Quantum K-theory and the Baxter Operator

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Submitted in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy
in the Graduate School of Arts and Sciences

COLUMBIA UNIVERSITY

2018

Abstract

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In this work, the connection between quantum K-theory and quantum integrable systems is studied. Using quasimap spaces the quantum equivariant K-theory of Nakajima quiver varieties is defined. For every tautological bundle in the K-theory there exists its one-parametric deformation, referred to as quantum tautological bundle. For specific cases of cotangent bundles to Grassmannians and flag varieties it is proved that the spectrum of operators of quantum multiplication by these quantum classes is governed by the Bethe ansatz equations for the inhomogeneous XXZ spin chain. It is also proved that each such operator corresponds to the universal elements of quantum group $\mathcal{U}_\hbar(\widehat{\mathfrak{sl}}_n)$. In particular, the Baxter operator for the XXZ spin chain is identified with the operator of quantum multiplication by the exterior algebra of the tautological bundle. An explicit universal combinatorial formula for this operator is found in the case of $\mathcal{U}_\hbar(\widehat{\mathfrak{sl}}_2)$. The relation between quantum line bundles and quantum dynamical Weyl group is shown. This thesis is based on works [37] and [24].

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Acknowledgments

First and foremost, I would like to thank my advisor Andrei Okounkov. He introduced me to new areas of mathematics and changed the way I view what I've known before. His precise advice and calm confidence always helped me when I felt completely lost. I am also grateful for the vibrant group of students and postdocs he created at Columbia University, for the opportunities he created for me to meet other mathematicians.

I would also like to thank my coauthors: Peter Koroteev, Andrey Smirnov and Anton Zeitlin. I had a good time working with you and definitely learned a lot from you. I would like to specifically thank Andrey Smirnov, as he was the first person to become interested in the subject of this thesis and helped me a lot.

I am thankful to Noah Arbesfeld, Alexander Braverman, Ivan Danilenko, Eugene Gorsky, Yakov Kononov, Dmitrii Korb, Melissa Liu, Davesh Maulik, Michael McBreen, Andrei Negut, Nikita Nekrasov, Anton Osinenko, Leonid Rybnikov, Alexander Shapiro, Changjian Su, Zijun Zhou for their interest in the subject and for sharing their perspectives with me. I would like to thank the Mathematics Department at Higher School of Economics, for forming my interest in math during my undergraduate studies.

I would like to thank all my friends in Columbia and New York. Fellow graduate students, old friends and new friends, you've made this such a fun and exciting time for me. The same can be said about my good old friends Edik, Fima, and Lenya.

I thank my family: my parents, all of my sisters and brothers, and my grandparents; for their joy, endless support, and never letting me feel alone.

Finally, I would like to thank Anya. For making me happy.

to my parents and my siblings

Chapter 1

Introduction

1.1 Overview

The deep connection between quantum integrable systems and quantum geometry, in particular quantum cohomology and K-theory, was observed in the pioneering works of N. Nekrasov and S. Shatashvili [33, 34]. It was noted, that the integrable systems associated with the quantum groups studied extensively in the 1980s by the Leningrad school, describe quantum geometry of the large class of symplectic algebraic varieties. These ideas were further developed by A. Braverman, D. Maulik, A. Okounkov [8,28,36] and other authors both in mathematics and physics, e.g. [7,15–17,32]. The substantial progress in this direction shed light on earlier papers of A. Givental, Y. P. Lee [19,25] and collaborators.

In general, such constructions should exist for all Nakajima varieties. The simplest nontrivial examples of such varieties are the cotangent bundles of Grassmannians $\mathbf{N}_{k,n} = T^*\mathbf{Gr}_{k,n}$. Results of [33, 34] suggest that certain *quantum deformations* of cohomology and K-theory rings of these varieties should be related to XXX and XXZ spin chains correspondingly. These are the integrable systems described by the Yangian $Y(\mathfrak{sl}_2)$ and the quantum group $\mathcal{U}_\hbar(\widehat{\mathfrak{sl}}_2)$ respectively. Moreover, [33, 34] conjecture that the operator of multiplication by the weighted exterior algebra of the tautological bundle in such deformed K-theory of $\mathbf{N}_{k,n}$ should be related to the so-called Baxter Q-operator for the XXZ-spin chain. These conjectures can be generalized to the case of cotangent bundles to all partial flags (the corresponding algebra will be $\mathcal{U}_\hbar(\widehat{\mathfrak{sl}}_n)$).

We define the quantum K-theory ring of any Nakajima quiver variety using the moduli space of quasimaps [1]. This construction in the case of $\mathbf{N}_{k,n}$ allows us to give a corrected formulation and a proof of the above conjecture. In fact, we discover that to relate quantum K-theory with spin chains it is not enough to consider the operators of quantum multiplication by classical K-theory classes. It turns out that *both* the multiplication in K-theory and the tautological classes should be deformed simultaneously.

We introduce elements $\hat{\tau}(z)$ in the quantum equivariant K-theory of a quiver variety, which we call *quantum tautological bundles*. In the classical limit $z \to 0$ these elements coincide with the corresponding classical tautological bundles τ in the standard equivariant K-theory. Among the main objects we study in this paper are the operators of *quantum multiplication by the quantum tautological bundles*. We show that the spectrum of these operators is described by the Bethe ansatz equations for the XXZ-spin chain (see Theorem 1.3.1 for Grassmannians and Theorem 6.2.1 for all partial flags).

We use the geometric action of $\mathcal{U}_\hbar(\widehat{\mathfrak{sl}}_2)$ on the equivariant K-theory of $\mathbf{N}_{k,n}$ (see e.g. [45]) to identify it with the space of quantum states for the XXZ model. Under this identification the Baxter operator [3] of the XXZ model coincides with the operator of quantum multiplication by the weighted quantum exterior algebra of the tautological bundle (see Theorem 1.5.1).

We conjecture that the quantum tautological bundles are represented by certain universal elements from $\mathcal{U}_\hbar(\widehat{\mathfrak{sl}}_2)$, which depend on z but do not depend on the parameters k and n of the Grassmannian. In particular, we explicitly find this universal element in the case of quantum exterior powers of the tautological bundle (see Theorem 1.6.2). Moreover, quantum multiplication by the determinant of the tautological

[1]Our definition of quantum K-theory ring is alternative to the one given in [19], where the moduli space of *stable maps* were used.

bundle coincides with the generator of the lattice part of the quantum affine dynamical Weyl group (see Theorem 1.6.3). This observation relates quantum K-theory with the theory of quantum dynamical groups of P. Etingof and A. Varchenko [11], which is a deformation of the standard quantum Weyl group [26].

In this chapter we mostly focus on the example of Grassmannians, as generalizations to flags are fairly straightforward, yet more tedious in terms of formulas and computations.

1.2 Classical K-theory of $\mathbf{N}_{k,n}$

Let $R = Hom(V, W)$ for $W = \mathbb{C}^n$ and $V = \mathbb{C}^k$ with $k \leq n$. Let $\mu^* : \mathfrak{gl}(V) \to Vect(R)$ be a map of Lie algebras induced by the canonical action of $GL(V)$ on R. The dual of this map is known as a moment map $\mu : T^*R \to \mathfrak{gl}(V)^*$. Our main object of study is the following hyperkähler quotient:

$$\mathbf{N}_{k,n} = T^*R /\!/\!/\!/ GL(V) = \mu^{-1}(0) /\!/ GL(V) = \mu^{-1}(0)_{ss}/GL(V),$$

where the symbol $\mu^{-1}(0)_{ss}$ denotes the intersection of $\mu^{-1}(0)$ with the semistable locus corresponding to injective elements in R. By construction, $\mathbf{N}_{k,n}$ is a smooth symplectic variety which, in fact, is isomorphic to the cotangent bundle over the Grassmanian of k-planes in n-space $V \subset W$. Set $\mathbf{N}(n) = \coprod_{k=0}^{n} \mathbf{N}_{k,n}$.

Note that $\mathbf{N}_{k,n}$ is naturally equipped with the following tautological bundles:

$$\mathcal{V} = (T^*R \times V) /\!/\!/\!/ GL(V), \quad \mathcal{W} = (T^*R \times W) /\!/\!/\!/ GL(V).$$

More generally, let $K_{GL(V)}(\cdot) = \Lambda[x_1^{\pm 1}, x_2^{\pm 1}, \cdots, x_k^{\pm 1}]$ be the ring of symmetric Laurent polynomials on k-variables. Every such polynomial $\tau \in K_{GL(V)}(\cdot)$ is a character of some virtual representation $\tau(V)$ of $GL(V)$ (tensorial polynomial in V and V^*).[2] We

[2]For example, the polynomial

$$\tau(x_1, \cdots, x_k) = (x_1 + \cdots + x_k)^2 - \sum_{1 \leq i_1 < i_2 < i_3 \leq k} x_{i_1}^{-1} x_{i_2}^{-1} x_{i_3}^{-1}$$

will denote the corresponding virtual tautological bundles on $\mathbf{N}_{k,n}$ by the same symbol τ:

$$\tau = (T^*R \times \tau(V))/\!/\!/\!/GL(V).$$

The tautological bundles τ can be uniquely represented by the symmetric Laurent polynomials in the corresponding Chern roots of $\mathcal{V}$ and thus there should be no confusion in our notations.

We set a framing torus $\mathsf{A} = \mathbb{C}^\times_{a_1} \times \cdots \times \mathbb{C}^\times_{a_n}$ to be a n-torus acting on W by scaling the coordinates with characters a_i. Let $\mathbb{C}^\times_\hbar$ be a one-torus acting on T^*R by scaling the cotangent directions with character $\hbar$. We adopt notation $\mathsf{T} = \mathsf{A} \times \mathbb{C}^\times_\hbar$.

The action of T on T^*R induces its action on $\mathbf{N}_{k,n}$. The fixed set $\mathbf{N}^{\mathsf{T}}_{k,n}$ consists of $n!/k!/(n-k)!$ isolated points representing the k-planes spanned by coordinate vectors. They are conveniently labeled by k-subsets $\mathbf{p} = \{x_1, \cdots, x_k\} \subset \{a_1, \cdots, a_n\}$. The fixed point set $\mathbf{N}(n)^{\mathsf{T}}$ therefore consists of 2^n points.

The equivariant K-theory $K_{\mathsf{T}}(\mathbf{N}(n))$ is a module over the ring of equivariant constants: $R = K_{\mathsf{T}}(\cdot) = \mathbb{Z}[a_1^{\pm}, \cdots, a_n^{\pm 1}, \hbar^{\pm 1}]$. The localized K-theory

$$K_{\mathsf{T}}(\mathbf{N}(n))_{loc} = K_{\mathsf{T}}(\mathbf{N}(n)) \bigotimes_R \mathcal{A} = \bigoplus_{k=0}^{n} K_{\mathsf{T}}(\mathbf{N}_{k,n}) \bigotimes_R \mathcal{A} \tag{1.1}$$

is an $\mathcal{A}$-vector space ($\mathcal{A} = \mathbb{Q}(a_1, \cdots, a_n, \hbar)$) of dimension 2^n spanned by the K-theory classes of fixed points $\mathcal{O}_{\mathbf{p}}$. Note, that the operators of tensor multiplication by tautological bundles in the equivariant K-theory are diagonal in the basis of fixed points:

$$\tau \otimes \mathcal{O}_{\mathbf{p}} = \tau(x_1, \cdots, x_k)\, \mathcal{O}_{\mathbf{p}} \quad \text{for} \quad \mathbf{p} = \{x_1, \cdots, x_k\} \subset \{a_1, \cdots, a_n\}. \tag{1.2}$$

This statement can be conveniently formulated for all fixed points and tautological bundles simultaneously.

Proposition 1.2.1. *The eigenvalues of the operators of multiplication by tautological bundles in $K_{\mathsf{T}}(\boldsymbol{N}_{k,n})$ are given by the values of the corresponding Laurent polynomials*

corresponds to $\tau(V) = V^{\otimes 2} - \Lambda^3 V^*$.

$\tau(s_1, \cdots, s_k)$ *evaluated at the solutions of the following equations:*

$$\prod_{j=1}^{n}(s_i - a_j) = 0, \quad i = 1 \cdots k \tag{1.3}$$

with $s_i \neq s_j$.

It is obvious that the solutions of this system with $s_i \neq s_j$ are in one-to-one correspondence with the k-subsets $\{x_1, \cdots, x_k\} \subset \{a_1, \cdots, a_n\}$ and, therefore, with the set of the fixed points $\mathbf{N}_{k,n}^{\mathsf{T}}$. Theorem 1.3.1 provides an elegant generalization of this statement to the case of the quantum K-theory. The system of equations (1.3) turns out to be the classical limit ($z \to 0$) of the so-called Bethe ansatz equations (1.4).

1.3 Quantum K-theory and Bethe ansatz

In Chapter 3 we use the moduli space of *quasimaps* to $\mathbf{N}_{k,n}$ to define certain associative, commutative, one-parametric deformation of its equivariant K-theory ring. We denote the deformed tensor product by $\circledast$ and call the corresponding ring *quantum K-theory* of $\mathbf{N}_{k,n}$. The word "deformation" here means that for two K-theory classes A, B we have

$$A \circledast B = A {\otimes} B + \sum_{d=1}^{\infty} A {\circledast_d} B z^d,$$

so that if the deformation parameter is equal to zero $z \to 0$ (this special case is usually referred to as *classical limit*), the quantum product $\circledast$ coincides with the classical tensor product $\otimes$. The definition of the quantum product follows closely the definition of the product in quantum cohomology: the classes $A \circledast_d B \in K_{\mathsf{T}}(\mathbf{N}_{k,n})$ (quantum corrections) are given by certain degree d curve counting in $\mathbf{N}_{k,n}$.

Next, for a tautological bundle $\tau \in K_{\mathsf{T}}(N_{k,n})$ as above, we define a deformation which will be referred to as *quantum tautological bundle*:

$$\hat{\tau}(z) = \tau + \sum_{d>0}^{\infty} \tau_d z^d \in K_{\mathsf{T}}(\mathbf{N}_{k,n})[[z]]$$

One of the goals of this paper is to study the spectrum of operators of *quantum* multiplication by *quantum* tautological bundles. The following Theorem is the generalization of Proposition 1.2.1 to the quantum level.

Theorem 1.3.1. *The eigenvalues of operators of quantum multiplication by* $\hat{\tau}(z)$ *are given by the values of the corresponding Laurent polynomials* $\tau(s_1, \cdots, s_k)$ *evaluated at the solutions of the following equations:*

$$\prod_{j=1}^{n} \frac{s_i - a_j}{\hbar a_j - s_i} = z\, \hbar^{-n/2} \prod_{\substack{j=1 \\ j\neq i}}^{k} \frac{s_i \hbar - s_j}{s_i - s_j \hbar}\,, \quad i = 1 \cdots k. \tag{1.4}$$

When $z = 0$ we return to the statement of Proposition 1.2.1.

1.4 XXZ model and Baxter Q-operator

A specialist can immediately recognize that (1.4) are nothing but the *Bethe ansatz* equations for the so-called XXZ spin chain. Let us briefly recall some basic facts about this quantum integrable system, see also [38] for a more detailed outline.

Let us consider a system of n interacting magnetic dipoles (usually refered to as *spins*) on a 1-dimensional periodic lattice. Each spin can take two possible configurations "up" and "down", such that the space of the quantum states of this system has dimension 2^n:

$$\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2. \tag{1.5}$$

In this system of spins only the neighboring ones (with labels i and $i+1$) can interact. The energy of the interaction is described by the following Hamiltonian:

$$H_2 = -\sum_{i=1}^{n} \sigma_x^i \otimes \sigma_x^{i+1} + \sigma_y^i \otimes \sigma_y^{i+1} + \Delta\, \sigma_z^i \otimes \sigma_z^{i+1}, \tag{1.6}$$

where $\Delta = \hbar^{1/2} + \hbar^{-1/2}$ is the parameter of anisotropy and σ_m^i are the standard Pauli matrices acting in the i-th factor of (1.5). The periodic boundary conditions are imposed by identifying the first with $(n+1)$-th spin space. Up to a gauge transformation such identification is given by a diagonal matrix. Modulo an irrelevant scalar this matrix can be chosen to be in the following form:

$$\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} : \quad \mathbb{C}^2_{(1)} \to \mathbb{C}^2_{(n+1)}.$$

This free parameter z, defining the periodic boundary condition will play the crucial role in this paper, namely it is the parameter of deformation in the quantum K-theory.

The quantum system of spins governed by the hamiltonian (1.6) is called the quantum XXZ spin chain. The most important feature of this model is its *integrability*, which implies the existence of mutually commuting higher Hamiltonians $H_n, \quad n = 1 \cdots \infty$. For example:

$$S_z = \sum_{i=1}^{n} \sigma_z^i$$

is the operator of total spin commuting with (1.6). This operator provides the grading on the space of states:

$$\mathcal{H} = \bigoplus_{k=0}^{n} \mathcal{H}_k, \quad \mathcal{H}_k = \{v \in \mathcal{H} : S_z(v) = kv\}.$$

Obviously $\dim \mathcal{H}_k = n!/(k!(n-k)!)$ thus this graded sum can be identified with (1.1).

The Hamiltonians of the XXZ model can be conveniently summed into the generating function, known as the Baxter Q-operator [3]:

$$Q(x) = \sum_{i=0}^{\infty} x^i H_i.$$

The physical problem is to find the joint spectrum of H_i. It is given by:

Theorem 1.4.1. *(*[38]*) The eigenvalues of the Baxter's operator $Q(x)$ are given by*

[3] To be precise, the "physical" hamiltonians of the XXZ model are related in a nontrivial way to the coefficients of the Baxter operator H_i. This distinction is of course irrelevant for the diagonalization problem.

$\prod_{i=0}^{k}(1+xs_i)$ *where* s_i *are the solutions of the Bethe equations (1.4).*

This means that the eigenvalues of H_i are given by the values of k-th elementary symmetric function evaluated on the solutions of Bethe equations. In view of Theorem 1.3.1 these are the same as the eigenvalues of operator of the quantum multiplication by the quantum i-th exterior power $\widehat{\Lambda^i \mathcal{V}}(z)$ in the quantum K-theory of $\mathbf{N}_{k,n}$.

1.5 Quantum group structure of the XXZ model and K-theory of $\mathbf{N}_{k,n}$

We see that the spectrum of quantum tautological bundles in quantum K-theory of $\mathbf{N}_{k,n}$ coincides with spectrum of observables in the XXZ model. Let us explain the connection between the quantum physics of the XXZ model and quantum geometry of $\mathbf{N}(n)$. It is well known that the symmetries of the XXZ spin chain are described by the *quantum affine group* $\mathcal{U}_\hbar(\widehat{\mathfrak{sl}}_2)$ (see e.g. [38]). In particular, the Hilbert space of quantum states of the XXZ model is an irreducible $\mathcal{U}_\hbar(\widehat{\mathfrak{sl}}_2)$-module:

$$\mathcal{H}_{XXZ} = \mathbb{C}^2(a_1) \otimes \mathbb{C}^2(a_2) \otimes \cdots \otimes \mathbb{C}^2(a_n), \tag{1.7}$$

where $\mathbb{C}^2(a_i)$ are the two-dimensional evaluation representations of $\mathcal{U}_\hbar(\widehat{\mathfrak{sl}}_2)$.

It was shown in [45] (see also [36] for an alternative construction) that there is a natural action of $\mathcal{U}_\hbar(\widehat{\mathfrak{sl}}_2)$ on the equivariant K-theory $K_{\mathsf{T}}(\mathbf{N}(n))$. In short, as a $\mathcal{U}_\hbar(\widehat{\mathfrak{sl}}_2)$-module the K-theory of $\mathbf{N}(n)$ is isomorphic to the Hilbert space of the XXZ spin chain : $K_{\mathsf{T}}(\mathbf{N}(n)) = \mathcal{H}_{XXZ}$. The evaluation parameters of representation a_i and the "Planck constant" $\hbar$ are the counterparts of the equivariant characters of the torus T on the side of equivariant K-theory. Our claim is that the ring of quantum tautological bundles coincides with the completed ring of the Hamiltonians of the XXZ spin chain under this isomorphism. For instance, let us consider the K-theory class of x-weighted exterior algebra of $\mathcal{V}$:

$$\tau_x = \bigoplus_{m=0}^{\infty} x^m \Lambda^m(\mathcal{V}) = \Lambda^\bullet_x \mathcal{V}.$$

Theorem 1.5.1. *The operator of quantum multiplication by the quantum tautological bundle $\hat{\tau}_x(z)$ coincides with the Baxter operator for the XXZ spin chain under the identification $K_{\mathsf{T}}(\boldsymbol{N}(n)) = \mathcal{H}_{XXZ}$ as $\mathcal{U}_\hbar(\widehat{\mathfrak{sl}}_2)$-modules:*

$$\hat{\tau}_x(z) = Q(x).$$

1.6 Universal formula and the dynamical quantum affine Weyl group

The operators of quantum multiplication by $\hat{\tau}(z)$ satisfy the following Theorem.

