On a triply-graded generalization of Khovanov homology

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ABSTRACT

On a triply-graded generalization of Khovanov homology

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In this thesis we study a certain generalization of Khovanov homology that unifies both the original theory due to M. Khovanov, referred to as the even Khovanov homology, and the odd Khovanov homology introduced by P. Ozsváth, Z. Szabó, and J. Rasmussen.

The generalized Khovanov complex is a variant of the formal Khovanov bracket introduced by Bar Natan, constructed in a certain 2-categorical extension of cobordisms, in which the disjoint union is a cubical 2-functor, but not a strict one. This allows us to twist the usual relations between cobordisms with signs or, more generally, other invertible scalars. We prove the homotopy type of the complex is a link invariant, and we show how both even and odd Khovanov homology can be recovered. Then we analyze other link homology theories arising from this construction such as a unified theory over the ring $\mathbb{Z}_\pi := \mathbb{Z}[\pi]/(\pi^2 - 1)$, and a variant of the algebra of dotted cobordisms, defined over $\Bbbk := \mathbb{Z}[X, Y, Z^{\pm 1}]/(X^2 = Y^2 = 1)$.

The generalized chain complex is bigraded, but the new grading does not make it a stronger invariant. However, it controls up to some extend signs in the complex, the property we use to prove several properties of the generalized Khovanov complex such as multiplicativity with respect to disjoint unions and connected sums of links, and the duality between complexes for a link and its mirror image. In particular, it follows the odd Khovanov homology of anticheiral links is self-dual. Finally, we explore Bockstein-type homological operations, proving the unified theory is a finer invariant than the even and odd Khovanov homology taken together.

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Chapter 1

Introduction

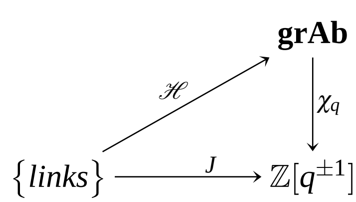
1.1 Background

The Jones polynomial $J_L(q) \in \mathbb{Z}[q^{\pm 1}]$ of a link L is a very powerful invariant that can be defined using two simple axioms:

1) $J_{\bigcirc} = q + q^{-1}$, and
2) $q^2 J_{\nearrow\!\!\!\!\nwarrow} - q^{-2} J_{\nwarrow\!\!\!\!\nearrow} = (q - q^{-1}) J_{)(}$,

where $\nearrow\!\!\!\!\nwarrow$, $\nwarrow\!\!\!\!\nearrow$, and $)($ represent three oriented link diagrams, which differ only locally as shown in the pictures. Despite its simple definition, the Jones polynomial is a result of a deep connection between links and representations of quantum groups, and its discovery put a new life to knot theory.

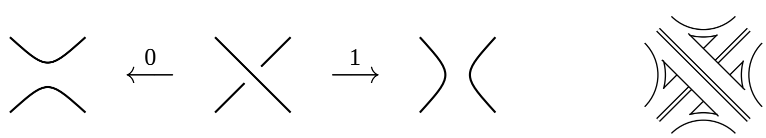
The next revolution took place at the end of the century with the paper of M. Khovanov [Kh99]: the Jones polynomial is only a shadow of another object, the *Khovanov homology* $\mathscr{H}(L)$. More precisely, the triangle to the right commutes, where we write **grAb** for the category of bigraded abelian groups, and χ_q stands for the *graded Euler characteristic*: $\chi_q(M) = \sum_{i,j}(-1)^i q^j \dim M^{i,j}$, where $M = \bigoplus M^{i,j}$ is the graded decomposition of M. For this reason we say the Khovavon homology is a *categorification* of the Jones polynomial.



It did not take much time to prove usefulness of this new invariant. For instance, the Khovanov homology detects the unknot [KM12] and unlinks [HN12], although the problem is still open for the Jones polynomial. The Lee deformation [Le05] leads to a spectral sequence, from which J. Rasmussen extracted a lower bound for the knot genus, giving a combinatorial proof of Milnor Conjecture [Ra04]. This raised a question, whether there were other link homology theories categorifying the Jones polynomial. D. Bar-Natan [BN05] described a very general construction that produces a link homology for any rank two Frobenius algebra satisfying some additional relations. Then M. Khovanov classified all theories that arise from Frobenius systems [Kh04], proving that one of the Bar-Natan's link homology theories, based on the algebra of dotted cobordisms, is universal.

When it seemed categorifications of the Jones polynomial were well understood, P. Ozsváth, J. Rasmussen and Z. Szabó published a paper with a distinct construction [ORS13] based on a *projective* TQFT. Their invariant also categorifies the Jones polynomial, but the algebra used in the construction is not cocommutative and even not coassociative. They call it *odd Khovanov homology*, because of similarity with the original constrution, which we now refer to as *even*. Both theories agree modulo 2, but they are not equivalent over $\mathbb{Z}$. In particular, results of A. Shumakovitch [Sh11] provide examples of pairs of knots that can be distinguished by one theory but not by the other. Moreover, it was proved by J. Bloom that the odd Khovanov homology is mutation invariant [Blo10], generalizing the similar result by S. Wehrli for even Khovanov homology with $\mathbb{Z}_2$ coefficients [Wh10].

Beyond the differences, both theories are constructed in a very similar way. First, given a link diagram D with n crossings we create 2^n pictures, by resolving each crossing horizontally (type 0 resolution) or vertically (type 1 resolution):



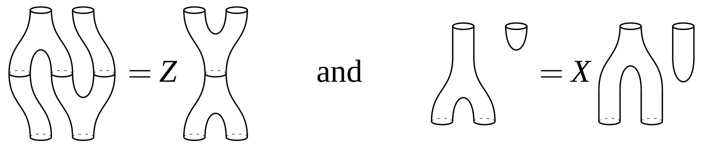
(1.1)

The picture of crossing highways is placed to the right to help to remember the naming convention: a resolution of a crossing can be seen as a right turn (assuming the traffic is on the right side). In type 0 we leave the lower highway, while in type 1 the upper one. We place all such pictures in vertices of an n-dimensional cube $\mathscr{I}(D)$, which edges we decorate with certain cobordisms, see Fig. 4.1 on page 53. This cube commutes and by applying a TQFT functor we obtain a commuting cube of abelian groups and homomorphisms, which can be collapsed to a chain complex (after changing signs of some maps). On the other hand, a projective TQFT from [ORS13] produces a cube that commutes only up to signs, which has to be fixed before collapsing. This can be always done, although it is kind of a mystery, why this is possible. This last step is exactly why the odd theory does not fit into Bar-Natan's framework. In his philosophy, one should construct a chain complex $Kh(D)$ in the additive extension of the category of cobordisms and proof its invariance before applying a TQFT functor. However, this approach does work for the odd theory: the image of $Kh(D)$ under a projective TQFT may not be a complex anymore.

We refine the Bar-Natan's construction by using cobordisms equipped with an additional structure, framed Morse functions $\tau\colon W \longrightarrow I$ that separate critical points [Ig87], which we call *chronologies*. Homotopies of these functions, which we refer to as *changes of chronologies*, equip the category of chronological cobordisms **ChCob** with a structure of a 2-category and we can express the projective functor from [ORS13] as a strict 2-functor $\mathscr{F}_{odd}\colon \mathbf{ChCob} \longrightarrow \mathbf{Ab}$ that maps a change of a chronology to scaling of a homomorphism by ± 1.[1] Because of a very simple 2-categorical structure in a category of R-modules, a 2-functor $\mathscr{F}\colon \mathbf{ChCob} \longrightarrow \mathbf{Mod}_R$ can be seen as an ordinary functor $\mathscr{F}\colon R\mathbf{ChCob} \longrightarrow \mathbf{Mod}_R$ from a *linearization* of **ChCob**: its morphisms are linear combinations of chronological cobordisms modulo chronological relations—a chronology can be changed at a cost of multiplication by a certain scalar. We construct in Chapter 3 a certain linearization $\Bbbk\mathbf{ChCob}$ of two-dimensional cobordisms, where $\Bbbk = \mathbb{Z}[X,Y,Z^{\pm 1}]/(X^2 = Y^2 = 1)$, that is rich enough to support the construction of both even

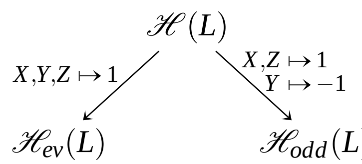
[1] One can regard **Ab** as a 2-category with 2-morphisms labeled with integers, where $n\colon f \Longrightarrow g$ exists if $g = nf$. This generalizes in a natural way to a category of modules over a commutative ring.

and odd Khovanov homology. For example, there are relations



(1.2)

By translating Bar-Natan's construction into this new framework, we obtain a complex $Kh(D)$ in $\Bbbk$**ChCob**, and we show it is a link invariant up to homotopy and certain local relations. In particular, it follows from contractibility of certain loops in the space of framed functions that we can always distribute signs over edges of the cube to make it commute. Beyond even and odd homology, we have found a number of link homology theories with parameters, especially the *covering homology* $\mathscr{H}(L)$, a sequence of graded modules over the ring of truncated polynomials $\mathbb{Z}[X,Y,Z^{\pm 1}]/(X^2 = Y^2 = 1)$, from which we can obtain both even and odd Khovanov homology as illustrated to the right. The specializations should be made at the level of chains. This construction was first described in [Put08], and improved later in [Put13]. Another example is given by chronological cobordisms with dots that generalizes the universal Bar-Natan's theory to the odd setting. By an analogy to the even case we show it induces the universal link homology in our framework (see Theorem 5.2.9). A motivation was to find an odd analog of Lee's deformation, but this goal has not been reached.



1.2 Overview of the results

The 2-category n**ChCob** of n-dimensional chronological cobordisms is interesting on its own. As we prohibit existence of two critical points at the same level, there are two disjoint union: the left-to-right and right-to-left, defined by taking the ordinary disjoint union of cobordisms and shifting all critical points of one of them over the critical points of the other, see Fig. 3.6 of page 32. We prove in Chapter 3 that one of the disjoint unions is a cubical 2-functor, so that n**ChCob** is a Gray monoidal category, a 2-categorical analog of a monoidal category. We also compute the presentation of 2**ChCob** in terms of generators and relations, see Section 3.2.2.

We are especially interested in a combinatorial description of 2-morphisms. We use it to find a linearization $\Bbbk$**ChCob**, which is an ordinary category. Not all cobordisms are free generators in this category, and we compute for every cobordism W the set of its annihilators.

Theorem 3.3.9. *Let* $\Bbbk = \mathbb{Z}[X,Y,Z^{\pm 1}]/(X^2 = Y^2 = 1)$ *and choose an embedded chronological cobordism* W *in* $\Bbbk$**ChCob**. *Write* $\mathrm{Aut}(W) := \{k \in \Bbbk \mid kW = W\}$. *Then*

$$\mathrm{Aut}(W) = \begin{cases} \{1\}, & \textit{if } W \textit{ has genus } 0 \textit{ and at most one closed component,} \\ \{1, XY\}, & \textit{otherwise.} \end{cases} \tag{1.3}$$

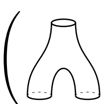
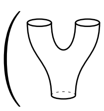
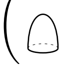
Having constructed the complex $Kh(D)$ we explore various chronological TQFT functors $\mathscr{F}\colon \Bbbk\mathbf{ChCob} \longrightarrow \mathbf{Mod}_{\Bbbk}$ that produce invariant link homology $H(\mathscr{F}Kh(D))$. They are encoded by *chronological Frobenius systems*, see Definition 3.3.10. The list contains the following functors:

1) $\mathscr{F}_{ev}$, computing the even Khovanov homology $\mathscr{H}_{ev}(L)$ of a link L,
2) $\mathscr{F}_{odd}$, computing the odd Khovanov homology $\mathscr{H}_{odd}(L)$,
3) $\mathscr{F}_{cov}$, leading to a sequence of $\Bbbk$-modules $\mathscr{H}_{\pi}(L)$ that covers the two cases above,
4) $\mathscr{F}_{\bullet}$, given by the algebra of cobordisms with dots, a 'chronological' variant of dotted cobordisms introduced by Bar-Natan [BN05, BN07].

The last one is the universal functor—every chronological Frobenius system can be obtained from the dotted algebra $(R_{\bullet}, A_{\bullet})$ either by a base change of by twisting the coalgebra structure, and the behavior of homology under these two operations is well understood.

Theorem 5.2.9. *Any homogeneous rank two chronological Frobenius system* (R,A) *is obtained from* $(R_{\bullet}, A_{\bullet})$ *by a base change and a twist. In particular,* $\mathscr{H}_{\bullet}(L) := H(\mathscr{F}_{\bullet}Kh(L))$ *is the most general link homology theory in our framework.*

Two-dimensional cobordisms admit a natural grading by $\mathbb{Z}\times\mathbb{Z}$: the first number counts merges and birth, whereas the other splits and deaths:

deg () $= (-1,0)$, deg () $= (0,-1)$, (1.4) deg () $= (1,0)$, deg () $= (0,1)$. (1.5)

There is an induced bigrading on the generalized Khovanov complex $Kh(D)$ that descends to homology. It does not provide new information (the homology can be nontrivial only when the two degree components are equal, see Section 5.4.1), but it gives us some control over the signs in the cube—the category $\Bbbk$**ChCob** is an example of a *graded monoidal category*: the disjoint union preserves composition of cobordisms up to a scalar that depends only on the degrees of the cobordisms involved. A similar property is satisfied by the connected sum of cobordisms. We use that in Section 4.2.4 to derive formulas for the generalized complexes for disjoint unions and connected sums of links.

Proposition 4.2.5. *Given link diagram D and D' there is an isomorphism of bigraded complexes $Kh(D \sqcup D') \cong Kh(D) \sqcup Kh(D')$, which is natural with respect to graded morphisms.*

Proposition 4.2.6. *Given link diagram D and D' there is an isomorphism of bigraded complexes $Kh(D \# D') \cong Kh(D) \# Kh(D')$.*

In the above, the operations $\sqcup$ and # for complexes are the additive extensions of the corresponding operations on cobordisms.

On the algebraic side, the disjoint union is translated into a tensor product over the ring of scalars, whereas the connected sum into a tensor product over an algebra $A' := \mathscr{F}(\bigcirc)\{-1,0\}$, where $\mathscr{F} : \Bbbk\mathbf{ChCob} \longrightarrow \mathbf{Mod}_{\Bbbk}$ is the chosen chronological TQFT. This requires the homology to admit a module structure over A'—it is given by placing a circle next to the link and gluing it with a merge cobordism, see Section 5.4.3. Defined a priori for link diagrams, the module structure depends only on the link component with which the circle is merged.

Proposition 5.4.10. *Given two link diagrams D, D', and a chronological TQFT there is an isomorphism of triply-graded complexes $\mathscr{F}Kh(D \sqcup D') \cong \mathscr{F}Kh(D) \underset{\Bbbk}{\otimes} \mathscr{F}Kh(D')$. Moreover, if the diagrams are based, then $\mathscr{F}Kh(D \# D') \cong \mathscr{F}Kh(D) \underset{A'}{\otimes} \mathscr{F}Kh(D')$.*

The covering homology $\mathscr{H}(L)$ has three parameters, X, Y, and Z, leading to eight link homology theories over integers. We show that all of them are isomorphism either to the even or odd homology, depending on the value of XY. This is done by introducing yet another grading on chronological cobordisms, which is not additive with respect to disjoint unions. Using it, we prove in Section 5.3 that the generalized chain complex splits into a bunch of isomorphic copies of the same complex over the ring $\mathbb{Z}_\pi := \mathbb{Z}[\pi]/(\pi^2-1)$. Here, both X and Z are set to 1, whereas Y acts as a multiplication by π.

Theorem 5.3.13 (Reduction of parameters)**.** *The generalized Khovanov complex $\mathscr{F}_A Kh(D)$, regarded as a complex of $\mathbb{Z}_\pi$-modules, decomposes into a direct sum of subcomplexes*

$$\mathscr{F}_A Kh(D) \cong \bigoplus_{(a,b)\in\mathbb{Z}_2\times\mathbb{Z}} \mathscr{F}_A Kh(D)_{a,b}, \tag{1.6}$$

each isomorphic to $\mathscr{F}_A Kh(D;\mathbb{Z}_\pi) \cong \mathscr{F}_\pi Kh(D)$.

In the above $\mathscr{F}_A$ is a functor induced by some chronological Frobenius algebra A in $\mathbf{Mod}_{\Bbbk}$, whereas $\mathscr{F}_\pi$ is its reduction to the ring $\mathbb{Z}_\pi$. This result is then used to prove the duality phenomenon for the generalized Khovanov homology: the complex $\mathscr{H}(L^!)$ for the mirror image $L^!$ of the link L is precisely the value of the derived $\mathrm{Hom}(\_,\mathbb{Z}_\pi)$ applied to $\mathscr{H}(L)$.

Theorem 5.3.17 (Duality for generalized Khovanov homology)**.** *Given a link diagram D and its mirror image $D^!$ there is an isomorphism of complexes*

$$\mathscr{F}_A Kh(D^!) \cong \mathscr{F}_A Kh(D)^*, \tag{1.7}$$

where $(C^*)^i := \mathrm{Hom}(C^{-i},\Bbbk)$ for a chain complex C. In particular, the odd Khovanov homology $\mathscr{H}_{odd}(L)$ of a link L is dual to $\mathscr{H}_{odd}(L^!)$, and similarly for $\mathscr{H}_\pi(L)$ and $\mathscr{H}_\pi(L^!)$.

This was already known for the even Khovanov homology, and computer-based calculation revealed a similar phenomenon for the odd homology theory. However, the latter is not clear from the construction: the algebra used to construct the odd Khovanov complex is not self-dual. In case of the covering theory the roles of X and Y are interchanged, and we use the result on reducing the parameters to switch X and Y back.

The two variants of Khovanov homology, even and odd, coincides when regarded with $\mathbb{Z}_2$ coefficients. Therefore, there are two Bockstein operations, β_e, β_o, and we prove not only they are distinct (which was already known), but we show all their alternating compositions are different and non-trivial. Thence, the algebra of homological operations contains a graded subalgebra with two generators in every degree. This subalgebra is already a finer invariant than the homology itself—there exist pairs of knots with 15 crossings that have isomorphic both even and odd homology, but the sum of the two Bocksteins acts differently, see Section 5.5.2.

The two Bockstein operations admit lifts to integral homologies (Section 5.5.3). Namely, each of the two Bocksteins is covered by a degree 1 operation between even and odd variants of the Khovanov homology:

$$\mathscr{H}_{ev}(L) \underset{\varphi_{oe}}{\overset{\varphi_{eo}}{\rightleftarrows}} \mathscr{H}_{odd}(L). \tag{1.8}$$

These lifts arise from exact sequences of coefficients for the unified homology $\mathscr{H}_\pi(L)$, so that their nontriviality implies $\mathscr{H}_\pi(L)$ is an interesting extension between the two variants of integral Khovanov homology. As in the $\mathbb{Z}_2$ case, the alternating compositions do not vanish. These operations have an interesting behavior with respect to the mirror image operation: even if they distinguish two links L and L', they might be trivial for $L^!$ and $L'^!$. This is possible, because torsion in $\mathscr{H}_{ev}(L^!)$ and $\mathscr{H}_{odd}(L^!)$ is shifted when compared to $\mathscr{H}_{ev}(L)$ and $\mathscr{H}_{odd}(L)$, leaving no space for φ_{eo} and φ_{oe}.

1.3 Outline

We begin the thesis with a brief description of graded monoidal categories and Gray 2-categories, a language we use to describe our construction. The key idea is that we do not want the monoidal product to preserve composition of morphisms, but we still want to have a control on how far from functoriality the product is. This is encoded by certain 2-morphisms in case of Gray 2-categories, and by certain scalars in graded monoidal categories. For instance, in the latter case

$$(f'\otimes g')\circ(f\otimes g) = \lambda\cdot(f'\otimes f)\otimes(g'\otimes g) \tag{1.9}$$

where all morphisms are homogeneous, and λ depends only on degrees of f and g'.

Chapter 3 describes chronological topological quantum field theories. The first section serves as a short introduction to the theory of framed functions, used later to decorate cobordisms. The 2-category **ChCob** is introduced in Section 3.2, whereas in the beginning of Section 3.3 we construct its linearization $\Bbbk$**ChCob**. We end this chapter with a proof of nondegeneracy of $\Bbbk$**ChCob**: most cobordisms have trivial 2-automorphisms.

The construction of the generalized Khovanov chain complex is described in Chapter 4. We explore here how $Kh(D)$ behaves under basic operations on link diagrams. In particular, we obtain the formulas for disjoint unions and connected sums of links in Section 4.2.4, and we analyze the relation between complexes for a link and its mirror image in Section 5.3.4. The proof of invariance is postponed to the end of this chapter—the formula for the connected sum simplifies the computation for the first Reidemeister move a lot.

The last chapter explores various TQFT functors that can be used to compute homology from $Kh(D)$. We recover here both the even and odd Khovanov homology, but we also describe the odd variant of dotted cobordisms showing that the homology theory they induce is the most we can get. This is also the place where we reduce the number of parameters in our theory, showing that the even and odd homology are the only invariants over integers. The chapter ends with a discussion on homological operations.

Chapter 2

Preliminaries

This chapter serves as a brief introduction to monoidal products in graded categories and 2-categories, which provide a framework for the construction of the generalized Khovanov homology. For more details, see [Be67, GPS95, Gr74].

2.1 Graded monoidal categories

2.1.1 The category of graded modules

Fix a group G and a unital commutative ring R. We say an R-module M is *graded by G* if it decomposes into a direct sum $M = \bigoplus_{a\in G} M_a$. An element $m \in M_a$ is *homogeneous of degree* $\deg m = a$. A homomorphism $f\colon M \longrightarrow N$ between graded modules is *graded* if $f(M_a) \subset N_a$ for every $a \in G$, and *homogeneous of degree $b \in Z(G)$* if $f(M_a) \subset N_{ba}$, where $Z(G)$ is the center of G. Clearly, a graded homomorphism is homogeneous of degree e, the neutral element of G.

The category of R-modules is symmetric monoidal: the monoidal product is given as the tensor product over R, and the free module of rank one is the unit for this product. This structure can be twisted in case of G-graded modules.

Definition 2.1.1. Choose a function $\lambda\colon G\times G \longrightarrow U(R)$ that is a group homomorphism in each variable, where $U(R)$ is the group of invertible elements in R. Define the *graded tensor product* for G-graded modules in the ordinary way, but for homogeneous homomorphisms f and g we define the product $f \otimes g$ by the formula

$$(f \otimes g)(m \otimes n) := \lambda(\deg g, \deg m) f(m) \otimes g(n). \tag{2.1}$$

There is a *braiding* $\sigma_{M,N} \colon M \otimes N \longrightarrow N \otimes M$, $m \otimes n \longmapsto \lambda(\deg m, \deg n) n \otimes m$, which is a *symmetry* if $\lambda(a,b)\lambda(b,a) = 1$ for all $a, b \in G$.

The graded tensor product generalizes the Koszul product ($G = \mathbb{Z}_2$ and $\lambda(a,b) = (-1)^{ab}$), and the anyonic braiding ($G = \mathbb{Z}$ and $\lambda(a,b) = \zeta^{ab}$ for some root of unity ζ).

Lemma 2.1.2. *The following hold*

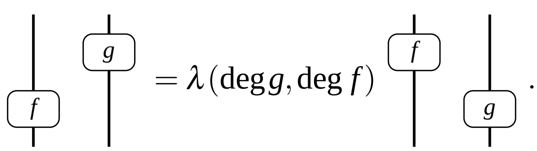
$$(f' \otimes g') \circ (f \otimes g) = \lambda(\deg g', \deg f)(f' \circ f) \otimes (g' \circ g), \text{ and} \tag{2.2}$$

$$\sigma_{M',N'} \circ (f \otimes g) = \lambda(\deg f, \deg g)(g \otimes f) \circ \sigma_{M,N} \tag{2.3}$$

for any homogeneous homomorphisms $M \xrightarrow{f} M' \xrightarrow{f'} M''$ *and* $N \xrightarrow{g} N' \xrightarrow{g'} N''$.

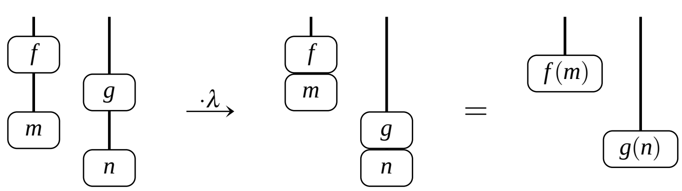
Proof. Straightforward. □

There is a nice graphical interpretation of the formulas (2.2) and (2.1). We represent a homomorphism $f \colon M \longrightarrow N$ by a box labeled f with two legs: one at the bottom, labeled with M, and one at the top, labeled with N. Composition of morphism is given by placing the boxes one over the other and a tensor product of homomorphisms by placing them side by side, the left on the higher level than the right one. Then we have the following relation for homogeneous morphisms f and g:



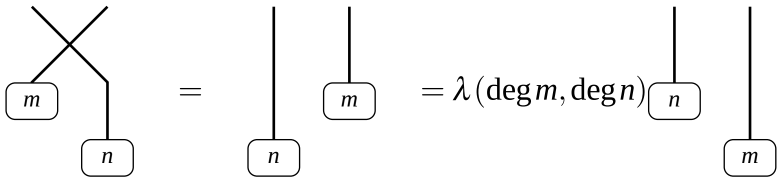
(2.4)

For example, (2.1) follows from the following simple calculation, where we represent an element $m \in M$ of a module M by a box with no input (think of it as a homomorphisms $R \longrightarrow M$ taking 1 to m):



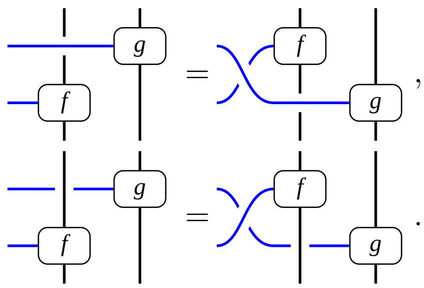
(2.5)

where the arrow $\xrightarrow{\cdot\lambda}$ indicates that the picture to the right must be scaled by $\lambda(\deg g, \deg m)$. If $\lambda(a,b)\lambda(b,a) = 1$ we can represent the symmetry $\sigma_{M,N}: M \otimes N \longrightarrow N \otimes M$ by a crossing:



(2.6)

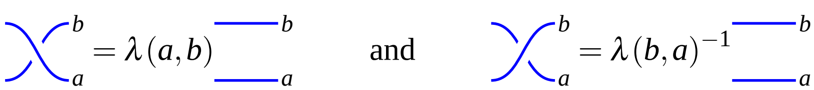
This does not work in the braided case. Indeed, one can first change the heights of boxes labeled m and n, which results in $\sigma(m \otimes n) = \lambda(\deg n, \deg m)^{-1} n \otimes m$. Comparing the two values we conclude it must be $\lambda(\deg n, \deg m)\lambda(\deg m, \deg n) = 1$. One solution to this issue is to add horizontal lines originating at all boxes and pointing leftwards, in which case the relation (2.4) appears in two versions



(2.7)

(2.8)

Decorating the horizontal lines with degrees of the boxes, we add untwisting relations



(2.9)

This can be done only at the left edge of the diagram. The product $f \otimes g$ is then represented by the diagram in which the line originating from g passes over the input for f, and we can represent σ by the positive crossing $\times$. However, the composition of boxes becomes more complicated—one cannot simply join two boxes, unless their horizontal lines pass all other lines in the same way. We shall not go deeper into the braided case, as all graded tensor products considered in this thesis are symmetric.

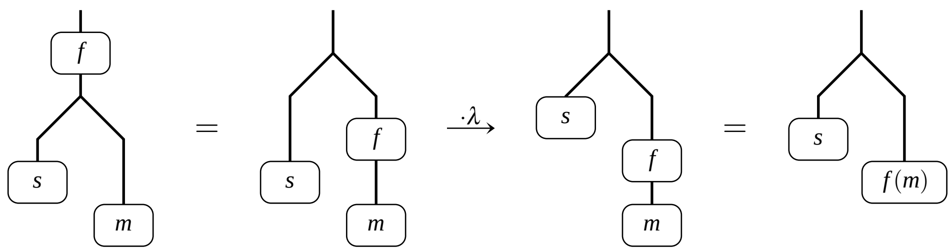
Definition 2.1.3. Choose a ring S that is a G-graded bimodule over R, and consider the category of G-graded modules over S. We say that

1) the ring S is *commutative* if $rs = \lambda(\deg r, \deg s) sr$ for homogeneous elements $r, s \in S$,

2) a G-graded bimodule M over S is *symmetric* if $sm = \lambda(\deg s, \deg m)ms$ for homogeneous elements $s \in S$, $m \in M$, and

3) a homogeneous function $f\colon M \longrightarrow N$ between G-graded bimodules over S is *right linear* if $f(ms) = f(m)s$, but *left linear* if $f(sm) = \lambda(\deg f, \deg s)sf(m)$ for a homogeneous element $s \in S$.

If we think of linearity as a commutativity of a map f with the action of S, then the last definition follows easily from the graphical calculus (notice that the actions of S are degree 0 maps):



(2.10)

With these conventions we can define a tensor product of G-graded bimodules $M \otimes_S N$ in the usual way, with actions of S given as $s(m \otimes n) := (sm) \otimes n$ and $(m \otimes n)s := m \otimes (ns)$. If both M and N are symmetric in the graded sense, so is $M \otimes_S N$.

2.1.2 A product in graded categories

Choose a ring R. An R-linear category $\mathbf{A}$ is *graded* by a group G, if

1) for every two objects $A, B \in \mathbf{A}$ the R-module of morphisms $\mathrm{Mor}(A,B)$ is graded by G, in which case we say that a morphism $f \in \mathrm{Mor}(A,B)_g$ is *homogeneous* of degree $\deg f = g$,

2) the above gradation is coherent with composition, i.e. $\deg(f \circ g) = \deg(f)\deg(g)$ for homogeneous morphisms f and g, and

3) there is a family of linear functors $\{g\}\colon \mathbf{A} \longrightarrow \mathbf{A}$ called *degree shifts*, parametrized by central elements $a \in Z(G)$, such that $\{e\} = \mathrm{id}_{\mathbf{A}}$, $\{a\} \circ \{b\} = \{ab\}$, and there are canonical module isomorphisms $\mathrm{Mor}(A\{a\}, A'\{a'\}) \approx \mathrm{Mor}(A, A')$ such that a morphism $f \in \mathrm{Mor}(A, A')_x$ is homogeneous of degree $\deg f = a'xa^{-1}$ when regarded as an element of $\mathrm{Mor}(A\{a\}, A'\{a'\})$.

The last condition is equivalent to a choice of degree a isomorphisms $i_a\colon A \longrightarrow A\{a\}$ satisfying $i_a \circ i_b = i_{ab}$ and $i_e = \mathrm{id}$, where $e \in G$ is the neutral element—think of i_a as the morphism that corresponds to $\mathrm{id}\colon A \longrightarrow A$.

Remark 2.1.4. The first two conditions say $\mathbf{A}$ is a category enriched over the category of G-graded R-modules. Any such category can be made a G-graded category by replacing its objects with symbols $A\{a\}$, where $A \in \mathrm{Ob}\mathbf{A}$ and $a \in Z(G)$. For morphisms we put $\mathrm{Mor}(A\{a\}, A'\{a'\}) := \mathrm{Mor}(A, A')$ with degrees shifted accordingly to match the definition of a graded category, and we set $A\{a\}\{b\} := A\{ab\}$.

We shall now define a graded product. For that we have to understand first how to construct a product of graded categories. Because we are dealing with categories enriched over $\mathbf{Mod}_R$, it certainly should not be just a Cartesian product (composition is not linear). Instead, choose a bihomomorphism $\lambda\colon G \times G \longrightarrow U(R)$ and define the product $\mathbf{A} \times_\lambda \mathbf{B}$ of G-graded categories $\mathbf{A}$ and $\mathbf{B}$ as follows:

1) objects are pairs (A, B) of objects $A \in \mathbf{A}$ and $B \in \mathbf{B}$,
2) morphisms $(A, B) \longrightarrow (A', B')$ are elements of the tensor products over the ring R:

$$\mathrm{Mor}((A, A'), (B, B')) := \mathrm{Mor}_{\mathbf{A}}(A, A') \otimes \mathrm{Mor}_{\mathbf{B}}(B, B'), \text{ and} \tag{2.11}$$

3) composition satisfies the *graded product rule*

$$(f' \otimes g') \circ (f \otimes g) := \lambda(\deg g', \deg f)(f' \circ f) \otimes (g' \circ g) \tag{2.12}$$

for homogeneous morphisms f, f', g, and g'.

The category $\mathbf{A} \times_\lambda \mathbf{B}$ does not admit an obvious additive structure, as all objects $(A, 0)$ and $(0, B)$ are isomorphic to the zero object $(0, 0)$, as each of them has a unique endomorphism—the zero map. Likewise for the naive degree shifts: the isomorphism $(A, B) \longrightarrow (A\{a\}, B\{a\})$ has degree a^2 instead of a.

Definition 2.1.5. A *graded product of type* λ in a G-graded category $\mathbf{A}$ consists of an object I and an associative degree-preserving linear functor $\boxtimes\colon \mathbf{A} \times_\lambda \mathbf{A} \longrightarrow \mathbf{A}$ such that $I \boxtimes (\_)$ and

$(\_)\boxtimes I$ are identity functors. In particular, $(f'\boxtimes g')\circ(f\boxtimes g)=\lambda(\deg g',\deg f)(f\circ f')\boxtimes(g'\circ g)$ for homogeneous morphisms f, f', g, and g'.

The category $\mathbf{Mod}_R$ admits a braiding $\sigma_{M,N}\colon M\otimes N\longrightarrow N\otimes M$, which induces a functor $P\colon \mathbf{A}\times_\lambda \mathbf{B}\longrightarrow \mathbf{B}\times_\lambda \mathbf{A}$ that maps (A,B) to (B,A), and $f\otimes g$ to $\sigma(f\otimes g)=\lambda(\deg f,\deg g)g\otimes f$.

Definition 2.1.6. A *braiding* in a graded monoidal category $\mathbf{A}$ is a natural transformation of functors $\sigma\colon \otimes\Longrightarrow\otimes\circ P$, i.e. a family of natural isomorphisms $\sigma_{A,B}\colon A\otimes B\longrightarrow B\otimes A$ such that $\sigma_{A',B'}\circ(f\otimes g)=\lambda(\deg f,\deg g)(g\otimes f)\circ\sigma_{A,B}$ for homogeneous morphisms $f\colon A\longrightarrow A'$ and $g\colon B\longrightarrow B'$.

The graded product interacts with the degree shift functors in a non-trivial way. Indeed, the product is functorial as long as we consider graded morphisms, and each homogeneous morphisms can be made graded after a suitable degree shift of its domain of a codomain. Hence, we should not expect the objects $A\{a\}\otimes B\{b\}$ and $(A\otimes B)\{ab\}$ to be equal. They are, however, canonically isomorphic via the isomorphism

$$A\{a\}\otimes B\{b\}\xrightarrow{i_a^{-1}\otimes i_b^{-1}}A\otimes B\xrightarrow{i_{ab}}(A\otimes B)\{ab\}. \tag{2.13}$$

Notice the choice of the first map: $i_a^{-1}\otimes i_b^{-1}$ is not the inverse of $i_a\otimes i_b$. The following lemma explains how the scaling coefficient appear in the formula for the graded product.

Lemma 2.1.7. *Choose an object X and a morphism $f\colon A\longrightarrow B$ of degree $\deg f=ab^{-1}$, where $a,b\in Z(G)$. Then the following diagrams commute*

$$\begin{array}{ccc} A\{a\}\otimes X\{x\} & \xrightarrow{i_{ax}\circ(i_a^{-1}\otimes i_x^{-1})} & (A\otimes X)\{ax\} \\ \Big\downarrow{\scriptstyle \overline{f}\otimes\mathrm{id}} & & \Big\downarrow{\scriptstyle \overline{f\otimes\mathrm{id}}} \\ B\{b\}\otimes X\{x\} & \xrightarrow{i_{bx}\circ(i_b^{-1}\otimes i_x^{-1})} & (B\otimes X)\{bx\} \end{array} \tag{2.14}$$

$$\begin{array}{ccc} X\{x\}\otimes A\{a\} & \xrightarrow{i_{xa}\circ(i_x^{-1}\otimes i_a^{-1})} & (X\otimes A)\{xa\} \\ \Big\downarrow{\scriptstyle \mathrm{id}\otimes\overline{f}} & & \Big\downarrow{\scriptstyle \lambda(ab^{-1},x)\overline{\mathrm{id}\otimes f}} \\ X\{x\}\otimes B\{b\} & \xrightarrow{i_{xb}\circ(i_x^{-1}\otimes i_b^{-1})} & (X\otimes B)\{xb\} \end{array} \tag{2.15}$$

where $\overline{\cdot}$ *denotes appropriate conjugations by canonical isomorphisms* $i_\bullet$.

