtration $0 \to \mathrm{triv} \to J \to \tau^\vee_{\mathrm{alg}}(1) \to 0$. The operator $\varepsilon_{\kappa\otimes\sigma,w}$ by definition is induced from the quotient morphism $J \to \tau^\vee_{\mathrm{alg}}(1)$, and is easily seen to commute with all $\mathbb{U}_p$-operators as well as unramified Hecke operators.

The commutativity of the connection $\nabla_{\kappa\otimes\sigma,w}$ with all the Hecke operators is a result of the functoriality of the Gauss–Manin connection, which says that for any map of abelian schemes

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & A' \\ \downarrow & \circlearrowleft & \downarrow \\ S & \xrightarrow{f} & R \end{array},$$

we have

$$\begin{array}{ccccc} f^*\mathcal{H}^1_{\mathrm{dR}}(A'/R) & \xrightarrow{f^*\nabla} & f^*\mathcal{H}^1_{\mathrm{dR}}(A'/R)\otimes f^*\Omega^1_R & \longrightarrow & f^*\mathcal{H}^1_{\mathrm{dR}}(A'/R)\otimes\Omega^1_S \\ \downarrow{\scriptstyle\varphi^*} & & \circlearrowleft & & \downarrow{\scriptstyle\varphi^*\otimes 1} \\ \mathcal{H}^1_{\mathrm{dR}}(A/S) & & \xrightarrow{\quad\nabla\quad} & & \mathcal{H}^1_{\mathrm{dR}}(A/S)\otimes\Omega^1_S \end{array}.$$

Let π be the universal isogeny $A \to A' = A/L$ over $\mathcal{C}_i(v)$. By the definition of the operator $U_{p,i}$, $1 \le i \le n-1$, in order to prove that it commutes with the connection $\nabla_{\kappa\otimes\sigma,w}$, we only need to show the following diagram commutes.

$$\begin{array}{ccc} p_2^*\mathcal{V}^{\dagger,r}_{\kappa\otimes\sigma,w} & \xrightarrow{\tilde{\pi}^*} & p_1^*\mathcal{V}^{\dagger,r}_{\kappa\otimes\sigma,w} \\ \downarrow{\scriptstyle p_2^*\nabla_{\kappa\otimes\sigma,w}} & & \downarrow{\scriptstyle p_1^*\nabla_{\kappa\otimes\sigma,w}} \\ p_2^*\mathcal{V}^{\dagger,r+1}_{\kappa\otimes\sigma,w}\otimes\Omega^1_{\mathcal{C}_i(v)} & \xrightarrow{\tilde{\pi}^*\otimes 1} & p_1^*\mathcal{V}^{\dagger,r+1}_{\kappa\otimes\sigma,w}\otimes\Omega^1_{\mathcal{C}_i(v)} \end{array}$$

Write a local section of $p_2^*\mathcal{V}^{\dagger,r+1}_{\kappa\otimes\sigma,w} = p_2^*\mathcal{T}^{\times}_{\mathcal{H},w}(v)\times^{\mathcal{Q}_w} V^r_{\kappa\otimes\sigma,w}$ as (α,u), with $u\in V^r_{\kappa\otimes\sigma,w}$ and α a w-compatible local basis of $p_2^*\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/\mathcal{Y}_{\mathrm{Iw}}(v))$. For any $g\in\mathcal{Q}_w$, $(\alpha,u)=(\alpha\circ g, g^{-1}\cdot u)$. Take $\gamma\in\Delta^-_{Q,w}$ such that $\pi^*\alpha\circ\gamma^{-1}\in p_1^*\mathcal{T}^{\times}_{\mathcal{H},w}(v)$. If D is a local section of the tangent bundle of $\mathcal{C}_i(v)$ then

$$\begin{aligned}\big(p_1^*\nabla_{\sigma,w}(D)\circ\tilde{\pi}^*\big)(\alpha,v) &= p_1^*\nabla_{\sigma,w}(D)\big((\pi^*\alpha\circ\gamma^{-1},\,\gamma\cdot v)\big)\\ &= \big(\pi^*\alpha\circ\gamma^{-1},\,D(\gamma\cdot v)+X(D,\pi^*\alpha\circ\gamma^{-1})\cdot\gamma\cdot v\big)\\ &= \big(\pi^*\alpha\circ\gamma^{-1},\,D(\gamma\cdot v)+X(D,\alpha\circ\gamma^{-1})\cdot\gamma\cdot v\big)\\ &= \tilde{\pi}^*\big(\alpha,\,\gamma^{-1}\cdot D(\gamma\cdot v)+\gamma^{-1}\cdot X(D,\alpha\circ\gamma^{-1})\cdot\gamma\cdot v\big)\\ &= \tilde{\pi}^*\Big(\alpha,\,Dv+\big(\gamma^{-1}D\gamma+\mathrm{Ad}(\gamma^{-1})X(D,\alpha\circ\gamma)\big)\cdot v\Big)\\ &= \tilde{\pi}^*\big(\alpha,\,Dv+X(D,\alpha)\cdot v\big)\\ &= \big(\tilde{\pi}^*\circ p_2^*\nabla_{\sigma,w}(D)\big)(\alpha,\,v),\end{aligned}$$

where the third equality follows from the functoriality of the Gauss–Manin connection. The commutativity of $\nabla_{\kappa\otimes\sigma,w}$ with other Hecke operators are shown similarly. □

Define two characters $\nu_{p,D}$, $\nu_{p,E}$ from $T^+_{\mathbf{G}}$ to $\mathbb{Q}^\times$, sending $t=\mathrm{diag}(p^{a_1},\cdots,p^{a_n},p^{a_0-a_1},\cdots,p^{a_0-a_n})$ to $\nu_{p,D}(t)=p^{a_0-2a_1}$, $\nu_{p,E}(t)=p^{a_0-2a_n}$, where $a_1\le\cdots\le a_n$, $a_0\ge 2a_n$. Both $\nu_{p,D}$ and $\nu_{p,E}$ are trivial on scalar matrices. Evaluated at $\gamma_{p,i}\in T^+_{\mathbf{G}}$ defined as (3.17), we have $\nu_{p,D}(\gamma_{p,i})=p^2$, $\nu_{p,E}(\gamma_{p,i})=1$ for $1\le i\le n-1$, and $\nu_{p,D}(\gamma_{p,n})=\nu_{p,E}(\gamma_{p,n})=p$. Let ℓ be an unramified prime. Define the character $\nu_\ell:\mathrm{GSp}(2n,\mathbb{Z}_\ell)\backslash\mathrm{GSp}(2n,\mathbb{Q}_\ell)/\mathrm{GSp}(2n,\mathbb{Z}_\ell)\to\mathbb{Q}^\times$, sending γ_ℓ to $|\nu(\gamma_\ell)|_\ell^{-1}$ where ν is the multiplier character.

Lemma 3.10.2.

$$\text{(i)}\quad \nu_{p,D}(\gamma_p)\cdot t^+U_{\gamma_p} = U_{\gamma_p}t^+, \qquad t^-U_{\gamma_p} = \nu_{p,E}(\gamma_p)\cdot U_{\gamma_p}t^-,$$

$$\text{(ii)}\quad \nu_\ell(\gamma_\ell)\cdot t^+T_{\gamma_\ell} = T_{\gamma_\ell}t^+, \qquad t^-T_{\gamma_\ell} = \nu_\ell(\gamma_\ell)\cdot T_{\gamma_\ell}t^-.$$

Proof. (ii) is obvious since the corresponding representations differ by a twist of the multiplier character. (i) is basically the same as (ii), but when defining the $\mathbb{U}_p$-operators, we renormalized the algebraic representations τ_{alg}, $\tau^\vee_{\mathrm{alg}}$ to the $\Delta^-_{I,w}$-modules τ, $\tau^\vee$ by twisting the characters χ_1, χ_2 to ensure the integrality. Therefore given $\gamma_p = \mathrm{diag}(p^{a_1},\cdots,p^{a_n},p^{a_0-a_1},\cdots,p^{a_0-a_n}) \in T^+_{\mathbf{G}}$, the commutators of U_{γ_p} with t^+, t^- should involve the similitude $\nu_p(\gamma_p)$, as well as the character $\chi_1(\gamma_p^\circ) = p^{-2a_1}$, $\chi_2(\gamma_p^\circ) = p^{2a_n}$ with $\gamma_p^\circ = \mathrm{diag}(p^{a_n},\cdots,p^{a_1})$. Explicitly the commutators are $\nu_{p,D}(\gamma_p) = \nu_p(\gamma_p)\cdot p^{-2a_1} = \nu_p(\gamma_p)\cdot\chi_1(\gamma_p^\circ)$, and $\nu_{p,E}(\gamma_p) = \nu_p(\gamma_p)\cdot p^{-2a_n} = \nu_p(\gamma_p)\cdot\chi_2(\gamma_p^\circ)^{-1}$. □

Corollary 3.10.3.

$$\text{(i)}\quad \nu_{p,D}(\gamma_p)\cdot D_{\kappa\otimes\sigma}U_{\gamma_p} = U_{\gamma_p}D_{\kappa\otimes\sigma}, \qquad E_{\kappa\otimes\sigma}U_{\gamma_p} = \nu_{p,E}(\gamma_p)\cdot U_{\gamma_p}E_{\kappa\otimes\sigma},$$

$$\text{(ii)}\quad \nu_\ell(\gamma_\ell)\cdot D_{\kappa\otimes\sigma}T_{\gamma_\ell} = T_{\gamma_\ell}D_{\kappa\otimes\sigma}, \qquad E_{\kappa\otimes\sigma}T_{\gamma_\ell} = \nu_\ell(\gamma_\ell)\cdot T_{\gamma_\ell}E_{\kappa\otimes\sigma}.$$

In particular, for the compact operator U_p we have

$$p^{2n-1}\cdot D_{\kappa\otimes\sigma}U_p = U_pD_{\kappa\otimes\sigma}, \qquad E_{\kappa\otimes\sigma}U_p = p\cdot U_pE_{\kappa\otimes\sigma}.$$

3.11 The slope decomposition

We consider the slope decomposition of the operator U_p acting on $N^{\dagger,\infty}_{\mathcal{U},w,v,\mathrm{cusp}} := \bigcup_{r\geq 0} N^{\dagger,r}_{\mathcal{U},w,v,\mathrm{cusp}}$. We have seen that each $N^{\dagger,r}_{\mathcal{U},w,v,\mathrm{cusp}}$ is a projective $\mathcal{A}(\mathcal{U})$-Banach module with the action of U_p being compact. Applying the Coleman–Riesz–Serre theory on the spectrum of compact operators as developed in [9], one can define the Fredohlm determinant $P_r(T) = \det\left(1 - TU_p|_{N^{\dagger,r}_{\mathcal{U},w,v,\mathrm{cusp}}}\right)$, which belongs to $\mathcal{A}(\mathcal{U})\{\{T\}\}$, the $\mathcal{A}(\mathcal{U})$-algebra of power series with convergence radius being infinity. Because of the integrality of the operator U_p, all the coefficients of $P_r(T)$ are power bounded, i.e. $P_r(T) \in \mathcal{A}(\mathcal{U})^\circ\{\{T\}\}$.

Proposition 3.11.1. *The sequence*

$$0 \longrightarrow N^{\dagger, r-1}_{\kappa, w, v, \mathrm{cusp}} \longrightarrow N^{\dagger, r}_{\kappa, w, v\, \mathrm{cusp}} \xrightarrow{\frac{1}{r!} E^r_{\kappa, w}} N^{\dagger, 0}_{\kappa \otimes \mathrm{Sym}^r \tau^\vee, w, v\, \mathrm{cusp}} \longrightarrow 0 \tag{3.27}$$

is exact.

Proof. Let $\eta : \mathfrak{X}_1(p^m)(v) \to \mathfrak{X}^\star(v)$ be as in §3.5. Combining the vanishing result (3.9) there and (3.12), we get the exact sequence of small formal Banach sheaves over $\mathfrak{X}^\star(v)$

$$0 \longrightarrow \eta_* \tilde{\mathfrak{V}}^{\dagger, r-1}_{\kappa, w}(-C) \longrightarrow \eta_* \tilde{\mathfrak{V}}^{\dagger, r}_{\kappa, w}(-C) \xrightarrow{\frac{1}{r!} E^r_{\kappa, w}} \eta_* \tilde{\mathfrak{V}}^{\dagger, 0}_{\kappa \otimes \mathrm{Sym}^r \tau^\vee, w}(-C) \longrightarrow 0.$$

Due to the smallness we know that the augmented Cěch complexes of the above sheaves are exact after inverting p [1, Theorem A.1.2.2]. Thus we deduce the exactness of the sequence

$$\begin{aligned} 0 \longrightarrow H^0\big(\mathcal{X}_1(p^m)(v), \tilde{\mathcal{V}}^{\dagger, r-1}_{\kappa, w}(-C)\big) \longrightarrow & H^0\big(\mathcal{X}_1(p^m)(v), \tilde{\mathcal{V}}^{\dagger, r}_{\kappa, w}(-C)\big) \\ & \xrightarrow{\frac{1}{r!} E^r_{\kappa, w}} H^0\big(\mathcal{X}_1(p^m)(v), \tilde{\mathcal{V}}^{\dagger, 0}_{\kappa \otimes \mathrm{Sym}^r \tau^\vee, w}(-C)\big) \longrightarrow 0. \end{aligned}$$

The proposition follows by taking the invariants of $I_n(\mathbb{Z}/p^m\mathbb{Z})$. □

Combining (3.27) and the equality

$$E^r_{\kappa, w} U_p = p^r U_p E^r_{\kappa, w},$$

we see that there exist $C_r(T) \in \mathcal{A}(\mathcal{U})^\circ\{\{T\}\}$ such that

$$P_r(T) = P_{r-1}(T) C_r(p^r T).$$

Therefore we can define $P_\infty(T) \in \mathcal{A}(\mathcal{U})^\circ\{\{T\}\}$ as the limit

$$P_\infty(T) := \lim_{r \to \infty} P_r(T).$$

Given $Q(T) \in \mathcal{A}(\mathcal{U})[T]$ dividing $P_\infty(T)$, one checks by definition [12, p.434-435] that for sufficiently large r, the resultant $\mathrm{Res}\big(Q(T), P_\infty(T)/P_r(T)\big)$ is a unit in $\mathcal{A}(\mathcal{U})$, so $Q(T)$ divides $P_r(T)$.

Now take $Q(T) \in \mathcal{A}(\mathcal{U})[T]$ whose constant term is 1 and the leading coefficient is a unit of $\mathcal{A}(U)$, such that $P_\infty(T) = Q(T)S(T)$ with $S(T)$ relatively prime to $Q(T)$. We call such a $Q(T)$ admissible for $N^{\dagger,\infty}_{\mathcal{U},w,v,\mathrm{cusp}}$. If $r >> 0$ we can apply [9, Thm.3.3] to get the slope decomposition

$$N^{\dagger,r}_{\mathcal{U},w,v,\mathrm{cusp}} = N^r_{Q,\mathcal{U},\mathrm{cusp}} \oplus F^r_{Q,\mathcal{U}}, \tag{3.28}$$

satisfying

(i) the direct summand $N^r_{Q,\mathcal{U},\mathrm{cusp}}$ is a projective $\mathcal{A}(\mathcal{U})$-Banach module of finite rank, and also we have $\det\left(1 - TU_p|_{N^r_{Q,\mathcal{U},\mathrm{cusp}}}\right) = Q(T)$,

(ii) the operator $Q^*(U_p)$ is invertible on $F^r_{Q,\mathcal{U}}$, where $Q^*(T) = T^{\deg Q}Q(1/T)$.

It is not difficult to see that the module $N^r_{Q,\mathcal{U},\mathrm{cusp}}$ is independent of r, as long as r is large enough, and we denote it by $N_{Q,\mathcal{U},\mathrm{cusp}}$. The subscripts w, v are omitted since all eigenvalues of U_p acting on $N_{Q,\mathcal{U},\mathrm{cusp}}$ are nonzero, and it follows from the property of increasing analyticity and overconvergence of the operator U_p, that the module does not depend on w, v. Elements in the finite rank projective $\mathcal{A}(\mathcal{U})$-Banach module $N_{Q,\mathcal{U},\mathrm{cusp}}$ are Q-finite slope families of cuspidal nearly overconvergnent forms, and we have the Q-finite slope projection

$$e_{Q,\mathcal{U}} : N^{\dagger,\infty}_{\mathcal{U},w,v,\mathrm{cusp}} \longrightarrow N_{Q,\mathcal{U},\mathrm{cusp}}.$$

3.12 p-adic splitting of $\mathcal{V}^{\dagger,r}_{\kappa,w}$ over ordinary locus

Let Y, X, $\mathfrak{X}$, $\mathfrak{X}(v)$, $\mathfrak{X}_{\mathrm{Iw}}(v)$, $\mathcal{X} = X_{\mathrm{rig}}$, $\mathcal{X}(v)$, $\mathcal{X}_{\mathrm{Iw}}(v)$ be defined as in §3.3. Over X (resp. Y) there is the semi-abelian scheme $\mathbf{p} : \mathcal{G} \to X$ (resp. the universal abelian scheme $\mathbf{p} : \mathcal{A} \to Y$). Denote by $\mathbf{p} : G_0 \to X_0$ (resp. $\mathbf{p} : A_0 \to Y_0$) the reduction modulo ϖ. Set $X_{0,\mathrm{ord}}$, $Y_{0,\mathrm{ord}}$ to be the ordinary locus of X_0, Y_0. Fix a lift $\varsigma : \mathcal{O}_K \to \mathcal{O}_K$ of the Frobenius of the residue field $k = \mathcal{O}_K/\varpi$. Let

$F : X_{0,\mathrm{ord}} \to X_{0,\mathrm{ord}}$ be the absolute Frobenius and consider the commutative diagram

$$\begin{array}{ccccc} X_{0,\mathrm{ord}} & \hookrightarrow & \mathfrak{X}(0) & \longrightarrow & \mathrm{Spf}(\mathcal{O}_K) \\ \downarrow F & & \downarrow u & & \downarrow \varsigma \\ X_{0,\mathrm{ord}} & \hookrightarrow & \mathfrak{X}(0) & \longrightarrow & \mathrm{Spf}(\mathcal{O}_K) \end{array},$$

where u is the lift of the absolute Frobenius defined by sending an ordinary semi-abelian scheme $\mathcal{G}$ to its quotient by the $\mathcal{G}[p]^o$, the connected part of $\mathcal{G}[p]$, and composing with the base change by ς. The isogeny $\mathcal{A} \to \mathcal{A}/\mathcal{A}[p]^o$ induces, by pullback, a morphism

$$\Phi : u^*\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/\mathfrak{Y}(0))^{\mathrm{can}} \longrightarrow \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/\mathfrak{Y}(0))^{\mathrm{can}}$$

of formal coherent sheaves over $\mathfrak{X}(0)$. By [37, Theorem 4.1], the locally free formal sheaf $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/\mathfrak{Y}(0))^{\mathrm{can}}$ of rank $2n$ has a unique Φ-stable locally free formal sub-sheaf $\mathfrak{U}_{\mathcal{H}}$ of rank n, over which Φ restricts to an isomorphism. This $\mathfrak{U}_{\mathcal{H}}$ gives rise to a splitting, called the unit-root splitting, of the Hodge filtration:

$$\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/\mathfrak{Y}(0))^{\mathrm{can}} = \omega(\mathcal{G}/\mathfrak{X}(0)) \oplus \mathfrak{U}_{\mathcal{H}}.$$

Moreover $\mathfrak{U}_{\mathcal{H}}$ is stable under the Gauss–Manin connection. The unit-root splitting pulls back to $\mathfrak{X}_{\mathrm{Iw}}(p)(0)$, and induces a projection $\mathfrak{J} \to \mathcal{O}_{\mathfrak{X}_{\mathrm{Iw}}(p)(0)}$. Taking the generic fibre we get the projection

$$H^0(\mathcal{X}_{\mathrm{Iw}}(0), \mathcal{V}^{\dagger,r}_{\kappa,w}) = H^0(\mathcal{X}_{\mathrm{Iw}}(0), \omega^{\dagger}_{\kappa,w} \otimes \mathrm{Sym}^r \mathcal{J}) \longrightarrow H^0(\mathcal{X}_{\mathrm{Iw}}(0), \omega^{\dagger}_{\kappa,w}). \tag{3.29}$$

The Igusa tower $\mathfrak{S}(p^\infty)$ defined in §3.9.5 is étale over $\mathfrak{X}_{\mathrm{Iw}}(0)$ with the group $T_n(\mathbb{Z}_p)$ acting on it. The space of p-adic forms of weight κ consists of functions on $\mathfrak{S}(p^\infty)$ that are κ'-invariant under the action of $T_n(\mathbb{Z}_p)$, i.e.

$$M^{p\text{-adic}}_{\kappa} = H^0(\mathfrak{S}(p^\infty), \mathcal{O}_{\mathfrak{S}(p^\infty)})[\kappa'].$$

Composing (3.25) with $r = 0$ and (3.29) we obtain the map

$$\xi_p : N^{\dagger,r}_{\kappa,w,v} \longrightarrow M^{p\text{-adic}}_{\kappa}[1/p],$$

sending nearly overconvergent forms to p-adic forms.

Let $\kappa \in \mathcal{W}(K)$ be an arithmetic weight with algebraic part κ_{alg} and finite part κ_{f}. Denote by $N^r_\kappa(\Gamma_1(N,p^m),K)$ the space of weight κ_{alg}, degree r nearly holomorphic Siegel modular forms of level $\Gamma_1(N,p^m)$ with nebentype κ_{f} at p.

Proposition 3.12.1. *The following restriction of ξ_p to classical nearly holomorphic Siegel modular forms*

$$\xi_{p,\mathrm{cl}} : N^r_\kappa\left(\Gamma_1(N,p^m),K\right) \hookrightarrow N^{\dagger,r}_{\kappa,w,v} \xrightarrow{\xi_p} M^{p\text{-}adic}_\kappa[1/p]$$

is injective.

Proof. Take $f \in \ker \xi_{p,\mathrm{cl}}$. Under the map $\phi : N^r_\kappa(\Gamma_1(N,p^m),K)\otimes_K \mathbb{C} \to N^r_\kappa(\mathbb{H}_n,\Gamma_1(N,p^m))$ defined as (2.13), the image $\phi(f)$ of f is a polynomial in $(\operatorname{Im} z)^{-1}$ with coefficients being holomorphic maps from the upper half space $\mathbb{H}_n$ to $W_{\kappa_{\mathrm{alg}}}(\mathbb{C})$. By definition ϕ is equivalent to the projection from $\mathcal{V}^r_\kappa$ to $\mathcal{V}^0_\kappa$ through the C^∞ splitting, given by the Hodge decomposition of $\mathcal{H}^1_{\mathrm{dR}}(A_{\mathbb{H}_n/}\mathbb{H}_n)\otimes C^\infty(\mathbb{H}_n,\mathbb{C})$. Let $S \subset \mathbb{H}_n$ be the subset consisting of ordinary CM points. It is analytically dense inside $\mathbb{H}_n$. At each point of S, the unit-root splitting agrees with the C^∞ splitting [39, Lemma 5.1.27]. Therefore $f \in \ker \xi_{p,\mathrm{cl}}$ implies that $\phi(f) = 0$ and f=0. □

In general it is conjectured that for all w-analytic weight κ, the map ξ_p is injective. The injectivity is proved in the $n = 1$ case.

Proposition 3.12.2. *([63, Proposition 3.2.4]) When $n = 1$, $\mathbf{G} = \mathrm{GL}(2)_{/\mathbb{Q}}$, the map*

$$\xi_p : N^{\dagger,r}_{\kappa,w,v} \to M^{p\text{-}adic}_\kappa[1/p]$$

is injective.

Below we replicate the proof given in [63] with more details.

Proof. Suppose that there exists a nonzero $f \in \ker \xi_p$. In the GL(2) case, we can identify $\mathcal{Y}_{\mathrm{Iw}}(v)$ with the open subset $\mathcal{Y}(v)$ of $\mathcal{Y}$. Let $\{\mathcal{U}_i\}$ be an admissible cover of $\mathcal{Y}(v)$, such that each $\mathcal{U}_i$

is an affinoid subdomain, and there is a basis (α_i, β_i) of $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/\mathcal{U}_i)$ giving rise to a section of $\mathcal{T}^{\times}_{\mathcal{H},w}(v) \to \mathcal{Y}(v)$ over $\mathcal{U}_i$. Denote by $\mathcal{U}_{\mathcal{H}}$ the rigid fibre of the formal invertible sheaf $\mathfrak{U}_{\mathcal{H}}$. After a refinement of $\{\mathcal{U}_i\}$ if necessary, we can assume that over $\mathcal{U}_{i,\mathrm{ord}} = \mathcal{U}_i \cap \mathcal{Y}(0)$ there is a section β'_i of $\mathcal{U}_{\mathcal{H}}$ such that (α_i, β'_i) gives a section of $\mathcal{T}^{\times}_{\mathcal{H},w}(v) \to \mathcal{Y}(v)$ over $\mathcal{U}_{i,\mathrm{ord}}$. Then there exists $\lambda_i \in \mathcal{A}(\mathcal{U}_{i,\mathrm{ord}})$ such that $(\alpha_i, \beta'_i) = (\alpha_i, \beta_i)\begin{pmatrix} 1 & \lambda_i \\ 0 & 1 \end{pmatrix}$. Evaluating f at $(\mathcal{A}/\mathcal{U}_i, (\alpha_i, \beta_i))$, we get $P_{f,i}(Y) \in \mathcal{A}(\mathcal{U}_i)[Y]_{\leq r}$. Then $\xi_p(f) = 0$ implies that $P_{f,i}(\lambda_i) = 0$, i.e. λ_i is algebraic over the function field of $\mathcal{U}_i$ for all i. Applying [7, Theorem 1] we know that there is $0 < v' < v$ such that $\lambda_i \in \mathcal{A}(\mathcal{U}_i \cap \mathcal{Y}(v'))$ for all i. It follows that there is an invertible subsheaf $\mathcal{U}'_{\mathcal{H}} \subset \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/\mathcal{Y}(v'))$ extending $\mathcal{U}_{\mathcal{H}} \subset \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/\mathcal{Y}(0))$. By the rigidity of analytic functions, one deduces that $\mathcal{U}'_{\mathcal{H}}$ is stable under the Gauss–Manin connection ∇. Now consider the convergent F-isocrystal $R^1\mathbf{p}_{\mathrm{rig},*}(\mathcal{A}/\mathcal{Y})$ over $Y_0/(\mathcal{O}_K, \varsigma)$, and we use $\mathcal{E}$ to denote its restriction to $Y_{0,\mathrm{ord}}/(\mathcal{O}_K, \varsigma)$. By definition $\mathcal{E}$ is an overconvergent F-isocrystal over $(Y_{0,\mathrm{ord}}, Y_0)/(\mathcal{O}_K, \varsigma)$. The inclusions $Y_0 \hookrightarrow \mathfrak{Y}(v')$, $Y_0 \hookrightarrow \mathfrak{Y}(v'/p)$ are both closed embeddings and fit into the following commutative diagram

$$\begin{array}{ccccccc} Y_{0,\mathrm{ord}} & \hookrightarrow & Y_0 & \hookrightarrow & \mathfrak{Y}(v'/p) & \longrightarrow & \mathrm{Spf}(\mathcal{O}_K) \\ \downarrow F & & \downarrow F & & \downarrow u & & \downarrow \sigma \\ Y_{0,\mathrm{ord}} & \hookrightarrow & Y_0 & \hookrightarrow & \mathfrak{Y}(v') & \longrightarrow & \mathrm{Spf}(\mathcal{O}_K) \end{array},$$

where u is the map defined by sending an abelian scheme A to A/H_1, its quotient by the level 1 canonical subgroup, and composing with the base change by ς. The isogeny $A \to A/H_1$ induces, by pullback, a morphism

$$\Phi : u^*\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/\mathfrak{Y}(pv')) \longrightarrow \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/\mathfrak{Y}(v')).$$

The triple $(\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/\mathcal{Y}(v')), \nabla, \Phi)$ is a $\mathfrak{Y}(v')$ realization of the overconvergent F-isocrystal $\mathcal{E}$ (cf. [4, §2.3.2]). The unit-root splitting $\mathfrak{U}_{\mathcal{H}}$ corresponds to a convergent sub-F-isocrystal $\mathcal{E}'$ of $\mathcal{E}$ over $Y_0/(\mathcal{O}_K, \varsigma)$. The extension $\mathcal{U}'_{\mathcal{H}}$ of $\mathcal{U}_{\mathcal{H}}$ over $\mathcal{Y}(v')$, stable under ∇, makes $\mathcal{E}'$ an overconvergent isocrystal over $(Y_{0,\mathrm{ord}}, Y_0)/\mathcal{O}_K$. By the discussion at the end of [4, §2.3.9], $\mathcal{E}'$ is actually a unit-root overconvergent F-isocrystal over $(Y_{0,\mathrm{ord}}, Y_0)/(\mathcal{O}_K, \varsigma)$. Then [14, Theorem 4.12] (cf. also Remark

4.15 there) says that the representation $\rho_{\mathcal{E}'} : \pi_1(Y_{\mathrm{ord},0}) \to \mathbb{Z}_p^\times$ associated to $\mathcal{E}'$ has finite local monodromy. However according to a theorem of Igusa [38, Theorem 4.3], the image of $\rho_{\mathcal{E}'}$ of the inertia group at each supersingular point of Y_0 surjects onto $\mathbb{Z}_p^\times$. □

3.13 Polynomial q-expansions and p-adic q-expansions

The embedding (3.24) induces, by restricition, the injective map

$$N_{\kappa,w,v}^{\dagger,r} \longrightarrow H^0(\mathfrak{S}(p^\infty), \mathrm{Sym}^r \mathfrak{J})[1/p]. \tag{3.30}$$

For each geometrically connected component $\mathfrak{S}(p^\infty)^\circ$, with the Mumford object constructed in §2.7, one can define a map

$$\iota : \mathrm{Spf}(\mathcal{O}_K[1/t][[N^{-1}S_{L_n,\geq 0}]]) \longrightarrow \mathfrak{S}(p^\infty)^\circ.$$

The canonical basis $(\omega_{\mathrm{can}}, \delta_{\mathrm{can}})$ induces an isomorphism $\iota^* \mathrm{Sym}^r \mathfrak{J} \simeq \mathcal{O}_K[1/t][[N^{-1}S_{L_n,\geq 0}]][\underline{Y}]_{\leq r}$, which, together with (3.30), defines a p-adic polymonial q-expansion map

$$\varepsilon_{\iota,q,\mathrm{poly}} : N_{\kappa,w,v}^{\dagger,r} \longrightarrow \mathcal{O}_K[[N^{-1}S_{L_n,\geq 0}]][\underline{Y}]_{\leq r}[1/p].$$

Remark 3.13.1. Note that the image of $\varepsilon_{\iota,q,\mathrm{poly}}$ are polynomials in $\underline{Y}$ with scalar coefficients, while the polynomial q-expansion $f(q, \underline{Y})$, defined as (2.19) for a classical nearly holomorphic form f of a classical weight κ, is a polynomial in $\underline{Y}$ with coefficients inside the representation W_κ. To obtain the polynomial q-expansion here from the polymonial q-expansion in (2.19), one simply applies the canonical map $\mathfrak{e}_{\mathrm{can}} : W_\kappa \to \mathbb{A}^1$, defined as the evaluation at the identity matrix in $\mathrm{GL}(n)$.

If c is the number of geometrically connected components of $\mathfrak{Y}_1(p^\infty)(0)$, we can choose $\iota_1, \cdots, \iota_c$ such that ι_j maps $\mathrm{Mum}_N(q)$ to the j-th component . We define the polynomial q-expansion map $\varepsilon_{q,\mathrm{poly}}$ as $\bigoplus_{j=1}^c \varepsilon_{\iota_j,q,\mathrm{poly}}$. Then it follows from the irreduciblity of the Igusa tower $\mathfrak{S}(p^\infty)$ [33, Corollary 8.17], that the map $\varepsilon_{q,\mathrm{poly}}$ is injective. Similarly we can define the polynomial q-expansion map for families of nearly overconvergent forms which is again injective.

Proposition 3.13.2. *The polynomial q-expansion maps*

$$\varepsilon_{q,\mathrm{poly}}: N^{\dagger,\infty}_{\kappa,w,v} \longrightarrow \left(\mathcal{O}_K[[N^{-1}S_{L_n,\geq 0}]][\underline{Y}][1/p]\right)^{\oplus c},$$
$$\varepsilon_{q,\mathrm{poly}}: N^{\dagger,\infty}_{\mathcal{U},w,v} \longrightarrow \left(\mathcal{A}(\mathcal{U})^{\circ}[[N^{-1}S_{L_n,\geq 0}]][\underline{Y}][1/p]\right)^{\oplus c}$$

are injective.

In §3.12 we defined a map $\xi_p : N^{\dagger,r}_{\kappa,w,v} \longrightarrow M^{p\text{-adic}}_{\kappa}[1/p]$ using the unit root splitting. Composing ξ_p with the q-expansion map for p-adic forms, we get the map

$$\varepsilon_{q,p\text{-adic}}: N^{\dagger,\infty}_{\kappa,w,v} \longrightarrow M^{p\text{-adic}}_{\kappa}[1/p] \longrightarrow \left(\mathcal{O}_K[[N^{-1}S_{L_n,\geq 0}]][1/p]\right)^{\oplus c}, \tag{3.31}$$

and call it the p-adic q-expansion of nearly overconvergent forms. Recall that in the construction of $\mathrm{Mum}_N(q)$ we defined a basis $(\omega_{\mathrm{can}}, \delta_{\mathrm{can}})$. The locally free sheaf spanned by the δ_{can} is exactly the unit-root part. Therefore $\varepsilon_{q,p\text{-adic}}$ is nothing but $\varepsilon_{q,\mathrm{poly}}|_{\underline{Y}=0}$. In the case when the map ξ_p is injective, the q-expansion $\varepsilon_{q,p\text{-adic}}$ will also be injective. For families we define the p-adic q-expansion simply as $\varepsilon_{q,\mathrm{poly}}|_{\underline{Y}=0}$.

Proposition 3.13.3. *Suppose that the subdomian $\mathcal{U} \subset \mathcal{W}$ is a closed ball centered at a classical point and $Q(T) \in \mathcal{A}(\mathcal{U})[T]$ is admissible for $N^{\dagger,\infty}_{\mathcal{U},w,v,\mathrm{cusp}}$. Then after being restricted to $N_{Q,\mathcal{U},\mathrm{cusp}}$, the p-adic q-expansion map*

$$\varepsilon_{q,p\text{-adic}}: N_{Q,\mathcal{U},\mathrm{cusp}} \longrightarrow \left(\mathcal{A}(\mathcal{U})^{\circ}[[N^{-1}S_{L_n,\geq 0}]][1/p]\right)^{\oplus c}$$

is injective.

