Extension of OKID to Output-Only System Identification

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Abstract

Observer/Kalman filter IDentification (OKID) is a successful approach for the estimation, from measured input-output data, of the linear state-space model describing the dynamic behavior of a structure. From such a mathematical model, it is possible to recover the modal parameters, which can be exploited to update a detailed numerical model of the structure, e.g. a Finite Element Model (FEM), to be used to predict the structural response to future excitation and to evaluate damage scenarios. This paper extends OKID to output-only system identification, i.e. to the case where only the response of the structure is measured and the input is unknown. The approach is suitable for structural health monitoring based on modal parameters, in particular for those civil infrastructures whose excitation is random in nature and in the way it is applied to the structure (e.g. wind, traffic) and therefore is difficult to measure. The paper rigorously proves the applicability of the OKID approach to the output-only case, presents the resulting new algorithms and demonstrates them via a numerical example.

I. INTRODUCTION

As the infrastructure system rapidly ages and maintenance and rehabilitation operations are more costly and urgent, Structural Health Monitoring (SHM) has become an active area of research in civil engineering. Several system identification techniques have been developed over the past few decades and their application is growing with the availability of instrumentation on civil infrastructures. The purpose of system identification is the estimation of a mathematical model for the structure under consideration from the time histories of its response to environmental disturbances (e.g. earthquake, wind, traffic, etc.). From such a model, one can then compute the modal parameters of the structure (natural frequencies, damping factors, mode shapes) and use them for SHM, directly to detect damaged areas or indirectly to refine or update more detailed numerical models of the structure (e.g. a Finite Element Model, FEM).

The modal parameters are classically identified from a mathematical model representative of the structure in operational conditions, which describes the relationship between the excitation applied to the structure and the structure's vibrational response. However, especially in the case of civil infrastructures, it is often difficult to measure and control the excitation without disrupting the normal operations. The difficulties arise from the random character of the excitation both in the nature (e.g. wind, traffic) and in the way it is applied to the structure (e.g. distribution of wind pressure). In response to these obstacles, a variety of system identification techniques have been developed using only structural response information and are typically referred to as stochastic identification or output-only system identification. The standard assumption made in the development of output-only identification techniques is that the input is white and stationary. The success of such techniques obviously depends on how close the excitation is to said assumption. In the frequency domain, when the input is white, the power spectral density of the output itself can be considered to represent the dynamical properties of the system. Stochastic techniques in the frequency domain such as peak-picking [1], [2], frequency domain decomposition [2], [3] and maximum likelihood identification [4], [5] have then been developed. In the case of time-domain analysis, it has been shown that the covariance functions of the system vibrational response uniquely characterize a dynamic system in a similar fashion to impulse response functions. The observation has led to the application to the stochastic case of classic system identification techniques that use impulse response functions [6]. Such techniques are referred to as *covariance-driven* stochastic identification techniques, as opposed to other time-domain techniques known as *data-driven* that directly use the recorded output time histories. Among the latter, several stochastic subspace methods are available, i.e. algorithms based on well-established linear algebra concepts and robust numerical techniques. A notable example is presented in [7] and a good general reference is [8]. A survey on output-only system identification methods is beyond the scope of this paper.

In this work, we propose several new algorithms for output-only identification by extending a technique which has proven to be very successful in the input-output case, namely Observer/Kalman filter Identification (OKID) [9], [10]. OKID was originally developed and distributed by NASA for the identification of lightly damped linear structures typical in aerospace applications [11]. Over the past twenty years it found many applications, also in civil engineering and SHM [12], [13]. The original OKID algorithm is known as OKID/ERA because it completes the identification process by Eigensystem Realization Algorithm (ERA, [14]) or some improved variants of it, e.g. ERA with Data Correlation (ERA-DC, [15]). Recently OKID was raised to the level of a general approach to linear system identification as it was demonstrated that it can be combined with many algorithms other than ERA [16]. In this paper, we rigorously prove that OKID can be extended to stochastic system identification, leading to O^{3}KID (Output-Only Observer/Kalman filter IDentification), and we derive the

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corresponding specialized equations and algorithms. In particular, we propose O^{3}KID/ERA-DC, i.e. an improved version of O^{3}KID/ERA [17]. After estimating the Markov parameters of the Kalman filter associated with the system to be identified, O^{3}KID/ERA-DC completes the identification with ERA-DC. Additionally, we present O^{3}KID/DIi and O^{3}KID/DPi, i.e. two algorithms based on the estimation of the Kalman filter output residuals followed by the Deterministic Intersection (DI, [18]) or Deterministic Projection (DP, [19]) methods. Indeed, many other algorithms could be formulated within the general framework provided by O^{3}KID. The above mentioned algorithms are demonstrated via a numerical example on a shear-type building.

II. PROBLEM DESCRIPTION

Like any linear dynamical system, the mathematical model of a structure can be represented in the following state-space form

$$x(k+1) = Ax(k) + Bu(k) + w'_p(k) \tag{1a}$$
$$y(k) = Cx(k) + Du(k) + w'_m(k) \tag{1b}$$

where $x \in \mathbb{R}^{n\times 1}$ is the state vector, $u \in \mathbb{R}^{m\times 1}$ is the input vector, $y \in \mathbb{R}^{q\times 1}$ is the output vector, $A \in \mathbb{R}^{n\times n}$ is the system matrix, $B \in \mathbb{R}^{n\times m}$ is the input matrix, $C \in \mathbb{R}^{q\times n}$ is the output matrix and $D \in \mathbb{R}^{q\times m}$ is the direct influence matrix. The vectors $w'_p \in \mathbb{R}^{n\times 1}$ and $w'_m \in \mathbb{R}^{q\times 1}$ represent noise in the process (e.g. noise in the input or unmodeled dynamics) and in the measurement (e.g. noise in the output measurement and, if $D \neq 0$, in the input, too). The standard assumption is that the process and measurement noises are zero-mean white stationary processes, uncorrelated with u and y, and with covariance $Q' \in \mathbb{R}^{n\times n}$ and $R' \in \mathbb{R}^{q\times q}$, respectively. When both input and output measurements are available (input-output system identification), OKID methods have been demonstrated to be very effective in identifying the matrices of the system [9], [12], [13].