Proof. Follows directly from the third condition for a graded product in Definition 2.1.5. $\square$

2.2 Monoidal 2-categories

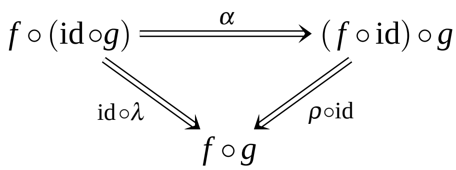
We have seen that for graded categories the composition rule for the product can be twisted by a certain function. A general framework for this type of constructions is provided by 2-categories and Gray monoidal products.

2.2.1 2-categories

The shortest way to define a 2-category is to say that it is a category enriched over **Cat**. This means the following:

- for every two objects A and B there is a category of morphisms $\mathrm{Mor}(A,B)$; morphism of this category are called 2-*morphisms* and composition is denoted by $\star$,
- the composition is given by functors $\circ_{A,B,C}\colon \mathrm{Mor}(B,C)\times\mathrm{Mor}(A,B)\longrightarrow\mathrm{Mor}(A,C)$,
- the identity morphisms are picked by functors $1\!\mathrm{l}_A\colon * \longrightarrow \mathrm{Mor}(A,A)$, where the category $*$ consists of a single object $*$ and a single morphism id_*,
- the associativity and unitarity axioms are replaced with three invertible 2-morphisms $\rho_f\colon f\circ\mathrm{id}_A \Longrightarrow f$, $\lambda_f\colon \mathrm{id}_B\circ f\Longrightarrow f$, and $\alpha_{f,g,h}\colon f\circ(g\circ h)\Longrightarrow(f\circ g)\circ h$ for any $f\in\mathrm{Mor}(A,B)$, $g\in\mathrm{Mor}(B,C)$, and $h\in\mathrm{Mor}(C,D)$, fitting into the commutative diagrams

$$\begin{array}{ccccc}
 & f\circ(g\circ(h\circ k)) & \xRightarrow{\ \mathrm{id}\circ\alpha\ } & f\circ((g\circ h)\circ k) & \\
 \swarrow{\scriptstyle\alpha} & & & & {\scriptstyle\alpha}\searrow \\
(f\circ g)\circ(h\circ k) & & & & (f\circ(g\circ h))\circ k \\
 & {\scriptstyle\alpha}\searrow & & \swarrow{\scriptstyle\alpha\circ\mathrm{id}} & \\
 & & ((f\circ g)\circ h)\circ k & &
\end{array} \tag{2.16}$$



(2.17)

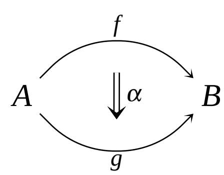
They are called the MacLane's coherence conditions [ML98].

A 2-category is *strict*, if all α, ρ and λ are identities. Otherwise, it is *weak*.

Example 2.2.1. Given two small categories $\mathbf{C}$ and $\mathbf{D}$ there is a category $[\mathbf{C} \longrightarrow \mathbf{D}]$ of functors from $\mathbf{C}$ to $\mathbf{D}$, where the role of morphisms is played by natural transformations. Therefore, we have a 2-category of all small categories. This 2-category is strict, because composition of functors is associative.

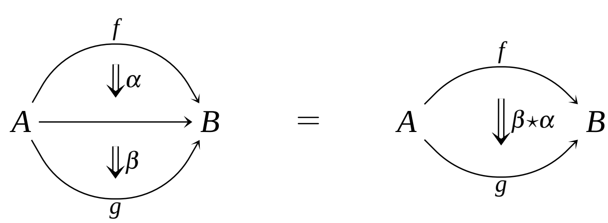
Example 2.2.2. Consider a category $\mathbf{Mod}_R$ of modules over a fixed commutative ring R. We can extend it to a 2-category with 2-morphisms given by elements of R as follows. Choose module homomorphisms $f, g \colon M \longrightarrow N$ and $r \in R$. We write $r \colon f \Longrightarrow g$ if $g(m) = f(rm)$ for any $m \in M$. Both compositions of 2-morphisms are given as multiplication in R. The 2-category defined this way is again strict.

If we represent objects by points on a plane and 1-morphisms by oriented edges, then 2-morphisms decorate regions. With this interpretation, a picture of a typical 2-morphisms looks as follows:



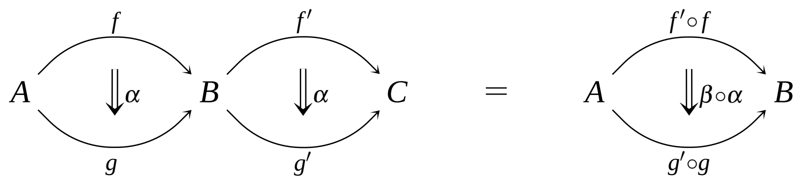
(2.18)

There are two ways of composing 2-morphisms: a *vertical* composition, induced by the internal composition in morphism categories $\mathrm{Mor}(A, B)$



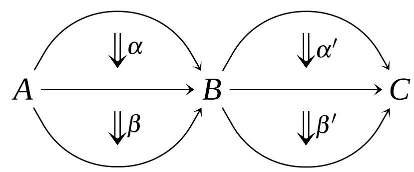
(2.19)

and a *horizontal* composition, given by the composition functors $\circ_{A,B,C}$



(2.20)

Moreover, the two ways of composing 2-morphisms are compatible, which means that the diagram



(2.21)

produces the same 2-morphism no matter whether we first compose the 2-morphisms vertically or horizontally. In other words,

$$(\beta' \star \alpha') \circ (\beta \star \alpha) = (\beta' \circ \beta) \star (\alpha' \circ \alpha). \tag{2.22}$$

This property, called the *interchange law*, together with the obvious associativity and unitarity axioms, is another way how to define a 2-category [Be67].

The higher structure of 2-categories affects a notion of a functor: we no longer assume that it preserves identities nor compositions of morphisms. Instead, both properties should hold up to some 2-morphisms, which are part of the data, subject to some coherence relations.[1]

Definition 2.2.3. A functor $F\colon \mathbf{C} \longrightarrow \mathbf{D}$ between 2-categories consists of a function of objects $F_0\colon \mathrm{Ob}\mathbf{C} \longrightarrow \mathrm{Ob}\mathbf{D}$, a collection of functors $F_{A,B}\colon \mathrm{Mor}(A,B) \longrightarrow \mathrm{Mor}(FA,FB)$, and 2-morphisms $\iota_A\colon \mathrm{id}_{FA} \Longrightarrow F(\mathrm{id}_A)$ and $\varphi_{f,g}\colon F(f)\circ F(g) \Longrightarrow F(f\circ g)$ satisfying certain coherence relations. A functor F is *strict*, if both 2-morphisms are equalities.

A famous result states that every 2-category can be strictified: every 2-category is equivalent to some strict 2-category. Hence, we do not have to care about weak 2-categories. On the other hand, this does not apply to functors: there are functors between strict 2-categories that cannot

[1] See [Be67] for details. The most general definition does not even assume invertibility of ι and φ, but we will never need such functors.

be replaced by strict ones. However, most functors used in this paper will be strict, with the only exception of cubical functors [GPS95].

Definition 2.2.4. A functor $F\colon \mathbf{C}_1 \times \cdots \times \mathbf{C}_r \longrightarrow \mathbf{D}$ between strict 2-categories[2] is *cubical* if the following conditions hold:

1) $F(\mathrm{id}_{A_1}, \ldots, \mathrm{id}_{A_r}) = \mathrm{id}_{F(A_1,\ldots,A_r)}$, and
2) $F(f_1 \circ g_1, \ldots, f_r \circ g_r) = F(f_1, \ldots, f_r) \circ F(g_1, \ldots, g_r)$ if there is k such that $g_i = \mathrm{id}$ and $f_j = \mathrm{id}$ for all $i < k < j$.

In other words, ι is the identity 2-morphism and so is φ, unless we have to 'permute' nontrivial morphisms f and g'.

In the case of a cubical functor, the coherence relations mentioned in Definition 2.2.3 reduce to two commuting diagrams of 2-morphisms

$$\begin{array}{ccc} F(\underline{f}) \circ F(\underline{g}) & \xRightarrow{F(\underline{\alpha})\circ F(\underline{\beta})} & F(\underline{f}') \circ F(\underline{g}') \\ \Big\Downarrow{\scriptstyle \varphi} & & \Big\Downarrow{\scriptstyle \varphi} \\ F(\underline{f} \circ \underline{g}) & \xRightarrow{F(\underline{\alpha}\circ\underline{\beta})} & F(\underline{f}' \circ \underline{g}') \end{array} \tag{2.23}$$

$$\begin{array}{ccc} F(\underline{f}) \circ F(\underline{g}) \circ F(\underline{h}) & \xRightarrow{\varphi\circ\mathrm{id}} & F(\underline{f} \circ \underline{g}) \circ F(\underline{h}) \\ \Big\Downarrow{\scriptstyle \mathrm{id}\circ\varphi} & & \Big\Downarrow{\scriptstyle \varphi} \\ F(\underline{f}) \circ F(\underline{g} \circ \underline{h}) & \xRightarrow{\varphi} & F(\underline{f} \circ \underline{g} \circ \underline{h}) \end{array} \tag{2.24}$$

where we used a shortcut notation $\underline{f} = (f_1, \ldots, f_r)$ for morphisms in a product of 2-categories, and similarly for 2-morphisms. The latter condition has the following interpretation when $r = 2$: whenever we have three pairs of morphisms, passing from a composition of values of F on them to the value of F on their composition requires two 'transpositions' of 'inner' arguments and it can be done in two different ways. The condition (2.24) says, it does not matter which way we choose.

[2] There is also a more general notion of a cubical functor between weak 2-categories.

2.2.2 Gray products

A Gray monoidal structure is the analogue of a strict monoidal one in the world of ordinary categories: there is a more general definition of a (weak) monoidal 2-category, but each such category is equivalent (in a monoidal sense) to a Gray-monoidal one [GPS95]. Because of that it is sometimes called a *semi-strict* monoidal 2-category [BaNe95, La05].

Definition 2.2.5. A *Gray monoidal structure* in a strict 2-category $\mathbf{C}$ consists of an associative cubical functor $\otimes\colon \mathbf{C}\times\mathbf{C}\longrightarrow\mathbf{C}$ and a unit object $I\in\mathbf{C}$ such that both $I\otimes(\_)$ and $(\_)\otimes I$ are identity 2-functors.

Example 2.2.6. Consider a (non-additive) subcategory $\mathbf{Mod}_R^h\subset\mathbf{Mod}_R$ of all G-graded R-modules and only homogeneous morphisms. The graded tensor product, when restricted to this subcategory, is a cubical functor: the 2-morphism $\varphi\colon (f'\otimes g')\circ(f\otimes g)\Longrightarrow(f'\circ f)\otimes(g'\circ g)$ is given as multiplication by $\lambda(\deg g',\deg f)$. This example shows that graded monoidal categories are very close to Gray categories.

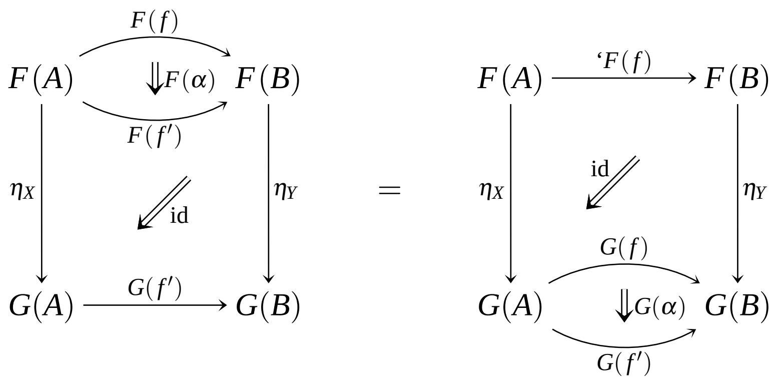
It is much harder to describe braiding in a monoidal 2-category: writing down all coherence conditions takes usually a few pages [BaNe95, KV94]. Since we will never use this notion in such generality, we provide here a very simplified version with all 2-morphisms being identities. That is why we call it a *strict braiding*.

Definition 2.2.7. A *strict braiding* in a Gray monoidal category $(\mathbf{C},\otimes,I)$ is a collection of isomorphisms $\sigma_{A,B}\colon A\otimes B\longrightarrow B\otimes A$ such that each $\sigma_{A,\_}$ and $\sigma_{\_,B}$ is a natural transformation and the triangle below commutes

$$\begin{array}{ccc} A\otimes B\otimes C & \xrightarrow{\mathrm{id}\otimes\sigma_{B,C}} & A\otimes C\otimes B \\ & {\scriptstyle\sigma_{A\otimes B,C}}\searrow \quad \swarrow{\scriptstyle\sigma_{A,C}\otimes\mathrm{id}} & \\ & C\otimes A\otimes B & \end{array} \tag{2.25}$$

for any object C. If in addition $\sigma_{A,B}\circ\sigma_{B,A}=\mathrm{id}$, we call σ a *strict symmetry*.

A natural transformation $\eta\colon F\longrightarrow G$ in a 2-categorical setting means a little more than a commutativity of squares. Indeed, it should be coherent with 2-morphisms, which can be translated as the following equality of 2-morphisms



(2.26)

for any 2-morphism $\alpha\colon f \Longrightarrow f'$.

Example 2.2.8. The category $\mathbf{Mod}_R^h$ from Example 2.2.6 is strictly braided, with the braiding isomorphism $\sigma_{A,B}(a \otimes b) := \lambda(\deg a, \deg b) b \otimes a$.

Chapter 3

Chronological Topological Quantum Field Theories

A Topological Quantum Field Theory is a monoidal functor $\mathscr{F}\colon n\mathbf{Cob} \longrightarrow \mathbf{A}$ from the category of $(n-1)$-dimensional manifolds and n-dimensional cobordisms between them to some symmetric monoidal abelian category $\mathbf{A}$, usually the category of vector spaces over a field, or bimodules over a fixed ring. Regarding a cobordism W as a spacetime, a timeless description of an evolution of the initial space Σ, a TQFT $\mathscr{F}$ provides certain quantitative information: how states of the space Σ (the vectors from $\mathscr{F}(\Sigma)$) changes during the evolution (the function $\mathscr{F}(W)$).

If we want to capture some information about catastrophes in this evolution, such as breaking a space into pieces or merging two components, we have to choose a time function. We allow smooth deformation of the time to keep the flexibility of TQFTs, but with some control at the same time. In particular, we are interesting on in the order, the *chronology*, of the catastrophes, which is preserved by small perturbations of the time function. This can be done by extending the category of cobordisms to a certain 2-category described in this chapter.

3.1 Framed functions

Definition 3.1.1. An *Igusa function* is a smooth function $f\colon W \longrightarrow \mathbb{R}$, such that at every point $p \in W$ one of the following conditions holds:

IF1: p is *regular*, i.e. the derivative df_p does not vanish, or

IF2: f has a *Morse singularity* (or A_1 singularity) at p, i.e. $df_p = 0$ but the Hessian $Hess_p(f)$ is nondegenerate, or

IF3: f has a *birth-death singularity* (or A_2 singularity) at p, i.e. $df_p = 0$ and $Hess_p(f)$ has a 1-dimensional kernel $N(p) \subset T_pW$, but $d^3 f_p$ is nonzero on $N(p)$.

Morse and birth-death singularities of a function f have the following local models:

$$f(x_1,\dots,x_n) = f(p) - x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots x_n^2, \tag{3.1}$$

$$f(x_1,\dots,x_n) = f(p) - x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots x_{n-1}^2 + x_n^3. \tag{3.2}$$

In the latter case the nullspace $N(p)$ of $Hess_p(f)$ is spanned by $\frac{\partial}{\partial x_n}$. The number $k = \mu(p)$ is called the *index* of p.

Igusa functions arise naturally if one considers homotopies between smooth functions: a generic function on W is Morse (conditions IF1 and IF2) and *separative* (critical points lie on different levels), but a space of such functions is not even connected. However, a transversality argument implies a generic homotopy f_t is separative Morse except finitely many moments $0 < t_1 < \dots < t_k < 1$, at which either two critical points are permuted or a birth-death singularity occurs [Ce68]; we refer to them as *events*. We can visualize them by drawing the singular locus $S(f) := \{(t, f_t(x)) \mid x \in \mathrm{crit}(f_t)\}$, see Fig. 3.1.

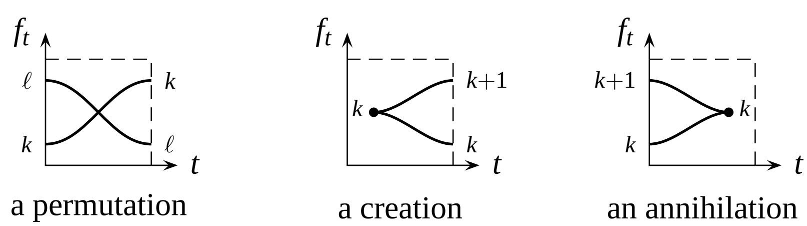


Figure 3.1: Singular loci for elementary homotopies of Igusa functions. Cusps represent A_2-singularities, and labels are the indicies of critical points.

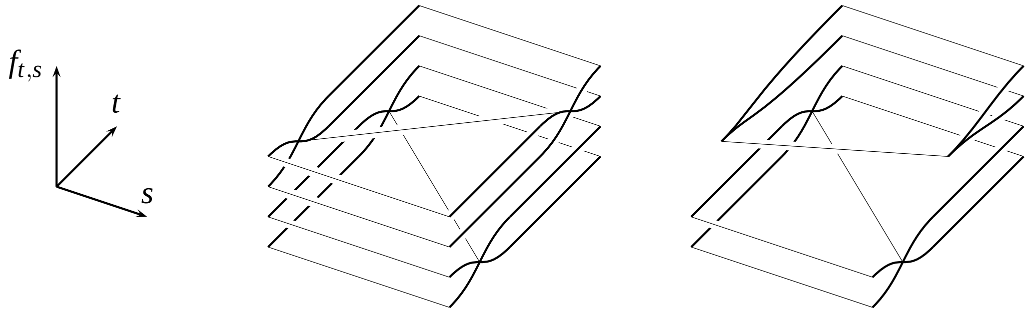
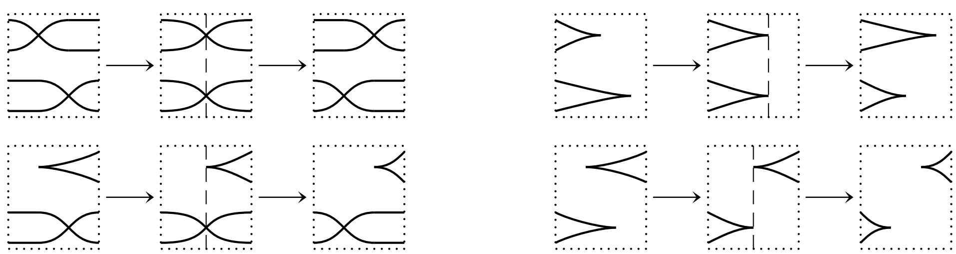


Figure 3.2: Exampes of singular loci, when two events occurs at the same times.

Choose a generic two-parameter family $f_{t,s}\colon W \longrightarrow \mathbb{R}$ of Igusa functions, $t,s \in I$. The path $t \longmapsto f_{t,s}$ is a generic homotopy of Igusa functions for all except finitely many $s \in I$, at which one of the situations described below occurs, see [Ig84, EM11].

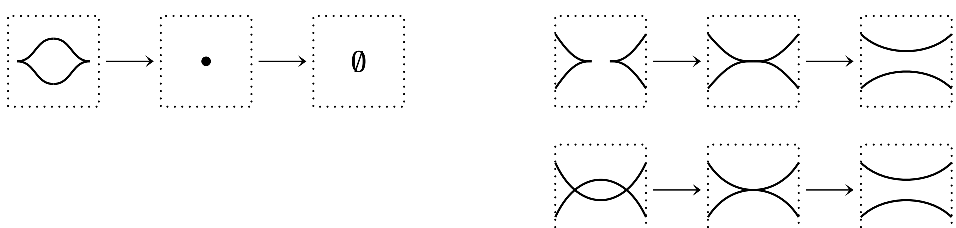
Case I Two events can occur at the same time t_i. For example, we have homotopies



(3.3)

where dashed lines indicate singular values of t. See also Fig. 3.2 for singular loci of the left two homotopies.

Case II A non-transverse event occurs, i.e. the singular set is not transverse to some level set $\{t = a\}$. Up to direction of the change, there are three such homotopies



(3.4)

and their singular loci are shown in Fig. 3.3.

Case III Either three Morse singularities or an A_2-singularity and a Morse one meet at the same

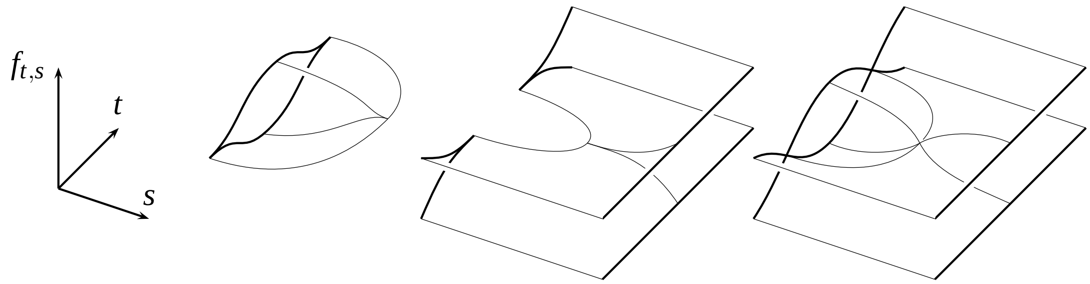
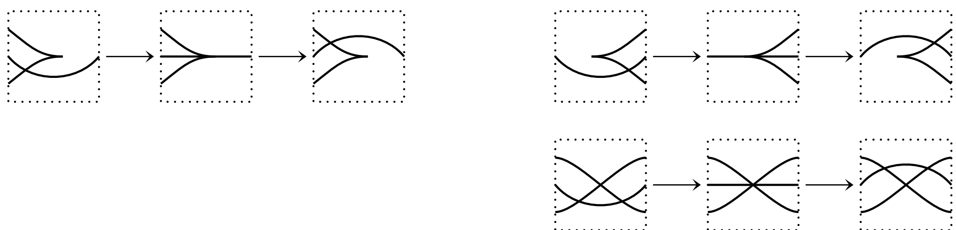


Figure 3.3: Singular loci of non-transverse events.

critical level. There are three types of such homotopies



(3.5)

with singular loci of two of them visualized in Fig. 3.4 (the case of an annihilation is symmetric to the one of a creation).

The space of Igusa functions is not simply connected, which is manifested by the lack of the dove tail singularity in the list above. Indeed, this singularity is modeled by a biquadratic polynomial and as such it cannot appear. We can fix this issue by adding a framing.[1] In fact, the space of framed functions is contractible [Ig87, Lu09, EM11], but we will not use this fact in this paper. The following definition comes from [EM11].

Given a Riemannian metric on W and a critical point $p \in W$ of an Igusa function $f\colon W \longrightarrow \mathbb{R}$ consider its Hessian as a linear map $Hess_p(f)\colon T_pW \longrightarrow T_pW$. Denote by $E^-(p)$ and $E^+(p)$ its negative and positive eigenspaces respectively.

Definition 3.1.2. Let $f\colon W \longrightarrow \mathbb{R}$ be an Igusa function. A *framing* on f is a choice of a Riemannian metric on W and an orthonormal frame $v_1, \dots, v_{\mu(p)}$ of $E^-(p)$ at every critical point

[1] Framed functions were introduced to overcome the problem of lost information, when replacing a manifold with a Morse function: although a Morse function decomposes W into cells, one cannot build W back, unless a parametrization of each cell is given. This is the additional information a framing provides [Ig87].

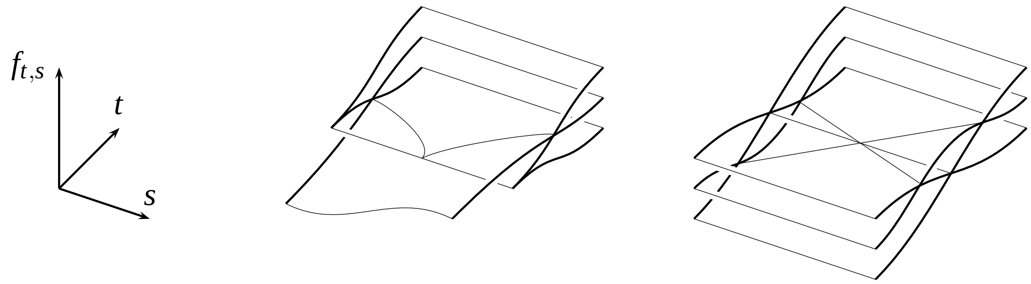


Figure 3.4: Singular locuces of exceptional events from the third group.

p. If p is an A_2 singularity, we add an extra vector $v_{\mu(p)+1} \in N(p)$ in the positive direction of $d^3\tau$.

The topology on the space of framed functions $\mathrm{Fun}^{\mathrm{fr}}(W)$ was described indirectly in [Ig87] by constructing a simplicial complex homotopy equivalent to this space. Here we only remind how homotopies look like, following [EM11].

Choose a smooth function $f\colon W \times I^m \longrightarrow \mathbb{R}$ such that each *slice* $f_{\underline{t}}\colon W \longrightarrow \mathbb{R}$ for $\underline{t} \in I^m$ is an Igusa function. Denote by $V \subset W \times I^m$ the set of critical points of all slice functions $f_{\underline{t}}$ and let Σ be the subset of all A_2 points. Genericly, V is an m-dimensional submanifold of $W \times I^m$, Σ has codimension 1 in V, and V is transverse to each slice $W \times \{\underline{t}\}$ at the set $V - \Sigma$, see [EM11]. Let $V - \Sigma = V^0 \cup \cdots \cup V^n$ and $\Sigma = \Sigma^0 \cup \cdots \cup \Sigma^{n-1}$ be decompositions of $V - \Sigma$ and Σ with respect to the index. Then

- Σ^k is the intersection of the closures of V^k and V^{k+1}, and
- for $z = (p, \underline{t}) \in V^k$ one has $T_pW = E^-(z) \oplus E^+(z)$, and
- for $z = (p, \underline{t}) \in \Sigma^k$ one has $T_pW = E^-(z) \oplus N(z) \oplus E^+(z)$,

where $E^{\pm}(z)$ is the positive of negative eigenspace of $Hess_p(f_{\underline{t}})$ and $N(z)$ is its nullspace. It follows that for $z_0 \in \Sigma^k$ and $z \in V^k$

$$\lim_{z \longrightarrow z_0} E^+(z) = N(z_0) \oplus E^+(z_0) \quad \text{and} \quad \lim_{z \longrightarrow z_0} E^-(z) = E^-(z_0), \tag{3.6}$$

whereas for $z_0 \in \Sigma^k$ and $z \in V^{k+1}$

$$\lim_{z \longrightarrow z_0} E^-(z) = E^-(z_0) \oplus N(z_0) \quad \text{and} \quad \lim_{z \longrightarrow z_0} E^+(z) = E^+(z_0). \tag{3.7}$$

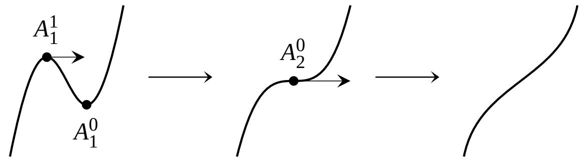


Figure 3.5: A cancelation of framed A_1 points.

A framing on $f\colon W \times I^m \longrightarrow I$ forms a collection of sections $(v_1, \ldots, v_n)$, where each v_k is defined only over the union $\Sigma^{k-1} \cup V^k \cup \cdots \cup \Sigma^{n-1} \cup V^n$, such that $v_k(z) \in N(z)$ for $z \in \Sigma^{k-1}$ and at $z \in V^k \cup \Sigma^k$ the vectors $v_1(z), \ldots, v_k(z)$ form an orthonormal frame of $E^-(z)$. In particular, when we approach a birth-death singularity, framings of canceling points agree with the framing of the limiting point, see Fig. 3.5. For more details see [EM11].

Theorem 3.1.3 (cf. [EM11, Lu09])**.** *For any compact manifold W, the space of framed Igusa functions* $\mathrm{Fun}^{\mathrm{fr}}(W)$ *is contractible.*

There is a natural action of $SO(k)$ on the set of all framings of a critical point of index k. The quotient by this action, one per each critical point, results in a much smaller space of functions, which is still simply connected.

Definition 3.1.4. An *orientation* of an Igusa function is a choice of an orientation of the negative eigenspace $E^-(p)$ at every critical point p. The space of oriented Igusa functions on W will be denoted by $\mathrm{Fun}^{\mathrm{or}}(W)$.

Theorem 3.1.5. $\mathrm{Fun}^{\mathrm{or}}(W)$ *is simply connected for any compact manifold W.*

Proof. Consider the canonical projection $\pi\colon \mathrm{Fun}^{\mathrm{fr}}(W) \longrightarrow \mathrm{Fun}^{\mathrm{or}}(W)$. It is easy to see that it has connected fibers (a product of $SO(k)$'s). Hence, if we can show it has a path-lifting property, then any loop γ can be lifted to a loop up to reparametrization (lift γ as a path and connect its endpoints in a fiber). Then a contracting homotopy upstairs descends to a contracting homotopy of γ.

Pick a path $\gamma\colon [0,1] \longrightarrow \mathrm{Fun}^{\mathrm{or}}(W)$. The compactness of $[0,1]$ implies the existence of a sequence $0 = t_0 < t_1 < \cdots < t_k = 1$ such that $\gamma|_{[t_{i-1},t_i]}$ looks like one of the homotopies listed in Fig. 3.1. Since π has connected fibers, it is enough to lift each of the three homotopies.

- If γ has only Morse singularities, for each critical point of $\gamma(0)$ choose any framing with a given orientation and transport it along the path.
- If γ has a birth singularity of index k at p, pick any framing at this point agreeing with its orientation. Then transport it along the path of points with index k and for the path of index $k+1$ add to the framing the additional vector coming from the nullspace $N(p)$.
- For a death singularity do the same but with the time reversed.

Hence, every path in $\mathrm{Fun}^{\mathrm{or}}(W)$ lifts to $\mathrm{Fun}^{\mathrm{fr}}(W)$. □

Remark 3.1.6. The group $SO(k)$ is not simply connected, and there is a choice for a path connecting the endpoints of the lift. In particular, $\pi_2(\mathrm{Fun}^{\mathrm{or}}(W))$ may be nontrivial. This is not a problem for us, as we never go beyond $\pi_1(\mathrm{Fun}^{\mathrm{or}}(W))$ in this thesis.

3.2 The category of chronological cobordisms

3.2.1 Basic definitions and operations

An $(n+1)$-manifold W is a *cobordism* between two oriented n-manifolds Σ_0 and Σ_1 if its boundary is diffeomorphic to $\Sigma_0 \sqcup -\Sigma_1$ (the minus sign stands for the opposite orientation of Σ_1). We will often write W_{in} and W_{out} for the components of ∂W identified with Σ_0 and $-\Sigma_1$ respectively, and call them the *input* and the *output* of W.

Given cobordisms W from Σ_0 to Σ_1 and W' from Σ_1 to Σ_2, one can glue them together along the orientation reversing diffeomorphism $W_{out} \approx \Sigma_1 \approx W'_{in}$ to obtain a cobordism $W'W$. Unfortunately, this operation is defined only up to a diffeomorphism, the issue we can address by considering cobordisms with *collars*. Namely, think of an n-manifold Σ as an open cylinders $\Sigma \times (-\varepsilon, \varepsilon)$ for a fixed small $\varepsilon > 0$, and a cobordisms from Σ_0 to Σ_1 as a manifold W with a pair of embeddings $\Sigma_0 \times [0,\varepsilon) \longrightarrow W \longleftarrow \Sigma_1 \times (-\varepsilon, 0]$. If W' is another cobordism from Σ_1 to Σ_2, then the gluing $W'W := W' \cup (\Sigma_1 \times (-\varepsilon, \varepsilon)) \cup W$ has a well-defined smooth structure.

Definition 3.2.1. A *chronological cobordism* is a cobordism W together with an oriented Morse function $\tau\colon W \longrightarrow I$ separating critical points, for which $\tau^{-1}([0,\varepsilon))$ and $\tau^{-1}((1-\varepsilon, 1])$ are the collars of W_{in} and W_{out} respectively, on which τ is the projection on the second factor. A homotopy of τ in the space of oriented Igusa functions is called a *change of a chronology*.

We are interested only in the order of critical points of the function τ. Therefore, we identify chronologies that differ by a change preserving the order.

Definition 3.2.2. Chronological cobordisms (W,τ) and (W,τ') are *equivalent* if there exists a path γ in $\mathrm{Fun}^{\mathrm{or}}(W)$ from τ to τ' such that each $\gamma_t\colon W \longrightarrow I$ is a Morse function that separates critical points.[2] In such case we write $(W,\tau) \sim (W,\tau')$ or $\tau \sim \tau'$.

Given cobordisms (W,τ) from Σ_0 to Σ_1 and (W,τ') from Σ_1 to Σ_2 we define a chronology τ'' on the gluing $W'W$ by concatenation:

$$\tau''(p) = \begin{cases} \frac{1}{2}\tau(p), & \text{for } p \in W, \\ \frac{1}{2}(\tau'(p)+1), & \text{for } p \in W'. \end{cases} \tag{3.8}$$

The assumed behavior of a chronology on collars of a cobordism guarantees τ'' is smooth. Hence, we have an associative and unital operation on equivalence classes of cobordisms, where units are given by cylinders $\Sigma \times I$ with the simplest chronology—the projection on I.

Recall that given two paths $\gamma,\gamma'\colon I \times X$ in a topological space X such that $\gamma(1) = \gamma'(0)$ we define their concatenation $\gamma' \star \gamma$ by the formula

$$(\gamma' \star \gamma)(t) := \begin{cases} \gamma(2t), & 0 \leqslant t \leqslant 1/2, \\ \gamma'(2t-1), & 1/2 \leqslant t \leqslant 1. \end{cases} \tag{3.9}$$

$$\begin{array}{ccc} (W,H_0) & \xrightarrow{\;H\;} & (W,H_1) \\ {\scriptstyle\gamma}\downarrow & \overset{\text{htpy}}{\swarrow\!\!\swarrow} & \downarrow{\scriptstyle\gamma'} \\ (W,H_0') & \xrightarrow{\;H_t'\;} & (W,H_1') \end{array}$$

Definition 3.2.3. Let $H,H'\colon W \times I \longrightarrow I$ be changes of chronologies such that (W,H_0) and (W,H_0') are equivalent chronological cobordisms, i.e. there is a path γ in $\mathrm{Fun}^{\mathrm{or}}(W)$ between H_0 and H_0'. We say H and H' are *equivalent* if there is a path γ' from H_1 to H_1' such that the paths $\gamma_t' \star H_t$ and $H_t' \star \gamma_t$ are homotopic in $\mathrm{Fun}^{\mathrm{or}}(W)$. In such case we write $H \sim H'$.

[2] We are allowed to deform not only the function τ, but also the chosen Riemannian structure on W. As shown in [Ig87] all Riemanian structures can be related by such deformations.

Remark 3.2.4. The connectivity of $\mathrm{Fun}^{\mathrm{or}}(W)$ implies the homotopy in the definition above always exists. Hence, any two changes $H, H' \colon W \times I \longrightarrow I$, for which $H_0 \sim H'_0$ and $H_1 \sim H'_1$, are equivalent.

We can *juxtapose* changes occurring on different regions of a cobordism. Formally, if H and H' are changes of chronologies on W and W' respectively, and cobordisms W and W' can be glued together, there is a change of a chronology on $W'W$ induced by the map

$$(H' \circ H)(p,t) := \begin{cases} H(p,t), & p \in W, \\ H'(p,t), & p \in W', \end{cases} \tag{3.10}$$

which may need to be smoothed. This operation is clearly associative and unital.

Concatenation of changes of chronologies is a bit cumbersome: we cannot combine homotopies $H, H' \colon W \times I \longrightarrow I$ if (W, H_1) and (W, H'_0) are only equivalent, as H_1 may not agree with H'_0. Instead, we define

$$(H' \star H)(p,t) := \begin{cases} H(p,3t), & 0 \leqslant t \leqslant 1/3, \\ \gamma(p, 2t-1), & 1/3 \leqslant t \leqslant 2/3, \\ H'(\varphi(p), 3t-2), & 2/3 \leqslant t \leqslant 1, \end{cases} \tag{3.11}$$

where γ is a path in $\mathrm{Fun}^{\mathrm{or}}(W)$ from H_1 to H'_0. Hence, $H' \star H$ is given as the following sequence of homotopies:

$$(W, H_0) \xrightarrow{H} (W, H_1) \xrightarrow{\gamma} (W, H'_0) \xrightarrow{H'} (W, H'_1). \tag{3.12}$$

This operation is well-defined up to equivalence due to Remark 3.2.4 (in particular, it does not depend on the path γ), and it is clearly associative and unital up to equivalence.