Proof. Take $F \in N_{Q,\mathcal{U},\mathrm{cusp}}$ with $\varepsilon_{q,p\text{-adic}}(F) = 0$. Then Proposition 3.12.1 implies that for each arithmetic weight $\kappa \in \mathcal{U}(\overline{\mathbb{Q}}_p)$ such that the specialization F_κ is a classical nearly holomorphic form, we have $F_\kappa = 0$. We reduce to show that the subset of $\mathcal{U}(\overline{\mathbb{Q}}_p)$ consisting of points κ with F_κ being classical is Zariski dense inside $\mathcal{U}$. By the construction of $N_{Q,\mathcal{U},\mathrm{cusp}}$, we know that $F \in N^{r}_{Q,\mathcal{U},\mathrm{cusp}}$

for some $r \in \mathbb{N}$. Then F can be written as (Corollary 3.7.5)

$$\eta F = F_0 + \theta D F_1 + \cdots + \theta^r D^r F_r$$

with $F_i \in N^{\dagger,0}_{\mathcal{U}\otimes \mathrm{Sym}^i \tau^\vee, w, v}$ and $\eta \in K[\mathrm{Log}_1, \cdots, \mathrm{Log}_n]$ nonzero. By Corollary 3.10.3 there is a bound, depending on Q and r, on the slopes of $F_0, F_1, \cdots, F_n$. Therefore if a classical weight $\kappa \in \mathcal{U}(\overline{\mathbb{Q}}_p)$ is outside the zeroes of η, and sufficiently regular with respect to that bound on slopes, then the classicity of $F_{0,\kappa}, \cdots, F_{n,\kappa}$ can be deduced from [1, Proposition 7.3.1] and [52], from which the classicity of F_κ follows. Since $\mathcal{U}$ is a closed ball centered at a classical point, such sufficiently regular classical points outside the zeroes of η are Zariski dense in $\mathcal{U}$. □

3.14 Families by q-expansions

Keep the assumption on $\mathcal{U}, Q$ as in Proposition 3.13.3. Let $\Sigma \subset \mathcal{U}(\overline{\mathbb{Q}}_p)$ be a Zariski dense subset consisting of arithmetic points. Define $N^{\Sigma,\mathrm{poly}}_{Q,\mathcal{U},\mathrm{cusp}}$ (resp. $N^{\Sigma}_{Q,\mathcal{U},\mathrm{cusp}}$) to be the sub-$\mathcal{A}(\mathcal{U})$-module of $\left(\mathcal{A}^\circ(\mathcal{U})[[N^{-1}S_{L,\geq 0}]][\underline{Y}][1/p]\right)^{\oplus c}$ (resp. $\left(\mathcal{A}^\circ(\mathcal{U})[[N^{-1}S_{L,\geq 0}]][1/p]\right)^{\oplus c}$) consisting of those elements whose specialization at almost all $\kappa \in \Sigma$ is contained in $\varepsilon_{q,\mathrm{poly}}(N_{Q_\kappa,\kappa,\mathrm{cusp}})$ (resp. $\varepsilon_{q,p\text{-adic}}(N_{Q_\kappa,\kappa,\mathrm{cusp}})$).

Proposition 3.14.1. *With $\mathcal{U}, Q$ as in Proposition 3.13.3, the polynomial q-expansion map induces an isomorphism from $N_{Q,\mathcal{U},\mathrm{cusp}}$ to $N^{\Sigma,\mathrm{poly}}_{Q,\mathcal{U},\mathrm{cusp}}$.*

Proof. We follow the argument of [65, Theorem 1.2.2], [31, Theorem 7.3.1]. Abbreviate $\mathcal{A}(\mathcal{U})$, $N_{Q,\mathcal{U}\,\mathrm{cusp}}$, $N_{Q_\kappa,\kappa,\mathrm{cusp}}$, $N^{\Sigma,\mathrm{poly}}_{Q,\mathcal{U},\mathrm{cusp}}$, as $\mathcal{A}$, N, N_κ, $N^{\Sigma,\mathrm{poly}}$. Let I be the set consisting of monomials $q^{\beta_i}\prod Y_{jk}^{a_{jk}}$, where $a_{jk} \in \mathbb{N}$, $1 \leq j \leq k \leq n$, and $\beta_i \in N^{-1}S_{L_n,\geq 0}$ with the subscript $1 \leq i \leq c$ meaning the i-th connected component. By taking coefficients there is a natural embedding of $\left(\mathcal{A}(\mathcal{U})^\circ[[N^{-1}S_{L_n,\geq 0}]][\underline{Y}][1/p]\right)^{\oplus c}$ into the direct product $\mathcal{A}^I$. Denote by $K(\mathcal{A})$ the fraction field of $\mathcal{A}$. The $\mathcal{A}$-module N is finite projective. Let $d = \mathrm{rank}_{\mathcal{A}}(N) = \dim_{K(\mathcal{A})}(N \otimes K(\mathcal{A})) < \infty$, and pick $F_1, \cdots, F_d \in N$ such that they span $N \otimes K(\mathcal{A})$ over $K(\mathcal{A})$. Write their images inside $\mathcal{A}^I$ under the polynomial q-expansion map as $\left(a(F_j, i)\right)_{i\in I}$, $1 \leq j \leq d$. Thanks to the injectivity of the map $\varepsilon_{q,\mathrm{poly}}$, we can choose $i_1, \cdots, i_d$ such that $D = \det\left(a(F_j, i_t)\right)_{1\leq j,t\leq d} \neq 0$. We claim that $DN^{\Sigma,\mathrm{poly}} \subset \varepsilon_{q,\mathrm{poly}}(N)$. Otherwise there exists $G = \left(a(G,i)\right)_{i\in I} \in DN^{\Sigma,\mathrm{poly}} \backslash \varepsilon_{q,\mathrm{poly}}(N)$. Subtracting from G

a linear combination of the $\varepsilon_{q,\text{poly}}(F_j)$'s, we get a nonzero $G' \in N^{\Sigma,\text{poly}}$ with $a(G,i_t)=0$ for all $1 \le t \le d$. Since Σ is Zariski dense there exists some $\kappa \in \Sigma$, such that specializing at κ, the vectors $\varepsilon_{q,\text{poly}}(F_1)_\kappa, \cdots, \varepsilon_{q,\text{poly}}(F_d)_\kappa$ and G'_κ are $\overline{\mathbb{Q}}_p$-linearly independent and $G'_\kappa = \varepsilon_{q,\text{poly}}(f)$ for some $f \in N_\kappa$. The injectivity of $\varepsilon_{q,\text{poly}}$ shows that $F_{1,\kappa}, \cdots, F_{d,\kappa}, f$ are linearly independent inside N_κ which is impossible. Therefore $N^{\Sigma,\text{poly}} = \varepsilon_{q,\text{poly}}(N) \otimes K(\mathcal{A}) \cap \mathcal{A}^I$. We also deduce that $N^{\Sigma,\text{poly}}$ is a finitely generated $\mathcal{A}$-module because $\mathcal{A}$ is noetherian. In fact $\mathcal{A}$ is a noetherian UFD and a Jacobson ring [8, §5.2.6 Theorem 1, 3]. Now take an arbitrary $G'' \in \varepsilon_{q,\text{poly}}(N) \otimes K(\mathcal{A}) \cap \mathcal{A}^I$, we want to prove that G'' actually lies inside $\varepsilon_{q,\text{poly}}(N)$. Since $\mathcal{A}$ is a UFD we can take some $\eta \in \mathcal{A}$ such that $\eta G'' \in \varepsilon_{q,\text{poly}}(N)$ and for any η' strictly divides η, we have $\eta' G'' \notin \varepsilon_{q,\text{poly}}(N)$. Take $F \in N$ such that $\eta G'' = \varepsilon_{q,\text{poly}}(F)$. If $\mathfrak{m}$ is a maximal ideal of $\mathcal{A}$ containing η, then the polynomial q-expansion $\varepsilon_{q,\text{poly}}(F_{\kappa_\mathfrak{m}}) = \eta(\kappa_\mathfrak{m}) G''_{\kappa_\mathfrak{m}} = 0$, which implies that $F_{\kappa_\mathfrak{m}} = 0$ and $F \in \mathfrak{m}N$ by Proposition 3.5.1. This shows that $F \in \bigcap_{\eta \in \mathfrak{m}} \mathfrak{m}N$. The $\mathcal{A}$-module N is finite projective so there exists $a_1, \cdots, a_l \in \mathcal{A}$ such that each localization N_{a_i} is free of finite rank over A_{a_i} which is still a noetherian UFD [47, Lemma (19.B)] and a Jacobson ring [59, Tag 00G6]. Let $\eta_1, \cdots, \eta_b$ be all the prime factors of η. Each $\eta_j A_{a_i}$ is a prime ideal that is the intersection of all maximal ideals in A_{a_i} containing η_j. It follows that $\sqrt{\eta} A_{a_i} = \bigcap_j \eta_j A_{a_i} = \bigcap_{\eta \in \mathfrak{m}, \mathfrak{m} \in \text{Max}(A_{a_i})} \mathfrak{m} A_{a_i}$ and $\sqrt{\eta} N_{a_i} = \bigcap_{\eta \in \mathfrak{m}, \mathfrak{m} \in \text{Max}(A_{a_i})} \mathfrak{m} N_{a_i}$. Then from $F \in \bigcap_{\eta \in \mathfrak{m}} \mathfrak{m}N$, we deduce that $F \in \sqrt{\eta} N_{a_i}$ for all i, and hence $F \in \sqrt{\eta} N$. By our choice of η this implies that η is a unit in $\mathcal{A}$. □

If we apply the same argument to $N^\Sigma_{Q,\mathcal{U},\text{cusp}}$, due to the lack of injectivity of the map $\varepsilon_{q,p\text{-adic}}$ at all points in $\mathcal{U}$, we only get a weaker result.

Proposition 3.14.2. *With $\mathcal{U}, Q$ as in Proposition 3.13.3, there exists a nonzero $\eta \in \mathcal{A}(\mathcal{U})$ such that $\eta N^\Sigma_{Q,\mathcal{U},\text{cusp}}$ belongs to $\varepsilon_{q,p\text{-adic}}(N_{Q,\mathcal{U},\text{cusp}})$.*

Part II

p-adic Standard L-Functions for Symplectic Groups

Chapter 4

Siegel Eisenstein Series and the local zeta integrals away form p

4.1 Preliminaries

4.1.1 Some notation

In part I, we used κ to denote a continuous character from $T_n(\mathbb{Z}_p)$ to $\overline{\mathbb{Q}}_p^\times$. From now on we denote such a character by $\underline{\tau}$, and save the symbol κ for a continuous character from $\mathbb{Z}_p^\times$ to $\overline{\mathbb{Q}}_p^\times$.

An element of $\mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times \times T_n(\mathbb{Z}_p), \overline{\mathbb{Q}}_p^\times)$ is called an arithmetic point, if it can be written as the product of an algebraic character and a finite order character. For an arithmetic point $(\kappa, \underline{\tau})$ of the weight space $\mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times \times T_n(\mathbb{Z}_p), \overline{\mathbb{Q}}_p^\times)$, with algebraic part $(\kappa_{\mathrm{alg}}, \underline{\tau}_{\mathrm{alg}})$ and finite order part $(\kappa_{\mathrm{f}}, \underline{\tau}_{\mathrm{f}})$, we write $\kappa_{\mathrm{alg}} = k$, $\underline{\tau}_{\mathrm{alg}} = \underline{t} = (t_1, \cdots, t_n)$ with $k, t_1, \cdots, t_n$ being integers, and $\kappa_{\mathrm{f}} = \chi$, $\underline{\tau}_{\mathrm{f}} = \underline{\psi} = (\psi_1, \cdots, \psi_n)$ with $\chi, \psi_1, \cdots, \psi_n$ being characters of $\mathbb{Z}_p^\times$ of finite order. We call an arithmetic point $(\kappa, \underline{\tau})$ admissible if $t_1 \geq \cdots \geq t_n \geq k \geq n+1$.

Since we are going to work with both $G = \mathrm{Sp}(2n)$ and $H = \mathrm{Sp}(4n)$, rather than τ used before, we use τ_n (resp. τ_{2n}) to denote the symmetric square of the standard representation of $\mathrm{GL}(n)$ (resp. $\mathrm{GL}(2n)$).

4.1.2 Adelic $\mathbb{U}_p$-operators and their normalizations

In part II we consider nearly holomorphic Siegel modular forms of level $\Gamma = \Gamma_1(N, p^m)$. It is easy to see that, after inverting p, all constructions in Chapter 2 still works.

We will also mostly work with adeles instead of using the upper half space. In §2.6 we have already seen, given a vector $w^* \in W^*_\sigma(\mathbb{C})$, there is the map

$$\varphi_G(\cdot, w^*) : C^\infty_\sigma(\mathbb{H}_n, \Gamma) \to C^\infty(\Gamma\backslash G(\mathbb{R})), \tag{4.1}$$

relating smooth maps from the Siegel upper half space to the vector space $W_\sigma(\mathbb{C})$, which satisfy certain translation properties for the action of a congruence subgroup Γ, to smooth functions on the real Lie group $G(\mathbb{R})$ that are invariant under the left translation by Γ. Moreover, for each $f \in C^\infty_\sigma(\mathbb{H}_n, \Gamma)$, the subspace $V_f = \{\varphi_G(f, w^*) : w^* \in W^*_\sigma(\mathbb{C})\}$ of $C^\infty(\Gamma\backslash G(\mathbb{R}))$ is closed under the action of the maximal compact subgroup $K_{G,\infty}$ by right translation, and is isomorphic to $W_\sigma(\mathbb{C})$ as a $K_{G,\infty}$-representation.

Let $\widehat{\Gamma}$ be the completion of Γ inside $G(\mathbb{A}_\mathrm{f})$. The strong approximation implies that

$$G(\mathbb{Q})\backslash G(\mathbb{A})/\widehat{\Gamma} \cong \Gamma\backslash G(\mathbb{R}).$$

Let $\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A})/\widehat{\Gamma})$ be the space of automorphic forms on $G(\mathbb{A})$ that are invariant under the right translation of $\widehat{\Gamma}$. For $\underline{t} \in X(T_n)_+$ we use $\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A})/\widehat{\Gamma})_{\underline{t}}$ to denote its $\underline{t}$-isotypic part as a $K_{G,\infty}$-representation. The composition of (2.13) with (4.1) (taking $w^* = \mathfrak{e}_{\mathrm{can}}$) gives the map

$$\varphi_G(\cdot, \mathfrak{e}_{\mathrm{can}}) : H^0(X_{G,\Gamma}, \mathcal{V}^r_{\underline{t}}) \otimes_{\mathbb{Q}(\zeta_N)} \mathbb{C} \longrightarrow N^r_{\underline{t}}(\mathbb{H}_n, \Gamma) \overset{\varphi_G(\cdot,\mathfrak{e}_{\mathrm{can}})}{\longrightarrow} \mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A})/\widehat{\Gamma})_{\underline{t}}. \tag{4.2}$$

For nearly holomorphic form of weight $\underline{t}$, we will regard it as an element of both $H^0(X_{G,\Gamma}, \mathcal{V}^r_{\underline{t}})$ and $\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A})/\widehat{\Gamma})_{\underline{t}}$, and the identification will always be via the map $\varphi_G(\cdot, \mathfrak{e}_{\mathrm{can}})$. Proposition 2.6.1 tells us how the geometrically defined differential operators become the action of $\mathfrak{q}^+_G$ through the map $\varphi_G(\cdot, \mathfrak{e}_{\mathrm{can}})$.

Next we consider the $\mathbb{U}_p$-operators. For each $\underline{a} \in \mathbb{Z}^n$ we define $\Delta\underline{a} := (a_1 - a_2, \cdots, a_{n-1} - a_n, a_n)$ and $p^{\underline{a}} := \mathrm{diag}(p^{a_1}, \cdots, p^{a_n}, p^{-a_1}, \cdots, p^{-a_n}) \in G(\mathbb{Q})$. Denote by C_n^+ be the subset of $\mathbb{Z}^n$ consisting of $\underline{a}$ such that $\Delta\underline{a} \geq 0$. The construction in §3.9 associates to each $\underline{a} \in C_n^+$ an (optimally normalized) operator $U_{p,\underline{a}}$ acting on $H^0(X_{G,\Gamma}, \mathcal{V}_{\underline{t}}^r)$. Now we define adelic $\mathbb{U}_p$-operators, with particular attention to the normalizations, such that the map $\varphi_G(\cdot, \mathfrak{e}_{\mathrm{can}})$ is $\mathbb{U}_p$-equivariant.

For $\underline{a} \in C_n^+$ we define the operator $U_{p,\underline{a}}$ acting on $\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))_{\underline{t}}$ as

$$U_{p,\underline{a}} := p^{\langle \underline{t} + 2\rho_{G,c}, \underline{a} \rangle} \int_{N_G(\mathbb{Z}_p)} R_p(up^{\underline{a}})\, du, \tag{4.3}$$

where $R_p(g)$ is the right translation by $g \in G(\mathbb{Q}_p)$. We use $\rho_{G,c}$ (resp. ρ_G, $\rho_{G,nc}$) to denote the half sum of positive compact (resp. positive, positive noncompact) roots of $\mathfrak{g}$ with respect to B_G. If $K_p \subset G(\mathbb{Z}_p)$ is an open compact subgroup containing $N_G(\mathbb{Z}_p)$, then as an action on $\pi_p^{K_p}$, the above defined $U_{p,\underline{a}}$ equals, up to scalar, the usual Hecke operator associated to the characteristic function of the compact open subset $K_p p^{\underline{a}} K_p$ of $G(\mathbb{Q}_p)$. Set $N_G(\underline{a})$ to be the set of representatives of the quotient $N_G(\mathbb{Z}_p)/p^{\underline{a}} N_G(\mathbb{Z}_p) p^{-\underline{a}}$. Then the action of $U_{p,\underline{a}}$ on $\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A})/\widehat{\Gamma})_{\underline{t}}$ can also be written as be

$$U_{p,\underline{a}} = p^{\langle \underline{t} - 2\rho_{G,nc}, \underline{a} \rangle} \sum_{u \in N_G(\underline{a})} R_p(up^{\underline{a}}). \tag{4.4}$$

It is not difficult to check that with the our definitions of the operator $U_{p,\underline{a}}$ on $H^0(X_{G,\Gamma}, \mathcal{V}_{\underline{t}}^r)$ and $\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A})/\widehat{\Gamma})_{\underline{t}}$, the map (4.2) is indeed $\mathbb{U}_p$-equivariant.

Remark 4.1.1. Note that although up to a scalar one may think of the adelic operator $U_{p,\underline{a}}$ as defined locally at the place p, the correct normalization for studying p-adic properties of these operators essentially depends on the $K_{G,\infty}$-type. This illustrates a common phenomenon in the study of p-adic automorphic forms that the place p and the archimedean place are closely related.

Then from the integrality of the geometrically defined $\mathbb{U}_p$-operators (§3.9.5 shows that they preserve the integral structure induced from the embedding to p-adic forms), we easily deduce

Proposition 4.1.2. *Given a weight $\underline{t}$ nearly holomorphic form $\varphi \in \mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))_{\underline{t}}$ (for example an automorphic form of $K_{G,\infty}$-type $\underline{t}$ inside a cuspidal automorphic representation whose* *archimedean component is a holomorphic discrete series), let $\mathbb{U}_p(\varphi)$ be the finite dimensional $\mathbb{C}$-vector space (viewed also as a $\overline{\mathbb{Q}}_p$-vector space by our fixed isomorphism between $\mathbb{C}$ and $\overline{\mathbb{Q}}_p$) spanned by $U_{p,\underline{a}}\varphi$, $\underline{a} \in C_n^+$. Then by our normalization for the $\mathbb{U}_p$-operators, for each $U_{p,\underline{a}}$ all of its eigenvalues on $\mathbb{U}_p(\varphi)$ are p-adic integers.*

For $1 \leq j \leq n$, we define the operator $U_{p,j}$ to be the one that corresponds to the element $\mathrm{diag}(pI_j, I_{n-j}, p^{-1}I_j, I_{n-j})$ inside $G(\mathbb{Q})$, and $U_p = \prod_{j=1}^n U_{p,j}$. Equivalently we can define $U_p = U_{p,\rho_G}$, the operator associated to $\rho_G = (n, n-1, \cdots, 1) \in C_n^+$. The above proposition tells us that, for a nearly holomorphic form φ, the limit

$$\lim_{r\to\infty} U_p^{r!}\varphi, \tag{4.5}$$

with respect to the usual p-adic topology of the finite dimensional $\overline{\mathbb{Q}}_p$-vector spaces $\mathbb{U}_p(\varphi)$, is well defined. We denote by $e\varphi$ this limit, which is called the ordinary projection of φ, because it is the projection of φ to the direct sum of the generalized eigenspaces of the $\mathbb{U}_p$-operators associated to eigenvalues that are all p-adic units. Although in the uniform definition (4.5) a limit with respect to the p-adic topology is involved, in each specific cases the ordinary projector is a $\mathbb{C}$-linear endomorphism of a finite dimensional vector space that can be written as a polynomial of U_p.

The following proposition follows from Corollary 3.10.3, and implies that ordinary nearly holomorphic forms must be holomorphic.

Proposition 4.1.3. *As maps from $H^0(X_{G,\Gamma}, \mathcal{V}^r_{\underline{t}})$ to $H^0(X_{G,\Gamma}, \mathcal{V}^{r-1}_{\underline{t}\otimes\tau_n^*})$, we have*

$$E_\sigma U_p = p^2 \cdot U_p E_\sigma. \tag{4.6}$$

4.2 Siegel Eisenstein series

Let k be an integer larger or equal to $n+1$ and ξ be a primitive Dirichlet character with conductor dividing Np^∞ such that the parity condition $\xi(-1) = (-1)^k$ holds. We record here some computation results of Shimura [54, 57] on the Fourier coefficients of certain holomorphic Siegel Eisenstein series of weight k, and put the formulas into a form that is ready for p-adic interpolation.

4.2.1 Siegel Eisenstein series on H and their Fourier coefficients

Take a primitive Dirichlet character ξ whose conductor divides Np^∞. For a complex number s we denote by $\xi_s = \xi|\cdot|^s \circ \det$ the character of $Q_H(\mathbb{A})$ sending $\begin{pmatrix} A & B \\ 0 & {}^{\mathrm{t}}A^{-1} \end{pmatrix}$ to $\xi(\det A)|\det A|^s$. Let $I_{Q_H}(s,\xi) = \mathrm{Ind}_{Q_H(\mathbb{A})}^{H(\mathbb{A})} \xi_s$ be the normalized induction consisting of smooth functions f on $H(\mathbb{A})$ that satisfy $f(qh) = \xi_s(q)\delta_{Q_H}^{1/2}(q)f(h)$ for all $h \in H(\mathbb{A})$ and $q \in Q_H(\mathbb{A})$. Here the modulus character δ_{Q_H} takes value $|\det A|^{\frac{2n+1}{2}}$ at $\begin{pmatrix} A & B \\ 0 & {}^{\mathrm{t}}A^{-1} \end{pmatrix}$. Similarly we define the local degenerate principal series $I_{Q_H,v}(s,\xi)$ for all places of $\mathbb{Q}$.

Given a section $f(s,\xi) \in I_{Q_H}(s,\xi)$, its associated Siegel Eisenstein series is defined as

$$E(h, f(s,\xi)) = \sum_{\gamma \in Q_H(\mathbb{Q})\backslash H(\mathbb{Q})} f(s,\xi)(\gamma h).$$

The sum is absolutely convergent for $\mathrm{Re}\,(s)$ sufficiently large and admits a meromorphic continuation.

We have already fixed an additive character $\mathbf{e}_{\mathbb{A}}$ of $\mathbb{Q}\backslash\mathbb{A}$ and a Haar measure on $\mathbb{A}$. If $x \in \mathrm{Sym}(2n,\mathbb{A})$ set $u(x)$ to be the element $\begin{pmatrix} I_{2n} & x \\ 0 & I_{2n} \end{pmatrix}$ of the unipotent radical $U_H(\mathbb{A}) \subset Q_H(\mathbb{A})$. For $\boldsymbol{\beta} \in \mathrm{Sym}(2n,\mathbb{Q})$ the $\boldsymbol{\beta}$-th Fourier coefficient for $E\big(\cdot, f(s,\xi)\big)$ is defined as

$$E_{\boldsymbol{\beta}}(h, f(s,\xi)) := \int_{\mathrm{Sym}(2n,\mathbb{Q})\backslash \mathrm{Sym}(2n,\mathbb{A})} E(u(x)h, f(s,\xi))\mathbf{e}_{\mathbb{A}}(-\mathrm{Tr}\,\boldsymbol{\beta}x)\,dx.$$

If $\det(\boldsymbol{\beta}) \neq 0$ and $f(s,\xi) = \otimes_v f_v(s,\xi)$ is factorizable, then

$$E_{\boldsymbol{\beta}}(h, f(s,\xi)) = \prod_v W_{\boldsymbol{\beta},v}(h, f(s,\xi)) \tag{4.7}$$

with

$$W_{\boldsymbol{\beta},v}(h, f_v(s,\xi)) = \int_{\mathrm{Sym}(2n,\mathbb{Q}_v)} f_v(s,\xi)(w_H u(\varsigma)h)\mathbf{e}_v(-\mathrm{Tr}\,\boldsymbol{\beta}\varsigma)\,d_v\varsigma$$

where $w_H = \begin{pmatrix} 0 & -I_{2n} \\ I_{2n} & 0 \end{pmatrix}$.

Let S_{f} be the set of finite places of $\mathbb{Q}$ dividing Np and S be the union of S_{f} with $\{\infty\}$. In the following, for $v \notin S$ we always take $f_v(s,\xi)$ to be the unique section $f_v^{\mathrm{ur}}(s,\xi) \in I_{Q_H,v}(s,\xi)$ that takes value 1 on $H(\mathcal{O}_v)$. For $v \in S_{\mathrm{f}}$ the section $f_v(s,\xi)$ we will consider is supported on the so-called "big cell" inside $H(\mathbb{Q}_v)$, i.e. $Q_H(\mathbb{Q}_v)w_H U_H(\mathbb{Q}_v)$. An element $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in H(\mathbb{Q}_v)$ belongs to the "big cell" if and only if $\det C \neq 0$. Given $\boldsymbol{z} = \boldsymbol{x} + i\boldsymbol{y} \in \mathbb{H}_{2n}$ we put $h_{\boldsymbol{z}} = 1_{\mathrm{f}} \cdot \begin{pmatrix} \sqrt{\boldsymbol{y}} & \boldsymbol{x}\sqrt{\boldsymbol{y}}^{-1} \\ 0 & \sqrt{\boldsymbol{y}}^{-1} \end{pmatrix}_\infty$. With $h = h_{\boldsymbol{z}}$ and at least one local section supported on the "big cell", (4.7) holds for all $\boldsymbol{\beta}$. Next we compute formulas for $W_{\boldsymbol{\beta},v}(h, f_v(s,\xi))$ place by place.

4.2.2 The ramified places

Let α_v be a compactly supported smooth function on $\mathrm{Sym}(2n,\mathbb{Q}_v)$. We define the section $f_v^{\alpha_v}(s,\xi) \in I_{Q_H,v}(s,\xi)$ as

$$f_v^{\alpha_v}(s,\xi)\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right) = \begin{cases} \xi^{-1}(\det C)|\det C|^{-(s+\frac{2n+1}{2})}\alpha_v(C^{-1}D) & \text{if } \det C \neq 0, \\ 0 & \text{if } \det C = 0. \end{cases} \tag{4.8}$$

An easy computation shows that

$$W_{\boldsymbol{\beta},v}(1_v, f_v^{\alpha_v}(s,\xi)) = \int_{\mathrm{Sym}(2n,\mathbb{Q}_v)} \alpha_v(\varsigma)\mathbf{e}_v(-\operatorname{Tr}\boldsymbol{\beta}\varsigma)\, d_v\varsigma = \widehat{\alpha}_v(\boldsymbol{\beta}). \tag{4.9}$$

Since the Fourier transform is an isomorphism on the space of compactly supported smooth functions on $\mathrm{Sym}(2n,\mathbb{Q}_v)$, the above formula gives us adequate flexibility in arranging, for our purpose of p-adic interpolation, the contribution of ramified places to the Fourier coefficients of the Siegel Eisenstein series. Later when choosing sections at p we will first decide what $\widehat{\alpha}_p$ should be and then get the corresponding $f_p^{\alpha_p}(s,\xi)$. Notice also that for such "big cell" sections, $W_{\boldsymbol{\beta},v}(1_v, f_v^{\alpha_v}(s,\xi))$ is independent of s and ξ.

In the following we always require the $\widehat{\alpha}_p$ to be supported on the following compact set

$$\left\{ b = \begin{pmatrix} b_1 & b_0 \\ {}^{\mathrm{t}}b_0 & b_2 \end{pmatrix} \in \mathrm{Sym}(2n, \mathbb{Z}_p) : b_1 \equiv 0 \mod p^2, \quad b_0 \in \mathrm{GL}(n, \mathbb{Z}_p) \right\}. \tag{4.10}$$

In particular under this requirement the Fourier coefficient $E_{\boldsymbol{\beta}}(h_{\boldsymbol{z}}, f(s,\xi))$ vanishes for all degenerate $\boldsymbol{\beta}$.

4.2.3 The unramified places

For $v \notin S$ we record here Shimura's calculation of $W_{\boldsymbol{\beta},v}(1_v, f_v^{\mathrm{ur}}(s,\xi))$ in the case when $\boldsymbol{\beta}$ is nondegenerate. Let val_v be the valuation of $\mathbb{Q}_v$ taking value 1 at the uniformizer and q_v be the cardinality of the residue field. Denote by $\mathrm{Sym}(2n, \mathcal{O}_v)^*$ the set of symmetric matrices $\eta \in \mathrm{Sym}(2n, \mathbb{Q}_v)$ such that $\mathrm{Tr}\, \eta\varsigma \in \mathcal{O}_v$ for all $\varsigma \in \mathrm{Sym}(2n, \mathcal{O}_v)$. Define

$$d_v(s,\xi) := L_v(s + \frac{2n+1}{2}, \xi) \prod_{j=1}^{n} L_v(2s + 2n + 1 - 2j, \xi^2).$$

With all data unramified at v we have

Theorem 4.2.1 ([57, Theorem 13.6, Proposition 14.9])**.** *The Fourier coefficient $W_{\boldsymbol{\beta},v}(1_v, f_v^{\mathrm{ur}}(s,\xi))$ vanishes unless $\boldsymbol{\beta}$ lies inside the intersection of $\mathrm{Sym}(2n,\mathbb{Q})$ with $\mathrm{Sym}(2n,\mathcal{O}_v)^*$. When it is nonvanishing, we have*

$$W_{\boldsymbol{\beta},v}(1_v, f_v^{\mathrm{ur}}(s,\xi)) = d_v(s,\xi) L_v(s + \frac{1}{2}, \xi\lambda_{\boldsymbol{\beta}}) \cdot g_{\boldsymbol{\beta},v}\left(\xi(q_v) q_v^{-(s+\frac{2n+1}{2})}\right). \tag{4.11}$$

Here $\lambda_{\boldsymbol{\beta}}(q_v) := \left(\frac{(-1)^n \det(2\boldsymbol{\beta})}{q_v}\right)$ and $g_{\boldsymbol{\beta},v}(t)$ is a polynomial with coefficients in $\mathbb{Z}$ whose constant term is 1 and degree is at most $4n \cdot \mathrm{val}_v\left(\det(2\boldsymbol{\beta})\right)$. In particular $g_{\boldsymbol{\beta},v}(t) = 1$ if $\det(2\boldsymbol{\beta}) \in \mathcal{O}_v^\times$.

What is relevant to us is the evaluation of $E(\cdot, f(s,\xi))$ at $s_0 = k - \frac{2n+1}{2}$ with $\xi(-1) = (-1)^k$ and $k \geq n+1$. In that case we have the parity $(-1)^{k-n} = \xi\lambda_{\boldsymbol{\beta}}(-1)$ so the special value $L(s_0 + \frac{1}{2}, \xi\lambda_{\boldsymbol{\beta}}) = L(k-n, \xi\lambda_{\boldsymbol{\beta}})$ belongs to the set of interpolation points of the p-adic Dirichlet L-function.