When the input measurements are not available but can be assumed to be white and stationary (standard assumption in the literature of output-only system identification for SHM), the system in (1) can be rewritten as

$$x(k+1) = Ax(k) + w_p(k) \tag{2a}$$
$$y(k) = Cx(k) + w_m(k) \tag{2b}$$

where $w_p \in \mathbb{R}^{n\times 1}$ and $w_m \in \mathbb{R}^{q\times 1}$ are zero-mean white stationary processes including the original process and measurement noises and the effect of the unknown input on the state equation $(Bu(k))$ and on the measurement equation $(Du(k))$. As a result w_p and w_m satisfy the same assumptions as the original process and measurement noise and will be referred to as such in the rest of the paper. Although they are assumed to be uncorrelated with the output of the system, notice that when $D \neq 0$ the input component embedded in w_p and w_m generally makes them correlated via a cross-covariance matrix $S \in \mathbb{R}^{n\times q}$. This is indeed the case with structures where the input is force (or pressure) and the measured output is acceleration. The auto-covariance matrices are denoted by $Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{q\times q}$, respectively.

The output-only system identification problem addressed in this paper can be stated as follows. Given a set of length l of output data

$$\{y(k)\} = \{y(0), y(1), y(2), \ldots, y(l-1)\} \tag{3}$$

measured from the system in (2) starting at some unknown initial state $x(0)$ and driven by w_p and w_m, the objective is to identify the state-space model (2), i.e. to find the matrices A and C. Neither the noise sequences $\{w_p(k)\}$ and $\{w_m(k)\}$ nor their covariance matrices Q, R, S are assumed to be known.

III. GENERALIZED O^{3}KID APPROACH

Similar to OKID (input-output Observer/Kalman filter IDentification), O^{3}KID (Output-Only Observer/Kalman filter IDentification) consists in two main steps. First a set of algebraic equations, referred to as the O^{3}KID core equation, is solved by least-squares (LS). Then the Kalman filter associated with the system under consideration and the noise statistics embedded in the data is identified. Thanks to the close mathematical relationship between the system and the associated Kalman filter, the identification of the latter also solves the original problem, yielding the desired system matrices. In the case of input-output identification, OKID provides the matrices A, B, C and D in (1) as well as the gain K of the associated Kalman filter. In the output-only case, only the matrices A and C can be identified for the system, as illustrated in (2). Nevertheless the approach in O^{3}KID is the same as for the input-output case and yields K as well. O^{3}KID provides valuable information for applications in structural health monitoring, as the modal parameters of the structure under consideration (natural frequencies, damping factors and mode shapes) can be computed from the identified matrices A and C. Also, the resulting Kalman filter can be used for optimal state estimation or output filtering.

In this section, the applicability of OKID to output-only system identification is rigorously proven and its core equation is specialized to the output-only case, leading to O^{3}KID. It is also highlighted how, similar to OKID, the identification of the Kalman filter underlying O^{3}KID is crucial for the identification of A and C, since the unique properties of the

Kalman filter provide the key link between stochastic data (noisy measurements) and deterministic information (Kalman filter Markov parameters). In other words, the identification of the Kalman filter associated with the desired structural model is not accessory to O^{3}KID, it really is at its core. The concept is further elaborated in section III-B.

A. O^{3}KID core equation

In state-space model identification, the main difficulty is that both the sequence of states $x(k)$ and the matrices A and C are unknown. The identification problem is then nonlinear. The keystone of O^{3}KID is the use of a state observer to estimate the actual system state and overcome the nonlinearity of the problem. Consider the following observer for the system in (2)

$$\hat{x}(k+1) = A\hat{x}(k) + K\big(y(k) - \hat{y}(k)\big) \tag{4a}$$

$$\hat{y}(k) = C\hat{x}(k) \tag{4b}$$

where $\hat{x} \in \mathbb{R}^{n\times 1}$ and $\hat{y} \in \mathbb{R}^{q\times 1}$ are the observer state and output and $K \in \mathbb{R}^{n\times q}$ is the observer gain. The observer in (4) is in the form of one-step-ahead predictor, i.e. it provides an estimate $\hat{x}(k+1)$ for the next state $x(k+1)$ from the current state estimate $\hat{x}(k)$ and output measurement $y(k)$. Since all the matrices in (4) are constant with time, the observer is a linear-time-invariant (LTI) model.

Defining the observer output residuals as

$$\varepsilon(k) = y(k) - \hat{y}(k) \tag{5}$$

and plugging (4b) into (4a) and (5) into (4b), the observer in (4) can be written in the equivalent form

$$\hat{x}(k+1) = \bar{A}\hat{x}(k) + Ky(k) \tag{6a}$$

$$y(k) = C\hat{x}(k) + \varepsilon(k) \tag{6b}$$

where $\bar{A} = A - KC$. In this paper, (6), or more precisely (23) which will be later derived from (6), is referred to as the *bar form* of the observer in (4).