Lemma 3.2.5. *Choose pairs of equivalent changes of chronologies $H \sim \widetilde{H}$ and $H' \sim \widetilde{H}'$ on a cobordism W such that H' and H can be concatenated. Then we can concatenate $\widetilde{H}'$ with $\widetilde{H}$, and $H' \star H \sim \widetilde{H}' \star \widetilde{H}$.*

Proof. The asserted equivalences guarantee $(W, H_i) \sim (W, \widetilde{H}_i)$ and $(W, H'_i) \sim (W, \widetilde{H}'_i)$ for $i = 0, 1$, and since H' can be concatenated with H, (W, H_1) and (W, H'_0) are equivalent as well.

Hence, we have a sequence of equivalences $(W,\widetilde{H}_1) \sim (W,H_1) \sim (W,H_0') \sim (W',\widetilde{H}_0')$, which shows $\widetilde{H}'$ and $\widetilde{H}$ can be composed together. The thesis follows, since the rectangle below

$$\begin{array}{ccccccc} (W,H_0) & \xRightarrow{H_t} & (W,H_1) & \xRightarrow{\omega_t} & (W,H_0') & \xRightarrow{H_t'} & (W,H_1') \\ \Big\Downarrow{\scriptstyle\gamma_t} & & & & & & \Big\Downarrow{\scriptstyle\gamma_t'} \\ (W,\widetilde{H}_0) & \xRightarrow{\widetilde{H}_t} & (W,\widetilde{H}_1) & \xRightarrow{\widetilde{\omega}_t} & (W,\widetilde{H}_0') & \xRightarrow{\widetilde{H}_t'} & (W,\widetilde{H}_1') \end{array} \tag{3.13}$$

commutes up to homotopy due to Remark 3.2.4, where γ, γ', ω, and $\widetilde{\omega}'$ are paths of Morse functions given by corresponding equivalences of chronological cobordisms. $\square$

All the above almost shows that chronological cobordisms form a 2-category—it remains to check the interchange law (2.22) holds; this follows immediately as concatenation and juxtaposition commute with each other. We state this as the following proposition.

Proposition 3.2.6. *There is a strict 2-category of chronological cobordisms n***ChCob** *with oriented manifolds of dimension $(n-1)$ as objects, equivalence classes of chronological cobordisms as morphisms, and homotopy classes of changes of chronologies as 2-morphisms. The composition of morphism is induced by gluing, and the two compositions of 2-morphisms are given as juxtaposition (the horizontal one) and concatenation (the vertical one).*

Remark 3.2.7. For a chronological cobordism W the set of critical points $crit(W)$ is linearly ordered by the chronology: we write $x<y$ if $\tau(x)<\tau(y)$. This order is invariant under equivalence of cobordisms, but it is affected by changes of chronologies.

One of the important operations on cobordisms is the *disjoint union*. For chronological cobordisms it has to be defined carefully: with the naive definition one might get two critical points at the same level, what is prohibited. Instead, we have to shift critical points of the left or the right cobordism below critical points of the other one, obtaining a 'left-then-right' and a 'right-then-left' disjoint unions, denoted by $\overleftarrow{\sqcup}$ or $\overrightarrow{\sqcup}$ respectively (see Fig. 3.6). Formally, we equip $(W,\tau)\overleftarrow{\sqcup}(W',\tau')$ with a chronology

$$\tau_r(p) := \begin{cases} \beta^1_{1/2}(\tau(p)), & p \in W, \\ \beta^{1/2}_0(\tau'(p)), & p \in W', \end{cases} \tag{3.14}$$

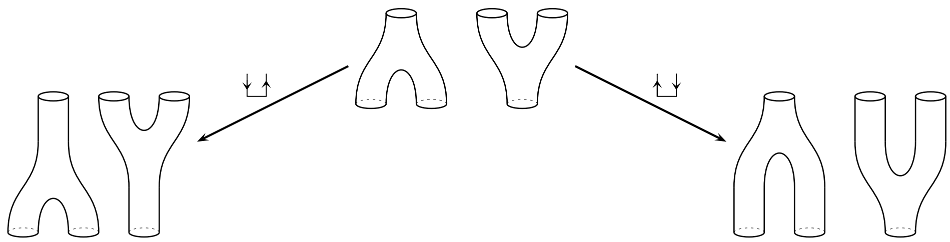
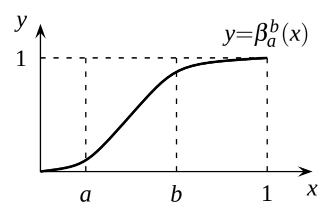


Figure 3.6: A disjoint union in a chronological setting requires a shift.

where $\beta_a^b\colon I \longrightarrow I$ is a perturbed ‘bump function’: an increasing function which is very close to 0 on the interval $[0,a]$ and very close to 1 on $[b,1]$. The chronology τ_ℓ on $(W,\tau)\overset{\downarrow\uparrow}{\sqcup}(W',\tau')$ is defined in a similar way. Finally, the formula (3.14) can be naturally extended to changes of chronologies—replace p with a pair (p,t).



This is the first place where we can see that chronological cobordisms indeed require a richer structure than just a category: the disjoint unions defined above are functorial only up to a change of a chronology $\sigma^{\sqcup}_{W,W'}\colon (W\overset{\downarrow\uparrow}{\sqcup}W',\tau_r) \Longrightarrow (W\overset{\uparrow\downarrow}{\sqcup}W',\tau_\ell)$ that pulls W below W'. This can be done by a linear interpolation of chronologies, $\sigma^{\sqcup}_{W,W'}(p,t) := (1-t)\tau_\ell(p)+t\tau_r(p)$.

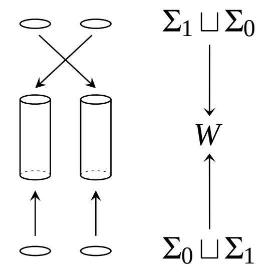
Theorem 3.2.8. *The 2-category n***ChCob** *is Gray monoidal. The monoidal product is induced by the ‘right-then-left’ disjoint union* $\overset{\uparrow\downarrow}{\sqcup}$ *and the unit is given by the empty manifold* $\emptyset$.

Proof. We have to check conditions from Definition 2.2.5. First, $\overset{\uparrow\downarrow}{\sqcup}$ is cubical. Indeed, the conditions from Definition 2.2.4 are trivially satisfied, as $\sigma^{\sqcup}_{W,W'}\colon W\overset{\downarrow\uparrow}{\sqcup}W' \Longrightarrow W\overset{\uparrow\downarrow}{\sqcup}W'$ does nothing if either W or W' has no critical points. Commutativity of the square (2.23) is given by a homotopy

$$h_s = \sigma^{\sqcup}_{W,W'}|_{[0,s]} \star \big((1-s)\alpha\overset{\downarrow\uparrow}{\sqcup}\beta + s\alpha\overset{\uparrow\downarrow}{\sqcup}\beta\big)\star\sigma^{\sqcup}_{W,W'}|_{[s,1]} \tag{3.15}$$

where $\sigma^{\sqcup}_{W,W'}|_{[a,b]}$ is a restriction of $\sigma^{\sqcup}_{W,W'}$ to $t\in[a,b]$. The homotopy h_s first shifts W and W' a bit towards their final position, then it applies the changes α and β on appropriate levels, and after that it shifts W and W' further to their final positions. Finally, commutativity of (2.24) follows easily: the two changes $\sigma^{\sqcup}\star(\sigma^{\sqcup}\circ\mathrm{id})$ and $\sigma^{\sqcup}\star(\mathrm{id}\circ\sigma^{\sqcup})$ are homotopic by a linear interpolation.

The unitarity condition is clear, and what remains is to check associativity. This follows directly from the way $\overrightarrow{\sqcup}$ is defined: the two chronologies on $W \overrightarrow{\sqcup} (W' \overrightarrow{\sqcup} W'')$ and $(W \overrightarrow{\sqcup} W') \overrightarrow{\sqcup} W''$ are homotopic by a reparametrization of the target interval I. $\square$



The ordinary category of cobordisms is not only monoidal, but it possesses a symmetry induced by a family of permutation diffeomorphisms $c\colon \Sigma_1 \sqcup \Sigma_0 \xrightarrow{\approx} \Sigma_0 \sqcup \Sigma_1$. Namely, take the cylinder $(\Sigma_0 \sqcup \Sigma_1) \times I$ with the standard inclusion as its input and the diffeomorphism c as its output (see the picture to the side). In case of chronological cobordisms, these permutation cylinders form natural transformations between unary functors $C \overrightarrow{\sqcup} (\_)$ and $(\_) \overrightarrow{\sqcup} C$, where C stands for any cylinder. This suggests the permutation cylinders equip n**ChCob** with a strict symmetry, see Definition 2.2.7. Indeed, commutativity of the triangle (2.25) follows easily from this construction.

Corollary 3.2.9. *The Gray monoidal category* $(n\mathbf{ChCob}, \overrightarrow{\sqcup}, \emptyset)$ *has a strict symmetry induced by permutation diffeomorphisms* $c\colon \Sigma_1 \sqcup \Sigma_0 \xrightarrow{\approx} \Sigma_0 \sqcup \Sigma_1$.

There is another operation on chronological cobordisms similar to the disjoint unions—the connected sum. Given chronological cobordisms W and W' remove vertical cylinders from them (verticality means the chronologies have no critical points on their small neighborhoods) and construct $W \# W'$ by identifying the cobordisms along the newly created boundary. Likewise for the disjoint unions, there are two connected sums of chronological cobordisms W and W', the 'left-then-right' $W \overrightarrow{\#} W'$ and the 'right-then-left' one $W \overleftarrow{\#} W'$, related by a change of a chronology $\sigma^{\#}_{W,W'}\colon W \overrightarrow{\#} W' \Longrightarrow W' \overleftarrow{\#} W$ that permutes the critical points.

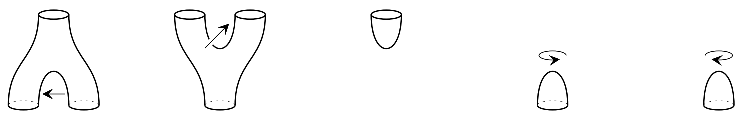
Let $n\mathbf{ChCob}_\circ$ be the category of nonempty manifolds with two distinguished points, and cobordisms between them, decorated with two vertical lines connecting the basepoints of the boundary manifolds. Then the connected sums are well-defined (choose small neighborhoods of the distinguished lines), and we have the following analog of Theorem 3.2.8.

Corollary 3.2.10. *The 2-category* $n\mathbf{ChCob}_\circ$ *is Gray monoidal. The product is induced by the 'right-then-left' connected sum* $\overleftarrow{\#}$ *and the unit is given by the* $(n-1)$*-dimensional ball.*

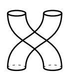
3.2.2 Dimension 2

We shall now focus on chronological cobordisms of dimension two. We start with a description of its low-level structure.

Proposition 3.2.11. 2**ChCob** *is a symmetric Gray monoidal category with objects freely generated by a circle* $\mathbb{S}^1$ *and morphisms freely generated by the following five cobordisms:*



a merge | a split | a birth | a positive death | a negative death (3.16)

with a twist  *acting as a strict symmetry.*

One should read the pictures above from bottom to top: the bottom circles form the input of a cobordism, the top ones form the output and the height function determines a chronology. Orientations of critical points are visualized by arrows.

Proof. Every 1-dimensional manifold is a family of circles, so that objects of 2**ChCob** are freely generated under the disjoint union by a single circle $\mathbb{S}^1$. Since all orientation preserving diffeomorphisms of $\mathbb{S}^1$ are isotopic to the identity, chronological cobordisms with no critical points are generated by a permutation of two circles, the symmetry of the monoidal structure. Morse theory provides a description of cobordisms with a single critical point. Since the order of critical points is fixed, it remains to analyze orientations of the critical points.

We need only one merge and one split—an orientation of the saddle point can be reversed by attaching a twist. The tangent space to a point of index 0 (a birth) is stable, so that there is only one choice for orientation (the empty frame), but it is unstable at points of index 2 (deaths). Hence, a choice of an orientation of a death is equivalent to an orientation of the tangent space, which can be either coherent with the orientation of the cobordism or not. ☐

We shall use Cerf theory (see Section 3.1) to describe 2-morphisms in terms of generators and relations. Most of them are easy to draw directly, but for some it will be useful to use other presentations. We shall describe now two of them—*movies*, and *surgery diagrams*.

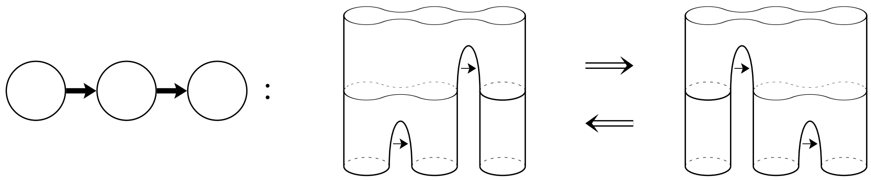
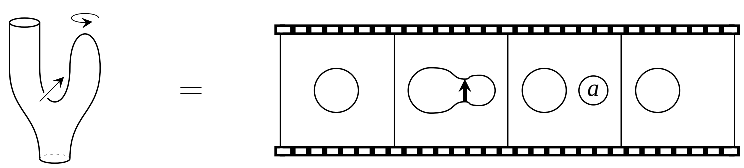


Figure 3.7: A two-arrow surgery diagram encodes a permutation of two saddle points.

A *movie presentation* of a chronological cobordism is a sequence of its regular levels, dense enough to capture all topological changes: such a sequence contains at least one regular level between any two critical ones. Two consecutive diagrams in the sequence differ in one of the following ways:

- they are isotopic, so that there is no critical level in between,
- one diagram is obtained form the other by a saddle move $\asymp \longrightarrow)($; this corresponds to a merge or a split,
- a one circle component appear (for a birth) or disappear (for a death).

We can add additional information to encode orientations of the critical points: an oriented chord for a saddle move, or a/c for a death oriented anti- or clockwise. We provide below one example.

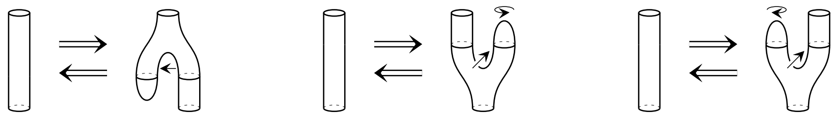


(3.17)

Movie presentations are a good way to visualize cobordisms. However, if a cobordism (W, τ) has only saddle points, a more compact description is given by its *surgery diagram*: a collection of circle with enumerated oriented chords between them. The circles illustrate the input of the cobordism W, whereas the chords represent 1-handles in the handle decomposition of W with respect to the chronology τ. Performing surgeries along the chords in the specified order results in a movie presentation of W. However, we can get more: a diagram with two chords encodes two chronological cobordisms, depending on the order of the chords, and a change that permutes the two points, see Fig. 3.7.

Proposition 3.2.12. *Changes of chronologies are generated under compositions and disjoint union by the following:*

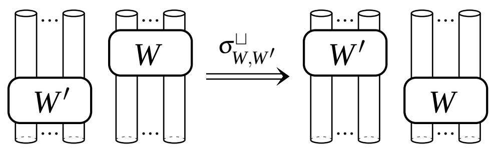
1) creation *and* annihilation *changes*



$$(3.18)$$

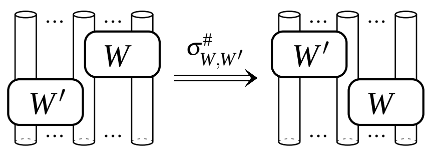
in which the orientations of deaths are determined by the monotonicity condition for $d^3\tau$ at an A_2 singularity (take the arrow at the saddle and rotate it towards the vertical cylinder),

2) *the* disjoint sum permutations



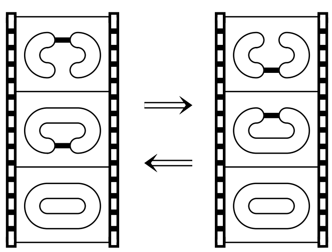
$$(3.19)$$

3) *the* connected sum permutations

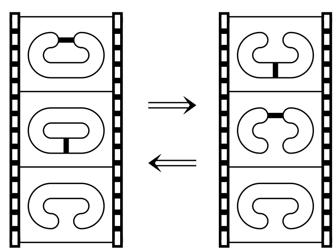


$$(3.20)$$

4) *the* exceptional permutation *changes, represented by the following movies*



and



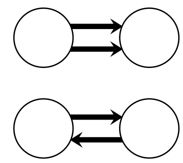
$$(3.21)$$

to which we refer respectively as a $\times$-change and a $\Diamond$-change, because of the shapes of cobordisms involved.

Proof. According to Cerf theory there are two types of changes:

- those generated by A_2-singularies, i.e. creation and annihilation changes, and
- those induced by homotopies H_t, such that H_{t_0} has two critical points at one level for some t_0.

In the latter case, we refer to the critical level of H_{t_0} as the *singular section* of H_t. Consider its components carrying the critical points—it is a four-valent graph Γ_H. Consider the connectivity of the graph[3]: the homotopy H represents a disjoint union permutation if ΓH has two components, a connected sum permutation if Γ_H is 2-connected, or one of the exceptional changes if Γ_H is 4-connected. □



Remark 3.2.13. There are two versions of the $\times$-change: its surgery diagram consists of two circles and two chords between them, which can either point to the same or to different circles. The two changes encodes by such diagrams are not equivalent, and up to a diffeomorphism each of them is its own inverse.

On the other hand, reversing one chord in a surgery diagram of a $\Diamond$-change results in the inverse change. Indeed, the only topological information we have is the order of chords induced by the arc connecting their heads (there is a natural orientation of the circle in the surgery diagram induced from the orientation of the underlying cobordism). This order may or may not coincide with the order of critical points, induced by the initial chronology, and the two cases lead to inverse permutation changes.

We shall now proceed to a description of *relations* between the generators of the set of 2-morphisms. These are given by homotopies of paths in the space of Igusa functions listed in Section 3.1. As before, not all of them can be easily drawn, especially the homotopies relating the two ways of switching levels of three critical points. We shall encode them with three-chord surgery diagrams—such a diagram represent six cobordisms, depending on the order of critical points, call them a, b, c, and six permutation changes between these cobordisms forming a hexagonal diagram

$$\begin{array}{ccccc} & W(b<a<c) & \Longrightarrow & W(b<c<a) & \\ \nearrow\!\!\!\nearrow & & & & \searrow\!\!\!\searrow \\ W(a<b<c) & & & & W(c<b<a) \\ \searrow\!\!\!\searrow & & & & \nearrow\!\!\!\nearrow \\ & W(a<c<b) & \Longrightarrow & W(c<a<b) & \end{array} \tag{3.22}$$

[3] A graph Γ is n-connected if at least n edges must be removed to split it into two components.

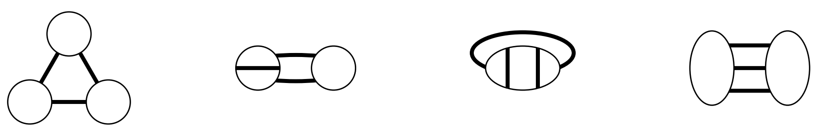
The notation $W(x{<}y{<}z)$ is used for the cobordisms with the point x at the lowest critical level, y in the middle, and z at the highest one. The relation imposed by the homotopy makes this hexagon commute.

Proposition 3.2.14. *The following is the complete set of relations among the generating changes of chronologies listed in Proposition 3.2.12:*

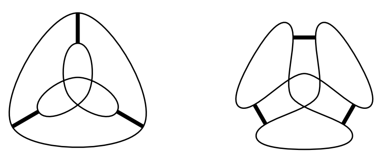
1) *the squares below commute for any cobordisms W, W', W'', and a 2-morphism α:*

$$\begin{array}{ccc} W \,{\downarrow}{\sqcup}{\uparrow}\, W' & \xrightarrow{\ \mathrm{id}\sqcup\alpha\ } & W \,{\downarrow}{\sqcup}{\uparrow}\, W'' \\ {\scriptstyle \sigma^{\sqcup}_{W,W'}}\Big\Downarrow & & \Big\Downarrow{\scriptstyle \sigma^{\sqcup}_{W,W''}} \\ W \,{\uparrow}{\sqcup}{\downarrow}\, W' & \xrightarrow{\ \mathrm{id}\sqcup\alpha\ } & W \,{\uparrow}{\sqcup}{\downarrow}\, W'' \end{array} \qquad \begin{array}{ccc} W \,\#^{\downarrow\uparrow}\, W' & \xrightarrow{\ \mathrm{id}\#\alpha\ } & W \,\#^{\downarrow\uparrow}\, W'' \\ {\scriptstyle \sigma^{\#}_{W,W'}}\Big\Downarrow & & \Big\Downarrow{\scriptstyle \sigma^{\#}_{W,W''}} \\ W \,\#^{\uparrow\downarrow}\, W' & \xrightarrow{\ \mathrm{id}\#\alpha\ } & W \,\#^{\uparrow\downarrow}\, W'' \end{array} \tag{3.23}$$

2) *hexagons encoded by the following surgery diagrams commute:*



(3.24)



(3.25)

where the crossings in the last two diagrams are the artifacts of projecting the diagrams to the plane (singular levels of the corresponding homotopies are not planar).

Proof. We shall analyze the three groups of homotopies from Section 3.1 on page 24.

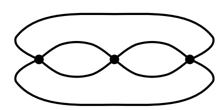
Group I: two changes occur simultaneously at different levels (3.3). This is the exchange law for 2-morphisms, so that this group does not introduce new relations.

Group II: nontransverse changes (3.4). These imply a change followed by its inverse is equivalent to the trivial one. Again, no interesting relations.

Group III: several critical points at the same level (3.5). This group introduces interesting relations between generating 2-morphisms. A homotopy $H_{s,t}$ from this group admits a *singular level*: the critical level of some H_{s_0,t_0} containing all the critical points (either three Morse singularities, or one Morse and one birth-death point). Denote by Γ_H the components of the singular level carrying the singularities; it is a graph with two types of vertices: 4-valent ones for Morse singularies, and 2-valent to birth-death singularities.

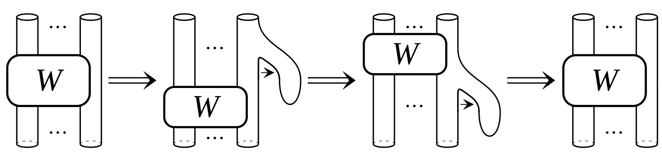
If the graph Γ_H is disconnected, it must have a component with a single 4-valent vertex. In such a case the homotopy H makes the left square in (3.23) commute, where the cobordism W contains the component of Γ_H with a single 4-vertex, and α is a change encoded by the other components (a creation or annihilation if the component contains one 2-valent vertex, or a permutation otherwise).

If Γ_H is 2-connected, break its two edges to obtain two components. The reverse operation is the connected sum—this shows a homotopy with such a graph makes the right square in (3.23) commute.



Finally, Γ_H can be 4-connected, which requires three 4-valent vertices. There is only one such graph, shown to the right. Take a look on a regular level of H_{s_0,t_0} just below the singular one—it is a collection of circles obtained from Γ_H by replacing a neighborhood of each vertex with two arcs. Join the arcs with a chord to obtain a three-chord surgery diagram for H. All such diagrams are listed in lines (3.24) and (3.25). □

Corollary 3.2.15. *The following sequence of homotopies represent a trivial change:*



(3.26)

for any cobordism W, not necessarily connected, and likewise for a split and a death.

Proof. Choose a creation change as α in the right squares in (3.23). Going around the square results in (3.26). □

3.2.3 Cobordisms embedded in $\mathbb{R}^2 \times I$

In the view of the construction of odd Khovanov homology it is unfortunate to have only one $\Diamond$-change up to inverse. One solution to this issue is to use cobordisms embedded in $\mathbb{R}^2 \times I$.

Definition 3.2.16. Define the 2-category $\mathbf{ChCob}^e$ as follows.

1) Objects are families of disjoint circles in a plane $\mathbb{R}^2$.

2) A morphism is a compact surface $W \subset \mathbb{R}^2 \times I$, such that the restriction $pr|_W$ of the projection $pr\colon \mathbb{R}^2 \times I \longrightarrow I$ to W is a separative Morse function. We call it a *chronology* on W and, as before, we orient critical points of $pr|_W$. Moreover, we assume that W is transverse to $\mathbb{R}^2 \times \partial I$ and that ∂W consists of two parts: the input $W \cap (\mathbb{R}^2 \times \{0\})$ and the output $W \cap (\mathbb{R}^2 \times \{1\})$ of W.

3) Finally, a 2-morphism is an *admissible* diffeotopy $\varphi\colon (\mathbb{R}^2 \times I) \times I \longrightarrow \mathbb{R}^2 \times I$, i.e. one that fixes boundary points, has compact support, and at every moment $t \in I$ the restriction $pr|_{\varphi_t(W)}$ is an Igusa function.

We call $\mathbf{ChCob}^e$ the 2-category of *embedded chronological cobordisms*.

Remark 3.2.17. We shall refer to orientations of deaths as *clockwise* or *anticlockwise* by comparing them with the standard orientation of $\mathbb{R}^2 \times \{t\} \subset \mathbb{R}^2 \times I$.

We shall identify cobordisms related by diffeotopies φ_t for which $pr|_{\varphi_t(W)}$ is separative Morse at every moment $t \in I$. In particular, this holds for the following families of deformations:

- *level-preserving* diffeotopies: $pr \circ \varphi_t = pr$ for every $t \in I$,
- *vertical* diffeotopies: $\varphi_t(p,z) = (p, h_t(z))$ for some diffeotopy h_t of the interval I.

Another important family consists of locally vertical diffeotopies—they are vertical only over a collection of disks, while constant beyond them.

Definition 3.2.18. Choose a family of disjoint unnested vertical tubes $C_1, \dots, C_r$ in $\mathbb{R}^2 \times I$ and an embedded chronological cobordism W that is vertical in their annular neighborhoods. A diffeotopy φ_t is *locally vertical* if it is vertical inside each tube C_i, but fixes all points outside them, except the small annular neighborhoods, in which we interpolate the two behaviors.

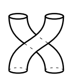
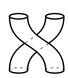
The requirement that W intersects each C_i in vertical lines implies that φ_t cannot create critical points. Moreover, each interpolation $(1-t)\varphi_1 + t\,\mathrm{id}$ induces a chronology on W, so that locally vertical diffeotopies can be 'straightened up' (compare this with Theorem 3.1.3).

Proposition 3.2.19. *Let φ_t and φ'_t be diffeotopies locally vertical with respect to the same family of tubes. If $\varphi_1 = \varphi'_1$, then they are homotopic in the space of admissible diffeotopies. In particular, a locally vertical diffeotopy φ_t satisfying $\varphi_1 = \mathrm{id}$ is trivial.*

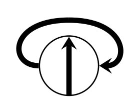
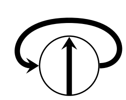
Proof. Take a linear homotopy $h_{t,s} := s\varphi_t + (1-s)\varphi'_t$. Because both φ_t and φ'_t are locally vertical, each $h_{t,s}$ is a diffeomorphism of $\mathbb{R}^2 \times I$ such that the restriction to $pr|_{h_{t,s}(W)}$ is a Morse function. $\square$

Given embedded cobordisms W and W' we define the 'left-then-right' and 'right-then-left' *disjoint unions* $W \overset{\downarrow\uparrow}{\sqcup} W'$ and $W \overset{\uparrow\downarrow}{\sqcup} W'$ by placing the cobordisms next to each other and pushing the critical points of W below or above those of W' respectively. The disjoint union permutation $\sigma_{W,W'}\colon W \overset{\downarrow\uparrow}{\sqcup} W' \Longrightarrow W \overset{\uparrow\downarrow}{\sqcup} W'$ is realized as a locally vertical diffeotopy, so that it equips $\overset{\uparrow\downarrow}{\sqcup}$ with a structure of a cubical functor.

Corollary 3.2.20. **ChCob**^{e} *is a Gray monoidal category, with a monoidal structure given by the right disjoint union* $\overset{\uparrow\downarrow}{\sqcup}$

Remark 3.2.21. This monoidal structure is strictly braided (see Definition 2.2.7) with a braiding induced by twists and . We will not use this fact in our paper.

The connected sum $W \# W'$ is formed from $W \overset{\uparrow\downarrow}{\sqcup} W'$ by performing a surgery along a vertical curtain in $\mathbb{R}^2 \times I$ with one edge on W and the other on W'. Again, there is some choice involved, and to make it a well defined operation one has to decorate objects and morphisms of **ChCob**^{e} with additional data, such as embedded arcs originating at the circles that go towards infinity.

The 2-category **ChCob**^{e} has a much richer structure than the one of abstract cobordisms. For instance, there are two kinds of merges, depending whether the input circles are nested or not, and likewise for splits. We shall usually ignore this additional structure except one case: we split $\Diamond$-changes into two groups using the intersection number of the two arrows in their surgery description (the two-arrow diagrams).

In other words, rotate the diagram to make the inner arrow points upwards, and check the direction of the outer one—it points either to the left or to the right as shown in the diagrams to the right, and the two changes encoded by the diagrams are not equivalent.

3.3 Chronological TQFTs

As shown in Example 2.2.2, the category $\mathbf{Mod}_R$ of G-graded R-modules admits a 2-categorical structure with 2-morphisms given by scaling. We define a *chronological TQFT* as a strict 2-functor $\mathscr{F}\colon n\mathbf{ChCob} \longrightarrow \mathbf{Mod}_R$ that intertwines the disjoint union with the graded tensor product. In particular, $\mathscr{F}$ assigns invertible elements from R to changes of chronologies, and it is worth examining this assignment separately. In this section we construct a *linearization* $R\mathbf{ChCob}^e$ of the 2-category of two-dimensional embedded chronological cobordisms. It is an ordinary category, which allows us to consider the chronological TQFTs as ordinary functors $\mathscr{F}\colon R\mathbf{ChCob}^e \longrightarrow \mathbf{Mod}_R$.

3.3.1 Linearization of cobordisms

Choose a function $\iota\colon 2\mathrm{Mor}(\mathbf{ChCob}^e) \longrightarrow R$, where R is a commutative ring, that is multiplicative with respect to both compositions of 2-morphisms and define a category $R\mathbf{ChCob}^e_\iota$ as follows:

1) the set of objects is not changed and it consists of families of circles in the plane $\mathbb{R}^2$, and
2) morphisms are finite linear combinations of chronological cobordisms $r_1W_1 + \ldots + r_kW_k$ with $r_i \in R$, modulo *chronological relations* $W' = \iota(\varphi)W$, one per every 2-morphism $\varphi\colon W \Longrightarrow W'$.

We extend the composition of cobordisms to formal sums in a linear way. The function ι can be considered as a part of a 2-functor $\mathbf{ChCob}^e \longrightarrow R\mathbf{ChCob}^e_\iota$, where 2-morphisms in the target category are scalings by elements of the ring R. We want this functor to be ‘faithful enough’ to support the construction of odd Khovanov homology. We start with a few observations.

Lemma 3.3.1. *For any function ι as above there is another one, $\hat{\iota}$, which assigns* 1 *to creations and annihilations, such that the linearizations $R\mathbf{ChCob}^e_\iota$ and $R\mathbf{ChCob}^e_{\hat{\iota}}$ are isomorphic.*

Proof. Each of the three creations (3.18) involve different generators. Hence, we can force the coefficients associated to them to be 1 by scaling births and deaths accordingly. ☐

Lemma 3.3.2. *We have $\iota(\sigma^{\sqcup}_{W,W'}) = \iota(\sigma^{\#}_{W,W'})$, whenever each of W and W' is a merge of a split.*

Proof. It follows from the right square in (3.23) for the cobordism W and the connected sum permutation $\alpha = \sigma^{\#}\colon M \# W' \Longrightarrow M \# W'$, where M is a merge. Indeed, commutativity of the square implies

$$\iota(\sigma^{\#}_{W,M})\iota(\sigma^{\sqcup}_{W,W'})\iota(\sigma^{\#}_{M,W'}) = \iota(\sigma^{\#}_{M,W'})\iota(\sigma^{\#}_{W,W'})\iota(\sigma^{\#}_{W,M}) \tag{3.27}$$

so that the middle terms must be equal. ☐

As a result, we have to specify ι only for disconnected union permutations and exceptional changes. Instead of finding the most general formula, and keeping in mind we want to regard embedded cobordisms as close to the abstract ones as possible, we shall define $\iota(\sigma^{\sqcup}_{W,W'})$ using the following *bidegree* $\deg W \in \mathbb{Z}\times\mathbb{Z}$, which counts critical points of the cobordism W as follows:

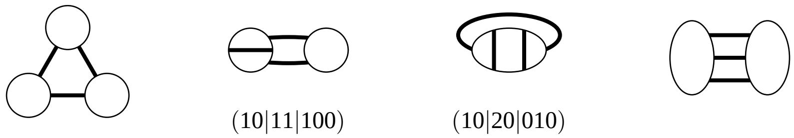
$$\deg W := (\#\text{births} - \#\text{merges}, \#\text{deaths} - \#\text{splits}). \tag{3.28}$$

The following result shows a connection between this bidegree with other topological properties of a cobordism.

Lemma 3.3.3. *Given a chronological cobordism W of degree $\deg W = (a,b)$ with n inputs and m outputs, $a+b = \chi(W)$ and $a+n = b+m$.*

Proof. Straightforward, by checking for generating cobordisms (3.16). ☐

It follows the bidegree is preserved by changes of chronologies, so that $R\mathbf{ChCob}^e$ is a graded category (after introducing formal degree shifts as in Remark 2.1.4). Our choice of ι is determined by the requirement that the disjoint union $\sqcup$ is a graded monoidal product in $R\mathbf{ChCob}^e$.



	$(10\vert 11\vert 100)$	$(10\vert 20\vert 010)$	
$(03\vert 00\vert 300)$	$(01\vert 20\vert 100)$	$(10\vert 02\vert 010)$	$(30\vert 00\vert 030)$
$(21\vert 00\vert 300)$	$(01\vert 02\vert 100)$	$(01\vert 11\vert 010)$	$(11\vert 00\vert 030)$

Table 3.1: Surgery diagrams of homotopies relating permutation changes. The numbers below each diagram count how many times various permutations occur: $\times$-changes with parallel or opposite arrows (the first group), $\Diamond$-changes with outer arrows oriented to the left or to the right (the second group) and the other changes grouped by the value of ι (respectively X, Y and Z). Different sequences correspond to different orientations of chords.

Proposition 3.3.4. *Choose invertible elements $X, Y, Z \in R$ such that $X^2 = Y^2 = 1$ and define ι on generating changes of chronologies by the following rules:*

1) *creations and annihilations are sent to equalities,*
2) *the coefficient associated to a disjoint union and connected sum permutation involving critical points of degrees (a,b) and (c,d) is given by $\lambda(a,b,c,d) = X^{ac}Y^{bd}Z^{ad-bc}$,*
3) *a $\times$-change is sent to Y, if the arrows point to the same circle, and to X otherwise, and*
4) *a $\Diamond$-change with a diagram in which the inner arrow is oriented upwards is sent to 1 or XY depending on whether the outer arrow is oriented to the left or to the right respectively.*

Then $\iota\colon \mathrm{2Mor}(\mathbf{ChCob}^e) \longrightarrow R$ is a well-defined multiplicative function.

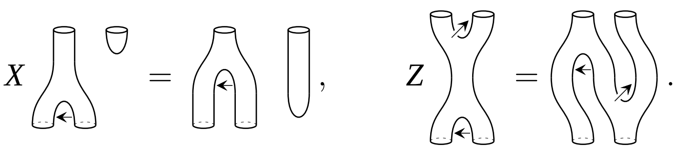
Proof. First, coherence of ι with the interchange law for 2-morphisms (2.22) follows from commutativity of R. Next, $\iota(\alpha)\iota(\alpha^{-1}) = 1$ for every elementary change α: this is trivial for creations and annihilations, and follows easily for disjoint union and connected sum permutations from the way λ is defined. If α is an exceptional permutations, $\iota(\alpha^{-1}) = \iota(\alpha)$ is a square root of 1.