Theorem 1.6.1. *The operators of quantum multiplication by $\hat{\tau}(z)$ are the universal elements in $\mathcal{U}_\hbar(\widehat{\mathfrak{sl}}_2)[[z]]$.*

As we explained above, the equivariant K-theory $K_{\mathsf{T}}(\mathbf{N}(n))$ is a natural $\mathcal{U}_\hbar(\widehat{\mathfrak{sl}}_2)$ module (1.7). By evaluating the universal element $\hat{\tau}(z)$ in $K_{\mathsf{T}}(\mathbf{N}(n))$ we obtain the operator of quantum multiplication by $\hat{\tau}(z)$ in this representation. Universality means that $\hat{\tau}(z) \in \mathcal{U}_\hbar(\widehat{\mathfrak{sl}}_2)[[z]]$ does not depend on n, i.e. on the choice of representation. Moreover, one can check that the operators of quantum multiplication by *classical* tautological classes τ do not have such a universal representation. This is an indication that the quantum bundles $\hat{\tau}(z)$ are more natural objects in the context of quantum K-theory.

To prove the universality thorem for the generic operators, we prove it for operators of quantum multiplication by the quantum exterior powers $\widehat{\Lambda^i \mathcal{V}}(z)$, since those are operators, whose eigenvalues are elementary symmetric functions of the solutions of Bethe equations.

Let E_r, F_r, H_r, K be the standard Drinfeld's generators of $\mathcal{U}_\hbar(\widehat{\mathfrak{sl}}_2)$, then the following theorem holds.

Theorem 1.6.2. *For arbitrary n and k the operator of quantum multiplication by the quantum l-th exterior power of tautological bundle is given by the following universal formula:*

$$\widehat{\Lambda^l \mathcal{V}}(z) = \Lambda^l \mathcal{V} + a_1(z)\, F_0 \Lambda^{l-1}\mathcal{V} E_{-1} + a_2(z)\, F_0^2 \Lambda^{l-2}\mathcal{V} E_{-1}^2 + \cdots + a_l(z) F_0^l E_{-1}^l$$

with

$$a_m(z) = \frac{(\hbar-1)^m \ \hbar^{\frac{m(m+1)}{2}} K^m}{(m)_\hbar! \prod\limits_{i=1}^{m} (1-(-1)^n z^{-1} \hbar^i K)}, \quad \textit{for} \quad (m)_\hbar = \frac{1-\hbar^m}{1-\hbar}, \quad (m)_\hbar! = (1)_\hbar \cdots (m)_\hbar.$$

Here $\widehat{\Lambda^i \mathcal{V}}(z)$ ($\Lambda^i \mathcal{V}$) stands for the operators of quantum (classical) multiplication by the quantum (classical) exterious powers. Note, that in the classical limit $z \to 0$ all $a_l(z)$ vanish and we obtain $\widehat{\Lambda^l \mathcal{V}}(0) = \Lambda^l \mathcal{V}$.

It is interesting to specialize this theorem to the case of the top exterior power: $\det(\mathcal{V}) = \mathcal{O}(1)$. It is well known that the operator of multiplication by the line bundle $\mathcal{O}(1)$ in the classical equivariant K-theory corresponds to the lattice part of quantum affine Weyl group of $\mathcal{U}_\hbar(\widehat{\mathfrak{sl}}_2)$. In particular, it acts on Drinfeld's generators by: $\mathcal{O}(1) F_m \mathcal{O}(1)^{-1} = F_{m-1}, \quad \mathcal{O}(1) F_m \mathcal{O}(1)^{-1} = E_{m+1}$ Using these equations and Theorem 1.6.2 we obtain:

$$\widehat{\mathcal{O}(1)}(z) = B(z)\, \mathcal{O}(1), \quad B(z) = \sum_{m=0}^{\infty} \frac{\hbar^{m(m+1)/2} (\hbar-1)^m K^m}{(m)_\hbar! \prod\limits_{i=1}^{m} (1-(-1)^n z^{-1} K \hbar^i)} F_0^m E_0^m$$

Up to a shift in the notations $B(z)$ coincides with the element of the universal enveloping algebra, defining the action of the quantum affine dynamical Weyl group $QW_{\mathcal{U}_\hbar(\widehat{\mathfrak{sl}}_2)}$ (see Proposition 14 in [11]). The dynamical parameter e^λ of the dynamical Weyl group in [11] is identified with the parameter of quantum deformation z in quantum K-theory. We conclude the following result:

Theorem 1.6.3. *The lattice element of the quantum dynamical Weyl group $QW_{\mathcal{U}_\hbar(\widehat{\mathfrak{sl}}_2)}$*

acts on $K_{\mathsf{T}}(\boldsymbol{N}(n))$ *as the operator of quantum multiplication by the quantum line bundle* $\widehat{\mathcal{O}(1)}(z)$.

1.7 Correspondence table

In short, the physics of the XXZ spin chain, quantum K-theory of $\mathbf{N}(n)$ and representation theory of $\mathcal{U}_\hbar(\widehat{\mathfrak{sl}}_2)$ are different languages describing the same object. The following table is a dictionary:

$XXZ-$spin chain	Geometry of $N_{k,n}$	Representation theory of $\mathcal{U}_\hbar(\widehat{\mathfrak{sl}}_2)$
$\mathcal{H}_{XXZ}$	$K_{\mathsf{T}}(\mathbf{N}(n))$	$\bigotimes_{i=1}^n \mathbb{C}^2(a_i)$
inhomogeneity parameters a_i	equivariant characters a_i	evaluation module parameters a_i
anisotropy parameter $\Delta = \hbar^{1/2} + \hbar^{-1/2}$	$\hbar = \mathsf{T}$ weight of symplectic form	$\hbar^{1/2}-$parameter of the quantum group
Transfer matrices, Baxter $\mathcal{Q}-$operators	generating function for quantum tautological bundles	weighted partial traces of of $R-$matrices
$z-$parameter of boundary condition	$z-$parameter of quantum deformation	$z-$ parameter of weight in the trace

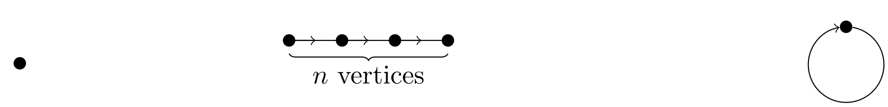
Chapter 2

Classical Constructions

In this chapter we give some of the background needed to understand the main theorems of this thesis. While it is almost impossible to give the full background needed, this is an attempt to produce a text that can be read with minimal need to go into other sources.

2.1 Quivers

The starting point for constructions considered in this text are quivers. A *quiver* is a collection of vertices (I) and edges (E) connecting them. Loops and multiple edges are permissible, and edges are oriented, although as we will see their orientation will not affect the construction in crucial ways, as all edges will be doubled. Here are some simple examples of quivers that will give rise to interesting examples in following constructions:



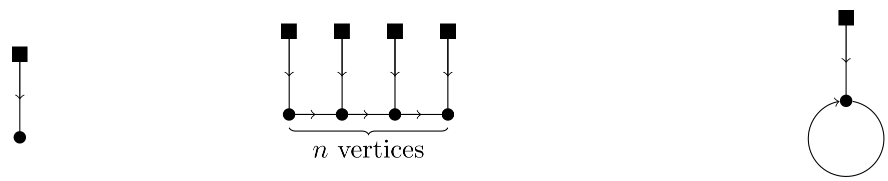
(a) One Vertex

(b) Type A

(c) Jordan Quiver

Figure 2.1: Examples of Quivers

We will consider *framed* quivers, that is quivers where all the vertices are doubled, and the each of the copies is has exactly one edge going from it to it's original and nothing else. It is convenient to put the copies above the originals and to denote them by squares. The framed quivers corresponding to the ones above will be:



(a) One Vertex

(b) Type A

(c) Jordan Quiver

Figure 2.2: Framed Quivers

Such combinatorial data can be used to construct a whole class of symplectic varieties, whose geometric invariants will carry natural representation-theoretical structures.

The first objects one considers are representations of framed quivers. A representation of a quiver is a collection of vector spaces $V_i, i \in I$ corresponding to original vertices, spaces $W_i, i \in I$ corresponding to their copies, and operators between them corresponding to edges of the framed quiver. Each representation has a dimension vector $(\mathbf{v}, \mathbf{w})$, so one can consider the space $\text{Rep}(\mathbf{v}, \mathbf{w})$ of representations of given dimensions. This space clearly has a vector space structure, as it is a sum of Hom-spaces corresponding to arrows of the quiver.

2.2 Nakajima Quiver Varieties

For any framed quiver and a dimension vector one can associate a Nakajima quiver variety. In this section we will give a brief construction of these varieties and consider some examples.

Consider the space $T^*\text{Rep}(\mathbf{v}, \mathbf{w})$. As any cotangent bundle, this space has a natural symplectic form on it. It also admits a natural symplectic action of the group $G = \prod GL(V_i)$. This action gives rise to a moment map $\mu : T^*\text{Rep}(\mathbf{v}, \mathbf{w}) \to \text{Lie}(G)^*$, defined by $\mu(\alpha_x) : u \to \langle \alpha, \vec{u}_x \rangle$ (here $u \in \text{Lie}(G)$, $\vec{u}$ is the vector field on X corresponding to the infinitesimal action of u, $\vec{u}_x$ is the value of this vector field at a point x, and α_x stands for a covector at x). Denote $L(\mathbf{v}, \mathbf{w}) = \mu^{-1}(0)$ to be the zero locus of this map. A *Nakajima quiver variety* is a GIT quotient of this locus by the group G:

$$X = L(\mathbf{v}, \mathbf{w}) /\!\!/ G.$$

Here a certain choice of stability is made, but we will skip in notation and will discuss it in specific examples.

The constructed variety has the following group of automorphisms acting on it:

$$\prod GL(Q_{ij}) \times \prod GL(W_i) \times \mathbb{C}^\times_\hbar.$$

Here Q_{ij} stands for the multipicity of edges between vertices i and j, i.e. in cases when there are multiple arrows between vertices i and j in the original quiver, the space $Q_{ij} \times Hom(V_i, V_j)$ is considered in the space of representations. The torus $\mathbb{C}_\hbar$ is a one dimensional torus scaling the cotangent directions with weight $\hbar$ and therefore the symplectic form with weight $\hbar^{-1}$. Denote by T the maximal torus of this group (this amounts to choosing a basis in the corresponding vector space). We will be primarily studying the equivariant K-theory of the variety X with respect to to the torus T. For a Nakajima quiver variety X one can define a set of tautological bundles on it $\mathcal{V}_i, \mathcal{W}_i, i \in I$. Tensorial polynomials of these bundles and their duals generate $K_\mathsf{T}(X)$ according to Kirwan's surjectivity theorem [27]. All bundles $\mathcal{W}_i$ are trivial. Let $(\cdot, \cdot)$ be a bilinear form on $K_\mathsf{T}(X)$ defined by the following formula

$$(\mathcal{F}, \mathcal{G}) = \chi(\mathcal{F} \otimes \mathcal{G} \otimes K^{-1/2}), \tag{2.1}$$

where K is the canonical class and χ is Euler characteristic. The reason for this extra shift of the classical bilinear form will be explained below.

Before going into more detail about the objects of our study, let us go through examples of constructed varieties, corresponding to some of the quivers shown in the previous section.

2.3 Examples

Example 1. Grassmannian

We start with the simplest example of the original quiver there is, that is just one vertex without any edges. The framed quiver is shown on picture (2.2a). Yet, as we are considering the space $T^*\mathrm{Rep}(\mathbf{v},\mathbf{w})$ we can instead of considering the cotangent bundle consider representations of the *double* quiver, that is the quiver with the same vertices, and each edge doubled with an opposite one. Assign the $\mathbf{v}$ dimension to be k and the $\mathbf{w}$ dimension to be n. In the considered example we have the following picture:

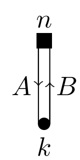


Figure 2.3: One Vertex Quiver

The moment map condition in this case can be written as $AB = 0$. The choice of stability condition can be made either $\mathrm{rk}A = k$ or $\mathrm{rk}B = k$. The geometry of the space will not depend on this choice, so we will choose $\mathrm{rk}B = k$, as it will be more convenient. In this case the variety can be naturally identified with the cotangent bundle to the Grassmannian of k-planes in an n-space $X = T^*\mathrm{Gr}(k,n)$ (the k-plane is the image of B and the n-plane is W). The group acting on this space will be $GL(n) \times \mathbb{C}^\times_\hbar$, where $GL(n)$ acts tautologically on the W, and $\mathbb{C}^\times_\hbar$ scales the fibers of the cotangent bundle. Let T denote a maximal torus of this group, as before. As we will be studying the equivariant K-theory of Nakajima quiver varieties, localization will be commonly used, so it will come in handy to write down what are the fixed points of the T action. In this example fixed points are classified by k-subsets of an n-set, i.e. by sets $\{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$. The tautological bundle in this case is the classical tautological bundle on the Grassmannian $\mathcal{V}$. In this case what we need to generate the whole K-theory are only wedge powers of this tautological bundle, i.e. $\mathcal{V}, \Lambda^2\mathcal{V}, \ldots, \Lambda^k\mathcal{V}$. The eigenvalues of operators of multiplication by these bundles at a fixed point are nothing but elementary symmetric functions of equivariant parameters corresponding to the subset classifying this point.

Example 2. Partial Flags

The second example we will consider will be a simplified version of the picture (2.2b) and a generalization of the previous example. Consider the figure (2.2b), where all the dimensions $\mathbf{w}_i$ are set to 0, except $\mathbf{w}_{n-1}$. Then, drawing the picture without the 0-dimensional vertices, and enumerating vertices from right to left we get the following picture:

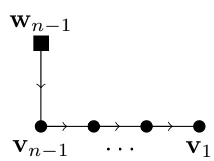


Figure 2.4: A_n Quiver

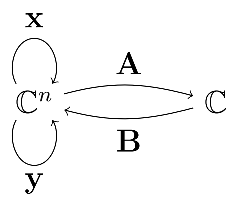
Analogously to the previous example, we can double the arrows and see that the stability condition can be chosen to imply surjectivity or injectivity of the morphisms. This means that we can choose all the arrows opposite to the ones shown on the picture to be surjective, then the constructed variety will obviously be the cotangent bundle to the space of partial flags of the form $\mathbb{C}^{\mathbf{v}_1} \subset \mathbb{C}^{\mathbf{v}_2} \subset \ldots \subset \mathbb{C}^{\mathbf{v}_{n-1}} \subset \mathbb{C}^{\mathbf{w}_{n-1}}$. It is also clear, that a different choice of stability condition will lead to a different partial flag variety. We can immediately see how this example generalizes the previous one, as Grassmannians are specific examples of partial flags. We can also see that fixed point of the torus action (the group in this case is $GL(\mathbf{w}_{n-1}) \times \mathbb{C}^{\times}_{\hbar}$) are classified by sequences of subsets of the form $\mathbf{V}_1 \subset \ldots \subset \mathbf{V}_{n-1} \subset \mathbf{W}_{n-1}$, where $|\mathbf{V}_i| = \mathbf{v}_i, \mathbf{W}_{n-1} = \{1, \ldots, \mathbf{w}_{n-1}\}$. Finally, the set of tautological bundles in this case will be $\{\mathcal{V}_1, \ldots, \mathcal{V}_{n-1}, \mathcal{W}_{n-1}\}$, where

$\mathcal{W}_{n-1}$ is trivial and $\mathcal{V}_1, \ldots, \mathcal{V}_{n-1}$ are tautological.

Example 3. Hilbert Scheme of points on $\mathbb{C}^2$

The last example we consider is the Jordan quiver.

After doubling all edges we get the following quiver:



In this case the stability condition can be chosen to be equivalent to $[\mathbf{x}, \mathbf{y}] = 0$, $\mathbf{A} = 0$, and $\mathbf{B}$ is a cyclic vector for $(\mathbf{x}, \mathbf{y})$. For a quiver representation satisfying this stability condition we can introduce a set of polynomials of two variables $I_{\mathbf{x},\mathbf{y},\mathbf{A}}$, constructed as:

$$I_{\mathbf{x},\mathbf{y},\mathbf{A}} := \{f \in \mathbb{C}[x, y] | f(\mathbf{x}, \mathbf{y})\mathbf{A} = 0\}.$$

It follows obviously from construction, that $I_{\mathbf{x},\mathbf{y},\mathbf{A}}$ is an ideal. In fact, since $\mathbf{A}$ is a cyclic vector for $\mathbf{x}$ and $\mathbf{y}$, the codimension of this ideal is exactly n. One can prove that such an assignment encompasses all ideals of codimension n of $\mathbb{C}[x, y]$ and is a bijection. The space of such ideals has a natural scheme structure, and is much better known by its other name: the Hilbert scheme of points on $\mathbb{C}^2$.

In this case it is natural to consider a slightly different torus action. Let the torus $\mathbb{C}^\times_{t_1}$ scale $\mathbf{x}$, and the torus $\mathbb{C}^\times_{t_2}$ scale $\mathbf{y}$. This corresponds to the natural torus action coming from the one on $\mathbb{C}^2$. The fixed points of this action can be classified by Young diagrams of size n.

This example is one of the most important Nakajima varieties, perhaps the most important there is, yet we do not address it in the following chapters of this thesis, and only deal with the first two. Hopefully, results given here will be generalized to this case in future projects.

Chapter 3

Quantum K-theory

In this chapter we construct the key objects used in this text, most importantly quasimaps, the quantum K-theory ring of a Nakajima quiver variety, the bare and the capped vertex.

3.1 Quasimaps

This section deals with the construction of quasimaps of different type and some of their properties.

Before stating the formal definition of quasimaps we first give some motivation for this definition. The role of the space of quasimaps is similar to that of stable maps in Gromov-Witten theory and so is the motivation. Consider a map f from a curve $\mathcal{C}$ to a Nakajima quiver variety X, perhaps the best example to keep in mind here is $T^*\mathrm{Gr}(k,n)$. In this case X carries two tautological bundles $\mathcal{V}$ and $\mathcal{W}$, with morphisms between them. This brings us to the following diagram:

$$\begin{array}{ccc} & & \mathcal{V} \leftrightarrows \mathcal{W} \\ & & \searrow \;\; \swarrow \\ \mathcal{C} & \xrightarrow{\;f\;} & X \end{array}$$

We pullback these bundles to the curve and denoting them $\mathscr{V} = f^*\mathcal{V}$ and $\mathscr{W} = f^*\mathcal{W}$, with a slight abuse of notation. One can see, that one can reconstruct the map from the data shown in this diagram:

$$\begin{array}{ccc} \mathscr{V} & \rightleftarrows & \mathscr{W} \\ & \searrow \quad \swarrow & \\ & \mathcal{C} & \end{array}$$

The arrows between $\mathscr{V}$ and $\mathscr{W}$ denote full rank sections of $Hom(\mathscr{V}, \mathscr{W})$ and $Hom(\mathscr{W}, \mathscr{V})$ correspondingly. This allows us to forget about the map, and just think about vector bundles on a curve instead.

Viewing maps to $T^*\mathrm{Gr}(k, n)$ in this way gives us a reasonable partial completion of the space of maps, where all we do is just lose the "full rank" condition, and allow the sections to have lower rank, at least in some points of the curve $\mathcal{C}$. This brings us to the formal definition of quasimaps, generalized for the case of any Nakajima quiver variety.

Definition 3.1.1. *A quasimap f from $\mathcal{C}$ to X*

$$f : \mathcal{C} \dashrightarrow X \tag{3.1}$$

Is a collection of vector bundles $\mathscr{V}_i$ on $\mathcal{C}$ of ranks $\mathbf{v}_i$ together with a section of the bundle

$$f \in H^0(\mathcal{C}, \mathscr{M} \oplus \mathscr{M}^* \otimes \hbar), \tag{3.2}$$

satisfying $\mu = 0$, where

$$\mathscr{M} = \sum_{i \in I} Hom(\mathscr{W}_i, \mathscr{V}_i) \oplus \sum_{i,j \in I} Q_{ij} \otimes Hom(\mathscr{V}_i, \mathscr{V}_j),$$

so that $\mathscr{W}_i$ are trivial bundles of rank $\mathbf{w}_i$ and μ is the moment map. Here $\hbar$ is a trivial line bundle with weight $\hbar$ introduced to have the action of T on the space of quasimaps. The degree of a quasimap is the vector of degrees of bundles $\mathscr{V}_i$.