Propagating (6) forward in time by $p-1$ time steps (via repeated substitution) and then shifting the time index backward by p, we obtain

$$\hat{x}(k) = \bar{A}^p\hat{x}(k-p) + Tv(k) \tag{7}$$

where

$$v(k) = \begin{bmatrix} y(k-1) \\ y(k-2) \\ \vdots \\ y(k-p) \end{bmatrix} \tag{8a}$$

$$T = \begin{bmatrix} K & \bar{A}K & \dots & \bar{A}^{p-2}K & \bar{A}^{p-1}K \end{bmatrix} \tag{8b}$$

As proven later, the observer in (6) is stable. Such a property guarantees that $\bar{A}^p$ becomes negligible for sufficiently large values of p ($p >> n$). Equation (7) yields then the following relation expressing the current state as a linear combination of solely past input and output values

$$\hat{x}(k) = Tv(k) \tag{9}$$

Plugging (9) into (6b), we obtain

$$y(k) = \Phi v(k) + \varepsilon(k) \tag{10}$$

where

$$\Phi = \begin{bmatrix} CK & C\bar{A}K & \dots & C\bar{A}^{p-2}K & C\bar{A}^{p-1}K \end{bmatrix} \tag{11}$$

Equation (10) relates input and output, without the state appearing explicitly. In the time-series literature it is known as an AutoRegressive (AR) model. Also, note that $\Phi \in \mathbb{R}^{q\times qp}$ contains the sequence of Markov parameters (or unit pulse response) of the observer in bar form. Equation (10) can be written for each time step $k = p, p+1, \dots, l-1$ of the measured data record, to obtain the following set of equations in matrix form

$$Y = \Phi V + E \tag{12}$$

where

$$Y = \begin{bmatrix} y(p) & y(p+1) & \dots & y(l-1) \end{bmatrix} \tag{13a}$$

$$V = \begin{bmatrix} v(p) & v(p+1) & \dots & v(l-1) \end{bmatrix} \tag{13b}$$

$$E = \begin{bmatrix} \varepsilon(p) & \varepsilon(p+1) & \dots & \varepsilon(l-1) \end{bmatrix} \tag{13c}$$

Equation (12) is at the core of O^{3}KID. Y and V are known (from measurements), Φ and E are not. By having $l-p>pq$ (more equations than unknowns) and considering E as an error term, it is possible to find the LS solution to (12)

$$\tilde{\Phi} = YV^T\left(VV^T\right)^{-1} = YV^\dagger \tag{14}$$

where † denotes the Moore-Penrose pseudoinverse of a matrix, as well as the corresponding LS residuals

$$\tilde{E} = Y - \tilde{\Phi}V \tag{15}$$

Post-multiplying (12) by V^T and replacing Φ and E with their LS estimates $\tilde{\Phi}$ and $\tilde{E}$, we obtain

$$YV^T = YV^T\left(VV^T\right)^{-1}VV^T + \tilde{E}V^T = YV^T + \tilde{E}V^T \tag{16}$$

which implies that $\tilde{E}V^T = 0$. From the definition of $v(k)$, we conclude that

$$\sum_{k=p}^{l-1}\tilde{\varepsilon}(k)y^T(k-j) = 0 \qquad j=1,2,...,p \tag{17}$$

Since the stated assumptions make the process in (2) stationary, then by the ergodic property we can estimate the ensemble average of each entry of the products in (17) between the current residual and the past outputs by their time average over a sufficiently long record. Assuming l is large and dividing (17) by $l-p$, we recognize the left-hand side as the time average of each entry of $\tilde{\varepsilon}(k)y^T(k-j)$. The ergodic property brings to the conclusion that, for all $k=p,p+1,\dots,l-1$,

$$\mathbb{E}\left[\tilde{\varepsilon}(k)y(k-j)^T\right] = 0 \qquad j=1,2,...,p \tag{18}$$

The residuals $\tilde{\varepsilon}$ of the LS problem (12) are then orthogonal to the past output values. This is the same property that uniquely characterizes the Kalman filter output residuals (see for example [10], [20]), which proves that the solution to the LS problem in (12) yields an estimate of the output residuals of the Kalman filter corresponding to the unknown system matrices A, C and noise statistics Q, R, S that generated the given output sequence $\{y(k)\}$. Among all the possible linear observers, the Kalman filter is optimal in the sense that it minimizes the expected value of the squared norm of the state estimate $\mathbb{E}\left[(x(k)-\hat{x}(k))^T(x(k)-\hat{x}(k))\right]$ at each time step k. Under the assumption of stationary noise (constant Q, R, S) and after a certain number of steps p after which the filter transient has vanished, the optimal gain becomes constant in time and is referred to as steady-state Kalman gain. Given the system matrices A, C and the noise covariance matrices Q, R, S, the steady-state Kalman gain K can be computed from the well-known algebraic Riccati equation [10], [20]. The choice of the letter K, usually reserved for the Kalman gain, for the gain of the observer in (4) is now justified.

As a corollary, $\tilde{\Phi}$ contains the estimates of the Markov parameters of the steady-state Kalman filter in its bar form. This is guaranteed by the fact that the steady-state Kalman filter is the unique LTI observer with output residuals orthogonal to the previous outputs. Since the unit pulse response is uniquely associated with a linear system, the unit pulse response estimated in $\tilde{\Phi}$ must correspond to the one of the steady-state Kalman filter associated with the system to be identified and the noise statistics Q, R, S embedded in the data[1]. The stability of the Kalman filter [20] also guarantees that the observer used to derive the O^{3}KID core equation is stable, as required for (9) to hold. For brevity, in this paper we often omit *steady-state* when referring to the Kalman filter at the core of O^{3}KID. Nevertheless it must be kept in mind that such Kalman filter has a time-invariant gain K. For the sake of clarity, it is also worth remarking that the Kalman filter in O^{3}KID is in the form of a one-step-ahead predictor, as opposed to other Kalman filter equations used for filtering or smoothing of signals [10].