The relations (3.26) and (3.23) follows the way λ is defined—it is a group homomorphism in each variable. Finally, it remains to check the relations given by the four planar diagrams in (3.24). For that see Tab. 3.1: the numbers below each diagram indicate how many times a particular elementary change occurs when we go around the hexagon (3.22). In each case, the product of values of ι is equal to 1. ☐

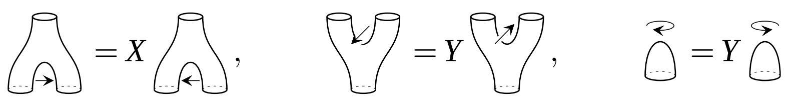
Corollary 3.3.5. *A choice of parameters $X, Y, Z \in R$ as above is equivalent to specifying a ring homomorphism $\Bbbk \longrightarrow R$, where $\Bbbk = \mathbb{Z}[X,Y,Z^{\pm 1}]/(X^2 = Y^2 = 1)$. Hence, there is a base change isomorphism R**ChCob**$^e \cong R \otimes \Bbbk$**ChCob**^{e} implying $\Bbbk$**ChCob**^{e} is the universal linearization of* **ChCob**^{e} *with respect to the function ι defined as in the proposition above.*

We gathered the values of ι for disjoint union and connected sum permutations in the table to the right, as we shall often use them. For instance, we have

$W \backslash W'$	birth	merge	split	death
birth	X	X	Z^{-1}	Z
merge	X	X	Z	Z^{-1}
split	Z	Z^{-1}	Y	Y
death	Z^{-1}	Z	Y	Y

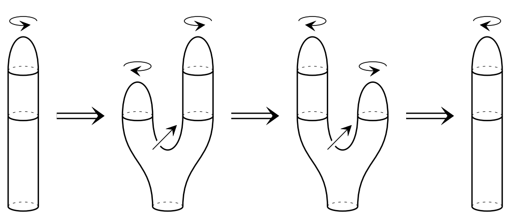


Corollary 3.3.6. *The following rules for reversing orientations hold:*



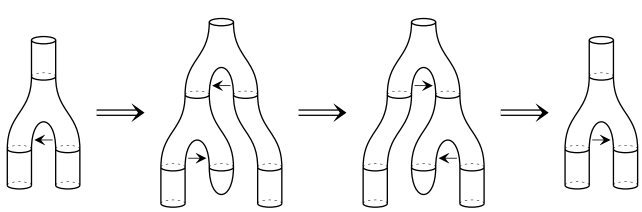
(3.29)

Proof. The last rule follows from the following change



(3.30)

and the first one from



(3.31)

Reversing an orientation of a split is done in a similar way. ☐

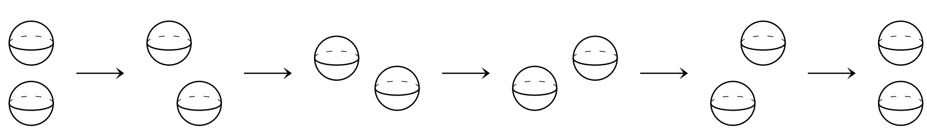
Remark 3.3.7. We shall usually omit the subscript, writing $R\mathbf{ChCob}^e$ for the linearized category. If the choice of ι is important, we shall write $R\mathbf{ChCob}^e_{abc}$ for the quotient by chronological relations with parameters X, Y, and Z set to a, b, and c accordingly.

Choose a change of a chronology $\varphi\colon W \Longrightarrow W'$ that is not a $\Diamond$-change. Despite φ being a diffeotopy of the ambient space, the value $\iota(\varphi)$ depends only the restriction of φ to the cobordism W, which is a change of a chronology in the abstract sense. Even more, given a diffeomorphic cobordism $\widetilde{W} \approx W$ and a corresponding change $\widetilde{\varphi}$ on $\widetilde{W}$, $\iota(\widetilde{\varphi}) = \iota(\varphi)$.

$\Diamond$-changes do not introduce essential relations in $R\mathbf{ChCob}^e$—they force a merge followed by a split to be annihilated by $(1-XY)$, a relation that is a consequence of the others, see Corollary 3.3.6. Hence, we can safely forget the ambient space and identify diffeomorphic cobordisms, obtaining another category, which we shall denote by $R\mathbf{ChCob}$. Formally, morphisms of $R\mathbf{ChCob}$ are R-linear combinations of diffeomorphism classes of chronological cobordisms modulo the relations induced by ι: we set $W' = \iota(\varphi)W$ for any embedding of W and W' into $\mathbb{R}^2 \times I$ and a diffeotopy $\varphi\colon W \Longrightarrow W'$.

Remark 3.3.8. One should not confuse $R\mathbf{ChCob}$ with a linearization of $2\mathbf{ChCob}$—in the latter one must have $X = Y$ not only because there is only one type, up to inverse, of a $\Diamond$-change, but this equality is also imposed by the additional relations coming from the non-planar diagrams (3.25). This is a reason why it is so difficult to extend the definition of odd Khovanov homology to virtual links, even if we restrict to those on orientable surfaces: the non-planar diagrams (3.25) encodes the cube of resolutions for the virtual Borromean rings, which are realized on a torus.

Because we identify in $R\mathbf{ChCob}$ diffeomorphic cobordisms, there exists cobordisms W such that $W = rW$ for some $r \in R$. Indeed, it is enough to find a nontrivial change of a chronology between cobordisms that are diffeomorphic, such as permuting two spheres:



(3.32)

Another example is a twice punctured torus—reverse orientations of both saddle points and then rotate the cobordism. We proof below that nothing more can happen in $\Bbbk$**ChCob**.

Theorem 3.3.9. *Let* $\Bbbk = \mathbb{Z}[X,Y,Z^{\pm 1}]/(X^2 = Y^2 = 1)$ *and choose an embedded chronological cobordism* W *in* $\Bbbk$**ChCob**. *Write* $\operatorname{Aut}(W) := \{k \in \Bbbk \mid kW = W\}$. *Then*

$$\operatorname{Aut}(W) = \begin{cases} \{1\}, & \text{if } W \text{ has genus } 0 \text{ and at most one closed component,} \\ \{1, XY\}, & \text{otherwise.} \end{cases} \tag{3.33}$$

Notice that elements of $\operatorname{Aut}(W)$ are invertible, since they are products of values of ι. A proof of this theorem is postponed to the end of this chapter.

3.3.2 Chronological Frobenius algebras

A chronological TQFT $\mathscr{F}\colon \Bbbk\mathbf{ChCob} \longrightarrow \mathbf{Mod}_R$ is determined by the pair $(\mathscr{F}(\emptyset), \mathscr{F}(\bigcirc))$ of two rings. With an analogy to ordinary TQFTs, we call this pair a chronological Frobenius system.

Definition 3.3.10. A *chronological Frobenius system* in the category $\mathbf{Mod}_R$ with a symmetric G-graded tensor product of type λ is a pair (S,A) of two R-modules such that S is a graded ring and A a symmetric S-bimodule, together with four homogeneous operations, a unit $\eta\colon S \longrightarrow A$, a counit $\varepsilon\colon A \longrightarrow S$, a multiplication $\mu\colon A \otimes_S A \longrightarrow A$, and a comultiplication $\Delta\colon A \longrightarrow A \otimes_S A$, subject to the following conditions:

$$\mu \circ (\mu \otimes \mathrm{id}) = \lambda(\deg\mu, \deg\mu)\mu \circ (\mathrm{id} \otimes \mu), \tag{3.34}$$

$$(\Delta \otimes \mathrm{id}) \circ \Delta = \lambda(\deg\Delta, \deg\Delta)(\mathrm{id} \otimes \Delta) \circ \Delta, \tag{3.35}$$

$$\mu \circ (\eta \otimes \mathrm{id}) = \mathrm{id}, \qquad (\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id}, \tag{3.36}$$

$$\mu \circ \sigma = \lambda(\deg\mu, \deg\mu)\mu, \qquad \sigma \circ \Delta = \lambda(\deg\Delta, \deg\Delta)\Delta, \tag{3.37}$$

$$(\mu \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta) = \lambda(\deg\mu, \deg\Delta)\Delta \circ \mu = (\mathrm{id} \otimes \mu) \circ (\Delta \otimes \mathrm{id}). \tag{3.38}$$

We call A a *chronological Frobenius algebra* over S.

Example 3.3.11. The case $G = \{1\}$ and $R = \mathbb{Z}$ with $X = Y = Z = 1$ recovers the usual notion of a Frobenius algebra. For instance, the Khovanov's Frobenius algebra is the ring $A := \mathbb{Z}[\alpha]/(\alpha^2)$ with a counit and a comultiplication defined as

$$\begin{cases} \varepsilon(1) = 0, \\ \varepsilon(\alpha) = 1, \end{cases} \qquad \begin{cases} \Delta(1) = \alpha \otimes 1 + 1 \otimes \alpha, \\ \Delta(\alpha) = \alpha \otimes \alpha. \end{cases} \tag{3.39}$$

Example 3.3.12. The choice $X = Z = 1$ and $Y = -1$ leads to superalgebras: $\mathbb{Z}_2$-graded abelian groups with the exterior product, i.e. $\lambda(a,b) = (-1)^{ab}$. For instance, in the ORS algebra [ORS13] $A = \bigwedge[a]$ is the exterior algebra on one generator in degree $1 \in \mathbb{Z}_2$; the tensor power $A^{\wedge s} \cong \bigwedge[a_1, \dots, a_s]$ has s generators, each in degree $1 \in \mathbb{Z}_2$. The product is induced by the quotient map $A^{\wedge 2} \longrightarrow A^{\wedge 2}/(a_1 - a_2) \cong A$, whereas the coproduct $A \cong A^{\wedge 2}/(a_1 - a_2) \longrightarrow A^{\wedge 2}$ sends $[w]$ into $(a_1 - a_2) \wedge w$. In the tensor notation $\Delta(1) = a \otimes 1 - 1 \otimes a$ and $\Delta(a) = a \otimes a$.

Example 3.3.13. The above examples can be generalized as follows. Take the ring $\Bbbk := \mathbb{Z}[X, Y, Z^{\pm 1}]/(X^2 = Y^2 = 1)$ as coefficients, $\mathbb{Z} \times \mathbb{Z}$ as the grading group, and the graded tensor product given by $\lambda(a,b,c,d) = X^{ac} Y^{bd} Z^{ad-bc}$. The algebra A is freely generated by v_+ and v_- in degrees $(1,0)$ and $(0,-1)$ respectively, and the operations are defines as follows

$$\mu\colon A \otimes A \longrightarrow A, \qquad \begin{cases} v_+ \otimes v_+ \longmapsto v_+, & v_- \otimes v_+ \longmapsto XZv_-, \\ v_+ \otimes v_- \longmapsto v_-, & v_- \otimes v_- \longmapsto 0, \end{cases} \tag{3.40}$$

$$\Delta\colon A \longrightarrow A \otimes A, \qquad \begin{cases} v_+ \longmapsto v_- \otimes v_+ + YZv_+ \otimes v_-, \\ v_- \longmapsto v_- \otimes v_-, \end{cases} \tag{3.41}$$

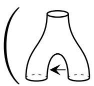
$$\eta\colon R \longrightarrow A, \qquad \left\{ 1 \longmapsto v_+, \right. \tag{3.42}$$

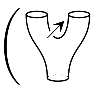
$$\varepsilon\colon A \longrightarrow R, \qquad \begin{cases} v_+ \longmapsto 0, \\ v_- \longmapsto 1. \end{cases} \tag{3.43}$$

Denote by $\mathbb{Z}_{ev}$ the ring of integers with the trivial $\Bbbk$-module structure, i.e. the generators X, Y, and Z act on $\mathbb{Z}$ as the identity, and $\mathbb{Z}_{odd}$ the case where Y acts as -1. Then $(\mathbb{Z}_{ev}, A \otimes \mathbb{Z}_{ev})$ is the Khovanov's Frobenius system, whereas $(\mathbb{Z}_{odd}, A \otimes \mathbb{Z}_{odd})$ is the ORS superalgebra from Example 3.3.12.

The conditions for a chronological Frobenius algebra reflect the chronological relations: (3.34), (3.35) and (3.38) are like the connected sum permutations changes, (3.36) mimics the creation and the annihilation changes, whereas (3.37) is the orientation reversion. Therefore, this is not a surprise that they give chronological TQFT functors.

Proposition 3.3.14. *Choose a chronological Frobenius system* (S, A) *in the category of* G*-graded modules* $\mathbf{Mod}_R$ *of type* λ_G*. Then there is a group homomorphism* $\psi\colon \mathbb{Z}\times\mathbb{Z} \longrightarrow G$*, a* $\Bbbk$*-algebra structure on* R*, and a* $\Bbbk$*-linear functor* $\mathscr{F}_A\colon \Bbbk\mathbf{ChCob} \longrightarrow \mathbf{Mod}_R$ *that sends a family of* s *circles to the tensor product* $A^{\otimes s}$ *and*

$\mathscr{F}_A$ () $= \big(\mu\colon A\otimes A \longrightarrow A\big),$ $\qquad$ $\mathscr{F}_A$ () $= \big(\eta\colon S \longrightarrow A\big),$ (3.44)

$\mathscr{F}_A$ () $= \big(\Delta\colon A \longrightarrow A\otimes A\big),$ $\qquad$ $\mathscr{F}_A$ () $= \big(\varepsilon\colon A \longrightarrow S\big).$ (3.45)

This functor is graded in the sense that $\deg\mathscr{F}(W) = \psi(\deg W)$ *for a cobordism* W*.*

Proof. The condition $\deg\mathscr{F}(W) = \psi(\deg W)$ requires $\psi(1,0) = \deg\eta$ and $\psi(0,1) = \deg\varepsilon$, while the ring homomorphism $\Bbbk \longrightarrow R$ is determined by λ_G as below:

$$X \longmapsto \lambda_G(\deg\mu, \deg\mu), \qquad Y \longmapsto \lambda_G(\deg\Delta, \deg\Delta), \qquad Z \longmapsto \lambda_G(\deg\mu, \deg\Delta).$$

It remains to check that $\mathscr{F}_A$ preserves the chronological relations. Most cases follow from (2.1) and conditions (3.34)–(3.38), with the exception of $\times$- and $\Diamond$-changes. The former follows from (3.37), as an $\times$-change adds a twist on one side of the cobordism. In the latter both cobordisms are equivalent, so it is enough to show that $1 - XY$ annihilates $\mu\circ\Delta$. This follows from (3.37): $\mu\circ\Delta = XY(\mu\circ\sigma)\circ(\sigma\circ\Delta) = XY\mu\circ\sigma^2\circ\Delta = XY\mu\circ\Delta$. ☐

We shall end this chapter with a proof of the nondegeneracy result for chronological cobordisms. For that we define a universal rank 2 Frobenus system, with scalars in a $\mathbb{Z}\times\mathbb{Z}$-graded commutative ring

$$R_U := {}^{\Bbbk[a,c,e,f,t,h]}\Big/\begin{pmatrix} (XY-1)h, (XY-1)t, af+ce, \\ ae+ceh+YZcft-1 \end{pmatrix} \tag{3.46}$$

where $\deg a = \deg e = (0,0)$, $\deg c = \deg f = (1,1)$, $\deg h = (-1,-1)$ and $\deg t = (-2,-2)$. The element $XY-1$ annihilates not only polynomials in h and t, but also c^2 and f^2 due to the graded commutativity, see Definition 2.1.3. Consider a rank two chronological Frobenius agebra A_U over R_U with the following operations:

$$\begin{cases} \mu(v_+ \otimes v_+) = v_+, & \mu(v_- \otimes v_+) = XZv_-, \\ \mu(v_+ \otimes v_-) = v_-, & \mu(v_- \otimes v_-) = hv_- + tv_+, \end{cases} \tag{3.47}$$

$$\begin{cases} \Delta(v_+) = (ft - YZ^{-1}eh)v_+ \otimes v_+ + e(v_- \otimes v_+ + YZv_+ \otimes v_-) + Z^2 fv_- \otimes v_-, \\ \Delta(v_-) = Z^{-2}etv_+ \otimes v_+ + ft(YZ^{-1}v_- \otimes v_+ + v_+ \otimes v_-) + (e+fh)v_- \otimes v_-, \end{cases} \tag{3.48}$$

$$\begin{cases} \eta(1) = v_+, \end{cases} \tag{3.49}$$

$$\begin{cases} \varepsilon(v_+) = c, \\ \varepsilon(v_-) = a. \end{cases} \tag{3.50}$$

It is a graded version of the system (R_4, A_4) in [Kh04] and it has the same universality property. The following proposition is proven in the same way as Proposition 4 in [Kh04].

Proposition 3.3.15. *Let (R', A') be a homogeneous chronological Frobenius system in $\mathbf{Mod}_{\Bbbk}$ of rank two. Then there is a unique graded ring homomorphism $R_U \longrightarrow R'$ such that $A' \cong A \otimes_{R_U} R'$.*

We are now ready to prove the nondegeneracy result for $\Bbbk\mathbf{ChCob}$.

Proof of Theorem 3.3.9. Given a chronological cobordism W we want to compute the group $\mathrm{Aut}(W) := \{k \in \Bbbk \mid kW = W\}$; its elements are invertible, as they are products of values of ι.

We shall first show that $\mathrm{Aut}(W)$ is a subgroup of $\{1, XY\}$. For that take a graded ring $R_1 = R_U/(X-Y, a, e, h) = \mathbb{Z}[X, Z^{\pm 1}, c, f, t]/(X^2 = XZcft = 1)$, and consider a chronological Frobenius system (R_1, A_1) with $A_1 = A_U \otimes R_1$. It has the following operations:

$$\begin{cases} \mu(v_+ \otimes v_+) = v_+, & \mu(v_- \otimes v_+) = XZv_-, \\ \mu(v_+ \otimes v_-) = v_-, & \mu(v_- \otimes v_-) = tv_+, \end{cases} \qquad \begin{cases} \eta(1) = v_+, \end{cases} \tag{3.51}$$

$$\begin{cases} \Delta(v_+) = ftv_+ \otimes v_+ + Z^2 fv_- \otimes v_-, \\ \Delta(v_-) = ftv_+ \otimes v_- + XZ^{-1}ftv_- \otimes v_+, \end{cases} \qquad \begin{cases} \varepsilon(v_+) = c, \\ \varepsilon(v_-) = 0. \end{cases} \tag{3.52}$$

In particular, $\mu(\Delta(v_+)) = (1+Z^2)ftv_+$. Since c, f, and t are invertible and polynomials in Z are not zero divisors, it follows $\mathscr{F}_1(W)$ is not a zero divisor for any closed surface W. This implies $\mathrm{Aut}(W)$ is a subgroup of $\{1, XY\}$. If $\partial W \neq \emptyset$, create a closed surface $\widehat{W}$ by capping its boundary components with births and deaths. Then $\mathrm{Aut}(W) \subset \mathrm{Aut}(\widehat{W})$, as every 2-morphism $\varphi\colon W \Longrightarrow W$ in $\mathbf{ChCob}^e$ extends to $\widehat{W}$ in a way that preserves the value of ι (juxtapose φ with the identity 2-morphisms on the caps).

Now assume W is a surface of genus 0 with at most one closed component. Choose the graded ring $R_2 := R_U/(c^2, a-1, e-1, h) \cong \Bbbk[c,t]/(c^2, (XY-1)t)$ and consider a chronological Frobenius system (R_2, A_2) with $A_2 = A_U \otimes R_2$. In particular, the unit and counit are given by formulas

$$\eta(1) = v_+, \qquad \varepsilon(v_+) = c, \qquad \varepsilon(v_-) = 1, \tag{3.53}$$

and a sphere is evaluated to c. Create $\widehat{W}$ by capping some inputs and outputs of W so that, up to a change of a chronology, $\widehat{W}$ is a disjoint union of caps and at most one spherical component. The homomorphism $\mathscr{F}_2(\widehat{W})\colon A^{\otimes k} \longrightarrow A^{\otimes \ell}$ takes $(v_-)^{\otimes k}$ to $(v_+)^{\otimes \ell}$ or $c(v_+)^{\otimes \ell}$, perhaps multiplied by a monomial in X, Y and Z. Since none of $r \in \Bbbk$ annihilates c, $(1-r)W = 0$ implies $r = 1$, which shows the group $\mathrm{Aut}(\widehat{W})$ is trivial. $\square$

Chapter 4

The generalized Khovanov complex

4.1 The construction of the Khovanov complex

We shall now give a detailed construction of the generalized Khovanov complex. For this section fix a link diagram D with n crossings, among which there are n_+ positive and n_- negative ones. We need to make a few choices: enumerate the crossings, and choose for each an arrow connecting the two arcs in the horizontal resolution, i.e. $\stackrel{\uparrow}{\asymp}$ or $\stackrel{\downarrow}{\asymp}$. Fig. 4.1 visualizes this construction for the trefoil knot.

4.1.1 The graded cube of resolutions

Most of the picture in Fig. 4.1 is occupied by resolutions of the trefoil diagram placed at vertices of a three-dimensional cube. For a general link diagram D with n crossings, take an n-dimensional cube and at each vertex $\xi = (\xi_1, \dots, \xi_n)$, $\xi_i \in \{0,1\}$, place the diagram D_ξ obtained from D by replacing each i-th crossing $\times$ with its horizontal $\asymp$, if $\xi_i = 0$, or vertical $)($, if $\xi_i = 1$, resolution.

Edges are encoded by sequences $\zeta = (\zeta_1, \dots, \zeta_n)$ with exactly one ζ_i being a star $*$. The star indicates direction of the edge: replacing it with 0 or 1 results in the source or the target vertex respectively. Choose an edge $\zeta\colon \xi \longrightarrow \xi'$ with $\zeta_i = *$ and let U be a small neighborhood

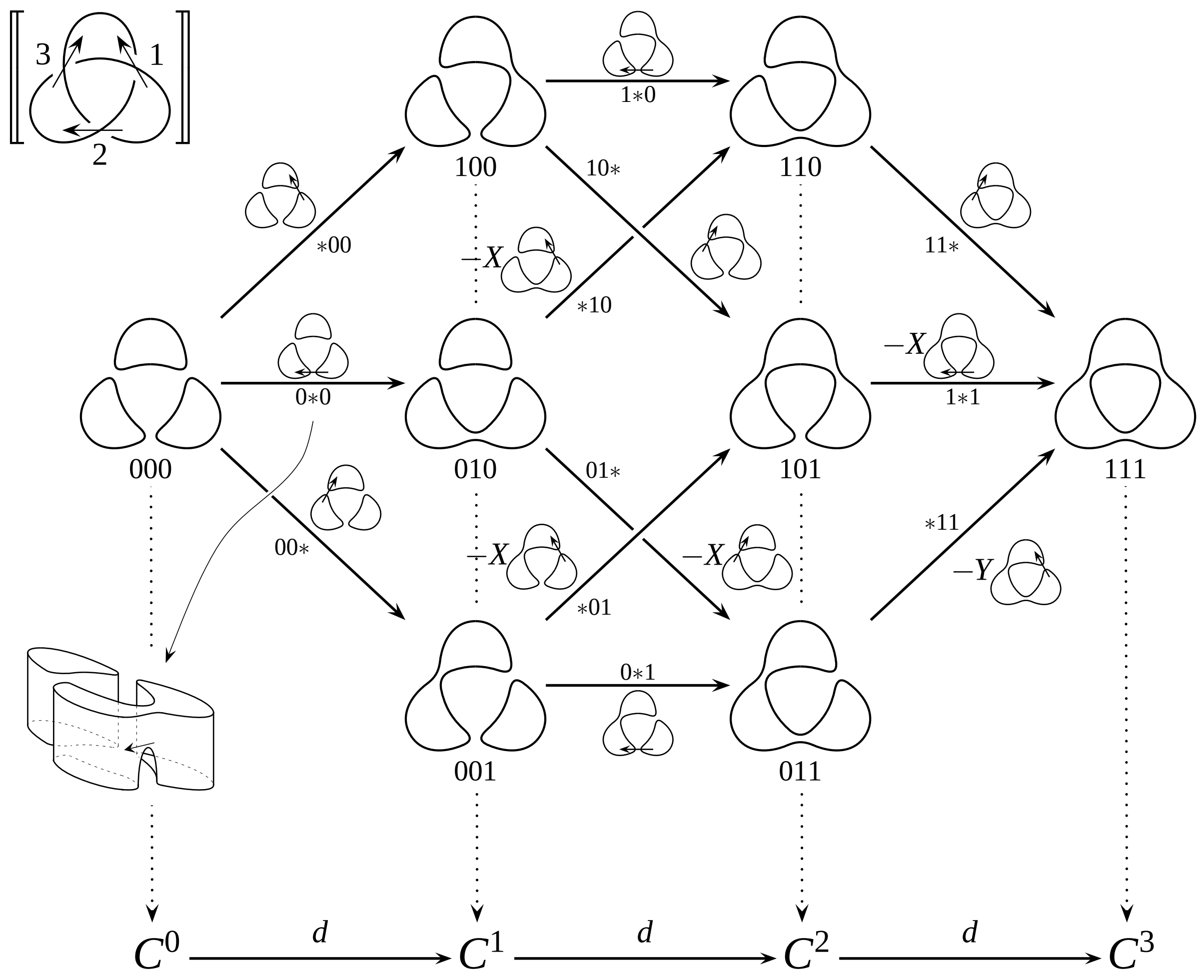
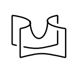
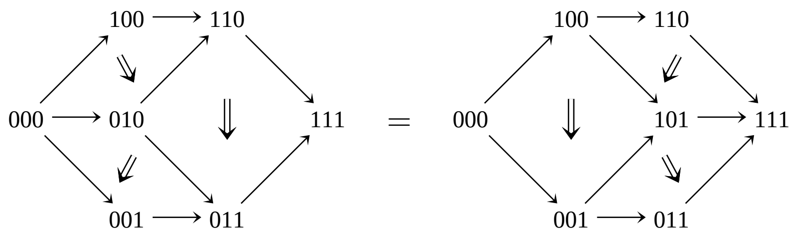


Figure 4.1: The Khovanov bracket for the trefoil.

of the i-th crossing. The edge ζ is decorated with a unique cobordism $D_\zeta \subset \mathbb{R}^2 \times I$ that has only one critical point—a cylinder $(D-U) \times I$ with a saddle  inserted over U. The small arrow over the crossing determines an *orientation* of the saddle point, so that we think of D_ζ as a chronological cobordism. In Fig. 4.1 the cobordisms are represented by their *surgery diagrams*, see Section 3.2.2. A 3D picture of the cobordism decorating the edge $\zeta = (0,*,0)$ is given in the left-bottom corner.

The above describe a diagram $\mathscr{I}(D)$ in the 2-category $\mathbf{ChCob}^e$: vertices are 1-manifolds (resolutions of the diagram D), edges are chronological cobordisms between these manifolds and faces are decorated with changes of chronologies. This diagram commutes in the 2-

categorical sense: a composition of 2-morphisms along any 3-dimensional subcube is trivial:



(4.1)

This follows from Proposition 3.2.19, as the two changes are locally vertical with respect to small tubes around the crossings of D. It follows that all cobordisms encoded by paths from the initial vertex to a given vertex ξ have the same number of merges and splits.

Lemma 4.1.1. *Denote by ℓ_ξ the number of circles in the vertex ξ of $\mathscr{I}_{\mathrm{gr}}(D)$. Then any cobordisms encoded by a directed path from the initial vertex to ξ has exactly $\frac{1}{2}(\|\xi\| + \ell_0 - \ell_\xi)$ merges and $\frac{1}{2}(\|\xi\| - \ell_0 + \ell_\xi)$ splits, where $\|\xi\| := \xi_1 + \cdots + \xi_n$ is the* weight *of the vertex ξ.*

Proof. The numbers ℓ_0 and ℓ_ξ count respectively the circles in the input and output of the cobordism encoded by a directed path from the initial vertex to ξ. Since $\|\xi\|$ counts both merges and splits, the thesis follows from Lemma 3.3.3. □

We refine the cube to the *graded cube of resolutions* $\mathscr{I}_{\mathrm{gr}}(D)$ by setting

$$\mathscr{I}_{\mathrm{gr}}(D)_\xi := D_\xi \left\{ \tfrac{\|\xi\|+\ell_0-\ell_\xi}{2}, \tfrac{\|\xi\|-\ell_0+\ell_\xi}{2} \right\}. \tag{4.2}$$

According to the lemma above, the edges of $\mathscr{I}_{\mathrm{gr}}(D)$ are decorated with graded morphisms.

Remark 4.1.2. Another choice for the degree shift of D_ξ is the pair $\left(\frac{\|\xi\|-\ell_\xi}{2}, \frac{\|\xi\|+\ell_\xi}{2}\right)$, which depends only on the vertex ξ. However, $\|\xi\| \pm \ell_\xi$ might not be even, in which case the quotients are not integer numbers. In the view of the formula $\lambda(a,b,c,d) = X^{ac}Y^{bd}Z^{ad-bc}$ introducing degree shifts by half-integers requires a choice of 4^{th} roots of parameters X, Y, and Z, so that we want to apply only integral degree shifts integral as long as there are morphisms that are not graded. In particular, we are interested in Lemma 2.1.7, in which case we want the degree shift of the object X be integral if $\deg f$ is nontrivial.

4.1.2 Sign assignments

Choose a commutative ring R and apply the function $\iota\colon 2\mathrm{Mor}(\mathbf{ChCob}^e) \longrightarrow R$ from Section 3.3.1 to faces of the cube $\mathscr{I}_{\mathrm{gr}}(D)$; the faces are now decorated with elements from R according to Tab. 4.1. They define a 2-cochain $\psi \in C^2(I^n;U(R))$, where $U(R)$ is the group of invertible elements in R. Here one must be careful with the two cases in the group under Z—the value of ψ is either Z or Z^{-1}, depending on the orientation of the face:

$$\psi\left(\begin{array}{ccccc} & & \bullet & & \\ & \overset{\text{merge}}{\nearrow} & & \overset{\text{split}}{\searrow} & \\ \bullet & & \circlearrowright & & \bullet \\ & \underset{\text{split}}{\searrow} & & \underset{\text{merge}}{\nearrow} & \\ & & \bullet & & \end{array}\right) = Z, \qquad \text{but} \qquad \psi\left(\begin{array}{ccccc} & & \bullet & & \\ & \overset{\text{split}}{\nearrow} & & \overset{\text{merge}}{\searrow} & \\ \bullet & & \circlearrowright & & \bullet \\ & \underset{\text{merge}}{\searrow} & & \underset{\text{split}}{\nearrow} & \\ & & \bullet & & \end{array}\right) = Z^{-1}. \tag{4.3}$$

We call ψ the *commutativity* 2-*cochain*.

We say $\varepsilon \in C^1(I^n;U(R))$ is a *sign assignment* if $d\varepsilon = -\psi$. This means the corrected cube $\mathscr{I}^{\varepsilon}_{\mathrm{gr}}(D)$ anticommutes, where $\mathscr{I}^{\varepsilon}_{\mathrm{gr}}(D)$ has the same vertices as $\mathscr{I}_{\mathrm{gr}}(D)$, but an edge ζ is decorated with $\mathscr{I}^{\varepsilon}_{\mathrm{gr}}(D)_\zeta := \varepsilon(\zeta)\cdot\mathscr{I}(D)_\zeta$. The existence of such a cochain follows easily.

Lemma 4.1.3. *The cochain ψ is a cocycle. Hence, $-\psi = d\varepsilon$ for some sign assignment ε.*

Proof. The 2-commutativity of faces (4.1) of any 3-dimensional subcube in $\mathscr{I}_{\mathrm{gr}}(D)$ implies that $d(-\psi) = d\psi = 1$. The existence of ε follows from the contractibility of I^n. $\square$

A sign assignment for a given cube is unique up to an isomorphism, where an isomorphisms of cubes $\eta\colon \mathscr{I} \longrightarrow \mathscr{I}'$ is a collection of invertible morphisms $\eta_\xi\colon \mathscr{I}_\xi \longrightarrow \mathscr{I}'_\xi$ such that the square commutes

$$\begin{array}{ccc} \mathscr{I}_\xi & \xrightarrow{\ \eta_\xi\ } & \mathscr{I}'_\xi \\ {\scriptstyle \mathscr{I}_\zeta}\big\downarrow & & \big\downarrow{\scriptstyle \mathscr{I}'_\zeta} \\ \mathscr{I}_{\xi'} & \xrightarrow{\ \eta_{\xi'}\ } & \mathscr{I}'_{\xi'} \end{array} \tag{4.4}$$

for every edge $\zeta\colon \xi \longrightarrow \xi'$.

Lemma 4.1.4. *Let ε and ε' be two sign assignments for $\mathscr{I}_{\mathrm{gr}}(D)$. Then the cubes $\mathscr{I}^{\varepsilon}_{\mathrm{gr}}(D)$ and $\mathscr{I}^{\varepsilon'}_{\mathrm{gr}}(D)$ are isomorphic.*

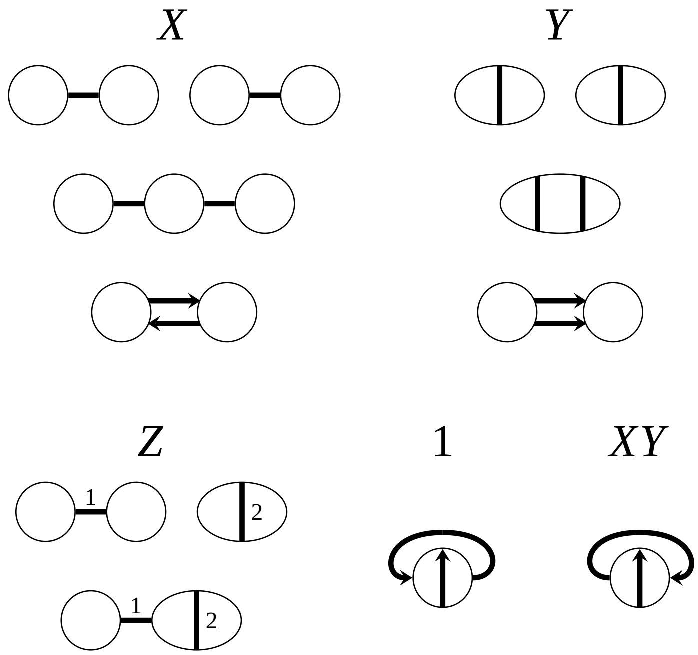


Table 4.1: Diagrams for faces that can appear in a cube of resolutions, grouped by values of the commutativity cochain ψ. All coefficients live in a commutative ring R, they are invertible, and $X^2 = Y^2 = 1$. Thin lines are the input circles and thick arrows visualize saddle points. Orientations of the arrows are omitted if ψ does not depend on them. The small numbers 1 and 2 in the two configurations placed under the letter Z indicate an initial order of critical points, see (4.3). For the opposite one, take Z^{-1}.

Proof. The equality $d\varepsilon = -\psi = d\varepsilon'$ and contractibility of I^n implies that $\varepsilon' = d\eta \cdot \varepsilon$ for some 0-cochain $\eta \in C^0(I^n; U(R))$. The family of morphisms $f_\xi := \eta(\xi) \cdot \mathrm{id}$ form then a desired isomorphism $f\colon \mathscr{I}^{\varepsilon}_{\mathrm{gr}}(D) \longrightarrow \mathscr{I}^{\varepsilon'}_{\mathrm{gr}}(D)$. $\square$

Finally, the isomorphism class of the cube $\mathscr{I}^{\varepsilon}_{\mathrm{gr}}(D)$ is independent of the direction of arrows at crossings of D—their only role is to keep a coherent choice of framing along the whole cube.

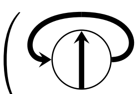
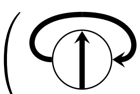
Lemma 4.1.5. *Let D_1, D_2 be diagrams of a link L with n crossings, which differ only in directions of arrows over crossings. Then for any sign assignment ε_1 for $\mathscr{I}_{\mathrm{gr}}(D_1)$ there exists a sign assignment ε_2 for $\mathscr{I}_{\mathrm{gr}}(D_2)$ such that $\mathscr{I}^{\varepsilon_1}_{\mathrm{gr}}(D_1) = \mathscr{I}^{\varepsilon_2}_{\mathrm{gr}}(D_2)$.*

Proof. Without loss of generality we may assume D_1 and D_2 differ only in the direction of the arrow at the i-th crossing. Reversing the arrow changes orientation of critical points of cobordisms at edges ζ with $\zeta_i = *$. Let ψ_i be the commutativity cocycle of the cube $\mathscr{I}_{\mathrm{gr}}(D_i)$. For a sign assignment ε_1 for $\mathscr{I}_{\mathrm{gr}}(D_1)$ we define

$$\varepsilon_2(\zeta) := \begin{cases} \varepsilon_1(\zeta), & \text{if } \zeta_i \neq *, \\ X\varepsilon_1(\zeta), & \text{if } \zeta_i = * \text{ and } D_\zeta \text{ is a merge,} \\ Y\varepsilon_1(\zeta), & \text{if } \zeta_i = * \text{ and } D_\zeta \text{ is a split.} \end{cases} \tag{4.5}$$

A direct computation shows $d\varepsilon_2 = -\psi_2$. Hence, ε_2 is the desired sign assignment for the cube $\mathscr{I}_{\mathrm{gr}}(D_2)$. ☐

There is one additional choice we made: we can exchange the values of ι on $\Diamond$-changes and still have a well-defined linearization R**ChCob**. The new commutativity cocycle $\overline{\psi}$ has the same values as ψ, except that

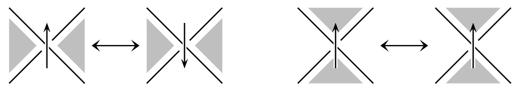
$\overline{\psi}\Big($  $\Big) = XY$ and $\overline{\psi}\Big($  $\Big) = 1.$ (4.6)

We shall now prove that the corrected cube of resolutions does not depend on which commutativity cocycle we choose. Unfortunately, there is a gap in the original proof from [ORS13], noticed by Cotton Seed: given a sign assignment ε with $d\varepsilon = \psi$ the authors of [ORS13] constructed $\overline{\varepsilon}$ with $d\overline{\varepsilon} = \overline{\psi}$, but an isomorphism of cubes $\mathscr{I}^{\varepsilon}(D) \cong \mathscr{I}^{\overline{\varepsilon}}(D)$ is missing. We found such an isoromorphism only when the cube $\mathscr{I}^{\varepsilon}_{\mathrm{gr}}(D)$ is regarded as a diagram in R**ChCob**,[1] which is enough for the odd theory, but leaves the case of nested theories open.