This definition follows the motivation given before. Note that this basically mimics the definition of a Nakajima quiver variety, but instead of vector spaces we consider vector bundles over a curve $\mathcal{C}$, and also drop the stability condition. One way to think about quasimaps is as maps to the pre-quotient stack $L(\mathbf{v},\mathbf{w})$. A *stable* quasimap, is a quasimap, which sends only a finite number of points outside of the semistable locus. For the rest of this manuscript all quasimaps are considered to be stable (and the space of quasimaps only involves stable ones).

For a point on the curve $p \in \mathcal{C}$ we have an evaluation map to the quotient stack $\mathrm{ev}_p : \mathsf{QM}^{\mathbf{d}} \to L(\mathbf{v},\mathbf{w})/G$ defined by $\mathrm{ev}_p(f) = f(p)$. Note that the quotient stack contains X as an open subset corresponding to locus of semistable points:

$$X = \mu_{ss}^{-1}(0)/G \subset L(\mathbf{v},\mathbf{w})/G.$$

A quasimap f is called nonsingular at p if $f(p) \subset X$. In short, we conclude that the open subset ${\mathsf{QM}^{\mathbf{d}}}_{\text{nonsing p}} \subset \mathsf{QM}^{\mathbf{d}}$ of quasimaps nonsingular at the given point p is endowed with a natural evaluation map:

$${\mathsf{QM}^{\mathbf{d}}}_{\text{nonsing } p} \xrightarrow{\mathrm{ev}_p} X \qquad (3.3)$$

that sends a quasimap to its value at p.

The moduli space of relative quasimaps ${\mathsf{QM}^{\mathbf{d}}}_{\text{relative}\, p}$ is a resolution of ev_p (or compactification), meaning we have a commutative diagram:

$$\begin{array}{ccccc} & & {\mathsf{QM}^{\mathbf{d}}}_{\text{relative}\, p} & & \\ & \nearrow & & \searrow^{\widetilde{\mathrm{ev}}_p} & \\ {\mathsf{QM}^{\mathbf{d}}}_{\text{nonsing}\, p} & & \xrightarrow{\quad \mathrm{ev}_p \quad} & & X \end{array}$$

with a **proper** evaluation map $\widetilde{\mathrm{ev}}_p$ from ${\mathsf{QM}^{\mathbf{d}}}_{\text{relative}\, p}$ to X. The construction of this resolution and the moduli space of relative quasimaps is relatively technical and this technique will not be directly used in any computations, so we address the reader to [35] for any details. It follows a similar construction of relative Gromov-Witten and Donaldson-Thomas moduli spaces. The main idea of this construction is to allow the base curve to change in cases, when the relative point becomes singular. When this happens we replace the relative point by a chain of non-rigid projective lines, such that the endpoint and all the nodes are not singular. Similarly, for nodal curves, we do not allow the singularity to be in the node, and if that happens we instead paste in a chain of non-rigid projective lines.

These moduli spaces have a natural action of maximal torus T, lifting its action from X. When there are at most two special (relative or marked) points and the original curve is $\mathbb{P}^1$ we extend T by additional torus $\mathbb{C}_q^\times$, which scales $\mathbb{P}^1$ such that the tangent space $T_0\mathbb{P}^1$ has character denoted by q. We call the full torus by $G = \mathsf{T} \times \mathbb{C}_q^\times$.

3.2 Picture Notations, Virtual Structure and Gluing Operator

In the theory of relative quasimaps it is to use picture notation, introduced by Okounkov in [35]. Here is some of it, which we will use in this manuscript:

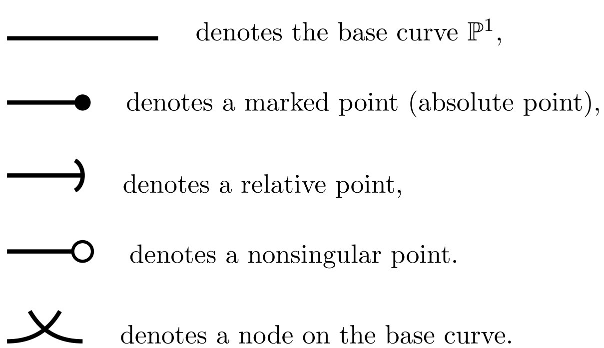


Figure 3.1: Quasimap Notations

The moduli spaces of quasimaps constructed above have a perfect deformation-obstruction theory [9]. This allows one to construct a tangent virtual bundle T^{vir}, a virtual structure sheaf $\hat{\mathcal{O}}_{\text{vir}}$ and a virtual canonical bundle. We will define multiplication in the quantum K-theory using this data. Without going into detail of the construction of this virtual sheaf, we state the formula of the reduced virtual tangent bundle. Let $(\{\mathscr{V}_i\}, \{W_i\})$ be the data defining a quasimap which is nonsingular at fixed point p. We define the fiber of the reduced virtual tangent bundle to $\mathsf{QM}^{\mathbf{d}}{}_{\text{nonsing}\, p}$ at this point to be equal to:

$$T^{\text{vir}}_{(\{\mathscr{V}_i\},\{\mathscr{W}_i\})}\mathsf{QM}^{\mathbf{d}}_{\text{nonsing p}} = H^\bullet(\mathscr{M} \oplus \hbar \mathscr{M}^*) - (1+\hbar)\bigoplus_i Ext^\bullet(\mathscr{V}_i, \mathscr{V}_i). \tag{3.4}$$

The symmetrized virtual structure sheaf is defined by:

$$\hat{\mathcal{O}}_{\text{vir}} = \mathcal{O}_{\text{vir}} \otimes \mathscr{K}^{1/2}_{\text{vir}} q^{\deg(\mathscr{P})/2}, \tag{3.5}$$

where $\mathscr{K}_{\text{vir}} = \det^{-1} T^{\text{vir}}\mathsf{QM}^{\mathbf{d}}$ is the virtual canonical bundle and $\mathscr{P}$ is the polarization bundle.

Since we will be using the symmetrized virtual structure sheaf we will need to adjust the standard bilinear form on K-theory. That is the reason to for the shift of the bilinear form in (2.1).

In order to construct the quantum product we need an important element in the theory of relative quasimaps, namely the gluing operator. This is the operator[1] $\mathbf{G} \in End(K_{\mathsf{T}}(X))[[z]]$ defined by:

$$\mathbf{G} = \sum_{\mathbf{d}=\overrightarrow{0}}^{\infty} z^{\mathbf{d}} ev_{p_1,p_2*}(\mathsf{QM}^{\mathbf{d}}_{\text{relative}\, p_1,p_2} \hat{\mathcal{O}}_{\text{vir}}) \in K^{\otimes 2}_{\mathsf{T}}(X)[[z]], \tag{3.6}$$

so that the corresponding picture is : .

It plays an important role in the degeneration formula, see e.g. [35]. Namely, let a smooth curve $\mathcal{C}_\varepsilon$ degenerate to a nodal curve:

$$\mathcal{C}_0 = \mathcal{C}_{0,1} \cup_p \mathcal{C}_{0,2}.$$

Here $\mathcal{C}_{0,1}$ and $\mathcal{C}_{0,2}$ are two different components that are glued to each other at point p. The degeneration formula counts quasimaps from $\mathcal{C}_\varepsilon$ in terms of relative quasimaps from $\mathcal{C}_{0,1}$ and $\mathcal{C}_{0,2}$, where the relative conditions are imposed at the gluing point p. The family of spaces $\mathsf{QM}(\mathcal{C}_\varepsilon \to X)$ is flat, which means that we can replace curve

[1] In fact, the gluing operator is a rational function of the quantum parameters $\mathbf{G} \in End(K_{\mathsf{T}}(X))(z)$.

counts for any $\mathcal{C}_\varepsilon$ by $\mathcal{C}_0$. In particular, we can replace counts of quasimaps from $\mathbb{P}^1$ by a degeneration of it, for example by two copies of $\mathbb{P}^1$ glued at a point.

The gluing operator $\mathbf{G} \in \mathrm{End} K_{\mathsf{T}}(X)[[z]]$ is the tool that allows us to replace quasimap counts on $\mathcal{C}_\varepsilon$ by counts on $\mathcal{C}_{0,1}$ and $\mathcal{C}_{0,2}$, so that the following degeneration formula holds:

$$\chi(\mathsf{QM}(\mathcal{C}_0 \to X), \hat{\mathcal{O}}_{\mathrm{vir}} z^{\mathbf{d}}) = \left(\mathbf{G}^{-1} \mathrm{ev}_{1,*}(\hat{\mathcal{O}}_{\mathrm{vir}} z^{\mathbf{d}}), \mathrm{ev}_{2,*}(\hat{\mathcal{O}}_{\mathrm{vir}} z^{\mathbf{d}})\right).$$

The corresponding picture interpretation is as follows:

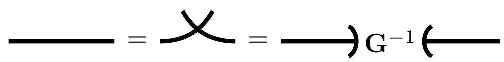


Figure 3.2: Gluing Formula

3.3 Quantum K-theory Ring

In this section we define multiplication and important objects of the quantum K-theory ring of X.

As a vector space quantum K-theory ring $QK_{\mathsf{T}}(X)$ is isomorphic to $K_{\mathsf{T}}(X) \otimes \mathbb{C}[[z_{\{i\}}]]$, $i \in I$. We will often use the following notation: for a vector $\mathbf{d} = (d_i)$,

$$z^{\mathbf{d}} = \prod_{i \in I} z_i^{d_i}.$$

Definition 3.3.1. *An element of the quantum K-theory*

$$\hat{\tau}(z) = \sum_{\mathbf{d}=\overrightarrow{0}}^{\infty} z^{\mathbf{d}} \mathrm{ev}_{p_2,*}\left(\mathsf{QM}^{\mathbf{d}}_{\mathrm{relative}\, p_2}, \widehat{\mathcal{O}}_{\mathrm{vir}} \tau(\mathcal{V}_i|_{p_1})\right) \in QK_{\mathsf{T}}(X) \tag{3.7}$$

is called quantum tautological class corresponding to τ. In picture notation it will be represented by:

Figure 3.3: Quantum Tautological Class

These classes evaluated at $z = 0$ are equal to the classical tautological classes on X ($\hat{\tau}(0) = \tau$).

For any element $\mathcal{F} \in K_{\mathsf{T}}(X)$ the following element

$$\sum_{\mathbf{d}=\overrightarrow{0}}^{\infty} z^{\mathbf{d}} \mathrm{ev}_{p_1,p_3*} \left(\mathsf{QM}^{\mathbf{d}}_{p_1,p_2,p_3}, \mathrm{ev}^*_{p_2}(\mathbf{G}^{-1}\mathcal{F})\widehat{\mathcal{O}}_{\mathrm{vir}} \right) \in K_{\mathsf{T}}(X)^{\otimes 2}[[z]] \tag{3.8}$$

can be made into an operator from the second copy of $K_{\mathsf{T}}(X)$ to the first copy by the bilinear form $(\cdot,\cdot)$ defined above. We define the operator of quantum multiplication by $\mathcal{F}$ to be this operator shifted by $\mathbf{G}^{-1}$, i.e

$$\mathcal{F}\circledast = \sum_{\mathbf{d}=\overrightarrow{0}}^{\infty} z^{\mathbf{d}} \mathrm{ev}_{p_1,p_3*} \left(\mathsf{QM}^{\mathbf{d}}_{p_1,p_2,p_3}, \mathrm{ev}^*_{p_2}(\mathbf{G}^{-1}\mathcal{F})\widehat{\mathcal{O}}_{\mathrm{vir}} \right) \mathbf{G}^{-1} \tag{3.9}$$

Definition 3.3.2. *The quantum equivariant K-theory ring of X is vector space $QK_{\mathsf{T}}(X) = K_{\mathsf{T}}(X)[[z]]$ endowed with multiplication (3.9).*

This product enjoys the following properties, similar to the product in quantum cohomology.

Theorem 3.3.3. *The quantum K-theory ring $QK_{\mathsf{T}}(X)$ is a commutative, associative and unital algebra.*

Proof. Commutativity of this algebra follows from the construction, by switching points p_2 and p_3. Associativity of this ring follows from the fact that operators of quantum multiplication by two different sheafs $\mathcal{F}$ and $\mathcal{G}$ commute. The picture proof of this is as follows:

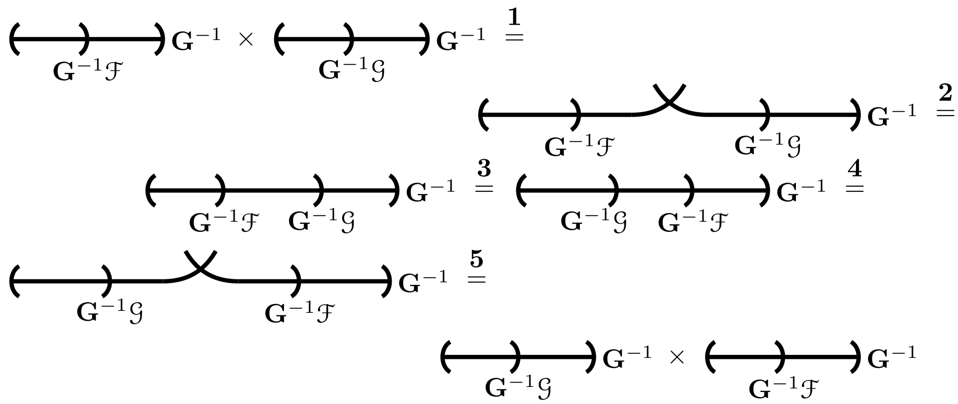


Figure 3.4: Commutativity

Here equalities **1** and **5** come from gluing formulas, **2** and **4** degeneration formulas and equality **3** is a deformation of the base curve. The existence and properties of multiplicative identity element in $QK_{\mathsf{T}}(X)$ is discussed in Proposition 3.3.5. □

The operators of quantum multiplication by the **quantum** tautological bundles obey the most natural properties. First, using Kirwan's K-theoretic surjectivity theorem, we have the following result.

Proposition 3.3.4. *Quantum tautological classes generate the quantum equivariant K-theory over the quantum equivariant K-theory of a point $QK_{\mathsf{T}}(\cdot) = \mathbb{C}[a_m^{\pm 1}][[z_i]]$ where a_m for $m = 1 \cdots \dim \mathsf{T}$ are the equivariant parameters of T.*

Proof. Since, by Kirwan's K-theoretic surjectivity theorem, classical K-theory is generated by tautological classes, the quantum K-theory will be generated by quantum tautological classes according to Nakayama's Lemma. □

Second, in contrast with quantum cohomology, the multiplicative identity of the quantum K-theory ring does not always coincide with the multiplicative identity of classical K-theory (i.e. the structure sheaf $\mathcal{O}_X$):

Proposition 3.3.5. *The multiplicative identity of $QK_{\mathsf{T}}(X)$ is given by $\hat{\mathbf{1}}(z)$ (i.e. the quantum tautological class for insertion $\tau = 1$).*

Proof. To show this, we start with tautology: $\mathbf{G} \times \mathbf{G}^{-1} = \mathbf{Id}$. The following proof can be again done in picture notation

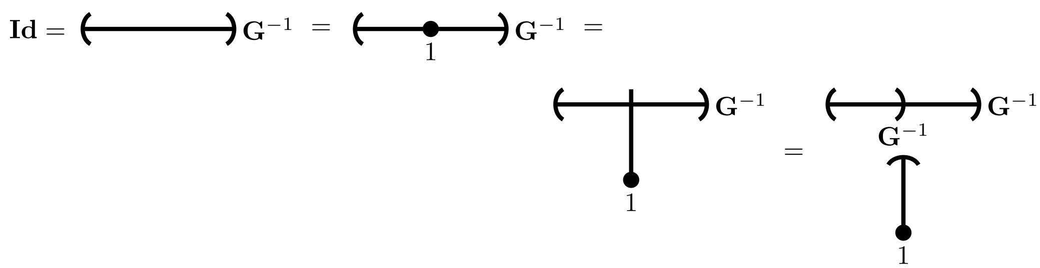


Figure 3.5: Quantum Identity

□

This proof also shows the essence of considering quantum tautological classes in K-theory in the first place. In an absolute analogous way one can show that multiplication by a quantum tautological class is the same as instead of considering the second point to be relative, one can plug in the corresponding descendant. In short:

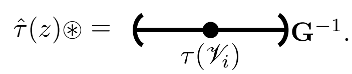


Figure 3.6: Quantum Tautological Multiplication

3.4 Vertex functions

The spaces $\mathsf{QM}^{\mathbf{d}}_{\mathrm{nonsing\,p_2}}$ and $\mathsf{QM}^{\mathbf{d}}_{\mathrm{relative\,p_2}}$ admit an action of an extra torus $\mathbb{C}_q$ which scales the original $\mathbb{P}^1$ keeping points p_1 and p_2 fixed. Set $\mathsf{T}_q = \mathsf{T} \times \mathbb{C}_q$ be the torus acting on these spaces.

Definition 3.4.1. *The element*

$$V^{(\tau)}(z) = \sum_{\mathbf{d}=\vec{0}}^{\infty} z^{\mathbf{d}} \mathrm{ev}_{p_2,*}\Big(QM^{\mathbf{d}}_{\mathrm{nonsing}\,p_2}, \widehat{\mathcal{O}}_{\mathrm{vir}} \tau(\mathscr{V}_i|_{p_1}) \Big) \in K_{\mathsf{T}_q}(X)_{loc}[[z]]$$

is called bare vertex with descendent τ. In picture notation it will be denoted by

Figure 3.7: Bare Vertex

The space $\mathsf{QM}^{\mathbf{d}}_{\text{nonsing}\,\text{p}_2}$ is not proper (the condition of non-singularity at a point is an open condition), but the set of T_q-fixed points is, hence the bare vertex is singular at $q = 1$.

Definition 3.4.2. *The element*

$$\hat{V}^{(\tau)}(z) = \sum_{\mathbf{d}=\vec{0}}^{\infty} z^{\mathbf{d}} \mathrm{ev}_{p_2,*}\Big(\mathsf{QM}^{\mathbf{d}}_{\text{relative}\,p_2}, \widehat{\mathcal{O}}_{\text{vir}} \tau(\mathscr{V}_i|_{p_1}) \Big) \in K_{\mathsf{T}_q}(X)[[z]]$$

is called capped vertex with descendent τ. In picture notation it will be represented by:

Figure 3.8: Capped Vertex

Note here, that the definition of the capped vertex and the definition of quantum tautological classes (3.3.1) are very similar, with the main difference being the spaces they live in. By definition, the quantum tautological classes can be obtained by taking a limit of the capped vertex: $\lim_{q \to 1} \hat{V}^{(\tau)}(z) = \hat{\tau}(z)$. The last limit exists as the coefficients of $\hat{V}^{(\tau)}(z)$ are Laurent polynomials in q, due to the properness of the evaluation map in the relative case.

In fact, the following strong statement is known about capped vertex functions.

Theorem 3.4.3. *Power series $\hat{V}^{(\tau)}(z)$ is a Taylor expansion of a rational function in quantum parameters z.*

Proof. There are two different proofs of this theorem: the first is based on large framing vanishing [43], the second originates from integral representations of solutions of quantum difference equations [2]. □

As a corollary, quantum tautological classes $\hat{\tau}(z)$ are rational functions of z.

3.5 Capping Operator and Difference Equation

The operator which relates capped and bare vertices, is known as capping operator and is defined as the following class in localized K-theory:

$$\Psi(z) = \sum_{\mathbf{d}=0}^{\infty} z^{\mathbf{d}} \mathrm{ev}_{p_1,p_2,*}\left(\mathsf{QM}^{\mathbf{d}}_{\substack{\text{relative p}_1\\ \text{nonsing p}_2}}, \widehat{\mathcal{O}}_{\mathrm{vir}}\right) \in K^{\otimes 2}_{\mathsf{T}_q}(X)_{loc}[[z]]. \tag{3.10}$$

The bilinear form makes it an operator acting from the second to the first copy of $K_{\mathsf{T}_q}(X)_{loc}[[z]]$. This operator satisfies the quantum difference equations. We summarize that in the Theorem below [35].