In summary, solving the O^{3}KID core equation (12) by LS yields an estimate for the sequence of the output residuals of the Kalman filter

$$\tilde{E} = \begin{bmatrix}\tilde{\varepsilon}(p) & \tilde{\varepsilon}(p+1) & \tilde{\varepsilon}(p+2) & \dots & \tilde{\varepsilon}(l-1)\end{bmatrix} \tag{20}$$

and for the sequence of Markov parameters of the Kalman filter in bar form

$$\tilde{\Phi} = \begin{bmatrix}\tilde{\Phi}_1 & \tilde{\Phi}_2 & \dots & \tilde{\Phi}_{p-1} & \tilde{\Phi}_p\end{bmatrix} \tag{21}$$

where $\tilde{\Phi}_j \in \mathbb{R}^{q\times q}$ is an estimate for the j^{th} Markov parameter $C\bar{A}^{j-1}K$.

It is important to remark that since the output residuals of the Kalman filter have the property of being white [10], [20], then the LS estimate of the Kalman filter Markov parameters is asymptotically unbiased. The O^{3}KID core equation yields

[1] For clarity of presentation, the observer underlying the O^{3}KID approach has been introduced in (4) in the typical form of a Kalman filter. This is justified by the fact that such observer is eventually proven to be the Kalman filter. For a more rigorous mathematical presentation, the observer in (4) could have been introduced in the most general form of LTI observer, i.e.

$$\hat{x}(k+1) = F\hat{x}(k) + Hy(k) \tag{19a}$$
$$\hat{y}(k) = C\hat{x}(k) \tag{19b}$$

The fact that the steady-state Kalman filter is the unique observer in the form of (19) featuring the above mentioned orthogonality property of the LS residuals would in the end prove that $F=A-KC$ and $H=K$, making (19) equivalent to (4) or (6).

then a *good* estimate for a deterministic piece of information on the system, filtering out the unknown noise w_p and w_m. The Kalman filter Markov parameters are deterministic in the sense that given the system and noise covariance matrices A, C, Q, R and S, the associated Kalman filter and its Markov parameters are uniquely and deterministically defined. Notice how the noise in the measurements prevents one from getting direct access to the system and it is necessary to pass through the associated Kalman filter. Another more intuitive interpretation of the role of the Kalman filter in O^{3}KID is given in section III-B.

Before jumping into the second step of O^{3}KID, it is worth noting also the following remarkable fact. Despite not knowing the system or the noise statistics (both necessary to find the corresponding Kalman filter), we managed to use the equation of the (unknown) Kalman filter to derive a relation between the measured input and output. Finally, the LS solution to the resulting set of equations confirms that the equation we started from was not just the equation of *an observer*, but that of *the Kalman filter*, which has a well defined and close relationship with the equation of the system to be identified. This is the essence of O^{3}KID.

B. Kalman filter identification via output residuals

The second step of O^{3}KID consists in the identification of the Kalman filter associated with the system to be identified and the noise statistics embedded in the data. Whereas the first step, i.e. solving the O^{3}KID core equation (12), is the same for all methods based on O^{3}KID, the latter differ by how they identify the Kalman filter. In this regard, O^{3}KID methods can be classified in two families: the ones identifying the Kalman filter from its Markov parameters and those accomplishing the same task from its output residuals. Although in the input-output case the first OKID algorithm to be developed belongs to the Markov parameter family (OKID/ERA, [9]) and the methods working on the output residuals are very recent (e.g. OKID/DIi and OKID/DPi, [16]), the latter have the merit of providing intuition on the applicability of OKID to output-only system identification. Therefore, they are presented first in this paper.

Recalling the definition of observer output residuals (5), (4) can be written as follows

$$\hat{x}(k+1) = A\hat{x}(k) + K\varepsilon(k) \tag{22a}$$

$$\hat{y}(k) = C\hat{x}(k) \tag{22b}$$

which is usually known in the literature as the *innovation form* of the Kalman filter. Equation (22) can also be looked at as the state-space model of a dynamic system with ε as the input and $\hat{y}$ as the output. Most importantly, no (unknown) noise term is present in (22). Therefore a new noise-free input-output identification problem can be formulated as follows. Given the sequences of ε and $\hat{y}$, identify the matrices of the system in (22), i.e. A, C and K. Yielding A and C, the solution to the new problem would solve also the original identification problem.

From the first step of O^{3}KID, an estimate for the sequence of $\varepsilon(k)$ for $k = p, p+1, \ldots, l-1$ is indeed available, as well as for the sequence of $\hat{y}(k)$, since $\varepsilon(k)$ and $\hat{y}(k)$ are related via the definition of observer output residuals (5). We indeed converted the original identification problem (2), characterized by the presence of unknown signals (w_p and w_m), into the simpler identification problem (22). *Any* method formulated for deterministic system identification can be used to address the new problem, solving the original problem at the same time. This gives rise to many O^{3}KID-based output-only identification algorithms, as many as the deterministic state-space model identification methods one can think of.