Lemma 4.1.6. *Given a link diagram D choose sign assignments ε and $\overline{\varepsilon}$ for the cube $\mathscr{I}_{\mathrm{gr}}(D)$ with respect to the cocycles ψ and $\overline{\psi}$ respectively. Then there is an isomorphism of cubes $\mathscr{I}^{\varepsilon}_{\mathrm{gr}}(D) \cong \mathscr{I}^{\overline{\varepsilon}}_{\mathrm{gr}}(D)$, regarded as diagrams in R**ChCob**.*

[1] This step requires us to enumerate circles in each resolution, since the disjoint union in R**ChCob** is not strictly symmetric. The cube, however, is independent of these choices: different orders of circles are related by canonical isomorphisms, which in turn induce an isomorphism of cubes.

Proof. Instead of constructing $\overline{\varepsilon}$ we shall alter the diagram D into D', so that $\delta\varepsilon = \overline{\psi}$ for D'. Color the diagram D black and white in a checkerboard fashion. Given a set of arrows orienting crossings, reverse every arrow between white regions:



(4.7)

to obtain a new decorated diagram D'. This operation preserves all the diagrams from Tab. 4.1, except the two shown in (4.6), which are exchanged. Hence, $\delta\varepsilon := \varepsilon$ for D'. We construct an isomorphism $s\colon \mathscr{I}^{\varepsilon}_{\mathrm{gr}}(D) \cong \mathscr{I}^{\overline{\varepsilon}}_{\mathrm{gr}}(D')$ as follows. The coloring of D induces a coloring of its resolutions D_ξ such that every circle is a boundary of a unique black region. Take the boundary circles of a black region and apply a half-twist to them; the component $s_\xi\colon D_\xi \longrightarrow D'_\xi$ is a composition of such half-twists for all black regions in D_ξ. It is an isomorphism of cubes, since what it does is exactly to reverse the arrows connecting white regions. ☐

4.1.3 Direct sums along diagonals and the complex

Motivated by [BN05] we construct the generalized Khovanov bracket in the *additive closure* $\mathrm{Mat}(R\mathbf{ChCob})$ of the category $R\mathbf{ChCob}$.

Definition 4.1.7. The *additive closure* $\mathrm{Mat}(\mathbf{C})$ of an R-linear category $\mathbf{C}$, where R is a fixed ring, is defined as follows:

- objects are formal direct sums $\bigoplus_{i=1}^{n} C_i$ of objects from $\mathbf{C}$,
- a morphism $F\colon \bigoplus_{i=1}^{n} A_i \longrightarrow \bigoplus_{j=1}^{m} B_j$ is a matrix $\big(F_{ij}\colon A_j \longrightarrow B_i\big)$ of morphisms from $\mathbf{C}$,
- the composition of morphisms $F \circ G$ mimics the formula for a product of matrices

$$(F \circ G)_{ij} := \sum_k F_{ik} \circ G_{kj}. \tag{4.8}$$

This category is R-linear with a natural action of R and addition defined as addition of matrices: $(F+G)_{ij} := F_{ij} + G_{ij}$.

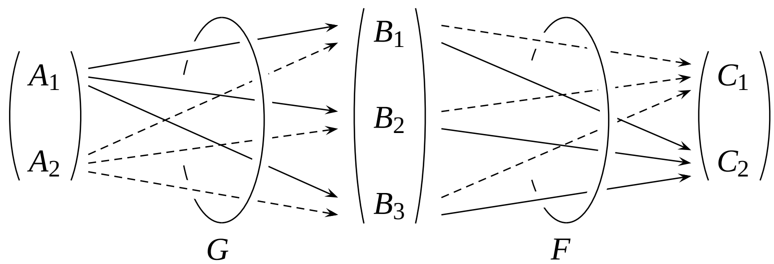


Figure 4.2: The composition of morphisms in the additive closure of a category. The component $(F\circ G)_{21}$ is indicated by solid lines.

We can represent objects of $\mathrm{Mat}(\mathbf{C})$ by finite sequences (vectors) of objects in $\mathbf{C}$ and morphisms between such sequences by bundles[2] (matrices) of morphisms in $\mathbf{C}$, see Fig. 4.2. It means each column in Fig. 4.1 forms a single object C^i, as indicated by dotted arrows going downwards, and all edges between two columns form a single morphism $d\colon C^i \longrightarrow C^{i+1}$. Because every square in $\mathscr{I}^{\varepsilon}_{\mathrm{gr}}(D)$ anticommutes, $d^2=0$.

Definition 4.1.8. The *generalized Khovanov bracket* $[\![D]\!]_\varepsilon$ is the chain complex defined as:

$$[\![D]\!]^i_\varepsilon := \bigoplus_{\|\xi\|=i} D_\xi\left\{\tfrac{\|\xi\|+\ell_0-\ell_\xi}{2}, \tfrac{\|\xi\|-\ell_0+\ell_\xi}{2}\right\}, \qquad d_\varepsilon|_{D_\xi} := \sum_{\zeta\colon \xi\to\xi'} \varepsilon(\zeta)\cdot D_\zeta, \tag{4.9}$$

where ℓ_ξ is the number of circles in the resolution D_ξ, $\|\xi\| := \xi_1+\ldots+\xi_n$, and ε is a sign assignment for the cube $\mathscr{I}_{\mathrm{gr}}(D)$. The *generalized Khovanov complex* $Kh(D)$ is obtained from $[\![D]\!]_\varepsilon$ by shifting both degrees: $Kh^i(D) := [\![D]\!]^{i+n_-}_\varepsilon\left\{\frac{n_+-\ell_0}{2}-n_-, \frac{n_++\ell_0}{2}-n_-\right\}$.

Proposition 4.1.9. *The isomorphism class of the Khovanov bracket $[\![D]\!]_\varepsilon$ depends only on the link diagram D.*

Proof. Changing the order of crossings results in a different parametrization of the cube $\mathscr{I}_{\mathrm{gr}}(D)$, but the chain objects $[\![D]\!]^i_\varepsilon$ are preserved and likewise for the differential. Independence of the other choices follows from Lemmas 4.1.5 and 4.1.4, as an isomorphism of anticommutative cubes descends to an isomorphism of complexes. □

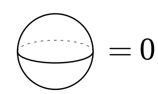
[2] In the colloquial sense, not the mathematical one.

The additional degree shift for $Kh^i(D)$ is not a pair of integers in general. Unfortunately, this cannot be fixed—we shall see later that these degree shifts are forced by Reidemeister moves. However, all differential in the generalized Khovanov bracket are already graded, and there is no need for roots of X, Y, or Z, see Remark 4.1.2. However, one must be careful with various constructions on the chain complex such as tensor product—because of that we shall first define them for the bracket, in case some new coefficients appear.

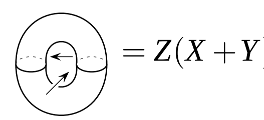
4.1.4 The statement of invariance

The Khovanov complex $Kh(D)$ is not a link invariant. For example, it depends on the number of crossings in a chosen diagram. This dependence disappears after passing to the homotopy category of complexes and imposing modified versions of Bar-Natan's *S*, *T* and *4Tu* relations [BN05], explained below.

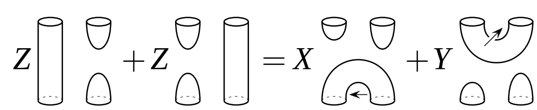
(*S*) The *S* relation replaces with 0 all cobordisms that have a sphere as a connected component. The number and orientations of critical points do not matter.



(*T*) The *T* relation allows us to remove a standard torus component at a cost of multiplying the cobordism with $Z(X+Y)$. Here, the standard torus is defined as a torus with four critical points and an arrow at the merge pointing to the circle originating on the left hand side of the split. The death is oriented clockwise.



(*4Tu*) The four tube relation *4Tu* involves four cobordisms from two circles to two circles. Each of them consists of a tube and two cups, but the position of the tube is different in each picture: for the first two cobordisms the tube is a vertical cylinder over one of the two circles, while in the remaining two cases it connects either the input or the output circles. Notice the choice of framing for saddle points and heights of caps (the left caps are smaller than the right ones). Again, all deaths are oriented clockwise.



The relations above, especially *T* and *4Tu*, are local. This means all other critical points can appear only above or below the pictures shown.[3] All relations are homogeneous—the degree of the standard torus is zero, whereas each cobordism involved in *4Tu* has degree $(-1,-1)$—so that they are coherent with changes of chronologies. Let $R\mathbf{ChCob}_{/\ell}$ be the quotient of $R\mathbf{ChCob}$ by these relations.

Theorem 4.1.10. *Given a link L with a diagram D, the homotopy type of the generalized Khovanov complex* $Kh(D)$*, regarded as a complex in* $R\mathbf{ChCob}_{/\ell}$*, is an invariant of L, i.e. complexes for two link diagrams related by any of the Reidemeister moves are chain homotopy equivalent.*

A proof of this theorem is postponed to the end of this section, after we explore certain properties of the complex.

4.2 Basic properties

We shall now explore how the Khovanov complex behaves under basic operations on links such as disjoint unions, connected sums, reversing orientation of its components or taking the mirror image. This behavior mimics the corresponding properties of the Jones polynomial and the Kauffman bracket.

4.2.1 Skein exact sequences

Choose a crossing in a link diagram D, and construct diagrams D^h and D^v by replacing the crossing with its horizontal and vertical resolutions; we shall denote the diagram D, D^h, and D^h by $\times$, $\asymp$, and $)($ respectively. Notice that both $\mathscr{I}_{\text{gr}}(\asymp)$ and $\mathscr{I}_{\text{gr}}()()$ are subcubes of $\mathscr{I}_{\text{gr}}(\times)$, although we have to shift the degree of the latter by $(1,0)$ or $(0,1)$, depending on the degree of the cobordisms between the initial states of $\asymp$ and $)($. The remaining edges of $\mathscr{I}_{\text{gr}}(\times)$ form a morphism of the cubes, which we shall denote by $\mathscr{I}_{\text{gr}}(\rtimes)$.

[3] This is exactly how the right disjoint union of chronological cobordisms sum behaves.

Recall that the *mapping cone* of a chain map $\psi\colon C \longrightarrow D$ is the chain complex $C(\psi)$ with

$$C(\psi)^i := C^{i+1} \oplus D^i, \qquad d = \begin{pmatrix} -d_C & 0 \\ \psi & d_D \end{pmatrix} \tag{4.10}$$

In case ψ is an *antichain* map, i.e. $d_D\psi + \psi d_C = 0$, take just d_C in the top left corner. The proposition below follows directly from Definition 4.1.8 and the discussion above.

Proposition 4.2.1. *Denote by $\ell_0(D)$ be the number of circles in the initial state for the link diagram D. The generalized Khovanov bracket satisfies the following equalities:*

(KB1) $\llbracket \emptyset \rrbracket = 0$,

(KB2) $\llbracket L \sqcup \bigcirc \rrbracket = \llbracket L \rrbracket \sqcup \bigcirc$,

(KB3a) $\llbracket \times \rrbracket = C\left(\llbracket \Join \rrbracket \colon \llbracket \asymp \rrbracket \longrightarrow \llbracket)(\rrbracket\{1,0\}\right)[1]$ *if* $\ell(\asymp) > \ell()()$, *and*

(KB3b) $\llbracket \times \rrbracket = C\left(\llbracket \Join \rrbracket \colon \llbracket \asymp \rrbracket \longrightarrow \llbracket)(\rrbracket\{0,1\}\right)[1]$ *if* $\ell(\asymp) < \ell()()$,

where the chain morphisms in the last two equalities are induced by cobordisms $\Join\colon \asymp \longrightarrow)($.

The notation $\llbracket L \rrbracket \sqcup \bigcirc$ in (KB2) is used for the operation of adding a disjoint circle: add a circle to each link diagram in $\llbracket L \rrbracket$ and a vertical cylinder to each summand of the differential.

(KB3a) and (KB3b) imply existence of an exact sequence that mimics the Jones skein relation. Say that a sequence $\ldots \longrightarrow A_i \longrightarrow A_{i+1} \longrightarrow A_{i+2} \longrightarrow \ldots$ in $\mathrm{Mat}(R\mathbf{ChCob})$ is *exact*, if its image under any additive functor $\mathscr{F}\colon \mathrm{Mat}(R\mathbf{ChCob}) \longrightarrow \mathbf{Mod}_R$ is exact.

Proposition 4.2.2. *Choose four link diagrams $\times_+$, $\times_-$, $)($, and $\asymp$, which differ only locally as visualized. Pick any orientation on the last diagram and set $e := n_-(\asymp) - n_-(\times_+)$ to be the difference in the number of negative crossings of the diagrams. Then there are short exact sequences*

$$0 \longrightarrow Kh(\asymp)\{\tfrac{3e}{2}+1, \tfrac{3e}{2}+1\}[e+1] \longrightarrow Kh(\times_+) \longrightarrow Kh()()\{\tfrac{1}{2}, \tfrac{1}{2}\} \longrightarrow 0,$$

$$0 \longrightarrow Kh()()\{-\tfrac{1}{2}, -\tfrac{1}{2}\} \longrightarrow Kh(\times_-) \longrightarrow Kh(\asymp)\{\tfrac{3e-1}{2}, \tfrac{3e-1}{2}\}[e-1] \longrightarrow 0,$$

and

$$0 \longrightarrow Kh()()[2]\{\tfrac{1}{2}, \tfrac{1}{2}\} \longrightarrow Kh(\times_-)[2]\{1,1\} \longrightarrow Kh(\times_+)\{-1,-1\} \longrightarrow Kh()()\{-\tfrac{1}{2}, -\tfrac{1}{2}\} \longrightarrow 0.$$

Proof. Assume that $\ell(\asymp) > \ell(\rangle\langle)$; the other case is proven in the same way. An additive functor preserves mapping cones, so that the following sequences are exact

$$0 \longrightarrow [\![\asymp]\!][1]\{0,1\} \longrightarrow [\![\times]\!] \longrightarrow [\![\rangle\langle]\!] \longrightarrow 0, \tag{4.11}$$

$$0 \longrightarrow [\![\rangle\langle]\!][1]\{1,0\} \longrightarrow [\![\times]\!] \longrightarrow [\![\asymp]\!] \longrightarrow 0. \tag{4.12}$$

We recover the first two sequences in the thesis by shifting the degrees accordingly. The last sequence results from combining the other two together. □

4.2.2 Reversing orientation of components

The generalized Khovanov complex $Kh(L)$ depends on the orientation of components of the link L in a well-understood way.

Given two links L, L' in $\mathbb{R}^3$ with diagrams D, D' we define the linking number $\operatorname{lk}(L,L')$ as the sum of signs of the crossings involving components of both D and D'. It is well-known that this number does not depend on the choice of the diagrams.

Proposition 4.2.3. *Given an oriented link L denote by $-L$ the same link with reversed orientation of all its components, and by L' the link in which the orientation of a single component L_0 is reversed. Then*

$$Kh(-L)^r \cong Kh(L)^r, \tag{4.13}$$

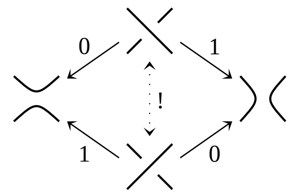
$$Kh(L')^r \cong Kh(L)^{r-2\ell}\{-3\ell,-3\ell\}, \tag{4.14}$$

where $\ell = \operatorname{lk}(L-L_0, L_0)$.

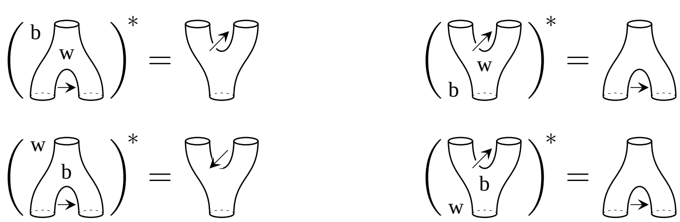
Proof. For (4.13) it is enough to see that after reversing orientation of all components the signs of crossings are the same. If we reverse the orientation only of one component L_0, then the signs of the crossings of L_0 with other components are changed. Hence, $n_+(L') = n_+(L) - 2\ell$ and $n_-(L') = n_-(L) + 2\ell$, which implies $\frac{n_+(L')}{2} - n_-(L') = \frac{n_+(L)}{2} - n_-(L) - 3\ell$. □

4.2.3 Mirror image

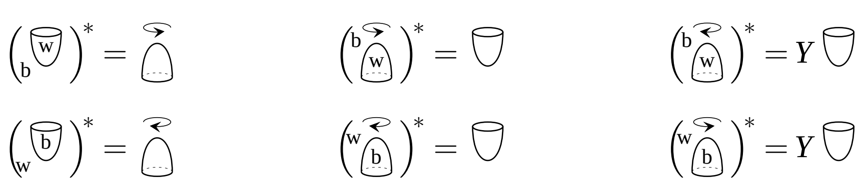
Given a link diagram D we form its mirror image $D^!$ by replacing every crossing with the other one—think about placing a mirror below the diagram. It follows the cube of resolutions of $D^!$ is a reflection of $\mathscr{I}(D)$: we start in the terminal state of D, which is the initial state of $D^!$, and proceed backwards (see the picture to the right). Formally the symmetry comes from a duality functor $(\_)^*\colon R\mathbf{ChCob} \longrightarrow R\mathbf{ChCob}$ induced by the vertical flip of $\mathbb{R}^2 \times I$. We must be careful with defining orientations of critical points in W^*: if p is a critical pints of W, its orientation determines an orientation of the stable part of T_pW^*. We choose for the unstable part the complementary orientation with respect to the outward orientation of the cobordism W. The only exception is a death, as there is only one orientation of births: if W is a negatively oriented death, we first rewrite it as a positively oriented death scaled by Y, and then we make the flip.



The convention for orientation of critical points can be described also diagrammatically in the following way. Color each region in the complement of W black or white, so that the unbounded region is white and regions with same colors do not meet. Then for saddle point p rotate the framing arrow in W^* clockwise, if the region below $p \in W$ is white, and anticlockwise otherwise:



Since we want the duality functor to be coherent with annihilations and creations, there is no choice left for births and deaths:



Flipping a cobordism permutes its degree components, $\deg W^* = (b,a)$ if $\deg W = (a,b)$, but it also intertwines the two disjoint unions, $(W \,\underline{\sqcup}\, W')^* = W^* \,\overline{\sqcup}\, W'^*$. Hence, in the linearized case, the roles of X and Y are exchanged, but the role of Z is preserved. Therefore, the flipping operation is a functor $(\_)^*\colon R\mathbf{ChCob}_{XYZ} \longrightarrow R\mathbf{ChCob}_{YXZ}$ between two different categories. It is coherent with all chronological relations, as well as with relations S, T and *4Tu*. We extend it to categories of complexes, by reflecting the homological grading, i.e. we set $(C^*)^i := (C^{-i})^*$.

Proposition 4.2.4. *The generalized Khovanov complexes of a link L and its mirror image $L^!$ are dual to each other, i.e. $Kh_{XYZ}(L^!) \cong Kh_{YXZ}(L)^*$, where Kh_{abc} stands for a Khovanov complex constructed in the category $R\mathbf{ChCob}_{/\ell}$ with chronological parameters X, Y and Z set to a, b and c respectively.*

Proof. Choose a diagram of L with n enumerated crossings and arrows over them. To obtain a diagram for $L^!$ replace first each crossing $\times$ with the opposite one $\times$, and rotate the arrows over crossings using the same convention as for $(\_)^*$: color regions black and white and rotate an arrow anticlockwise, when it is placed over a white region, and clockwise otherwise. With this choice of diagrams $\mathscr{I}_{\mathrm{gr}}(L^!) = \mathscr{I}_{\mathrm{gr}}(L)^*$, which follows directly from the construction of the cube of resolutions. Moreover, a sign assignment $\varepsilon \in C^1(I^n; U(R))$ for $\mathscr{I}(L)$ is automatically a sign assignment for $\mathscr{I}(L)^*$. Therefore, $(\llbracket L \rrbracket_\varepsilon)^* = \llbracket L^! \rrbracket_\varepsilon[-n]$ and the proposition follows. $\square$

4.2.4 Disjoint union and connected sum of links

Given complexes C and C' in a graded monoidal category, which differentials $d\colon C_i \longrightarrow C_{i+1}$ and $d'\colon C'_i \longrightarrow C'_{i+1}$ are morphisms of degree zero, we can form their tensor product $C \otimes C'$ as follows:

$$(C \otimes C')_i := \bigoplus_{p+q=i} C_p \otimes C'_q, \tag{4.15}$$

$$d|_{C_p \otimes C'_q} := d \otimes \mathrm{id} + (-1)^p \,\mathrm{id} \otimes d'. \tag{4.16}$$

In case of Khovanov complexes, $Kh^p(D) \mathbin{\upharpoonleft\!\sqcup\!\downharpoonright} Kh^q(D')$ is a collection of diagrams $D_\xi \sqcup D'_{\xi'}$ with $\|\xi\| = p$ and $\|\xi'\| = q$, which can be seen as resolutions of the disjoint union $D \sqcup D'$. However, $d \mathbin{\upharpoonleft\!\sqcup\!\downharpoonright} \mathrm{id}$ and $\mathrm{id} \mathbin{\upharpoonleft\!\sqcup\!\downharpoonright} d$ do not commute in $Kh(D \sqcup D')$. The reason they do in $Kh(D) \mathbin{\upharpoonleft\!\sqcup\!\downharpoonright} Kh(D')$ is the way the degree shifts are applied: the objects $D_\xi\{a,b\} \sqcup D'_{\xi'}\{a',b'\}$ and $(D_\xi \sqcup D'_{\xi'})\{a+a',b+b'\}$ are isomorphic but not in a canonical way, see Lemma 2.1.7.

Proposition 4.2.5. *Given link diagram D and D' there is an isomorphism of bigraded complexes $Kh(D \sqcup D') \cong Kh(D) \mathbin{\upharpoonleft\!\sqcup\!\downharpoonright} Kh(D')$, which is natural with respect to graded morphisms.*

Proof. Choose graded cubes of resolutions $\mathscr{I}^{\varepsilon}_{\mathrm{gr}}(D)$ and $\mathscr{I}^{\varepsilon'}_{\mathrm{gr}}(D')$ corrected by certain sign assignments ε and ε'. Their tensor product is a commutative cube with vertices isomorphic to vertices of the cube $\mathscr{I}_{\mathrm{gr}}(D \sqcup D')$:

$$\begin{aligned}(\mathscr{I}_{\mathrm{gr}}(D) \mathbin{\upharpoonleft\!\sqcup\!\downharpoonright} \mathscr{I}_{\mathrm{gr}}(D'))(\xi\xi') &= D_\xi\{a,b\} \sqcup D_{\xi'}\{a',b'\} \\ &\cong (D_\xi \sqcup D'_{\xi'})\{a+a',b+b'\} = \mathscr{I}_{\mathrm{gr}}(D \sqcup D')(\xi\xi') \end{aligned} \tag{4.17}$$

where $\xi\xi'$ is the concatenation of the sequences ξ and ξ'. The isomorphism in the middle is given by the composition of canonical isomorphisms $i_{a+a',b+b'} \circ (i^{-1}_{a,b} \sqcup i^{-1}_{a',b'})$. It does not commute with edge-morphisms in the cubes. Following Lemma 2.1.7 define a sign assignment $\varepsilon \mathbin{\upharpoonleft\!\sqcup\!\downharpoonright} \varepsilon'$ for $\mathscr{I}_{\mathrm{gr}}(D \sqcup D')$ as follows:

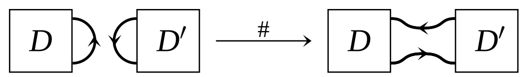
- given an edge ζ in $\mathscr{I}_{\mathrm{gr}}(D)$ and a vertex ξ' in $\mathscr{I}_{\mathrm{gr}}(D')$, set $(\varepsilon \mathbin{\upharpoonleft\!\sqcup\!\downharpoonright} \varepsilon')(\zeta\xi') = \varepsilon(\zeta)$, and
- given a vertex ξ in $\mathscr{I}_{\mathrm{gr}}(D)$ with diagram D_ξ shifted by $(a,b) \in \mathbb{Z} \times \mathbb{Z}$, and an edge ζ' in $\mathscr{I}_{\mathrm{gr}}(D')$, set

$$(\varepsilon \mathbin{\upharpoonleft\!\sqcup\!\downharpoonright} \varepsilon')(\xi\zeta') = \begin{cases} (-1)^{\|\xi\|} X^a Z^b \varepsilon'(\zeta'), & \text{if } D'_{\zeta'} \text{ is a merge,} \\ (-1)^{\|\xi\|} Y^b Z^{-a} \varepsilon'(\zeta'), & \text{if } D'_{\zeta'} \text{ is a split.} \end{cases} \tag{4.18}$$

Then $\varepsilon \mathbin{\upharpoonleft\!\sqcup\!\downharpoonright} \varepsilon'$ is a sign assignment for $\mathscr{I}_{\mathrm{gr}}(D \sqcup D')$ and (4.17) is an isomorphism of cubes. Indeed, if S is a face spanned by edges of $\mathscr{I}_{\mathrm{gr}}(D)$ or $\mathscr{I}_{\mathrm{gr}}(D')$, then $d(\varepsilon \mathbin{\upharpoonleft\!\sqcup\!\downharpoonright} \varepsilon')(S)$ is equal to $d\varepsilon(S)$ or $d\varepsilon(S')$ respectively, In a mixed case the two-arrow diagram for S is disjoint, and $\psi(S)$ agrees with the coefficient given in Lemma 2.1.7.

The above results in an isomorphism $[\![D]\!]_{\varepsilon} \vec{\sqcup} [\![D']\!]_{\varepsilon'} \cong [\![D \sqcup D']\!]_{\varepsilon \vec{\sqcup} \varepsilon'}$. Since a global degree shift does not affect the differential, the thesis follows. ☐

There is a similar formula in case of another operation of links. The *connected sum* $D\#D'$ of two oriented link diagrams with basepoints D and D' is given by cutting the diagrams at the basepoints and gluing them in such a way that their orientations agree:



This operation depends only on the link components that carry the basepoints. Clearly, resolutions of $D\#D'$ are exactly the connected sums of resolutions of D and D': $(D\#D')_{\xi\xi'} = D_{\xi}\#D_{\xi'}$. Since the right connected sum $\vec{\#}$ behaves like the right disjoint union $\vec{\sqcup}$, especially the Lemma 2.1.7 holds, we have the following result.

Proposition 4.2.6. *Given link diagram D and D' there is an isomorphism of bigraded complexes $Kh(D\#D') \cong Kh(D)\#Kh(D')$.*

Proof. Given graded cubes $\mathscr{I}^{\varepsilon}_{\mathrm{gr}}(D)$ and $\mathscr{I}^{\varepsilon'}_{\mathrm{gr}}(D')$ we show that $\varepsilon \vec{\#} \varepsilon'$, defined as in the proof of Proposition 4.2.5, is a sign assignment for the cube $\mathscr{I}_{\mathrm{gr}}(D\#D')$, resulting in an isomorphism of generalized Khovanov brackets $[\![D]\!]_{\varepsilon} \vec{\#} [\![D']\!]_{\varepsilon'} \cong [\![D\#D']\!]_{\varepsilon \vec{\#} \varepsilon'}$. ☐

4.3 Invariance

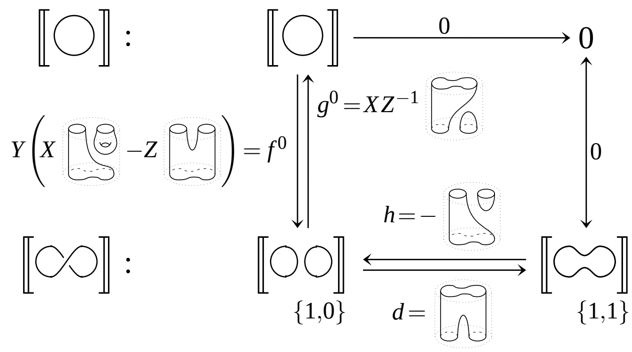
We finish this chapter with a proof of the Invariance Theorem 4.1.10. We shall draw many pictures of chronological cobordisms, and for simplicity some details will be omitted. Here are our conventions to keep:

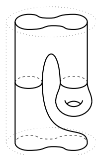
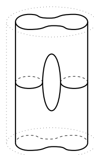
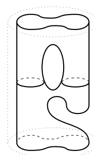
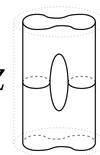
1) all deaths are oriented clockwise, and
2) arrows orienting saddles are directed either to the right or to the front.

In particular, we can cancel at no cost a merge or a split with a birth or a death respectively on its right-hand side, while a left-hand cancellation costs a multiplication by X or Y.

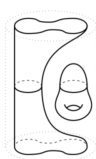
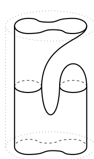
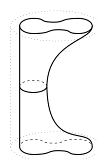
4.3.1 The first Reidemeister move

The first Reidemeister move relates two link diagrams ⟲ and ⟳, where we draw here only the essential piece of the diagrams. In the view of Proposition 4.2.6 it is enough to show that $Kh(\infty)$ and $Kh(\bigcirc)$ are homotopy equivalent. We define chain homotopy equivalences $f\colon [\![\bigcirc]\!] \rightleftarrows [\![\infty]\!] :g$ and a chain homotopy $h\colon [\![\infty]\!] \longrightarrow [\![\infty]\!]$ as in the diagram to the right. Clearly, $g\colon [\![⟳]\!] \longrightarrow [\![⟲]\!]$ is a chain map, but for $f\colon [\![⟲]\!] \longrightarrow [\![⟳]\!]$ we have to do the following short computation:

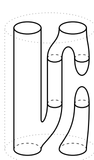
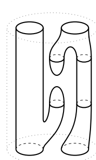
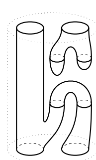
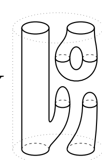


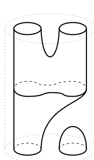
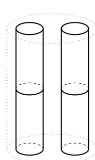
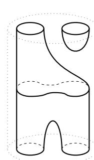
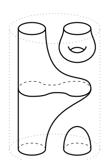
$$df^0 = XY$$  $$-YZ$$  $$= YZ$$  $$-YZ$$  $$= 0. \tag{4.19}$$

The maps f and g are homotopy equivalences inverse to each other. Indeed, the relation T implies:

$$g^0 f^0 = YZ^{-1}$$  $$-XY$$  $$= (Y(X+Y) - XY)$$  $$= \mathrm{id}, \tag{4.20}$$

whereas *4Tu* makes $f^0 g^0 - \mathrm{id} = hd$:

$$0 = \quad Z$$  $$+Z$$  $$-X$$  $$-Y$$ 

$$= YZ$$  $$+XZ$$  $$-XZ$$  $$-$$  $$= -XZ(f^0 g^0 - \mathrm{id} - hd).$$

After expanding $f^0 g^0$ we can see that the last cobordism should appear with the coefficient $-XY$. The equality holds, because the cobordism has a handle, hence, it is annihilated by

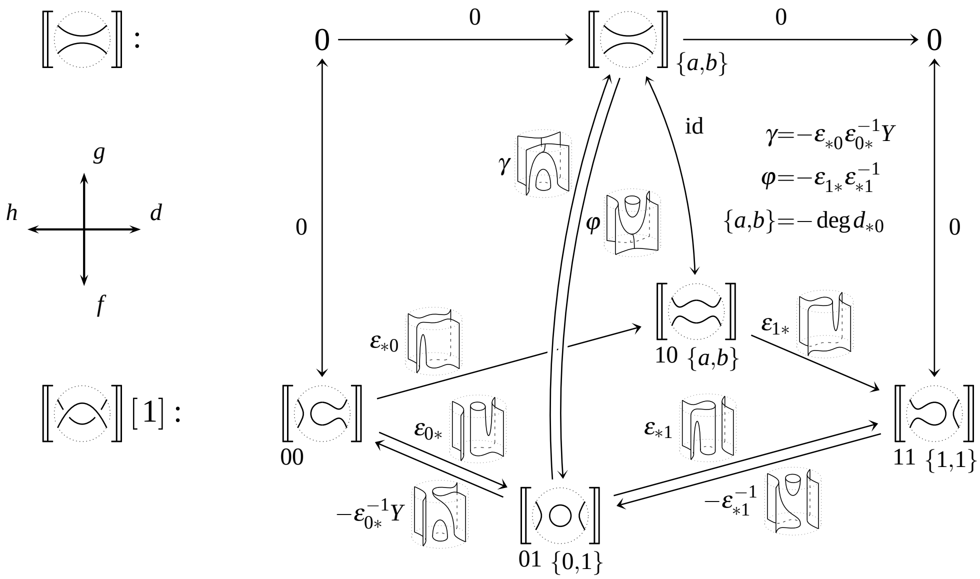


Figure 4.3: Invariance under the R_2 move.

$(1-XY)$. The remaining equality $dh = -\mathrm{id}$ follows from the chronological relations (remove the birth).

4.3.2 The second Reidemeister move

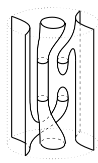
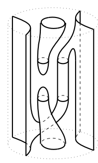
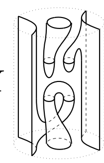
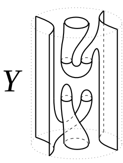
To show that $Kh(\,)$ and $Kh(\,)$ are homotopy equivalent, it is enough to find homotopy equivalences between $[\![\,]\!][-1]$ and $[\![\,]\!]\{a,b\}$, where (a,b) is the degree shift of the initial resolution $_0$ regarded as a vertex of the cube $\mathscr{I}_{\mathrm{gr}}(\,)$. Regard the cube $\mathscr{I}_{\mathrm{gr}}(\,) \cong \mathscr{I}_{\mathrm{gr}}(\,)$ as a subcube of $\mathscr{I}_{\mathrm{gr}}(\,)$, so that a sign assignment ε for the latter induces a sign assignment ε' for the first.

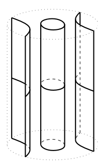
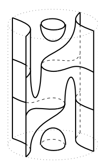
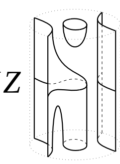
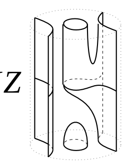
Consider $[\![\,]\!]$ as the total complex of $[\![\,]\!] \longrightarrow [\![\,]\!] \oplus [\![\,]\!] \longrightarrow [\![\,]\!]$, where we omitted degree shifts for clarity. We construct chain maps $f\colon [\![\,]\!] \rightleftarrows [\![\,]\!] :g$ be defining maps between certain pieces of bicomplexes, see Fig. 4.3. The nontrivial components of f and g are compositions $f^0 := h_{*1}d_{1*}$ and $g^0 := d_{*0}h_{0*}$. They are graded after the degree shifts are applied.

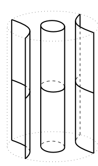
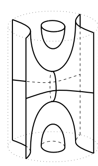
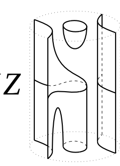
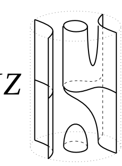
The choice of coefficients for the component of h guarantees that it anticommutes with the differentials, and chronological relations imply $h_{0*}d_{0*} + \mathrm{id} = 0$ and $d_{*1}h_{*1} + \mathrm{id} = 0$. Hence, both f^0 and g^0 are chain maps, and so are the induced morphisms f and g. Since any cobordism with a spherical component is 0, $gf = \mathrm{id}$ and $hf = 0$. It remains to show that h is a chain homotopy between fg and the identity morphism. The only nontrivial case is in the middle, were we have to check the matrix equality

$$\begin{pmatrix} g^0 f^0 & f^0 \\ g^0 & I \end{pmatrix} - \begin{pmatrix} \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{pmatrix} = \begin{pmatrix} h_{*1}d_{*1} + d_{0*}h_{0*} & h_{*1}d_{1*} \\ d_{*0}h_{0*} & 0 \end{pmatrix}. \tag{4.21}$$

It follows from definitions of f^0 and g^0 and the *4Tu* relation:

$0 \;=\; Z$  $+\; Z$  $-X$  $-\; Y$ 

$= XZ$  $+XYZ$  $-XZ$  $-XYZ$ 

$= XZ$  $-\gamma\varphi XZ$  $-XZ$  $-XYZ$ 

$$= XZ(-f^0 g^0 + \mathrm{id} + h_{*1}d_{*1} + d_{0*}h_{0*}).$$

The coefficient X in the first term appears, because the birth is canceled with a merge from the left hand side. The same happens in the last two terms, but in the third one we also have to reverse an orientation of the lower merge. Finally, to modify the second term, we first used chronological relations and then anticommutativity of the lower square in Fig. 4.3 (erase the caps to see compositions of differentials).

4.3.3 The third Reidemeister move

The case of the third move is the simplest one, despite dealing with the largest complex. This is because it can be derived from the invariance under the second move, as it is done in the case of the Kauffman bracket, using invariance of mapping cones under inclusions into strong deformation retracts.