4.2.4 The archimedean place

For an integer $k \geq n+1$ satisfying $\xi(-1) = (-1)^k$ we consider the canonical section $f^k_\infty(s,\xi) \in I_{Q_H,\infty}(s,\xi)$ defined as

$$f^k_\infty(s,\xi)(h) = j(h,i)^{-k}|j(h,i)|^{k-(s+\frac{2n+1}{2})}$$

where $j(h,i) = \det(\mu(h,i)) = \det(Ci+D)$ for $h = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. It gives rise to a Siegel Eisenstein series of scalar weight k. Then

$$\begin{aligned} & W_{\boldsymbol{\beta},\infty}(h_{\boldsymbol{z}}, f^k_\infty(s,\xi)) \\ = & \int_{\mathrm{Sym}(2n,\mathbb{R})} \det\left(\sqrt{\boldsymbol{y}}i + (\boldsymbol{x}+\varsigma)\sqrt{\boldsymbol{y}}^{-1}\right)^{-k} \left|\det\left(\sqrt{\boldsymbol{y}}i + (\boldsymbol{x}+\varsigma)\sqrt{\boldsymbol{y}}^{-1}\right)\right|^{k-(s+\frac{2n+1}{2})} \mathbf{e}_\infty(-\operatorname{Tr}\boldsymbol{\beta}\varsigma)\,d\varsigma \\ = & \,\mathbf{e}_\infty(\operatorname{Tr}\boldsymbol{\beta}\boldsymbol{x})(\det\boldsymbol{y})^{\frac{1}{2}(s+\frac{2n+1}{2})}\xi_{2n}\left(\boldsymbol{y},\boldsymbol{\beta};\frac{1}{2}(s+\frac{2n+1}{2})+\frac{k}{2},\frac{1}{2}(s+\frac{2n+1}{2})-\frac{k}{2}\right), \end{aligned}$$

where for $h_1, h_2 \in \mathrm{Sym}(2n,\mathbb{R})$ and $s_1, s_2 \in \mathbb{C}$ the function ξ_{2n} is defined as

$$\xi_{2n}(h_1,h_2;s_1,s_2) := \int_{\mathrm{Sym}(2n,\mathbb{R})} \det(\varsigma + ih_1)^{-s_1}\det(\varsigma - ih_1)^{-s_2}\mathbf{e}_\infty(\operatorname{Tr} h_2\varsigma)d\varsigma.$$

The function $\xi_{2n}(h_1,h_2;s_1,s_2)$ is studied by Shimura in full generality [54]. Before stating the result we define the Gamma function

$$\Gamma_m(s) := \pi^{\frac{m(m-1)}{4}} \prod_{j=0}^{m-1} \Gamma(s - \frac{j}{2}).$$

Theorem 4.2.2 (Theorem 4.2, *loc. cit*)**.** *Let r_+ (resp. r_-) be the number of positive (resp. negative) eigenvalues of $\boldsymbol{\beta}$ and $r = 2n - r_+ - r_-$. Set $\delta_+(\boldsymbol{\beta}\boldsymbol{y})$ (resp. $\delta_-(\boldsymbol{\beta}\boldsymbol{y})$) to be the product of* *all positive eigenvalues (resp. absolute values of negative eigenvalues) of $\boldsymbol{\beta y}$.*

$$\begin{aligned}\xi_{2n}(\boldsymbol{y},\boldsymbol{\beta};s_1,s_2) = & 2^{2n+\frac{r_+r_-}{2}+2s_1(r_+-n)+2s_2(r_--n)+\frac{r(2n+1)}{2}}(-1)^{n(s_1-s_2)}\pi^{r_+s_1+r_-s_2-\frac{r_+r_-}{2}+\frac{r(r+1)}{2}} \\ & \times (\det \boldsymbol{y})^{\frac{2n+1}{2}-(s_1+s_2)}\delta_+(\boldsymbol{\beta y})^{s_1-\frac{2n+1}{2}+\frac{r_-}{4}}\delta_-(\boldsymbol{\beta y})^{s_2-\frac{2n+1}{2}+\frac{r_+}{4}} \\ & \times \frac{\Gamma_r(s_1+s_2-\frac{2n+1}{2})}{\Gamma_{2n-r_-}(s_1)\Gamma_{2n-r_+}(s_2)}\omega(2\pi\boldsymbol{y},\boldsymbol{\beta};s_1,s_2)\end{aligned}$$

Here $\omega(2\pi\boldsymbol{y},\boldsymbol{\beta};s_1,s_2)$ is a holomorphic function in s_1,s_2, and if $\boldsymbol{\beta}$ is strictly positive definite

$$\omega(2\pi\boldsymbol{y},\boldsymbol{\beta};s_1,0) = 2^{-n(2n+1)}\mathbf{e}_\infty(i\operatorname{Tr}\boldsymbol{\beta y}).$$

The value $W_{\boldsymbol{\beta},\infty}(h_{\boldsymbol{z}},f^k_\infty(s,\xi))$ we are interested in is at $s_0 = k-\frac{2n+1}{2}$, which corresponds to the evaluation of $\xi_{2n}(\boldsymbol{y},\boldsymbol{\beta};s_1,s_2)$ at $s_1=k, s_2=0$. Look at the term $\frac{\Gamma_r(s_1+s_2-\frac{2n+1}{2})}{\Gamma_{2n-r_-}(s_1)\Gamma_{2n-r_+}(s_2)}$. By our requirement on $\widehat{\alpha}_p$ only nondegenerate $\boldsymbol{\beta}$'s need to be considered, for which $r=0$ and the numerator is 1. Meanwhile the function $\Gamma_{2n-r_+}(s_2)$ in the denominator has a pole at $s_2=0$ unless $r_+=2n$. Hence for nondegenerate $\boldsymbol{\beta}$ the value $W_{\boldsymbol{\beta},\infty}(h_{\boldsymbol{z}},f^k_\infty(k-\frac{2n+1}{2},\xi))$ is nonvanishing only if $\boldsymbol{\beta}$ is strictly positive definite. For those $\boldsymbol{\beta}$'s we have

$$W_{\boldsymbol{\beta},\infty}(h_{\boldsymbol{z}},f^k_\infty(k-\frac{2n+1}{2},\xi)) = (-1)^{nk}\frac{2^{2n}}{\Gamma_{2n}(k)}\pi^{2nk}(\det 2\boldsymbol{\beta})^{k-\frac{2n+1}{2}}(\det \boldsymbol{y})^{\frac{k}{2}}\mathbf{e}_\infty(\operatorname{Tr}\boldsymbol{\beta z}).$$

4.2.5 Summary

Let $d^S(s,\xi) = \prod_{v\notin S} d_v(s,\xi)$ and we normalize the Siegel Eisenstein series as

$$E^*(h,f(s,\xi)) = d^S(s,\xi)^{-1}E(h,f(s,\xi)).$$

Let $\mathrm{Sym}(2n,\mathbb{Q})_{>0}$ be the subset of $\mathrm{Sym}(2n,\mathbb{Q})$ consisting of (strictly) positive definite elements, and $\Sigma_{p,+}$ be the subset of $\mathrm{Sym}(2n,\mathbb{Q})_{>0}$ consisting of elements that belong to both the set (4.10) and $\mathrm{Sym}(2n,\mathcal{O}_v)^*$ for all $v\notin S$. Use $\alpha_{S_{\mathrm{f}}}$ to denote the collection of the Schwartz functions α_v on

$\mathrm{Sym}(2n,\mathbb{Q}_v)$ for $v\in S_{\mathrm{f}}$ with $\widehat{\alpha}_p$ always assumed to be supported on the set (4.10). Put

$$f^{k,\alpha_{S_{\mathrm{f}}}}(k-\frac{2n+1}{2},\xi)=\bigotimes_{v\notin S}f_v^{\mathrm{ur}}(k-\frac{2n+1}{2},\xi)\otimes\bigotimes_{v\in S_{\mathrm{f}}}f_v^{\alpha_v}(k-\frac{2n+1}{2},\xi)\otimes f_\infty^k(k-\frac{2n+1}{2},\xi),\quad(4.12)$$

which is a section inside $I_{Q_H}(k-\frac{2n+1}{2},\xi)$.

Combining results from the previous three sections we know that the normalized Siegel Eisenstein series $E^*(\cdot,f^{k,\alpha_{S_{\mathrm{f}}}})$ on $H(\mathbb{A})$ is holomorphic of weight k with Fourier coefficients supported on $\Sigma_{p,+}$. Put $g^S_{\boldsymbol{\beta}}(k,\xi^{-1})=\prod_{v\notin S}g_{\boldsymbol{\beta},v}(\xi(q_v)q_v^{-k})$. For $\boldsymbol{\beta}\in\Sigma_{p,+}$ there is the formula

$$\begin{aligned}&E^*_{\boldsymbol{\beta}}\left(h_{\boldsymbol{z}},f^{k,\alpha_{S_{\mathrm{f}}}}(k-\frac{2n+1}{2},\xi)\right)\\&=(-1)^{nk}\frac{2^{2n}}{\Gamma_{2n}(k)}\pi^{2nk}L^S(k-n,\lambda_{\boldsymbol{\beta}}\xi)g^S_{\boldsymbol{\beta}}(k,\xi^{-1})\prod_{v\in S_{\mathrm{f}}}\widehat{\alpha}_v(\boldsymbol{\beta})(\det 2\boldsymbol{\beta})^{k-\frac{2n+1}{2}}(\det\boldsymbol{y})^{\frac{k}{2}}\mathbf{e}_\infty(\mathrm{Tr}\,\boldsymbol{\beta}\boldsymbol{z}).\end{aligned}\quad(4.13)$$

Implementing the q-expansion principle, with suitable $\alpha_{S_{\mathrm{f}}}$, one can deduce the algebraicity of $E^*(\cdot,f^{k,\alpha_{S_{\mathrm{f}}}}(k-\frac{2n+1}{2},\xi))$, i.e. up to an explicit normalization factor it lies inside the image under the map (4.2) of algebraic global sections.

We modify (4.13) into a form that is more convenient for later p-adic interpolation. Under our parity condition on k and ξ, the functional equation for Dirichlet L-functions indicates

$$L(k-n,\lambda_{\boldsymbol{\beta}}\xi)=\frac{(2\pi i)^{k-n}}{2\Gamma(k-n)C^{k-n-1}_{\lambda_{\boldsymbol{\beta}}\xi}G(\lambda^{-1}_{\boldsymbol{\beta}}\xi^{-1})}L(1-k+n,\lambda^{-1}_{\boldsymbol{\beta}}\xi^{-1}).\quad(4.14)$$

Now write ξ as the product $\phi^{-1}\chi^{\circ-1}$ of two primitive characters, where the conductor of ϕ (resp. χ°) divides N (is a power of p). We write χ to mean the character associated to χ° taking value 0 at p. When there is no need to emphasize the primitivity of χ° we also simply write χ. Set $\phi_{\boldsymbol{\beta}}=\lambda^{-1}_{\boldsymbol{\beta}}\phi$ whose conductor is prime to p. Using the relation $G(\phi_{\boldsymbol{\beta}}\chi)=\phi_{\boldsymbol{\beta}}(C_\chi)\chi(C_{\phi_{\boldsymbol{\beta}}})G(\phi_{\boldsymbol{\beta}})G(\chi)$

and (4.14) we get from (4.13)

$$
\begin{aligned}
&E^*_{\boldsymbol\beta}\left(h_{\boldsymbol z}, f^{k,\alpha_{S_{\rm f}}}(k-\frac{2n+1}{2},\phi^{-1}\chi^{\circ-1})\right)\\
=&\frac{2^{k+n-1}(\pi i)^{2nk+k-n}}{\Gamma(k-n)\Gamma_{2n}(k)\phi(C_\chi)C_\chi^{k-n-1}G(\chi)}\cdot\frac{\lambda_{\boldsymbol\beta}(C_\chi)L_p(1-k+n,\phi_{\boldsymbol\beta}\chi^\circ)}{L_p(k-n,\phi_{\boldsymbol\beta}^{-1}\chi^{\circ-1})}\\
&\times\frac{\det(2\boldsymbol\beta)^{1/2}}{G(\phi_{\boldsymbol\beta})}\cdot\chi^{-1}(C_{\phi_{\boldsymbol\beta}})C_{\phi_{\boldsymbol\beta}}^{-k+n+1}\cdot L_N(k-n,\phi_{\boldsymbol\beta}^{-1}\chi^{-1})^{-1}\cdot L^p(1-k+n,\phi_{\boldsymbol\beta}\chi)\\
&\times g^S_{\boldsymbol\beta}(k,\phi\chi)\cdot\prod_{v\in S_{\rm f}}\widehat\alpha_v(\boldsymbol\beta)\det(2\boldsymbol\beta)^{k-n-1}\cdot(\det\boldsymbol y)^{\frac{k}{2}}\mathbf e_\infty(\operatorname{Tr}\boldsymbol\beta\boldsymbol z).
\end{aligned}
$$

For readers who are familiar with p-adic interpolation, it is noticeable that the above formula has been grouped into factors each of which is ready for p-adic interpolation with respect to k and χ, with the possible exception of the term $\frac{L_p(1-k+n,\phi_{\boldsymbol\beta}\chi^\circ)}{\lambda_{\boldsymbol\beta}(C_\chi)L_p(k-n,\phi_{\boldsymbol\beta}^{-1}\chi^{\circ-1})}$, especially the term $\lambda_{\boldsymbol\beta}(C_\chi)$. This term depends both on k,χ and $\boldsymbol\beta$ and in general does not admit p-adic interpolation. However by our requirement on $\widehat\alpha_p$, it suffices to consider only $\boldsymbol\beta=\begin{pmatrix}\beta_1&\beta_0\\ {}^{\rm t}\beta_0&\beta_2\end{pmatrix}$ that lies inside $\Sigma_{p,+}$. For such a $\boldsymbol\beta$ it is easy to see that $\det\boldsymbol\beta$ is a p-adic integer and $\det\boldsymbol\beta\equiv(-1)^n(\det\beta_0)^2 \mod p$. Thus $\lambda_{\boldsymbol\beta}(p)=1$. Let c_χ be the integer such that $C_\chi=p^{c_\chi}$. Define

$$
A_{n,\phi,k,\chi}:=\frac{2^{k+n-1}(\pi i)^{2nk+k-n}}{\Gamma(k-n)\Gamma_{2n}(k)}\cdot\frac{L_p(1-k+n,\phi\chi^\circ)}{L_p(k-n,\phi^{-1}\chi^{\circ-1})}\left(\left(\phi(p)p^{k-n-1}\right)^{c_\chi}G(\chi)\right)^{-1}. \tag{4.15}
$$

Proposition 4.2.3. *For $\boldsymbol\beta\in\Sigma_{p,+}$ we have*

$$
\begin{aligned}
&E^*_{\boldsymbol\beta}\left(h_{\boldsymbol z}, f^{k,\alpha_{S_{\rm f}}}(k-\frac{2n+1}{2},\phi^{-1}\chi^{\circ-1})\right)\\
=&A_{n,\phi,k,\chi}\cdot\frac{\det(2\boldsymbol\beta)^{1/2}}{G(\phi_{\boldsymbol\beta})}\cdot\chi^{-1}(C_{\phi_{\boldsymbol\beta}})C_{\phi_{\boldsymbol\beta}}^{-k+n+1}\cdot L_N(k-n,\phi_{\boldsymbol\beta}^{-1}\chi^{-1})^{-1}\cdot L^p(1-k+n,\phi_{\boldsymbol\beta}\chi)\\
&\times g^S_{\boldsymbol\beta}(k,\phi\chi)\cdot\prod_{v\in S_{\rm f}}\widehat\alpha_v(\boldsymbol\beta)\det(2\boldsymbol\beta)^{k-n-1}\cdot(\det\boldsymbol y)^{\frac{k}{2}}\mathbf e_\infty(\operatorname{Tr}\boldsymbol\beta\boldsymbol z).
\end{aligned}
$$

One can observe that on the RHS of the equality, the term $A_{n,\phi,k,\chi}$ is independent of $\boldsymbol\beta$ and other terms admit p-adic interpolations with respect to k,χ for suitably chosen $\alpha_{S_{\rm f}}$ (cf. §5.2).

From now on we fix a primitive Dirichlet character ϕ whose conductor divides N, and we will sometimes omit N and ϕ from some notation that actually depends on them. Proposition 4.2.3 basically gives us a one-variable family of Siegel Eisenstein series on H where the variable is κ. What we want is an $(n+1)$-variable cuspidal family on $G \times G$, whose members are the restrictions to $G \times G$ of Siegel Eisenstein series on H, and its pairing with an n-variable family on $G \times G$ will give the desired $(n+1)$-variable p-adic L-function. Constructing this $(n+1)$-variable family boils down to selecting sections $f_{\kappa,\underline{\tau}}$ inside $I_{Q_H}(k - \frac{2n+1}{2}, \phi^{-1}\chi^{\circ -1})$ for each admissible $(\kappa, \underline{\tau})$. It is no surprise that for all $v \notin S$ we set $f_{\kappa,\underline{\tau},v}$ to be the unramified section $f_v^{\mathrm{ur}}(k - \frac{2n+1}{2}, \phi^{-1}\chi^{\circ -1})$. For $v \in S_{\mathrm{f}}$ we consider the "big cell" sections. Thus what we need to select is the collection of Schwartz functions $\alpha_{\kappa,\underline{\tau},S_{\mathrm{f}}}$ and the archimedean section $f_{\kappa,\underline{\tau},\infty}$.

After recalling the doubling method formula in §4.3, we make the choice of $\alpha_{\kappa,\underline{\tau},N}$ and compute the corresponding local zeta integrals in §4.4. Regarding the archimedean place, we make a choice of $f_{\kappa,\underline{\tau},\infty}$ and show the nonvanishing of the resulting archimedean zeta integral in §4.5. The place p will be treated in the next chapter. Based on the two criteria in the introduction, i.e. nonvanishing local zeta integrals and p-adically interpolatable q-expansions, all choices are completely natural.

4.3 Doubling method for symplectic groups

Let us briefly recall the formulas of the doubling method. We have fixed the rank $2n$ free $\mathbb{Z}$-module $\mathbf{L}_n$ with a symplectic pairing and $G = G(\mathbf{L}_n)$. Let $\mathbf{V}_n = V_n \oplus V_n^*$ be the polarized symplectic space over $\mathbb{Q}$ with basis $e_1, \cdots, e_n, f_1, \cdots, f_n$ obtained from $\mathbf{L}_n$ by tensoring with $\mathbb{Q}$. Take another copy of $\mathbf{V}_n$ with basis $e_1', \cdots, e_n', f_1', \cdots, f_n'$, and put $\mathbf{V}_{2n} = \mathbf{V}_n \oplus \mathbf{V}_n$ with the induced symplectic pairing. Elements in $H = G(\mathbf{V}_{2n})$ will be written in matrix form with respect to the basis $e_1, \cdots, e_n, e_1', \cdots, e_n', f_1, \cdots, f_n, f_1', \cdots, f_n'$. Then there is the (holomorphic) embedding ι of

$G \times G$ into H given by

$$\iota : G \times G \hookrightarrow H$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \longmapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & a' & 0 & b' \\ c & 0 & d & 0 \\ 0 & c' & 0 & d' \end{pmatrix}.$$

Fix the map ϑ from $\mathbf{V}_n$ into itself whose matrix is $\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$ with respect to our fixed basis. It does not preserve the symplectic pairing but has the similitude -1. Let $\mathbf{V}_{2n}^d = \{(v, \vartheta(v)) : v \in \mathbf{V}_n\}$ and $\mathbf{V}_{2n,d} = \{(v, -\vartheta(v)) : v \in \mathbf{V}_n\}$ which are both maximal isotropic subspaces of $\mathbf{V}_{2n}$. The doubling Siegel parabolic P_H is defined to be the stabilizer of $\mathbf{V}_{2n}^d$. The standard Siegel parabolic Q_H is the stabilizer of the maximal isotropic subspace $V_n \oplus V_n$ and we have

$$P_H = \mathcal{S}^{-1} Q_H \mathcal{S} \quad \text{with} \quad \mathcal{S} = \begin{pmatrix} I_n & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 \\ 0 & I_n & I_n & 0 \\ I_n & 0 & 0 & I_n \end{pmatrix}.$$

For each section $f(s,\xi) \in I_{Q_H}(s,\xi)$ we set

$$f^d(s,\xi)(h) = f(s,\xi)(\mathcal{S}h) \tag{4.16}$$

for $h \in H(\mathbb{A})$. Then $f^d(s,\xi)$ lies inside $I_{P_H}(s,\xi)$ and $E(\cdot, f(s,\xi)) = E(\cdot, f^d(s,\xi))$. For an element $g \in G$ we define g^ϑ to be $\vartheta g \vartheta \in G$. This conjugation by ϑ is called the MVW involution. The MVW involution of an irreducible smooth representation of $G(\mathbb{Q}_v)$ is isomorphic to its contragredient [48, p. 91].

Given an irreducible cuspidal automorphic representation $\pi \subset \mathcal{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A}))$ of $G(\mathbb{A})$ and its complex conjugation $\overline{\pi} \subset \mathcal{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A}))$, which is isomorphic to the contragredient of π, we fix isomorphisms $\pi \cong \bigotimes'_v \pi_v$ and $\overline{\pi} \cong \bigotimes'_v \tilde{\pi}_v$ such that for factorizable $\varphi_1, \varphi_2 \in \pi$ with images $\bigotimes_v \varphi_{1,v} \in \bigotimes'_v \pi_v$ and $\bigotimes_v \overline{\varphi}_{2,v} \in \bigotimes'_v \tilde{\pi}_v$, we have

$$\langle \varphi_1, \overline{\varphi}_2 \rangle = \prod_v \left\langle \varphi_{1,v}, \overline{\varphi}_{2,v} \right\rangle_v,$$

where the pairing on the left hand side is the bi-$\mathbb{C}$-linear Petersson inner product with respect to our fixed Haar measure on $G(\mathbb{A})$ and the pairing on the right hand side is the natural pairing between π_v and its contragredient $\tilde{\pi}_v$.

For $\varphi \in \pi$ we define its MVW involution φ^ϑ by $\varphi^\vartheta(g) = \varphi(g^\vartheta)$, and we know that φ^ϑ lies inside $\overline{\pi}$ due to the multiplicity one theorem [2].

For a local section $f_v(s,\xi) \in I_{Q_H,v}(s,\xi)$ we define the operator

$$\begin{aligned} T_{f_v(s,\xi)} : \pi &\longrightarrow \pi \\ \varphi &\longmapsto \left(T_{f_v(s,\xi)}\varphi\right)(g) = \int_{G(\mathbb{Q}_v)} f_v^d(s,\xi)(\iota(g'_v,1))\varphi(gg'_v)d_vg'_v. \end{aligned}$$

Certainly in order for $T_{f_v(s,\xi)}$ to be well defined we must address convergence issues. The absolute convergence can be proved for $s \in \mathbb{C}$ with $\mathrm{Re}\,(s)$ sufficiently large. In our applications a meromorphic continuation always exists and we use it to define $T_{f_v(s,\xi)}$ for general $s \in \mathbb{C}$. In fact when $v \mid N$ or when $v = p$ and $\chi\psi_1, \cdots, \chi\psi_n$ are all nontrivial, by our choices the function $f^d_{\kappa,\underline{\tau},v}(\iota(\cdot,1))$ on $G(\mathbb{Q}_v)$ is compactly supported. When $v = \infty$ the absolute convergence follows from the fact that π_∞ is a discrete series as discussed in [46]. The only place we need to be careful with the convergence issue is the computation in §5.7, i.e. the local zeta integral at p with some of $\chi\psi_1, \cdots, \chi\psi_n$ being trivial.

The doubling local zeta integral is defined as (purely locally)

$$\begin{aligned} Z_v(f_v(s,\xi),\cdot,\cdot) : \pi_v \times \tilde{\pi}_v &\longrightarrow \mathbb{C} \\ (v_1,\tilde{v}_2) &\longmapsto Z_v(f_v(s,\xi),v_1,\tilde{v}_2) = \int_{G(\mathbb{Q}_v)} f_v^d(s,\xi)(\iota(g_v,1)) \left\langle \pi_v(g_v)v_1, \tilde{v}_2 \right\rangle_v d_vg_v. \end{aligned} \tag{4.17}$$

As a pairing between $I_{Q_H,v}(s,\xi)$ and $\pi_v \times \tilde{\pi}_v$, the doubling local zeta integral has the equivariance property that for $(g_1, g_2) \in G(\mathbb{Q}_v) \times G(\mathbb{Q}_v)$,

$$Z_v\left(R_p(\iota(g_1, g_2^{\vartheta}))f(s,\xi), \pi_v(g_1)v_1, \tilde{\pi}_v(g_2)\tilde{v}_2\right) = Z_v(f_v(s,\xi), v_1, \tilde{v}_2). \tag{4.18}$$

Remark 4.3.1. The standard notation for the zeta integral should be written as $Z_v(f_v^d(s,\xi), v_1, \tilde{v}_2)$. In our construction we always use $f_v(s,\xi)$ for computing the Fourier coefficients of $E^*(\cdot, f_v(s,\xi)) = E^*(\cdot, f_v^d(s,\xi))$ while the zeta integral is always computed with $f_v^d(s,\xi)$. The notation in (4.17) is more convenient for us here, and should cause no confusion.

Theorem 4.3.2 ([24,51,58]). *Suppose $f(s,\xi) = \bigotimes_{s \notin S} f_v^{\mathrm{ur}}(s,\xi) \otimes \bigotimes_{v \in S} f_v(s,\xi)$ is a section inside to $I_{Q_H}(s,\xi)$. If $\varphi \in \pi^{K_G^S}$ with $K_G^S = \prod_{v \notin S} G(\mathcal{O}_v)$, then*

$$\langle E^*\left(\iota(\cdot, g), f(s,\xi)\right), \overline{\varphi}\rangle = L^S(s+\frac{1}{2}, \pi \times \xi) \cdot \left(\prod_{v \in S} T_{f_v(s,\xi)}\overline{\varphi}\right)(g^{\vartheta}).$$

Equivalently for all factorizable $\varphi_1, \varphi_2 \in \pi^{K_G^S}$,

$$\left\langle E^*(\cdot, f(s,\xi))|_{G\times G}, \, \overline{\varphi}_1 \otimes \varphi_2^{\vartheta}\right\rangle = L^S(s+\frac{1}{2}, \pi \times \xi) \cdot \prod_{v \in S} \frac{Z_v(f_v(s,\xi), \overline{\varphi}_{1,v}, \varphi_{2,v})}{\left\langle \overline{\varphi}_{1,v}, \, \varphi_{2,v}\right\rangle_v} \left\langle \overline{\varphi}_1, \varphi_2 \right\rangle.$$

Remark 4.3.3. Our formulation of the doubling method aligns with those of [24, 58] where if the Siegel Eisenstein series on H is holomorphic its restriction to $G \times G$ is still holomorphic on both factors, because the embedding $\iota : G \times G \hookrightarrow H$ corresponds to the holomorphic embedding of the Siegel upper half spaces $\mathbb{H}_n \times \mathbb{H}_n \hookrightarrow \mathbb{H}_{2n}$ sending (z_1, z_2) to $\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}$. However it differs from the standard formulation in the study of the doubling method from the point of view of theta correspondence, where the embedding is equivalent to ι with a conjugation by ϑ on the second factor. The translation from the standard formulation to ours here depends on the choice of the map ϑ from $\mathbf{V}_n$ to itself with similitude -1.

4.4 The "volume sections" at places dividing N

In this section we make the choices for $\alpha_{\kappa,\underline{\tau},N}$ and $f_{\kappa,\underline{\tau},\infty}$. With our choices we compute the local zeta integrals for the doubling method for $v \mid N$, and show the nonvanishing of the archimedean zeta integral. In the next section we treat the place p. Based on the two criteria in the introduction, i.e. nonvanishing local zeta integrals and p-adically interpolatable q-expansions, all choices are completely natural.

For a place $v \mid N$ we pick a very simple so-called "volume section" that gives simple Fourier coefficients and easily computed local zeta integrals. Moreover it makes the restriction of the resulting Siegel Eisenstein series to $G \times G$ cuspidal when the archimedean section is taken to be f^k_∞. The cuspidality fact is crucial for us to apply Hida theory on G.

Define the Schwartz function $\alpha^{\mathrm{vol}}_v : \mathrm{Sym}(2n, \mathbb{Q}_v) \to \mathbb{C}$ to be the characteristic function of the compact open subset $\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} + N\,\mathrm{Sym}(2n, \mathcal{O}_v)$ of $\mathrm{Sym}(2n, \mathbb{Q}_v)$. The "volume section" inside $I_{Q_H,v}(s,\xi)$ is defined as $f^{\mathrm{vol}}_v(s,\xi) = f^{\alpha^{\mathrm{vol}}_v}_v(s,\xi)$. It gives the Fourier coefficient

$$W_{\boldsymbol{\beta},v}(1_v, f^{\mathrm{vol}}_v(s,\xi)) = \widehat{\alpha}^{\mathrm{vol}}_v(\boldsymbol{\beta}) = |N|_v^{n(2n+1)} \mathbf{e}_v(-2\,\mathrm{Tr}\,\beta_0) \cdot \mathbb{1}_{N^{-1}\,\mathrm{Sym}(2n,\mathcal{O}_v)^*}(\boldsymbol{\beta})$$

for $\boldsymbol{\beta} = \begin{pmatrix} \beta_1 & \beta_0 \\ {}^{\mathrm{t}}\beta_0 & \beta_2 \end{pmatrix}$, where $\mathbb{1}_{N^{-1}\,\mathrm{Sym}(2n,\mathcal{O}_v)^*}$ is the characteristic function of the set $N^{-1}\,\mathrm{Sym}(2n,\mathcal{O}_v)^*$. The "volume section" $f^{\mathrm{vol}}_{\kappa,\underline{\tau},v}$ is independent of $\underline{\tau}$ and its corresponding Fourier coefficient is a p-adic integer independent of both κ and $\underline{\tau}$.

Next we compute the local zeta integral. Let $\Gamma(N)_v$ be the open compact subgroup of $G(\mathbb{Q}_v)$ consisting of elements in $G(\mathcal{O}_v)$ whose reduction modulo N is 1.

Proposition 4.4.1. *Suppose $\varphi \in \pi$ is invariant under right translation by $\Gamma(N)_v$. Then*

$$T_{f^{\mathrm{vol}}_v(s,\xi)}\varphi = \xi_v(-1)^n \mathrm{vol}(\Gamma(N)_v) \cdot \varphi.$$

Proof. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\mathbb{Q}_v)$ we have

$$\mathcal{S}\iota(g,1) = \begin{pmatrix} I_n & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 \\ 0 & I_n & I_n & 0 \\ I_n & 0 & 0 & I_n \end{pmatrix} \begin{pmatrix} a & 0 & b & 0 \\ 0 & I_n & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & I_n \end{pmatrix} = \begin{pmatrix} a & 0 & b & 0 \\ 0 & I_n & 0 & 0 \\ c & I_n & d & 0 \\ a & 0 & b & I_n \end{pmatrix}. \tag{4.19}$$

It is contained inside the support of $f_v^{\mathrm{vol}}(s,\xi)$ if and only if $\det \begin{pmatrix} c & I_n \\ a & 0 \end{pmatrix} \neq 0$ and $\begin{pmatrix} c & I_n \\ a & 0 \end{pmatrix}^{-1} \begin{pmatrix} d & 0 \\ b & I_n \end{pmatrix} = \begin{pmatrix} a^{-1}b & a^{-1} \\ d - ca^{-1}b & -ca^{-1} \end{pmatrix}$ belongs to $\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} + N\,\mathrm{Sym}(2n, \mathcal{O}_v)$. Therefore

$$f_v^{d,\mathrm{vol}}(s,\xi)(g) = \begin{cases} \xi_v(-1)^n & \text{if } g \in \Gamma(N)_v, \\ 0 & \text{otherwise,} \end{cases}$$

and the proposition follows. □

For an admissible $(\kappa, \underline{\tau})$ we set $f_{\kappa,\underline{\tau},v} = f_{\kappa,\underline{\tau},v}^{\mathrm{vol}} = f_v^{\mathrm{vol}}(k - \frac{2n+1}{2}, \phi^{-1}\chi^{\circ -1})$ and use $f_{\kappa,\underline{\tau},N}^{\mathrm{vol}}$ to denote the product of local sections $\bigotimes_{v|N} f_{\kappa,\underline{\tau},v}^{\mathrm{vol}}$. We also put $\widehat{\alpha}_N^{\mathrm{vol}} = \prod_{v|N} \widehat{\alpha}_v^{\mathrm{vol}}$.

Before moving to the archimedean place, we record here the following theorem due to Garrett concerning the cuspidality of the restriction to $G \times G$ of the Siegel Eisenstein series.

Theorem 4.4.2 ([25, p. 465-473]). *Let $f(s,\xi)$ be a factorizable section inside $I_{Q_H}(s,\xi)$ with $f_v(s,\xi) = f_v^{\mathrm{vol}}(s,\xi)$ for some finite place v and $f_\infty(s,\xi) = f_\infty^k(s,\xi)$, $k > 2n+1$. Then the evaluation at $s = k - \frac{2n+1}{2}$ of the restriction of the Siegel Eisenstein series $E(\cdot, f(s,\xi))|_{G\times G}$ is a cuspidal holomorphic Siegel modular form of scalar weight k on $G \times G$.*

4.5 The archimedean sections

We select a section $f_{\kappa,\underline{\tau},\infty}$ from $I_{Q_H,\infty}(k-\frac{2n+1}{2},\phi^{-1}\chi^{\circ -1})$ for each admissible $(\kappa,\underline{\tau})=(k\cdot\chi,\underline{t}\cdot\underline{\psi})$ with κ satisfying the parity condition $\phi\chi(-1)=(-1)^k$. Denote by $\mathcal{D}_{\underline{t}}$ the holomorphic discrete series $(\mathfrak{g},K_{G,\infty})$-module whose lowest $K_{G,\infty}$-type is of highest weight $\underline{t}$, and by $\mathcal{D}_{\underline{t}}(\underline{t})$ the lowest $K_{G,\infty}$-type inside $\mathcal{D}_{\underline{t}}$. Let $\widetilde{\mathcal{D}_{\underline{t}}}$ be the contragredient of $\mathcal{D}_{\underline{t}}$ and $\widetilde{\mathcal{D}_{\underline{t}}}(-\underline{t})$ be its highest $K_{G,\infty}$-type.

In our application of the doubling method formula, the cuspidal automorphic forms φ on $G(\mathbb{A})$ we consider are those coming from global sections of the automorphic sheaf $\omega_{\underline{t}}=\mathcal{V}^0_{\underline{t}}$ over Shimura varieties of certain level through the map (4.2). Thus the archimedean factor π_∞ is a holomorphic discrete series and φ_∞ lies inside its lowest $K_{G,\infty}$-type. The nonvanishing condition we put on $f_{\kappa,\underline{\tau},\infty}$ is that for all such φ, the $(-\underline{t})$-isotypic part of $T_{f_{\kappa,\underline{\tau},\infty}}\overline{\varphi}$ is nontrivial, or equivalently the map $Z_\infty(f_{\kappa,\underline{\tau},\infty},\cdot,\cdot):\widetilde{\mathcal{D}_{\underline{t}}}(-\underline{t})\times\mathcal{D}_{\underline{t}}(\underline{t})\to\mathbb{C}$ is nonzero.

For the case $t_1=\cdots=t_n=k$ the very canonical choice for the archimedean section is f^k_∞. The corresponding local zeta integral is computed in [56] and the results clearly imply the nonvanishing. From Proposition 4.2.3 one sees that f^k_∞ also satisfies the condition that after dividing an explicit scalar, its Fourier coefficients are all algebraic.

In order for $E^*(\cdot,f_{\kappa,\underline{\tau}})$ to be algebraic it is natural to consider sections obtained by applying operators constructed from $\mathfrak{q}^+_H$ to f^k_∞, because then our discussion in §2.6 shows that the resulting Siegel Eisenstein series can be obtained by applying the (geometrically defined) differential operators to $E^*(\cdot,f^{k,\alpha_{S_{\mathrm{f}}}})$, and the differential operators have an algebraic structure as well as formulas on q-expansions.

Recall that we have fixed a basis $\widehat{\mu}^+_{ij}$, $1\le i\le j\le 2n$ for the Lie algebra $\mathfrak{q}^+_H$. Putting $\widehat{\mu}^+_{ij}=\widehat{\mu}^+_{ji}$ for $i>j$, we let $\widehat{\mu}^+_H$ be the symmetric $2n\times 2n$ matrix whose (i,j) entry is $\widehat{\mu}^+_{ij}$. Write $\widehat{\mu}^+_H=\begin{pmatrix}\widehat{\mu}^+_1 & \widehat{\mu}^+_0\\ {}^{\mathrm{t}}\widehat{\mu}^+_0 & \widehat{\mu}^+_2\end{pmatrix}$ in $n\times n$ blocks.

Inspired by [28], we define the following archimedean section

$$f_{\kappa,\underline{\tau},\infty}=\prod_{l=1}^{n}\det\nolimits_l\left(-\frac{1}{4\pi i}\widehat{\mu}^+_0\right)^{t_l-t_{l+1}}\cdot f^k_\infty, \tag{4.20}$$

where we put $t_{n+1} = k$ and for a matrix A we use $\det_l(A)$ to denote the determinant of its upper left $l \times l$ minor. The rest of this section is devoted to proving the following proposition stating that this $f_{\kappa,\underline{\tau},\infty}$ satisfies the nonvanishing condition. The strategy for making this selection will manifest in the proof.

Proposition 4.5.1. *With $f_{\kappa,\underline{\tau},\infty}$ defined as in (4.20), the map*

$$Z_\infty(f_{\kappa,\underline{\tau},\infty}, \cdot, \cdot) : \widetilde{\mathcal{D}_{\underline{t}}}(-\underline{t}) \times \mathcal{D}_{\underline{t}}(\underline{t}) \longrightarrow \mathbb{C} \tag{4.21}$$

is nonzero. Let $v_{\underline{t}} \in \mathcal{D}_{\underline{t}}(\underline{t})$ be a nonzero vector of highest weight and $v_{\underline{t}}^\vee \in \widetilde{\mathcal{D}_{\underline{t}}}(-\underline{t})$ be its dual vector. Then the number $\frac{Z_\infty(f_{\kappa,\underline{\tau},\infty}, v_{\underline{t}}^\vee, v_{\underline{t}})}{\langle v_{\underline{t}}^\vee, v_{\underline{t}} \rangle}$ is nonzero

Proof. Let $U(\mathfrak{h}_\mathbb{C}) \cdot f_\infty^{d,k}$ be the sub-$(\mathfrak{h}_\mathbb{R}, K_{H,\infty})$-module of $I_{P_H,\infty}(k - \frac{2n+1}{2}, \phi^{-1}\chi^{\circ -1})$ generated by $f_\infty^{d,k}$. As explained above due to the algebraicity consideration we want to pick our $f_{\kappa,\underline{\tau},\infty}^d$ from $U(\mathfrak{h}_\mathbb{C}) \cdot f_\infty^{d,k}$. Regarding $U(\mathfrak{h}_\mathbb{C}) \cdot f_\infty^{d,k}$ as a representation of the compact group $K_{G,\infty} \times K_{G,\infty}$, we prove that in the decomposition of $U(\mathfrak{h}_\mathbb{C}) \cdot f_\infty^{d,k}|_{K_{G,\infty} \times K_{G,\infty}}$, there is a unique piece $\sigma_{k,\underline{t}}$ which pairs nontrivially with $\widetilde{\mathcal{D}_{\underline{t}}}(-\underline{t}) \times \mathcal{D}_{\underline{t}}(\underline{t})$ under the zeta integral. Then we finish the proof by showing that $f_{\kappa,\underline{\tau},\infty}^d$ has a nonzero projection into $\sigma_{k,\underline{t}}$.