In this paper, we demonstrate via examples two possible choices, namely the Deterministic Intersection (DI) and the Deterministic Projection (DP) methods. In the literature of subspace methods, several intersection and projection algorithms have been developed [8]. In the examples of this paper, we refer to the DI algorithm in [8], [18] and the DP algorithm in [8], [19]. The Matlab® codes of both algorithms are provided in [8][2]. The DI and DP methods are considered deterministic because their formulation is based on purely deterministic state-space models (with no process or measurement noise). The resulting O^{3}KID-based algorithms are referred to as O^{3}KID/DIi and O^{3}KID/DPi, where the lower-case letter i indicates that the underlying Kalman filter is identified in its innovation form (22), distinguishing them from the following variant.

From (6), plugging (5) into (6b), we can write

$$\hat{x}(k+1) = \bar{A}\hat{x}(k) + Ky(k) \tag{23a}$$

$$\hat{y}(k) = C\hat{x}(k) \tag{23b}$$

which has already been referred to as the *bar form* of the Kalman filter. Similarly to (22), notice how the dynamic system in (23) is purely deterministic, with input $y(k)$ and output $\hat{y}(k)$ that are both known (from measurement and from estimation, respectively). Again, *any* method for deterministic system identification can be applied to find $\bar{A}$, C and K, from which A can be recovered as $A = \bar{A} + KC$. Such variant, when coupled with DI and DP to identify the Kalman filter in bar form, gives rise to O^{3}KID/DIb and O^{3}KID/DPb. Their input-output versions, OKID/DIb and OKID/DPb, were presented in detail in [16], but numerical and experimental examples suggest their performances are inferior to the algorithms based on the

[2]Reference and Matlab® code also available at `http://homes.esat.kuleuven.be/~smc/sysid/software/`.

innovation form of the Kalman filter. Intuitively, the direct identification of A is preferable to the identification of $\bar{A}$ and subsequent recovery of A from the estimated K and C.

Both the DI and DP methods are based on well-established concepts and techniques from linear algebra. The property at the core of the DI method is that the state history of the system to be identified (in this case, the Kalman filter in (22) can be found as the intersection of the row space of two data matrices. The computation of the intersection requires two singular value decompositions (SVD). Once Kalman state sequence is known, ordinary LS can be applied to find the matrices A, C and K. The key observation in the DP method is of geometrical nature. The dimension of the projection of the row space of a block Hankel output ($\hat{y}$) matrix on the orthogonal complement of the row space of a block Hankel input (ε) matrix provides the order of the system and its column space yields an estimate of the extended observability matrix. From the latter, the matrices A and C can be found. If desired, K can be identified as well, through an additional step. Note that both the DI and DP algorithms provided with [8], which are the ones used in this paper, attempt to identify also a direct influence matrix [3], which is known to be 0 for the Kalman filter in (22). As verified by numerical examples, the direct influence matrix identified by DP or DI is indeed negligible. The DI and DP algorithms provided with [8], can then be applied without modification to realize O^{3}KID/DIi and O^{3}KID/DPi.

It is worth adding that other algorithms based on the same approach can be devised simply by replacing DI and DP by other deterministic methods. For instance, one could use the subspace Algorithm 1 and Algorithm 2 in [8] or the algorithms from the superspace family [21], [22], [23]. Also note that the approach based on the Kalman output residuals highlights the following aspect of O^{3}KID. The O^{3}KID core equation (12) allows one to convert the original stochastic identification problem, complicated by the presence of unknown signals (w_p and w_m), into a new simpler deterministic identification problem whose solution includes the solution to the original problem. The intuitive interpretation is that the Kalman filter underlying (12) optimally filters the noise out of the problem, paralleling the same central role that the Kalman filter has in classic signal estimation [16]. Since the record used to construct (12) is finite, the filtering action is not exact and neither are the resulting estimates of A and C. Indeed, due to the stochastic nature of the problem addressed in this paper, an infinite record of output data would be necessary to aim to exact identification of the system in (2). The fact that a Kalman filter exists for the system in (1) as well as for the system in (2) indeed inspired the extension of the OKID approach to the output-only case and led to O^{3}KID.

C. Kalman filter identification via Markov parameters

The Kalman filter at the core of the O^{3}KID approach can be identified from its Markov parameters, whose estimate $\tilde{\Phi}$ is obtained from the LS solution of the O^{3}KID core equation. This is indeed the approach of the traditional OKID method in the input-output case, which completes the identification of the system matrices via ERA or ERA-DC. Before applying such algorithms, the following preliminary operation has to be done on the estimated Kalman filter Markov parameters. The latter refer to the Kalman filter in the bar form (23). With simple algebraic manipulation, it is possible to demonstrate that the sequence $\Phi_j = C\bar{A}^{j-1}K$ can be converted into the sequence $\Psi_j = CA^{j-1}K$, which corresponds to the Markov parameters of the Kalman filter in its innovation form (22). Additionally, it can be noticed that the sequence of estimated $C\bar{A}^{j-1}K$ is finite ($j = 1,2,...,p$) only in appearance. For $j > p$, $C\bar{A}^{j-1}K$ can be considered to be equal to 0 as assumed in the derivation of the O^{3}KID core equation. In other words, the sequence of estimated Markov parameters $\tilde{\Phi}_j$ can be extended to an arbitrary value $j = N > p$ simply by padding $\tilde{\Phi}$ in (21) with zeros. The estimate for the Markov parameters of the Kalman filter in innovation form can then be computed as

$$\tilde{\Psi}_1 = \tilde{\Phi}_1 \tag{25a}$$

$$\tilde{\Psi}_j = \tilde{\Phi}_j + \sum_{h=1}^{j-1} \tilde{\Phi}_h \tilde{\Psi}_{j-h} \qquad \text{for } j = 2,3,\dots,N \tag{25b}$$

From the Markov parameters $\Psi_j = CA^{j-1}K$, $j = 1,2,\dots,N$, ERA or ERA-DC can be applied to identify the desired matrices A, C and K. It is worth noting how the Markov parameters $\Phi_j = C\bar{A}^{j-1}K$ could also have been used as an input to ERA or ERA-DC, leading to the identification of $\bar{A}$, C and K, from which A could then be recovered as $A = \bar{A} + KC$. However, similar considerations to what observed for the identification of the Kalman filter via output residuals apply and the direct identification of A turns out to be more accurate.