Definition 4.3.1. We say that a chain complex D is a *strong deformation retract* of a chain complex C if there are chain maps $f\colon D \longrightarrow C$ and $g\colon C \longrightarrow D$ such that $gf = \mathrm{id}$ and $fg - \mathrm{id} = dh + hd$ for a homotopy h such that $hf = 0$.[4] The chain map f is called an *inclusion into a deformation retract*.

Both chain maps $Kh($$) \longrightarrow Kh($$)$ and $Kh($$) \longrightarrow Kh($$)$ described in previous sections are inclusions into deformation retracts. Indeed, the equality $hf = 0$ in the case of the first Reidemeister move is trivial, whereas for the latter it follows from the S relation.

Lemma 4.3.2. *The homotopy type of a mapping cone is preserved under compositions with inclusions into strong deformation retracts. More precisely, given a pair of strong deformation retracts*

$$C_a \underset{f_a}{\overset{g_a}{\rightleftarrows}} D_a \qquad \textit{and} \qquad C_b \underset{f_b}{\overset{g_b}{\rightleftarrows}} D_b \tag{4.22}$$

and a chain map $\psi\colon C_a \longrightarrow C_b$, the mapping cones $C(\psi f_a)$ and $C(f_b \psi)$ are strong deformation retracts of $C(\psi)$.

Proof. Let h be the homotopy associated to the retract D. Then there is a diagram with commuting squares

$$\begin{array}{llcccl}
C(\psi f_a): & \cdots \longrightarrow & D_a^r \oplus C_b^{r-1} & \xrightarrow{\ d\ } & D_a^{r+1} \oplus C_b^r & \longrightarrow \cdots \\
& & \tilde{g}_a^r \big\uparrow \big\downarrow \tilde{f}_a^r & & \tilde{g}_a^{r+1} \big\uparrow \big\downarrow \tilde{f}_a^{r+1} & \\
C(\psi): & \cdots \longrightarrow & C_a^r \oplus C_b^{r-1} & \underset{h}{\overset{d}{\rightleftarrows}} & C_a^{r+1} \oplus C_b^r & \longrightarrow \cdots
\end{array} \tag{4.23}$$

with morphisms $\tilde{f}_a$, $\tilde{g}_a$, and $\tilde{h}$ given by matrices

$$\tilde{f}_a^r = \begin{pmatrix} f_a & 0 \\ 0 & \mathrm{id} \end{pmatrix}, \qquad \tilde{g}_a^r = \begin{pmatrix} g_a & 0 \\ -\psi h & \mathrm{id} \end{pmatrix}, \qquad \tilde{h}^r = \begin{pmatrix} -h & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.24}$$

A quick computation shows $\tilde{h}\tilde{f}_a = 0$, $\tilde{g}_a \tilde{f}_a = \mathrm{id}$, and $\tilde{f}_a \tilde{g}_a - \mathrm{id} = d\tilde{h} + \tilde{h}d$, which proves $C(\psi f_a)$ is a strong deformation retract of $C(\psi)$. The other case is shown in a similar way. □

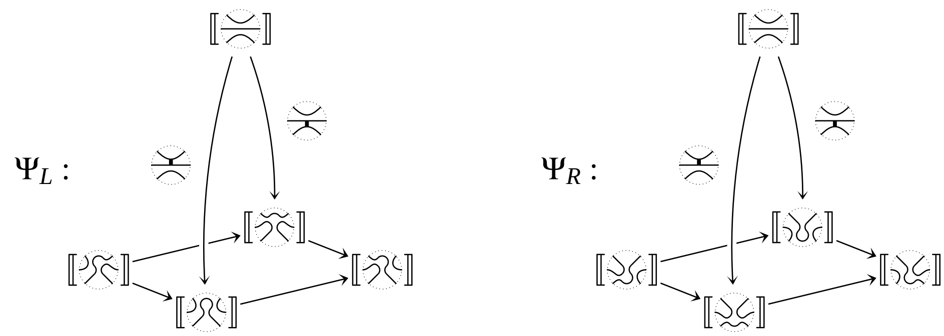
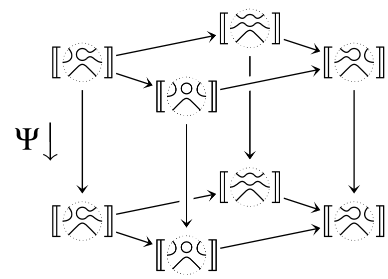


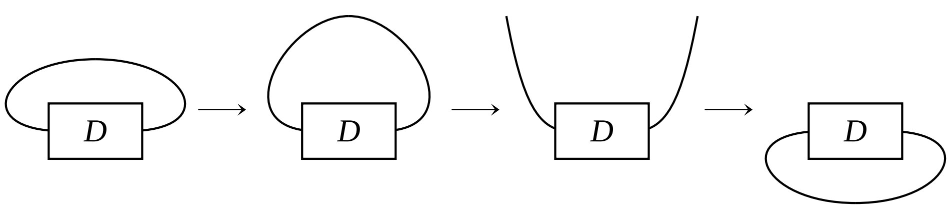
Figure 4.4: Morphisms describing complexes for the two link diagrams in the R_3 move.



Consider now $[\![\ ]\!]$ as the mapping cone of the chain map $\Psi = [\![\ ]\!]\colon [\![\ ]\!] \longrightarrow [\![\ ]\!]$ given by the four vertical morphisms shown to the right. Lemma 4.3.2 implies $[\![\ ]\!]$ is homotopy equivalent to the mapping cone of $\Psi_L = \Psi \circ f$ given in Fig 4.4, where f is the chain map from the proof of invariance under the second Reidemeister move. For the same reason $[\![\ ]\!]$ is homotopy equivalent to Ψ_R. Since link diagrams and are isotopic, the mapping cone complexes $C(\Psi_L)$ and $C(\Psi_R)$ are isomorphic.

4.3.4 The isotopy through infinity

There is one more move to consider, if we regard links in $\mathbb{S}^3$: the isotopy through infinity



(4.25)

denoted by R_∞. For the complex $Kh(D)$ to be invariant under this move we have to check how the faces decorated with $\Diamond$-changes, i.e. those with $\psi = 1$ or $\psi = XY$, behave. Indeed, these are the cases where the embedded structure was used to determine which arrow is inner.

[4] We will often omit the composition sign $\circ$.

The R_∞-move turns the circle inside out, moving the inner arrow outside, but one can easily see the diagram does not change. Hence, the naive isomorphism, which identifies the circles in resolutions of D and D' by the isotopy, works.

Corollary 4.3.3. *The homotopy type of the complex $Kh(L)$ is an invariant of a link L in $\mathbb{S}^3$.*

Chapter 5

Homology

Although the complex $Kh(L)$ is an invariant of the link L, it is a difficult problem to determine whether two complexes in $R\mathbf{ChCob}_{/\ell}$ are homotopy equivalent. One can obtain a partial answer, by applying a functor $\mathscr{F}: R\mathbf{ChCob}_{/\ell} \longrightarrow \mathbf{A}$ to some abelian category $\mathbf{A}$. Such a functor extends naturally to categories of complexes $\mathscr{F}: \mathbf{Kom}(R\mathbf{ChCob}_{/\ell}) \longrightarrow \mathbf{Kom}(\mathbf{A})$ and the homology $H(\mathscr{F}Kh(L))$ is an invariant of the link L. If $\mathbf{A} = \mathbf{Mod}_R$ is the category of modules, such functors are determined by chronological Frobenius algebras, see Section 3.3.2.

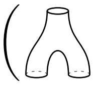
In this chapter we provide a few examples of chronological Frobenius algebras that lead to link homology theories, including both even and odd Khovanov homology. Among them is the algebra given by cobordisms with dots, which is universal in our framework.

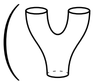
5.1 Recovering even and odd Khovanov homology

We begin with a few examples of chronological TQFT functors producing link homology theories.

Example 5.1.1 (Khovanov homology)**.** Recall the Khovanov's Frobenius algebra from Example 3.3.11. The induced functor $\mathscr{F}_{ev}: \Bbbk\mathbf{ChCob} \longrightarrow \mathbf{Mod}_{\mathbb{Z}}$ associates to a circle $\bigcirc$ the free abelian group A with two generators v_+ and v_- of degrees respectively $(1,0)$ and $(0,-1)$.

Cobordisms are translated into the following group homomorphisms:

$\mathscr{F}_{ev}\Bigg($  $\Bigg)\colon A\otimes A\longrightarrow A, \qquad \begin{cases} v_+\otimes v_+ \longmapsto v_+, & v_-\otimes v_+ \longmapsto v_-, \\ v_+\otimes v_- \longmapsto v_-, & v_-\otimes v_- \longmapsto 0, \end{cases}$ (5.1)

$\mathscr{F}_{ev}\Bigg($  $\Bigg)\colon A\longrightarrow A\otimes A, \qquad \begin{cases} v_+ \longmapsto v_-\otimes v_+ + v_+\otimes v_-, \\ v_- \longmapsto v_-\otimes v_-, \end{cases}$ (5.2)

$\mathscr{F}_{ev}\Bigg($  $\Bigg)\colon R\longrightarrow A, \qquad \begin{cases} 1\longmapsto v_+, \end{cases}$ (5.3)

$\mathscr{F}_{ev}\Bigg($  $\Bigg)\colon A\longrightarrow R, \qquad \begin{cases} v_+ \longmapsto 0, \\ v_- \longmapsto 1. \end{cases}$ (5.4)

Compatibility with the three relations S, T and *4Tu* is easy to check [BN05]. The resulting homology $\mathscr{H}_{ev}(L) := H(\mathscr{F}_{ev}Kh(L))$ is the categorification of the Jones polynomial from [Kh99].

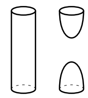
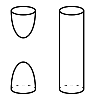
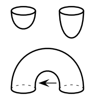
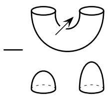
Example 5.1.2 (Odd Khovanov homology)**.** To obtain odd Khovanov homology, take the ORS superalgebra from Example 3.3.12 instead. It induces a functor $\mathscr{F}_{odd}\colon \Bbbk\mathbf{ChCob}\longrightarrow \mathbf{Mod}_{\mathbb{Z}}$ that maps a family of s circles to the exterior algebra on s generators $\Lambda_s = \bigwedge[a_1,\dots,a_s]$. We define the degree of $a_{i_1}\wedge\cdots\wedge a_{i_r}$ as $(s-r,-r)$, where the commutativity is controlled by the second number. In particular, $\deg(1)=(s,0)$.

Merging two circles identifies appropriate generators, while a split translates into a map

$$\Lambda_s \big/ (a_i - a_j) \ni [w] \longmapsto (a_i - a_j)\wedge w \in \Lambda_s, \tag{5.5}$$

assuming the i-th circle in the target configuration is to the left of the framing arrow and the j-th one is to the right. A birth is an inclusion of algebras and a clockwise death of an i-th circle is the Kronecker delta function $a_j \longmapsto \delta_{i,j}$ wedged with identity, i.e. it strips off a_i from the element w from the left hand side, if it is present, or sends w to 0 otherwise.

One can directly check that $\mathscr{F}_{odd}$ defined in this way is a strict 2-functor. It is shown in [ORS13] that $\mathscr{H}_{odd}(L) := H(\mathscr{F}_{odd}Kh(L))$ is an invariant of a link L. We grade Λ_s by setting $\deg(a_{i_1}\wedge\cdots\wedge a_{i_r}) = s-2r$, which makes $\mathscr{F}_{odd}$ a degree-preserving functor. Both a sphere and a torus evaluate to zero ($a_i - a_j$ becomes 0 after merging i-th and j-th circles) and *4Tu* follows from the table below.

				$-$ 
1	0	0	0	0
a_1	0	1	1	0
a_2	1	0	1	0
$a_1 \wedge a_2$	$-a_1$	a_2	0	$a_2 - a_1$

Therefore, invariance of $\mathscr{H}_{odd}(L) := \mathscr{F}_{odd}Kh(L)$ also follows from Theorem 4.1.10.

The two examples can be easily unified using the *covering* chronological Frobenius system from Example 3.3.13.

Example 5.1.3 (Covering Khovanov homology)**.** The covering Frobenius algebra is defined over $\Bbbk = \mathbb{Z}[X,Y,Z^{\pm 1}]/(X^2 = Y^2 = 1)$, inducing a functor $\mathscr{F}_{cov}\colon \Bbbk\mathbf{ChCob} \longrightarrow \mathbf{Mod}_{\Bbbk}$ that assigns to a circle the free module A with two generators, v_+ of degree $(1,0)$ and v_- of degree $(0,-1)$. The operations are defined in the following way:

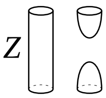
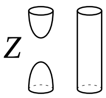
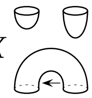
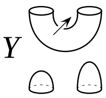
$$\mu\colon A\otimes A \longrightarrow A, \qquad \begin{cases} v_+\otimes v_+ \longmapsto v_+, & v_-\otimes v_+ \longmapsto XZv_-, \\ v_+\otimes v_- \longmapsto v_-, & v_-\otimes v_- \longmapsto 0, \end{cases} \tag{5.6}$$

$$\Delta\colon A \longrightarrow A\otimes A, \qquad \begin{cases} v_+ \longmapsto v_-\otimes v_+ + YZv_+\otimes v_-, \\ v_- \longmapsto v_-\otimes v_-, \end{cases} \tag{5.7}$$

$$\eta\colon \Bbbk \longrightarrow A, \qquad \begin{cases} 1 \longmapsto v_+, \end{cases} \tag{5.8}$$

$$\varepsilon\colon A \longrightarrow \Bbbk, \qquad \begin{cases} v_+ \longmapsto 0, \\ v_- \longmapsto 1. \end{cases} \tag{5.9}$$

It is clear that $\mathscr{F}_{cov}$ satisfies the sphere relation, and a direct calculation shows that a standard torus evaluates to $Z(X+Y)$. Finally, the *4Tu* relation follows from the table below.

	Z 	Z 	X 	Y 
$v_+ \otimes v_+$	0	0	0	0
$v_+ \otimes v_-$	$Xv_+ \otimes v_+$	0	$Xv_+ \otimes v_+$	0
$v_- \otimes v_+$	0	$Zv_+ \otimes v_+$	$Zv_+ \otimes v_+$	0
$v_- \otimes v_-$	$Yv_- \otimes v_+$	$Zv_+ \otimes v_-$	0	$Yv_- \otimes v_+ + Zv_+ \otimes v_-$

We call the invariant $\mathscr{H}_{cov}(L) := H(\mathscr{F}_{cov}Kh(L))$ the *covering Khovanov homology* of L.

The following proposition explains the name *covering homology*. Recall that we distinguished two $\mathbb{k}$-module structures on the ring of integers: $\mathbb{Z}_{ev}$, on which all monomials act as the identity, and $\mathbb{Z}_{odd}$, in which case Y acts as -1.

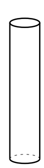
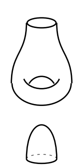
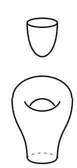
Proposition 5.1.4. *For any link L there are isomorphisms*

$$\mathscr{H}_{ev}(L) \cong \mathscr{H}_{cov}(L;\mathbb{Z}_{ev}) \qquad \textit{and} \qquad \mathscr{H}_{odd}(L) \cong \mathscr{H}_{cov}(L;\mathbb{Z}_{odd}), \tag{5.10}$$

where $\mathscr{H}_{cov}(L;M) := H(\mathscr{F}_{cov}Kh(L) \otimes M)$ for any $\mathbb{k}$-module M.

Proof. The first isomorphism follows directly from the construction: replacing X, Y and Z with 1's in the definition of the algebra A results in the Khovanov algebra. For the second one it is enough to show that functors $\mathscr{F}_{cov}(\_) \otimes \mathbb{Z}_{odd}$ and $\mathscr{F}_{odd}$ are equivalent. This follows from applying an isomorphism $i\colon A^{\otimes s} \otimes \mathbb{Z}_{odd} \longrightarrow \Lambda_s$ that sends any v_+ into 1 and v_- at the i-th position to a_i. Comparing the two definitions, one can easily see that $\mathscr{F}_{odd}(M) = i \circ (\mathscr{F}_{cov}(M) \otimes \mathbb{Z}_{odd}) \circ i^{-1}$ for any generating cobordism M. □

Example 5.1.5. One of the consequences of the *4Tu* relation is the following equality

$Z(X+Y)$  $=$  $+$  , (5.11)

called a neck-cutting relation. Again, we omitted the orienting arrows, but the convention is to orient all death clockwise, merges with arrow pointing leftwards, and splits with arrows pointing to the back. If we impose the relation $X+Y=0$, we can use (5.11) to move handles freely between components of a cobordism (up to multiplication by XZ^a). A similar theory over the two-element field $\mathbb{F}_2$ was analyzed in [BN05], suggesting we have found its lift to $\mathbb{Z}$ in the odd setting. Namely, we have an algebra $A_H := \mathrm{Mor}(\bigcirc,\bigcirc)$ over the ring $R_H := \mathbb{Z}[H,X,Z^{\pm 1}]/(2H,X^2-1)$, where H has degree $(-1,-1)$ and represents a handle. Unfortunately, H is a torsion element, as it is annihilated by $1-XY = 1+X^2 = 2$. One can check that A_H is a free module generated by v_+ and v_- of degrees $(1,0)$ and $(0,-1)$ respectively, with multiplication and comultiplication given by the formulas

$$\mu\colon A_H\otimes A_H \longrightarrow A_H, \qquad \begin{cases} v_+\otimes v_+ \longmapsto v_+, & v_-\otimes v_+ \longmapsto XZv_-, \\ v_+\otimes v_- \longmapsto v_-, & v_-\otimes v_- \longmapsto Hv_-, \end{cases} \tag{5.12}$$

$$\Delta\colon A_H \longrightarrow A_H\otimes A_H, \qquad \begin{cases} v_+ \longmapsto v_-\otimes v_+ + XZv_+\otimes v_- - HXZ^{-1}v_+\otimes v_+, \\ v_- \longmapsto v_-\otimes v_-. \end{cases} \tag{5.13}$$

The generator v_+ is represented by a death followed by a birth and v_- by a vertical cylinder. In tensor products, each v_+ is represented by a birth and all other circles are boundaries of a single component built from splits only (or a single death, if there is no v_-). See [BN05] for details.

5.2 The algebra of cobordisms with dots

5.2.1 The neck cutting relation for chronological cobordisms

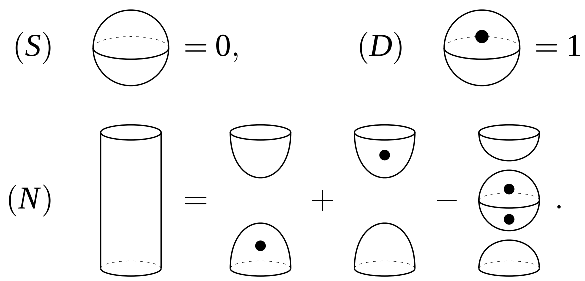
A very generic example of a chronological Frobenius algebra is given by the tautological functor $\mathrm{Mor}(\Sigma,\_)$, where Σ is any object of $\Bbbk\mathbf{ChCob}$.

Proposition 5.2.1. *Given an object $\Sigma\in\Bbbk\mathbf{ChCob}$, the group of morphisms $\mathrm{Mor}(\Sigma,\emptyset)$ is a ring with multiplication induced by the right disjoint sum and $\mathrm{Mor}(\Sigma,\bigcirc)$ is a chronological Frobenius algebra over $\mathrm{Mor}(\Sigma,\emptyset)$.*

The case $\Sigma=\bigcirc$ was analyzed in Example 5.1.5 under the assumption $X+Y=0$, in which case $\mathrm{Mor}(\bigcirc,\bigcirc)$ was a free rank 2 module over $\mathrm{Mor}(\bigcirc,\emptyset)\cong\mathbb{Z}[H,X,Z^{\pm 1}]/(2H,X^2-1)$.

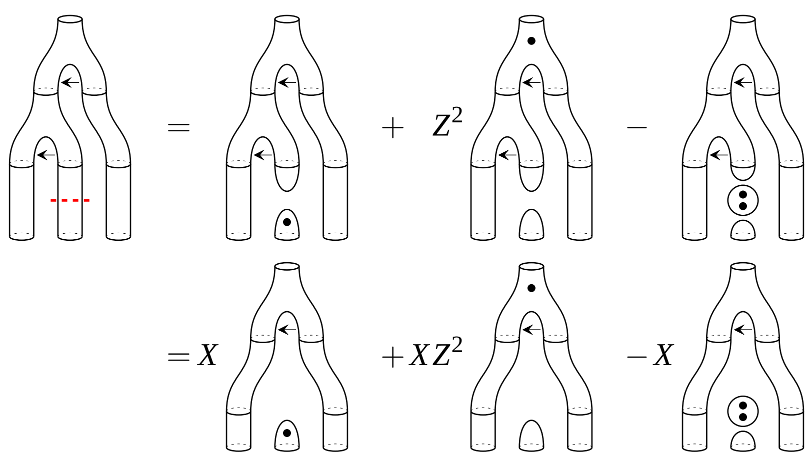
However, the rank of $\mathrm{Mor}(\Sigma, \bigcirc)$ over $\mathrm{Mor}(\Sigma, \emptyset)$ is in general infinite, but the neck-cutting relation (5.11) suggests a way how to reduce it to the finite case.

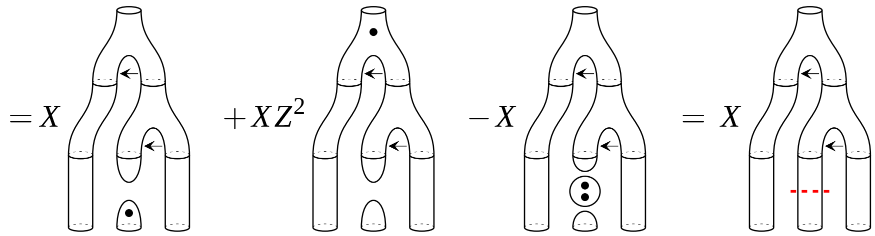
Definition 5.2.2. The category $\Bbbk\mathbf{ChCob}_\bullet$ consists of chronological cobordisms with dots on regular levels. A single dot has a degree $(-1,-1)$ and two dots cannot lie on the same level. In addition to chronological relations, we allow dots to move past other dots and critical points at the cost specified by λ, and we impose the following three local relations:



where all deaths are oriented clockwise.

Dots are a part of the chronological structure and one can think of them as ‘infinitesimal’ handles, which are ‘frozen’, so that a dot is not annihilated by $1-XY$. But a cobordism with two dots on one component is, because permuting two dots costs XY. All relations are homogeneous, thence coherent with changes of chronologies. Even more: the neck cutting relation N together with the cubical structure of the disjoint sum determines all coefficients for changes of chronologies, except the $\Diamond$-change. For example,

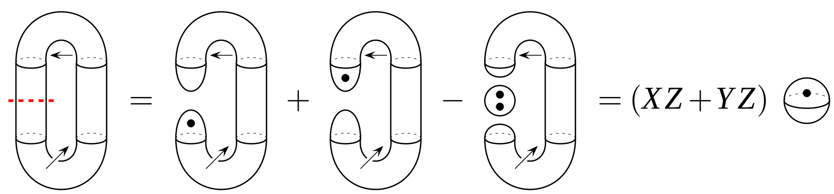




where we moved dots in the middle pictures from the birth to the top by the cost of Z^2. Dotted cobordisms satisfy also the other relations from $\Bbbk\mathbf{ChCob}_{/\ell}$. Hence, we can think of $\Bbbk\mathbf{ChCob}_{\bullet}$ as an abelian extension of $\Bbbk\mathbf{ChCob}_{/\ell}$.

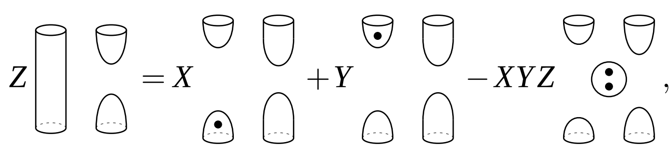
Lemma 5.2.3. *Relations T and 4Tu follow from S, D and N. Therefore, there is a natural functor* $\Bbbk\mathbf{ChCob}_{/\ell} \longrightarrow \Bbbk\mathbf{ChCob}_{\bullet}$.

Proof. For the T relation take a standard torus and cut its handle. In the resulting expression, one term has a sphere as its component and the other two can be reduced to dotted spheres by changing chronologies:

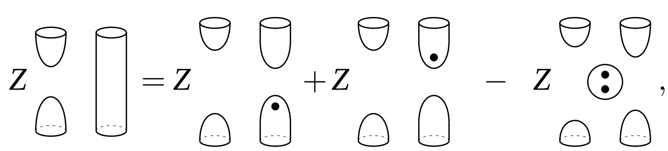


(5.14)

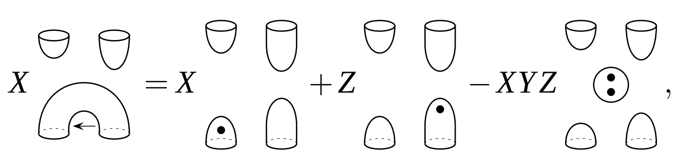
The *4Tu* relation is proved in a similar way, by cutting the unique tube in each term. Again, by changing chronologies we can reduce each term to four caps, with left caps smaller than the right ones, possibly with a two-dotted sphere in the middle:



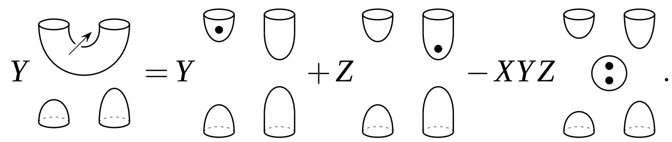
(5.15)



(5.16)



(5.17)

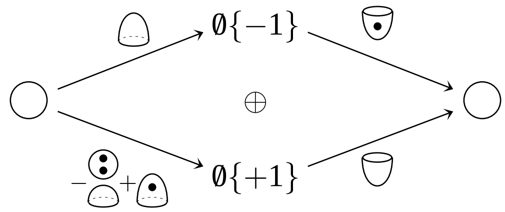


(5.18)

Because a two-dotted sphere is annihilated by $(XY-1)$, the sum of right hand sides of (5.15) and (5.16) is equal to the sum of right hand sides of (5.17) and (5.18). ☐

The additive closure $\mathrm{Mat}(\Bbbk\mathbf{ChCob_\bullet})$ is equivalent to a category of finitely generated free graded symmetric bimodules over a certain ring. This follows from the proposition below.

Proposition 5.2.4 (Delooping)**.** *The following two morphisms*



(5.19)

form a pair of inverse isomorphisms in the additive closure $\mathrm{Mat}(\Bbbk\mathbf{ChCob_\bullet})$.

Proof. Call the left map f and the right one g. The equality $g\circ f=\mathrm{id}$ is exactly the neck-cutting relation N, whereas the other composition is the identity 2×2 matrix—this follows directly from relations D and S. ☐

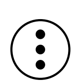
Corollary 5.2.5. *The tautological functor* $\mathrm{Mor}(\emptyset,\_)\colon \Bbbk\mathbf{ChCob_\bullet}\longrightarrow\mathbf{Mod}_R$ *is full and faithful, where* $R:=\mathrm{Mor}(\emptyset,\emptyset)$. *Hence, we can identify* $\Bbbk\mathbf{ChCob_\bullet}$ *with the category of finitely generated free graded symmetric* $\mathrm{Mor}(\emptyset,\emptyset)$*-bimodules.*

We shall now compute a presentation of the ring $\mathrm{Mor}(\emptyset,\emptyset)$.

Proposition 5.2.6. *There is an isomorphism of graded commutative rings*

$$\mathrm{Mor}(\emptyset,\emptyset)\cong R_\bullet := {}^{\Bbbk[h,t]}\big/_{\big((XY-1)t,\,(XY-1)h\big)},\tag{5.20}$$

where $\deg h=(-1,-1)$ *and* $\deg t=(-2,-2)$*, such that*

 $\longmapsto h$ *and*  $\longmapsto XZt+h^2.$ (5.21)

Proof. It is enough to show that the above defines a homomorphism—it is clearly invertible if it exists. We begin with constructing a graded monoidal functor $\mathscr{F}_\bullet\colon \Bbbk\mathbf{ChCob}_\bullet \longrightarrow \mathbf{Mod}_{R_\bullet}$. For that take a free rank two symmetric bimodule $A_\bullet = R_\bullet v_+ \oplus R_\bullet v_-$ with $\deg v_+ = (1,0)$ and $\deg v_- = (0,-1)$ as usual. This module is a chronological Frobenius algebra with operations

$$\mu\colon A_\bullet \otimes A_\bullet \longrightarrow A_\bullet, \qquad \begin{cases} v_+\otimes v_+ \longmapsto v_+, & v_-\otimes v_+ \longmapsto XZv_-,\\ v_+\otimes v_- \longmapsto v_-, & v_-\otimes v_- \longmapsto tv_+ + hv_-,\end{cases} \tag{5.22}$$

$$\Delta\colon A_\bullet \longrightarrow A_\bullet \otimes A_\bullet, \qquad \begin{cases} v_+ \longmapsto v_-\otimes v_+ + YZv_+\otimes v_- - YZ^{-1}hv_+\otimes v_+,\\ v_- \longmapsto v_-\otimes v_- + Z^{-2}tv_+\otimes v_+,\end{cases} \tag{5.23}$$

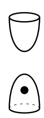
$$\eta\colon R_\bullet \longrightarrow A_\bullet, \qquad \begin{cases} 1 \longmapsto v_+,\end{cases} \tag{5.24}$$

$$\varepsilon\colon A_\bullet \longrightarrow R_\bullet, \qquad \begin{cases} v_+ \longmapsto 0,\\ v_- \longmapsto 1.\end{cases} \tag{5.25}$$

These tell us how to define $\mathscr{F}_\bullet$ on all generators except one, a cylinder decorated with a dot. Associate to it the following homomorphism:

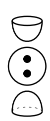
$$\theta\colon A_\bullet \longrightarrow A_\bullet, \qquad \begin{cases} v_+ \longmapsto v_-,\\ v_- \longmapsto XZ^{-1}(tv_+ + hv_-) = v_+tXZ + v_-h.\end{cases} \tag{5.26}$$

Clearly, $\varepsilon\circ\eta = 0$ and $\varepsilon\circ\theta\circ\eta = 1$, so that $\mathscr{F}_\bullet$ preserves relations S and T. It remains to show that $\mathscr{F}_\bullet$ is also coherent with the neck-cutting relation N. This follows from computing the terms on the right hand side of N:



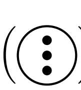
$$: A_\bullet \longrightarrow A_\bullet, \qquad \begin{cases} v_+ \longmapsto v_+,\\ v_- \longmapsto v_+\cdot h,\end{cases} \tag{5.27}$$

$$: A_\bullet \longrightarrow A_\bullet, \qquad \begin{cases} v_+ \longmapsto 0,\\ v_- \longmapsto v_-,\end{cases} \tag{5.28}$$



$$: A_\bullet \longrightarrow A_\bullet, \qquad \begin{cases} v_+ \longmapsto 0,\\ v_- \longmapsto v_+\cdot h.\end{cases} \tag{5.29}$$

Summing the first two and subtracting the last homomorphism results in the identity on $A_\bullet$. The functor $\mathscr{F}_\bullet$ induces a homomorphism $\varphi\colon \mathrm{Mor}(\emptyset,\emptyset) \longrightarrow R_\bullet$ by associating an element from the ring to any closed surface with dots. In particular, we compute

$\varphi\Big($  $\Big) = h$ and $\varphi\Big($  $\Big) = XZt + h^2,$ (5.30) which is the desired homomorphism. ☐

Remark 5.2.7. Similarly to the even case, dotted cobordisms lead us to a deformation of the odd theory, although both t and h are torsion elements: $2t = 2h = 0$ if $XY = -1$. In particular, we cannot set $t = 1$ to obtain the Lee deformation, unless we work with $\mathbb{Z}_2$ coefficients.

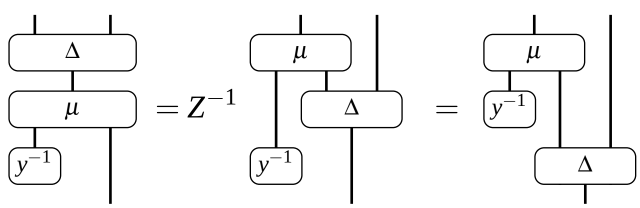
5.2.2 Universality of dotted cobordisms

The homology theory defined by the algebra $A_\bullet$ is universal: it carries the most information among all chronological Frobenius algebras producing link homology. The proof follows the argument from [Kh04] and it is based on the following observation.

Given a chronological Frobenius algebra A over a ring R and an invertible element $y \in A$ of degree $(1,0)$, we can twist its coalgebra structure by y as follows:

$$\varepsilon'(a) := \varepsilon(ya), \qquad \Delta'(a) := \Delta(y^{-1}a). \tag{5.31}$$

If Δ and ε are homogeneous, so are their twisted version Δ' and ε'. The degrees are not changed. Because $\deg y = -\deg\mu$, there is an equality $\Delta(y^{-1}a) = y^{-1}\Delta(a)$:



(5.32)

Lemma 5.2.8 (cf. [Kh04])**.** *Assume that $\mathscr{F}$ and $\mathscr{F}'$ are two functors induced by an algebra A and its twisted version A'. Then the complexes $\mathscr{F}Kh(L)$ and $\mathscr{F}'Kh(L')$ are isomorphic.*

Proof. Consider cubes $\mathscr{F}\mathscr{I}^\varepsilon(L)$ and $\mathscr{F}'\mathscr{I}^\varepsilon(L)$, both corrected by a sign assignment ε. They have the same R-modules in vertices and the only difference is in edges labeled with comultiplications. The isomorphism is constructed inductively, starting with the identity homomorphism on the initial vertex $(0,\dots,0)$ and applying the following rule at every face:

$$\begin{array}{ccc} \mathscr{F}\mathscr{I}_{\xi} & \xrightarrow{f} & \mathscr{F}'\mathscr{I}_{\xi'} \\ \downarrow \mu & & \downarrow \mu \\ \mathscr{F}\mathscr{I}_{\xi'} & \xrightarrow{f} & \mathscr{F}'\mathscr{I}_{\xi} \end{array} \qquad \begin{array}{ccc} \mathscr{F}\mathscr{I}_{\xi} & \xrightarrow{f} & \mathscr{F}'\mathscr{I}_{\xi'} \\ \downarrow \Delta & & \downarrow \Delta \\ \mathscr{F}\mathscr{I}_{\xi'} & \xrightarrow{y^{-1}\cdot f} & \mathscr{F}'\mathscr{I}_{\xi} \end{array} \tag{5.33}$$

where in the case of a split we multiply by y^{-1} the element from the copy of A corresponding to the circle that appears to the left of the split. □

We constructed in Section 3.3 the universal rank 2 chronological Frobenius system (R_U, A_U). It appears that this system can be twisted to the algebra of dotted cobordisms $(R_\bullet, A_\bullet)$, showing the latter produces universal Khovanov homology in our framework.

Theorem 5.2.9. *Any homogeneous rank two chronological Frobenius system (R,A) is obtained from $(R_\bullet, A_\bullet)$ by a base change and a twist. In particular, $\mathscr{H}_\bullet(L) := H(\mathscr{F}_\bullet Kh(L))$ is the most general link homology theory in our framework.*

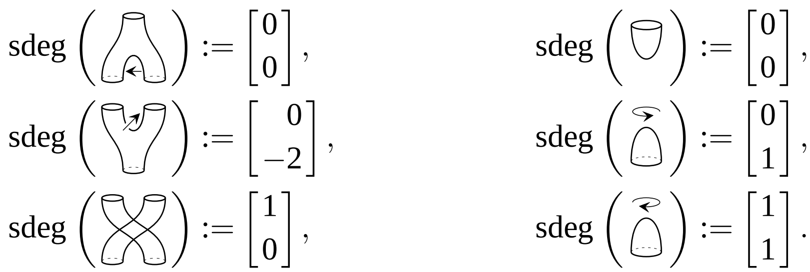
Proof. An element $y = ev_+ + YZfv_- \in A_U$ is invertible and of degree $(1,0)$, with an inverse $y^{-1} = (a+ch)v_+ - YZcv_-$. The dotted algebra $A_\bullet$ arises as the twisting of (R_U, A_U) by this element. □

5.3 Reduction of scalars

The universal link homology theory has coefficients in the ring $\Bbbk$ with three generators, two of which can be eliminated by introducing a new grading to the construction. The grading decomposes the generalized Khovanov homology over $\Bbbk$ into homogeneous pieces, all isomorphic to each other. For this reason we call it a *splitting grading*.