Theorem 3.5.1. *1)The capped vertex with descendent τ is a result of applying of the capping operator to the bare vertex*

$$\hat{V}^{(\tau)}(z) = \Psi(z) V^{(\tau)}(z). \tag{3.11}$$

his equation can be represented by the following picture notation:

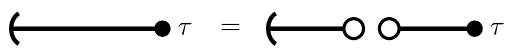


Figure 3.9: Capping Operator

2) The capping operator $\Psi(z)$ is the fundamental solution of the quantum difference equation:

$$\Psi(q^{\mathcal{L}_i} z) = \mathbf{M}_{\mathcal{L}_i}(z)\Psi(z)\mathcal{L}_i^{-1}, \tag{3.12}$$

where $\mathcal{L}_i = \det(V_i)$, $\mathcal{L}$ is the operator of classical multiplication by the corresponding line bundle and $(q^{\mathcal{L}} z)^{\mathbf{d}} = q^{\langle \mathcal{L}, \mathbf{d}\rangle} z^{\mathbf{d}}$, where $\mathbf{d} \in H_2(X, \mathbb{Z}), \mathcal{L} \in Pic(X)$. The matrix $\mathbf{M}_{\mathcal{L}_i}(z)$ is

$$\mathbf{M}_{\mathcal{L}_i}(z) = \sum_{\mathbf{d}=\vec{0}}^{\infty} z^{\mathbf{d}} \mathrm{ev}_* \left(QM^{\mathbf{d}}_{\text{relative}\, p_1,p_2}, \widehat{\mathcal{O}}_{\mathrm{vir}} \det H^{\bullet}\left(\mathscr{V}_i \otimes \pi^*(\mathcal{O}_{p_1})\right)\right) \mathbf{G}^{-1}, \tag{3.13}$$

where π is a projection from destabilization curve $\mathcal{C} \to \mathbb{P}^1$ and $\mathcal{O}_{p_1}$ is a class of point

$p_1 \in \mathbb{P}^1$.

Remark. The explicit form of operator $\mathbf{M}_{\mathcal{L}_i}$ is known for arbitrary Nakajima varieties. It is constructed in terms of representation theory terms of quantum loop algebra associated with a quiver [36].

Operators $\mathbf{M}_{\mathcal{L}_i}(z)$ turn out to be closely related to quantum tautological line bundles as the following Theorem suggests.

Theorem 3.5.2. *In the limit $q \to 1$ operators $\mathbf{M}_{\mathcal{L}}(z)$ coincide with the operators of quantum multiplication on the corresponding quantum tautological bundles:*

$$\lim_{q\to 1} \mathbf{M}_{\mathcal{L}_i}(z) = \widehat{\mathcal{L}}_i(z). \tag{3.14}$$

Proof. In the definition of the operator $\mathbf{M}_{\mathcal{L}_i}(z)$ the moduli space in question is the space of quasimaps from $\mathbb{P}^1$, relative to the points p_1 and p_2, with the evaluation map mapping to two copies of $K(X)$ by evaluating the map at p_1 and p_2. The bilinear form makes it an operator on $K(X)$ same way it did with quantum multiplication. The map π is the projection from the destabilization $\mathcal{C}$ of $\mathbb{P}^1$ on which the relative quasimap is defined onto the rigid $\mathbb{P}^1$. The specialization $q = 1$ corresponds to the non $\mathbb{C}_q^\times$-equivariant case. In this case any two points on $\mathbb{P}^1$ are isomorphic, so we can replace the sheaf $\pi^*(\mathcal{O}_{p_1})$ with $\pi^*(\mathcal{O}_{p_3})$, where p_3 is any other point on the rigid $\mathbb{P}^1$. After such a replacement, $\det H^\bullet(\mathscr{V}_i \otimes \pi^*(\mathcal{O}_{p_1}))$ will become $\det \mathscr{V}_i|_{p_3}$, since the point p_3 is not a relative point and, therefore, the map π is trivial in it. In picture notation this operator can be represented by

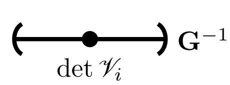


Figure 3.10: Quantum Line Bundle

This brings the proof to the final step, which is just a reiteration of the proof of Proposition 3.3.5 □

We will use this fact to compute the formula for the eigenvalues of the operators $\hat{\tau}(z)$. From now on we assume that the fixed points set X^{T} is a finite. The classes of fixed points are eigenvectors of classical multiplication in $K_{\mathsf{T}}(X)$. We are interested in the z-deformation of this construction.

Let us introduce the following notation. The eigenvalues of $\widehat{\mathcal{L}}_i(z)$ are $\lambda_{\mathbf{p},i}(z)$, so that $\lambda_{\mathbf{p},i}(0) = \lambda^0_{\mathbf{p},i}$, the eigenvalue of the classical multiplication on $\mathcal{L}_i$, corresponding to a fixed point $\mathbf{p} \in X^{\mathsf{T}}$. Let $l_{\mathbf{p},i} = \frac{\lambda_{\mathbf{p},i}(z)}{\lambda^0_{\mathbf{p},i}}$ be the normalized eigenvalue.

Lemma 3.5.3. *The following function*

$$f(t) = \exp\Big(\frac{1}{q-1}\int d_q t \ln l(t)\Big),$$

where $\int d_q t f(t) = (1-q)\sum_{n=0}^{\infty} f(tq^n)$ *is the standard Jackson q-integral, satisfies*

$$f(qt) = l(t) f(t)\,.$$

We denote

$$F_{\mathbf{p}}(z) = \exp\Big(\frac{1}{q-1}\sum_{i\in I}\int d_q z_i \ln \lambda_{\mathbf{p},i}(z)\Big) \tag{3.15}$$

Let us formulate an omnibus theorem concerning the solutions of the system of difference equations and eigenvalues of quantum multiplication operators.

Theorem 3.5.4. *1. The operator $\Psi(0)$ is the identity operator.*

2. Let $\Psi_{\boldsymbol{p}}(z)$ be the $\boldsymbol{p}$-th column of the matrix $\Psi(z)$. In the limit $q \mapsto 1$ the capping operator has the following asymptotic

$$\Psi_{\boldsymbol{p}}(z) = F_{\mathbf{p}}(z)\Big(\psi_{\boldsymbol{p}}(z) + \cdots\Big), \tag{3.16}$$

where $\psi_{\boldsymbol{p}}(z)$ are the column eigenvectors of the operators of quantum multiplication corresponding to the fixed point $\boldsymbol{p}$ and dots stand for the terms vanishing in the limit $q \to 1$.

3. The identity element in the quantum K-theory decomposes in the following manner

$$\hat{\mathbf{1}}(z) = \sum_{\boldsymbol{p}} v_{\boldsymbol{p}}(z)\psi_{\boldsymbol{p}}(z). \tag{3.17}$$

4. The coefficients of the bare vertex function have the following $q \to 1$ asymptotic in the fixed points basis

$$V^{(\tau)}_{\boldsymbol{p}}(z) = F_{\mathbf{p}}(z)^{-1}(\tau_{\boldsymbol{p}}(z)v_{\boldsymbol{p}}(z) + \cdots), \tag{3.18}$$

where $\tau_{\boldsymbol{p}}(z)$ denotes the eigenvalue of the operator of quantum multiplication by quantum tautological bundle $\hat{\tau}(z)$ for the eigenvector $\psi_{\boldsymbol{p}}(z)$, dots stand for the terms vanishing in the limit $q \to 1$.

The full proof of this theorem will be given in the following section, but before going into that, note that part (4) of the Theorem above immediately implies that the eigenvalues of the operator of quantum multiplication by $\hat{\tau}(z)$ can be computed from the asymptotics of the bare vertex functions.

Corollary 3.5.5. *The following expression:*

$$\tau_{\boldsymbol{p}}(z) = \lim_{q\to 1} \frac{V^{(\tau)}_{\boldsymbol{p}}(z)}{V^{(1)}_{\boldsymbol{p}}(z)} \tag{3.19}$$

gives the eigenvalues of the operator of quantum multiplication by $\hat{\tau}(z)$ corresponding to a fixed point $\boldsymbol{p} \in X^{\mathsf{T}}$.

3.6 Proof of Theorem 3.5.4

Part 1

The first part of this theorem is an obvious corollary from Theorem 3.5.1, that does not involve the quantum parameter at all.

By definition, the capping operator is a $K_{\mathsf{T}_q}(X)^{\otimes 2}$ -valued power series of the form:

$$\Psi(z) = \Psi_0 + \Psi_1 z + \Psi_2 z^2 + \dots,$$

and the term Ψ_0 is the term in question. It is obvious now that at $z = 0$ the quantum difference equation (3.12) holds trivially because $\mathbf{M}_{\mathcal{L}_i}(0) = \mathcal{L}_i$. The higher terms Ψ_i for $i > 1$ are fixed by (3.12) and can be computed explicitly by solving corresponding linear problem using known operator $\mathbf{M}_{\mathcal{L}_i}(z)$.

Part 2

The proposition follows from Theorem 3.5.2. Indeed, by this theorem $\psi_{\mathbf{p}}(z)$ is the eigenvector of $\mathbf{M}_{\mathcal{L}_i}(z)$ in the limit $q \to 1$. Thus, substituting (3.16) into the qunatum difference equation, for the leading term we obtain:

$$\lambda_{\mathbf{p},i}(z)\psi_{\mathbf{p}}(qz) = \lambda_{\mathbf{p},i}(z)\psi_{\mathbf{p}}(z),$$

which is an identity in $q \mapsto 1$ limit. Collecting the terms denoted by dots we find that they must vanish at $q \to 1$.

Part 3

Recall that quantum tautological classes in the quantum K-theory can be obtained from the capped vertex by taking the limit $q \to 1$, in particular $\hat{\mathbf{1}}(z) = \lim\limits_{q\to 1} \hat{V}^{(1)}(z)$ and thus:

$$\hat{\mathbf{1}}(z) = \lim_{q\to 1} \Psi(z) V^{(1)}(z) = \lim_{q\to 1} \sum_{\mathbf{p}} \Psi_{\mathbf{p}}(z) V^{(1)}_{\mathbf{p}}(z).$$

This means that exponentially divergent terms from the capping operator are canceled by the corresponding contributions from $V^{(1)}_{\mathbf{p}}(z)$. However, due to the linear independence of the eigenvectors $\psi_{\mathbf{p}}(z)$ it is possible only if these coefficients have the form:

$$V^{(1)}_{\mathbf{p}}(z) = w_{\mathbf{p}}(z) \exp\Big(-\frac{1}{q-1} \int d_q z \ln(\lambda_{\mathbf{p},i}(z))\Big).$$

Taking $v_{\mathbf{p}}(z) = w_{\mathbf{p}}(z, q = 1)$, we obtain the desired result.

Part 4

Following the argument of Part 3, we obtain that the following limit exists:

$$\hat{\tau}(z) = \lim_{q \to 1} \Psi(z) V^{(\tau)}(z) = \lim_{q \to 1} \sum_{\mathbf{p}} \Psi_{\mathbf{p}}(z) V_{\mathbf{p}}^{(\tau)}(z).$$

Thus, the coefficients of the descendent vertex must be of the form:

$$V_{\mathbf{p}}^{(\tau)}(z) = w_{\mathbf{p}}(z, q) \exp\Big(- \frac{1}{q-1} \int d_q z \ln(l_{\mathbf{p}}(z)) \Big)$$

for some $w_{\mathbf{p}}(z, q)$ regular at $q = 1$. This means that $\hat{\tau}(z) = \sum_{\mathbf{p}} w_{\mathbf{p}}(z, 1) \psi_{\mathbf{p}}(z)$. However, at the same time

$$\hat{\tau}(z) = \hat{\tau}(z) \circledast \hat{1}(z) = \hat{\tau}(z) \circledast \sum_{\mathbf{p}} v_{\mathbf{p}}(z) \psi_{\mathbf{p}}(z) = \sum_{\mathbf{p}} \tau_{\mathbf{p}}(z) v_{\mathbf{p}}(z) \psi_{\mathbf{p}}(z)$$

and the result follows from linear independence of $\psi_{\mathbf{p}}(z)$.

Chapter 4

Computations for Grassmannians

This chapter is dedicated to the specific case of cotangent bundles to Grassmannians.

Constructed as a simplest Nakajima variety in Section 2.3, cotangent bundles to Grassmannians have a highly non-trivial quantum K-theory that carries interesting connections to quantum integrable systems and representation theory (this will be dealt with in further sections). Note, that the next case considered will fully generalize this one, yet the formulas in this example are somewhat more neat and easier to understand.

Let us remind ourselves, that we are considering the following quiver:

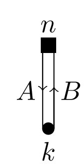


Figure 4.1: One Vertex Quiver

where n, k are the assigned dimensions ($n \geq k$). There is only one vertex in this case, so we drop the enumeration of vertices. The corresponding variety is $T^*\mathrm{Gr}(k,n)$, which we will denote by $\mathbf{N}_{k,n}$ for short.

4.1 Computing the Vertex

It will be convenient to adopt the following notations:

$$\phi(x) = \prod_{i=0}^{\infty}(1 - q^i x), \quad \{x\}_d = \frac{(\hbar/x, q)_d}{(q/x, q)_d}(-q^{1/2}\hbar^{-1/2})^d, \text{ where } (x, q)_d = \frac{\varphi(x)}{\varphi(q^d x)}.$$

Proposition 4.1.1. *Let $\boldsymbol{p} = \{x_1, \cdots, x_k\} \subset \{a_1, \cdots, a_n\}$ be a k-subset defining a torus fixed point $\boldsymbol{p} \in \boldsymbol{N}_{k,n}^T$. Then the coefficient of the vertex function for this point is given by:*

$$V_{\boldsymbol{p}}^{(\tau)}(z) = \sum_{d_1,\dots,d_k \in \mathbb{Z}_{\geq 0}} z^d q^{nd/2} \prod_{i,j=1}^{k} \{x_i/x_j\}_{d_i - d_j}^{-1} \prod_{i=1}^{k}\prod_{j=1}^{n}\{x_i/a_j\}_{d_i} \tau(x_1 q^{-d_1}, \cdots, x_k q^{-d_k}),$$

where $d = \sum_{i=1}^k d_i$.

Proof. If $\mathcal{V}$, $\mathcal{W}$ are the tautological bundles on $\mathbf{N}_{k,n}$, then the tangent bundle has the form:

$$T\mathbf{N}_{k,n} = \mathcal{P} + \hbar\mathcal{P}^* \text{ for } \mathcal{P} = \mathcal{W}^* \otimes \mathcal{V} - \mathcal{V}^* \otimes \mathcal{V}.$$

Recall that the degree d quasimaps to $\mathbf{N}_{k,n}$ are given by a pair of bundles: rank k, degree d bundle $\mathscr{V}$ and rank n trivial bundle $\mathscr{W}$ on $\mathbb{P}^1$. Let us consider a set of T_q-fixed points $(\mathscr{V}, \mathscr{W}) \in (\mathsf{QM}^d_{\text{nonsing}\, p_2})^{\mathsf{T}_q}$ such that the value of the evaluation map at p_2 is $\mathbf{p} \in \mathbf{N}_{k,n}^T$. The virtual tangent space is a representation of the torus T_q. We denote its T_q-character by:

$$\chi(d) = \text{char}_{\mathsf{T}_q}\Big(T^{vir}_{(\mathscr{V},\mathscr{W})}\mathsf{QM}^d\Big). \tag{4.1}$$

Localization in K-theory gives the following formula for the equivariant pushforward:

$$V_{\mathbf{p}}^{(\tau)}(z) = \sum_{d=0}^{\infty} \sum_{(\mathscr{V},\mathscr{W}) \in (\mathsf{QM}^d_{\text{nonsing}\, p_2})^{\mathsf{T}_q}} \hat{s}(\chi(d))\, z^d q^{\deg(\mathscr{P})/2} \tau(\mathscr{V}|_{p_1}),$$

where the sum runs over the T_q-fixed quasimaps which take value $\mathbf{p}$ at the nonsingular point p_2. We use notation $\hat{s}$ for the roof function defined by

$$\hat{s}(x) = \frac{1}{x^{1/2} - x^{-1/2}}, \quad \hat{s}(x+y) = \hat{s}(x)\hat{s}(y).$$

Note, that the tangent weights contribute to vertex via the roof function $\hat{s}(x)$ because the symmetrized virtual structure sheaf (3.5) is defined together with a shift on the square root of canonical bundle $\mathscr{K}_{\text{vir}}^{1/2}$. Thus, our goal is to compute (6.1). The *reduced* virtual tangent space to $\mathsf{QM}^d_{\text{nonsing}\, p_2}$ at such point is given by[1]:

$$T^{vir}_{(\mathscr{V},\mathscr{W})}\mathsf{QM}^d = H^\bullet(\mathscr{P} \oplus \hbar\, \mathscr{P}^*) - T_{\mathbf{p}}\mathbf{N}_{k,n}, \tag{4.2}$$

where $\mathscr{P}$ is the polarization bundle $\mathscr{P} = \mathscr{W}^* \otimes \mathscr{V} - \mathscr{V}^* \otimes \mathscr{V}$. The following Lemma will drastically simplify the computation of the contribution of $\text{char}_{\mathsf{T}_q}\Big(T^{vir}_{(\mathscr{V},\mathscr{W})}QM^d\Big)$ to the localization formula.

Lemma 4.1.2. *Let $\mathscr{P}$ be a polarization bundle on $\mathbb{P}^1$ corresponding to a T_q-fixed point on $\mathsf{QM}^d_{nonsing\, p_2}$. It splits into a sum of T_q-equivariant line bundles $\mathscr{P} = \bigoplus_i a_i q^{-d_i}\mathcal{O}(d_i)$ with for some characters a_i of the framing torus T. The cohomology of these line bundles have the following character T_q-modules:*

$$char_{\mathsf{T}_q}\Big(H^\bullet(a_i q^{-d_i}\mathcal{O}(d_i))\Big) = a_i \frac{q^{-d_i-1}-1}{q^{-1}-1} = \begin{cases} a_i + a_i q^{-1} + \cdots + a_i q^{-d_i} & \text{if} \quad d_i > 0 \\ 0 & \text{if} \quad d_i = -1 \\ -a_i q - a_i q^2 - \cdots - a_i q^{-d_i-1} & \text{if} \quad d_i < -1. \end{cases}$$

Proof. It is clear that the tautological bundles $\mathscr{V}$ and $\mathscr{W}$ representing T_q-fixed quasimap split into the sum of line bundles equivalently. It means that $\mathscr{P} = \bigoplus_i x_i\mathcal{O}(d_i)$ for some T_q -characters x_i. Since the quasimap is nonsingular at $p_2 = \infty$ the corresponding section should not vanish at p_2. The only such section of $\mathcal{O}(d_i)$ is z^{d_i}. The torus T_q acts on sections by $z \to qz$. By assumption, this section must be T_q-fixed. It is possible

[1]We use the reduced virtual tangent space which differs from standard one by subtracting $T_{\mathbf{p}}\mathbf{N}_{k,n}$. This term does not depend on d and thus produces a simple multiple in the vertex function. This is the multiple normalizing the vertex such that $V^{(\tau)}_{\mathbf{p}}(0) = \tau$.

only if $x_i = a_i q^{-d_i}$ for some character a_i of framing torus A, which does not act on $\mathbb{P}^1$. Lastly, if $d_i \geq 0$ then only zeroth cohomology group $H^0(\mathcal{O}(d_i))$ is nontrivial and is spanned by global sections $1, z, \cdots, z^{d_i}$. Thus, (4.3) follows trivially. For $d_i < 0$ applying the Serre duality one obtains same result. □

By Lemma 4.1.2, the polarization bundle representing a fixed point on the moduli space of quasimaps splits to a sum of linear subbundles. By multiplicativity of the roof function it is enough to compute the contribution of a single line subbundle $x\mathcal{O}(d) \subset \mathscr{P}(d)$ to the weight of the fixed point. According to Lemma 4.1.2 the contribution of such bundle to (4.2) is given by:

$$x\frac{q^{-d-1}-1}{q^{-1}-1} + x^{-1}\hbar\frac{q^{d-1}-1}{q^{-1}-1} - x - x^{-1}\hbar =$$
$$\begin{cases} xq^{-1}(1+q^{-1}+\cdots+q^{-d+1}) - x^{-1}\hbar(1+q+\cdots+q^{d-1}) & \text{if} \quad d>0, \\ x^{-1}\hbar q^{-1}(1+q^{-1}+\cdots+q^{d+1}) - x(1+q+\cdots+q^{-d-1}) & \text{if} \quad d<0. \end{cases} \tag{4.3}$$

Applying the roof function we find:

$$\begin{cases} \hat{s}\Big(xq^{-1}(1+q^{-1}+\cdots+q^{-d+1}) - x^{-1}\hbar(1+q+\cdots+q^{d-1})\Big) = \{x\}_d & \text{if} \quad d>0, \\ \hat{s}\Big(x^{-1}\hbar q^{-1}(1+q^{-1}+\cdots+q^{d+1}) - x(1+q+\cdots+q^{-d-1})\Big) = \{x\}_d & \text{if} \quad d<0. \end{cases}$$

It is clear that the fixed points on the moduli space of quasimaps taking value $\mathbf{p} = \{x_1, \cdots, x_k\}$ on the nonsingular point, correspond to the bundles of the form:

$$\mathscr{V} = \mathcal{O}(d_1)q^{-d_1}x_1 \oplus \cdots \mathcal{O}(d_k)q^{-d_k}x_k, \quad \mathscr{W} = \mathcal{O}a_1 \oplus \cdots \oplus \mathcal{O}a_n, \tag{4.4}$$

where $d_1 + \cdots + d_k = d$. Thus, the terms $\mathscr{W}^* \otimes \mathscr{V}$ and $-\mathscr{V}^* \otimes \mathscr{V}$ in the polarization produce the following contributions:

$$\mathscr{W}^* \otimes \mathscr{V} \longrightarrow \prod_{j=1}^{n} \{x_i/a_j\}_{d_i} \qquad -\mathscr{V}^* \otimes \mathscr{V} \longrightarrow \prod_{i,j=1}^{k} \{x_i/x_j\}^{-1}_{d_i-d_j}.$$

Note, that $\deg(\mathscr{P}) = nd$. That gives the polarization term $q^{nd/2}$ in the vertex. Finally, from (4.4) we obtain $\tau(\mathscr{V}|_{p_1}) = \tau(x_1 q^{-d_1}, \cdots, x_k q^{-d_k})$ concluding the computation. $\square$

4.2 Integral representation

As well as standard q-hypergeometric series, the vertex function has a Mellin - Barnes type integral representation. Indeed, note that the following Proposition is true.