We start by introducing several unitarizable irreducible $(\mathfrak{h}_\mathbb{R}, K_{H,\infty})$-modules whose $K_{H,\infty}$-finite parts are isomorphic to $U(\mathfrak{h}_\mathbb{C}) \cdot f_\infty^{d,k}$ or its contragradient when the parameters are within the range relevant to us here. Let (σ, W_σ) be a finite dimensional algebraic representation of $\mathrm{GL}(2n)$. Then $W_\sigma(\mathbb{C})$ is a $K_{H,\infty}$-representation. Define the $H(\mathbb{R})$-representation

$$\mathcal{O}(H(\mathbb{R}), K_{H,\infty}, \sigma) = \left\{ \begin{matrix} \text{analytic functions } f : H(\mathbb{R}) \to W_\sigma(\mathbb{C}) \text{ that are annihilated} \\ \text{by the action of } \mathfrak{q}_H^- \text{ on the right, and } f(hk) = \sigma^{-1}(k) f(g) \\ \text{for all } k \in K_{H,\infty}, h \in H(\mathbb{R}) \end{matrix} \right\}$$

with $H(\mathbb{R})$ acting by left inverse translation. Let $\mathcal{O}^{\mathrm{f}}(H(\mathbb{R}), K_{H,\infty}, \sigma)$ be the $(\mathfrak{h}_\mathbb{R}, K_{H,\infty})$-module which is the subspace of $\mathcal{O}(H(\mathbb{R}), K_{H,\infty}, \sigma)$ spanned by $K_{H,\infty}$-finite vectors. Let $\mathcal{O}(\mathbb{H}_{2n}, \sigma)$ be the space of W_σ-valued holomorphic functions on $\mathbb{H}_{2n}$ with $h = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} \in H(\mathbb{R})$ acting on

$f \in \mathcal{O}(\mathbb{H}_{2n}, \sigma)$ via

$$(h \cdot f)(\boldsymbol{z}) = \sigma\left({}^{\mathrm{t}}(C\boldsymbol{z}+D)\right) f\left((A\boldsymbol{z}+B)(C\boldsymbol{z}+D)^{-1}\right).$$

It is easily seen that $\mathcal{O}(H(\mathbb{R}), K_{H,\infty}, \sigma)$ is isomorphic to $\mathcal{O}(\mathbb{H}_{2n}, \sigma)$ (cf. Remark 2.6.2). One can also check that the $\mathfrak{h}_{\mathbb{C}}$-module $\mathcal{O}^{\mathrm{f}}(H(\mathbb{R}), K_{H,\infty}, \sigma)$ is isomorphic to the base change to $\mathbb{C}$ of the $\mathfrak{h}$-module V_σ defined in §2.3, and that the formulas there show that it has a unique highest $K_{H,\infty}$-type σ which is contained inside every sub-representation.

Let $W_{2k,0}$ be the real vector space of dimension $2k$ with a positive definite symmetric pairing and $O(2k,0)$ be the associated orthogonal group. The action of $O(2k,0) \times H(\mathbb{R})$ on the Schrödinger model $\mathcal{S}(W_{2k,0} \otimes \mathbf{V}_{2n,d,\mathbb{R}})$, the space of Schwartz functions on $W_{2k,0} \otimes \mathbf{V}_{2n,d,\mathbb{R}}$, of its Weil representation with respect to the polarization $\mathbf{V}_{2n} = \mathbf{V}_{2n}^d \oplus \mathbf{V}_{2n,d}$ is given by

$$\begin{aligned}
&(\omega(a)\mathfrak{s})(x) = \mathfrak{s}({}^{\mathrm{t}}ax), && a \in O(2k,0),\\
&(\omega(m)\mathfrak{s})(x) = \det\left(m|_{V^d_{2n,\mathbb{R}}}\right)^k \mathfrak{s}({}^{\mathrm{t}}mx), && m \in P(\mathbf{V}^d_{2n,\mathbb{R}}) \cap P(\mathbf{V}_{2n,d,\mathbb{R}}),\\
&(\omega(u)\mathfrak{s})(x) = \mathbf{e}_\infty(\langle -u(x), x\rangle/2)\mathfrak{s}(x), && u \in N(\mathbf{V}^d_{2n,\mathbb{R}}),\\
&(\omega(w)\mathfrak{s})(x) = i^{2nk}\int_{W_{2k,0}\otimes \mathbf{V}_{2n,d,\mathbb{R}}} \mathbf{e}_\infty(\langle y, wx\rangle)\mathfrak{s}(y)dy.
\end{aligned}$$

Here for an isotropic subspace V, $P(V)$ is the stabilizer of V and $N(V)$ is the unipotent radical of $P(V)$. The element w in $H(\mathbb{R})$ is the one sending $(v, \vartheta(v))$ to $(v, -\vartheta(v))$ and $(v, -\vartheta(v))$ to $-(v, \vartheta(v))$ for $v \in \mathbf{V}_n$.

Let $\Theta_{2k,0}(0) = \mathcal{S}(W_{2k,0} \otimes \mathbf{V}_{2n,d,\mathbb{R}})^{O(2k,0)}$ be the theta lift of the trivial representation from $O(2k,0)$ to $H(\mathbb{R})$. The morphism

$$\begin{aligned}
\Phi : \mathcal{S}(W_{2k,0} \otimes \mathbf{V}_{2n,d,\mathbb{R}}) &\longrightarrow I_{P_H,\infty}(k - \frac{2n+1}{2}, \mathrm{Sign}^k)\\
\mathfrak{s} &\longmapsto \Phi(\mathfrak{s})(g) := (\omega(g)\mathfrak{s})(0)
\end{aligned}$$

embeds $\Theta_{2k,0}(0)$ into the degenerate principal series [42, Theorem 3]. We denote by $R^d_{2k,0}$ the image of $\Theta_{2k,0}(0)$ inside $I_{P_H,\infty}(k - \frac{2n+1}{2}, \mathrm{Sign}^k)$ and $R_{2k,0}$ be the sub-$H(\mathbb{R})$-representation of $I_{Q_H,\infty}(k - \frac{2n+1}{2}, \mathrm{Sign}^k)$, which corresponds to $R^d_{2k,0}$ via (4.16).

The representation $\Theta_{2k,0}(0)$ is unitary and embeds into $\mathcal{O}(\mathbb{H}_{2n},-k)$ through the map

$$\mathfrak{s}\longmapsto\int_{W_{2k,0}\otimes\mathbf{V}_{2n,d,\mathbb{R}}}\mathbf{e}_\infty(\operatorname{Tr}x^{\mathrm{t}}x\boldsymbol{z})\mathfrak{s}(x)dx.$$

Therefore $\Theta_{2k,0}(0)$ is irreducible. If $k\geq n$ (we have always assumed $k\geq n+1$) the image is dense [36, p. 3]. It follows that $\mathcal{O}^{\mathrm{f}}(H(\mathbb{R}),K_{H,\infty},-k)$ is irreducible, so isomorphic to the Verma module $U(\mathfrak{h}_\mathbb{C})\otimes_{U(\mathfrak{k}_{H,\mathbb{C}}\oplus\mathfrak{q}_H^+)}\det^{-k}$ of highest weight $-k$.

We use the superscript MVW to denote the MVW-involution, i.e. conjugation by ϑ, of the above defined representations. In our case, thanks to the irreducibility, the MVW-involution is isomorphic to the contragredient representation. By using $-W_{2k,0}$ we define $\Theta_{0,2k}(0)$ and $R_{0,2k}\subset I_{P_H,\infty}(k-\frac{2n+1}{2},\mathrm{Sign}^k)$. It is easily seen that $\Theta_{0,2k}(0)\cong\Theta_{2k,0}(0)^{\mathrm{MVW}}$. The $K_{H,\infty}$-finite part of $R_{0,2k}$ will be denoted as $R^{\mathrm{f}}_{0,2k}$.

The degenerate principal $I_{Q_H,\infty}(k-\frac{2n+1}{2},\mathrm{Sign}^k)$ is $K_{H,\infty}$-multiplicity free [26]. Both $U(\mathfrak{h}_\mathbb{C})\cdot f^{d,k}_\infty$ and $R^{\mathrm{f}}_{0,2k}$ are irreducible $\mathfrak{h}_\mathbb{C}$-submodules of the degenerate principal series and contain the $K_{H,\infty}$-type of scalar weight k. Hence they must be equal to each other, and we are reduced to studying the $\mathfrak{h}_\mathbb{C}$-module $R^{\mathrm{f}}_{0,2k}$, which by the above discussion is isomorphic to $\mathcal{O}^{\mathrm{f}}(H(\mathbb{R}),K_{H,\infty},-k)^{\mathrm{MVW}}$ and the Verma module $M_k=U(\mathfrak{h}_\mathbb{C})\otimes_{U(\mathfrak{k}_{H,\mathbb{C}}\oplus\mathfrak{q}_H^-)}\det^k$ of lowest weight k. Regarding its decomposition as a $\mathfrak{g}_\mathbb{C}\times\mathfrak{g}_\mathbb{C}$-module there is the following theorem.

Theorem 4.5.2 ([35, Proposition 2.2, Corollary 2.3])**.** *If $k\geq n+1$, then*

$$\mathcal{O}^{\mathrm{f}}(H(\mathbb{R}),K_{H,\infty},-k)|_{\mathfrak{g}_\mathbb{C}\times\mathfrak{g}_\mathbb{C}}=\bigoplus_{r=0}^{\infty}\mathcal{O}^{\mathrm{f}}\left(G(\mathbb{R})\times G(\mathbb{R}),K_{G,\infty}\times K_{G,\infty},\det{}^{-k}\otimes\mathrm{Sym}^r(\mathfrak{q}_H^-/\mathfrak{q}_G^-\times\mathfrak{q}_G^-)\right).$$

Applying the decomposition results on algebraic $\mathrm{GL}(n)$-representations [55, Theorem 2.A], we have

$$\mathrm{Sym}^r(\mathfrak{q}_H^+/\mathfrak{q}_G^+\times\mathfrak{q}_G^+)\cong\bigoplus_{\substack{a_1\geq\cdots\geq a_n\geq0\\|\underline{a}|=r}}W_{\underline{a}}(\mathbb{C})\boxtimes W_{\underline{a}}(\mathbb{C})$$

as $K_{G,\infty}\times K_{G,\infty}$-representations, where $|\underline{a}|=a_1+\cdots+a_n$. Let $\underline{a}'=(-a_n,\cdots,-a_1)$. When $t_n\geq n+1$ the $(\mathfrak{g}_\mathbb{R},K_{G,\infty})$-module $\mathcal{O}^{\mathrm{f}}(\mathbb{H}_n,\underline{a}')^{\mathrm{MVW}}$ gives the holomorphic discrete series $\mathcal{D}_{\underline{a}}$ of lowest $K_{G,\infty}$-type $\underline{a}$. Since we have always assumed $k\geq n+1$ we obtain the the multiplicity free decomposition

$$R_{0,2k}|_{\mathfrak{g}_\mathbb{C}\times\mathfrak{g}_\mathbb{C}} \cong \bigoplus_{a_1\geq\cdots\geq a_n\geq k} \mathcal{D}_{\underline{a}} \boxtimes \mathcal{D}_{\underline{a}}. \tag{4.22}$$

Let $\sigma_{k,\underline{t}}$ be the unique $K_{G,\infty}\times K_{G,\infty}$-sub-representation of $R_{0,2k}$ that corresponds to $\mathcal{D}_{\underline{t}}(\underline{t})\boxtimes\mathcal{D}_{\underline{t}}(\underline{t})$ under the above isomorphism. Now due to the equivariance property (4.18) it is clear that the zeta integral pairing

$$Z_\infty : R_{0,2k}\times\left(\widetilde{\mathcal{D}_{\underline{t}}}(-\underline{t})\times\mathcal{D}_{\underline{t}}(\underline{t})\right)\longrightarrow\mathbb{C} \tag{4.23}$$

factors through $\sigma_{k,\underline{t}}$.

Lemma 4.5.3. *The pairing* (4.23) *is nontrivial.*

Proof. Since the representation of $G(\mathbb{R})$ we are considering is discrete series, the arguments in [46] demonstrates the equivalence between the nontriviality of (4.23) and the nonvanishing of the theta lift of $\mathcal{D}_{\underline{t}}$ from $G(\mathbb{R})$ to $O(0,2k)$. The nonvanishing of this theta lift is easily seen from [36, Theorem (6.13)] or from (4.22) plus the doubling seesaw. □

Thus a section inside $R_{0,2k}$ pairs nontrivially with $\widetilde{\mathcal{D}_{\underline{t}}}(-\underline{t})\times\mathcal{D}_{\underline{t}}(\underline{t})$ by the zeta integral if and only if its projection to $\sigma_{k,\underline{t}}$ is nontrivial. Once we know that the projection of $f^d_{\kappa,\underline{\tau},\infty}$ to $\sigma_{k,\underline{t}}$ is nonzero, we can deduce the nonvanishing of the map (4.21), as well as that of the number $\frac{Z_\infty(f_{\kappa,\underline{\tau},\infty},v^\vee_{\underline{t}},v_{\underline{t}})}{\langle v^\vee_{\underline{t}},v_{\underline{t}}\rangle}$ in the statement of Proposition 4.5.1 since by [55, Theorem 2.A] and the definition of $f^d_{\kappa,\underline{\tau},\infty}$, its projection to $\sigma_{k,\underline{t}}$ is the highest weight vector on both factors. Therefore the last step is to prove the following lemma.

Lemma 4.5.4. *The section $f^d_{\kappa,\underline{\tau},\infty}$ projects nontrivially onto $\sigma_{k,\underline{t}}$.*

Proof. Let v_k be the lowest weight vector of the Verma module $M_k = U(\mathfrak{h}_\mathbb{C})\otimes_{U(\mathfrak{k}_{H,\mathbb{C}}\oplus\mathfrak{q}^-_H)}\det^k$. Under the isomorphism between $R_{0,2k}$ and M_k the section $f^{d,k}_\infty$ corresponds to v_k. Therefore by the definition of $f^d_{\kappa,\underline{\tau},\infty}$, what we need to show is that $\prod_{l=1}^n \det_l\left(\widehat{\mu}^+_0\right)^{t_l-t_{l+1}}\cdot v_k$ has a nontrivial projection onto the lowest $\mathfrak{k}_{G,\mathbb{C}}\times\mathfrak{k}_{G,\mathbb{C}}$-type of the $\mathcal{D}_{\underline{t}}\boxtimes\mathcal{D}_{\underline{t}}$-isotypic component of $M_k|_{\mathfrak{g}_\mathbb{C}\times\mathfrak{g}_\mathbb{C}}$. The universal enveloping algebra $U(\mathfrak{h}_\mathbb{C})$ comes with a natural grading $\bigcup_{r\geq0}U_r(\mathfrak{h}_\mathbb{C})$, where $U_r(\mathfrak{h}_\mathbb{C})$ is spanned by elements that can be written as a product of no more than r vectors in $\mathfrak{h}_\mathbb{C}$. Viewing M_k

as a $\mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}}$ module, it has the natural filtration $\bigcup_{r\geq 0} M_{k,r}$, with $M_{k,r}$ being the module generated by v_k under the action of $U(\mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}})$ and $U_r(\mathfrak{h}_{\mathbb{C}})$. Let $\mathfrak{q}_i^+$ be the Lie subalgebra of the abelian Lie algebra of $\mathfrak{q}_H^+$ spanned by entries of $\widehat{\mu}_i^+$, for $i = 0, 1, 2$. Consider the morphism of $\mathbb{C}$-vector spaces

$$\begin{aligned} U(\mathfrak{q}_1^+ \oplus \mathfrak{q}_2^+) \otimes_{\mathbb{C}} U_r(\mathfrak{q}_0^+) &\longrightarrow M_{k,r} \\ \alpha \otimes \beta &\longmapsto \alpha\beta \cdot v_k. \end{aligned} \tag{4.24}$$

It is injective by the PBW theorem, and the image contains $U_r(\mathfrak{h}_{\mathbb{C}}) \cdot v_k$. From the relation $[[(\mathfrak{k}_{G,\mathbb{C}} \oplus \mathfrak{q}_G^-) \times (\mathfrak{k}_{G,\mathbb{C}} \oplus \mathfrak{q}_G^-), \mathfrak{q}_0^+], \mathfrak{q}_0^+] \subset \mathfrak{q}_H^+$ we see that $\big((\mathfrak{k}_{G,\mathbb{C}} \oplus \mathfrak{q}_G^-) \times (\mathfrak{k}_{G,\mathbb{C}} \oplus \mathfrak{q}_G^-)\big) \cdot U_r(\mathfrak{q}_0^+) \cdot v_k$ is contained in $U(\mathfrak{q}_1^+ \oplus \mathfrak{q}_2^+) U_r(\mathfrak{q}_0^+) \cdot v_k$. Therefore the image of (4.24) is stable under the action of $(\mathfrak{k}_{G,\mathbb{C}} \oplus \mathfrak{q}_G^-) \times (\mathfrak{k}_{G,\mathbb{C}} \oplus \mathfrak{q}_G^-)$, and (4.24) is a bijection, which implies that

$$\prod_{l=1}^{n} \det_l \left(\widehat{\mu}_0^+\right)^{t_l - t_{l+1}} \cdot v_k \notin M_{k,|\underline{t}|-nk-1}. \tag{4.25}$$

At the same time the bijection (4.24) gives

$$\begin{aligned} M_{k,r}/M_{k,r-1} &\cong U(\mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}}) \otimes_{U\left((\mathfrak{k}_{G,\mathbb{C}}\oplus\mathfrak{q}_G^-)\times(\mathfrak{k}_{G,\mathbb{C}}\oplus\mathfrak{q}_G^-)\right)} \left(\mathrm{Sym}^r(\mathfrak{q}_H^+/\mathfrak{q}_G^+ \times \mathfrak{q}_G^+) \otimes \det{}^k\right) \\ &\cong \bigoplus_{\substack{a_1 \geq \cdots \geq a_n \geq 0 \\ |\underline{a}|=r}} \left(U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{k}_{G,\mathbb{C}}\oplus\mathfrak{q}_G^-)} W_{\underline{a}+k}\right) \boxtimes \left(U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{k}_{G,\mathbb{C}}\oplus\mathfrak{q}_G^-)} W_{\underline{a}+k}\right) \\ &\cong \bigoplus_{\substack{a_1 \geq \cdots \geq a_n \geq 0 \\ |\underline{a}|=r}} \mathcal{D}_{\underline{a}+k} \boxtimes \mathcal{D}_{\underline{a}+k}. \end{aligned} \tag{4.26}$$

The vector $\prod_{l=1}^{n} \det_l \left(\widehat{\mu}_0^+\right)^{t_l - t_{l+1}} \cdot v_k$ belongs to the $\underline{t} \times \underline{t}$-isotypic part of $M_k|_{\mathfrak{k}_{G,\mathbb{C}} \times \mathfrak{k}_{G,\mathbb{C}}}$, so its image in $M_{k,|\underline{t}|-nk}/M_{k,|\underline{t}|-nk-1}$, which is nonzero by (4.25), lands inside $\mathcal{D}_{\underline{t}}(\underline{t}) \boxtimes \mathcal{D}_{\underline{t}}(\underline{t})$ under the isomorphism (4.26). Now we can conclude that $\prod_{l=1}^{n} \det_l \left(\widehat{\mu}_0^+\right)^{t_l - t_{l+1}} \cdot v_k$ projects nontrivially to the lowest $\mathfrak{k}_{G,\mathbb{C}} \times \mathfrak{k}_{G,\mathbb{C}}$-type of the $\mathcal{D}_{\underline{t}} \boxtimes \mathcal{D}_{\underline{t}}$-isotypic component of $M_k|_{\mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}}}$. □

□

4.6 The q-expansions

For a Schwartz function α_p on $\mathrm{Sym}(2n,\mathbb{Q}_p)$ whose Fourier transform is supported on the compact set (4.10) and takes values inside a number field on $\mathrm{Sym}(2n,\mathbb{Q})$, set

$$f^{\alpha_p}_{\kappa,\underline{\tau}} = \bigotimes_{v\notin S} f^{\mathrm{ur}}_v(k-\frac{2n+1}{2},\phi^{-1}\chi^{\circ-1})\otimes f^{\mathrm{vol}}_{\kappa,\underline{\tau},N}\otimes f^{\alpha_p}_p(k-\frac{2n+1}{2},\phi^{-1}\chi^{\circ-1})\otimes f_{\kappa,\underline{\tau},\infty}.$$

From the discussion in Chapter 2, we see that the Siegel Eisenstein series $E^*(\cdot,f^{\alpha_p}_{\kappa,\underline{\tau}})$ on H and its restriction to $G\times G$ are both nearly holomorphic of degree less or equal to $|\underline{t}|-nk$. Since the archimedean section $f_{\kappa,\underline{\tau},\infty}$ belongs to the $\underline{t}\boxtimes\underline{t}$-isotypic component of $I_{Q_H,\infty}(k-\frac{2n+1}{2},\mathrm{Sign}^k)|_{K_{G,\infty}\times K_{G,\infty}}$ and is of weight $(\underline{t},\underline{t})$, we know that the form $A^{-1}_{n,\phi,k,\chi}\cdot E^*(\cdot,f^{\alpha_p}_{\kappa,\underline{\tau}})|_{G\times G}$ lies inside the image of the embedding

$$H^0(X_{G,\Gamma}\times X_{G,\Gamma},\mathcal{V}^{|\underline{t}|-nk}_{\underline{t}}\boxtimes\mathcal{V}^{|\underline{t}|-nk}_{\underline{t}})\otimes_{\mathbb{Q}(\zeta_N)}F\xrightarrow{\varphi_{G\times G}(\cdot,\mathfrak{c}_{\mathrm{can}})}\mathcal{A}(G(\mathbb{Q})\times G(\mathbb{Q})\backslash G(\mathbb{A})\times G(\mathbb{A})/\widehat{\Gamma}\times\widehat{\Gamma})_{\underline{t}\boxtimes\underline{t}},$$

where $\Gamma=\Gamma_1(N,p^m)$ with m sufficiently large, and F is a sufficiently large number field . We denote by $\mathcal{E}^{\alpha_p}_{\kappa,\underline{\tau}}$ the global section of $\mathcal{V}^{|\underline{t}|-nk}_{\underline{t}}\boxtimes\mathcal{V}^{|\underline{t}|-nk}_{\underline{t}}$ over $X_{G,\Gamma}\times X_{G,\Gamma}$ which is mapped to $A^{-1}_{n,\phi,k,\chi}\cdot E^*(\cdot,f^{\alpha_p}_{\kappa,\underline{\tau}})|_{G\times G}$, and consider the ($p$-adic) q-expansion, defined as (3.31), of the nearly holomorphic form $\mathcal{E}^{\alpha_p}_{\kappa,\underline{\tau}}$. Let $\Sigma_{N,p,+}$ be the intersection of the set $\Sigma_{p,+}$ and $\bigcap_{v|N}N^{-1}\,\mathrm{Sym}(2n,\mathcal{O}_v)^*$.

Proposition 4.6.1. *Suppose $(\kappa,\underline{\tau})\in\mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times\times T_n(\mathbb{Z}_p),\overline{\mathbb{Q}}_p^\times)$ is an admissible point satisfying the parity condition $\phi\chi(-1)=(-1)^k$. Then*

$$\varepsilon_{q,p\text{-adic}}(\mathcal{E}^{\alpha_p}_{\kappa,\underline{\tau}})=\sum_{\beta_1,\beta_2\in N^{-1}\,\mathrm{Sym}(n,\mathbb{Z})^*}\;\sum_{\boldsymbol{\beta}=\begin{pmatrix}\beta_1&\beta_0\\ {}^{\mathrm{t}}\beta_0&\beta_2\end{pmatrix}\in\Sigma_{N,p,+}}\mathfrak{c}^{\alpha_p}_{\kappa,\underline{\tau}}(\boldsymbol{\beta})q^{\beta_1}q^{\beta_2},\tag{4.27}$$

with

$$\mathfrak{c}^{\alpha_p}_{\kappa,\underline{\tau}}(\boldsymbol{\beta}) = \frac{\det(2\boldsymbol{\beta})^{1/2}}{G(\phi_{\boldsymbol{\beta}})}\widehat{\alpha}^{\mathrm{vol}}_N(\boldsymbol{\beta}) \cdot \chi^{-1}(C_{\phi_{\boldsymbol{\beta}}})C_{\phi_{\boldsymbol{\beta}}}^{-k+n+1} \cdot L_N(k-n, \phi_{\boldsymbol{\beta}}^{-1}\chi^{-1})^{-1} \cdot L^p(1-k+n, \phi_{\boldsymbol{\beta}}\chi)$$
$$\times\, g^S_{\boldsymbol{\beta}}(k, \phi\chi) \cdot \widehat{\alpha}_p(\boldsymbol{\beta}) \prod_{l=1}^{n} \det\nolimits_l(-2\beta_0)^{t_l - t_{l+1}} \det(2\boldsymbol{\beta})^{k-n-1}.$$

Proof. The proof is straightforward. All we need to be careful about is to be precise with all representations and maps involved here, instead of looking at isomorphim classes or working up to scalars. We use the symbol $\underline{\tau}_k$ to mean an arithmetic character of $T_n(\mathbb{Z}_p)$ with algebraic part equal to the scalar weight $k = \kappa_{\mathrm{alg}}$. Let $E^{\alpha_p}_{\kappa,\underline{\tau}_k}$ be the inverse image of $A^{-1}_{n,\phi,k,\chi} \cdot E^*(\cdot, f^{\alpha_p}_{\kappa,\underline{\tau}_k})$ under the map $\varphi_H(\cdot, \mathfrak{e}_{\mathrm{can}})$, which is a global section of the sheaf $\omega_k = \mathcal{V}^0_k$ over $X_{H,\Gamma}$. It follows from the definition of polynomial q-expansions, the canonical test object carried by $\mathbb{H}_{2n}$ and Proposition 4.2.3 that

$$E^{\alpha_p}_{\kappa,\underline{\tau}_k}(q, \underline{Y}) = \sum_{\boldsymbol{\beta}\in\Sigma_{N,p,+}} \mathfrak{c}^{\alpha_p}_{\kappa,\underline{\tau}_k}(\boldsymbol{\beta}) \cdot v_k q^{\boldsymbol{\beta}},$$

where v_k is a basis of the representation $\det^k$. Let $\underline{\boldsymbol{X}} = (\boldsymbol{X}_{ij})_{\leq i,j\leq 2n}$ be the basis of the representation τ_{2n} defined as in the paragraphs above Proposition 2.6.1, and we write it in $n \times n$ blocks as $\begin{pmatrix} \underline{X}_1 & \underline{X}_0 \\ {}^{\mathrm{t}}\underline{X}_0 & \underline{X}_2 \end{pmatrix}$. Applying Proposition 2.7.1 we get

$$(D^e_k E^{\alpha_p}_{\kappa,\underline{\tau}_k})(q, 0) = \sum_{\boldsymbol{\beta}\in\Sigma_{N,p,+}} \mathfrak{c}^{\alpha_p}_{\kappa,\underline{\tau}_k}(\boldsymbol{\beta}) \cdot v_k \otimes \left(\sum_{1\leq i\leq j\leq 2n} (2-\delta_{ij})\boldsymbol{\beta}_{ij}\boldsymbol{X}_{ij}\right)^e q^{\boldsymbol{\beta}}. \tag{4.28}$$

Let $\tau_{2n,0}$ be the direct summand of $\tau_{2n}|_{\mathrm{GL}(n)\times\mathrm{GL}(n)}$ generated by entries of $\underline{X}_0$. For $\underline{a} \in X(T_n)_+$ with $|\underline{a}| = e$ and $a_n \geq 0$, put $a_{n+1} = 0$ and fix the morphism of $\mathrm{GL}(n) \times \mathrm{GL}(n)$-representations

$$\det\nolimits^k \otimes \mathrm{Sym}^e\, \tau_{2n,0} \longrightarrow W_{\underline{a}+k} \boxtimes W_{\underline{a}+k} \tag{4.29}$$

sending $v_k \otimes \prod_{l=1}^n \det_l(\underline{X}_0)^{a_l - a_{l+1}}$ to the vector $w_{\underline{a}+k} \boxtimes w_{\underline{a}+k}$. Here for each $\underline{b} \in X(T_n)_+$ the function $w_{\underline{b}} : \mathrm{GL}(n)/N_n \to \mathbb{A}^1$ is defined as $w_{\underline{b}}(g) = \det(g)^{-b_1} \prod_{l=1}^{n-1} \det_{n-l}(g)^{b_l - b_{l+1}}$. Recall that

$V^r_{k\otimes \mathrm{Sym}^e\tau_{2n}} = \det^k\otimes \mathrm{Sym}^e\tau_{2n}[\underline{\boldsymbol{Y}}]_{\leq r}$. Similarly to $\underline{\boldsymbol{X}}$ we write $\underline{\boldsymbol{Y}} = \begin{pmatrix} \underline{Y}_1 & \underline{Y}_0 \\ {}^{\mathrm{t}}\underline{Y}_0 & \underline{Y}_2 \end{pmatrix}$. It is easy to check that modulo $\underline{X}_1$, $\underline{X}_2$, $\underline{Y}_0$ gives rise to a $Q_G\times Q_G$-representation morphism from $V^r_{k\otimes \mathrm{Sym}^e\tau_{2n}}|_{Q_G\times Q_G}$ to $\mathrm{Sym}^e\tau_{2n,0}\otimes(V^r_k\boxtimes V^r_k)$ which, when composed with (4.29), gives

$$\pi_{k,\underline{a}} : V^r_{k\otimes \mathrm{Sym}^e\tau_{2n}}\Big|_{Q_G\times Q_G} \longrightarrow V^r_{\underline{a}+k}\boxtimes V^r_{\underline{a}+k}.$$

Given Proposition 2.5.2, 2.6.1 it is tautological to check that we have the following commutative diagram

$$\begin{array}{ccc}
H^0(X_{H,\Gamma},\omega_k) & \xrightarrow{\varphi_H(\cdot,\mathfrak{e}_{\mathrm{can}})} & \mathcal{A}(H(\mathbb{Q})\backslash H(\mathbb{A})) \\
\big\downarrow D^e_k & & \big\downarrow \prod_{l=1}^n \det_l\left(-\frac{1}{4\pi i}\widehat{\mu}^+_0\right)^{a_l-a_{l+1}} \\
H^0(X_{H,\Gamma},\mathcal{V}^e_{k\otimes\mathrm{Sym}^e\tau_{2n}}) & \xrightarrow{\varphi_H(\cdot,\mathfrak{e}_{\mathrm{can}}\otimes w_{\underline{a}'}(\underline{X}^*_0))} & \mathcal{A}(H(\mathbb{Q})\backslash H(\mathbb{A})) \\
\big\downarrow & & \big\downarrow \\
H^0(X_{G,\Gamma}\times X_{G,\Gamma},\iota^*\mathcal{V}^e_{k\otimes\mathrm{Sym}^e\tau_{2n}}) & & \\
\big\downarrow \pi_{k,\underline{a}} & & \\
H^0(X_{G,\Gamma}\times X_{G,\Gamma},\mathcal{V}^e_{\underline{a}+k}\boxtimes\mathcal{V}^e_{\underline{a}+k}) & \xrightarrow{\varphi_{G\times G}(\cdot,\mathfrak{e}_{\mathrm{can}})} & \mathcal{A}(G(\mathbb{Q})\times G(\mathbb{Q})\backslash G(\mathbb{A})\times G(\mathbb{A})),
\end{array}$$

where $\underline{a}' = (-a_n,\cdots,-a_1)$. Thus the ($p$-adic) q-expansion of $\mathcal{E}^{\alpha_p}_{\kappa,\underline{\tau}}$ is obtained from applying $w_{\underline{a}'}(\underline{X}^*_0)$, with $\underline{a} = \underline{t}-k$, to (4.28) and setting $\underline{\boldsymbol{Y}}$ to zero, $q^{\boldsymbol{\beta}}$ to $q^{\beta_1}q^{\beta_2}$, i.e.

$$\varepsilon_{q,p\text{-adic}}(\mathcal{E}^{\alpha_p}_{\kappa,\underline{\tau}}) = \sum_{\boldsymbol{\beta}=\left(\begin{smallmatrix}\beta_1 & \beta_0\\ {}^{\mathrm{t}}\beta_0 & \beta_2\end{smallmatrix}\right)\in\Sigma_{N,p,+}} \prod_{l=1}^n \det_l(-2\beta_0)^{t_l-t_{l+1}}\mathfrak{c}^{\alpha_p}_{\kappa,\underline{\tau}_k}(\boldsymbol{\beta})q^{\beta_1}q^{\beta_2},$$

which is exactly (4.27). □

Chapter 5

The measure $\mu_{\mathcal{E},q\text{-exp}}$ and local zeta integrals at p

We review briefly the theory of p-adic measures, and then pick suitable $\widehat{\alpha}_{\kappa,\underline{\tau},p}$ such that the $\varepsilon_{q,p\text{-adic}}(\mathcal{E}_{\kappa,\underline{\tau}}^{\alpha_{\kappa,\underline{\tau},p}})$'s amalgamate into an element of $\mathcal{M}eas\left(\mathbb{Z}_p^\times \times T_n(\mathbb{Z}_p), \mathcal{O}_F[[N^{-1}\operatorname{Sym}(n,\mathbb{Z})_{>0}^{*\oplus 2}]]\right)$, where F is a finite extension of $\mathbb{Q}_p$ containing all N-th roots of unity. Then we retrieve $f_{p,\kappa,\underline{\tau}}$ from $\widehat{\alpha}_{p,\kappa,\underline{\tau}}$ and carry out local computations at p.