Both ERA and ERA-DC are well-established algorithms [10], [14], [15]. Suffices here to say that they rely on the SVD of matrices with the Markov parameters Ψ_j arranged according to a a block Hankel structure. Such matrices can be expressed

[3] DI and DP are deterministic state-space system identification methods, hence they are formulated to identify a model in the following standard form

$$x'(k+1) = A'x'(k) + B'u'(k) \tag{24a}$$

$$y'(k) = C'x'(k) + D'u'(k) \tag{24b}$$

Comparing (24) with (22), the matrices A', B' and C' returned by DI and DP correspond to A, K and C. D' is expected to be null.

as the product of the controllability and observability matrices of the system to be identified (in the case of this paper, the Kalman filter in (22)).

IV. ALGORITHM

The detailed steps for O3KID are described below. The user can choose between identifying the underlying Kalman filter from its Markov parameters or from its output residuals. Furthermore, within these two families, any deterministic state-space model identification algorithm can be chosen other than the ones illustrated in this paper (DI, DP, ERA, ERA-DC).

The input to the following O3KID-based algorithms is the sequence of measured system output $\{y(k)\}$ of length l in (3). The output of the algorithm is the set of matrices A, C and K.

Comprehensive O3KID algorithm

1) construct the matrices Y and V in (13a) and (13b), choosing p sufficiently larger than the assumed order n of the system (2) to be identified (typically, 20 times larger)
2) compute

$$\tilde{\Phi} = YV^{\dagger}$$

Algorithms based on Kalman filter output residuals

3) compute

$$\begin{bmatrix}\tilde{y}(p) & \tilde{y}(p+1) & \dots & \tilde{y}(l-1)\end{bmatrix} = \tilde{\Phi}V$$
$$\begin{bmatrix}\tilde{\varepsilon}(p) & \tilde{\varepsilon}(p+1) & \dots & \tilde{\varepsilon}(l-1)\end{bmatrix} = Y - \tilde{\Phi}V$$

4) execute, with input $\tilde{\varepsilon}(k)$ and $\tilde{y}(k)$, $k = p, p+1, \dots, l-1$, any of the following algorithms
   - DI [8], [18] for O3KID/DIi
   - DP [8], [19] for O3KID/DPi
   - any other algorithm for deterministic state-space model identification from arbitrary excitation
5) with reference to (24), read the output matrices A', C' and B' (discard D')
6) complete the identification by taking $A = A'$, $C = C'$, $K = B'$

Algorithms based on Kalman filter Markov parameters

3) partition $\tilde{\Phi}$ in submatrices of q columns each

$$\tilde{\Phi} = \begin{bmatrix}\tilde{\Phi}_1 & \tilde{\Phi}_2 & \dots & \tilde{\Phi}_{p-1} & \tilde{\Phi}_p\end{bmatrix}$$

4) compute $\tilde{\Psi}_j$ for $j = 1, 2, \dots, N$ from (25) with $\tilde{\Phi}_j = 0$ for $j > p$, choosing N such that the sequence $\tilde{\Psi}_j$ covers a significant portion of the impulse response of the system
5) execute, with input $\tilde{\Psi}_j$, $j = 1, 2, ..., N$, any of the following algorithms
   - ERA [10], [14] for O3KID/ERA
   - ERA-DC [10], [15] for O3KID/ERA-DC
   - any other algorithm for deterministic state-space model identification from unit pulse response (Markov parameters)
6) read the output matrices A, C and K

V. EXAMPLE

The following example aims to illustrate the correctness and effectiveness of the output-only identification approach proposed in this paper.

Consider the lumped model of a four-story shear-type building, shown in Figure 1, with each mass equal to $m = 0.259$ kips-sec^2/in and each lateral spring of stiffness $k = 122.889$ kips/in. The building is also supposed to have viscous damping, quantified by a modal damping factor of 0.01 for each of the 4 vibration modes. The assumption of modal damping makes it possible to recover the modal parameters (natural frequencies, damping factors and mode shapes) from the identified state-space model without conceptual difficulties. The excitation is provided in correspondence of the third floor as a lateral force modeled as a white process normally distributed with zero mean and standard deviation of 1. The input then satisfies the assumptions made in section II. The gaussian noise that might affect the input channel is not explicitly modeled, since it can be considered included in the unmeasured excitation mentioned above. With regard to the structural response, the lateral acceleration at each floor is measured, for a total of 4 outputs. Zero-mean gaussian noise is present in each output channel, with standard deviation equal to 10% of the corresponding true output standard deviation σ_{y_i}. The complete discrete-time

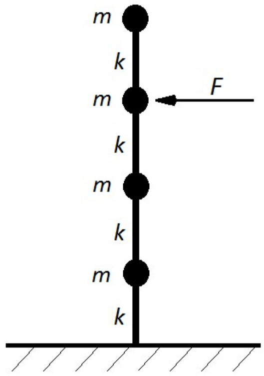


Fig. 1: **Lumped model of four-story shear-type building.**

state-space model of the structure in Figure 1 is therefore in the form of (1) with $n = 8$, $m = 1$ and $q = 4$. The process noise w'_p is not explicitly modeled ($Q' = 0$), whereas the covariance of the measurement noise w'_m is the diagonal matrix