Proposition 4.2.1.

$$V_{\boldsymbol{p}}^{(\tau)}(z) =$$
$$\frac{1}{2\pi i \alpha_p} \int\limits_{C_p} \prod_{i=1}^{k} \frac{ds_i}{s_i} \, e^{-\frac{\ln(z_\sharp)\ln(s_i)}{\ln(q)}} \prod_{i,j=1}^{k} \frac{\varphi\left(\frac{s_i}{s_j}\right)}{\varphi\left(\frac{q}{\hbar}\frac{s_i}{s_j}\right)} \prod_{i=1}^{n}\prod_{j=1}^{k} \frac{\varphi\left(\frac{q}{\hbar}\frac{s_j}{a_i}\right)}{\varphi\left(\frac{s_j}{a_i}\right)} \tau(s_1, \cdots, s_k). \quad (4.5)$$

Here the contour of integration C_p corresponding to a fixed point $p = \{x_1, \cdots, x_k\} \subset \{a_1, \cdots, a_n\}$ is a positively oriented contour enclosing the poles at $s_i = q^{-d_i} x_i$ for $i = 1, \ldots, k$, $d_i \in \mathbb{Z}_{\geq 0}$. We also used a shifted degree counting parameter $z_\sharp = (-1)^n \hbar^{n/2} z$ and α_p is a normalization constant:

$$\alpha_p = \prod_{i,j=1}^{k} \frac{\varphi\left(\frac{x_i}{x_j}\right)}{\varphi\left(\frac{q x_i}{\hbar x_j}\right)} \prod_{i=1}^{n}\prod_{j=1}^{k} \frac{\varphi\left(\frac{q x_j}{\hbar a_i}\right)}{\varphi\left(\frac{x_j}{a_i}\right)} \prod_{i=1}^{k} e^{-\frac{\ln(z_\sharp)\ln(x_i)}{\ln(q)}}.$$

Proof. In order to prove this Proposition one has to evaluate residues at zeroes of the function $\varphi\left(\frac{s_j}{a_i}\right)$. One can see, for example, that the contribution of one fraction $\frac{\varphi\left(\frac{q}{\hbar}\frac{s_j}{a_i}\right)}{\varphi\left(\frac{s_j}{a_i}\right)}$ is:

$$\frac{\left(1 - q^{-d_j+1}\frac{x_j}{\hbar a_i}\right)\left(1 - q^{-d_j+2}\frac{x_j}{\hbar a_i}\right)\cdots\left(1 - \frac{x_j}{\hbar a_i}\right)}{\left(1 - q^{-d_j}\frac{x_j}{a_i}\right)\left(1 - q^{-d_j+1}\frac{x_j}{a_i}\right)\cdots\left(1 - q^{-1}\frac{x_j}{a_i}\right)} = (-1)^{d_j}\left(\frac{q}{\hbar}\right)^{d_j/2}\{x_j/a_i\}_{d_j}.$$

These extra coefficients will provide shifts in the z-variable and provide necessary extra q-contributions. Combining contributions from all other terms we obtain the statement of the Proposition. □

The integral representation is convenient for computation of the $q \to 1$ asymptotical behavior of the vertex function. It is well known that in this limit a single term in the q-hypergeometric series dominates. In other words, in this limit, the integral (4.5) converges to its saddle point approximation and we arrive to the following proposition.

Proposition 4.2.2. *At $q \to 1$ the saddle point of the integral (4.5) is determined by Bethe equations:*

$$\prod_{j\neq i} \frac{s_i - s_j\hbar}{s_i\hbar - s_j} \prod_{j=1}^{n} \frac{s_i - a_j}{a_j\hbar - s_i} = z\hbar^{-n/2}, \quad i = 1, \ldots, k. \tag{4.6}$$

Proof. Let Φ denote the integrand in (4.5). The saddle point is defined by the equations: $s_i\partial_{s_i} \ln(\Phi) = 0$ for $i = 1, \ldots, k$. Let us prove the following Lemma.

Lemma 4.2.3. *Asymptotically, in the limit $q \to 1$ we have*

$$x\frac{\partial \ln \varphi(x)}{\partial x} = -\frac{\ln(1-x)}{\ln(q)} + o(\ln(q)).$$

Proof. To show that, one has to expand the expression for $x\frac{\partial \ln \varphi(x)}{\partial x}$:

$$x\frac{\partial \ln \varphi(x)}{\partial x} = -\sum_{i=0}^{\infty} \frac{q^i x}{1-q^i x} = -\sum_{i=0}^{\infty}\sum_{m=0}^{\infty} q^{i(m+1)}x^{m+1} = -\sum_{m=0}^{\infty} \frac{x^{m+1}}{1-q^{m+1}} =$$
$$-\sum_{m=0}^{\infty} \frac{x^{m+1}}{1-e^{(m+1)\ln(q)}} = -\frac{\ln(1-x)}{\ln(q)} + o(\ln(q)).$$

□

Hence, we obtain the following equations for the saddle point:

$$-z_\sharp + \sum_{j=1}^{n} \Big(\ln(1-\frac{s_i}{a_j}) - \ln(1-\frac{s_i}{\hbar a_j}) \Big) +$$
$$\sum_{j\neq i} \Big(-\ln(1-\frac{s_i}{s_j}) + \ln(1-\frac{s_j}{s_i}) + \ln(1-\frac{s_i}{\hbar s_j}) - \ln(1-\frac{s_j}{\hbar s_i}) \Big) = 0,$$

which after exponentiation gives (4.6). □

Corollary 4.2.4. *The eigenvalues $\tau_{\boldsymbol{p}}(z)$ of the operator of quantum multiplication by the quantum tautological bundle $\hat{\tau}(z)$ are given by $\tau(s_1, \cdots, s_k)$, evaluated at the solutions of the Bethe equations (4.6).*

Proof. By (3.19) the eigenvalue is given by the limit:

$$\tau_{\mathbf{p}}(z) = \lim_{q \to 1} \frac{V_{\mathbf{p}}^{(\tau)}(z)}{V_{\mathbf{p}}^{(1)}(z)}.$$

Again, let $\Phi^{(\tau)}(s_1, \cdots, s_k)$ denote the integrand in (4.5). In the limit $q \to 1$ the vertex functions are divergent with leading term given by the saddle point approximation. By the previous proposition is means that

$$\tau_{\mathbf{p}}(z) = \frac{\Phi^{(\tau)}(s_1, \cdots, s_k)}{\Phi^{(1)}(s_1, \cdots, s_k)},$$

where s_i satisfy the saddle point equations (4.6). The latter ratio is $\tau(s_1, \cdots, s_k)$. □

Chapter 5

Representation theory of $\mathcal{U}_\hbar(\widehat{\mathfrak{sl}}_2)$ and spin chains

5.1 Algebra $\mathcal{U}_\hbar(\widehat{\mathfrak{sl}}_2)$

Let us recall that the algebra $\mathcal{U}_\hbar(\widehat{\mathfrak{sl}}_2)$ with zero central charge is an associative algebra with 1 generated over $\mathbb{C}(\hbar^{1/2})$ by elements E_k, F_k, H_m, K ($k \in \mathbb{Z}$, $m \in \mathbb{Z} \setminus \{0\}$) subject to the following relations:

$$\begin{aligned}
&KK^{-1} = K^{-1}K = 1 \\
&[H_m, H_n] = [H_m, K^{\pm 1}] = 0 \\
&KE_mK^{-1} = \hbar E_m, KF_mK^{-1} = \hbar^{-1}F_m \\
&[E_m, F_l] = \frac{\psi^+_{m+l} - \psi^-_{m+l}}{\hbar^{1/2} - \hbar^{-1/2}} \\
&[H_k, E_l] = \frac{[2k]_{\hbar^{1/2}}}{k}E_{k+l},\ [H_k, F_l] = -\frac{[2k]_{\hbar^{1/2}}}{k}F_{k+l}
\end{aligned} \tag{5.1}$$

where $[n]_\hbar = \frac{\hbar^n - \hbar^{-n}}{\hbar - \hbar^{-1}}$ and

$$\begin{aligned}
&\sum_{m=0}^{\infty} \psi^+_m z^{-m} = K \exp\Big((\hbar^{1/2} - \hbar^{-1/2}) \sum_{k=1}^{\infty} H_k z^{-k}\Big) \\
&\sum_{m=0}^{\infty} \psi^-_{-m} z^{m} = K^{-1} \exp\Big(- (\hbar^{1/2} - \hbar^{-1/2}) \sum_{k=1}^{\infty} H_{-k} z^{k}\Big)
\end{aligned}$$

5.2 Construction of the action

Here we recall a basic construction from geometric representation theory of affine Lie algebras. More details can be found in [30, 45]. Let $\mathbf{Gr}(k,n)$ be the Grassmannian of k-planes in $\mathbb{C}^n$. We think of it as the zero section of the cotangent bundle $\mathbf{Gr}_{k,n} \subset \mathbf{N}_{k,n}$. We set $\mathbf{G}^k_{k+1} = \mathbf{Gr}_{k,n} \times \mathbf{Gr}_{k+1,n}$ and $\mathfrak{M}^k_{k+1} = T^*\mathbf{G}^k_{k+1} = \mathbf{N}_{k,n} \times \mathbf{N}_{k+1,n}$. Let π_1, π_2 be natural projections from $\mathfrak{M}^k_{k+1}$ to the first and second factor respectively. We consider a $\mathrm{GL}(n)$-orbit in $\mathbf{G}^k_{k+1}$:

$$\mathbf{O}^k_{k+1} = \{V_1 \times V_2 \in \mathbf{G}^k_{k+1} : \dim V_1 = k, \quad \dim V_2 = k+1, \quad V_1 \subset V_2\}$$

Let $\mathfrak{B}^k_{k+1}$ be a Lagrangian subvariety of $\mathfrak{M}^k_{k+1}$ given by the conormal to $\mathbf{O}^k_{k+1}$. As in previous sections we set $\mathcal{V}_1, \mathcal{V}_2$ to be the tautological bundles on $\mathbf{N}_{k,n}$ and $\mathbf{N}_{k+1,n}$ and $\mathcal{W}$ to be a trivial rank n bundle on these varieties. We define the set of bundles $e_e, f_r \in K_{\mathsf{T}}(\mathfrak{B}^k_{k+1})$ labeled by $r \in \mathbb{Z}$:

$$\begin{aligned} e_r &= (-1)^{\mathrm{rk}(\mathcal{W})+1} (\mathcal{V}_2/\mathcal{V}_1)^{r+\mathrm{rk}(\mathcal{W})-2\mathrm{rk}(\mathcal{V}_1)} \otimes \hbar^{-\mathrm{rk}(\mathcal{V}_1)/2}, \\ f_r &= \frac{\det \mathcal{V}_1^2}{\det \mathcal{W}} \otimes (\mathcal{V}_2/\mathcal{V}_1)^r \otimes \hbar^{(\mathrm{rk}(\mathcal{W})-2\mathrm{rk}(\mathcal{V}_1)-1)/2}, \end{aligned}$$

where $\hbar$ stands for trivial line bundle with the corresponding equivariant structure. These line bundles define the correspondences $E_r, F_r \in \mathrm{End}(K_{\mathsf{T}}(\mathbf{N}(n)))$ acting by rising and lowering the Grassmannian index k:

$$K_{\mathsf{T}}(\mathbf{N}_{k+1,n}) \xrightarrow{E_r} K_{\mathsf{T}}(\mathbf{N}_{k,n}), \quad K_{\mathsf{T}}(\mathbf{N}_{k,n}) \xrightarrow{F_r} K_{\mathsf{T}}(\mathbf{N}_{k+1,n}).$$

Explicitly, these operators are defined by:

$$E_r(\alpha) = \pi_{1,*}(\pi_2^*(\alpha) \otimes e_r) \in K_{\mathsf{T}}(\mathbf{N}_{k-1,n}), \quad F_r(\alpha) = \pi_{2,*}(\pi_1^*(\beta) \otimes f_r) \in K_{\mathsf{T}}(\mathbf{N}_{k+1,n})$$

for a class $\alpha \in K_{\mathsf{T}}(\mathbf{N}_{k,n})$. Let us further consider the following complex of equivariant bundles on $\mathbf{N}_{k,n}$:

$$C^{\bullet} = \hbar\mathcal{V} \to \mathcal{W} \to \hbar^{-1}\mathcal{V}^{\vee}$$

and define

$$\psi^{+}(z) = K\Lambda^{\bullet}_{z^{-1}}(C^{\bullet}), \quad \psi^{-}(z) = K^{-1}\Lambda^{\bullet}_{z}(C^{\bullet *}), \quad \text{with} \quad K = \hbar^{(rk(\mathcal{V}^{\vee})-rk(\mathcal{V}))/2}.$$

These classes determine the map $K_{\mathsf{T}}(\mathbf{N}_{k,n}) \stackrel{\psi^{\pm}(z)}{\longrightarrow} K_{\mathsf{T}}(\mathbf{N}_{k,n})$ acting by tensor multiplication $\alpha \to \psi^{\pm}(z) \otimes \alpha$.

Theorem 5.2.1. *For all n the geometric action of $K, E_r, F_r, \psi^{\pm}(z)$ endows $K_{\mathsf{T}}(\boldsymbol{N}(n))$ with a structure of an $\mathcal{U}_{\hbar}(\widehat{\mathfrak{sl}}_2)$ module. As an $\mathcal{U}_{\hbar}(\widehat{\mathfrak{sl}}_2)$ module it is isomorphic to the tensor product of n 2-dimensional evaluation modules:*

$$K_{\mathsf{T}}(\boldsymbol{N}(n)) = \mathbb{C}^2(a_1) \otimes \cdots \otimes \mathbb{C}^2(a_n)$$

where a_i and $\hbar$ are the equivariant parameters of T.

The definition of the evaluation representation $\mathbb{C}^2(a)$, in the following denoted as $\pi_1(a)$ along with all other necessary information about evaluation representations of quantum groups will be given in the next section in a more suitable realization of $\mathcal{U}_{\hbar}(\widehat{\mathfrak{sl}}_2)$.

However, the explicit formulas for the action of $\mathcal{U}_{\hbar}(\widehat{\mathfrak{sl}}_2)$ on $K_{\mathsf{T}}(\mathbf{N}(n))$ can be computed as a simple exercise on localization in equivariant K-theory. For future references we give this formulas in the basis of fixed points:

Proposition 5.2.2. *In the basis of fixed points $\mathcal{O}_{\boldsymbol{p}}$ for $\boldsymbol{p} = \{i_1, \cdots, i_k\} \subset \boldsymbol{n} =$*

$\{1, 2, \cdots, n\}$ *we have:*

$$K(\mathcal{O}_{\boldsymbol{p}}) = \hbar^{(n-2|\boldsymbol{p}|)/2}\mathcal{O}_{\boldsymbol{p}},$$

$$H_m(\mathcal{O}_{\boldsymbol{p}}) = \frac{[m]_{\hbar^{1/2}}}{m}\Big(\hbar^{-m/2}\sum_{i\in\boldsymbol{n}\setminus\boldsymbol{p}} a_i^{-m} - \hbar^{m/2}\sum_{i\in\boldsymbol{p}} a_i^{-m}\Big)\mathcal{O}_{\boldsymbol{p}},$$

$$E_r(\mathcal{O}_{\boldsymbol{q}}) = \sum_{s\in\boldsymbol{q}}\left(a_s^{-r-1}\frac{\prod\limits_{j\in\boldsymbol{q}\setminus\{s\}}(a_j - \hbar a_s)}{\prod\limits_{j\in\boldsymbol{n}\setminus\boldsymbol{q}}(a_j - a_s)}\right)\mathcal{O}_{\boldsymbol{q}\setminus\{s\}}, \tag{5.2}$$

$$F_r(\mathcal{O}_{\boldsymbol{p}}) = \sum_{s\in\boldsymbol{n}\setminus\boldsymbol{p}}\left(\hbar^{(n-2k-1)/2}a_s^{-r+1}\frac{\prod\limits_{j\in\boldsymbol{n}\setminus(\boldsymbol{p}\cup\{s\})}(a_j - a_s\hbar^{-1})}{\prod\limits_{j\in\boldsymbol{p}}(a_j - a_s)}\right)\mathcal{O}_{\boldsymbol{p}\cup\{s\}},$$

where $[n]_\hbar = \frac{\hbar^n - \hbar^{-n}}{\hbar - \hbar^{-1}}$.

5.3 Chevalley generators

In the previous section we constructed the action of $\mathcal{U}_\hbar(\widehat{\mathfrak{sl}}_2)$ on $K_{\mathsf{T}}(\mathbf{N}_{k,n})$ in its Drinfeld realization. In this section we give a different description. Quantum affine algebra $\mathcal{U}_\hbar(\widehat{\mathfrak{sl}}_2)$ can be constructed via the following Chevalley generators: $e_{\pm\alpha}$, $e_{\pm(\delta-\alpha)}$, $K_\alpha^{\pm 1} = \hbar^{h_\alpha/2}$, $k_{\delta-\alpha}^{\pm 1} = \hbar^{h_{\delta-\alpha}/2}$, satisfying the commutation relations

$$\begin{aligned} k_\gamma k_\gamma^{-1} &= k_\gamma^{-1}k_\gamma = 1\ , & [k_\gamma^{\pm 1}, k_{\gamma'}^{\pm 1}] = 0\ ,\\ k_\gamma e_{\pm\alpha}k_\gamma^{-1} &= \hbar^{\pm(\gamma,\alpha)/2}e_{\pm\alpha}\ , & k_\gamma e_{\pm(\delta-\alpha)}k_\gamma^{-1} = \hbar^{\pm(\gamma,\delta-\alpha)/2}e_{\pm(\delta-\alpha)},\end{aligned}$$

$$\begin{aligned} [e_\alpha, e_{-\delta+\alpha}] &= 0\ , & [e_{-\alpha}, e_{\delta-\alpha}] = 0\ ,\\ [e_\alpha, e_{-\alpha}] &= [h_\alpha]_{\sqrt{\hbar}}\ , & [e_{\delta-\alpha}, e_{-\delta+\alpha}] = [h_{\delta-\alpha}]_{\sqrt{\hbar}}\ ,\end{aligned} \tag{5.3}$$

$$[e_{\pm\alpha},[e_{\pm\alpha},[e_{\pm\alpha},e_{\pm(\delta-\alpha)}]_{\sqrt{\hbar}}]_{\sqrt{\hbar}}]_{\sqrt{\hbar}} \;=\; 0\;,$$
$$[[[e_{\pm\alpha},e_{\pm(\delta-\alpha)}]_{\sqrt{\hbar}},e_{\pm(\delta-\alpha)}]_{\sqrt{\hbar}},e_{\pm(\delta-\alpha)}]_{\sqrt{\hbar}} \;=\; 0\;,$$

where $(\gamma=\alpha,\delta-\alpha)$, (α,α)=$-(\delta-\alpha,\;\alpha)$ and $[h_\beta]_{\sqrt{\hbar}}:=(k_\beta-k_\beta^{-1})/(\sqrt{\hbar}-\sqrt{\hbar}^{-1})$. The brackets $[\cdot,\cdot]$ and $[\cdot,\cdot]_h$ are the $\hbar$-commutator:

$$[e_\beta,e_{\beta'}]_{\sqrt{\hbar}} \;=\; e_\beta e_{\beta'}-\hbar^{(\beta,\beta')/2}e_{\beta'}e_\beta\;. \tag{5.4}$$

From now on for simplicity, we assume that (α,α)=2. It is important to mention that Serre relations remain intact under the transformation $\hbar\to\hbar^{-1}$.