5.1 p-adic measures

Suppose that Y is a compact and totally disconnected topological space. Let R be a p-adic ring, i.e. $R = \varprojlim R/p^nR$, and M be a p-adically complete R-module. Denote by $\mathscr{C}(Y,R)$ the R-algebra of continuous R-valued functions on Y. An M-valued p-adic measure on Y is a continuous R-linear map

$$\begin{aligned} \mu : \mathscr{C}(Y,R) &\longrightarrow M \\ f &\longmapsto \mu(f) = \int_Y f\, d\mu, \end{aligned}$$

where the topology on $\mathscr{C}(Y,R)$ is the topology of uniform convergence. The set of M-valued p-adic measures on Y is a p-adically complete R-module and is denoted as $\mathcal{M}eas(Y,M)$. For an

R-algebra R', which is also p-adically complete, since $\mathscr{C}(Y,R') = \mathscr{C}(Y,R)\widehat{\otimes}R'$, there is a natural map $\mathcal{M}eas(Y,M) \to \mathcal{M}eas(Y, M\widehat{\otimes}R')$ and we view $\mathcal{M}eas(Y,M)$ as a subset of $\mathcal{M}eas(Y, M\widehat{\otimes}R')$ if $R \to R'$ is injective. From definition it is easily seen that we have the following maps

$$\begin{aligned} Y &\longrightarrow \mathcal{M}eas(Y,R) \\ y &\longmapsto \delta_y(f) := f(y), \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} \mathcal{M}eas(Y,M) \times \mathscr{C}(Y,R) &\longrightarrow \mathcal{M}eas(Y,M) \\ (\mu, h) &\longmapsto \mu_h(f) := \int_Y fh\, d\mu. \end{aligned} \tag{5.2}$$

Moreover if we assume that Y is equipped with the structure of an abelian group (written multiplicatively), then we can define the convolution on $\mathcal{M}eas(Y,R)$ as

$$\begin{aligned} \mathcal{M}eas(Y,R) \times \mathcal{M}eas(Y,R) &\longrightarrow \mathcal{M}eas(Y,R) \\ (\mu_1, \mu_2) &\longmapsto \mu_1 * \mu_2(f) := \int_Y \int_Y f(yz)\, d\mu_1(y) d\mu_2(z). \end{aligned} \tag{5.3}$$

If $f \in \mathrm{Hom}_{\mathrm{cont}}(Y, R^\times)$ is a multiplicative character, we have

$$\int_Y f\, d(\mu_1 * \mu_2) = \left(\int_Y f\, d\mu_1\right)\left(\int_Y f\, d\mu_2\right). \tag{5.4}$$

5.2 The p-adic measure $\mu_{\mathcal{E},q\text{-exp}}$ and the section $f_{\kappa,\underline{\tau},p}$

Now take $Y = \mathbb{Z}_p^\times \times T_n(\mathbb{Z}_p)$ and $R = \mathcal{O}_F$. The goal is to select the Schwartz function $\widehat{\alpha}_{\kappa,\underline{\tau},p}$ and construct an element $\mu_{\mathcal{E},q\text{-exp}}$ inside the space $\mathcal{M}eas\left(\mathbb{Z}_p^\times \times T_n(\mathbb{Z}_p), \mathcal{O}_F[[N^{-1}\,\mathrm{Sym}(n,\mathbb{Z})^{*\oplus 2}_{>0}]]\right)$ such that

$$\int_{\mathbb{Z}_p^\times \times T_n(\mathbb{Z}_p)} (\kappa, \underline{\tau})\, d\mu_{\mathcal{E},q\text{-exp}} = \varepsilon_{q,p\text{-adic}}(\mathcal{E}_{\kappa,\underline{\tau}}^{\alpha_{\kappa,\underline{\tau},p}}).$$

By definition it is enough to construct, for each $\boldsymbol{\beta} \in \Sigma_{N,p,+}$, a measure $\mu_{\mathcal{E},\boldsymbol{\beta}} \in \mathcal{M}eas(Y, \mathcal{O}_F)$ with the property that

$$\begin{aligned}\int_{\mathbb{Z}_p^\times \times T_n(\mathbb{Z}_p)} (\kappa, \underline{\tau})\, d\mu_{\mathcal{E},\boldsymbol{\beta}} = \mathfrak{c}_{\kappa,\underline{\tau}}^{\alpha_{\kappa,\underline{\tau},p}}(\boldsymbol{\beta}) = \frac{\det(2\boldsymbol{\beta})^{1/2}}{G(\phi_{\boldsymbol{\beta}})} \widehat{\alpha}_N^{\mathrm{vol}}(\boldsymbol{\beta}) \cdot \chi^{-1}(C_{\phi_{\boldsymbol{\beta}}}) C_{\phi_{\boldsymbol{\beta}}}^{-k+n+1} \cdot L_N(k-n, \phi_{\boldsymbol{\beta}}^{-1}\chi^{-1})^{-1} \\ \times L^p(1-k+n, \phi_{\boldsymbol{\beta}}\chi) \cdot g_{\boldsymbol{\beta}}^S(k, \phi\chi) \\ \times \widehat{\alpha}_{\kappa,\underline{\tau},p}(\boldsymbol{\beta}) \prod_{l=1}^n \det{}_l(-2\beta_0)^{t_l - t_{l+1}} \det(2\boldsymbol{\beta})^{k-n-1}.\end{aligned} \tag{5.5}$$

Because of (5.4) we can deal with the RHS of (5.5) term by term.

The first term $\frac{\det(2\boldsymbol{\beta})^{1/2}}{G(\phi_{\boldsymbol{\beta}})} \widehat{\alpha}_N^{\mathrm{vol}}(\boldsymbol{\beta})$ is a constant inside $\mathcal{O}_F$. The second term is interpolated by the measure $C_{\phi_{\boldsymbol{\beta}}}^{n+1} \cdot \delta_{(C_{\phi_{\boldsymbol{\beta}}}^{-1}, \mathrm{id})}$, where id is the unity of $T_n(\mathbb{Z}_p)$. Both of the term $L_N(k-n, \phi_{\boldsymbol{\beta}}^{-1}\chi^{-1})^{-1}$ and the term $g_{\boldsymbol{\beta}}^S(k, \phi\chi)$ can be written as $\mathcal{O}_F$-linear combinations of $\chi^{-1}(m)m^{-k}$ with some positive integers m prime to p. For each m the measure $\delta_{(m^{-1}, \mathrm{id})}$ interpolates $\chi^{-1}(m)m^{-k}$.

Regarding the term $L^p(1-k+n, \phi_{\boldsymbol{\beta}}\chi) = (1 - \phi_{\boldsymbol{\beta}}\chi^\circ(p)p^{k-n-1}) L(1-k+n, \phi_{\boldsymbol{\beta}}\chi^\circ)$, there is the following theorem on the existence of p-adic Dirichlet L-functions.

Theorem 5.2.1 (Kubota–Leopoldt, [31, Theorem 4.4.1])**.** *Given a nontrivial primitive Dirichlet character ξ with conductor prime to p, there is a unique measure $\mu_\xi \in \mathcal{M}eas\left(\mathbb{Z}_p^\times, \mathbb{Z}_p[\xi]\right)$ such that for all integers $j \geq 1$ and finite order characters $\chi \in \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, \mathbb{C}^\times)$,*

$$\int_{\mathbb{Z}_p^\times} \chi(y) y^j \, d\mu_\xi(y) = (1 - \xi\chi^\circ(p)p^{j-1}) L(1-j, \xi\chi^\circ).$$

As for the trivial character, for each fixed prime ℓ prime to p, there is a unique measure $\mu_\ell \in \mathcal{M}eas\left(\mathbb{Z}_p^\times, \mathbb{Z}_p\right)$ such that for all j and χ as before,

$$\int_{\mathbb{Z}_p^\times} \chi(y) y^j \, d\mu_\ell(y) = (1 - \chi(\ell)^{-1}\ell^{-j})(1 - \chi^\circ(p)p^{j-1}) L(1-j, \chi^\circ).$$

For simplicity we assume that $\phi^2 \neq 1$ from now on, so that $\phi_{\boldsymbol{\beta}}$ will always be nontrivial. Without this assumption, for a fixed prime ℓ prime to p, we can interpolate $(1 - \chi(\ell)^{-1}\ell^{-k+n}) \cdot \mathcal{E}_{\kappa,\underline{\tau}}$ instead of $\mathcal{E}_{\kappa,\underline{\tau}}$. Then everything in the following goes the same, and we get the measure $\mu_{\mathcal{C},\ell,\phi,\beta_1,\beta_2}$ as described in Remark 1.0.2.

Let $h_n(y) = y^{-n}$. Using (5.2) we get the measure $\mu_{\phi_{\boldsymbol{\beta}},h_n}$ on $\mathbb{Z}_p^\times$ with $\mu_{\phi_{\boldsymbol{\beta}},h_n}(\kappa) = L^p(1-k+n, \phi_{\boldsymbol{\beta}}\chi)$, whose direct product with the measure δ_{id} on $T_n(\mathbb{Z}_p)$ gives the desired p-adic interpolation of $L^p(1-k+n, \phi_{\boldsymbol{\beta}}\chi)$.

It remains to treat the term $\widehat{\alpha}_{\kappa,\underline{\tau},p}(\boldsymbol{\beta}) \prod_{l=1}^n \det_l(-2\beta_0)^{t_l - t_{l+1}} \det(2\boldsymbol{\beta})^{k-n-1}$ by selecting suitable $\widehat{\alpha}_{\kappa,\underline{\tau},p}$. Due to the density of polynomial functions inside $\mathscr{C}(\mathbb{Z}_p^\times \times T_n(\mathbb{Z}_p), F)$, the measure interpolating this expression must be $\det(2\boldsymbol{\beta})^{-n-1} \cdot \delta_{(b_0, b_1, \cdots, b_n)}$, where $b_0 = \det(2\boldsymbol{\beta}) \det(-2\beta_0)^{-1}$, $b_1 = \det_1(-2\beta_0)$, $b_l = \det_{l-1}(-2\beta_0)^{-1} \det_l(-2\beta_0)$ for $2 \le l \le n$, and we must require all the $\det_l(-2\beta_0)$ to lie inside $\mathbb{Z}_p^\times$. Accordingly we see that a natural choice of the Schwartz function $\widehat{\alpha}_{\kappa,\underline{\tau},p}$ on $\mathrm{Sym}(2n, \mathbb{Q}_p)$ is

$$\widehat{\alpha}_{\kappa,\underline{\tau},p}\left(\begin{pmatrix} \beta_1 & \beta_0 \\ {}^{\mathrm{t}}\beta_0 & \beta_2 \end{pmatrix}\right)$$
$$= \mathbb{1}_{p^2 \,\mathrm{Sym}(n,\mathbb{Z}_p)^*}(\beta_1) \mathbb{1}_{\mathrm{Sym}(n,\mathbb{Z}_p)^*}(\beta_2) \prod_{l=1}^n \mathbb{1}_{\mathrm{GL}_l(\mathbb{Z}_p)}\left((2\beta_0)_l\right) \cdot \chi(\det(2\boldsymbol{\beta})) \prod_{l=1}^n \psi_l \psi_{l+1}^{-1}(\det_l(-2\beta_0)),$$

where (similar to how we have put $t_{n+1} = k$) we set $\psi_{n+1} = \chi$, and $(2\beta_0)_l$ stands for the upper left $l \times l$ minor of $2\beta_0$. In fact the only freedom in the choice is to vary the support.

The inverse Fourier transform of the above defined $\widehat{\alpha}_{\kappa,\underline{\tau},p}$ gives $\alpha_{\kappa,\underline{\tau},p}$, and our choice of $f_{\kappa,\underline{\tau},p}$ is the "big cell" section $f_{\kappa,\underline{\tau}}^{\alpha_{\kappa,\underline{\tau},p}}(s,\xi) \in I_{Q_H,p}(s,\xi)$ associated to $\alpha_{\kappa,\underline{\tau},p}$, evaluated at $s = k - \frac{2n+1}{2}$ and $\xi = \phi^{-1}\chi^{\circ-1}$. Now it is clear that the desired measure $\mu_{\mathcal{E},\boldsymbol{\beta}}$ in (5.5) exists. One also notices that its evaluation at $(\kappa, \underline{\tau})$ with $\phi\chi(-1) \neq (-1)^k$ is 0.

So far for all admissible $(\kappa, \underline{\tau})$ satisfying $\phi\chi(-1) = (-1)^k$, we have made our choices of $f_{\kappa,\underline{\tau},v} \in I_{Q_H,v}(k - \frac{2n+1}{2}, \phi^{-1}\chi^{\circ-1})$ for all places v. From now on we write $f_{\kappa,\underline{\tau}}$ to mean the product of all the local sections we have selected for admissible $(\kappa, \underline{\tau})$ if $\phi\chi(-1) = (-1)^k$, and simply 0 if the parity condition does not hold. We denote by $\mathcal{E}_{\kappa,\underline{\tau}}$ the global section of $\mathcal{V}_{\underline{t}}^{|\underline{t}|-nk} \boxtimes \mathcal{V}_{\underline{t}}^{|\underline{t}|-nk}$ over $X_{G,\Gamma} \times X_{G,\Gamma}$ that is mapped to $A_{n,\phi,k,\chi}^{-1} \cdot E^*(\cdot, f_{\kappa,\underline{\tau}})|_{G \times G}$ by the map $\varphi_{G\times G}(\cdot, \mathfrak{e}_{\mathrm{can}})$.

Theorem 5.2.2. *There is a measure* $\mu_{\mathcal{E},q\text{-}\mathrm{exp}} \in \mathscr{M}eas\left(\mathbb{Z}_p^\times \times T_n(\mathbb{Z}_p), \mathcal{O}_F[[N^{-1}\,\mathrm{Sym}(n,\mathbb{Z})_{>0}^{*\oplus 2}]]\right)$ *sat-*

isfying

$$\int_{\mathbb{Z}_p^\times\times T_n(\mathbb{Z}_p)}(\kappa,\underline{\tau})\,d\mu_{\mathcal{E},q\text{-exp}}=\varepsilon_{q,p\text{-adic}}(\mathcal{E}_{\kappa,\underline{\tau}})$$

for all admissible $(\kappa,\underline{\tau})\in\mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times\times T_n(\mathbb{Z}_p),\overline{\mathbb{Q}}_p^\times)$.

Explicit computation results on the local zeta integrals for $v\nmid p\infty$ have been obtained in Theorem 4.3.2 and Proposition 4.4.1. For the archimedean place we the nonvanishing result is shown in Proposition 4.5.1. It remains to carry out the local computations at p, which occupy the rest of this section. All the results are summarized in the following Proposition 5.2.3, which gives the interpolation properties of the $(n+1)$-variable p-adic L-function we will finally construct.

By the reasoning near the end of §4.1.2 we can define $(e\times 1)\mathcal{E}_{\kappa,\underline{\tau}}$, the ordinary projection of $\mathcal{E}_{\kappa,\underline{\tau}}$ on the first factor. For an irreducible cuspidal automorphic representation π of $G(\mathbb{A})$ with $\pi_\infty\cong\mathcal{D}_{\underline{t}}$, denote by $\pi_{\underline{t}}^{\widehat{\Gamma}_1(N,p^m),\underline{\psi}}$ the subspace of π consisting of automorphic forms whose archimedean components, under an isomorphism $\pi\cong\bigotimes'_v\pi_v$ are the highest weight vector inside the lowest $K_{G,\infty}$-type $\underline{t}$, invariant under the right translation of $\widehat{\Gamma}_1(N,p^m)$, and acted on by the character $\underline{\psi}$ by the group $T_G(\mathbb{Z}_p)$.

Proposition 5.2.3. *Let* $\varphi\in\pi_{\underline{t}}^{\widehat{\Gamma}_1(N,p^m),\underline{\psi}}$ *be a weight* $\underline{t}$ *ordinary cuspidal Siegel modular form. Regarding the Petersson inner product of* $\overline{\varphi}$ *with the automorphic form* $\varphi_{G\times G}((e\times 1)\mathcal{E}_{\kappa,\underline{\tau}},\mathfrak{e}_{\mathrm{can}})$ *on its first factor, we have*

$$\begin{aligned}&\left\langle\varphi_{G\times G}((e\times 1)\mathcal{E}_{\kappa,\underline{\tau}},\mathfrak{e}_{\mathrm{can}})(\cdot,g),\,\overline{\varphi}\right\rangle\\ =\;&\phi(-1)^n\mathrm{vol}\left(\widehat{\Gamma}(N)\right)\frac{p^{n^2}(p-1)^n}{\prod_{l=1}^n(p^{2l}-1)}\cdot\frac{\Gamma(k-n)\Gamma_{2n}(k)}{2^{k+n-1}(\pi i)^{2nk+k-n}}\cdot\frac{Z_\infty(f_{\kappa,\underline{\tau},\infty},v_{\underline{t}}^\vee,v_{\underline{t}})}{\langle v_{\underline{t}}^\vee,v_{\underline{t}}\rangle}\\ &\times E_p(k-n,\pi\times\phi^{-1}\chi^{-1})\cdot L^{Np\infty}(k-n,\pi\times\phi^{-1}\chi^{-1})\cdot eW(\varphi)(g),\end{aligned}$$

where the modified Euler factor $E_p(k-n,\pi\times\phi^{-1}\chi^{-1})$ *is defined by* (1.3)*, and the operator* $W:\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))\to\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ *is defined as*

$$W(\varphi)(g):=\left(\int_{N_G(\mathbb{Z}_p)}R_p(u)\overline{\varphi}^{\vartheta}du\right)(g)=\int_{N_G(\mathbb{Z}_p)}\overline{\varphi}(\vartheta gu\vartheta)\,du.\tag{5.6}$$

Thanks to the multiplicity one theorem for symplectic groups, the operator W preserves π and $\pi_{\underline{t}}^{\widehat{\Gamma}_1(N,p^m),\underline{\psi}}$. However this W is not $\mathbb{C}$-linear.

In the unitary case such local zeta integrals are calculated in [19,64]. The restrictive conditions in [64] amount to $c_{\chi\psi_1} > c_{\chi\psi_2} > \cdots > c_{\chi\psi_n}$ here. Computations in [19] are done in a different way from ours below, applying the Godement–Jacquet local functional equation, and without considering the ordinary projection.

5.3 An observation of Böcherer–Schmidt

The first step of the calculation is to compute the inverse Fourier transform of $\widehat{\alpha}_{\kappa,\underline{\tau},p}$. However this computation is in fact not very convenient because of the term $\chi(\det(2\boldsymbol{\beta}))$. The observation of Böcherer–Schmidt is that, for computing local zeta integrals, instead of using $\widehat{\alpha}_{\kappa,\underline{\tau},p}$, we may use the Schwartz function modified from it by changing $\chi(\det(2\boldsymbol{\beta}))$ to $\chi(-1)^n\chi^2(\det(2\beta_0))$, i.e.

$$\begin{aligned}
&\widehat{\alpha}'_{\kappa,\underline{\tau},p}\left(\begin{pmatrix}\beta_1 & \beta_0\\ {}^{\mathrm{t}}\beta_0 & \beta_2\end{pmatrix}\right)\\
&=\mathbb{1}_{p^2\,\mathrm{Sym}(n,\mathbb{Z}_p)^*}(\beta_1)\mathbb{1}_{\mathrm{Sym}(n,\mathbb{Z}_p)^*}(\beta_2)\prod_{l=1}^{n}\mathbb{1}_{\mathrm{GL}_l(\mathbb{Z}_p)}((2\beta_0)_l)\cdot\chi(-1)^n\chi^2(\det(2\beta_0))\prod_{l=1}^{n}\psi_l\psi_{l+1}^{-1}(\det_l(-2\beta_0))\\
&=\mathbb{1}_{p^2\,\mathrm{Sym}(n,\mathbb{Z}_p)^*}(\beta_1)\mathbb{1}_{\mathrm{Sym}(n,\mathbb{Z}_p)^*}(\beta_2)\prod_{l=1}^{n}\mathbb{1}_{\mathrm{GL}_l(\mathbb{Z}_p)}((2\beta_0)_l)\cdot\chi(-1)^n\prod_{l=1}^{n}\psi'_l\psi'^{-1}_{l+1}(\det_l(-2\beta_0)),
\end{aligned}\tag{5.7}$$

where $\psi'_l=\chi\psi_l$ if $1\le l\le n$ and ψ'_{n+1} is the trivial character. Let $f'_{\kappa,\underline{\tau},p}\in I_{Q_H,p}(k-\frac{2n+1}{2},\phi^{-1}\chi^{\circ-1})$ be the section associated to $\alpha'_{\kappa,\underline{\tau},p}$, the inverse Fourier transform of $\widehat{\alpha}'_{\kappa,\underline{\tau},p}$.

Recall that we have defined the adelic $\mathbb{U}_p$-operators in (4.3) and (4.4). For $\underline{a}\in C_n^+$, with the embedding $\iota: G\times G\hookrightarrow H$, we can make $U_{p,\underline{a}}$ act on smooth functions on $H(\mathbb{A})$ simply by the formula (4.4) on the first factor. We use $U_{p,\underline{a}}\times 1$ to denote this action, and it is easily seen to be compatible with restriction by ι and the operator $U_{p,\underline{a}}\times 1$ on $G\times G$. The operator $U_{p,n}$ is the one with $\underline{a}=(1,\cdots,1)$,

Proposition 5.3.1. *If m is a positive integer such that the conductor of χ divides p^{2m}, then*

$$(U_{p,n}^m \times 1)E^*(\cdot, f_{\kappa,\underline{\tau}}) = (U_{p,n}^m \times 1)E^*(\cdot, f'_{\kappa,\underline{\tau}}).$$

Proof. Let $E^{*,p}_{\boldsymbol{\beta}}(h_{\boldsymbol{z}}, f_{\kappa,\underline{\tau}})$ be $E^{*}_{\boldsymbol{\beta}}(h_{\boldsymbol{z}}, f_{\kappa,\underline{\tau}})$ with the factor $W_{\boldsymbol{\beta},p}(1_p, f_{\kappa,\underline{\tau},p})$ removed. The $\boldsymbol{\beta}$-th Fourier coefficient of $(U_{p,n}^m \times 1)E^*(\cdot, f_{\kappa,\underline{\tau}})$ at $h_{\boldsymbol{z}}$ is equal to $E^{*,p}_{\boldsymbol{\beta}}(h_{\boldsymbol{z}}, f_{\kappa,\underline{\tau}})W_{\boldsymbol{\beta},p}(1_p, (U_{p,n}^m \times 1) f_{\kappa,\underline{\tau},p})$. We define similarly $E^{*,p}_{\boldsymbol{\beta}}(h_{\boldsymbol{z}}, f'_{\kappa,\underline{\tau}})$, and it is obvious that $E^{*,p}_{\boldsymbol{\beta}}(h_{\boldsymbol{z}}, f_{\kappa,\underline{\tau}}) = E^{*,p}_{\boldsymbol{\beta}}(h_{\boldsymbol{z}}, f'_{\kappa,\underline{\tau}})$. Therefore all we need to show is that

$$W_{\boldsymbol{\beta},p}(1_p, (U_{p,n}^m \times 1) f_{\kappa,\underline{\tau},p}) = W_{\boldsymbol{\beta},p}(1_p, (U_{p,n}^m \times 1) f'_{\kappa,\underline{\tau},p}) \tag{5.8}$$

for all $\boldsymbol{\beta} \in \mathrm{Sym}(2n, \mathbb{Q})$. Let $S_n = \mathrm{Sym}(n, \mathbb{Q}_p)$, $M_n = M_n(\mathbb{Q}_p)$ and for element $\varsigma \in S_{2n}$ we write it in $n \times n$ blocks as $\begin{pmatrix} \varsigma_1 & \varsigma_0 \\ {}^{\mathrm{t}}\varsigma_0 & \varsigma_1 \end{pmatrix}$. One easily computes

$$\begin{aligned}
& p^{-(|\underline{t}|-n(n+1))m} W_{\boldsymbol{\beta},p}(1_p, (U_{p,n}^m \times 1) f_{\kappa,\underline{\tau},p}) \\
&= \sum_{u \in S_n(\mathbb{Z}/p^{2m}\mathbb{Z})} \int_{S_{2n}} f_{\kappa,\underline{\tau},p}\left(w u(\varsigma) \iota \left(\begin{pmatrix} p^m & u p^{-m} \\ 0 & p^{-m} \end{pmatrix}, 1 \right) \right) \mathbf{e}_p(-\operatorname{Tr} \boldsymbol{\beta}\varsigma)\, d\varsigma \\
&= \sum_{u \in S_n(\mathbb{Z}/p^{2m}\mathbb{Z})} \int_{M_n} \int_{S_n} \int_{S_n} f_{\kappa,\underline{\tau},p} \left(\begin{pmatrix} 0 & 0 & -p^{-m} & 0 \\ 0 & 0 & 0 & -1 \\ p^m & 0 & (\varsigma_1 + u) p^{-m} & \varsigma_0 \\ 0 & 1 & {}^{\mathrm{t}}\varsigma_0 p^{-m} & \varsigma_2 \end{pmatrix} \right) \mathbf{e}_p(-\operatorname{Tr} \boldsymbol{\beta}\varsigma)\, d\varsigma_1\, d\varsigma_2\, d\varsigma_0 \\
&= \left(\phi(p) p^k\right)^{nm} \sum_{u \in S_n(\mathbb{Z}/p^{2m}\mathbb{Z})} \int_{M_n} \int_{S_n} \int_{S_n} \alpha_{\kappa,\underline{\tau},p} \left(\begin{pmatrix} (\varsigma_1 + u) p^{-2m} & \varsigma_0 p^{-m} \\ {}^{\mathrm{t}}\varsigma_0 p^{-m} & \varsigma_2 \end{pmatrix} \right) \mathbf{e}_p(-\operatorname{Tr} \boldsymbol{\beta}\varsigma)\, d\varsigma_1\, d\varsigma_2\, d\varsigma_0 \\
&= \left(\phi(p) p^{k-2n-1}\right)^{nm} \sum_{u \in S_n(\mathbb{Z}/p^{2m}\mathbb{Z})} \mathbf{e}_p(\operatorname{Tr} \beta_1 u) \int_{S_{2n}} \alpha_{\kappa,\underline{\tau},p}(\varsigma) \mathbf{e}_p\left(-\operatorname{Tr} \begin{pmatrix} \beta_1 p^{2m} & \beta_0 p^m \\ {}^{\mathrm{t}}\beta_0 p^m & \beta_2 \end{pmatrix} \varsigma \right) d\varsigma \\
&= \left(\phi(p) p^{k-n}\right)^{nm} \mathbb{1}_{\mathrm{Sym}(n,\mathbb{Z}_p)}(\beta_1) \widehat{\alpha}_{\kappa,\underline{\tau},p} \left(\begin{pmatrix} \beta_1 p^{2m} & \beta_0 p^m \\ {}^{\mathrm{t}}\beta_0 p^m & \beta_2 \end{pmatrix} \right),
\end{aligned}$$

and similarly

$$p^{-(|\underline{t}|-n(n+1))m}W_{\boldsymbol{\beta},p}(1_p,(U_{p,n}^m\times 1)f'_{\kappa,\underline{\tau},p})=\left(\phi(p)p^{k-n}\right)^{nm}\mathbb{1}_{\mathrm{Sym}(n,\mathbb{Z}_p)}(\beta_1)\widehat{\alpha}'_{\kappa,\underline{\tau},p}\left(\begin{pmatrix}\beta_1p^{2m} & \beta_0p^m\\ {}^{\mathrm{t}}\beta_0p^m & \beta_2\end{pmatrix}\right).$$

It is easily seen that if β_1, β_0p^m and β_2 are all integral, then

$$\det\left(\begin{pmatrix}\beta_1p^{2m} & \beta_0p^m\\ {}^{\mathrm{t}}\beta_0p^m & \beta_2\end{pmatrix}\right)\equiv(-1)^n\det\left(\beta_0p^m\right)^2\mod p^{2m},$$

so when the conductor of χ divides p^{2m}, the functions $\widehat{\alpha}_{\kappa,\underline{\tau},p}$ and $\widehat{\alpha}'_{\kappa,\underline{\tau},p}$ take the same value at such $\begin{pmatrix}\beta_1p^{2m} & \beta_0p^m\\ {}^{\mathrm{t}}\beta_0p^m & \beta_2\end{pmatrix}$, and (5.8) is true for all $\boldsymbol{\beta}\in\mathrm{Sym}(2n,\mathbb{Q})$. □

Due to the above proposition we know that for $\underline{a}\in C_n^+$ with $\Delta\underline{a}\gg 0$ and $\varphi\in\pi$ we have

$$\left\langle(U_{p,\underline{a}}\times 1)E^*\left(\iota(\cdot,g),f_{\kappa,\underline{\tau}}\right),\overline{\varphi}\right\rangle=\left\langle(U_{p,\underline{a}}\times 1)E^*\left(\iota(\cdot,g),f'_{\kappa,\underline{\tau}}\right),\overline{\varphi}\right\rangle,$$

where the Petersson inner product is taken on the first factor of the restricted Siegel Eisenstein series. We will compute the local zeta integral for $f'_{\kappa,\underline{\tau},p}$.

5.4 The inverse Fourier transform of $\widehat{\alpha}'_{\kappa,\underline{\tau},p}$

In this subsection we regard characters of $\mathbb{Z}_p^\times$ (including the trivial character) as functions on $\mathbb{Q}_p$ by making them take the value 0 outside $\mathbb{Z}_p^\times$. Given characters of $\mathbb{Z}_p^\times$ of finite order $\underline{\xi}=(\xi_1,\cdots,\xi_n)$ whose conductors are $p^{c_{\xi_1}},\cdots,p^{c_{\xi_n}}$, for each $1\le l\le n$, define the Schwartz function $\Psi_{\underline{\xi},l}$ on $M_l(\mathbb{Q}_p)$ as

$$\Psi_{\underline{\xi},l}(x)=\mathbb{1}_{M_l(\mathbb{Z}_p)}(x)\cdot\prod_{j=1}^{l-1}\xi_j\xi_{j+1}^{-1}(\det_j(-x))\cdot\xi_l(\det_l(-x)).$$

Denote by $\mathcal{F}^{-1}\Psi_{\underline{\xi},l}$ the inverse Fourier transform of $\Psi_{\underline{\xi},l}$. First we give an inductive formula for $\mathcal{F}^{-1}\Psi_{\underline{\xi},l}$. Set

$$\Phi_{\underline{\xi},l}(\varsigma) = \begin{cases} \xi_l^{-1}\left(p^{c_{\xi_l}}\varsigma_l\right)\mathcal{F}^{-1}\Psi_{\underline{\xi},l-1}(\varsigma') & \text{if } \varsigma \in \begin{pmatrix} I_{l-1} & \mathbb{Z}_p^{l-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \varsigma' & \\ & \varsigma_l \end{pmatrix}\begin{pmatrix} I_{l-1} & 0 \\ {}^{\mathrm{t}}\mathbb{Z}_p^{l-1} & 1 \end{pmatrix} \\ & \text{with } \varsigma' \in M_{l-1}(\mathbb{Q}_p),\ \varsigma_l \in \mathbb{Q}_p, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\Phi'_{\underline{\xi},l}(\varsigma) = \begin{cases} \mathcal{F}^{-1}\Psi_{\underline{\xi},l-1}(\varsigma') & \text{if } \varsigma \in \begin{pmatrix} \varsigma' & \mathbb{Z}_p^{l-1} \\ {}^{\mathrm{t}}\mathbb{Z}_p^{l-1} & \mathbb{Z}_p \end{pmatrix} \text{ with } \varsigma' \in M_{l-1}(\mathbb{Q}_p), \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 5.4.1. *We have*

1. *if ξ_l is nontrivial, then*

$$\mathcal{F}^{-1}\Psi_{\underline{\xi},l}(\varsigma) = p^{-lc_{\xi_l}}G(\xi_l)\Phi_{\underline{\xi},l}(\varsigma), \tag{5.9}$$

2. *if ξ_l is the trivial character, then*

$$\mathcal{F}^{-1}\Psi_{\underline{\xi},l}(\varsigma) = -p^{-l}\Phi_{\underline{\xi},l}(\varsigma) + (1-p^{-1})\Phi'_{\underline{\xi},l}(\varsigma). \tag{5.10}$$

Proof. Write $\varsigma = \begin{pmatrix} \varsigma' & \eta \\ {}^{\mathrm{t}}\mu & \lambda \end{pmatrix}$ and $x = \begin{pmatrix} x' & y \\ {}^{\mathrm{t}}z & w \end{pmatrix}$ with $\varsigma', x' \in M_{l-1}(\mathbb{Q}_p)$ and $\lambda, w \in \mathbb{Q}_p$. Then

$$\begin{aligned}
&\mathcal{F}^{-1}\Psi_{\underline{\xi},l}(\varsigma) \\
&= \int_{M_l(\mathbb{Q}_p)} \Psi_{\underline{\xi},l}(x)\mathbf{e}_p\left(\mathrm{Tr}\,{}^{\mathrm{t}}x\varsigma\right)\,dx \\
&= \int_{M_{l-1}(\mathbb{Z}_p)\times\mathbb{Z}_p^{l-1}\times\mathbb{Z}_p^{l-1}\times\mathbb{Z}_p} \Psi_{\underline{\xi},l}\left(\begin{pmatrix} x' & y \\ {}^{\mathrm{t}}z & w \end{pmatrix}\right)\mathbf{e}_p\left(\mathrm{Tr}({}^{\mathrm{t}}x'\varsigma' + {}^{\mathrm{t}}z\mu + {}^{\mathrm{t}}y\eta + w\lambda)\right)\,dx'\,dy\,dz\,dw
\end{aligned}$$

$$= \int_{M_{l-1}(\mathbb{Z}_p)} \Psi_{\underline{\xi},l-1}(x')\mathbf{e}_p\left(\mathrm{Tr}\,{}^{\mathrm{t}}x'\varsigma'\right) \int_{\mathbb{Z}_p^{l-1}\times\mathbb{Z}_p^{l-1}\times\mathbb{Z}_p} \xi_l(-w+{}^{\mathrm{t}}zx'^{-1}y)\mathbf{e}_p\left(\mathrm{Tr}({}^{\mathrm{t}}z\mu+{}^{\mathrm{t}}y\eta+w\lambda)\right)\,dy\,dz\,dw\,dx'$$
$$= \int_{M_{l-1}(\mathbb{Z}_p)} \Psi_{\underline{\xi},l-1}(x')\mathbf{e}_p\left(\mathrm{Tr}\,{}^{\mathrm{t}}x'\varsigma'\right) \int_{\mathbb{Z}_p^{l-1}\times\mathbb{Z}_p^{l-1}} \mathbf{e}_p\left(\mathrm{Tr}({}^{\mathrm{t}}z\mu+{}^{\mathrm{t}}y\eta+{}^{\mathrm{t}}zx'^{-1}y{}^{\mathrm{t}}\lambda)\right) \int_{\mathbb{Z}_p} \xi_l(-w)\mathbf{e}_p\left(\mathrm{Tr}(w\lambda)\right)\,dw\,dy\,dz\,dx'.$$

First assume that ξ_l is nontrivial. Then

$$\int_{\mathbb{Z}_p} \xi_l(-w)\mathbf{e}_p\left(\mathrm{Tr}(w\lambda)\right)\,dw = p^{-c_{\xi_l}}G(\xi_l)\xi_l^{-1}(\det(p^{c_{\xi_l}}\lambda)). \tag{5.11}$$

Hence $\mathcal{F}^{-1}\Psi_{\underline{\xi}}(\varsigma)$ is 0 unless λ belongs to $p^{-c_{\xi_l}}\mathbb{Z}_p^\times$. Suppose $\lambda \in p^{-c_{\xi_l}}\mathbb{Z}_p^\times$, and we write

$$\begin{pmatrix} \varsigma' & \eta \\ {}^{\mathrm{t}}\mu & \lambda \end{pmatrix} = \begin{pmatrix} 1 & \eta\lambda^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{\varsigma}' & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \lambda^{-1}{}^{\mathrm{t}}\mu & 1 \end{pmatrix}$$

with $\tilde{\varsigma}' = \varsigma' - \eta\lambda^{-1}{}^{\mathrm{t}}\mu$. Then

$$\begin{aligned}
&\int_{M_{l-1}(\mathbb{Z}_p)} \Psi_{\underline{\xi},l-1}(x')\mathbf{e}_p\left(\mathrm{Tr}\,{}^{\mathrm{t}}x'\varsigma'\right) \int_{\mathbb{Z}_p^{l-1}\times\mathbb{Z}_p^{l-1}} \mathbf{e}_p\left(\mathrm{Tr}({}^{\mathrm{t}}z\mu+{}^{\mathrm{t}}y\eta+{}^{\mathrm{t}}zx'^{-1}y\lambda)\right)\,dy\,dz\,dx' \\
=&\int_{M_{l-1}(\mathbb{Z}_p)} \Psi_{\underline{\xi},l-1}(x')\mathbf{e}_p\left(\mathrm{Tr}\,{}^{\mathrm{t}}x'\tilde{\varsigma}'\right) \int_{\mathbb{Z}_p^{l-1}} \mathbf{e}_p\left(\mathrm{Tr}(\lambda^{-1}{}^{\mathrm{t}}\eta x'(\mu+x'^{-1}y\lambda))\right) \int_{\mathbb{Z}_p^{l-1}} \mathbf{e}_p\left(\mathrm{Tr}({}^{\mathrm{t}}z(\mu+x'^{-1}y\lambda))\right)\,dz\,dy\,dx' \\
=&\,p^{-(l-1)c_{\xi_l}}\mathbb{1}_{\mathbb{Z}_p^{l-1}}(\lambda^{-1}{}^{\mathrm{t}}\mu)\mathbb{1}_{\mathbb{Z}_p^{l-1}}(\eta\lambda^{-1})\cdot\int_{M_{l-1}(\mathbb{Z}_p)} \Psi_{\underline{\xi},l-1}(x')\mathbf{e}_p\left(\mathrm{Tr}\,{}^{\mathrm{t}}x'\tilde{\varsigma}'\right)\,dx'.
\end{aligned} \tag{5.12}$$

Combining (5.11) and (5.12) we get (5.9). Now if ξ_l is the trivial character, then

$$\int_{\mathbb{Z}_p} \xi_l(-w)\mathbf{e}_p\left(\mathrm{Tr}(w\lambda)\right)\,dw = -p^{-1}\mathbb{1}_{p^{-1}\mathbb{Z}_p^\times}(\lambda) + (1-p^{-1})\mathbb{1}_{\mathbb{Z}_p}(\lambda). \tag{5.13}$$

When $\lambda \in p^{-1}\mathbb{Z}_p^\times$, (5.12) holds with c_{ξ_l} replaced by 1. When $\lambda \in \mathbb{Z}_p$,

$$\begin{aligned}
&\int_{M_{l-1}(\mathbb{Z}_p)} \Psi_{\underline{\xi},l-1}(x')\mathbf{e}_p\left(\mathrm{Tr}\,{}^{\mathrm{t}}x'\varsigma'\right) \int_{\mathbb{Z}_p^{l-1}\times\mathbb{Z}_p^{l-1}} \mathbf{e}_p\left(\mathrm{Tr}({}^{\mathrm{t}}z\mu+{}^{\mathrm{t}}y\eta+{}^{\mathrm{t}}zx'^{-1}y\lambda)\right)\,dy\,dz\,dx' \\
=&\int_{M_{l-1}(\mathbb{Z}_p)} \Psi_{\underline{\xi},l-1}(x')\mathbf{e}_p\left(\mathrm{Tr}\,{}^{\mathrm{t}}x'\varsigma'\right) \int_{\mathbb{Z}_p^{l-1}\times\mathbb{Z}_p^{l-1}} \mathbf{e}_p\left(\mathrm{Tr}({}^{\mathrm{t}}z\mu+{}^{\mathrm{t}}y\eta)\right)\,dy\,dz\,dx' \\
=&\,\mathcal{F}^{-1}\Psi_{\underline{\xi},l-1}(\varsigma')\cdot\mathbb{1}_{\mathbb{Z}_p^{l-1}}(\eta)\mathbb{1}_{\mathbb{Z}_p^{l-1}}(\mu).
\end{aligned} \tag{5.14}$$

We see that (5.10) follows from (5.13),(5.12) (with c_{ξ_l} replaced by 1) and (5.14). □

Recall that for an n-tuple of integers $\underline{c} = (c_1, \cdots, c_n)$ we have defined $p^{\underline{c}}$ to be the element $\mathrm{diag}(p^{c_1}, \cdots, p^{c_n}, p^{-c_1}, \cdots, p^{-c_n})$ inside $G(\mathbb{Q}_p)$, so $p^{\underline{c_{\chi\psi}}}$ gives a diagonal matrix in $G(\mathbb{Q}_p)$. When $\xi_1, \cdots, \xi_n$ are all nontrivial, the induction formula in Proposition 5.4.1 easily gives formulas for $\mathcal{F}^{-1}\Psi_{\underline{\xi},l}$, and hence formulas for the section $f'^{d}_{\kappa,\underline{\tau},p}$.