5.3.1 An additional grading in $\Bbbk$ChCob

The category $\Bbbk$**ChCob** of chronological cobordisms admits an additional grading by the group $\mathbb{Z}_2 \times \mathbb{Z}$, which we shall refer to as the *splitting degree*, given by a function sdeg with the following values on the generating cobordisms:

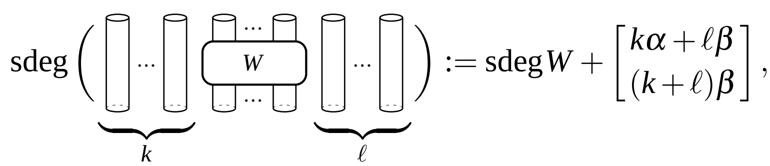


(5.34)

(5.35)

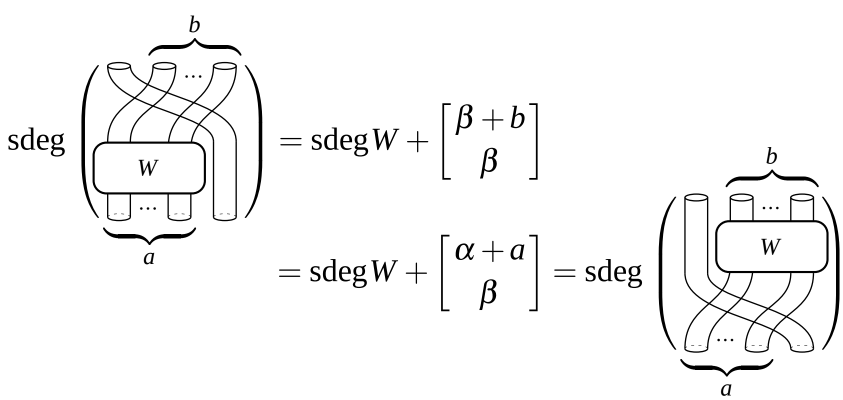
(5.36)

In addition, we grade the coefficient ring $\Bbbk$ by letting $\operatorname{sdeg} X = \operatorname{sdeg} Y = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\operatorname{sdeg} Z = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. We use the vertical notation for elements of $\mathbb{Z}_2 \times \mathbb{Z}$ to distinguish it from the bidegree defined in Section 3.3.1. This degree is not additive with respect to the disjoint union; instead we set



(5.37)

assuming $\deg W = (\alpha, \beta)$. The above formula is clearly additive with respect to composition of cobordisms, and it is coherent with the symmetry:

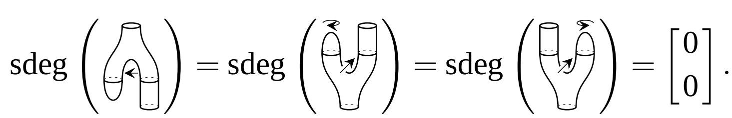


(5.38)

where $\deg W = (\alpha, \beta)$, and the equality $\alpha + a = \beta + b$ follows from Lemma 3.3.3.

Proposition 5.3.1. *The splitting degree is coherent with chronological relations.*

Proof. Creation and annihilation changes preserve the degree, as we can directly compute



(5.39)

Choose cobordisms $W_i\colon a_i\mathbb{S}^1 \longrightarrow b_i\mathbb{S}^1$ for $i = 1,2$. If $\deg W_i = (m_i, s_i)$, we have

$$\operatorname{sdeg}(W_1 \mathbin{{\uparrow}\!\sqcup\!{\downarrow}} W_2) = \operatorname{sdeg}(W_1 \sqcup C_{b_2\mathbb{S}^1}) + \operatorname{sdeg}(C_{a_1\mathbb{S}^1} \sqcup W_2), \tag{5.40}$$

$$\operatorname{sdeg}(W_1 \mathbin{{\downarrow}\!\sqcup\!{\uparrow}} W_2) = \operatorname{sdeg}(W_1 \sqcup C_{a_2\mathbb{S}^1}) + \operatorname{sdeg}(C_{b_1\mathbb{S}^1} \sqcup W_2), \tag{5.41}$$

where $C_{n\mathbb{S}^1}$ is a disjoint union of n vertical tubes. Using the formula (5.37) we compute

$$\begin{aligned} \operatorname{sdeg}(W_1 \mathbin{{\uparrow}\!\sqcup\!{\downarrow}} W_2) - \operatorname{sdeg}(W_1 \mathbin{{\downarrow}\!\sqcup\!{\uparrow}} W_2) &= \begin{bmatrix} (b_2 - a_2)s_1 \\ (b_2 - a_2)s_1) \end{bmatrix} + \begin{bmatrix} (a_1 - b_1)m_2 \\ (a_1 - b_1)s_2) \end{bmatrix} \\ &= \begin{bmatrix} s_1s_2 + m_1m_2 \\ s_1m_2 - m_1s_2 \end{bmatrix} = \operatorname{sdeg}\left(X^{m_1m_2}Y^{s_1s_2}Z^{m_1s_2 - s_1m_2}\right), \end{aligned} \tag{5.42}$$

which shows that $W_1 \mathbin{{\uparrow}\!\sqcup\!{\downarrow}} W_2$ and $\lambda(\deg W_1, \deg W_2) W_1 \mathbin{{\downarrow}\!\sqcup\!{\uparrow}} W_2$ have the same degree. Likewise the degrees of $W_1 \# W_2$ and $\lambda(\deg W_1, \deg W_2) W_1 \# W_2$ are equal—the same computation works, with a_i and b_i replaced by $a_i - 1$ and $b_i - 1$. □

Remark 5.3.2. As before, we extend $\Bbbk$**ChCob** to a graded category by introducing formal degree shifts by elemenst of $\mathbb{Z}_2 \times \mathbb{Z}$. Hence, $\Bbbk$**ChCob** is graded by $\mathbb{Z}_2 \times \mathbb{Z}^3$, the first two components coming from sdeg and the last two from deg. We write $\Bbbk$**ChCob**$_0$ for the subcategory formed by graded morphisms.

Let $\Bbbk_0 \subset \Bbbk$ be the subring of degree zero elements. It is generated by XY, and as such it is isomorphic to $\mathbb{Z}_\pi := \mathbb{Z}[\pi]/(\pi^2 - 1)$. On the other hand, there is a ring epimorphism $\Bbbk \longrightarrow \mathbb{Z}_\pi$ sending both X and Z to 1, and Y to π, resulting in a functor $\Bbbk$**ChCob** $\longrightarrow \mathbb{Z}_\pi$**ChCob**.

Lemma 5.3.3. *The pair of $\Bbbk_0$-linear functors $I\colon \mathbb{Z}_\pi$**ChCob** $\rightleftarrows \Bbbk$**ChCob**$_0$ $:P$,*

$$I(\Sigma) := \Sigma\{0,0\}, \qquad\qquad P(\Sigma\{a,b\}) := \Sigma,$$

$$I(\pi^k W) := X^{a+k}Y^kZ^bW, \quad \operatorname{sdeg} W = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right], \qquad\qquad P(X^pY^qZ^rW) := \pi^q W,$$

is an equivalence of categories.

Proof. Clearly $PI = \mathrm{id}$, and morphisms $\Sigma\{a,b\} \xrightarrow{\cdot X^aZ^b} \Sigma$ form an isomorphism $\mathrm{id} \cong IP$. □

We shall use this result later to show that the two generalized Khovanov complexes, either with coefficients in $R = \Bbbk$ or $R = \mathbb{Z}_\pi$, carry the same amount of information.

5.3.2 Decomposition of a chronological TQFT

From now on let $\mathbf{Mod}_{\Bbbk}$ stand for the category of $\Bbbk$-modules that admit a $\mathbb{Z}_2 \times \mathbb{Z}$-grading compatible with the grading of $\Bbbk$, and we write $\mathbf{Mod}_{\Bbbk,0}$ for the subcategory formed by linear maps that preserve the new degree. Again, the new grading is not additive with respect to the tensor product, but instead we set

$$\operatorname{sdeg}(m \otimes n) := \operatorname{sdeg}(m) + \operatorname{sdeg}(n) + \begin{bmatrix} \beta \|n\| \\ \beta \|n\| \end{bmatrix}, \tag{5.43}$$

for homogeneous $m \in M$ and $n \in N$, where $\deg m = (\alpha, \beta)$ and $\|n\|$ is the *weight* of n: the difference of the two components of deg (e.g. $\|m\| = \alpha - \beta$). The name is motivated by the behavior of the symmetry isomorphism—it is homogeneous only when restricted to submodules supported in a single weight.

Lemma 5.3.4. *The associator $(M_1 \otimes M_2) \otimes M_3 \longrightarrow M_1 \otimes (M_2 \otimes M_3)$ preserves the splitting degree. Moreover, if M_1 and M_2 are supported in weights w_1 and w_2 respectively, then the symmetry isomorphism $\tau\colon M_1 \otimes M_2 \longrightarrow M_2 \otimes M_1$ is homogeneous of degree* $\operatorname{sdeg} \tau = \left[\begin{smallmatrix} w_1 w_2 \\ 0 \end{smallmatrix}\right]$.

Proof. Choose homogeneous $m_i \in M_i$, $i = 1, 2, 3$ of bidegrees $\deg m_i = (\alpha_i, \beta_i)$. Using formula (5.43) we compute

$$\begin{aligned}
\operatorname{sdeg}((m_1 \otimes m_2) \otimes m_3) &= \operatorname{sdeg}(m_1 \otimes m_2) + \operatorname{sdeg}(m_3) + \begin{bmatrix} (\beta_1 + \beta_2)\|m_3\| \\ (\beta_1 + \beta_2)\|m_3\| \end{bmatrix} \\
&= \operatorname{sdeg}(m_1) + \operatorname{sdeg}(m_2) + \operatorname{sdeg}(m_3) + \begin{bmatrix} \beta_1\|m_2\| + \beta_1\|m_3\| + \beta_2\|m_3\| \\ \beta_1\|m_2\| + \beta_1\|m_3\| + \beta_2\|m_3\| \end{bmatrix} \\
&= \operatorname{sdeg}(m_1) + \operatorname{sdeg}(m_2 \otimes m_3) + \begin{bmatrix} \beta_1\|m_2 \otimes m_3\| \\ \beta_1\|m_2 \otimes m_3\| \end{bmatrix} \\
&= \operatorname{sdeg}(m_1 \otimes (m_2 \otimes m_3)).
\end{aligned}$$

For the second statement we compute $\tau(m_1\otimes m_2) = X^{\alpha_1\alpha_2}Y^{\beta_1\beta_2}Z^{\alpha_1\beta_2-\beta_1\alpha_2}m_2\otimes m_1$, so that

$$\begin{aligned}
&\operatorname{sdeg}(\tau(m_1\otimes m_2)) - \operatorname{sdeg}(m_1\otimes m_2) = \\
&\quad = \begin{bmatrix}\alpha_1\alpha_2+\beta_1\beta_2\\ \beta_1\alpha_2-\alpha_1\beta_2\end{bmatrix} + \operatorname{sdeg}(m_2\otimes m_1) - \operatorname{sdeg}(m_1\otimes m_2)\\
&\quad = \begin{bmatrix}\alpha_1\alpha_2+\beta_1\beta_2\\ \beta_1\alpha_2-\alpha_1\beta_2\end{bmatrix} + \begin{bmatrix}\beta_2 w_1\\ \beta_2 w_1\end{bmatrix} - \begin{bmatrix}\beta_1 w_2\\ \beta_1 w_2\end{bmatrix}\\
&\quad = \begin{bmatrix}(\alpha_1-\beta_1)(\alpha_2-\beta_2)\\ 0\end{bmatrix} = \begin{bmatrix}w_1w_2\\ 0\end{bmatrix}.
\end{aligned}$$

□

Lemma 5.3.5. *Choose a homogeneous map $f\colon M \longrightarrow N$ and two modules M' and M'' supported in weights k and ℓ respectively. Then*

$$\operatorname{sdeg}(\mathrm{id}_{M'}\otimes f\otimes \mathrm{id}_{M''}) = \operatorname{sdeg} f + \begin{bmatrix}k\alpha+\ell\beta\\ (k+\ell)\beta\end{bmatrix}, \tag{5.44}$$

where $\deg f = (\alpha,\beta)$*. In particular, a tensor product of homogeneous maps is homogeneous, and so is the graded tensor product relation* (2.2)*.*

Proof. Pick homogeneous $m_1\in M'$, $m_2\in M$ and $m_3\in M''$, each in bidegree $\deg(m_i) = (\alpha_i,\beta_i)$. Then

$$\begin{aligned}
&\operatorname{sdeg}((\mathrm{id}\otimes f\otimes\mathrm{id})(m_1\otimes m_2\otimes m_3)) - \operatorname{sdeg}(m_1\otimes m_2\otimes m_3)\\
&\quad = \operatorname{sdeg}\left(X^{\alpha_1\alpha}Y^{\beta_1\beta}Z^{\alpha_1\beta-\beta_1\alpha}m_1\otimes f(m_2)\otimes m_3\right) - \operatorname{sdeg}(m_1\otimes m_2\otimes m_3)\\
&\quad = \operatorname{sdeg} f + \begin{bmatrix}\beta_1\beta+\alpha_1\alpha\\ \beta_1\alpha-\beta\alpha_1\end{bmatrix} + \begin{bmatrix}\beta_1(\alpha-\beta)+\beta(\alpha_3-\beta_3)\\ \beta_1(\alpha-\beta)+\beta(\alpha_3-\beta_3)\end{bmatrix} = \operatorname{sdeg} f + \begin{bmatrix}k\alpha+\ell\beta\\ (k+\ell)\beta\end{bmatrix}.
\end{aligned}$$

The last statement follows from a direct computation, as in Proposition 5.3.1. □

Consider now the covering Frobenius algebra $A = \Bbbk v_+ \oplus \Bbbk v_-$ from Example 3.3.13. Its generators have weights $\|v\|_+ = \|v\|_- = 1$, implying that

$$\operatorname{sdeg}(\mathrm{id}_{A^{\otimes k}}\otimes f\otimes \mathrm{id}_{A^{\otimes\ell}}) = \operatorname{sdeg} f + \begin{bmatrix}k\alpha+\ell\beta\\ (k+\ell)\beta\end{bmatrix}, \tag{5.45}$$

which is similar to formula (5.37). We define the splitting degree on A by setting $\operatorname{sdeg} v_+ = \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$ and $\operatorname{sdeg} v_- = \left[\begin{smallmatrix}1\\-1\end{smallmatrix}\right]$; Table 5.1 contains degrees of generators of $A^{\otimes 2}$.

Generator	$v_+ \otimes v_+$	$v_+ \otimes v_-$	$v_- \otimes v_+$	$v_- \otimes v_-$
deg:	$(2,0)$	$(1,-1)$	$(1,-1)$	$(0,-2)$
sdeg:	$\begin{bmatrix}0\\0\end{bmatrix}$	$\begin{bmatrix}1\\-1\end{bmatrix}$	$\begin{bmatrix}0\\-2\end{bmatrix}$	$\begin{bmatrix}1\\-3\end{bmatrix}$

Table 5.1: Degrees of generators of the second power of A.

Lemma 5.3.6. *A generator $v = v_k \otimes \cdots \otimes v_1 \in A^{\otimes k}$ is homogeneous of degree* $\operatorname{sdeg} v = \left[\begin{smallmatrix} a \\ a \end{smallmatrix}\right]$ *with $a = -\sum_{v_i = v_-} i$.*

Proof. The lemma follows from an easy induction argument and is left to the reader. □

Proposition 5.3.7. *The functor $\mathscr{F}_A \colon \Bbbk\mathbf{ChCob} \longrightarrow \mathbf{Mod}_{\Bbbk}$ preserves the splitting degree.*

Proof. In the view of Lemma 5.3.4 and formula (5.45) it is enough to check that the four operations of the chronological Frobenius algebra A have the same degrees as the corresponding cobordisms. This follows directly from the expressions for these maps and Lemma 5.3.6. □

The same results hold if we replace A with the algebra $A_\bullet$ of cobordisms with dots. Despite the ring of scalars $R_\bullet$ begin graded, each of its homogeneous components has weight 0. In particular, sdeg is additive with respect to multiplication, see formula (5.44), so that all elements of $A_\bullet$ have weight 1. Then, both the formula (5.45) and Lemma 5.3.6 hold for $A_\bullet$. However, one must be careful when proving the analogue of Proposition 5.3.7: although the left action of R on A is graded, $\operatorname{sdeg}(r \cdot v_\pm) \neq \operatorname{sdeg} r + \operatorname{sdeg} v_\pm$. For instance, $\operatorname{sdeg}(hv_+) = \operatorname{sdeg} h + \operatorname{sdeg} v_+ - \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$. On the other hand, the splitting degree is additive with respect to the right action of $R_\bullet$.

There is an equivalence between categories $\mathbf{Mod}_{\Bbbk,0}$ and $\mathbf{Mod}_{\mathbb{Z}_\pi}$ similar to the one from Lemma 5.3.3. Extracting the degree 0 component M_0 of a $\Bbbk$-module M results in a functor $r \colon \mathbf{Mod}_{\Bbbk,0} \longrightarrow \mathbf{Mod}_{\mathbb{Z}_\pi}$. On the other hand, given a $\mathbb{Z}_\pi$-module N one creates a $\Bbbk$-module $i(N) := \bigoplus_{(a,b)\in\mathbb{Z}_2\times\mathbb{Z}} N$, where $(XY)\cdot n = \pi n$, while X and Z permute the copies of N in $i(N)$.

Lemma 5.3.8. *The pair of functors $i \colon \mathbf{Mod}_{\mathbb{Z}_\pi} \rightleftarrows \mathbf{Mod}_{\Bbbk,0} \colon r$ is an equivalence of categories.*

Proof. Straightforward. $\square$

The two equivalences intertwine $\mathscr{F}_A\colon \Bbbk\mathbf{ChCob}_0 \to \mathbf{Mod}_{\Bbbk,0}$ and $\mathscr{F}_\pi\colon \mathbb{Z}_\pi\mathbf{ChCob} \to \mathbf{Mod}_{\mathbb{Z}_\pi}$, which we shall use in the next section to reduce the number of parameters in the generalized Khovanov homology with no loss of information.

5.3.3 Decomposition of homology

Choose a link diagram D. Its corrected cube of resolutions $\mathscr{I}^{\varepsilon}_{\mathrm{gr}}(D)$ anticommutes, so that any two oriented paths between two vertices encode chronological cobordisms of the same splitting degree. Therefore, we can make $\mathscr{I}^{\varepsilon}_{\mathrm{gr}}(D)$ graded with respect to this new degree by applying certain degree shifts to its vertices, as we did for the $\mathbb{Z}\times\mathbb{Z}$-grading. As a result we obtain a new grading in homology, and our goal is to show it is a link invariant.

Theorem 5.3.9. *The homotopy type of the graded generalized Khovanov complex is a link invariant. In particular, the generalized Khovanov homology $\mathscr{H}(L)$ admits a $\mathbb{Z}_2\times\mathbb{Z}$-grading coherent with the action of $\Bbbk$.*

For Theorem 5.3.9 to make sense, the relations *S*, *T* and *4Tu* must be homogeneous. This follows from a direct computation. The next step is to show that all isomorphisms involved in the proof of invariance from Section 4.1 are homogeneous—this is enough, as any homogeneous isomorphism can be made graded by scaling it with some monomial X^aZ^b. We first show that the grading does not depend on the extra choices made in the construction of the generalized Khovanov bracket. The key tool is the following result.

Lemma 5.3.10. *Suppose there is a commutative square in $\Bbbk\mathbf{ChCob}$*

$$\begin{array}{ccc} \Sigma_0 & \xrightarrow{f_0} & \Sigma_1 \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ \Sigma_0' & \xrightarrow{f_1} & \Sigma_1' \end{array} \tag{5.46}$$

where each morphism is a chronological cobordism scaled by an invertible element from $\Bbbk$. If each f_i is graded with respect to the splitting degree, $\operatorname{sdeg} g = \operatorname{sdeg} g'$.

Proof. It is enough to show that the composition $f_1 g = g' f_0$ does not vanish. This follows from Theorem 3.3.9. $\square$

Sign assignments. Given two sign assignments ε_1 and ε_2 of the cube $\mathscr{I}_{\mathrm{gr}}(D)$, the corrected cubes $\mathscr{I}^{\varepsilon_1}_{\mathrm{gr}}(D)$ and $\mathscr{I}^{\varepsilon_2}_{\mathrm{gr}}(D)$ are isomorphic via a family of morphisms $f_\xi := \nu(\xi)\,\mathrm{id}$, where $\nu \in C^0(I^n;\Bbbk^*)$ is a cochain such that $\varepsilon 1_2 = \delta\nu \cdot \varepsilon_1$. Hence, each f_ξ is a homogeneous map, and their degrees are equal due to Lemma 5.3.10.

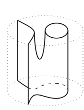
Arrows over crossings. Independence of this decoration follows trivially. Indeed, given link diagrams D and D' that differ only in the direction of arrows decorating the crossings and a sign assignment ε for $\mathscr{I}(D)$, we have constructed a sign assignment ε' for $\mathscr{I}(D')$ such that $\mathscr{I}^{\varepsilon}(D) = \mathscr{I}^{\varepsilon\lambda^{-1}}(D')$.

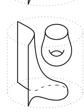
Orderings on crossings and circles. A change in enumeration of crossings permutes only the summands of $[\![D]\!]$. On the other hand, each component of the isomorphism of cubes that reorders circles in vertices is a composition of twists. Hence, it is homogeneous, and we again use Lemma 5.3.10 to deduce all components have the same splitting degree.

Corollary 5.3.11. *The isomorphism class of the graded generalized Khovanov bracket $[\![D]\!]$ depends only on the link diagram D.*

We shall now proceed to Reidemeister moves.

Reidemeister I. Consider the chain homotopy equivalences $f\colon [\![$  $]\!] \rightleftarrows [\![$  $]\!] : g$ as defined in Section 4.3.1. It follows directly from Lemma 5.3.10 that g induces a homogeneous chain map, and for f we have to check that the degrees of its two components are equal. Indeed,

sdeg$\Big(Z$  $\Big) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \end{bmatrix},$ (5.47)

sdeg$\Big(X$  $\Big) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ -3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$ (5.48)

after forgetting the circles not shown in the diagrams, and placing the circle drawn in full as the first one.

Reidemeister II. Consider the chain homotopy equivalences $f\colon [\![\;]\!] \rightleftarrows [\![\;]\!] : g$ as defined in Section 4.3.2, and choose degree shifts such that the identity components $[\![\;]\!] \rightleftarrows [\![\;]\!]$ are graded. Then $f_{01} = h_{*1}d_{1*}$ and $g_{01} = d_{*0}h_{0*}$ are also graded, because the differentials and homotopies are such.

Reidemeister III. Invariance under the last move followed from a strictly algebraic argument: the complex $[\![\;]\!]$ is the mapping cone of the chain map $[\![\;]\!]\colon [\![\;]\!] \longrightarrow [\![\;]\!]$, and composing it with the chain homotopy equivalence $f\colon [\![\;]\!] \longrightarrow [\![\;]\!]$ does not change the homotopy type of the mapping cone. As the map f is graded, so are the homotopy equivalences

$$[\![\;]\!] \simeq C([\![\;]\!] \longrightarrow [\![\;]\!]) \simeq C([\![\;]\!] \longrightarrow [\![\;]\!]) \simeq [\![\;]\!]. \tag{5.49}$$

This ends the proof of Theorem 5.3.9. □

Corollary 5.3.12. *The generalized Khovanov complexes $Kh(D)$ and $Kh_\pi(D)$, constructed in $\Bbbk$**ChCob**$_0$ and $\mathbb{Z}_\pi$**ChCob** respectively, are equivalent link invariants, i.e. $Kh(D) \simeq Kh(D')$ for link diagrams D and D' if and only if $Kh_\pi(D) \simeq Kh_\pi(D')$.*

Proof. Follows directly from Lemma 5.3.3. □

We are now ready to prove the main result of this section.

Theorem 5.3.13 (The reduction of parameters)**.** *The generalized Khovanov complex $\mathscr{F}_A Kh(D)$, regarded as a complex of $\mathbb{Z}_\pi$-modules, decomposes into a direct sum of subcomplexes*

$$\mathscr{F}_A Kh(D) \cong \bigoplus_{(a,b)\in\mathbb{Z}_2\times\mathbb{Z}} \mathscr{F}_A Kh(D)_{a,b}, \tag{5.50}$$

each isomorphic to $\mathscr{F}_A Kh(D;\mathbb{Z}_\pi) \cong \mathscr{F}_\pi Kh_\pi(D)$. The action of $\Bbbk$ is given by isomorphisms

$$\begin{cases} X\colon \mathscr{F}_A Kh(D)_{a,b} \xrightarrow{\ \mathrm{id}\ } \mathscr{F}_A Kh(D)_{a+1,b}, \\ Y\colon \mathscr{F}_A Kh(D)_{a,b} \xrightarrow{\ \cdot\pi\ } \mathscr{F}_A Kh(D)_{a+1,b}, \\ Z\colon \mathscr{F}_A Kh(D)_{a,b} \xrightarrow{\ \mathrm{id}\ } \mathscr{F}_A Kh(D)_{a,b+1}. \end{cases} \tag{5.51}$$

In particular, $\mathscr{H}(L) \cong i(\mathscr{H}_\pi(L))$, where $i\colon \mathbf{Mod}_{\mathbb{Z}_\pi} \longrightarrow \mathbf{Mod}_\Bbbk$ is the equivalence functor from Lemma 5.3.8.

Proof. The decomposition follows from Theorem 5.3.9, so it remains to compute the degree zero subcomplex. First, $r(M)$ is naturally isomorphic to $M \otimes \mathbb{Z}_\pi$ via $m \longmapsto m \otimes 1$. Indeed, this map is linear over $\mathbb{Z}_\pi$, and its inverse sends $m \otimes 1$, with $\operatorname{sdeg}(m) = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$, into $X^a Z^b m$. Hence, $\mathscr{F}_A Kh(D)_{0,0}$ is naturally isomorphic to $\mathscr{F}_A Kh(D) \otimes \mathbb{Z}_\pi = \mathscr{F}_A Kh(D; \mathbb{Z}_\pi)$. □

Given a graded ring automorphism $\varphi \in \operatorname{Aut}_0(\Bbbk)$ we can replace the chronological parameters X, Y, and Z with its images under φ, resulting in a graded category $\Bbbk_\varphi\mathbf{ChCob}_0$ and a chronological TQFT $\mathscr{F}_\varphi \colon \Bbbk_\varphi\mathbf{ChCob}_0 \longrightarrow \mathbf{Mod}_{\Bbbk,0}$. As before, given a link diagram D we can construct the generalized Khovanov complex $Kh_\varphi(D)$ in $\operatorname{Mat}(\Bbbk_\varphi\mathbf{ChCob})$. In the view of Corollary 5.3.12, the complexes $\mathscr{F}_A Kh(D)$ and $\mathscr{F}_\varphi Kh_\varphi(D)$ are equivalent link invariants if $\varphi(XY) = XY$. We shall now show they are in fact isomorphic.

Proposition 5.3.14. *Assume $\varphi(XY) = XY$. Then the complexes of $\Bbbk$-modules $\mathscr{F}Kh(D)$ and $\mathscr{F}_\varphi Kh_\varphi(D)$ are isomorphic for any link diagram D.*

Proof. Decompose the complexes as in Theorem 5.3.13. Then $\mathscr{F}Kh(D)_{0,0}$ and $\mathscr{F}_\varphi Kh_\varphi(D)_{0,0}$ are complexes of free $\mathbb{Z}_\pi$-modules, and φ induces an isomorphism between them. Indeed, π acts on both complexes as multiplication by $XY = \varphi(XY)$. Thence, it is enough to extend the equality in a $\Bbbk$-linear way. Explicitly,

$$\mathscr{F}Kh(D) \ni u \longmapsto \left(\tfrac{\varphi(X)}{X}\right)^a \left(\tfrac{\varphi(Z)}{Z}\right)^b u \in \mathscr{F}_\varphi Kh_\varphi(D) \tag{5.52}$$

for a generator $u = v_{i_1} \otimes \ldots \otimes v_{i_k}$ of degree $\operatorname{sdeg}(u) = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$.[1] □

Denote by $\Bbbk_\varphi$ the ring $\Bbbk$ with a module structure twisted by φ, i.e. $k \cdot x := \varphi(k)x$. Every $\Bbbk$-module structure on $\mathbb{Z}$ can be obtained by taking a tensor product $\Bbbk_\varphi \otimes \mathbb{Z}_{ev}$ or $\Bbbk_\varphi \otimes \mathbb{Z}_{odd}$ for an automorphism φ fixing XY. For instance, if $\varphi(X) = -X$ and likewise for Y and Z, then each parameter acts on $\mathbb{Z}' := \Bbbk_\varphi \otimes \mathbb{Z}_{ev}$ as -1.

Corollary 5.3.15. *Given a $\Bbbk$-module structure on $\mathbb{Z}$, the homology $\mathscr{H}(L;\mathbb{Z})$ is either the even Khovanov homology, if XY acts on $\mathbb{Z}$ as identity, or the odd Khovanov homology otherwise.*

[1] Here, $\operatorname{sdeg}(u)$ is the degree of u as an element of graded $\mathscr{F}Kh$, and it can be different as when u is regarded as an element $A^{\otimes k}$.

Remark 5.3.16. The even and odd Khovanov homology are not equivalent. Hence, the condition on φ in Proposition 5.3.14 is necessary.

5.3.4 Duality for mirror links revisited

The behavior of the generalized Khovanov complex $Kh(L)$ under taking the mirror image was disappointing: the dual complex is in another category, with the role of X and Y interchanged. However, homology of both complexes exists in the same category of $\Bbbk$-modules, and we ask for a true duality of the generalized Khovanov homology. The main issue is that a chronological Frobenius algebra cannot be self-dual: an isomorphism $A^* \cong A$ cannot exist unless $X = Y$. For instance, taking as A the covering algebra we have

$$\Delta^*(v_+^* \otimes v_-^*) = YZv_+^*, \quad \text{and} \tag{5.53}$$

$$\mu^*(v_-^*) = v_+^* \otimes v_-^* + XZv_-^* \otimes v_+^*. \tag{5.54}$$

We shall use the splitting from Theorem 5.3.13 to exchange X and Y back. In the following we write $\overline{\Bbbk}$ for the module $\Bbbk$ with exchanged actions of X and Y.

Theorem 5.3.17 (Duality for generalized Khovanov homology)**.** *Given a link diagram D and its mirror image $D^!$ there is an isomorphism of complexes*

$$\mathscr{F}_A Kh(D^!) \cong \mathscr{F}_A Kh(D)^*, \tag{5.55}$$

where $(C^*)^i := \operatorname{Hom}(C^{-i}, \Bbbk)$ for a chain complex C. In particular, the odd Khovanov homology $\mathscr{H}_{odd}(L)$ of a link L is dual to $\mathscr{H}_{odd}(L^!)$, and similarly for $\mathscr{H}_\pi(L)$ and $\mathscr{H}_\pi(L^!)$.

Proof. Propositions 4.2.4 and 5.3.14 give a sequence of isomorphisms

$$\mathscr{F}_A Kh(D^!) \cong \mathscr{F}_{\overline{\Bbbk}\otimes A} Kh_{\overline{\Bbbk}}(D^!) \cong \mathscr{F}_A Kh(D)^*. \tag{5.56}$$

The cases of $\mathscr{H}_{odd}$ and $\mathscr{H}_\pi$ follows from an isomorphism $\operatorname{Hom}_{\Bbbk}(F,\Bbbk) \otimes R \cong \operatorname{Hom}_R(F \otimes R, R)$ that holds for any free module F and a ring homomorphism $\Bbbk \longrightarrow R$. □

The duality isomorphism (5.55) is given explicitly as

$$\mathscr{F}(D^!_{\zeta}) \ni u \longmapsto (XY)^a u^* \in \mathscr{F}(D_{\bar{\zeta}})^*, \tag{5.57}$$

where $u = v_{i_1} \otimes \ldots \otimes v_{i_k}$ has degree $\operatorname{sdeg} u = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$. For the other version of Khovanov homology, simply replace XY with either π, for the unified homology, or (-1) for the odd one. Note the role of the splitting degree: although it does not descend directly to $\mathscr{H}_{odd}(L)$ nor $\mathscr{H}_{\pi}(L)$, it controls the duality isomorphism.

5.4 Properties of the generalized Khovanov homology

We shall now go quickly through the properties of the generalized Khovanov homology derived from the properties of the Khovanov complex described in Section 4.2.

5.4.1 Decategorification and a reduction to a bigraded theory

Given a chronological TQFT functor $\mathscr{F} : \Bbbk\mathbf{ChCob} \longrightarrow \mathbf{Mod}_{\Bbbk}$ that preserves the relations *S*, *T*, and *4Tu*, the homology $\mathscr{H}(L) := H^*(\mathscr{F}Kh(L))$ of a link L is a sequence of bigraded modules. Hence, its graded Euler characteristic is a two-variable polynomial

$$P_L(r,s) := \sum_{i,j,k} (-1)^i r^j s^k \operatorname{rk} \mathscr{H}^{i,j,k}(L). \tag{5.58}$$

Unfortunately, this polynomial is not stronger than the Jones polynomial. Indeed, the third exact sequence from Proposition 4.2.2 implies the following equality

$$(rs)P_{\times} - (rs)^{-1}P_{\times} = ((rs)^{1/2} - (rs)^{-1/2})P_{)(}, \tag{5.59}$$

so that $P_L(r,s) = J_L(\sqrt{rs})\,P_{\bigcirc}$. It appears that the extra grading on homology is degenerate as well. It is enough to show this for the Frobenius system $(R_{\bullet}, A_{\bullet})$, as we proved its universality.

Proposition 5.4.1. *The homology group* $\mathscr{H}^{i,j,k}_{\bullet}(L)$ *is trivial, unless* $j = k$.

Proof. The proposition follows trivially for the unknot, since $\mathscr{H}_\bullet(\bigcirc) \cong R_\bullet\{-\frac{1}{2}, -\frac{1}{2}\} \oplus R_\bullet\{\frac{1}{2}, \frac{1}{2}\}$ and all homogeneous elements of $R_\bullet$ have degrees of the form (i,i). The general case follows then from Proposition 4.2.2 using induction on complexity of link diagrams. ☐

The above proposition can be also proven directly by checking that all homogeneous generators of $\mathscr{F}_\bullet Kh(D)$ have degrees of the form (i,i).

Remark 5.4.2. In spite of the proposition above, the additional grading is useful. For instance, it is a key component in understanding the isomorphism between a tensor product of complexes for two links and the complex for their disjoint union.

5.4.2 Homological thinness and quasi-alternating links

Define the class of quasi-alternating links as the smallest family of links satisfying the following two conditions:

1) the unknot is quasi-alternating, and
2) if a link admits a diagram with a crossing $\times$, such that its both resolutions $\asymp$ and $)($ are quasi-alternating and $\det(\times) = \det(\asymp) + \det()()$, then the link is also quasi-alternating,

where $\det(L)$ is the *determinant* of a link L, defined below. All alternating links are quasi-alternating, and it was shown in [OM07] that they are *homologically thin*: the Khovanov homology of such a link L is supported in two diagonals with $2i - j = \sigma(L) \pm 1$, where $\sigma(L)$ is the *signature* of a link L. It is not a surprise that the same holds for the generalized Khovanov homology.

We shall now define the determinant and the signature of a link. For that color regions of the link diagram D black and white in a checkerboard fashion. We split the crossings of D in two types and assign to them *incidence numbers* $\mu(c)$ as in Fig. 5.1. We define the *total incidence number* $\mu(D)$ as the sum of the incidence numbers over all crossings of type II.

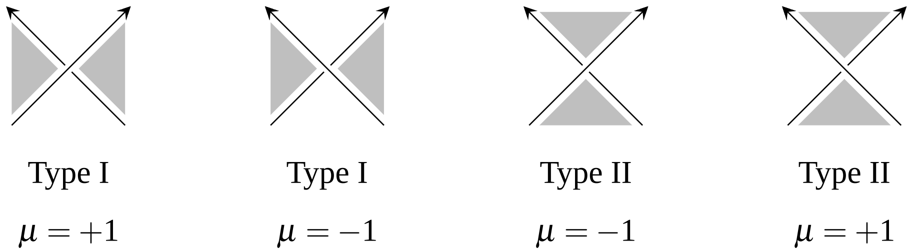


Figure 5.1: Types and incidence numbers of crossings in a link diagram colored black and white.

Let $\mathscr{W}$ be a free abelian group generated by the white regions, and define a *Goeritz form* $\mathfrak{g}\colon \mathscr{W} \times \mathscr{W} \longrightarrow \mathbb{Z}$ as a symmetric bilinear function given on generators as below:

$$\mathfrak{g}(R,R') - \begin{cases} -\sum_{c\in R\cap R'} \mu(c), & \text{if } R \neq R', \\ -\sum_{R''\neq R} \}(R,R''), & \text{if } R = R'. \end{cases} \tag{5.60}$$

The matrix $\tilde{G}$ for $\mathfrak{g}$ is singular. The *Goeritz matrix* G of the diagram D is obtained from $\tilde{G}$ by removing the first row and the first column. We use it to define the *signature* $\sigma(D) := \sigma(G_D) - \mu(D)$ and the *determinant* $\det D := |\det G|$ of a link diagram D. It is a classical result that both numbers depend only on the link, but not on its actual diagram used for calculations.

We shall use the following result of Ozsváth, Manolescu and Murasugi, for the signature of a link. As before, the symbols $\diagup\!\!\!\!\diagdown$, $)($, and $\asymp$ denote three link diagrams that differ only locally as seen in the pictures. Choose any orientation for the last diagram.