The Hopf algebra structure on $\mathcal{U}_\hbar(\widehat{\mathfrak{sl}}_2)$ is given by the following coproduct $\Delta_{\sqrt{\hbar}}$ and antipode $S_{\sqrt{\hbar}}$:

$$\begin{aligned}
\Delta_{\sqrt{\hbar}}(k_\gamma^{\pm1}) &= k_\gamma^{\pm1}\otimes k_\gamma^{\pm1}\;, & S_{\sqrt{\hbar}}(k_\gamma^{\pm1}) &= k_\gamma^{\mp1}\;,\\
\Delta_{\sqrt{\hbar}}(e_\beta) &= e_\beta\otimes 1+k_\beta^{-1}\otimes e_\beta\;, & S_{\sqrt{\hbar}}(e_\beta) &= -k_\beta e_\beta\;,\\
\Delta_{\sqrt{\hbar}}(e_{-\beta}) &= e_{-\beta}\otimes k_\beta+1\otimes e_{-\beta}\;, & S_{\sqrt{\hbar}}(e_{-\beta}) &= -e_{-\beta}k_\beta^{-1}\;,
\end{aligned} \tag{5.5}$$

where $\beta=\alpha,\,\delta-\alpha$.

Notice, that our definition corresponds to the standard one (see e.g. [22]) when $\sqrt{\hbar}=q$, standard deformation parameter for quantum groups.

The relation between the Chevalley relaization and the Drinfeld one is given via the following formulas:

$$\begin{aligned}
&e_{\alpha_0}=F_0K^{-1},\quad e_{\alpha_1}=E_{-1},\quad e_{-\alpha_0}=KE_0,\quad e_{-\alpha_1}=F_1,\\
&k_{\alpha_1}=k_{\alpha_0}^{-1}=K.
\end{aligned} \tag{5.6}$$

The celebrated universal R-matrix is an element in the tensor product $\mathbf{b}_+\otimes\mathbf{b}_-$, where $\mathbf{b}_\pm$ are upper and lower Borel subalgebras of $\mathcal{U}_{\sqrt{\hbar}}(\widehat{\mathfrak{sl}}_2)$, which satisfies the following relations with respect to the coproduct $\Delta_{\sqrt{\hbar}}$ and opposite coproduct $\tilde{\Delta}_{\sqrt{\hbar}}=\sigma\Delta_{\sqrt{\hbar}}$

and $\sigma(a\otimes b)=b\otimes a$:

$$\begin{aligned}\tilde{\Delta}_{\sqrt{\hbar}}(a)&=R\Delta_{\sqrt{\hbar}}(a)R^{-1} \qquad && \forall\ a\in U_{\sqrt{\hbar}}(\widehat{\mathfrak{sl}}_2)\ ,\\ (\Delta_{\sqrt{\hbar}}\otimes \mathrm{id})R&=R^{13}R^{23}\ , && (\mathrm{id}\otimes\Delta_{\sqrt{\hbar}})R=R^{13}R^{12}.\end{aligned} \tag{5.7}$$

The relations above can be understood in the following way: $R^{12}=\sum a_i\otimes b_i\otimes \mathrm{id}$, $R^{13}=\sum a_i\otimes \mathrm{id}\otimes b_i$, $R^{23}=\sum \mathrm{id}\otimes a_i\otimes b_i$ if R has the form $R=\sum a_i\otimes b_i$ For more information on the structure of the R-matrix, see Appendix.

5.4 Q-operator from oscillator representations.

In this section we describe the oscillator representations of $\mathbf{b}_-$ (sometimes called prefundamental representations [12]), which serve as building blocks for evaluation modules, which are defined in this section as well. Namely, all finite-dimensional evaluation representations of $\mathbf{b}_-$ (see below) can be reproduced within the Grothendieck ring generated by prefumdamental representations, as we will see below.

First, we introduce the deformed oscillator algebra:

$$\hbar^{1/2}\mathcal{E}_+\mathcal{E}_- - \hbar^{-1/2}\mathcal{E}_-\mathcal{E}_+ = \frac{1}{\hbar^{1/2}-\hbar^{-1/2}}, \quad [H,\mathcal{E}_\pm]=\pm 2\mathcal{E}_\pm. \tag{5.8}$$

The representations of $\mathbf{b}_-$, which we are interested in, can be described by the following homomorphisms:

$$\rho_\pm(x):\begin{cases} h_\alpha\to \pm H,\\ h_{\delta-\alpha}\to \mp H,\\ e_{-\alpha}\to \mathcal{E}_\mp,\\ e_{\alpha-\delta}\to x\mathcal{E}_\pm,\end{cases} \tag{5.9}$$

so that the space of representations is given by Fock spaces

$$\rho_\pm(x)=\{\mathrm{span}\{\mathcal{E}_\mp^k|0\rangle_\pm\};\ \mathcal{E}_\pm|0\rangle_\pm=0,\ H|0\rangle_\pm=0\} \tag{5.10}$$

We are interested in the decomposition of the tensor product

$$\rho_-(x\hbar^{-\frac{n+1}{2}}) \otimes \rho_+(x\hbar^{\frac{n+1}{2}}) \tag{5.11}$$

for $n \in \mathbb{Z}$. In order to describe the components of the decomposition of this tensor product in the Grothendieck ring we need to introduce the evaluation representations $\pi_n^+(x)$ of $\mathbf{b}_-$, associated with Verma modules of $\mathcal{U}_\hbar(\widehat{\mathfrak{sl}}_2)$:

$$\pi_n^+(x) : \begin{cases} h_\alpha \to \pm\mathcal{H}, \\ h_{\delta-\alpha} \to \mp\mathcal{H}, \\ e_{-\alpha} \to \mathcal{F}, \\ e_{\alpha-\delta} \to x\mathcal{E}, \end{cases} \tag{5.12}$$

so that $\pi_n^+(x) = \{\text{span}(\mathcal{F}^k v_0);\ \mathcal{E}v_0 = 0, \mathcal{H}v_0 = nv_0\}$ and therefore

$$\begin{aligned} \mathcal{H}v_k &= (n-2k)v_k, \\ \mathcal{F}v_k &= v_{k+1}, \\ \mathcal{E}v_k &= [k]_{\sqrt{\hbar}}[n-k+1]_{\sqrt{\hbar}}v_{k-1}. \end{aligned} \tag{5.13}$$

Denote the representations corresponding to finite dimensional modules of $\mathcal{U}_\hbar(\widehat{\mathfrak{sl}}_2)$ as follows: $\pi_n(x) \equiv \pi_n^+(x)/\pi_{-n-2}^+(x)$. These are known as *evaluation representations*. Let us also denote 1-dimensional representations of $\mathbf{b}_-$ with eigenvalue of h_α equial to s as ω_s. It is clear that $\omega_s \otimes \omega_{s'} = \omega_{s+s'}$. Then we have the following Proposition.

Proposition 5.4.1. *Decomposition of the product (5.11) in the Grothendieck ring of* $\mathbf{b}_-$ *is given by the following simple expresion:*

$$\rho_-(x\hbar^{-\frac{n+1}{2}}) \cdot \rho_+(x\hbar^{\frac{n+1}{2}}) = \omega_{-n}(1-\omega_{-2})^{-1}\pi_n^+(x), \tag{5.14}$$

where $(1-\omega_{-2})$ *is understood as the geometric series expansion.*

Proof. Here we give a sketch of the proof, for the details we refer to the paper [4]. Let us write the coproduct of the generators of $\mathbf{b}_-$ in the tensor product:

$$\begin{aligned}
&\bar{\mathcal{H}} \equiv \Delta_{\rho_-(xy^{-1})\otimes\rho_-(xy)}(h_\alpha) = 1\otimes H - H\otimes 1 \\
&\bar{\mathcal{F}} \equiv \Delta_{\rho_-(xy^{-1})\otimes\rho_-(xy)}(e_{-\alpha}) = \mathcal{E}_+\otimes\sqrt{\hbar}^H + 1\otimes\mathcal{E}_- = b_+ + a_+ \\
&\bar{\mathcal{E}} \equiv= x^{-1}\Delta_{\rho_-(xy^{-1})\otimes\rho_-(xy)}(e_{\alpha-\delta}) = y^{-1}\mathcal{E}_-\otimes\sqrt{\hbar}^{-H} + 1\otimes y\mathcal{E}_+ = b_- + a_-,
\end{aligned} \tag{5.15}$$

where we decomposed each of the coproducts into two terms denoted as $b_\pm, a_\pm$ in the order of their appearance and y stands for $\hbar^{\frac{n+1}{2}}$. Then the following commutation relations are satisfied:

$$a_{\sigma_1}b_{\sigma_2} = \hbar^{\sigma_1\sigma_2}b_{\sigma_2}a_{\sigma_1}, \tag{5.16}$$

$$\sqrt{\hbar}a_-a_+ - \sqrt{\hbar}^{-1}a_+a_- = \frac{y^2}{\sqrt{\hbar}-\sqrt{\hbar}^{-1}}, \quad \sqrt{\hbar}b_+b_- - \sqrt{\hbar}^{-1}b_-b_+ = \frac{y^{-2}}{\sqrt{\hbar}-\sqrt{\hbar}^{-1}} \tag{5.17}$$

Then, introducing the basis vectors $|\rho_k^{(m)}\rangle = (a_+ + b_+)^k(a_+ - \gamma b_+)^m|0\rangle_-\otimes|0\rangle_+$. One can show that for generic γ they span the total space $\rho_-(x\hbar^{-\frac{n+1}{2}})\otimes\rho_+(x\hbar^{\frac{n+1}{2}})$. These vectors are of special nature, namely:

$$\begin{aligned}
&\bar{\mathcal{H}}|\rho_k^{(m)}\rangle = -2(k+m)|\rho_k^{(m)}\rangle, \\
&\bar{\mathcal{F}}|\rho_k^{(m)}\rangle = |\rho_{k+1}^{(m)}\rangle, \\
&\bar{\mathcal{E}}|\rho_k^{(m)}\rangle = [k]_{\sqrt{\hbar}}[n-k+1]_{\sqrt{\hbar}}|\rho_{k-1}^{(m)}\rangle + c_k^m|\rho_k^{(m-1)}\rangle.
\end{aligned} \tag{5.18}$$

We see that up to a shift in $\mathcal{H}$ and extra coefficients c_k^m this gives the decomposition in terms of representations. On the level of the Grothendieck group the latter problem is irrelevant and the former problem can be corrected by multiplication on 1-dimensional representations of appropriate weight, forming the geometric series. □

Now we describe the traces of R-matrices in those representations. First of all, we assume that whenever we write the universal R-matrix, its $\mathbf{b}_+$-part is considered in some finite-dimensional representation, so that the following trace

$$\tilde{Q}^f_\pm(x) = tr_{\rho_\pm(x)}\Big[(I\otimes\rho_\pm(x))R(I\otimes\sqrt{\hbar}^{fh_\alpha})\Big] \tag{5.19}$$

is well-defined for f being positive. We want to normalize this operator, so that at $x=0$ it was equal to 1, therefore, we have to introduce

$$\begin{aligned}&\mathcal{Z}_\pm(h_\alpha) = tr_{\rho_\pm(x)}((I\otimes\rho_\pm(x))K\otimes\sqrt{\hbar}^{fh_\alpha}) = \\ &tr_{\rho_\pm(x)}((I\otimes\rho_\pm(x))\sqrt{\hbar}^{\frac{(h_\alpha+2f)\otimes h_\alpha}{2}}) = \sum_{s=0}^{\infty}\sqrt{\hbar}^{-(h_\alpha+2f)s} = \frac{1}{1-\hbar^{-\frac{h_\alpha}{2}+f}},\end{aligned} \tag{5.20}$$

so we denote $Q^f_\pm(x) = \mathcal{Z}^{-1}_\pm(h_\alpha)\tilde{Q}^f_\pm(x)$. Then we have the following theorem, which is a consequence of the Proposition we proved above.

Theorem 5.4.2. *The product of two Q-operators gives a trace of the R-matrix in $\pi^+_n(x)$ representation:*

$$\hbar^{\frac{n(h_\alpha+2f)}{4}}Q^f_+(\hbar^{\frac{n+1}{2}}x)Q^f_-(\hbar^{-\frac{n+1}{2}}x) = tr_{\pi^+_n(x)}\Big[R(I\otimes\sqrt{\hbar}^{fh_\alpha})\Big](1-h^{-\frac{h_\alpha}{2}+f}) \tag{5.21}$$

Proof. To prove that, it is enough to see that

$$tr_{\omega_s}\Big[R(I\otimes\hbar^{fh_\alpha/2})\Big] = \hbar^{\frac{s}{4}(h_\alpha+2f)}. \tag{5.22}$$

Then formula (5.14) corrected by normalization coefficients gives the result of the Theorem. □

Let us introduce the following notation:

$$\begin{aligned}&T^{f,+}_n(x) \equiv tr_{\pi^+_n(x)}\Big[R(I\otimes\hbar^{fh_\alpha/2})\Big],\\ &T^f_n(x) \equiv tr_{\pi_n(x)}\Big[R(I\otimes\hbar^{fh_\alpha/2})\Big] = T^{f,+}_n(x) - T^{f,+}_{-n-2}(x).\end{aligned} \tag{5.23}$$

There are two direct consequences of the Theorem we proved. One is known as the *quantum Wronskian relation* between $Q^f_\pm(x)$-operators, illustrating that they are not independent. The second, known as Baxter TQ-relation, expresses the dependence of T_1^f on $Q_\pm^f$- operators.

Proposition 5.4.3. *The following relations hold:*

$$\hbar^{h_\alpha/4+f/2}Q_+^f(\hbar^{1/2}x)Q_-^f(\hbar^{-1/2}x)-\hbar^{-h_\alpha/4-f/2}Q_+^f(\hbar^{-1/2}x)Q_-^f(\hbar^{1/2}x)= \hbar^{h_\alpha/4+f/2}-\hbar^{-h_\alpha/4-f/2} \quad (5.24)$$

$$T_1^f(x)Q_\pm^f(x)=\hbar^{\pm(h_\alpha/4+f/2)}Q_\pm^f(\hbar x)+\hbar^{\mp(h_\alpha/4+f/2)}Q_\pm^f(\hbar^{-1}x) \quad (5.25)$$

Proof. To prove the first relation it is enough to use the statement of the Theorem in the case of $n=0$ and use the fact that $T_0(x)=1$. The second follows from the Theorem above, when $n=1$, and the application of the Wronskian relation. □

5.5 XXZ spin chains and algebraic Bethe ansatz

One can consider the evaluation representations $\pi_k(x)$ for the upper Borel subalgebra $\mathbf{b}_+$ as well, namely,

$$\pi_n(x): e_\alpha\to\mathcal{E},\ e_{\delta-\alpha}\to x\mathcal{F},\ h_\alpha\to\mathcal{H},\quad h_{\delta-\alpha}\to-\mathcal{H}, \quad (5.26)$$

where $\mathcal{E},\mathcal{F},\mathcal{H}$ satisfy the standard commutation relations of $\mathcal{U}_{\sqrt{\hbar}}(sl(2))$ and their action on the corresponding $n+1$-dimensional module is given by the formulas (5.13). The normalized universal R-matrix with $\mathbf{b}_+$ being represented via evaluation homomorphism and $\mathbf{b}_-$ considered in 2-dimensional representation π_1, is given by the following simple expression, see e.g. [23]:

$$(\pi_n(x)\otimes\pi_1(1))R=\phi_n(x)\begin{pmatrix}\hbar^{\frac{\mathcal{H}}{4}}-\sqrt{\hbar}^{-1}x\hbar^{-\frac{\mathcal{H}}{4}} & (\hbar^{1/2}-\hbar^{-1/2})\mathcal{F}\hbar^{-\frac{\mathcal{H}}{4}}\\ x(\hbar^{1/2}-\hbar^{-1/2})\mathcal{E}\hbar^{\frac{\mathcal{H}}{4}} & \hbar^{-\frac{\mathcal{H}}{4}}-\sqrt{\hbar}^{-1}x\hbar^{\frac{\mathcal{H}}{4}}\end{pmatrix}, \quad (5.27)$$

where $\phi_n(x) = \exp(\sum_{k=1}^{\infty} \frac{\sqrt{\hbar}^{n+1}+\sqrt{\hbar}^{-n-1}}{1+\hbar^k}\frac{x^k}{k})$. For convenience it is sometimes useful to consider the variable u, so that $u^{-2} = x$. Then the normalized and Cartan-conjugated R-matrix transforms into the following symmetric expression:

$$\mathcal{L}(u) = u\hbar^{1/4}\phi_n(u^{-2})^{-1}u^H\Big[(\pi_1(u^{-2})\otimes\pi_1(1))R\Big]u^{-H} = \\ \begin{pmatrix} uk\hbar^{1/4} - \hbar^{-1/4}u^{-1}k^{-1} & \hbar^{-1/2}u^{-1}f \\ \hbar^{-1/2}u^{-1}e & \hbar^{1/4}uk^{-1} - \hbar^{-1/4}u^{-1}k \end{pmatrix}. \tag{5.28}$$

Here $e = \mathcal{E}\hbar^{\frac{\mathcal{H}}{4}}(\sqrt{\hbar} - \sqrt{\hbar}^{-1})$, $f = \hbar^{-\frac{\mathcal{H}}{4}}\mathcal{F}(\sqrt{\hbar} - \sqrt{\hbar}^{-1})$, $k = \hbar^{\frac{\mathcal{H}}{4}}$, which satisfy simple commutation relations:

$$ke = \sqrt{\hbar}ek, \quad kf = \sqrt{\hbar}^{-1}fk, \quad ef - fe = (\hbar^{1/2} - \hbar^{-1/2})(k^2 - k^{-2}) \tag{5.29}$$

The operator $\mathcal{L}(u)$ is known as the $\mathcal{L}$*-operator* for the XXZ spin chain and the reason is as follows.

Consider the tensor product $\pi_1(\xi_1^2/u^2)\otimes\cdots\otimes\pi_1(\xi_n^2/u^2)\otimes\pi_1(1)$, where each of the 2-dimensional evaluation modules is refered as *site* of the lattice and the last site is considered as auxilliary. Note that each of $\pi_1(\xi_1^2/u^2) \cong \mathbb{C}^2 = \mathrm{span}_{\mathbb{C}}(\nu_0, \nu_1)$, where ν_0 and ν_1 are correspondingly highest and lowest weight vectors with respect to representations of e, f, k-algebra. Then the *monodromy matrix* acting in this space is the following product:

$$\mathcal{T}(u) \equiv \mathcal{L}_1(u/\xi_1)\ldots\mathcal{L}_N(u/\xi_n)\begin{pmatrix} Z & 0 \\ 0 & Z^{-1}\end{pmatrix} \tag{5.30}$$

where the i-th $\mathcal{L}$-operator acts in the tensor product of the i-th cite and the auxilliary module. Here Z is some fixed free complex parameter. It is clear that

$$[tr\mathcal{T}(u_1), tr\mathcal{T}(u_2)] = 0 \tag{5.31}$$

because of the relation $(\Delta \otimes I)R = R^{13}R^{23}$. Namely, up to multiplication by the function from (5.28) $tr\mathcal{T}(z)$ coincides with $T_1^f(x)$ from the previous section, where $e^f = Z$ and $\mathbf{b}_+$ is represented in $\pi_1(\xi_1^2) \otimes \cdots \otimes \pi_1(\xi_n^2)$. Then the Yang-Baxter equation $R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}$ implies the commutativity condition on the traces.

The relation (5.31) is known as *integrability condition*, and the operator-valued coefficients of the expansion of $\log(tr\mathcal{T}(u))$ are known as the Hamiltonians of the XXZ model. The so-called Algebraic Bethe Ansatz provides an effective method for finding eigenvalues and eigenvectors of $tr\mathcal{T}(u)$ and therefore, solve the problem of simultaneous diagonalization of Hamiltonians.

Below we illustrate the key steps of the method (for the details we refer the reader to section 4.2 of [38]).

It is convenient to write down the expression for $\mathcal{T}(u)$ as follows:

$$\mathcal{T}(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}, \tag{5.32}$$

so that $tr(\mathcal{T}(u)) = A(u) + D(u)$. Then the Yang-Baxter equation produces commutation relations between A, B and D, B for different values of parameter z. Denoting the highest weight vector in the product of our N cites as

$$\Omega_+ \equiv \nu_0 \otimes ... \otimes \nu_0 \tag{5.33}$$

we arrive to the following Theorem, see e.g. [38].