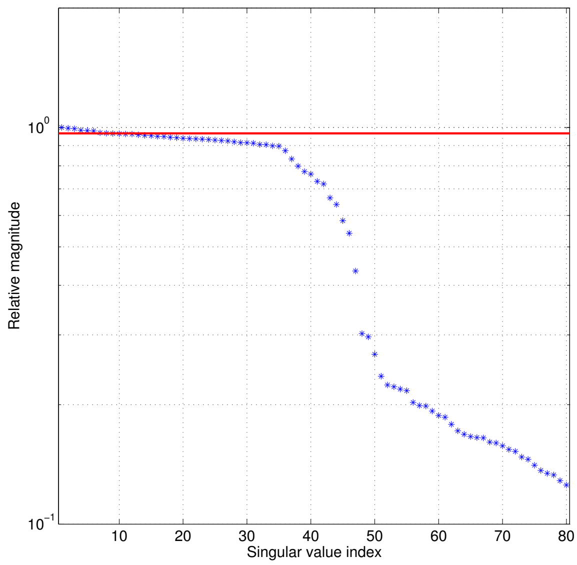
$$R' = 0.1^2 \begin{bmatrix} \sigma_{y_1}^2 0 & 0 & 0 & \\ 0 & \sigma_{y_2}^2 & 0 & 0 \\ 0 & 0 & \sigma_{y_3}^2 & 0 \\ 0 & 0 & 0 & \sigma_{y_4}^2 \end{bmatrix} \tag{27}$$

Only the output (corrupted by noise) is assumed to be measured. The contribution of the input to the state and measurement dynamics is absorbed in the noise terms w_p and w_m in (2), which then have covariance matrices $Q = BB^T$ and $R = DD^T + R'$, respectively. The model to be identified consists in the matrices A and C.

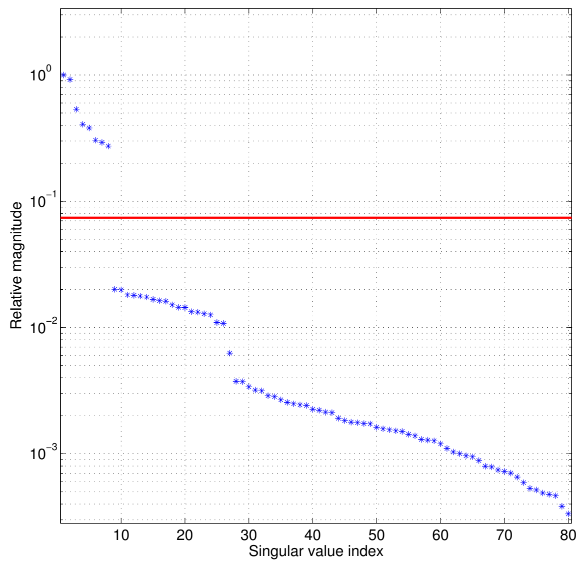
The example is based on a Monte Carlo simulations of 100 runs. Independent sequences for w_p and w_m are generated for each run and the corresponding output sequences obtained from (2) are fed to the illustrated algorithms to identify the matrices A and C. In each of the four O^{3}KID-based algorithms, the SVD of a data matrix needs to be performed. The system order is revealed at this stage by the number of singular values that can be considered different from zero. In deterministic system identification (noise-free input and output data), the separation between non-zero and zero singular values is an extremely simple task since the latter are indeed numerically zero. The presence of noise w_p and w_m prevents the singular values from being exactly zero and the user is asked to select the singular values that are sufficiently small to be considered zero and discarded. Figure 2 shows the singular value plots obtained for each O^{3}KID-based algorithm for a single run of the Monte Carlo simulation. Note how for O^{3}KID/DPi, O^{3}KID/ERA and O^{3}KID/ERA-DC there is a clear gap between the 8 largest singular values and the others. The order $n = 8$ is then correctly selected without difficulty. In contrast, the singular value plot in O^{3}KID/DIi does not reveal the order of the system as clearly. In the Monte Carlo simulation, in order to compare the results from the different algorithms, the number of non-zero singular values chosen in O^{3}KID/DIi is forced to 8.

From the identified matrices A and C, the modal parameters of the structures can be computed. Tables I and II report the average of the natural frequencies and damping factors identified in each run of the Monte Carlo simulation. All the four O^{3}KID algorithms are executed with $p = 100$ in the O^{3}KID equation, $i = 20$ in DI and DP and $N = 200$ in ERA and ERA-DC. As expected from the analysis of its singular value plot, the performance of O^{3}KID/DIi is inferior to the other algorithms. Remarkably, in most of the runs, O^{3}KID/DIi is able to identify the correct modes. However, the lower accuracy and precision, in particular for the first mode, reflect the fact that in Figure 2a the singular values corresponding to true and spurious modes are very close to each other, so that the latter affect the former. The other O^{3}KID algorithms provide estimates of natural frequencies and damping factors that are extremely close to the true values even for the first mode. Both the average estimation error and the standard deviation are below 1%. Note how O^{3}KID/DPi performs significantly better in the estimation of damping factors than O^{3}KID/ERA and O^{3}KID/ERA-DC. The latter provide very similar estimates for the modal parameters (the difference is associated with digits beyond those reported in the tables), but ERA-DC performs consistently better than ERA.