Corollary 5.4.2. *As a function on $G(\mathbb{Q}_p)$ the smooth function $f'^{d}_{\kappa,\underline{\tau},p}(\iota(\cdot,1))$ is supported on the compact open subset*

$$N_G^-(\mathbb{Z}_p)p^{\underline{c_{\chi\psi}}}\begin{pmatrix} p^{-1}I_n & 0 \\ 0 & pI_n \end{pmatrix} B_G(\mathbb{Z}_p)\begin{pmatrix} pI_n & 0 \\ 0 & p^{-1}I_n \end{pmatrix},$$

and takes the value

$$\prod_{l=1}^{n}\psi_l(x_l)\cdot p^{-n(n+1)-\sum_{l=1}^{n} l c_{\chi\psi_l}}\left(p^k\phi(p)\right)^{\sum_{l=1}^{n} c_{\chi\psi_l}}\prod_{l=1}^{n} G(\chi\psi_l)$$

at the element $u^- p^{\underline{c_{\chi\psi}}}\mathrm{diag}(x_1, \cdots, x_n, x_1^{-1}, \cdots, x_n^{-1})u$, with $x_l \in \mathbb{Z}_p^\times$, $u^- \in N_G^-(\mathbb{Z}_p)$ and $u \in \begin{pmatrix} p^{-1}I_n & 0 \\ 0 & pI_n \end{pmatrix} N_G(\mathbb{Z}_p)\begin{pmatrix} pI_n & 0 \\ 0 & p^{-1}I_n \end{pmatrix}$.

Proof. Write $g \in G(\mathbb{Q})$ as $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, using (4.19) we get

$$\begin{aligned}
f'^{d}_{\kappa,\underline{\tau},p}(\iota(g,1)) &= f'_{\kappa,\underline{\tau},p}(\mathcal{S}\iota(g,1)) \\
&= \phi_p\chi_p\left(\det\begin{pmatrix} c & 1 \\ a & 0 \end{pmatrix}\right)\left|\det\begin{pmatrix} c & 1 \\ a & 0 \end{pmatrix}\right|_p^{-k}\alpha'_{\kappa,\underline{\tau},p}\left(\begin{pmatrix} a^{-1}b & a^{-1} \\ d-ca^{-1}b & -ca^{-1} \end{pmatrix}\right) \\
&= \chi(-1)^n\chi^{-1}\left(\det(a)|\det(a)|_p\right)\phi^{-1}\left(|\det(a)|_p\right)|\det(a)|_p^{-k}p^{-n(n+1)} \\
&\quad\cdot \mathbb{1}_{p^{-2}\mathrm{Sym}(n,\mathbb{Z}_p)}(a^{-1}b)\mathbb{1}_{\mathrm{Sym}(n,\mathbb{Z}_p)}(ca^{-1})\cdot\chi(-1)^n\mathcal{F}^{-1}\Psi_{\chi\underline{\psi}}(a^{-1}),
\end{aligned}$$

and the statement follows by applying Proposition 5.4.1. □

We will say that an admissible point $(\kappa, \underline{\tau})$ belongs to the ramified cases if none of the characters $\chi\psi_1, \cdots, \chi\psi_n$ is trivial.

5.5 The $\mathbb{U}_p$-operators and the theory of Jacquet modules

Before starting the computation of the zeta integrals at p ,we state some facts that follow easily from the theory of Jacquet modules and are useful in the study of p-adic automorphic forms of finite slopes. One can also consult the treatment in [33, §5.1].

Let $\pi \subset \mathcal{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A}))$ be an irreducible cuspidal automorphic representation with a fixed isomorphism $\pi \cong \bigotimes'_v \pi_v$. Assume that $\pi_\infty \cong \mathcal{D}_{\underline{t}}$. For each $\varphi \in \pi$ its ordinary projection $e\varphi$ is defined by the discussion in §4.1.2. Put $\pi_{\mathrm{ord}} = e\pi$. By Proposition 4.6 we know that π_{ord} is contained inside the subspace of holomorphic forms inside π.

The facts we show below and will be of use later are: if π_{ord} is nonzero, then π_p is isomorphic to a composition factor of certain principal series, and the projection of π_{ord} to π_p is one dimensional, and the action of the $\mathbb{U}_p$-operators on $\bigcap_{\underline{a}\in C_n^+} U_{p,\underline{a}}(\pi_p)$, the intersection of the images of all the $\mathbb{U}_p$-operators acting on π_p, is semisimple.

Given an admissible representation Π of $G(\mathbb{Q}_p)$, define $U_{p,\underline{a},\mathrm{loc}} = \int_{N_G(\mathbb{Z}_p)} \Pi(up^{\underline{a}})\, du$ (in a purely local situation we do not care about the normalization). Let $\Pi(N_G(\mathbb{Q}_p))$ be the subspace of Π spanned by $\Pi(u)v - v$ for all $u \in N_G, v \in \Pi$. The Jacquet module $\Pi_{N_G(\mathbb{Q}_p)}$ is defined to be the quotient of Π by $\Pi(N_G(\mathbb{Q}_p))$.

It follows from Jacquet's Lemma [10, Theorem 4.1.2, Proposition 4.1.4] that the restriction of the projection $\Pi \to \Pi_{N_G(\mathbb{Q}_p)}$ to $\bigcap_{\underline{a}\in C_n^+} U_{p,\underline{a},\mathrm{loc}}(\Pi)$ is an isomorphism of $T_G(\mathbb{Z}_p)$-representations. It is also easy to check that the action of $U_{p,\underline{a}}, \underline{a} \in C_n^+$ on $U_{p,\underline{a},\mathrm{loc}}(\Pi)$ translates to the action of $p^{\underline{a}} \in T_G(\mathbb{Q}_p)$ on the Jacquet module $\Pi_{N_G(\mathbb{Q}_p)}$. Let δ_{B_G} be the modulus character associated to B_G. It takes the value $\prod_{j=1}^n |x_j|_p^{2(n+1-j)}$ on $\mathrm{diag}(x_1, \cdots, x_n, x_1^{-1}, \cdots, x_n^{-1}) \in B_G(\mathbb{Q}_p)$. There is the Frobenius reciprocity indicating $\mathrm{Hom}_{G(\mathbb{Q}_p)}\left(\Pi, \mathrm{Ind}_{B_G(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \underline{\theta}\right) \cong \mathrm{Hom}_{T_G(\mathbb{Q}_p)}\left(\Pi_{N_G(\mathbb{Q}_p)}, \underline{\theta}\delta_{B_G}^{1/2}\right)$ where $\underline{\theta} = (\theta_1, \cdots, \theta_n)$ is a character of $T_G(\mathbb{Q}_p)$ and $\mathrm{Ind}_{B_G(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \underline{\theta}$ is the normalized induction. Therefore one concludes that as long as the operator $U_p = U_{p,\rho_G}$ acting on π has a nonzero eigenvalue, the $G(\mathbb{Q}_p)$-representation π_p can be embedded into a principal series representation $\mathrm{Ind}_{B_G(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \underline{\theta}$ for some $\underline{\theta}$. More precisely we have the following proposition.

Proposition 5.5.1. *Suppose that there are* $\mathfrak{a}_1, \cdots, \mathfrak{a}_n \in \mathcal{O}_{\overline{\mathbb{Q}}_p} \backslash \{0\}$ *and an automorphic form* $\varphi \in \pi_{\underline{t}}^{\widehat{\Gamma}_1(N,p^m),\underline{\psi}}$ *on which the operator* $U_{p,\underline{a}}$ *acts by* $\prod_{j=1}^n \mathfrak{a}_j^{a_j}$ *for all* $\underline{a} \in C_n^+$. *Let* $\underline{\theta}$ *be the character of* $T_G(\mathbb{Q}_p)$ *whose restriction to* $T_G(\mathbb{Z}_p)$ *is* $\underline{\psi}$ *and* $\theta_j(p) = \alpha_j = p^{-(t_j-j)}\mathfrak{a}_j$. *Then* π_p *can be embedded into the principal series representation* $\mathrm{Ind}_{B_G(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \underline{\theta}$.

Notice that when π_{ord} is nonzero, the p-adic evaluations of the above defined $\alpha_1, \cdots, \alpha_n, \alpha_1^{-1}, \cdots, \alpha_n^{-1}$ are pairwise distinct, and are among $\pm(t_1-1), \cdots, \pm(t_n-n)$.

The information regarding the $\mathbb{U}_p$-operators acting on $\bigcap_{\underline{a}\in C_n^+} U_{p,\underline{a}}(\pi_p)$ can be deduced from the knowledge of the action of $T_G(\mathbb{Q}_p)$ on $\left(\mathrm{Ind}_{B_G(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \underline{\theta}\right)_{N_G(\mathbb{Q}_p)}$, the Jacquet module of the principal series that contains π_p. According to [10, Proposition 6.3.1, Proposition 6.3.3], the composition series of $\left(\mathrm{Ind}_{B_G(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \underline{\theta}\right)_{N_G(\mathbb{Q}_p)}$ consists of $|W_G|$ characters of $T_G(\mathbb{Q}_p)$, which are $(\underline{\theta}\circ w)\cdot\delta_{B_G}^{1/2}$, $w \in W_G$, where W_G is the Weyl group of G with respect to its maximal torus T_G. The nontriviality of π_{ord} implies that these $|W_G|$ characters are pairwise distinct, so the $T_G(\mathbb{Q}_p)$-action on $\left(\mathrm{Ind}_{B_G(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \underline{\theta}\right)_{N_G(\mathbb{Q}_p)}$, as well as $\pi_{p,N_G(\mathbb{Q}_p)}$, is semisimple. By a simple examination of the corresponding p-adic valuations one also sees that there is only one w in W_G having the property that for all $\underline{a} \in C_n^+$ the number $\underline{\theta}(w(p^{\underline{a}}))\delta_{B_G}^{1/2}(p^{\underline{a}})$ has p-adic valuation less or equal to $-\langle \underline{t} + 2\rho_{G,c}, \underline{a}\rangle$.

Proposition 5.5.2. *If* π_{ord} *is nonzero, then the action of* $\mathbb{U}_p$*-operators on* $\bigcap_{\underline{a}\in C_n^+} U_{p,\underline{a}}(\pi_p)$ *is semisimple. Let* $\pi_{p,\mathrm{ord}}$ *be the image of the projection of* π_{ord} *to* π_p. *Then* $\pi_{p,\mathrm{ord}}$ *is one dimensional.*

From now on when π_{ord} is nonzero, we put $\mathfrak{a}_1, \cdots, \mathfrak{a}_n \in \mathcal{O}_{\overline{\mathbb{Q}}_p}^\times$ to be the p-adic integers such that the $\mathbb{U}_p$-operator $U_{p,\underline{a}}$ acts on π_{ord} by $\prod_{j=1}^n \mathfrak{a}_j^{a_j}$ for $\underline{a} = (a_1, \cdots, a_n) \in C_n^+$. We will also assume that the group $T_G(\mathbb{Z}_p)$ acts on π_{ord} by the character $\underline{\psi}$. For $1 \le j \le n$, the number α_j and the character θ_j of $\mathbb{Q}_p^\times$ are defined from $\mathfrak{a}_j$ and ψ_j as above, i.e. $\alpha_j = p^{-(t_j-j)}\mathfrak{a}_j$ and $\theta_j|_{\mathbb{Z}_p^\times} = \psi_j$ with $\theta_j(p) = \alpha_j$.

5.6 The proof of Prop 5.2.3 for the ramified cases

Proof (the ramified cases). Assume that $\chi\psi_1, \cdots, \chi\psi_n$ are all nontrivial, and $\varphi \in \pi_{\underline{t}}^{\Gamma_1(N,p^m)}$ is ordinary. The computation is straightforward. For $\Delta\underline{a} \gg 0$, by definition of the operator $U_{p,\underline{a}}$ (4.3) and Proposition 5.3.1,

$$\begin{aligned}\left(T_{(U_{p,\underline{a}}\times 1)f_{\kappa,\underline{\tau},p}}\overline{\varphi}\right)(g^{\vartheta}) &= \left(T_{(U_{p,\underline{a}}\times 1)f'_{\kappa,\underline{\tau},p}}\overline{\varphi}\right)(g^{\vartheta}) \\ &= p^{\left\langle \underline{t}+2\rho_{G,c},\,\underline{a}\right\rangle}\int_{G(\mathbb{Q}_p)}\int_{N_G(\mathbb{Z}_p)} f'^{d}_{\kappa,\underline{\tau},p}(\iota(g'up^{\underline{a}},1))\overline{\varphi}(g^{\vartheta}g')du\,dg' \\ &= p^{\left\langle \underline{t}+2\rho_{G,c},\,\underline{a}\right\rangle}\int_{G(\mathbb{Q}_p)} f'^{d}_{\kappa,\underline{\tau},p}(\iota(g',1))\overline{\varphi}(g^{\vartheta}g'p^{-\underline{a}})dg.\end{aligned}$$

Abbreviate the scalar $p^{-\sum_{l=1}^{n} l\cdot c_{\chi\psi_l}}\left(p^k\phi(p)\right)^{\sum_{l=1}^{n} c_{\chi\psi_l}}\prod_{l=1}^{n} G(\chi\psi_l)$ as $b_{k,\phi,\chi\underline{\psi}}$. Then applying Corollary 5.4.2 we get

$$\begin{aligned}&b_{k,\phi,\chi\underline{\psi}}^{-1}\mathrm{vol}\left(N_G^{-}(\mathbb{Z}_p)B_G(\mathbb{Z}_p)\right)^{-1}\cdot\left(T_{(U_{p,\underline{a}}\times 1)f_{\kappa,\underline{\tau},p}}\overline{\varphi}\right)(g^{\vartheta}) \\ &= p^{\left\langle \underline{t}+2\rho_{G,c},\,\underline{a}\right\rangle-\left\langle 2\rho_G,\,\underline{c_{\chi\psi}}\right\rangle}\int_{N_G^{-}(\mathbb{Z}_p)}\int_{N_G(\mathbb{Z}_p)}\overline{\varphi}\left(g^{\vartheta}u^{-}p^{\underline{c_{\chi\psi}}}\begin{pmatrix}p^{-1}I_n & 0\\ 0 & pI_n\end{pmatrix}u\begin{pmatrix}pI_n & 0\\ 0 & p^{-1}I_n\end{pmatrix}p^{-\underline{a}}\right)du\,du^{-} \\ &= p^{\left\langle \underline{t}+2\rho_{G,c},\,\underline{a}\right\rangle-\left\langle 2\rho_G,\,\underline{c_{\chi\psi}}\right\rangle}\int_{N_G^{-}(\mathbb{Z}_p)}\overline{\varphi}\left(g^{\vartheta}u^{-}p^{\underline{c_{\chi\psi}}-\underline{a}}\right)du^{-} \\ &= p^{\left\langle \underline{t}+2\rho_{G,c},\,\underline{a}\right\rangle-\left\langle 2\rho_G,\,\underline{c_{\chi\psi}}\right\rangle}\int_{N_G(\mathbb{Z}_p)}\overline{\varphi}\left(\vartheta gup^{\underline{a}-\underline{c_{\chi\psi}}}\vartheta\right)du \\ &= p^{\left\langle \underline{t}-2\rho_{G,nc},\,\underline{c_{\chi\psi}}\right\rangle}\left(U_{p,\underline{a}-\underline{c_{\chi\psi}}}\overline{\varphi}^{\vartheta}\right)(g)\end{aligned}$$

Using $\mathrm{vol}\left(N_G^{-}(\mathbb{Z}_p)B_G(\mathbb{Z}_p)\right)=\frac{p^{n^2}(p-1)^n}{\prod_{l=1}^{n}(p^{2l}-1)}$, we get

$$\left(T_{(U_{p,\underline{a}}\times 1)f_{\kappa,\underline{\tau},p}}\overline{\varphi}\right)(g^{\vartheta}) = b_{k,\phi,\chi\underline{\psi}}\frac{p^{n^2}(p-1)^n}{\prod_{l=1}^{n}(p^{2l}-1)}\,p^{\left\langle \underline{t}-2\rho_{G,nc},\,\underline{c_{\chi\psi}}\right\rangle}\cdot\left(U_{p,\underline{a}-\underline{c_{\chi\psi}}}\overline{\varphi}^{\vartheta}\right)(g). \tag{5.15}$$

The automorphic form $\overline{\varphi}^{\vartheta}\in\pi$ in general is not fixed by $N_G(\mathbb{Z}_p)$, and $W(\varphi)$ by definition equals its average over $N_G(\mathbb{Z}_p)$. We have

$$(U_{p,\underline{a}}\overline{\varphi}^{\vartheta})(g)=\int_{N_G(\mathbb{Z}_p)}\overline{\varphi}^{\vartheta}\left(gup^{\underline{a}}\right)du=\int_{N_G(\mathbb{Z}_p)}\int_{N_G(\mathbb{Z}_p)}\overline{\varphi}^{\vartheta}\left(gup^{\underline{a}}u'\right)du\,du'=\left(U_{p,\underline{a}}W(\varphi)\right)(g),$$

so (5.15) becomes

$$\left(T_{(U_{p,\underline{a}}\times 1)f_{\kappa,\underline{\tau},p}}\overline{\varphi}\right)(g^{\vartheta}) = b_{k,\phi,\chi\underline{\psi}}\frac{p^{n^2}(p-1)^n}{\prod_{l=1}^n(p^{2l}-1)}\,p^{\left\langle \underline{t}-2\rho_{G,nc},\,\underline{c_{\chi\psi}}\right\rangle}\cdot\left(U_{p,\underline{a}-\underline{c_{\chi\psi}}}W(\varphi)\right)(g),$$

which, together with Theorem 4.3.2, Proposition 4.4.1, the fact that an ordinary nearly holomorphic form must be holomorphic and Proposition 5.5.2, implies Proposition 5.2.3 in the ramified case. □

5.7 The proof of Prop 5.2.3 for general cases

We first state a proposition whose proof is postponed to the end of §6.2.

Proposition 5.7.1. *For each admissible point* $(\kappa,\underline{\tau})$ *the nearly holomorphic form* $(e\times 1)\mathcal{E}_{\kappa,\underline{\tau}}$ *is ordinary on both factors.*

The idea of the proof is simple. The statement is true in the ramified cases by results in §5.6. The admissible points belonging to the ramified cases are Zariski sense inside the weight space $\mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^{\times}\times T_n(\mathbb{Z}_p),\overline{\mathbb{Q}}_p^{\times})$ and the statement for the general cases follows from a p-adic family argument.

Another proposition that will be useful for us verifies the nonvanishing of the ordinary projection of $W(\varphi)$ for a nonzero ordinary Siegel modular form φ.

Proposition 5.7.2. *If* $\varphi\in\pi_{\underline{t}}^{\widehat{\Gamma}_1(N,p^m),\underline{\psi}}$ *is nonzero ordinary, then* $eW(\varphi)$ *is nonzero.*

Proof. Take $\varphi'\in\pi$ invariant under the right translation of $N_G^-(\mathbb{Z}_p)$. We consider the Petersson inner product of $eW(\varphi)$ with φ'^{ϑ}.

$$\begin{aligned}\left\langle U_{p,\underline{a}}W(\varphi),\varphi'^{\vartheta}\right\rangle =&p^{\left\langle \underline{t}+2\rho_{G,c},\,\underline{a}\right\rangle}\int_{G(\mathbb{Q})\backslash G(\mathbb{A})\times N_G(\mathbb{Z}_p)\times N_G(\mathbb{Z}_p)}\overline{\varphi}(\vartheta gup^{\underline{a}}u'\vartheta)\varphi'(\vartheta g\vartheta)\,dg\,du\,du'\\ =&p^{\left\langle \underline{t}+2\rho_{G,c},\,\underline{a}\right\rangle}\int_{G(\mathbb{Q})\backslash G(\mathbb{A})\times N_G(\mathbb{Z}_p)}\overline{\varphi}(\vartheta gup^{\underline{a}}\vartheta)\varphi'(\vartheta g\vartheta)\,dgdu.\end{aligned}$$

Making the change of variable $g\mapsto\vartheta g\vartheta$, we get

$$\left\langle U_{p,\underline{a}}W(\varphi),\varphi'^{\vartheta}\right\rangle =p^{\left\langle \underline{t}+2\rho_{G,c},\,\underline{a}\right\rangle}\int_{G(\mathbb{Q})\backslash G(\mathbb{A})\times N_G(\mathbb{Z}_p)}\overline{\varphi}(g\vartheta up^{\underline{a}}\vartheta)\varphi'(g)\,dg\,du$$

$$
\begin{aligned}
&=p^{\langle \underline{t}+2\rho_{G,c},\, \underline{a}\rangle} \int_{G(\mathbb{Q})\backslash G(\mathbb{A})\times N_G(\mathbb{Z}_p)} \overline{\varphi}(g)\varphi(g\vartheta p^{-\underline{a}}u\vartheta)\, dg\, du \\
&=p^{\langle \underline{t}+2\rho_{G,c},\, \underline{a}\rangle} \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \overline{\varphi}(g)\varphi(gp^{\underline{a}})\, dg \\
&=p^{\langle \underline{t}+2\rho_{G,c},\, \underline{a}\rangle} \int_{G(\mathbb{Q})\backslash G(\mathbb{A})\times N_G(\mathbb{Z}_p)} \overline{\varphi}(gu)\varphi(gp^{\underline{a}})\, dg\, du \\
&=p^{\langle \underline{t}+2\rho_{G,c},\, \underline{a}\rangle} \int_{G(\mathbb{Q})\backslash G(\mathbb{A})\times N_G(\mathbb{Z}_p)} \overline{\varphi}(g)\varphi(gup^{\underline{a}})\, dg\, du \\
&=\left\langle \overline{\varphi}, U_{p,\underline{a}}\varphi' \right\rangle,
\end{aligned}
$$

from which it follows that

$$
\left\langle eW(\varphi), \varphi'^{\vartheta} \right\rangle = \lim_{r\to\infty} \left\langle U_p^{r!} W(\varphi), \varphi' \right\rangle = \lim_{r\to\infty} \left\langle \overline{\varphi}, U_p^{r!} \varphi' \right\rangle = \left\langle \overline{\varphi}, e\varphi' \right\rangle.
$$

For fixed φ and φ', there are finite dimensional subspaces (viewed as both over $\mathbb{C}$ and $\overline{\mathbb{Q}}_p$) of π and $\overline{\pi}$ which contain all the automorphic forms appearing in the above identity, and we regard the Petersson inner product as a bi-$\overline{\mathbb{Q}}_p$-linear pairing between them. Hence the limits with respect to the p-adic topology are well defined and commute with the Petersson inner product.

Now take $\varphi' = R_p(p^{\underline{c}})\varphi$ with $\underline{c} \in C_n^+$ and $\Delta\underline{c}$ sufficiently large such that $R_p(p^{\underline{c}})\varphi$ is fixed by $N_G^-(\mathbb{Z}_p)$. Combining the above computation and the fact $U_{p,\underline{a}} R_p(p^{\underline{c}}) = U_{p,\underline{a}+\underline{c}}$, we see that

$$
\left\langle eW(\varphi), (R_p(p^{\underline{c}})\varphi)^{\vartheta} \right\rangle = \left\langle \overline{\varphi}, U_{p,\underline{c}}\varphi \right\rangle \neq 0,
$$

and the nonvanishing of $eW(\varphi)$ follows. □

Now we begin the proof of Proposition 5.2.3 for general cases.

Proof (general cases). Assume that $\varphi \in \pi_{\underline{t}}^{\widehat{\Gamma}_1(N,p^m),\underline{\psi}}$ is nonzero ordinary. Let $W(\varphi)^p$ be the image of $W(\varphi)$ under the map $\pi \xrightarrow{\sim} \bigotimes' \pi_v \to \bigotimes_{v\neq p} \pi_v$. By the doubling method formula Theorem 4.3.2 and Proposition 4.4.1, 4.6, 5.7.1, we deduce that the image of the automorphic form $\left\langle \varphi_{G\times G}((e\times 1)\mathcal{E}_{\kappa,\underline{\tau}}, \mathfrak{e}_{\mathrm{can}})(\cdot, g), \overline{\varphi} \right\rangle$ in $\bigotimes' \pi_v$ lies inside $W(\varphi)^p \otimes \pi_{p,\mathrm{ord}}$. By by Proposition 5.5.2, we know that $W(\varphi)^p \otimes \pi_{p,\mathrm{ord}}$ is a one dimensional $\mathbb{C}$-vector space, so the nonvanishing of $eW(\varphi)$ implies that there exists a complex number $C_{\phi,\kappa,\underline{\tau},\pi} \in \mathbb{C} \cong \overline{\mathbb{Q}}_p$ such that

$$\left\langle \varphi_{G\times G}((e\times 1)\mathcal{E}_{\kappa,\underline{\tau}}, \mathfrak{e}_{\mathrm{can}})(\cdot, g), \overline{\varphi}\right\rangle = C_{\phi,\kappa,\underline{\tau},\pi} \cdot eW(\varphi)(g).$$

Let

$$B_{\phi,\kappa,\underline{\tau},\pi} = A^{-1}_{n,\phi,k,\chi} \cdot \phi(-1)^n \mathrm{vol}\left(\widehat{\Gamma}(N)\right) \cdot \frac{Z_\infty(f_{\kappa,\underline{\tau},\infty}, v^\vee_{\underline{t}}, v_{\underline{t}})}{\langle v^\vee_{\underline{t}}, v_{\underline{t}}\rangle} \cdot L^{Np\infty}(k-n, \pi\times\phi^{-1}\chi^{-1}),$$

where $A_{n,\phi,k,\chi}$ is defined as (4.15). This $B_{\phi,\kappa,\underline{\tau},\pi}$ is a finite complex number because of the absolute convergence of the archimedean zeta integral and the fact that the partial standard L-function $L^{Np\infty}(s,\pi\times\phi^{-1}\chi^{-1})$ does not have a pole at $k-n$. Let $\alpha_1,\cdots,\alpha_n$ and $\underline{\theta} = (\theta_1,\cdots,\theta_n)$ be the invariants associated to π_p at the end of §5.5. Define

$$R_p(s,\theta_j,\phi^{-1}) := \frac{1-(\chi\psi_j)^\circ(p)\cdot\phi(p)\alpha_j^{-1}p^{s-1}}{1-(\chi\psi_j)^\circ(p)\cdot\phi(p)^{-1}\alpha_j p^{-s}} \cdot \left(\phi(p)\alpha_j^{-1}p^{s-1}\right)^{c_{\chi\psi_j}} G(\chi\psi_j),$$

where by convention $(\chi\psi_j)^\circ(p) = \begin{cases} 1 & \text{if } \chi\psi_j \text{ is trivial} \\ 0 & \text{otherwise} \end{cases}$. The ordinarity condition on π implies that $R_p(s,\theta_j,\phi^{-1})$, $1\le j\le n$, dose not have a pole at $s=k-n$. Our goal is to show that

$$C_{\phi,\kappa,\underline{\tau},\pi} = B_{\phi,\kappa,\underline{\tau},\pi} \cdot \mathrm{vol}\left(B_G(\mathbb{Z}_p)N^-_G(\mathbb{Z}_p)\right)\prod_{j=1}^n R_p(k-n,\theta_j,\phi^{-1}). \tag{5.16}$$

Let $f'^d_{\kappa,\underline{\tau},p}(s) = f_p^{d,\alpha'_{\kappa,\underline{\tau},p}}(s-\frac{1}{2},\phi^{-1}\chi^{-1})$, the "big cell" section inside $I_{P_H,p}(s-\frac{1}{2},\phi^{-1}\chi^{\circ-1})$ (defined as (4.8)) , associated to the Schwartz function $\alpha'_{\kappa,\underline{\tau},p}$ whose Fourier transform is (5.7). We have $f'^d_{\kappa,\underline{\tau},p} = f'^d_{\kappa,\underline{\tau},p}(k-n)$. We add the parameter s here due to convergence consideration, because in general $f_{\kappa,\underline{\tau},p}(\iota(\cdot,1))$ is not compactly supported. In the following we assume $\mathrm{Re}\,(s)\gg 0$ whenever necessary, and the computation results will be easily seen to admit meromorphic continuations with respect to s.