Theorem 5.5.1. *Vectors $B(v_1)\dots B(v_k)\Omega_+$ are the eigenvectors of $tr(\mathcal{T}(u))$ with eigenvalues*

$$\begin{gathered}\Lambda(u|\{v_i\}, Z) = \\ \alpha(u)\prod_{i=1}^{k}\frac{v_i u^{-1}\sqrt{\hbar} - v_i^{-1}u\sqrt{\hbar}^{-1}}{v_i u^{-1} - v_i^{-1}u} + \delta(u)\prod_{i=1}^{k}\frac{v_i^{-1}u\sqrt{\hbar} - v_i u^{-1}\sqrt{\hbar}^{-1}}{v_i^{-1}u - v_i u^{-1}}\end{gathered} \tag{5.34}$$

so that

$$\alpha(u) = Z\prod_{i=1}^{n}(u\xi_i^{-1}\sqrt{\hbar} - u^{-1}\xi_i\sqrt{\hbar}^{-1}), \quad \delta(u) = Z^{-1}\prod_{i=1}^{n}(u\xi_i^{-1} - u^{-1}\xi_i) \tag{5.35}$$

are the eigenvalues of $A(u)$ and $D(u)$ on Ω and numbers v_i satisfy the Bethe equations:

$$\prod_{\alpha=1}^{n}\frac{v_i\xi_\alpha{}^{-1}\sqrt{\hbar} - v_i^{-1}\xi_\alpha\sqrt{\hbar}^{-1}}{v_i\xi_\alpha^{-1} - v_i^{-1}\xi_\alpha} = Z^{-2}\prod_{j=1, i\neq j}^{k}\frac{v_iv_j^{-1}\sqrt{\hbar} - v_i^{-1}v_j\sqrt{\hbar}^{-1}}{v_iv_j^{-1}\sqrt{\hbar}^{-1} - v_i^{-1}v_j\sqrt{\hbar}} \tag{5.36}$$

For our considerations we will redenote $\xi_\alpha^2 \equiv a_\alpha$, $v_i^2 = s_i$, and we remind that $u^{-2} = x$ so that we can rewrite

$$tr\mathcal{T}(u) =$$
$$\hbar^{\frac{n}{4}}\prod_{i=1}^{n}\frac{u^n}{a_i^{n/2}}\Big[Z\hbar^{\frac{h_\alpha}{4}}\prod_{i=1}^{n}g_i(x\hbar^{-1})\frac{Q(\hbar x)}{Q(x)} + Z^{-1}\hbar^{\frac{-h_\alpha}{4}}\prod_{i=1}^{n}g_i(x)\frac{Q(\hbar^{-1}x)}{Q(x)}\Big], \tag{5.37}$$

where

$$g_i(x) = (1 - a_ix) \tag{5.38}$$

and the eigenvalues of the operator $Q(x)$ on $P(s_1, \ldots, s_k) \equiv B(v_1)\ldots B(v_k)\Omega$ are given by

$$\prod_{i=1}^{k}(1 - xs_i) \tag{5.39}$$

The Bethe ansatz equations in the new variables are as follows:

$$\prod_{\alpha=1}^{n}\frac{\sqrt{\hbar} - a_\alpha s_i^{-1}\sqrt{\hbar}^{-1}}{1 - a_\alpha s_i^{-1}} = Z^{-2}\prod_{j=1,\ i\neq j}^{k}\frac{s_j^{-1}\sqrt{\hbar} - s_i^{-1}\sqrt{\hbar}^{-1}}{s_j^{-1}\sqrt{\hbar}^{-1} - s_i^{-1}\sqrt{\hbar}} \tag{5.40}$$

Now we want to relate operator $Q(x)$ with the operators $Q^f_\pm(x)$ from previous section. It is clear that they differ by multiplication on a certain function, so that $Q(x) =$

$F(x)Q_+^f(x)$ for $f = \log_{\sqrt{\hbar}} Z$. If $T_1^f(x) = t(x)\mathcal{T}(u)$, then we have the folowing system of functional equations:

$$t(x)G(\hbar^{-1}x)F(\hbar x) = F(x), \quad t(x)G(x)F(\hbar^{-1}x) = F(x), \tag{5.41}$$

$$G(x) = \prod_{i=1}^{N} g_i(x). \tag{5.42}$$

While we already know the solution for $t(x)$ from the explicit normalization of the $\mathcal{L}$-operator, the solution for F can be derived from the two equations above:

$$F(\hbar^{-2}x) = \frac{G(\hbar^{-2}x)}{G(\hbar^{-1}x)} F(x) \tag{5.43}$$

Representing $F(x) = \prod_{i=1}^{n} f_i(x)$, this equation reduces to $f_i(\hbar^{-2}x) = \frac{g_i(\hbar^{-1}x)}{g_i(\hbar^{-2}x)} f_i(x)$ and therefore

$$f_i(\hbar^{-2}x) = \frac{1 - a_i x\hbar^{-2}}{1 - a_i x\hbar^{-1}} f_i(x) \tag{5.44}$$

Hence,

$$f_i(x) = \frac{\prod_{k=1}^{\infty}(1 - a_i x\hbar^{2k-2})}{\prod_{k=1}^{\infty}(1 - a_i x\hbar^{2k-1})} = \frac{(a_i x\hbar^{-2}, \hbar^2)_\infty}{(a_i x\hbar^{-1}, \hbar^2)_\infty} \tag{5.45}$$

Therefore we have the following Proposition.

Proposition 5.5.2. *The operator*

$$\mathcal{Q}_+(x) = \prod_{i=1}^{n} \frac{(a_i x\hbar^{-2}, \hbar^2)_\infty}{(a_i x\hbar^{-1}, \hbar^2)_\infty} Q_+^{\log_{\sqrt{\hbar}} Z}(x) \tag{5.46}$$

has eigenvalues $\prod_{i=1}^{s}\Big(1 - x s_i\Big)$ *on the vectors* $P(s_1, \dots s_k)$*, provided Bethe ansatz equations*

$$\prod_{p=1}^{n} \frac{\hbar^{-1}a_p - s_i}{a_p - s_i} = Z^{-2}\hbar^{-\frac{n}{2}} \prod_{j=1,\ i\neq j}^{k} \frac{s_i - s_j\hbar^{-1}}{s_i\hbar^{-1} - s_j} \tag{5.47}$$

are satisfied.

One can reproduce $Q_-^{\log Z}(x)$ from algebraic Bethe ansatz as well. In order to do that, one has to use lowest weight vector $\Omega_- = \nu_1 \otimes \cdots \otimes \nu_n$. It is annihilated by $B(u)$-operators and the space of states is spanned by the operators $C(v_1)\dots C(v_k)$ acting on Ω_-. The following Theorem is an analogue of Theorems (5.5.1) and (5.5.2).

Theorem 5.5.3. *Vectors $P_-(s_1,\dots s_k) = C(v_1)\dots C(v_k)\Omega_-$, where $s_i = v_i^2$ are the eigenvectors for $tr(\mathcal{T}(u))$ with eigenvectors $\Lambda(u|\{v_i\}, Z^{-1})$ (see (5.34)) , so that l_i satisfy Bethe equations*

$$\prod_{p=1}^{n} \frac{\hbar^{-1}a_p - s_i}{a_p - s_i} = Z^2\hbar^{-\frac{n}{2}} \prod_{j=1,\ i\neq j}^{k} \frac{s_i - s_j\hbar^{-1}}{s_i\hbar^{-1} - s_j} \tag{5.48}$$

The operator

$$\mathcal{Q}_-(x) = \prod_{i=1}^{n} \frac{(a_i x\hbar^{-2}, \hbar^2)_\infty}{(a_i x\hbar^{-1}, \hbar^2)_\infty} Q_-^{\log_{\sqrt{\hbar}} Z}(x) \tag{5.49}$$

has eigenvalues $\prod_{i=1}^{n}(1 - xs_i)$ on the vectors $P_-(s_1,\dots s_k)$.

The following Proposition gives a normalized version of the quantum Wronskian relation (see Proposition 5.4.3).

Theorem 5.5.4. *The Wronskian relation between $\mathcal{Q}_\pm$-operators reads as follows:*

$$\begin{aligned} &Z\hbar^{h_\alpha/4}\mathcal{Q}_+(\hbar^{1/2}x)\mathcal{Q}_-(\hbar^{-1/2}x) - Z^{-1}\hbar^{-h_\alpha/4}\mathcal{Q}_+(\hbar^{-1/2}x)\mathcal{Q}_-(\hbar^{1/2}x) = \\ &(Z\hbar^{h_\alpha/4} - \hbar^{-h_\alpha/4}Z^{-1})\prod_{i=1}^{n}(1 - a_i\hbar^{-1/2}x). \end{aligned} \tag{5.50}$$

5.6 Explicit expression for the Q-operator via simple root generators

In order to represent operator $\mathcal{Q}_+(x)$ via Chevalley generators of $\mathbf{b}_+$ on $\pi_1(a_1) \otimes \cdots \otimes \pi_1(a_n)$, we have to understand how to compute traces of weighted products of $\mathcal{E}_\pm$ in

ρ_+ representation.

Our first ingredient is to compute $tr(e^{\alpha H}\mathcal{E}_+^k\mathcal{E}_-^m)$ for any k, m and positive α.

Lemma 5.6.1. *i) Trace $tr(e^{\alpha H}\mathcal{E}_+^k\mathcal{E}_-^m)$ is zero if $k \neq m$.*
ii) Assuming ${}_+\langle 0|$ is dual to the a vacuum vector $c_k = {}_+\langle 0|\mathcal{E}_+^k\mathcal{E}_-^k|0\rangle_+ = \frac{\hbar^{-k(k+1)/4}[k]_{\sqrt{\hbar}}!}{(\hbar^{1/2}-\hbar^{-1/2})^k}$

Proof. Part i) follows from the cyclic property of the trace and the commutation relations of generators. To prove part ii), one should use that

$$\mathcal{E}_+\mathcal{E}_-^k = c\sqrt{\hbar}^{-1}\mathcal{E}_-^{k-1} + \hbar^{-1}\mathcal{E}_-\mathcal{E}_+\mathcal{E}_-^{k-1} = c\sqrt{\hbar}^{-1}\frac{1-\hbar^{-k}}{1-\hbar^{-1}}\mathcal{E}_-^{k-1} + \hbar^{-k}\mathcal{E}_-^k\mathcal{E}_+ = \\ c\hbar^{-k/2}[k]_{\sqrt{\hbar}}\mathcal{E}_-^{k-1} + \hbar^{-k}\mathcal{E}_-^k\mathcal{E}_+, \tag{5.51}$$

where $c = (\hbar^{1/2} - \hbar^{-1/2})^{-1}$. Therefore $c_k = c\hbar^{-k/2}[k]_{\sqrt{\hbar}}c_{k-1}$ and this implies ii). □

Finally, one can write the trace of the desired expression:

$$tr(e^{\alpha H}\mathcal{E}_+^m\mathcal{E}_-^m) = \sum_{k=0}^{\infty} {}_+\langle 0|\mathcal{E}_+^k e^{\alpha H}\mathcal{E}_+^m\mathcal{E}_-^m\mathcal{E}_-^k|0\rangle_+ \frac{\hbar^{\frac{k(k+1)}{4}}}{c^k[k]_{\sqrt{\hbar}}!}. \tag{5.52}$$

Let us express every summand in the expression above as σ_k. Then

$$\sigma_k = \frac{e^{-2\alpha k}\hbar^{\frac{k(k+1)}{4}}}{c^k[k]_{\sqrt{\hbar}}!}\,{}_+\langle 0|\mathcal{E}_+^{k+m}\mathcal{E}_-^{k+m}|0\rangle_+ = \\ \frac{e^{-2\alpha k}\hbar^{\frac{k(k+1)}{4}}}{c^k[k]_{\sqrt{\hbar}}!}c^{k+m}[k+m]_{\sqrt{\hbar}}!\hbar^{-(k+m)(k+m+1)/4} = \\ \hbar^{-\frac{m(m+1)}{4}}\hbar^{\frac{(m+k)(m+k-1)}{4}}\hbar^{-\frac{k(k-1)}{4}}c^m\frac{(k+m)_{\hbar^{-1}}!}{(k)_{\hbar^{-1}}!}x^k = \hbar^{-m/2}c^m\frac{(k+m)_{\hbar^{-1}}!}{(k)_{\hbar^{-1}}!}x^k$$

where we used the fact that $[s]_{\sqrt{\hbar}}! = \hbar^{\frac{s(s-1)}{4}}(s)_{\hbar^{-1}}$ and $x = e^{-2\alpha}$. In order to sum over k, let us use the quantum binomial formula:

$$\sum_{k\geq 0}\frac{(k+m)_{\hbar^{-1}}!}{(k)_{\hbar^{-1}}!(m)_{\hbar^{-1}}!}x^k = \prod_{k=0}^{m}\frac{1}{1-\hbar^{-k}x}. \tag{5.53}$$

As a result we obtain the following Proposition.

Proposition 5.6.2. *The weighted trace of the product of oscillator operators is given* *by the following formula:*

$$tr(e^{\alpha H}\mathcal{E}_+^m\mathcal{E}_-^m) = \frac{\hbar^{-m/2}(m)_{\hbar^{-1}}!}{(\hbar^{1/2}-\hbar^{-1/2})^m}\prod_{k=0}^{m}\frac{1}{1-\hbar^{-k}e^{-2\alpha}}. \tag{5.54}$$

Now we are ready to write the formula for the $\mathcal{Q}_+$-operator via the trace formula we introduced. Notice, that in the expression for $Q_+^f(x)$ we are taking the trace of the following expression:

$$\exp_{\hbar^{-1}}(x(\hbar^{1/2}-\hbar^{-1/2})(e_{\delta-\alpha}\otimes\mathcal{E}_+))R_0(x)\exp_{\hbar^{-1}}((\hbar^{1/2}-\hbar^{-1/2})(e_\alpha\otimes\mathcal{E}_-))K(1\otimes\hbar^{fH/2})$$

Notice that all other q-exponential terms vanish, since $\tilde{e}_{-\delta} = -\frac{\sqrt{\hbar}x}{\hbar^{1/2}-\hbar^{-1/2}}$ is constant in the oscilaltor representation, and therefore all its commutators vanish. In order to express $R_0(x)$ appropriately, we have to calculate the $\tilde{e}_{-k\delta}$ generators:

$$\begin{aligned}&\mathcal{E}(u) = (\hbar^{1/2}-\hbar^{-1/2})\sum_{k\geq 1}\tilde{e}_{-k\delta}u^k = \\ &\log(1+(\hbar^{1/2}-\hbar^{-1/2})\tilde{e}'_{-\delta}u) = \log(1-xu) = -\sum_{k\geq 1}\frac{(\sqrt{\hbar}xu)^k}{k}\end{aligned} \tag{5.55}$$

Therefore $\tilde{e}_{-k\delta} = -\frac{(\hbar^{1/2}x)^k}{k(\hbar^{1/2}-\hbar^{-1/2})}$. Hence

$$R_0(x) = \exp\Big(-\sum_{k\geq 0}\frac{(\sqrt{\hbar}x)^k}{[2k]_{\sqrt{\hbar}}}(\tilde{e}_{k\delta}\otimes 1)\Big) \tag{5.56}$$

Let us combine that with the normalizing function for the Q-operator and denote it $W(x)$. Notice that $W(x)$ is independent of z. Also, let us denote $U = \hbar^{(h_\alpha\otimes 1)/4}Z$, so that $K\otimes Z^H = U^{1\otimes H}$. Recall, that the normalization factor is $\mathcal{Z}_+(h_\alpha) = (1-U^{-2})^{-1}$. Therefore, one can write the expression for $Q(x)$ as follows:

$$\mathcal{Q}_+(x) = 1+(1-U^{-2})\sum_{m=1}^{\infty}\tilde{W}_m^Z x^m, \tag{5.57}$$

$$\tilde{W}_m^Z = \sum_{k=0}^{m}\frac{(\hbar^{1/2}-\hbar^{-1/2})^k\hbar^{-k/2}}{(k)_{\hbar^{-1}}!}\prod_{i=0}^{k}\frac{1}{(1-\hbar^{-i}U^{-2})}e_{\delta-\alpha}^k W_{m-k}e_\alpha^k.$$

Therefore we have the following Theorem.

Theorem 5.6.3. *Z-dependence of the operator $\mathcal{Q}_+(x)$ acting on the representation $\pi_1(a_1)\otimes\cdots\otimes\pi_N(a_N)$ can be expressed as follows:*

$$\mathcal{Q}_+(x) = 1 + \sum_{m=1}^{\infty} W_m^Z x^m,$$
$$W_m^Z = \sum_{k=0}^{m} \frac{(\hbar^{1/2}-\hbar^{-1/2})^k \hbar^{-k/2}}{(k)_{\hbar^{-1}}! \prod_{i=1}^{k}(1-\hbar^{-i}\hbar^{-h_\alpha/2}Z^{-2})} e_{\delta-\alpha}^k W_{m-k} e_\alpha^k, \tag{5.58}$$

where $W(x) = \sum_{m=1}^{\infty} x^m W_m$ is the limit $Z\to 0$, which corresponds to the diagonal operator with eigenvalues $\prod_i(1-a_i x)$ in the standard basis of the representation.

Switching to the Drinfeld basis (5.6), we obtain the following result.

Corollary 5.6.4. *In the Drinfeld basis the formula for W_m^Z reads as follows:*

$$W_m^Z = \sum_{k=0}^{m} \frac{(1-\hbar^{-1})^k \hbar^{-k(k+1)/2} K^{-k}}{(k)_{\hbar^{-1}}! \prod_{i=1}^{k}(1-\hbar^{-i}K^{-1}Z^{-2})} F_0^k W_{m-k} E_{-1}^k. \tag{5.59}$$

A similar formula can be written for $\mathcal{Q}_-(x)$.

5.7 Geometric realization of the Q-operator

In this subsection we will relate quantum tautological bundles to the Baxter $\mathcal{Q}_+$-operator, which will allow us to write combinatorial formula for some of them on $K_\mathsf{T}(\mathbf{N}(n))$ in the basis of fixed points. Recall from Chapter 4 that the eigenvalues of $\hat{\tau}(z)$ are given by $\tau(s_1,\dots,s_n)$ evaluated at solutions of Bethe equations from Chapter 4, which differ from the ones, obtained in section by a substitution $\hbar\to\hbar^{-1}$ and $Z^2\to(-1)^n z$. Moreover, we know from the section 7.3.7 of [36] that the quantum tautological bundle corresponding to the top wedge power of the tautological bundle is expressed via the following formula.

$$\widehat{\mathcal{O}(1)}(z) = B(z)\,\mathcal{O}(1), \quad B(z) = \sum_{m=0}^{\infty} \frac{\hbar^{m(m+1)/2}(\hbar-1)^m K^m}{[m]_\hbar! \prod\limits_{i=1}^{m}(1-(-1)^n z^{-1}K\hbar^i)} F_0^m E_0^m \tag{5.60}$$

This immediately leads to the following result.

Theorem 5.7.1. *The operator $\mathcal{Q}_+(x)$ upon the transformation $\hbar \to \hbar^{-1}$, $K \to K^{-1}$ and identification $z = Z^2$ is equal to the operator of quantum multiplication by the quantum weighted exterior algebra of tautological bundle $\sum_{i=0}^{k}(-1)^i \widehat{\Lambda^i \mathcal{V}} x^i$ in the basis of fixed points. Moreover, the following combinatorial formula, explicitly expressing the dependence of quantum exterior powers tautological bundles on the deformation parameter, is is valid:*

$$\widehat{\Lambda^m \mathcal{V}} \;=\; \sum_{k=0}^{m} \frac{(\hbar-1)^k \hbar^{k(k+1)/2} K^k}{(k)_\hbar! \prod_{i=1}^{k}(1-(-1)^n \hbar^i K z^{-1})} F_0^k (\Lambda^{m-k}\mathcal{V}) E_{-1}^k \tag{5.61}$$

Proof. First of all, the eigenvalues of the corresponding operators coincide. Therefore, we have to show that these operators are diagonalized in the same basis. At the same time, each of the $\widehat{\Lambda^m \mathcal{V}}$ on H-level m coincide with the corresponding operators W_m^Z. This means that the corresponding eignevectors (Bethe vectors) are the same. □

Chapter 6

Computations for Partial Flags

In this chapter we will study in detail and apply the formalism which we have developed in the previous chapters to the case when Nakajima quiver variety X is the cotangent bundle to the space of partial flags. In other words, we are interested in studying quantum K-theory of the following quiver of type A_n (see Example 2 in Section 2.3)

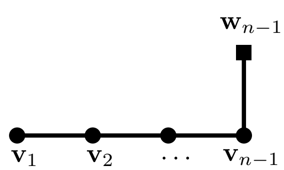


Figure 6.1: A_n Quiver

The special case when $\mathbf{v}_i = i$, $\mathbf{w}_{n-1} = n$ is known as complete flag variety, which we denote as $\mathbb{F}l_n$. It will be convenient to introduce the following notation: $\mathbf{v}'_i = \mathbf{v}_{i+1} - \mathbf{v}_{i-1}$, for $i = 2, \ldots, n-2$, $\mathbf{v}'_{n-1} = \mathbf{w}_{n-1} - \mathbf{v}_{n-2}$, $\mathbf{v}'_1 = \mathbf{v}_2$.