Lemma 5.4.3 (cf. [OM07, Mu65])**.** *Suppose* $\det()(), \det(\asymp) > 0$, *and* $\det(\diagup\!\!\!\!\diagdown) = \det()() + \det(\asymp)$. *Then*

$$\sigma()() - \sigma(\diagup\!\!\!\!\diagdown) = 1, \text{ and} \tag{5.61}$$

$$\sigma(\asymp) - \sigma(\diagup\!\!\!\!\diagdown) = e, \tag{5.62}$$

where $e = n_-(\asymp) - n_-(\diagup\!\!\!\!\diagdown)$ *is the difference between the numbers of negative crossings in the diagrams* $\asymp$ *and* $\diagup\!\!\!\!\diagdown$.

Consider now the generalized Khovanov homology with a single grading by $\delta = 2i - j - k$, where i stands for the homological grading and (j,k) for the bigrading. We can transform the exact sequences from Proposition 4.2.2 to a simpler form.

Proposition 5.4.4. *Choose a link diagram with a crossing* $\times$*, and suppose* $\det(\asymp), \det()() > 0$ *and* $\det(\times) = \det(\asymp) + \det()()$*, where we pick any orientation for the diagram* $\asymp$*. Then there is an exact sequence*

$$\ldots \longrightarrow \mathscr{H}^{i-\frac{\sigma()()}{2}}()() \longrightarrow \mathscr{H}^{i-\frac{\sigma(\times)}{2}}(\times) \longrightarrow \mathscr{H}^{i-\frac{\sigma(\asymp)}{2}}(\asymp) \longrightarrow \mathscr{H}^{i-1-\frac{\sigma()()}{2}}()() \longrightarrow \ldots \quad (5.63)$$

Proof. It follows from Proposition 4.2.2 and Lemma 5.4.3. □

Corollary 5.4.5. *Quasi-alternating links are homologically thin with respect to the covering Khovanov homology* $\mathscr{H}_{cov}$.

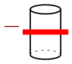
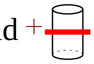
Proof. The unknot is clearly homologically thin. Next, it follows directly from the definition of a quasi-alternating link, that all of them have positive determinants. The desired result follows then from Proposition 5.4.4, since if both $\asymp$ and $)($ are homologically thin, so must be the third link due to the exactness of the sequence. □

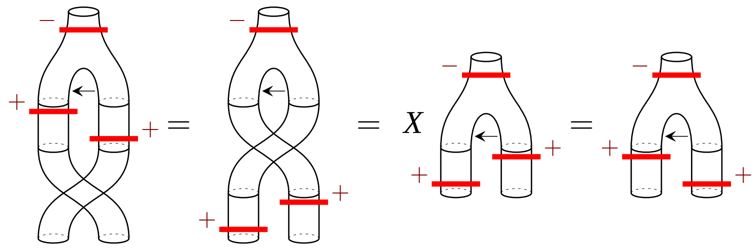
5.4.3 Module structure

A chronological TQFT $\mathscr{F}$ intertwines the disjoint union with the tensor product over $\Bbbk$, and the connected sum with a tensor product over the algebra $A' := \mathscr{F}(\bigcirc)\{1,0\}$; the degree shift is to make A' an associative algebra.

Lemma 5.4.6. *Given a chronological TQFT* $\mathscr{F} : \Bbbk\mathbf{ChCob} \longrightarrow \mathbf{Mod}_{\Bbbk}$ *let* $A := \mathscr{F}(\bigcirc)$*. Then the algebra* $A' := A\{-1,0\}$ *is commutative, associative and unital.*

Proof. It is enough to check the corresponding relations at the level of cobordisms—the lemma then follows by transforming the pictures by $\mathscr{F}$. We shall show only commutativity, leaving the other two properties as an exercise. For that, we shall draw the canonical isomorphisms as

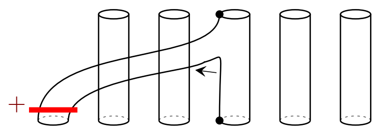
: $A \longrightarrow A\{-1,0\}$ and : $A\{-1,0\} \longrightarrow A$. Then, the commutativity follows directly from chronological relations:



(5.64)

□

Definition 5.4.7. Given a disjoint union of k circles $k\bigcirc$ with a basepoint b define the left A'-module structure on $\mathscr{F}(k\bigcirc)$ by merging $\bigcirc\{-1,0\}$ with the basepointed circle from the left hand side:



(5.65)

Likewise we define the right action of A', by merging $\bigcirc\{-1,0\}$ from the right hand side.

We check directly that both actions are associative and they commute. Moreover, $\mathscr{F}(k\bigcirc)$ is a symmetric bimodule in the graded sense: given homogeneous $x \in \mathscr{F}(k\bigcirc)$ and $a \in A'$ we have $x \cdot a = \lambda(\deg x, \deg a) a \cdot x$.

Choose a based link diagram D and construct its cube of resolutions $\mathscr{I}_{\mathrm{gr}}(D)$. According to the above vertices of the cube $\mathscr{F}\mathscr{I}_{\mathrm{gr}}(D)$ are bimodules over the algebra A'.

Lemma 5.4.8. *Edge morphisms in the cube $\mathscr{F}\mathscr{I}_{\mathrm{gr}}(D)$ are bimodule homomorphisms.*

Proof. Both the edge morphisms and the actions of A' are graded, so that $d_\zeta(a \cdot x) = a \cdot d_\zeta x$ for $a \in A'$, $x \in \mathscr{F}Kh(D)$ and any edge morphism d_ζ, and likewise for the right action. □

Corollary 5.4.9. *For a based link diagram D the image of the generalized Khovanov complex $\mathscr{F}Kh(D)$ under a chronological TQFT $\mathscr{F}$ is a chain complex of symmetric A'-bimodules. Therefore, its homology $\mathscr{H}(D)$ is a sequence of symmetric graded A'-bimodules.*

In Section 4.2.4 we described decomposition of generalized Khovanov complexes for disjoint union and connected sums of links.

Proposition 5.4.10. *Given two link diagrams D, D', and a chronological TQFT there is an isomorphism of complexes* $\mathscr{F}Kh(D \sqcup D') \cong \mathscr{F}Kh(D) \underset{\Bbbk}{\otimes} \mathscr{F}Kh(D')$. *Moreover, if the diagrams are based, then* $\mathscr{F}Kh(D\#D') \cong \mathscr{F}Kh(D) \underset{A'}{\otimes} \mathscr{F}Kh(D')$.

Proof. The first isomorphism follows from Proposition 4.2.5 and monoidality of $\mathscr{F}$. The second is a consequence of Proposition 4.2.6 and Corollary 5.4.9. □

Remark 5.4.11. The sign assignment $\varepsilon \otimes \varepsilon'$ we chose for $\mathscr{I}_{\mathrm{gr}}(D \sqcup D')$ implies that for $x \otimes y \in \mathscr{F}Kh(D \sqcup D') \cong \mathscr{F}Kh(D) \otimes \mathscr{F}Kh(D')$ we have

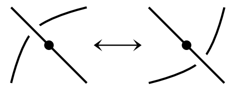
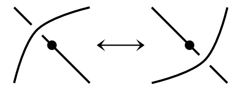
$$d(x \otimes y) = \begin{cases} dx \otimes y + (-1)^i X^a Z^b x \otimes dy, & \text{if } d \otimes \mathrm{id} \text{ is a merge,} \\ dx \otimes y + (-1)^i Y^a Z^b x \otimes dy, & \text{if } \mathrm{id} \otimes d \text{ is a split,} \end{cases} \tag{5.66}$$

where (a,b) is the degree of x regarded as an element of $\mathscr{F}[\![D]\!]^i$. We can choose another isomorphism between $Kh(D) \otimes Kh(D')$ and $Kh(D \sqcup D')$ by considering all degree shifts at once—in such case the pair (a,b) in the formula (5.66) would be the degree of $x \in \mathscr{F}Kh(D)$, which seems a better choice. However, a and b could be half-integers, which would require us to choose square roots on X, Y, and Z, see Remark 4.1.2. A similar formula holds for the differential in $\mathscr{F}Kh(D\#D')$.

The bimodule structure on homology is a link invariant. Indeed, one can always perform Reidemeister moves beyond a neighborhood of the basepoint, perhaps using the isotopy through infinity (4.25), in which case the chain homotopy equivalences from Section 4.3 commute with the actions of A'. In particular, we can move the basepoint freely along a component of a link.

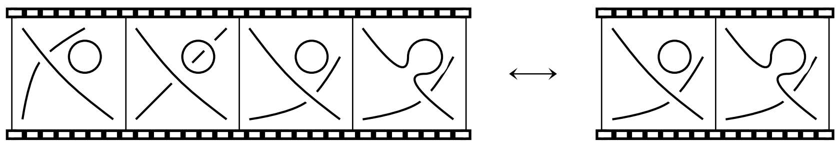
On the other hand, moving a basepoint to a different component of a link may change the module structure. Following [HN12] we choose a basepoint on every component of a link, which results in a bimodule structure on $\mathscr{F}Kh(L)$ over the algebra $(A')^{\otimes c}$, where c is the number of components of the link. Again, this structure descends to homology, but we have to work harder to proof this structure is invariant under Reidemeister move: with more that one basepoint we cannot avoid passing them through crossings.

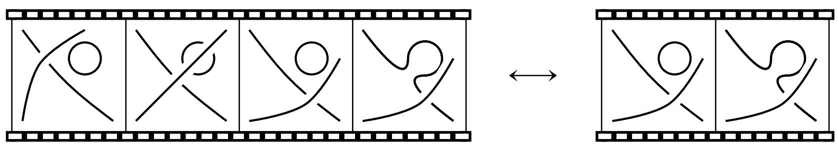
Proposition 5.4.12. *Given a based link diagram D, the bimodule structure on $\mathscr{F}Kh(D)$ is preserved up to isomorphism when the basepoint is moved through a crossing:*

 *and*  (5.67)

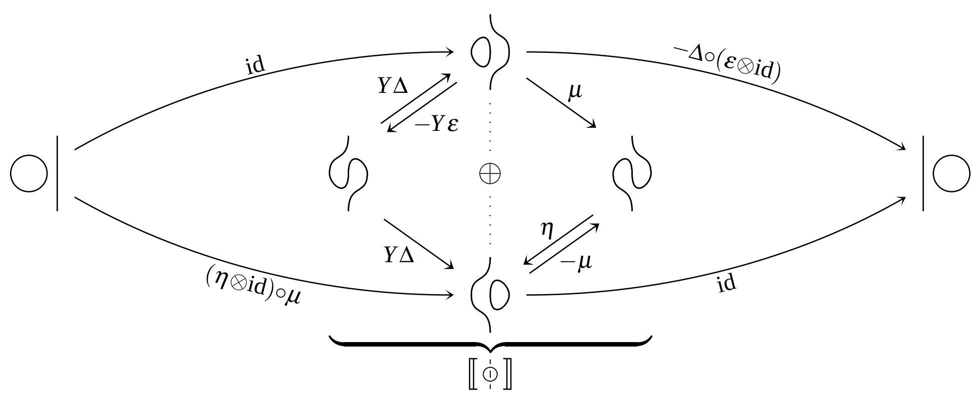
In particular, given a link L with c components, the bimodule structure on $\mathscr{H}(L)$ over $(A')^{\otimes c}$ is an invariant of L.

Proof. Invariance of the bimodule structure under passing a dot through a crossing is equivalent to saying that cobordisms in the following pairs

 (5.68)

 (5.69)

induce isomorphic operations on complexes. We shall prove only the case the link diagram is the unknot—the general case then follows from Proposition 4.2.6. We start by computing the homomorphisms induced by passing a circle over another one—it is given by a composition of the chain homotopy equivalences used in the proof of invariance of the chain complex under the second Reidemeister move. We provide the appropriate diagram below, where the whole circle being moved is visible, but only the essential fragment of the other one.



The left homomorphism is the inclusion $f\colon [\![\, \bigcirc | \,]\!] \longrightarrow [\![\, \dot{\phi} \,]\!]$, and the right one is the retraction $g\colon [\![\, \dot{\phi} \,]\!] \longrightarrow [\![\, | \bigcirc \,]\!]$. The two backward maps in the middle complex are pieces of chain homotopies (compare with Fig. 4.3). We make the following conventions:

- the two crossings in $\dot{\phi}$ are decorated with arrows pointing inwards,
- when enumerating circles, the one being moved is always put first.

With this choice we have to permute the outputs of a comultiplication, that is why the coefficients Y appear. We compute directly $(\eta \otimes \mathrm{id}) \circ \mu - \Delta \circ (\varepsilon \otimes \mathrm{id}) = \varphi \otimes \mathrm{id}$, where

$$\varphi(v_+) = v_+ \quad \text{and} \quad \varphi(v_-) = v_+h - v_-. \tag{5.70}$$

Notice that it is graded and $\varphi^2 = \mathrm{id}$. It remains to check it is a homomorphism of rings, which follows from the following computation:

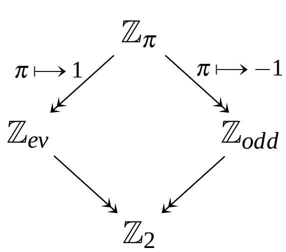
$$\begin{aligned}
&\mu((v_+h - v_-) \otimes (v_+h - v_-)) \\
&\qquad = \mu(v_+h \otimes v_+h) - \mu(v_+h \otimes v_-) - \mu(v_- \otimes v_+h) + \mu(v_- \otimes v_-) \\
&\qquad = hv_+h - hv_- - XZv_-h + tv_+ + hv_- \\
&\qquad = tv_+ + h(v_+h - v_-) = \varphi(tv_+ + hv_-).
\end{aligned}$$

The other case, when a circle is moved below another one, results exactly in the same map. □

5.5 Homological operations

5.5.1 A pullback description of covering homology

As we proved in Section 5.3, $\mathbb{Z}_\pi = \mathbb{Z}[\pi]/(\pi^2 - 1)$ is the universal ring of coefficients in our framework. Both $\mathbb{Z}_{ev}$ and $\mathbb{Z}_{odd}$ arise as quotients of this ring, and both projects to $\mathbb{Z}_2$ in a unique way. Notice, that $\mathbb{Z}_2$ admits a unique $\mathbb{Z}_\pi$-module structure, as π is invertible. Thence we obtain a commuting square diagram shown to the left. In fact, it is a pullback square in the category of rings.



Lemma 5.5.1. *There is an isomorphism of rings* $\mathbb{Z}_\pi \cong \{(a,b) \in \mathbb{Z}^2 \mid a \equiv b \mod 2\}$*, such that the projections on the first and on the second factors are exactly* $\mathbb{Z}_{ev}$ *and* $\mathbb{Z}_{odd}$*.*

Proof. The desired isomorphism maps 1 to $(1,1)$ and π to $(1,-1)$. This map is injective, since $(a+b, a-b) = (0,0)$ implies $a = b = 0$, and surjective, as (a,b) with $a \equiv b \mod 2$ is an image of $\frac{a+b}{2} + \frac{a-b}{2}\pi$. To finish the proof, notice that the action of π preserves the first factor, but negates the second. □

Corollary 5.5.2. *Given a link diagram D, the chain complex* $\mathscr{F}_\pi Kh(D)$ *is a pullback of the even and odd Khovanov complexes over their reductions modulo 2. In particular, it is a subcomplex of the direct sum* $\mathscr{F}_{ev}Kh(D) \oplus \mathscr{F}_{odd}Kh(D)$*.*

Proof. The chain complex $\mathscr{F}_\pi Kh(D)$ is a sequence of free $\mathbb{Z}_\pi$-modules, and the functor $M \otimes (\_)$ is exact if M is a free module. □

Therefore, we can see $\mathscr{H}_\pi(L)$ as a derived pullback of the diagram

$$\mathscr{H}_{ev}(L) \longrightarrow \mathscr{H}_{\mathbb{Z}/2}(L) \longleftarrow \mathscr{H}_{odd}(L). \tag{5.71}$$

On the other hand, the pullback the kernel of the map $\mathscr{F}_\pi Kh(D) \longrightarrow \mathscr{F}_{ev}Kh(D)$ is the subcomplex of $\mathscr{F}_{odd}Kh(D)$ formed by elements with even coefficients, which is isomorphism to the odd Khovanov complex. Likewise, the kernel of $\mathscr{F}_\pi Kh(D) \longrightarrow \mathscr{F}_{odd}Kh(D)$ is isomorphic to $\mathscr{F}_{ev}Kh(D)$. Hence, there are short exact sequences

$$0 \longrightarrow \mathscr{F}_{odd}Kh(D) \longrightarrow \mathscr{F}_\pi Kh(D) \longrightarrow \mathscr{F}_{ev}Kh(D) \longrightarrow 0, \text{ and} \tag{5.72}$$

$$0 \longrightarrow \mathscr{F}_{ev}Kh(D) \longrightarrow \mathscr{F}_\pi Kh(D) \longrightarrow \mathscr{F}_{odd}Kh(D) \longrightarrow 0. \tag{5.73}$$

Hence, $\mathscr{F}_\pi Kh(D)$ is an extension between the two theories. We shall show in the subsequent sections that its homology is a stronger invariant than both even and odd homology together. However, the difference is very subtle.

Proposition 5.5.3. *Let* $f\colon \mathscr{F}_{ev}Kh(L) \longrightarrow \mathscr{F}_{ev}Kh(L')$ *and* $g\colon \mathscr{F}_{odd}Kh(L) \longrightarrow \mathscr{F}_{odd}Kh(L')$ *be quasi-isomorphisms*[2] *that agree modulo 2. Then* $\mathscr{H}_\pi(L) \cong \mathscr{H}_\pi(L')$*.*

[2] A chain map is a *quasi-isomorphism* if it induces an isomorphism on homology.

Proof. The pullback (f,g) of the chain maps f and g is the desired isomorphism, which follows from the 5-lemma applied to the exact sequence (5.72). ☐

5.5.2 Bockstein operations in Khovanov homology

Since odd and even differentials agree modulo 2, there are at least three Bockstein operations for $\mathscr{H}_{\mathbb{Z}/2}(L)$:

1) the *even Bockstein*, $\beta_e\colon \mathscr{H}^i_{\mathbb{Z}/2}(L) \longrightarrow \mathscr{H}^{i+1}_{\mathbb{Z}/2}(L)$, $\beta_e[x] = \left[\frac{1}{2}d_{ev}x\right]$,
2) the *odd Bockstein*, $\beta_o\colon \mathscr{H}^i_{\mathbb{Z}/2}(L) \longrightarrow \mathscr{H}^{i+1}_{\mathbb{Z}/2}(L)$, $\beta_o[x] = \left[\frac{1}{2}d_{odd}x\right]$, and
3) the *mixed Bockstein*, $\beta := \beta_e + \beta_o$.

The last one arise from the short exact sequence of coefficients

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2[\pi]/(\pi^2-1) \longrightarrow \mathbb{Z}_2 \longrightarrow 0. \tag{5.74}$$

Indeed, from the pullback description of $\mathscr{F}_\pi Kh(L)$ we have $d_\pi = \frac{1}{2}(d_{ev}+d_{odd}) + \frac{\pi}{2}(d_{ev}-d_{odd})$, so that $\beta[x] = \left[\frac{1}{2}d_\pi x\right] = \left[\frac{1}{2}d_{ev}x + \frac{1}{2}d_{odd}x\right]$. To see there is nothing more, use the free resolution

$$0 \longleftarrow \mathbb{Z}/2 \longleftarrow \mathbb{Z}_\pi \xleftarrow{\begin{bmatrix}1+\pi & 1-\pi\end{bmatrix}} \mathbb{Z}_\pi^2 \xleftarrow{\begin{bmatrix}1-\pi & 0\\ 0 & 1+\pi\end{bmatrix}} \mathbb{Z}_\pi^2 \xleftarrow{\begin{bmatrix}1+\pi & 0\\ 0 & 1-\pi\end{bmatrix}} \mathbb{Z}_\pi^2 \longleftarrow \dots \tag{5.75}$$

to compute $\mathrm{Ext}^1_{\mathbb{Z}_\pi}(\mathbb{Z}/2,\mathbb{Z}/2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$. In particular, the sum of any two of the three Bocksteins results in the third.

Our goal is to show that these operations are independent of each other, which results in an infinite subalgebra of the graded Hopf algebra of homological operations.

Example 5.5.4. Choose a chain complex C with a differential of degree 1. Due to the Universal Coefficient Theorem, the homology of C with coefficients in $\mathbb{Z}_2$ is given as the direct sum $H^i(C,\mathbb{Z}_2) \cong H^i(C)\otimes\mathbb{Z}_2 \oplus Tor(H^{i+1}(C),\mathbb{Z}_2)$. The Bockstein homomorphism pairs elements of order 2 from the *Tor*-summand with modulo 2 reduction of homology class they come from. Therefore, the odd Bockstein is trivial for alternating links, as odd Khovanov homology of such links consists of free groups [ORS13]. On the other hand, the even Khovanov homology of $(2,n)$-torus knots has Z_2 summands.

Example 5.5.5. Consider the $(3,4)$-torus knot, labeled as 8_{19} in the Rolfsen's table [Ro76]. Its even and odd Khovanov homology are presented in Tab. 5.2. An analysis of positions of $\mathbb{Z}_2$ summands results in the following table, where the horizontal arrows illustrate nontrivial contributions to the Bockstein homomorphisms.

	0	1	2	3	4	5
17						$\mathbb{F}_2$
15						$\mathbb{F}_2$
13				$\mathbb{F}_2 \xrightarrow{\beta_o}$	$\mathbb{F}_2$	
11			$\mathbb{F}_2 \xrightarrow{\beta_e}$	$\mathbb{F}_2 \xrightarrow{\beta_o}$	$\mathbb{F}_2$	
9			$\mathbb{F}_2$			
7	$\mathbb{F}_2$					
5	$\mathbb{F}_2$					

In particular, the composition $\beta_o\beta_e$ does not vanish on the generator $u \in \mathscr{H}^{2,11}_{\mathbb{Z}/2}(8_{19})$. In particular, $\beta^2(u) \neq 0$, so that the two Bockstein homomorphisms do not commute with each other.

Example 5.5.6. The $(3,5)$-torus knot, labeled as 10_{124} in the Rolfsen's table [Ro76], admits a class of the opposite property. The table of Bockstein operations

	0	1	2	3	4	5	6	7
21								$\mathbb{F}_2$
19						$\mathbb{F}_2 \xrightarrow{\beta_o}$	$\mathbb{F}_2 \xrightarrow{\beta_e}$	$\mathbb{F}_2$
17						$\mathbb{F}_2 \xrightarrow{\beta_o}$	$\mathbb{F}_2$	
15				$\mathbb{F}_2 \xrightarrow{\beta_o}$	$\mathbb{F}_2$			
13			$\mathbb{F}_2 \xrightarrow{\beta_e}$	$\mathbb{F}_2 \xrightarrow{\beta_o}$	$\mathbb{F}_2$			
11			$\mathbb{F}_2$					
9	$\mathbb{F}_2$							
7	$\mathbb{F}_2$							

reveals $\beta_e\beta_o$ does no vanish on the generator $u \in \mathscr{H}^{5,19}_{\mathbb{Z}/2}(10_{124})$.

The ring $\mathbb{Z}_2$ is a field. Hence, by the Künneth formula and Proposition 4.2.5, the homology of a disjoint union splits $\mathscr{H}_{\mathbb{Z}/2}(L \sqcup L') \cong \mathscr{H}_{\mathbb{Z}/2}(L) \otimes \mathscr{H}_{\mathbb{Z}/2}(L')$. We say a homological operation $\theta_L \colon \mathscr{H}^i_{\mathbb{Z}/2}(L) \longrightarrow \mathscr{H}^{i+d}_{\mathbb{Z}/2}(L)$ is *primitive* if $\theta_{L\sqcup L'} = \theta_L \otimes \mathrm{id} + \mathrm{id} \otimes \theta_{L'}$.

Lemma 5.5.7. *The odd and even Bocksteins are differentials, $\beta_e^2 = \beta_o^2 = 0$, and $\beta^2 = [\beta_e, \beta_o]$ is their commutator. Moreover, all three operations are primitive.*

Proof. Straightforward from the definition and Proposition 4.2.5. ☐

Hence, there are at most three nontrivial operations in each degree generated by the Bocksteins: the alternating compositions $\underbrace{\cdots\beta_o\beta_e}_{n} = \beta^{n-1}\beta_e$ and $\underbrace{\cdots\beta_e\beta_o}_{n} = \beta^{n-1}\beta_o$, together with their sum β^n.

Theorem 5.5.8. *All three operations listed above are different.*

Proof. It is enough to show that $\beta^n\beta_e \neq 0$ for every n. Indeed, $0 \neq \beta^{n+1}\beta_e = \beta^n\beta_o\beta_e$ implies both $\beta^n \neq 0$ and $\beta^n\beta_o \neq 0$.

Choose a link L with a class $u \in \mathscr{H}_{\mathbb{Z}/2}(L)$ such that $\beta_o\beta_e(u) \neq 0$ but $\beta_o(u) = \beta_e\beta_o\beta_e(u) = 0$. For instance, we can take as L the $(3,4)$-torus knot with the generator $u \in \mathscr{H}^{2,11}_{\mathbb{Z}/2}(L)$. We shall prove the proposition by induction on n. For that, assume a link L' admits a class $v \in \mathscr{H}_{\mathbb{Z}/2}(L')$ such that $\beta^{2n-1}\beta_e(v) \neq 0$. Notice that $\beta^{2n+1}\beta_e = \beta_o \cdots \beta_o\beta_e$, i.e. the last operation is the odd Bockstein homomorphism. Then for the class $u \otimes v \in \mathscr{H}_{\mathbb{Z}/2}(L \sqcup L')$ we have

$$\beta^{2n+1}\beta_e(u\otimes v) = u\otimes\beta^{2n+1}\beta_e(v) + \beta_e(u)\otimes\beta^{2n}(v) + \beta_o\beta_e(u)\otimes\beta^{2n-1}\beta_e(v), \tag{5.76}$$

and the third term is nonzero. Since all three terms live in different summands of $\mathscr{H}_{\mathbb{Z}/2}(L \sqcup L')$, it must be $\beta^{2n+1}\beta_e(u\otimes v) \neq 0$. ☐

The ranks of even and odd Bockstein homomorphisms are fully determined by the even and odd Khovanov homology. Therefore, they do not provide new information. However, the mixed Bockstein β is different: Alexander Shumakovitch found with a help from a computer eight pairs of knots with the same even and odd Khovanov homology, but different ranks of β^2. All of them have 15 crossings:

$15^n_{23106} \leftrightarrow 15^n_{56014}$	$15^n_{23432} \leftrightarrow 15^n_{56014}$	$15^n_{23106} \leftrightarrow 15^n_{56014}$	$15^n_{23432} \leftrightarrow 15^n_{56014}$
$15^n_{44028} \leftrightarrow \overline{15}^n_{50224}$	$15^n_{44028} \leftrightarrow \overline{15}^n_{50224}$	$15^n_{73047} \leftrightarrow \overline{15}^n_{91280}$	$15^n_{73047} \leftrightarrow \overline{15}^n_{91280}$

	0	1	2	3	4	5
17						$\mathbb{Z}$
15						$\mathbb{Z}$
13				$\mathbb{Z}$	$\mathbb{Z}$	
11				$\mathbb{Z}_2$	$\mathbb{Z}$	
9			$\mathbb{Z}$			
7	$\mathbb{Z}$					
5	$\mathbb{Z}$					

	0	1	2	3	4	5
17						$\mathbb{Z}$
15						$\mathbb{Z}\oplus\mathbb{Z}_3$
13					$\mathbb{Z}_2$	$\mathbb{Z}_3$
11			$\mathbb{Z}$		$\mathbb{Z}_2$	
9			$\mathbb{Z}$			
7	$\mathbb{Z}$					
5	$\mathbb{Z}$					

Table 5.2: Even and odd Khovanov homology of the knot 8_{19}.

	0	1	2	3	4	5	6	7
21								$\mathbb{Z}$
19						$\mathbb{Z}$		$\mathbb{Z}_2$
17						$\mathbb{Z}$	$\mathbb{Z}$	
15				$\mathbb{Z}$	$\mathbb{Z}$			
13				$\mathbb{Z}_2$	$\mathbb{Z}$			
11			$\mathbb{Z}$					
9	$\mathbb{Z}$							
7	$\mathbb{Z}$							

	0	1	2	3	4	5	6	7
21								$\mathbb{Z}$
19							$\mathbb{Z}_2$	$\mathbb{Z}$
17						$\mathbb{Z}_3$	$\mathbb{Z}_2$	
15					$\mathbb{Z}_2$	$\mathbb{Z}_3$		
13			$\mathbb{Z}$		$\mathbb{Z}_2$			
11			$\mathbb{Z}$					
9	$\mathbb{Z}$							
7	$\mathbb{Z}$							

Table 5.3: Even and odd Khovanov homology of the knot 10_{124}.

Here, $\overline{K}$ represents a mirror image of the link K. Existence of such pairs can be explained by the observation that Bockstein homomorphisms are described by a noncanonical splitting $H^i(C,\mathbb{Z}_2)\cong H^i(C)\otimes\mathbb{Z}_2\oplus Tor(H^{i+1}(C),\mathbb{Z}_2)$, and in case of Khovanov homology the two splittings, one for the even and one for the odd version, do not coincide. In other words, we cannot pick isomorphisms for even and odd Khovanov homology that agree over $\mathbb{Z}_2$.

Corollary 5.5.9. *The unified homology $\mathscr{H}_\pi(L)$ is a stronger invariant than $\mathscr{H}_{ev}(L)\oplus\mathscr{H}_{odd}(L)$.*

5.5.3 Integral lifts

Regarding $\mathscr{H}_\pi(L)$ as an extension of even and odd Khovanov homology leads to homological operations between integral Khovanov homologies of a link:

1) $\varphi_{eo}\colon \mathscr{H}^i_{ev}(L) \longrightarrow \mathscr{H}^{i+1}_{odd}(L)$, the boundary homomorphism from the sequence (5.72), and
2) $\varphi_{oe}\colon \mathscr{H}^i_{odd}(L) \longrightarrow \mathscr{H}^{i+1}_{ev}(L)$, the boundary homomorphism from the sequence (5.73).

Tensoring the exact sequences (5.72) and (5.73) with $\mathbb{Z}_2$ reveals that φ_{eo} and φ_{oe} are integral lifts of Bocksteins homomorphisms from previous section.

Proposition 5.5.10. *Given a link L there are commuting squares*

$$\begin{array}{ccc} \mathscr{H}^i_{ev}(L) & \xrightarrow{\varphi_{eo}} & \mathscr{H}^{i+1}_{odd}(L) \\ \downarrow & & \downarrow \\ \mathscr{H}^i_{\mathbb{Z}/2}(L) & \xrightarrow{\beta_o} & \mathscr{H}^{i+1}_{\mathbb{Z}/2}(L) \end{array} \qquad \begin{array}{ccc} \mathscr{H}^i_{odd}(L) & \xrightarrow{\varphi_{oe}} & \mathscr{H}^{i+1}_{ev}(L) \\ \downarrow & & \downarrow \\ \mathscr{H}^i_{\mathbb{Z}/2}(L) & \xrightarrow{\beta_e} & \mathscr{H}^{i+1}_{\mathbb{Z}/2}(L) \end{array} \tag{5.77}$$

Proof. We start with computing the formula φ_{eo} using the pullback description of $\mathscr{F}_\pi Kh(L)$. Pick a cocycle $x \in \mathscr{F}_{ev}Kh(L)$; it is covered by $(x,x) \in \mathscr{F}_\pi Kh(L)$, where we identity even and odd chain groups in the natural way. Then $d_\pi(x,x) = (0, d_{odd}x)$ is the image of $\frac{1}{2}d_{odd}x$, as the inclusion of the odd homology takes a chain y into $(0,2y)$. Notice that the division makes sense, as over $\mathbb{Z}_2$ we have $d_{odd}x \equiv d_{ev}x = 0$. Hence, $\varphi_{eo}[x] = \left[\frac{1}{2}d_{odd}x\right]$, which agrees modulo 2 with $\beta_o([x]\otimes\mathbb{Z}_2)$. The case of φ_{oe} is proven likewise. □

Corollary 5.5.11. *All alternating compositions $\cdots\varphi_{eo}\varphi_{oe}$ and $\cdots\varphi_{oe}\varphi_{eo}$ are nontrivial.*

Proof. In the view of Proposition 5.5.10 it is enough to find a link L and a class $a \in \mathscr{H}_{odd}(L)$ such that the composition $\cdots\beta_o\beta_e$ does not vanish on the $\mathbb{Z}_2$-reduction $\bar{a} \in \mathscr{H}_{\mathbb{Z}/2}(L)$. This follows from the proof of Theorem 5.5.8, since the generator $u \in \mathscr{H}^{2,11}_{\mathbb{Z}/2}(8_{19})$ has an integral lift to odd Khovanov homology. □

The compositions $\theta_e := \varphi_{oe}\varphi_{eo}$ and $\theta_o := \varphi_{eo}\varphi_{oe}$ are certain degree two operations on even and odd Khovanov homology. Computations due to Alexander Shumakovitch revealed 9 pairs of knots with 14 crossings that have same odd Khovanov homology, but different ranks of θ_o:

$$13^n_{1002} \leftrightarrow 14^n_{6487} \qquad \overline{13}^n_{141} \leftrightarrow \overline{14}^n_{2551} \qquad 13^n_{651} \leftrightarrow 14^n_{16550}$$

$$13^n_{661} \leftrightarrow 14^n_{16550} \qquad 14^n_{12393} \leftrightarrow 14^n_{12532} \qquad 14^n_{1346} \leftrightarrow 14^n_{7711}$$
$$14^n_{5293} \leftrightarrow 14^n_{12516} \qquad 14^n_{5373} \leftrightarrow 14^n_{12516} \qquad 14^n_{6632} \leftrightarrow \overline{14}^n_{21021}$$

Remark 5.5.12. The operations θ_o and θ_e have a very interesting behavour with respect to taking mirror images of links: they are pairs of knots that can be distinguished by ranks of the operations, but not their mirror images. For instance, θ_o is trivial for a mirror image of any of the knots above, and the pairs become indistinguishable. This can be explained by the observation that a knot may be thin, although its mirror image is not, see the case of 13^n_{1002} and 14^n_{6487} in Tab. 5.4.

Further computation showed that among 201,702 knots with at most 15 crossings there are 99 pairs with the same both even and odd Khovanov homology, which have different homological operations: θ_o for 95 pairs, and θ_e for the other 4 pairs.

	-6	-5	-4	-3	-2	-1	0	1	2	3	4
11											$\mathbb{Z}$
9										$\mathbb{Z}^2$	$\mathbb{Z}$
7									$\mathbb{Z}^4$	$\mathbb{Z}^2$	
5								$\mathbb{Z}^5$	$\mathbb{Z}^4 \oplus \mathbb{Z}_2$		
3							$\mathbb{Z}^6$	$\mathbb{Z}^5 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$		
1						$\mathbb{Z}^7$	$\mathbb{Z}^6 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$			
-1					$\mathbb{Z}^6$	$\mathbb{Z}^7 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$				
-3				$\mathbb{Z}^5$	$\mathbb{Z}^6 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$					
-5			$\mathbb{Z}^4$	$\mathbb{Z}^5$	$\mathbb{Z}_2$						
-7		$\mathbb{Z}^2$	$\mathbb{Z}^4$								
-9	$\mathbb{Z}$	$\mathbb{Z}^2$									
-11	$\mathbb{Z}$										

	-4	-3	-2	-1	0	1	2	3	4	5	6
11											$\mathbb{Z}$
9										$\mathbb{Z}^2$	$\mathbb{Z}$
7									$\mathbb{Z}^4$	$\mathbb{Z}^2$	
5								$\mathbb{Z}^5 \oplus \mathbb{Z}_2$	$\mathbb{Z}^4$		
3							$\mathbb{Z}^6 \oplus \mathbb{Z}_2$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2$			
1						$\mathbb{Z}^7 \oplus \mathbb{Z}_2$	$\mathbb{Z}^6 \oplus \mathbb{Z}_2$				
-1					$\mathbb{Z}^6 \oplus \mathbb{Z}_2$	$\mathbb{Z}^7 \oplus \mathbb{Z}_2$					
-3				$\mathbb{Z}^5 \oplus \mathbb{Z}_2$	$\mathbb{Z}^6 \oplus \mathbb{Z}_2$						
-5			$\mathbb{Z}^4$	$\mathbb{Z}^5 \oplus \mathbb{Z}_2$							
-7		$\mathbb{Z}^2$	$\mathbb{Z}^4$								
-9	$\mathbb{Z}$	$\mathbb{Z}^2$									
-11	$\mathbb{Z}$										

Table 5.4: The odd Khovanov homology for knots 13^n_{1002} and 14^n_{6487} (the upper table) and their mirror images (the lower one). The solid blue arrows indicates the place where the operation θ_o is surjective for both knots, whereas at the dashed red arrow θ_o is surjective for 13^n_{1002}, but a zero map for 14^n_{6487}. Notice there is no place for a nontrivial θ_o in the lower table.

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