For $\underline{c}\in C_n^+$, we write $\left\langle\varphi_{G\times G}\left((U_{p,\underline{c}}\times 1)\mathcal{E}_{\kappa,\underline{\tau}},\mathfrak{e}_{\mathrm{can}}\right)(\cdot,g),\overline{\varphi}\right\rangle$ as $\varphi_{\underline{c}}(g)$. Given $\varphi'\in\pi_{\underline{t}}^{N_G^-(\mathbb{Z}_p)}$, we have

$$\begin{aligned}
\left\langle \varphi_{\underline{c}}, \varphi'^{\vartheta} \right\rangle &= B_{\phi,\kappa,\underline{\tau},\pi} \lim_{s\to k-n} p^{\langle \underline{t}+2\rho_{G,c},\, \underline{c}\rangle} \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \int_{G(\mathbb{Q}_p)} \int_{N_G(\mathbb{Z}_p)} f'^{d}_{\kappa,\underline{\tau},p}(s)(\iota(g' u p^{\underline{c}}, 1)) \overline{\varphi}(g^{\vartheta} g') \varphi'(g^{\vartheta})\, du\, dg'\, dg \\
&= B_{\phi,\kappa,\underline{\tau},\pi} \lim_{s\to k-n} p^{\langle \underline{t}+2\rho_{G,c},\, \underline{c}\rangle} \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \int_{G(\mathbb{Q}_p)} f'^{d}_{\kappa,\underline{\tau},p}(\iota(g', 1)) \overline{\varphi}(g^{\vartheta} g' p^{-\underline{c}}) \varphi'(g^{\vartheta})\, dg'\, dg \\
&= B_{\phi,\kappa,\underline{\tau},\pi} \lim_{s\to k-n} p^{\langle \underline{t}+2\rho_{G,c},\, \underline{c}\rangle} \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \int_{G(\mathbb{Q}_p)} f'^{d}_{\kappa,\underline{\tau},p}(\iota(g', 1)) \overline{\varphi}(g) \varphi'(g p^{\underline{c}} g'^{-1})\, dg'\, dg \\
&= B_{\phi,\kappa,\underline{\tau},\pi} \lim_{s\to k-n} p^{\langle \underline{t}+2\rho_{G,c},\, \underline{c}\rangle} \mathrm{vol}\left(Q_G(\mathbb{Z}_p) U_G^-(\mathbb{Z}_p)\right) \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \overline{\varphi}(g) \int_{\mathrm{GL}(n,\mathbb{Q}_p)} p^{-n(n+1)} \cdot \eta_{\kappa,\underline{\tau},n}(s,a) \\
&\quad \cdot \int_{U_G(\mathbb{Z}_p)\times U_G^-(\mathbb{Z}_p)} \varphi'\left(g p^{\underline{c}} \begin{pmatrix} p^{-1} I_n & 0 \\ 0 & p I_n \end{pmatrix} u \begin{pmatrix} p I_n & 0 \\ 0 & p^{-1} I_n \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & {}^{\mathrm{t}}a^{-1} \end{pmatrix} u^- \right) du\, du^-\, da\, dg, \\
&= B_{\phi,\kappa,\underline{\tau},\pi} \lim_{s\to k-n} \frac{\mathrm{vol}\left(B_G(\mathbb{Z}_p) N_G^-(\mathbb{Z}_p)\right)}{\mathrm{vol}\left(B_n(\mathbb{Z}_p) N_n^-(\mathbb{Z}_p)\right)} \cdot p^{\langle \underline{t}+2\rho_{G,c},\underline{c}\rangle} \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \overline{\varphi}(g) \int_{N_G(\mathbb{Z}_p)\times \mathrm{GL}(n,\mathbb{Q}_p)} \eta_{\kappa,\underline{\tau},n}(s,a) \\
&\quad \cdot \varphi'\left(g u p^{\underline{c}} \begin{pmatrix} a & 0 \\ 0 & {}^{\mathrm{t}}a^{-1} \end{pmatrix} \right) du\, da\, dg,
\end{aligned}$$

where for $1 \le l \le n$ we define the Schwartz function $\eta_{\kappa,\underline{\tau},l}(s,\cdot)$ on $M_l(\mathbb{Q}_p)$, supported on $\mathrm{GL}(n,\mathbb{Q}_p)$, as

$$\eta_{\kappa,\underline{\tau},l}(s,a) = \chi(\det(a)|\det(a)|_p) \cdot \phi(|\det(a)|_p) \cdot |\det(a)|_p^{s-1} \cdot \mathcal{F}^{-1}\Psi_{\chi\underline{\psi},l}(a) \tag{5.17}$$

for $a \in M_l(\mathbb{Q}_p)$. Define the operator

$$\mathcal{T}_{\kappa,\underline{\tau},p}(s) = \int_{\mathrm{GL}(n,\mathbb{Q}_p)} \eta_{\kappa,\underline{\tau},n}(s,a) \pi_p \left(\begin{pmatrix} a & 0 \\ 0 & {}^{\mathrm{t}}a^{-1} \end{pmatrix} \right) da$$

(certainly in general as an operator acting on π or a model of π_p, its absolute convergence requires $\mathrm{Re}\,(s)$ to be sufficiently large). Then

$$\left\langle \varphi_{\underline{c}},\, \varphi'^{\vartheta} \right\rangle = B_{\phi,\kappa,\underline{\tau},\pi} \frac{\mathrm{vol}\left(B_G(\mathbb{Z}_p) N_G^-(\mathbb{Z}_p)\right)}{\mathrm{vol}\left(B_n(\mathbb{Z}_p) N_n^-(\mathbb{Z}_p)\right)} \lim_{s\to k-n} \left\langle \overline{\varphi},\, U_{p,\underline{c}} \mathcal{T}_{\kappa,\underline{\tau},p}(s) \varphi' \right\rangle .$$

At the same time it follows from the computation in the last proposition that

$$\left\langle U_{p,\underline{c}}W(\varphi),\, \varphi'^{\vartheta}\right\rangle = \left\langle \overline{\varphi},\, U_{p,\underline{c}}\varphi'\right\rangle.$$

There exists a polynomial $R(X) \in \mathbb{C}[X]$ such that $(e\times 1)\mathcal{E}_{\kappa,\underline{\tau}} = (R(U_p)\times 1)\mathcal{E}_{\kappa,\underline{\tau}}$ and $eW(\varphi) = R(U_p)\varphi$. Thus we have

$$\left\langle \varphi_{G\times G}((e\times 1)\mathcal{E}_{\kappa,\underline{\tau}}, \mathfrak{e}_{\mathrm{can}}),\, \overline{\varphi}\otimes\varphi'^{\vartheta}\right\rangle = B_{\phi,\kappa,\underline{\tau},\pi}\frac{\mathrm{vol}\left(B_G(\mathbb{Z}_p)N_G^-(\mathbb{Z}_p)\right)}{\mathrm{vol}\left(B_n(\mathbb{Z}_p)N_n^-(\mathbb{Z}_p)\right)}\lim_{s\to k-n}\left\langle \overline{\varphi},\, R(U_p)\mathcal{T}_{\kappa,\underline{\tau},p}(s)\varphi'\right\rangle,$$

$$\left\langle eW(\varphi),\, \varphi'^{\vartheta}\right\rangle = \left\langle \overline{\varphi},\, R(U_p)\varphi'\right\rangle.$$

Now one sees that in order to verify (5.16), it suffices to show that there exists some $\varphi' \in \pi^{N_G^-(\mathbb{Z}_p)}$ with $e\varphi' \neq 0$, such that, as a function in s, $\left\langle \overline{\varphi},\, Q(U_p)\mathcal{T}_{\kappa,\underline{\tau},p}(s)\varphi'\right\rangle$ admits a meromorphic continuation and

$$\lim_{s\to k-n}\frac{\left\langle \overline{\varphi},\, R(U_p)\mathcal{T}_{\kappa,\underline{\tau},p}(s)\varphi'\right\rangle}{\left\langle \overline{\varphi},\, R(U_p)\varphi'\right\rangle} = \mathrm{vol}\left(B_n(\mathbb{Z}_p)N_n^-(\mathbb{Z}_p)\right)\prod_{j=1}^{n}R_p(k-n,\theta_j,\phi^{-1}). \tag{5.18}$$

If we fix φ' it is not difficult to check that there exists an open compact subgroup $K_p \subset G(\mathbb{Z}_p)$ such that π^{K_p} contains $U_{p,\underline{c}}\varphi'$, $U_{p,\underline{c}}\mathcal{T}_{\kappa,\underline{\tau},p}(s)\varphi'$ for all $\underline{c} \in C_n^+$ and $s \in \mathbb{C}$ with $\mathrm{Re}\,(s)$ large enough. Therefore we can assume that the polynomial $R(X)$ satisfies $e\varphi' = R(U_p)\varphi'$ and $e\mathcal{T}_{\kappa,\underline{\tau},p}(s)\varphi' = R(U_p)\mathcal{T}_{\kappa,\underline{\tau},p}(s)\varphi'$. In this case the value of the left hand side of (5.18) dose not change if we replace φ by any $\varphi'' \in \pi$ with $\left\langle \overline{\varphi''},\, e\varphi'\right\rangle \neq 0$. Thus we have a big freedom in choosing φ' and φ'' to compute the left hand side of (5.18). The requirements on φ' and φ'' are $\varphi' \in \pi_{\underline{t}}^{N_G^-(\mathbb{Z}_p)}$ and $\left\langle \overline{\varphi''},\, e\varphi'\right\rangle \neq 0$. It is also clear that the computation can be reduced to a local situation using any model of π_p.

Let $\mathfrak{f}_{N_G^-} \in \mathrm{Ind}_{B_G(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\underline{\theta}$ be the section supported $B_G(\mathbb{Q}_p)N_G^-(\mathbb{Z}_p)$ and taking the value 1 on $N_G^-(\mathbb{Z}_p)$. Fix an open compact subgroup K_p of $G(\mathbb{Z}_p)$ sufficiently small such that the vectors $U_{p,\underline{c}}\mathfrak{f}_{N_G^-}$, $U_{p,\underline{c}}\mathcal{T}_{\kappa,\underline{\tau},p}(s)\mathfrak{f}_{N_G^-}$, with $\underline{c} \in C_n^+$, $\mathrm{Re}\,(s) \gg 0$, are all fixed by the right translation of K_p and the restriction of $\underline{\theta}$ to $B_G(\mathbb{Q}_p)\cap K_p$ is the trivial character. Let $\tilde{\mathfrak{f}}_{K_p} \in \mathrm{Ind}_{B_G(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\underline{\theta}^{-1}$ be the section supported on $B_G(\mathbb{Q}_p)K_p$ and taking the value 1 on K_p . (5.18) will follow from the equality

$$\frac{\left\langle \tilde{\mathfrak{f}}_{K_p},\, U_{p,\underline{c}}\mathcal{T}_{\kappa,\underline{\tau},p}(s)\mathfrak{f}_{N_G^-}\right\rangle}{\left\langle \tilde{\mathfrak{f}}_{K_p},\, U_{p,\underline{c}}\mathfrak{f}_{N_G^-}\right\rangle} = \mathrm{vol}\left(B_n(\mathbb{Z}_p)N_n^-(\mathbb{Z}_p)\right)\prod_{j=1}^{n} R_p(s,\theta_j,\phi^{-1}) \tag{5.19}$$

for all $\underline{c} \in C_n^+$. Note that although π_p is a sub-representation of $\mathrm{Ind}_{B_G(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\underline{\theta}$ and in general they are not equal, by the discussion in §5.5, under the normalization for the $\mathbb{U}_p$-operators associated to $\underline{t}$, the ordinary subspace in $\mathrm{Ind}_{B_G(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\underline{\theta}$ is one dimensional and certainly coincides with that of π_p.

The pairing between a section $\mathfrak{f} \in \mathrm{Ind}_{B_G(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\underline{\theta}$ and a section $\tilde{\mathfrak{f}} \in \mathrm{Ind}_{B_G(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\underline{\theta}^{-1}$ is given as $\langle \mathfrak{f},\tilde{\mathfrak{f}}\rangle = \int_{G(\mathbb{Z}_p)}\mathfrak{f}(g)\tilde{\mathfrak{f}}(g)\,dg$. We have

$$\begin{aligned}
\frac{\left\langle \tilde{\mathfrak{f}}_{K_p},\, U_{p,\underline{c}}\mathcal{T}_{\kappa,\underline{\tau},p}(s)\mathfrak{f}_{N_G^-}\right\rangle}{\left\langle \tilde{\mathfrak{f}}_{K_p},\, U_{p,\underline{c}}\mathfrak{f}_{N_G^-}\right\rangle} &= \frac{\left(U_{p,\underline{c}}\mathcal{T}_{\kappa,\underline{\tau},p}(s)\mathfrak{f}_{N_G^-}\right)(1)}{\left(U_{p,\underline{c}}\mathfrak{f}_{N_G^-}\right)(1)}\\
&= \frac{\int_{\mathrm{GL}(n,\mathbb{Q}_p)}\int_{N_G(\mathbb{Z}_p)}\mathfrak{f}_{N_G^-}\left(up^{\underline{c}}\left(\begin{smallmatrix} a & 0\\ 0 & {}^{\mathrm{t}}a^{-1}\end{smallmatrix}\right)\right)\eta_{\kappa,\underline{\tau},n}(s,a)\,du\,da}{\int_{N_G(\mathbb{Z}_p)}\mathfrak{f}_{N_G^-}\left(up^{\underline{c}}\right)}\\
&= \int_{\mathrm{GL}(n,\mathbb{Q}_p)}\mathfrak{f}_{N_G^-}\left(\begin{pmatrix} a & 0\\ 0 & {}^{\mathrm{t}}a^{-1}\end{pmatrix}\right)\eta_{\kappa,\underline{\tau},n}(s,a)\,da\\
&= \int_{\mathrm{GL}(n,\mathbb{Q}_p)}\mathfrak{w}_n(a)\,\eta_{\kappa,\underline{\tau},n}(s,a)\,da,
\end{aligned}$$

where for $1 \le l \le n$ we define $\mathfrak{w}_l : \mathrm{GL}(l,\mathbb{Q}_p) \to \mathbb{C}$ to be the smooth function supported on $B_n(\mathbb{Q}_p)N_n^-(\mathbb{Z}_p)$ such that $\mathfrak{w}_l(bu^-) = \prod_{j=1}^{l}\theta_j(b_j)|b_j|_p^{l+1-j}$ for $b \in \mathrm{diag}(b_1,\cdots,b_l)N_l(\mathbb{Q}_p)$ and $u^- \in N_l^-(\mathbb{Z}_p)$. The desired equality (5.19) can be deduced from the induction relation

$$\begin{aligned}
&\int_{GL(l,\mathbb{Q}_p)}\mathfrak{w}_l(a)\eta_{\kappa,\underline{\tau},l}(s,a)\,da\\
&= \mathrm{vol}\left(B_{l-1,1}(\mathbb{Z}_p)N_{l-1,1}^-(\mathbb{Z}_p)\right)R_p(s,\theta_l,\phi^{-1})\int_{GL(l-1,\mathbb{Q}_p)}\mathfrak{w}_{l-1}(a)\eta_{\kappa,\underline{\tau},l-1}(s,a)\,da,
\end{aligned} \tag{5.20}$$

where $B_{l-1,1}(\mathbb{Z}_p) = \begin{pmatrix} \mathrm{GL}(l-1,\mathbb{Z}_p) & \mathbb{Z}_p^{l-1}\\ 0 & \mathbb{Z}_p^\times\end{pmatrix}$ and $N_{l-1,1}(\mathbb{Z}_p) = \begin{pmatrix} I_{l-1} & 0\\ {}^{\mathrm{t}}\mathbb{Z}_p^{l-1} & 1\end{pmatrix}$.

From the definition of $\mathcal{F}^{-1}\Psi_{\chi\underline{\psi},l}$, we see that it is invariant under the right (resp. left) translation of $N_l^-(\mathbb{Z}_p)$ (resp. $N_l(\mathbb{Z}_p)$). By the definition of $\mathfrak{w}_l$ and that of $\eta_{\kappa,\underline{\tau},l}$ (5.17), we have

$$
\begin{aligned}
&\operatorname{vol}\left(B_{l-1,1}(\mathbb{Z}_p)N_{l-1,1}^-(\mathbb{Z}_p)\right)^{-1}\int_{GL(l,\mathbb{Q}_p)}\mathfrak{w}_l(a)\eta_{\kappa,\underline{\tau},l}(s,a)\,da\\
&=\int_{\mathrm{GL}(l-1,\mathbb{Q}_p)\times\mathbb{Q}_p^\times\times N_{l-1,1}(\mathbb{Q}_p)\times N_{l-1,1}^-(\mathbb{Z}_p)}\mathfrak{w}_l\left(u\begin{pmatrix}a' & 0\\ 0 & a_l\end{pmatrix}u^-\right)\left|\det(a')\right|_p^{-1}|a_l|_p^{l-1}\\
&\quad\cdot\eta_{\kappa,\underline{\tau},l}\left(s,u\begin{pmatrix}a' & 0\\ 0 & a_l\end{pmatrix}u^-\right)da'\,da_l\,du\,du^- \qquad (5.21)\\
&=\int_{\mathrm{GL}(l-1,\mathbb{Q}_p)\times\mathbb{Q}_p^\times\times\mathbb{Q}_p^{l-1}}\left(\phi(p)^{-1}\alpha_l p^{-s}\right)^{\mathrm{val}_p(a_l)}\chi\psi_l(a_l|a_l|_p)\cdot\mathfrak{w}_{l-1}(a')\\
&\quad\cdot\chi\left(\det(a')|\det(a')|_p\right)\cdot\phi(\det(a')|_p)\cdot|\det(a')|_p^{s-1}\cdot\mathcal{F}^{-1}\Psi_{\chi\underline{\psi},l}\left(\begin{pmatrix}a' & y\\ 0 & a_l\end{pmatrix}\right)da'\,da_l\,dy.
\end{aligned}
$$

Next we split the proof of (5.20) into two cases depending on whether the character $\chi\psi_l$ is trivial or not. First we look at the case when $\chi\psi_l$ is trivial. Using Proposition 5.4.1 we get

$$
\mathcal{F}^{-1}\Psi_{\chi\underline{\psi},l}\left(\begin{pmatrix}a' & y\\ 0 & a_l\end{pmatrix}\right)=\left(-p^{-l}\mathbb{1}_{p^{-1}\mathbb{Z}_p^\times}(a_l)\mathbb{1}_{a_l\mathbb{Z}_p^{l-1}}(y)+(1-p^{-1})\mathbb{1}_{\mathbb{Z}_p}(a_l)\mathbb{1}_{\mathbb{Z}_p^{l-1}}(y)\right)\mathcal{F}^{-1}\Psi_{\chi\underline{\psi},l-1}(a'),
$$

and (5.21) becomes

$$
\begin{aligned}
&\operatorname{vol}\left(B_{l-1,1}(\mathbb{Z}_p)N_{l-1,1}^-(\mathbb{Z}_p)\right)^{-1}\int_{\mathrm{GL}(l,\mathbb{Q}_p)}\mathfrak{w}_l(a)\eta_{\kappa,\underline{\tau},l}(s,a)\,da\\
&=\int_{\mathbb{Q}_p^\times\times\mathbb{Q}_p^{l-1}}\left(\phi(p)^{-1}\alpha_l p^{-s}\right)^{\mathrm{val}_p(a_l)}\left(-p^{-l}\mathbb{1}_{p^{-1}\mathbb{Z}_p^\times}(a_l)\mathbb{1}_{p^{-1}\mathbb{Z}_p^{l-1}}(y)+(1-p^{-1})\mathbb{1}_{\mathbb{Z}_p}(a_l)\mathbb{1}_{\mathbb{Z}_p^{l-1}}(y)\right)da_l\,dy\\
&\quad\cdot\int_{\mathrm{GL}(l-1,\mathbb{Q}_p)}\mathfrak{w}_{l-1}(a')\eta_{\kappa,\underline{\tau},l-1}(s,a')\,da'\\
&=\left(-p^{-1}\cdot\phi(p)\alpha_l^{-1}p^s+(1-p^{-1})\sum_{j=0}^\infty\left(\phi(p)^{-1}\alpha_l p^{-s}\right)^j\right)\cdot\int_{\mathrm{GL}(l-1,\mathbb{Q}_p)}\mathfrak{w}_{l-1}(a')\eta_{\kappa,\underline{\tau},l-1}(s,a')\,da'\\
&=\frac{1-\phi(p)\alpha_l^{-1}p^{s-1}}{1-\phi(p)^{-1}\alpha_l p^{-s}}\cdot\int_{\mathrm{GL}(l-1,\mathbb{Q}_p)}\mathfrak{w}_{l-1}(a')\eta_{\kappa,\underline{\tau},l-1}(s,a')\,da',
\end{aligned}
$$

which is exactly (5.20) in the case when $\chi\psi_l$ is trivial. Now assume that $\chi\psi_l$ is nontrivial. Again using Proposition 5.4.1 we get

$$\mathcal{F}^{-1}\Psi_{\chi\underline{\psi},l}\left(\begin{pmatrix} a' & y \\ 0 & a_l \end{pmatrix}\right) = p^{-lc_{\chi\psi_l}} G(\chi\psi_l)\cdot \mathbb{1}_{p^{-c_{\chi\psi_l}}\mathbb{Z}_p^\times}(a_l)\mathbb{1}_{a_l\mathbb{Z}_p^{l-1}}(y)\cdot(\chi\psi_l(p^{c_{\chi\psi_l}}a_l))^{-1}\cdot\mathcal{F}^{-1}\Psi_{\chi\underline{\psi},l-1}(a'),$$

which together with (5.21) gives

$$\begin{aligned}
&\operatorname{vol}\left(B_{l-1,1}(\mathbb{Z}_p)N^-_{l-1,1}(\mathbb{Z}_p)\right)^{-1}\int_{\mathrm{GL}(l,\mathbb{Q}_p)}\mathfrak{w}_l(a)\eta_{\kappa,\underline{\tau},l}(s,a)\,da\\
&=p^{-lc_{\chi\psi_l}}G(\chi\psi_l)\cdot\operatorname{vol}\left(p^{-c_{\chi\psi_l}}\mathbb{Z}_p^{l-1}\right)\cdot\left(\phi(p)^{-1}\alpha_l p^{-s}\right)^{-c_{\chi\psi_l}}\cdot\int_{\mathrm{GL}(l-1,\mathbb{Q}_p)}\mathfrak{w}_{l-1}(a')\eta_{\kappa,\underline{\tau},l-1}(s,a')\,da'\\
&=\left(\phi(p)\alpha_l^{-1}p^{s-1}\right)^{c_{\chi\psi_l}}G(\chi\psi_l)\cdot\int_{\mathrm{GL}(l-1,\mathbb{Q}_p)}\mathfrak{w}_{l-1}(a')\eta_{\kappa,\underline{\tau},l-1}(s,a')\,da',
\end{aligned}$$

and proves (5.20) in the case when $\chi\psi_l$ is nontrivial. ☐

Chapter 6

The measure $\mu_{\mathcal{E},\mathrm{ord},\underline{\imath}}$ and the construction of the p-adic L-function

From the previously constructed measure $\mu_{\mathcal{E},q\text{-exp}}$ on $\mathbb{Z}_p^\times \times T_n(\mathbb{Z}_p)$, we apply Hida theory to produce, for each character $\underline{\imath}$ of $T_n(\mathbb{Z}/p\mathbb{Z})$, a measure $\mu_{\mathcal{E},\mathrm{ord},\underline{\imath}}$ on $\mathbb{Z}_p^\times$, valued in n-variable Hida families of $G \times G$.

6.1 Brief review of Hida theory for G

Usually Hida theory is formulated with $\mathbf{G}$ instead of G, but it should be clear that by restricting to a connected component we get a good theory for G.

The Igusa tower

Let $Y_{\mathbf{G},N}$ be the Siegel moduli scheme defined over $\mathbb{Z}_p$, parametrizing principally polarized abelian schemes (A, λ) of dimension n with a principal level N structure ψ_N over $\mathrm{Spec}(\mathbb{Z}_p)$, and $X_{\mathbf{G},N}$ be a smooth toroidal compactification of $Y_{\mathbf{G},N}$ with boundary C, over which there is the semi-abelian scheme $\mathcal{G} \to X_{\mathbf{G},N}$ extending the universal abelian scheme $\mathcal{A} \to Y_{\mathbf{G},N}$. Let $\mathrm{Ha} = \mathrm{Ha}(\mathcal{G}[p^\infty])$ be the Hasse invariant, which is a global section of the invertible sheaf $(\det \omega(\mathcal{G}/X_{\mathbf{G},N}))^{\otimes p-1}$ over the reduction $X_{\mathbf{G},N/\mathbb{F}_p}$. The push-forward of $\det \omega(\mathcal{G}/X_{\mathbf{G},N})$ to the minimal compactification $X^*_{\mathbf{G},N}$ is ample. For a sufficiently large integer c we can lift Ha^c to a section over $X_{\mathbf{G},N}$, and we denote by E such a lift.

Now let F be a finite extension of $\mathbb{Q}_p$ containing all the N-th roots of unity, and $X_{G,N}$ be a connected component of the base change of $X_{\mathbf{G},N}$ to $\mathcal{O}_F$. Define $S = X_{G,N}[1/E]$ and $S_l = S \otimes_{\mathrm{Spec}(\mathbb{Z}_p)} \mathrm{Spec}(\mathbb{Z}/p^l\mathbb{Z})$. Let $T_{l,m} = \underline{\mathrm{Isom}}_{S_l}\left((\mathcal{G}[p^m])^{D,\text{ét}}, (\mathbb{Z}/p^m\mathbb{Z})^n\right)$ where the superscript D means the Cartier dual. The scheme $T_{l,m}$ is étale over S_l with Galois group $\mathrm{GL}_n(\mathbb{Z}/p^m\mathbb{Z})$. The inverse system $\cdots \to T_{l,m} \to T_{l,m-1} \to \cdots \to T_{l,1} \to S_l$ is called the Igusa tower. By abuse of notation the pullback of the divisor C to $T_{m,l}$ will also be written as C.

6.1.1 p-adic (cuspidal) Siegel modular forms

Define

$$V_{l,m} := H^0\left(T_{l,m}, \mathcal{O}_{T_{l,m}}(-C)\right)^{N_n(\mathbb{Z}/p^m\mathbb{Z})},$$

and set $V_{l,\infty} = \varinjlim_m V_{l,m}$. By taking the inverse and direct limits of $V_{l,\infty}$ one defines

$$V = \varprojlim_l V_{l,\infty}, \qquad \mathcal{V} = \varinjlim_l V_{l,\infty}.$$

Elements in V are called (cuspidal) p-adic Siegel modular forms (of tame principal level N) . The space $\mathcal{V}$ will be used to construct Hida families. We also define the space V' in the same way as V but without requiring the cuspidality condition. The evaluation at the Mumford object (whose construction is explained in §2.7) defines the q-expansion map

$$\varepsilon_{q,l} : V'_{l,\infty} \longrightarrow \mathcal{O}_F/p^l\mathcal{O}_F[[N^{-1}\,\mathrm{Sym}(n,\mathbb{Z})^*_{\geq 0}]],$$

and the p-adic q-expansion map for p-adic Siegel modular forms

$$\varepsilon_{q,p\text{-adic}} : V' \longrightarrow \mathcal{O}_F[[N^{-1}\,\mathrm{Sym}(n,\mathbb{Z})^*_{\geq 0}]]. \tag{6.1}$$

The injectivity of $\varepsilon_{q,l}$ and $\varepsilon_{q,p\text{-adic}}$ follows from the irreducibility of the Igusa tower $\varprojlim_m T_{1,m}$ [21, V.7], and is called the q-expansion principle for p-adic Siegel modular forms.

For each continuous character $\underline{\tau} \in \mathrm{Hom}_{\mathrm{cont}}(T_n(\mathbb{Z}_p), \overline{\mathbb{Q}}_p^\times)$ (also called a p-adic weight), let $V[\underline{\tau}]$ (resp. $\mathcal{V}[\underline{\tau}]$) be the $\underline{\tau}$-isotypic part of $V \otimes_{\mathcal{O}_F} \mathcal{O}_{F(\underline{\tau})}$ (resp. $\mathcal{V} \otimes_{\mathcal{O}_F} \mathcal{O}_{F(\underline{\tau})}$) under the action of $T_n(\mathbb{Z}_p)$, where $F(\underline{\tau})$ is the field obtained by adjoining to F the values of the character $\underline{\tau}$. Elements inside the space $V[\underline{\tau}]$ are called (cuspidal) p-adic Siegel modular forms of (p-adic) weight $\underline{\tau}$. Thanks to the Hodge–Tate map

$$(\mathcal{G}[p^\infty])^{D,\text{ét}} \otimes_{\mathbb{Z}_p} \mathcal{O}_{S_l} \xrightarrow{\sim} \omega(\mathcal{G}/S_l), \tag{6.2}$$

for an algebraic weight $\underline{t}$, there is the canonical embedding

$$H^0(X_{G,N}, \omega_{\underline{t}}(-C)) \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^l\mathbb{Z} \hookrightarrow H^0(S_l, \omega_{\underline{t}}(-C)) \hookrightarrow V_{l,\infty}[\underline{t}].$$

The cuspidality condition guarantees that the following standard condition for Hida theory is satisfied,

$$\text{(Hyp)} \qquad H^0(S, \omega_{\underline{t}}(-C)) \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^l\mathbb{Z} \xrightarrow{\sim} H^0(S_l, \omega_{\underline{t}}(-C))$$

for all dominant algebraic weight $\underline{t}$, from which the density theorem follows, saying that the space of classical forms $\bigoplus_{\underline{t} \gg 0} H^0(X_{G,N}, \omega_{\underline{t}}(-C))[1/p] \cap V$ is dense inside V [32, §3.5].

The action of $\mathbb{U}_p$-operators can be defined for $V[\underline{\tau}]$, $\mathcal{V}[\underline{\tau}]$ via algebraic correspondence (cf. §3.9.5 or [33, §8.3]), and is compatible with all the $\mathbb{U}_p$-operators we have defined before (in fact it is the $\mathbb{U}_p$-action on V that has a canonical normalization, and the normalizations of the $\mathbb{U}_p$-action in other circumstances are chosen to agree with it). Recall that we have set $U_p = U_{p,\rho_G}$ to be the operator associated to $\rho_G = (n, n-1, \cdots, 1) \in C_n^+$. By the discussion on §4.1.2, the ordinary projector

$$e = \lim_{r \to \infty} U_p^{r!}$$

is well defined on $\bigoplus_{\underline{t} \geq 0} H^0(X_{G,N}, \omega_{\underline{t}}(-C))$. Then the density theorem indicates that the operator e extends to V and $\mathcal{V}$. It projects the spaces V and $\mathcal{V}$ to their subspaces where all the eigenvalues of $\mathbb{U}_p$-operators are p-adic units. Put

$$V_{\text{ord}} = eV, \qquad \mathcal{V}^*_{\text{ord}} = \text{Hom}_{\mathcal{O}_F}(e\mathcal{V}, F/\mathcal{O}_F).$$

The group $T_n(\mathbb{Z}_p)$ naturally acts on both V_{ord} and $\mathcal{V}^*_{\text{ord}}$ and equip them with an $\mathcal{O}_F[[T_n(\mathbb{Z}_p)]]$-module structure. Besides (Hyp) the other two conditions for the axiomatic vertical control theorem are

(C) $e(Ef) = Ee(f)$ for all $f \in H^0(S_1, \omega_{\underline{t}})$.

(F) $\dim_F eH^0\left(X_{G,N}, \omega_{\underline{t}} \otimes \det^k \omega(\mathcal{G}/X_{G,N})\right)$ is bounded independent of k.

The condition (C) can be easily checked using the q-expansion principle and the condition (F) follows from results in [61].

6.1.2 Hida families and the vertical control theorem

The group $T_n(\mathbb{Z}_p)$ decomposes as $\Gamma_{T_n} \times T_n(\mathbb{Z}/p\mathbb{Z})$ with Γ_{T_n} being the p-profinite part. Set $\Lambda_n = \mathcal{O}_F[[\Gamma_{T_n}]]$. The $\mathcal{O}_F[[T_n(\mathbb{Z}_p)]]$-module of Hida families of cuspidal p-adic Siegel modular forms of tame principal level N is defined as

$$\mathcal{M}_{\text{ord}} = \text{Hom}_{\Lambda_n}\left(\mathcal{V}^*_{\text{ord}}, \Lambda_n\right). \tag{6.3}$$

Given $\underline{\tau} \in \text{Hom}(T_n(\mathbb{Z}_p), \overline{\mathbb{Q}}_p^\times)$ put $\mathbf{p}_{\underline{\tau}} : \mathcal{O}_F[[T_n(\mathbb{Z}_p)]] \to \mathcal{O}_{F(\underline{\tau})}$ to be the map sending $\gamma \in T_n(\mathbb{Z}_p)$ to $\underline{\tau}(\gamma)$.

Theorem 6.1.1 (Vertical Control Theorem [33, Theorem 8.13])**.** *As a Λ_n-module, the space $\mathcal{M}_{\text{ord}}$ of Hida families is free of finite rank. For each p-adic weight $\underline{\tau}$ we have the Hecke equivariant isomorphism $\mathcal{M}_{\text{ord}} \otimes_{\mathcal{O}_F[[T_n(\mathbb{Z}_p)]], \mathbf{p}_{\underline{\tau}}} \mathcal{O}_{F(\underline{\tau})} \cong V_{\text{ord}}[\underline{\tau}]$. When $\underline{t}$ is a sufficiently regular algebraic weight, $V_{\text{ord}}[\underline{t}] = eH^0(X_{G,N}, \omega_{\underline{t}}(-C))$.*

The unramified Hecke operators and $\mathbb{U}_p$-operators act on $\mathcal{M}_{\text{ord}}$ and we denote by $\mathbb{T}^N_{\text{ord}}$ the subalgebra of $\text{End}_{\mathcal{O}_F[[T_n(\mathbb{Z}_p)]]}(\mathcal{M}_{\text{ord}})$ generated by them. The natural map $\text{Spec}(\mathbb{T}^N_{\text{ord}}) \to \text{Spec}(\mathcal{O}_F[[T_n(\mathbb{Z}_p)]])$ is called the weight projection map.

The finite group $T_n(\mathbb{Z}/p\mathbb{Z})$ acts on $\mathcal{M}_{\mathrm{ord}}$ and we have the decomposition of free Λ_n-modules

$$\mathcal{M}_{\mathrm{ord}} = \bigoplus_{\underline{\imath} \in \mathrm{Hom}(T_n(\mathbb{Z}/p\mathbb{Z}), \mu_{p-1})} \mathcal{M}_{\mathrm{ord}, \underline{\imath}},$$

such that $T_n(\mathbb{Z}/p\mathbb{Z})$ acts on $\mathcal{M}_{\mathrm{ord},\underline{\imath}}$ by the character $\underline{\imath}$.

6.1.3 The spaces $\mathscr{M}eas(T_n(\mathbb{Z}_p), V_{\mathrm{ord}})^\natural$ and $\mathscr{M}eas(T_n(\mathbb{Z}_p), V_{\mathrm{ord}}^{\Delta})^\natural$

The group $T_n(\mathbb{Z}_p)$ acts on itself by multiplication and induces a natural $\mathcal{O}_F[[T_n(\mathbb{Z}_p)]]$-module structure on the space $\mathscr{C}(T_n(\mathbb{Z}_p), \mathcal{O}_F)$. We define $\mathscr{M}eas(T_n(\mathbb{Z}_p), V')^\natural$ to be the subspace of $\mathscr{M}eas(T_n(\mathbb{Z}_p), V')$ consisting of continuous maps $\mathscr{C}(T_n(\mathbb{Z}_p), \mathcal{O}_F) \to V'$ that are not only $\mathcal{O}_F$-linear but further $\mathcal{O}_F[[T_n(\mathbb{Z}_p)]]$-linear. An equivalent description for the elements of the subspace $\mathscr{M}eas(T_n(\mathbb{Z}_p), V')^\natural$ is that the evaluations at all $\underline{\tau} \in \mathrm{Hom}_{\mathrm{cont}}(T_n(\mathbb{Z}_p), \overline{\mathbb{Q}}_p^{\times})$ belong to $V'[\underline{\tau}]$. Let $\mathscr{M}eas(T_n(\mathbb{Z}_p), V_{\mathrm{ord}})^\natural$ be the ordinary cuspidal part of $\mathscr{M}eas(T_n(\mathbb{Z}_p), V')^\natural$. For each character $\underline{\imath}$ of $T_n(\mathbb{Z}/p\mathbb{Z})$, we construct a morphism $\Phi_{\underline{\imath}}$ mapping $\mathscr{M}eas(T_n(\mathbb{Z}_p), V_{\mathrm{ord}})^\natural$ into the space of Hida families.