Remark. In principle, in the computations below one could add extra framings to vertices to study the most generic situation in the setting of A_n quiver, but we shall refrain from doing it in this work to make calculations more transparent and simple.

6.1 Bare vertex for partial flags

The key for computing the bare vertex is the localization theorem in K-theory, which gives the following formula for the equivariant pushforward, which constituetes bare vertex $V^{(\tau)}_{\mathbf{p}}(z)$:

$$V^{(\tau)}_{\mathbf{p}}(z) = \sum_{\mathbf{d}\in\mathbb{Z}^n_{\geq 0}} \sum_{(\mathscr{V},\mathscr{W})\in(\mathsf{QM}^{\mathbf{d}}_{\text{nonsing}\,p_2})^{\mathsf{T}}} \hat{s}(\chi(\mathbf{d}))\, z^{\mathbf{d}} q^{\deg(\mathscr{P})/2} \tau(\mathscr{V}|_{p_1}).$$

Here the sum runs over the T-fixed quasimaps which take value $\mathbf{p}$ at the nonsingular point p_2. We use notation $\hat{s}$ for the roof function defined by

$$\hat{s}(x) = \frac{1}{x^{1/2} - x^{-1/2}}, \quad \hat{s}(x+y) = \hat{s}(x)\hat{s}(y).$$

and it is applied to the virtual tangent bundle:

$$\chi(\mathbf{d}) = \text{char}_{\mathsf{T}}\Big(T^{vir}_{\{(\mathscr{V}_i\},\ \mathscr{W}_{n-1})}\mathsf{QM}^{\mathbf{d}}\Big). \tag{6.1}$$

The condition $\mathbf{d} \in \mathbb{Z}^n_{\geq 0}$ is determined by stability conditions, which characterize all allowed degrees for quasimaps.

According to chapter 4, in order to compute localization contributions one has to write down the term for a single line bundle of $\mathscr{P} = \oplus x_i q^{-d_i}\mathcal{O}(d_i)$ in $\chi(\mathbf{d})$. It will be convenient to adopt the following notations:

$$\varphi(x) = \prod_{i=0}^{\infty}(1-q^i x), \quad \{x\}_d = \frac{(\hbar/x,q)_d}{(q/x,q)_d}(-q^{1/2}\hbar^{-1/2})^d, \ \text{ where } \ (x,q)_d = \frac{\varphi(x)}{\varphi(q^d x)}.$$

The following statement is true (for the proof see section 4.1).

Lemma 6.1.1. *The contribution of equivariant line bundle $xq^{-d}\mathcal{O}(d) \subset \mathcal{P}$ to $\chi(\mathbf{d})$ is $\{x\}_d$.*

To compute the vertex function we will also need to classify fixed points of $\mathsf{QM}^{\mathbf{d}}_{\text{nonsing}\,p_2}$. Such a point is described by the data $(\{\mathscr{V}_i\},\{\mathscr{W}_{n-1}\})$, where $\deg\mathscr{V}_i = d_i, \deg\mathscr{W}_{n-1} = 0$.

Each bundle $\mathscr{V}_i$ can be decomposed into a sum of line bundles $\mathscr{V}_i = \mathcal{O}(d_{i,1}) \oplus \ldots \oplus \mathcal{O}(d_{i,\mathbf{v}_i})$ (here $d_i = d_{i,1} + \ldots + d_{i,\mathbf{v}_i}$). For a stable quasimap with such data to exist the collection of $d_{i,j}$ must satisfy the following conditions

- $d_{i,j} \geq 0$,
- for each $i = 1, \ldots, n-2$ there should exist a subset in $\{d_{i+1,1}, \ldots d_{i+1,\mathbf{v}_{i+1}}\}$ of cardinality $\mathbf{v}_i$ $\{d_{i+1,j_1}, \ldots d_{i+1,j_{\mathbf{v}_i}}\}$, such that $d_{i,k} \geq d_{i+1,j_k}$.

To summarize, we will denote collections satisfying such conditions $d_{i,j} \in C$.

Now we are ready to sum up contributions for the entire vertex function.

Proposition 6.1.2. *Let* $\boldsymbol{p} = \mathbf{V}_1 \subset \ldots \subset \mathbf{V}_{n-1} \subset \{a_1, \cdots, a_{\mathbf{w}_{n-1}}\}$ $(\mathbf{V}_i = \{x_{i,1}, \ldots x_{i,\mathbf{v}_i}\})$ *be a chain of subsets defining a torus fixed point* $\boldsymbol{p} \in X^{\mathsf{T}}$. *Then the coefficient of the vertex function for this point is given by:*

$$V_{\boldsymbol{p}}^{(\tau)}(z) = \sum_{d_{i,j} \in C} z^{\mathbf{d}} q^{N(\mathbf{d})/2}\, EHG \quad \tau(x_{i,j} q^{-d_{i,j}}),$$

where $\mathbf{d} = (d_1, \ldots, d_{n-1}), d_i = \sum_{j=1}^{\mathbf{v}_i} d_{i,j}, N(\mathbf{d}) = \mathbf{v}_i' d_i$,

$$E = \prod_{i=1}^{n-1} \prod_{j,k=1}^{\mathbf{v}_i} \{x_{i,j}/x_{i,k}\}^{-1}_{d_{i,j} - d_{i,k}},$$

$$G = \prod_{j=1}^{\mathbf{v}_{n-1}} \prod_{k=1}^{\mathbf{w}_{n-1}} \{x_{n-1,j}/a_k\}_{d_{n-1,j}},$$

$$H = \prod_{i=1}^{n-2} \prod_{j=1}^{\mathbf{v}_i} \prod_{k=1}^{\mathbf{v}_{i+1}} \{x_{i,j}/x_{i+1,k}\}_{d_{i,j} - d_{i+1,k}}.$$

Proof. For the proof we need to gather all contributions $\mathscr{P}$, which separate into 3 types:

$$\mathscr{P} = \mathscr{W}_{n-1}^* \otimes \mathscr{V}_{n-1} + \sum_{i=1}^{n-2} \mathscr{V}_{i+1}^* \otimes \mathscr{V}_i - \sum_{i=1}^{n-1} \mathscr{V}_i^* \otimes \mathscr{V}_i$$

so that their input in the localization formula is as follows.

$$\mathscr{W}_{n-1}^* \otimes \mathscr{V}_{n-1} \longrightarrow \prod_{j=1}^{\mathbf{v}_{n-1}} \prod_{k=1}^{\mathbf{w}_{n-1}} \{x_{n-1,j}/a_k\}_{d_{n-1,j}},$$

$$\mathscr{V}_{i+1}^* \otimes \mathscr{V}_i \longrightarrow \prod_{j=1}^{\mathbf{v}_i} \prod_{k=1}^{\mathbf{v}_{i+1}} \{x_{i,j}/x_{i+1,k}\}_{d_{i,j}-d_{i+1,k}},$$

$$\mathscr{V}_i^* \otimes \mathscr{V}_i \longrightarrow \prod_{i=1}^{n-1} \prod_{j,k=1}^{\mathbf{v}_i} \{x_{i,j}/x_{i,k}\}_{d_{i,j}-d_{i,k}}^{-1},$$

Note, that $\deg(\mathscr{P}) = N(\mathbf{d})$. That gives the polarization term $q^{N(\mathbf{d})/2}$ in the vertex. $\square$

The same formula for the vertex can an be obtained using the following intergal representation [1,2]. It is very useful for a lot of apllications, in particular for computing the eigenvalues $\tau_{\mathbf{p}}(z)$.

Proposition 6.1.3. *The bare vertex function is given by*

$$V_{\boldsymbol{p}}^{(\tau)}(z) = \frac{1}{2\pi i \alpha_{\boldsymbol{p}}} \int\limits_{C_{\mathbf{p}}} \prod_{i=1}^{n-1} \prod_{j=1}^{\mathbf{v}_i} e^{-\frac{\ln(z_i^\sharp)\ln(s_{i,j})}{\ln(q)}} E_{\text{int}} G_{\text{int}} H_{\text{int}} \tau(s_1, \cdots, s_k) \prod_{i=1}^{n-1} \prod_{j=1}^{\mathbf{v}_i} \frac{ds_{i,j}}{s_{i,j}}, \tag{6.2}$$

where

$$E_{\text{int}} = \prod_{i=1}^{n-1} \prod_{j,k=1}^{\mathbf{v}_i} \frac{\varphi\left(\frac{s_{i,j}}{s_{i,k}}\right)}{\varphi\left(\frac{q}{\hbar}\frac{s_{i,j}}{s_{i,k}}\right)},$$

$$G_{\text{int}} = \prod_{j=1}^{\mathbf{w}_{n-1}} \prod_{k=1}^{\mathbf{v}_{n-1}} \frac{\varphi\left(\frac{q}{\hbar}\frac{s_{n-1,k}}{a_j}\right)}{\varphi\left(\frac{s_{n-1,k}}{a_j}\right)},$$

$$H_{\text{int}} = \prod_{i=1}^{n-2} \prod_{j=1}^{\mathbf{v}_{i+1}} \prod_{k=1}^{\mathbf{v}_i} \frac{\varphi\left(\frac{q}{\hbar}\frac{s_{i,k}}{s_{i+1,j}}\right)}{\varphi\left(\frac{s_{i,k}}{s_{i+1,j}}\right)},$$

$$\alpha_{\boldsymbol{p}} = \prod_{i=1}^{n-1} \prod_{j=1}^{\mathbf{v}_i} e^{-\frac{\ln(z_i^\sharp)\ln(s_{i,j})}{\ln(q)}} E_{\text{int}} G_{\text{int}} H_{\text{int}}\Big|_{s_{i,j}=x_{i,j}},$$

and the contour $C_{\mathbf{p}}$ runs around points corresponding to chamber C and the shifted *variable* $z^\sharp = z(-\hbar^{1/2})^{\det(\mathscr{P})}$ [1].

6.2 Bethe Equations and Baxter Operators

We are now ready to compute the eigenvalues of the operators corresponding to the tautological bundles.

Theorem 6.2.1. *The eigenvalues of* $\hat{\tau}(z)\circledast$ *is given by* $\tau(s_{i,k})$*, where* $s_{i,k}$ *satify Bethe equations:*

$$\begin{aligned}
\prod_{j=1}^{\mathbf{v}_2} \frac{s_{1,k}-s_{2,j}}{s_{1,k}-\hbar s_{2,j}} &= z_1(-\hbar^{1/2})^{-\mathbf{v}'_1} \prod_{\substack{j=1\\ j\neq k}}^{\mathbf{v}_1} \frac{s_{1,j}-s_{1,k}\hbar}{s_{1,j}\hbar - s_{1,k}}, \\
\prod_{j=1}^{\mathbf{v}_{i+1}} \frac{s_{i,k}-s_{i+1,j}}{s_{i,k}-\hbar s_{i+1,j}} \prod_{j=1}^{\mathbf{v}_{i-1}} \frac{s_{i-1,j}-\hbar s_{i,k}}{s_{i-1,j}-s_{i,k}} &= z_i(-\hbar^{1/2})^{-\mathbf{v}'_i} \prod_{\substack{j=1\\ j\neq k}}^{\mathbf{v}_i} \frac{s_{i,j}-s_{i,k}\hbar}{s_{i,j}\hbar - s_{i,k}}, \qquad (6.3)\\
\prod_{j=1}^{\mathbf{w}_{n-1}} \frac{s_{n-1,k}-a_j}{s_{n-1,k}-\hbar a_j} \prod_{j=1}^{\mathbf{v}_{n-2}} \frac{s_{n-2,j}-\hbar s_{n-1,k}}{s_{n-2,j}-s_{n-1,k}} &= z_{n-1}(-\hbar^{1/2})^{-\mathbf{v}'_{n-1}} \prod_{\substack{j=1\\ j\neq k}}^{\mathbf{v}_{n-1}} \frac{s_{n-1,j}-s_{n-1,k}\hbar}{s_{n-1,j}\hbar - s_{n-1,k}},
\end{aligned}$$

where $k = 1, \ldots, \boldsymbol{v}_i$ *for* $i = 1, \ldots, \boldsymbol{v}_{n-1}$.

Proof. There are several ways of obtaining these equations. One way corresponds to the study of asymptotics of (3.19) as it was done in section 4. However, there is a shortcut recently provided by [2]. One regards TX as an element in $K_{\prod_i GL(V_i)\times GL(W_{n-1})}(pt)$, so that a_j are coordinates of the torus acting on W_{n-1} and by $s_{i,k}$ are coordinates of the

[1] Note that here we are using the notation defined for z for $(-\hbar^{1/2})$, i.e.

$$z^\sharp = \prod_{i=1}^{n-1} z_i^\sharp,$$

$$z_i^\sharp = z_i(-\hbar^{1/2})^{\mathbf{v}'_i}.$$

torus acting on V_i. In this case we have

$$TX = T(T^*\mathrm{Rep}(\mathbf{v},\mathbf{w})) - \sum_{i\in I}(1+\hbar)\mathrm{End}(V_i) = \tag{6.4}$$
$$\sum_{i=1}^{n-2}\sum_{k=1}^{\mathbf{v}_i}\sum_{j=1}^{\mathbf{v}_{i+1}}\left(\frac{s_{i,k}}{s_{i+1,j}}+\frac{s_{i+1,j}\hbar}{s_{i,k}}\right)+\sum_{k=1}^{\mathbf{v}_{n-1}}\sum_{j=1}^{\mathbf{w}_{n-1}}\left(\frac{s_{n-1,k}}{a_j}+\frac{a_j\hbar}{s_{n-1,k}}\right)-(1+\hbar)\sum_{i\in I}\sum_{j,k=1}^{\mathbf{v}_i}\frac{s_{i,j}}{s_{i,k}}.$$

To get Bethe equations we need to use the following formula

$$\widehat{a}\left(s_{i,k}\frac{\partial}{\partial s_{i,k}}TX\right) = z_i,$$

where $\widehat{a}\left(\sum n_i x_i\right) = \prod\left(x_i^{1/2} - x_i^{-1/2}\right)^{n_i}$. □

The equations (6.3) are Bethe ansatz equations for the periodic anisotropic $\mathfrak{gl}(n)$ XXZ spin chain on $\mathbf{w}_{n-1}$ sites with twist parameters $z_1,\ldots,z_{n-1}$, impurities (shifts of spectral parameters) $a_1,\ldots,a_{\mathbf{w}_{n-1}}$, and quantum parameter $\hbar$, see e.g. [6, 38].

Let us consider the quantum tautological bundles $\widehat{\Lambda^k V_i}(z)$, $k=1,\ldots,\mathbf{v}_i$. It is useful to construct a generating function for those, namely

$$\mathbf{Q}_i(u) = \sum_{k=0}^{\mathbf{v}_i}(-1)^k u^{\mathbf{v}_i-k}\hbar^{\frac{ik}{2}}\widehat{\Lambda^k V_i}(z). \tag{6.5}$$

The seemingly strange $\hbar$ weights will be necessary in Section 4. In the integrable system literature these operators are known as Baxter operators [3, 38]. The following Theorem is a consequence of (3.19).

Proposition 6.2.2. *The eigenvalues of the operator $\mathbf{Q}_i(u)$ are the following polynomials in u:*

$$Q_i(u) = \prod_{k=1}^{\mathbf{v_i}}(u - \hbar^{\frac{i}{2}}s_{i,k}), \tag{6.6}$$

so that the coefficients are elementary symmetric functions in $s_{i,k}$ for fixed i.

Remark. To obtain the full Hilbert space of a $\mathfrak{gl}(n)$ XXZ model one has to consider a disjoint union of all partial flag varieties with framing W_{n-1} fixed, so that in the basis of fixed points the classical equivariant K-theory can be expressed as a tensor product $\mathbb{C}^n(a_1)\otimes\mathbb{C}^n(a_2)\otimes\ldots\mathbb{C}^n(a_{\mathbf{w}_{n-1}})$, where each of $\mathbb{C}^n(a_i)$ is an evaluation representation of $U_\hbar(\widehat{\mathfrak{gl}}(n))$, see e.g. [30]. There is a special interesting question regarding universal formulas for operators $\mathbf{Q}_i(u)$ which we used in [37] for $\mathfrak{gl}(2)$ model, corresponding to prefunadamental representations of the Borel subalgebra of $U_\hbar(\widehat{\mathfrak{gl}}(n))$ [12].

6.3 Compact limit

Simple form of the presentation for the bare vertex computed in this section, allows us to perform quantum K-theory computations in the case of merely partial flag varieties, removing the cotangent bundle part. That, as we shall see, corresponds to a properly defined limit $\hbar\to\infty$.

First of all, let us note, that following along the lines of Sec. 3 one can construct quantum tautological bundles corresponding to K-theory of partial flag varieties by simply counting only those quasimaps whose image does not belong to the fiber. The following Proposition gives the recipe to compute bare vertices and the spectra of quantum tautological bundles in this case.

Proposition 6.3.1. *1. In the integral formula for the bare vertex* (6.2) *we take the limit $\hbar\to\infty$, keeping $\{z^\sharp\}$ fixed as the new family of Kähler parameters.*

2. The Bethe ansatz equations, characterizing the eigenvalues of quantum tautological *bundles are as follows.*

$$\prod_{j=1}^{\mathbf{w}_{n-1}} \frac{s_{n-1,k}-a_j}{a_j} \prod_{j=1}^{\mathbf{v}_{n-2}} \frac{s_{n-1,k}}{s_{n-2,j}-s_{n-1,k}} = z^{\sharp}{}_{n-1} \prod_{j=1,j\neq k}^{\mathbf{v}_{n-1}} \frac{-s_{n-1,k}}{s_{n-1,j}}\,, \quad k=1,\dots,\mathbf{v}_{n-1}\,,$$
$$\prod_{j=1}^{\mathbf{v}_{i+1}} \frac{s_{i,k}-s_{i+1,j}}{s_{i+1,j}} \prod_{j=1}^{\mathbf{v}_{i-1}} \frac{s_{i,k}}{s_{i-1,j}-s_{i,k}} = z^{\sharp}{}_{i} \prod_{j=1,j\neq k}^{\mathbf{v}_{i}} \frac{-s_{i,k}}{s_{i,j}}\,, \quad k=1,\dots,\mathbf{v}_{i}\,, \quad i=2,\dots,n-2\,,$$
$$\prod_{j=1}^{\mathbf{v}_{2}} \frac{s_{1,k}-s_{2,j}}{s_{2,j}} = z^{\sharp}{}_{1} \prod_{j=1,j\neq k}^{\mathbf{v}_{1}} \frac{-s_{1,k}}{s_{1,j}}\,, \quad k=1,\dots,\mathbf{v}_{1}\,. \tag{6.7}$$

Proof. When applying the Localization to compute the bare vertex for the cotangent bundle to partial flags we can brake up the terms in pairs of the form $(\omega, \omega^{-1}\hbar)$. The latter corresponds to the cotangent fiber. The contribution of such a pair to the vertex will be equal to:

$$\frac{1}{\omega^{1/2}-\omega^{-1/2}} \frac{1}{(\hbar\omega^{-1})^{1/2}-(\hbar\omega^{-1})^{-1/2}} = \frac{1}{1-\omega^{-1}} \frac{-\hbar^{1/2}}{1-\hbar^{-1}\omega^{-1}}.$$

Therefore after rescaling by $(-\hbar^{1/2})$, which corresponds to expressing z to $z^{\sharp}$ will be equal to $\frac{1}{1-\omega^{-1}}$ in the $\hbar \to \infty$ limit, that is exactly the contribution of ω in the case of the partial flag variety. One can check that the resulting sum is indeed finite by looking at the intergal formula for the vertex (6.2). Namely, the integrand in the expression for the vertex after fiber removal is as follows:

$$E_{\text{int}} \to \prod_{i=1}^{n-1} \prod_{j,k=1}^{\mathbf{v}_i} \varphi\left(\frac{s_{i,j}}{s_{i,k}}\right),$$

$$G_{\text{int}} \to \prod_{j=1}^{\mathbf{w}_{n-1}} \prod_{k=1}^{\mathbf{v}_{n-1}} \frac{1}{\varphi\left(\frac{s_{n-1,k}}{a_j}\right)},$$

$$H_{\text{int}} \to \prod_{i=1}^{n-2} \prod_{j=1}^{\mathbf{v}_{i+1}} \prod_{k=1}^{\mathbf{v}_i} \frac{1}{\varphi\left(\frac{s_{i,k}}{s_{i+1,j}}\right)}.$$

In order to obtain the corresponding Bethe equations, one can again compute $q \to 1$

asymptotics or just simply evaluating the limit $\hbar \to \infty$ of (6.3) while expresing z in terms of $z_\sharp$. $\square$

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