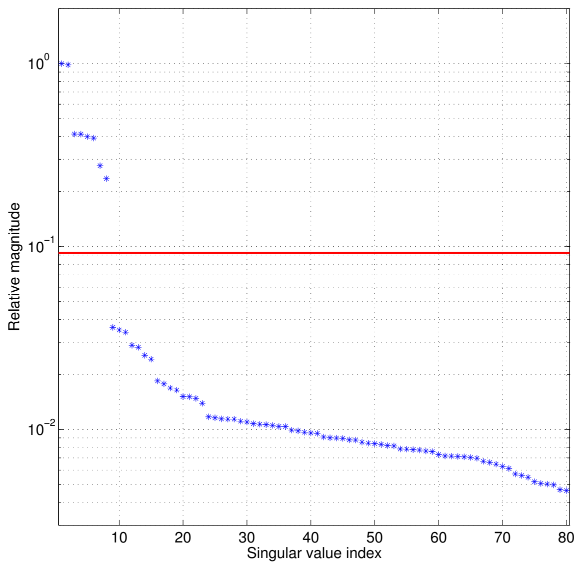
In summary, the numerical example shows the correctness of the proposed approach to output-only system identification and suggests that O^{3}KID/DPi, which relies on the estimation of the Kalman output residuals, can provide better performance than the algorithms based on the original formulation via the Kalman filter Markov parameters (ERA and ERA-DC).



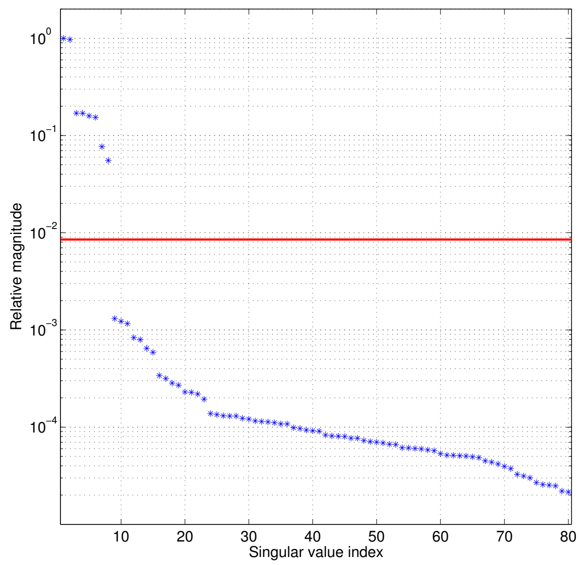
(a) **DIi.**



(b) **DPi.**



(c) **ERA (80 largest singular values).**



(d) **ERA-DC (80 largest singular values).**

Fig. 2: **Singular value plots for different O^{3}KID algorithms.**

VI. CONCLUSIONS

The well-established OKID method for system identification from input-output data has been extended to the case where only the output measurements are available and the input can be considered to be a white process. Not only the traditional OKID algorithms completing the identification via the Kalman filter Markov parameters by ERA or ERA-DC have been specialized to the output-only case, but also new algorithms based on the estimation of the Kalman output residuals and on final identification by subspace methods such as Deterministic Intersection and Deterministic Projection have been formulated. All of the proposed algorithms have been illustrated via a numerical example, which demonstrates how O^{3}KID is an effective approach for output-only system identification. Similar to the family of subspace methods, OKID and O^{3}KID provide then a unified framework for system identification and a useful tool for structural health monitoring. The results of the proposed algorithms can also be refined via optimization techniques similar to the one proposed in [13] to enhance the potential of the approach.

TABLE I: Identified natural frequencies (Hz) of the structure in Fig. 1 (Monte Carlo simulation, average $\bar{f}_i$ and standard deviation σ_i^f, $i = 1,2,3,4$, over 100 runs).

Method	$\bar{f}_1$	σ_1^f	$\bar{f}_2$	σ_2^f	$\bar{f}_3$	σ_3^f	$\bar{f}_4$	σ_4^f
True	1.395	–	3.972	–	5.945	–	7.012	–
O^3KID/DIi	1.406	0.024	3.974	0.009	5.946	0.012	7.013	0.012
O^3KID/DPi	1.401	0.005	3.974	0.008	5.945	0.011	7.014	0.011
O^3KID/ERA	1.397	0.007	3.973	0.009	5.947	0.012	7.013	0.012
O^3KID/ERA-DC	1.397	0.007	3.973	0.009	5.947	0.012	7.013	0.012
N4SID	1.396	0.008	3.973	0.008	5.946	0.011	7.014	0.011

TABLE II: Identified damping factors of the structure in Fig. 1 (Monte Carlo simulation, average $\bar{\zeta}_i$ and standard deviation σ_i^ζ, $i = 1,2,3,4$, over 100 runs).

Method	$\bar{\zeta}_1$	σ_1^ζ	$\bar{\zeta}_2$	σ_2^ζ	$\bar{\zeta}_3$	σ_3^ζ	$\bar{\zeta}_4$	σ_4^ζ
True	0.0100	–	0.0100	–	0.0100	–	0.0100	0
O^3KID/DIi	0.0510	0.0390	0.0118	0.0028	0.0107	0.0020	0.0104	0.0017
O^3KID/DPi	0.0111	0.0045	0.0101	0.0020	0.0101	0.0018	0.0099	0.0015
O^3KID/ERA	0.0230	0.0064	0.0078	0.0022	0.0091	0.0019	0.0095	0.0017
O^3KID/ERA-DC	0.0230	0.0064	0.0078	0.0022	0.0091	0.0019	0.0095	0.0017
N4SID	0.0115	0.0050	0.0104	0.0021	0.0102	0.0018	0.0100	0.0015

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