Unfolding the definitions one easily sees that there is a natural pairing $V_{\mathrm{ord}} \times \mathcal{V}_{\mathrm{ord}}^* \xrightarrow{\langle , \rangle} \mathcal{O}_F$ such that the following diagram commutes if $\underline{\imath} = \underline{\tau} \mid_{T_n(\mathbb{Z}/p\mathbb{Z})}$

$$\begin{array}{ccc}
\mathcal{M}_{\mathrm{ord},\underline{\imath}} \times \mathcal{V}_{\mathrm{ord}}^* & \longrightarrow & \Lambda_n \\
\Big\downarrow {\scriptstyle \mathbf{s}_{\underline{\tau}} \times 1} & & \Big\downarrow {\scriptstyle \mathbf{p}_{\underline{\tau}}} \\
V_{\mathrm{ord}}[\underline{\tau}] \times \mathcal{V}_{\mathrm{ord}}^* \hookrightarrow \left(V_{\mathrm{ord}} \otimes_{\mathcal{O}_F} \mathcal{O}_{F(\underline{\tau})}\right) \times \mathcal{V}_{\mathrm{ord}}^* & \xrightarrow{\langle , \rangle} & \mathcal{O}_{F(\underline{\tau})},
\end{array}$$

where $\mathbf{s}_{\underline{\tau}}$ is the specialization map

$$\mathbf{s}_{\underline{\tau}} : \mathcal{M}_{\mathrm{ord}} \longrightarrow \mathcal{M}_{\mathrm{ord}} \otimes_{\mathcal{O}_F[[T_n(\mathbb{Z}_p)]], \mathbf{p}_{\underline{\tau}}} \mathcal{O}_{F(\underline{\tau})} \xrightarrow{\sim} V_{\mathrm{ord}}[\underline{\tau}]. \tag{6.4}$$

This pairing induces an $\mathcal{O}_F[[T_n(\mathbb{Z}_p)]]$-linear pairing

$$\mathscr{M}eas(T_n(\mathbb{Z}_p), V_{\mathrm{ord}})^\natural \times \mathcal{V}_{\mathrm{ord}}^* \longrightarrow \mathscr{M}eas(T_n(\mathbb{Z}_p), \mathcal{O}_F), \tag{6.5}$$

where the $\mathcal{O}_F[[T_n(\mathbb{Z}_p)]]$-module structure on $\mathscr{M}eas(T_n(\mathbb{Z}_p), \mathcal{O}_F)$ comes from that of $\mathscr{C}(T_n(\mathbb{Z}_p), \mathcal{O}_F)$.

Now fix a character $\underline{\imath}$ of the finite group $T_n(\mathbb{Z}/p\mathbb{Z})$. Let $\mathfrak{u}$ be a generator of $1+p\mathbb{Z}_p$ and we associate to it the p-adic logarithm function $\log_{\mathfrak{u}} : 1+p\mathbb{Z}_p \to \mathbb{Z}_p$ such that the value at $\mathfrak{u}$ is 1 and we extend $\log_{\mathfrak{u}}$ to $\mathbb{Z}_p^\times$ by requiring it to take value 0 on μ_{p-1} ($\mathbb{Z}_p^\times$ canonically decomposes as $\mu_{p-1}\times(1+p\mathbb{Z}_p)$). Denote by γ_i the element of $T_n(\mathbb{Z}_p)$ whose i-th component is $\mathfrak{u}$ and other components are 1. Then $\gamma_1,\cdots,\gamma_n$ topologically generate Γ_{T_n}. The p-adic Mellin transform with respect to $\underline{\imath}$ is the map

$$\begin{aligned} \mathcal{M}eas(T_n(\mathbb{Z}_p),\mathcal{O}_F) &\longrightarrow \Lambda_n \\ \mu &\longmapsto \int_{T_n(\mathbb{Z}_p)} \underline{\imath}(x_1,\cdots,x_n)\prod_{i=1}^n \gamma_i^{\log_{\mathfrak{u}}(x_i)}\,d\mu(x_1,\cdots,x_n), \end{aligned} \tag{6.6}$$

where $\gamma_i^{\log_{\mathfrak{u}}(x_i)}$ is the element $\sum_{m=0}^\infty \binom{\log_{\mathfrak{u}} x_i}{m}(\gamma_i-1)^m \in \Lambda_n$ with the binomial coefficient $\binom{\log_{\mathfrak{u}} x_i}{m}$ defined as $\frac{\log_{\mathfrak{u}} x_i(\log_{\mathfrak{u}} x_i-1)\cdots(\log_{\mathfrak{u}} x_i-m+1)}{m!}$. One can check that this p-adic Mellin transform with respect to $\underline{\imath}$ is Λ_n-linear. Combining it with (6.5) we get a Λ_n-linear pairing

$$\mathcal{M}eas(T_n(\mathbb{Z}_p),V_{\mathrm{ord}})^\natural \times \mathcal{V}_{\mathrm{ord}}^* \longrightarrow \Lambda_n,$$

and therefore the desired morphism of Λ_n-modules

$$\Phi_{\underline{\imath}} : \mathcal{M}eas(T_n(\mathbb{Z}_p),V_{\mathrm{ord}})^\natural \to \mathcal{M}_{\mathrm{ord},\underline{\imath}}. \tag{6.7}$$

Moreover for each point $\underline{\tau} \in \mathrm{Hom}_{\mathrm{cont}}(T_n(\mathbb{Z}_p),\overline{\mathbb{Q}}_p^\times)$ whose restriction to $T_n(\mathbb{Z}/p\mathbb{Z})$ is $\underline{\imath}$ and $\mu \in \mathcal{M}eas(T_n(\mathbb{Z}_p),V_{\mathrm{ord}})^\natural$, we have

$$\int_{T_n(\mathbb{Z}_p)} \underline{\tau}\,d\mu = \mathbf{s}_{\underline{\tau}} \circ \Phi_{\underline{\imath}}(\mu).$$

For our applications we define the $\mathcal{O}_F[[T_n(\mathbb{Z}_p)]]$-module V^Δ, which as an $\mathcal{O}_F$-module is the subspace of $V\otimes_{\mathcal{O}_F} V$ generated by the elements killed by $\gamma\otimes 1-1\otimes\gamma$ for all $\gamma\in T_n(\mathbb{Z}_p)$. The action of $T_n(\mathbb{Z}_p)$ on V^Δ via either factor agrees with the other, so V^Δ has a well-defined $\mathcal{O}_F[[T_n(\mathbb{Z}_p)]]$-

module structure. Denote by $V^{\Delta}_{\mathrm{ord}}$ the sub-$\mathcal{O}_F[[T_n(\mathbb{Z}_p)]]$-module of V^{Δ} obtained by taking the ordinary projection on both factors, and we define the $\mathcal{O}_F[[T_n(\mathbb{Z}_p)]]$-module $\mathscr{M}eas(T_n(\mathbb{Z}_p), V^{\Delta}_{\mathrm{ord}})^{\natural}$ to be the space of continuous $\mathcal{O}_F[[T_n(\mathbb{Z}_p)]]$-linear maps from $\mathscr{C}(T_n(\mathbb{Z}_p), \mathcal{O}_F)$ to $V^{\Delta}_{\mathrm{ord}}$. Through the same argument as above we see that there exists a canonical $\mathcal{O}_F[[T_n(\mathbb{Z}_p)]]$-linear pairing

$$V^{\Delta}_{\mathrm{ord}} \times \left(\mathcal{V}^*_{\mathrm{ord}} \otimes_{\mathcal{O}_F[[T_n(\mathbb{Z}_p)]]} \mathcal{V}^*_{\mathrm{ord}}\right) \longrightarrow \mathcal{O}_F$$

whose restriction to either factor agrees with the previous pairing $V_{\mathrm{ord}} \times \mathcal{V}^*_{\mathrm{ord}} \xrightarrow{\langle , \rangle} \mathcal{O}_F$. It induces a morphism of Λ_n-modules

$$\Phi^{\Delta}_{\underline{\imath}} : \mathscr{M}eas(T_n(\mathbb{Z}_p), V^{\Delta}_{\mathrm{ord}})^{\natural} \to \mathcal{M}_{\mathrm{ord},\underline{\imath}} \otimes_{\Lambda_n} \mathcal{M}_{\mathrm{ord},\underline{\imath}}, \tag{6.8}$$

with the property

$$\int_{T_n(\mathbb{Z}_p)} \underline{\tau}\, d\mu = (\mathbf{s}_{\underline{\tau}} \times \mathbf{s}_{\underline{\tau}}) \circ \Phi^{\Delta}_{\underline{\imath}}(\mu)$$

for all $\underline{\tau} \in \mathrm{Hom}_{\mathrm{cont}}(T_n(\mathbb{Z}_p), \overline{\mathbb{Q}}_p^{\times})$ whose restriction to $T_n(\mathbb{Z}/p\mathbb{Z})$ is $\underline{\imath}$ and $\mu \in \mathscr{M}eas(T_n(\mathbb{Z}_p), V^{\Delta}_{\mathrm{ord}})^{\natural}$.

6.1.4 The q-expansions of Hida families

For each $\beta \in N^{-1}\,\mathrm{Sym}(n,\mathbb{Z})^*_{>0}$, the maps $\varepsilon_{q,\beta} : V_{l,\infty} \to \mathcal{O}_F/p^l\mathcal{O}_F$, $l \geq 1$, of taking the β-th coefficient of the q-expansion patch to an $\mathcal{O}_F$-linear map $\varepsilon_{q,\beta} : \mathcal{V} \to F/\mathcal{O}_F$, which gives an element of $\mathcal{V}^*_{\mathrm{ord}}$. Thus by definition there is a Λ_n-linear map

$$\varepsilon_{q,\beta} : \mathcal{M}_{\mathrm{ord}} \longrightarrow \Lambda_n$$

which makes the following diagram

$$\begin{array}{ccccc}
\mathcal{M}_{\mathrm{ord},\underline{\imath}} & & \xrightarrow{\qquad\qquad \varepsilon_{q,\beta} \qquad\qquad} & & \Lambda_n \\
\Big\downarrow \mathbf{s}_{\underline{\tau}} & & & & \Big\downarrow \mathbf{p}_{\underline{\tau}} \\
V_{\mathrm{ord}}[\underline{\tau}] & \xrightarrow{\varepsilon_{q,p\text{-adic}}} & \mathcal{O}_{F(\underline{\tau})}[[N^{-1}\,\mathrm{Sym}(n,\mathbb{Z})^*_{>0}]] & \xrightarrow{\beta\text{-th coefficient}} & \mathcal{O}_{F(\underline{\tau})}
\end{array}$$

commute for $\underline{\tau}$ that restricts to $\underline{\imath}$ on $T_n(\mathbb{Z}/p\mathbb{Z})$. From $\varepsilon_{q,\beta}$, for $(\beta_1,\beta_2) \in N^{-1}\operatorname{Sym}(n,\mathbb{Z})^{*\oplus 2}_{>0}$ we define the Λ_n-linear map

$$\varepsilon_{q,\beta_1,\beta_2} : \mathcal{M}_{\mathrm{ord}} \otimes_{\mathcal{O}_F[[T_n(\mathbb{Z}_p)]]} \mathcal{M}_{\mathrm{ord}} \longrightarrow \Lambda_n.$$

6.2 Construct $\mu_{\mathcal{E},\mathrm{ord},\underline{\imath}}$ from $\mu_{\mathcal{E},q\text{-}\mathrm{exp}}$

6.2.1 Embedding nearly holomorphic forms into p-adic forms

Let $T_{\infty,m}$ be the formal scheme $\varinjlim_l T_{l,m}$ defined over $\mathcal{O}_F$. When $m=0$ the formal scheme $T_{\infty,0}$ is the completion of $S = X_{G,N}[1/E]$ along its special fibre. Over $T_{\infty,0}$ the Hodge filtration admits a splitting

$$\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/Y_{G,N})^{\mathrm{can}}\big|_{T_{\infty,0}} = \omega(\mathcal{G}/T_{\infty,0}) \oplus \mathcal{U}_{\mathcal{H}},$$

called the unit root splitting (cf. §3.12). Take the generic fibre $T_{\mathrm{rig},m}$ of the formal scheme $T_{\infty,m}$. It is a rigid analytic subspace of the rigid analytic space $X^{\mathrm{an}}_{G,\Gamma(Np^m)}$ associated to the scheme $X_{G,\Gamma(Np^m)}$ over F. Pulling back the unit root splitting from level $\Gamma(N)$ to $\Gamma(Np^m)$ yields a projection $\mathcal{V}^r_{\underline{t}} \to \omega_{\underline{t}}$ of coherent sheaves over the rigid analytic space $T_{\mathrm{rig},m}$, from which one gets, combining with the Hodge–Tate map (6.2), the following Hecke equivariant map

$$\iota_{p\text{-adic}} : H^0(X_{G,\Gamma_1(N,p^m)}, \mathcal{V}^r_{\underline{t}})[\underline{\psi}] \longrightarrow H^0(T_{\infty,m}, \omega_{\underline{t}})^{N_n(\mathbb{Z}/p^m\mathbb{Z})}[\underline{\psi}][1/p] \longrightarrow V'[\underline{\tau}][1/p],$$

where $\underline{\tau} \in \operatorname{Hom}_{\mathrm{cont}}(T_n(\mathbb{Z}_p) \times \mathbb{Z}_p^\times, \overline{\mathbb{Q}}_p^\times)$ is an arithmetic weight with algebraic part $\underline{t}$ dominant and finite part $\underline{\psi}$ valued in $\mu_{(p-1)p^{m-1}}$. The symbol $[\underline{\psi}]$ means the $\underline{\psi}$ equivariant part under the natural action of $T_n(\mathbb{Z}_p)$. The injectivity of $\iota_{p\text{-adic}}$ is shown in Proposition 3.12.1.

The map $\iota_{p\text{-adic}}$ embeds nearly holomorphic forms into the space of p-adic forms Hecke equivariantly and gives an integral structure to the space $H^0(X_{G,\Gamma_1(N,p^m)}, \mathcal{V}^r_{\underline{t}})$ which is preserved by the $\mathbb{U}_p$-operators. Moreover we have

Proposition 6.2.1. *$eH^0(X_{G,\Gamma_1(N,p^m)}, \mathcal{V}^r_{\underline{t}}) = eH^0(X_{G,\Gamma_1(N,p^m)}, \omega_{\underline{t}})$.*

Proof. Proposition 4.6 says that the composition $E_{\underline{t}}e$ is 0, or equivalently the image of e is killed by the operator $E_{\underline{t}}$, so holomorphic. $\square$

6.2.2 The measure $\mu_{\mathcal{E},\mathrm{ord},\underline{\imath}}$

As explained in §3.13, the composition of $\iota_{p\text{-adic}}$ with (6.1) is exactly the (p-adic) q-expansion map for nearly holomorphic forms defined in (2.20). Now Proposition 4.6.1 together with the q-expansion principle implies that $\iota_{p\text{-adic}}(\mathcal{E}_{\kappa,\underline{\tau}})$ lies inside $V'[\underline{\tau}]$ for all admissible $(\kappa,\underline{\tau})$. One direct corollary of the q-expansion principle is that the space V' of p-adic forms (of tame principal level N) is a closed subspace, under the induced topology, of the space $\mathcal{O}_F[[N^{-1}\,\mathrm{Sym}(n,\mathbb{Z})^*_{>0}]]$. Then the density of all the admissible points inside $\mathrm{Hom}_{\mathrm{cont}}(T_n(\mathbb{Z}_p)\times\mathbb{Z}_p^\times,\overline{\mathbb{Q}}_p^\times)$ with respect to the p-adic topology indicates that the measure $\mu_{\mathcal{E},q\text{-exp}}$ in Theorem 5.2.2 belongs to the image of the embedding of $\mathscr{M}eas\left(\mathbb{Z}_p^\times\times T_n(\mathbb{Z}_p),V'^{\Delta}\right)$ into $\mathscr{M}eas\left(T_n(\mathbb{Z}_p)\times\mathbb{Z}_p^\times,\mathcal{O}_F[[N^{-1}\,\mathrm{Sym}(n,\mathbb{Z})^{*\oplus 2}_{\geq 0}]]\right)$, induced by the q-expansion map.

This is not sufficient for us. Before we continue we must make sure that $\mu_{\mathcal{E},q\text{-exp}}$ actually is contained in the image of the cuspidal part. Thanks to the cuspidality result Theorem 4.4.2 we know that $\iota_{p\text{-adic}}(\mathcal{E}_{\kappa,\underline{\tau}})$ is cuspidal if $t_1=t_2=\cdots=t_n=k>2n+1$. The Zariski density of such points guarantees that $\mu_{\mathcal{E},q\text{-exp}}$ lies inside the image of the injective map

$$\mathscr{M}eas\left(\mathbb{Z}_p^\times\times T_n(\mathbb{Z}_p),V^{\Delta}\right)\longrightarrow\mathscr{M}eas\left(T_n(\mathbb{Z}_p)\times\mathbb{Z}_p^\times,\mathcal{O}_F[[N^{-1}\,\mathrm{Sym}(n,\mathbb{Z})^{*\oplus 2}_{>0}]]\right),$$

and we denote by $\mu_{\mathcal{E}}$ the preimage of $\mu_{\mathcal{E},q\text{-exp}}$.

Now by applying the ordinary projection $e\times e: V^{\Delta}\to V^{\Delta}_{\mathrm{ord}}$ to $\mu_{\mathcal{E}}$, we obtain the measure $\mu_{\mathcal{E},\mathrm{ord}}$ inside $\mathscr{M}eas\left(\mathbb{Z}_p^\times\times T_n(\mathbb{Z}_p),V^{\Delta}_{\mathrm{ord}}\right)=\mathscr{M}eas\left(\mathbb{Z}_p^\times,\mathscr{M}eas(T_n(\mathbb{Z}_p),V^{\Delta}_{\mathrm{ord}})\right)$. Using (6.8), we define

$$\mu_{\mathcal{E},\mathrm{ord},\underline{\imath}}=\Phi^{\Delta}_{\underline{\imath}}\left(\mu_{\mathcal{E},\mathrm{ord}}\right).$$

This $\mu_{\mathcal{E},\mathrm{ord},\underline{\imath}}$ lies inside $\mathscr{M}eas(\mathbb{Z}_p^\times,\mathcal{M}_{\mathrm{ord},\underline{\imath}}\otimes_{\Lambda_n}\mathcal{M}_{\mathrm{ord},\underline{\imath}})$ and satisfies

$$(\mathbf{s}_{\underline{\tau}}\times\mathbf{s}_{\underline{\tau}})\left(\int_{\mathbb{Z}_p^\times}\kappa\,d\mu_{\mathcal{E},\mathrm{ord},\underline{\imath}}\right)=(e\times e)\mathcal{E}_{\kappa,\underline{\tau}}$$

for all admissible $(\kappa, \underline{\tau})$ such that the restriction of $\underline{\tau}$ to $T_n(\mathbb{Z}/p\mathbb{Z})$ is $\underline{\imath}$.

Before we ending this section we give the proof of Proposition 5.7.1.

Proof of 5.7.1. We show $(e\times e)(\mu_{\mathcal{E}}) - (e\times 1)(\mu_{\mathcal{E}}) = 0$, which can be implied by the vanishing of its image $\nu_{\beta_1,\beta_2} \in \mathcal{M}eas(\mathbb{Z}_p^\times \times T_n(\mathbb{Z}_p), \mathcal{O}_F)$ under the map

$$\begin{aligned}\mathcal{M}eas\left(\mathbb{Z}_p^\times \times T_n(\mathbb{Z}_p), V^\Delta\right) &\hookrightarrow \mathcal{M}eas\left(T_n(\mathbb{Z}_p)\times\mathbb{Z}_p^\times, \mathcal{O}_F[[N^{-1}\operatorname{Sym}(n,\mathbb{Z})^{*\oplus 2}_{>0}]]\right)\\ &\xrightarrow{(\beta_1,\beta_2)\text{-th coefficient}} \mathcal{M}eas(\mathbb{Z}_p^\times \times T_n(\mathbb{Z}_p), \mathcal{O}_F),\end{aligned}$$

for all $(\beta_1,\beta_2) \in N^{-1}\operatorname{Sym}(n,\mathbb{Z})^{*\oplus 2}_{>0}$. The p-adic Mellin transform (defined similarly as (6.6)) gives an isomorphism between $\mathcal{M}eas(\mathbb{Z}_p^\times \times T_n(\mathbb{Z}_p), \mathcal{O}_F)$ and $\mathcal{O}_F[[\mathbb{Z}_p^\times \times T_n(\mathbb{Z}_p)]]$. It is not difficult to see that the vanishing of ν_{β_1,β_1} follows from the Zariski density of the subset inside $\operatorname{Hom}_{\text{cont}}\left(\mathbb{Z}_p^\times \times T_n(\mathbb{Z}_p), \overline{\mathbb{Q}}_p^\times\right)$ consisting of those points (by (5.15) including all admissible points with $\chi\psi_1, \cdots, \chi\psi_n$ nontrivial) at which the evaluations of ν_{β_1,β_2} are zero. □

6.3 The p-adic L-function for ordinary families and its interpolation properties

The p-adic L-function for a given ordinary family of Hecke eigensystems is constructed by projecting the Hida-family-valued measure $\mu_{\mathcal{E},\text{ord},\underline{\imath}}$ to the corresponding eigenspace for that ordinary family and then taking a nonvanishing Fourier coefficient.

The universal ordinary Hecke algebra $\mathbb{T}^N_{\text{ord}}$ of tame principal level N is finite torsion free over Λ_n, and reduced because of Proposition 5.5.2.

Given a point $x \in \operatorname{Spec}(\mathbb{T}^N_{\text{ord}})(\overline{\mathbb{Q}}_p)$ whose projection to the weight space $\underline{\tau} \in \operatorname{Hom}_{\text{cont}}(T_n(\mathbb{Z}_p), \overline{\mathbb{Q}}_p^\times)$ is arithmetic with dominant algebraic part $\underline{t} \in X(T_n)_+$, define $\mathfrak{S}_x$ to be the finite dimensional $F(\underline{\tau})$-vector space consisting of cuspidal holomorphic Siegel modular forms which are contained in $H^0(X_{G,\Gamma_1(N,p^m)}, \omega_{\underline{t}}(-C))[\underline{\psi}]$ for some m, and belong to the eigenspace parametrized by x for the unramified Hecke operators and $\mathbb{U}_p$-operators. The space $\mathfrak{S}_x$ is stable under the operator eW, the composition of the ordinary projector and the operator W defined as (5.6). Let $\mathfrak{a}_{x,j} \in \mathcal{O}_{\overline{\mathbb{Q}}_p}^{\times}$, $1 \leq j \leq n$, be the p-adic integers such that for each $\underline{a} = (a_1, \cdots, a_n) \in C_n^+$, the eigenvalue of the operator $U_{p,\underline{a}}$ parametrized by x is given by $\prod_{j=1}^{n} \mathfrak{a}_{x,j}^{a_j}$. If $\pi \subset \mathcal{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A}))$ is an irreducible cuspidal automorphic representation generated by an element inside $\mathfrak{S}_x$, then for $v \nmid Np$, it is clear that the isomorphism class of π_v is completely determined by x. At the same time the isomorphism class of the component π_p is also determined by $\underline{\psi} = \underline{\tau}_{\mathrm{f}}$ and $\mathfrak{a}_{x,1}, \cdots, \mathfrak{a}_{x,n}$ (see §5.5). Thus the isomorphism class of the $G(\mathbb{A}^N)$-representation $\bigotimes'_{v \nmid N} \pi_v$ is determined by x and we denote it by π_x^N. Set $\alpha_{x,j} = p^{-(t_j - j)} \mathfrak{a}_{x,j}$.

To π_x^N and Dirichlet characters ϕ, χ, we associate the partial standard L-function $L^{Np\infty}(s, \pi_x^N \otimes \phi^{-1}\chi^{-1})$, and the modified Euler factor at p

$$E_p(s, \pi_x^N \times \phi^{-1}\chi^{-1}) = \frac{\left(1 - \chi^\circ(p) \cdot \phi(p) p^{s-1}\right) \prod_{j=1}^{n} \left(1 - (\chi\psi_j)^\circ(p) \cdot \phi(p) \alpha_{x,j}^{-1} p^{s-1}\right)}{\left(1 - \chi^\circ(p) \cdot \phi(p)^{-1} p^{-s}\right) \prod_{j=1}^{n} \left(1 - (\chi\psi_j)^\circ(p) \cdot \phi(p)^{-1} \alpha_{x,j} p^{-s}\right)}$$
$$\times \left(\phi(p) p^{s-1}\right)^{c_\chi} G(\chi) \prod_{j=1}^{n} \left(\phi(p) \alpha_{x,j}^{-1} p^{s-1}\right)^{c_{\chi\psi_j}} G(\chi\psi_j).$$

Let $\mathcal{C}$ be a geometrically irreducible component of $\mathrm{Spec}(\mathbb{T}^N_{\mathrm{ord}} \otimes_{\mathcal{O}_F} F)$. Set $F_{\mathcal{C}}$ to be the function field of $\mathcal{C}$ and $\mathbb{I}_{\mathcal{C}}$ to be the integral closure of Λ_n inside $F_{\mathcal{C}}$. Denote by $\lambda_{\mathcal{C}} : \mathbb{T}^N_{\mathrm{ord}} \to \mathbb{I}_{\mathcal{C}}$ the homomorphism of Λ_n-algebras corresponding to $\mathcal{C}$. The group $T_n(\mathbb{Z}/p\mathbb{Z})$ acts on $\mathbb{T}^N_{\mathrm{ord}}$ and its action on $\mathbb{I}_{\mathcal{C}}$ is by a character $\underline{\iota}_{\mathcal{C}}$.

There is an isomorphism of $F_{\mathcal{C}}$-algebras

$$\mathbb{T}^N_{\mathrm{ord}} \otimes_{\Lambda_n} F_{\mathcal{C}} = F_{\mathcal{C}} \oplus R_{\mathcal{C}}$$

such that the projection of $\mathbb{T}^N_{\mathrm{ord}} \otimes_{\Lambda_n} \mathbb{I}_{\mathcal{C}}$ onto the first factor coincides with $\lambda_{\mathcal{C}}$. Define $\mathbb{1}_{\mathcal{C}} \in \mathbb{T}^N_{\mathrm{ord}} \otimes_{\Lambda_n} F_{\mathcal{C}}$ to be the idempotent corresponding to the first factor. For a finite extension F' of F, write $\Lambda_{n,F'}$ (resp. $\mathbb{T}^N_{\mathrm{ord},F'}$, $\mathbb{I}_{\mathcal{C},F'}$) to be the base change of Λ_n (resp. $\mathbb{T}^N_{\mathrm{ord}}$, $\mathbb{I}_{\mathcal{C}}$) from $\mathcal{O}_F$ to F'. If the weight projection map $\Lambda_{n,F'} \to \mathbb{T}^N_{\mathrm{ord},F'}$ is étale at the point $x \in \mathcal{C}(F')$, put $x' \in \mathrm{Spec}(\mathbb{T}^N_{\mathrm{ord}} \otimes_{\Lambda_n} \mathbb{I}_{\mathcal{C},F',x})$ to be the maximal ideal generated by $T \otimes 1 - 1 \otimes \lambda_{\mathcal{C}}(T)$ for all $T \in \mathbb{T}^N_{\mathrm{ord}}$ and $1 \otimes a$ for all a

inside the maximal ideal corresponding to x. It follows from [59, Tag 00UE, Tag 00U8] that $\left(\mathbb{T}^N_{\mathrm{ord}} \otimes_{\Lambda_n} \mathbb{I}_{\mathcal{C},F',x}\right)_{x'} = \mathbb{I}_{\mathcal{C},F',x}$, so the localization map $\mathbb{T}^N_{\mathrm{ord}} \otimes_{\Lambda_n} \mathbb{I}_{\mathcal{C},F',x} \to \left(\mathbb{T}^N_{\mathrm{ord}} \otimes_{\Lambda_n} \mathbb{I}_{\mathcal{C},F',x}\right)_{x'}$ is surjective, and there exists the decomposition of $\mathbb{I}_{\mathcal{C},F',x}$-algebras

$$\mathbb{T}^N_{\mathrm{ord}} \otimes_{\Lambda_n} \mathbb{I}_{\mathcal{C},F',x} = \mathbb{I}_{\mathcal{C},F',x} \oplus R'_{\mathcal{C},x}$$

with the first projection being $\lambda_{\mathcal{C}}$. Thus the projector $\mathbb{1}_{\mathcal{C}}$ lies inside $\mathbb{T}^N_{\mathrm{ord}} \otimes_{\Lambda_n} \mathbb{I}_{\mathcal{C},F',x}$ as long as the weight projection map is étale at $x \in \mathcal{C}(F')$.

Now applying the Hecke projector $\mathbb{1}_{\mathcal{C}}$ to the measure $\mu_{\mathcal{E},\mathrm{ord},\underline{\mathfrak{t}}_{\mathcal{C}}}$ constructed in §6.2.2 gives an element inside $\mathscr{M}eas(\mathbb{Z}_p^\times, \mathcal{M}_{\mathrm{ord},\underline{\mathfrak{t}}} \otimes_{\Lambda_n} \mathcal{M}_{\mathrm{ord},\underline{\mathfrak{t}}}) \otimes_{\Lambda_n} F_{\mathcal{C}}$ on which the Hecke operators act by $\lambda_{\mathcal{C}}$. Suppose that the point $x \in \mathcal{C}(F')$ projects to an arithmetic point $\underline{\mathfrak{r}}$ in the weight space whose algebraic part is dominant and the weight projection map is étale at x. Let $\mathbf{s}_x : \mathcal{M}_{\mathrm{ord}} \otimes_{\Lambda_n} \mathbb{I}_{\mathcal{C},F',x} \to V_{\mathrm{ord}}[\underline{\mathfrak{r}}] \otimes_{\mathcal{O}_{F(\underline{\mathfrak{r}})}} \mathcal{O}_{F'}$ be the specialization map defined from (6.4) by extension of scalars. Fix an orthogonal basis $\mathfrak{s}_x = \{\varphi_1, \cdots, \varphi_d\}$ of the vector space $\mathfrak{S}_x$, i.e. $\varphi_1, \cdots, \varphi_d$ span $\mathfrak{S}_x$ and satisfy $\langle \varphi_i, \overline{\varphi_j} \rangle = 0$ if $i \neq j$. Then for each arithmetic $\kappa \in \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, \overline{\mathbb{Q}}_p^\times)$ with $t_1 \geq \cdots \geq t_n \geq k \geq n+1$ and $\kappa(-1) = \phi(-1)$, we know by construction that the specialization at x of the Hida family $\mathbb{1}_{\mathcal{C}} \int_{\mathbb{Z}_p^\times} \kappa \, d\mu_{\mathcal{E},\mathrm{ord},\underline{\mathfrak{t}}_{\mathcal{C}}}$ is a classical cuspidal holomorphic Siegel modular form on $G \times G$. By Proposition 5.2.3 we have

$$\begin{aligned} &\mathbf{s}_x \left(\mathbb{1}_{\mathcal{C}} \int_{\mathbb{Z}_p^\times} \kappa \, d\mu_{\mathcal{E},\mathrm{ord},\underline{\mathfrak{t}}_{\mathcal{C}}} \right) \\ &= \phi(-1)^n \mathrm{vol}\left(\widehat{\Gamma}(N)\right) \frac{p^{n^2}(p-1)^n}{\prod_{l=1}^n (p^{2l}-1)} \cdot \frac{\Gamma(k-n)\Gamma_{2n}(k)}{2^{k+n-1}(\pi i)^{2nk+k-n}} \cdot \frac{Z_\infty(f_{\kappa,\underline{\mathfrak{r}},\infty}, v_{\underline{t}}^\vee, v_{\underline{t}})}{\langle v_{\underline{t}}^\vee, v_{\underline{t}} \rangle} \\ &\quad \times E_p(k-n, \pi_x^N \times \phi^{-1}\chi^{-1}) \cdot L^{Np\infty}(k-n, \pi_x^N \times \phi^{-1}\chi^{-1}) \cdot \sum_{\varphi \in \mathfrak{s}_x} \frac{\varphi \otimes eW(\varphi)}{\langle \varphi, \overline{\varphi} \rangle}, \end{aligned} \tag{6.9}$$

where $v_{\underline{t}}$ is the highest weight vector inside the lowest $K_{G,\infty}$-type of the holomorphic discrete series $\mathcal{D}_{\underline{t}}$ and $v_{\underline{t}}^\vee$ is taken to be its dual vector.

For each $(\beta_1, \beta_2) \in N^{-1}\,\mathrm{Sym}(n,\mathbb{Z})^{*\oplus 2}_{>0}$, define (recall that for simplicity we have assumed $\phi^2 \neq 0$)

$$\mu_{\mathcal{C},\phi,\beta_1,\beta_2} = \varepsilon_{q,\beta_1,\beta_2}\left(\mathbb{1}_{\mathcal{C}} \cdot \mu_{\mathcal{E},\mathrm{ord},\underline{\mathfrak{t}}_{\mathcal{C}}}\right) \in \mathscr{M}eas(\mathbb{Z}_p^\times, \Lambda_n) \otimes_{\Lambda_n} F_{\mathcal{C}}.$$

Contrary to the case of $\mathrm{GL}(2)_{/\mathbb{Q}}$, where for an algebraic eigenform the first Fourier coefficient always has the smallest p-adic evaluation, in our situation there is no such canonical choice for β_1, β_2. By construction we know that the measure $\mu_{\mathcal{C},\phi,\beta_1,\beta_2}$ vanishes at all $\kappa \in \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, \overline{\mathbb{Q}}_p^\times)$ with $\kappa(-1) \neq \phi(-1)$.

Theorem 6.3.1. *Assume that the weight projection map* $\mathrm{Spec}(\mathbb{T}^N_{\mathrm{ord}}) \to \mathrm{Spec}(\mathcal{O}_F[[T_n(\mathbb{Z}_p)]])$ *is étale at the point* $x \in \mathcal{C}(\overline{\mathbb{Q}}_p)$. *Then the measure* $\mu_{\mathcal{C},\phi,\beta_1,\beta_2} \in \mathcal{M}eas(\mathbb{Z}_p^\times, \Lambda_n) \otimes_{\Lambda_n} F_{\mathcal{C}}$ *has no poles at* x. *Let* $\underline{\tau}$ *be the projection of* x *to the weight space* $\mathrm{Hom}_{\mathrm{cont}}(T_n(\mathbb{Z}_p), \overline{\mathbb{Q}}_p^\times)$. *For* $\kappa \in \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, \overline{\mathbb{Q}}_p^\times)$ *with* $\kappa(-1) = \phi(-1)$, *if* $(\kappa, \underline{\tau})$ *is admissible, i.e. arithmetic with* $t_1 \geq \cdots t_n \geq k \geq n+1$, *then we have*

$$\left(\int_{\mathbb{Z}_p^\times} \kappa \, d\mu_{\mathcal{C},\phi,\beta_1,\beta_2}\right)(x) = \phi(-1)^n \mathrm{vol}\left(\widehat{\Gamma}(N)\right) \frac{p^{n^2}(p-1)^n}{\prod_{l=1}^n (p^{2l}-1)} \cdot \frac{\Gamma(k-n)\Gamma_{2n}(k)}{2^{k+n-1}(\pi i)^{2nk+k-n}}$$
$$\times \frac{Z_\infty(f_{\kappa,\underline{\tau},\infty}, v_{\underline{t}}^\vee, v_{\underline{t}})}{\langle v_{\underline{t}}^\vee, v_{\underline{t}}\rangle} \cdot \sum_{\varphi \in \mathfrak{s}_x} \frac{\mathfrak{c}(\varphi, \beta_1)\mathfrak{c}(eW(\varphi), \beta_2)}{\langle \varphi, \overline{\varphi}\rangle}$$
$$\times E_p(k-n, \pi_x^N \times \phi^{-1}\chi^{-1}) \cdot L^{Np\infty}(k-n, \pi_x^N \times \phi^{-1}\chi^{-1}).$$

Here $\mathfrak{c}(\cdot, \beta_i)$ *stands for the* β_i*-th Fourier coefficient,* $i = 1, 2$.

The nonvanishing of the archimedean zeta integral term $Z_\infty(f_{\kappa,\underline{\tau},\infty}, v_{\underline{t}}^\vee, v_{\underline{t}})$ is guaranteed by Proposition 4.5.1, and the nonvanishing of $eW(\varphi)$, the ordinary projection of $\varphi \in \mathfrak{s}_x$, follows from Proposition 5.7.2.

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