A Conformity Test for Cointegration

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October 1995

1994-95 Discussion Paper Series No. 750

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October 9, 1995

Abstract

This paper formulates a conformity test for cointegration in the context of a VAR specification for a multivariate $I(1)$ process. The test statistic is a function of the characterisitic roots of the sample covariance matrix of a linear transformation of the cointegral vector; the latter is obtained from the unrestricted estimator of the underlying parameters of the VAR. It is further shown that this test procedure is also applicable to the case where the $I(1)$ process is a $MIMA(k)$, i.e. a multivariate integrated moving average process, the moving average being of order $k < \infty$.

The test statistic, under the null of cointegration, has a normal limiting distribution.

Key Words: Cointegration, Cointegration test, characteristic roots, VAR, $MIMA(k)$.

1 Introduction and Summary

Let $\{X_{t\cdot} : t \in \mathcal{N}\}$ be a stochastic sequence defined on some probability space $(\Omega, \mathcal{A}, \mathcal{P})$. If the sequence is taken to be $I(1)$, in the sense that $\{(I - L)X_{t\cdot} : t \in \mathcal{N}\}$ is **strictly stationary**, the question often arises as to whether the sequence in question is cointegrated. The latter means, in this context that there exists a $q \times r$ matrix B of rank $r \leq q$ such that $X_{t\cdot}B$ is (strictly) stationary. A number of tests have been proposed in the literature, some formal some informal. The somewhat informal tests involve running a regression of one of the elements on the others and using the Dickey-Fuller test, Dickey and Fuller (1979), (1981), to test the hypothesis of cointegration. A more formal test is given in Phillips and Ouliaris (1990) and a set of such tests are given in Johansen (1988), (1991), to mention but a few. All such tests employ an indirect approach in that they explore an implication of the cointegration hypothesis beyond the property that constitutes its definition.

In this paper we explore a conformity test for cointegration, and give the limiting distribution of the test statistic. We place the discussion in the VAR context of cointegration popularized by Johansen as noted above, but the results are equally applicable to contexts that are less constrained. Finally, the test statistic is shown to be asymptotically normal and thus tests for cointegration may be carried out in standard fashion, in contrast with other procedures that require special tabulations.

2 Notation and Problem Formulation

Consider the standard VAR

$$X_{t\cdot}\Pi(L) = \sum_{j=0}^{n} X_{t-j\cdot}\Pi_j = \epsilon_{t\cdot}, \quad t \geq 1, \quad \Pi_0 = I_q, \quad X_{-t\cdot} = 0, \quad t \geq 0, \quad (1)$$

where $X_{t\cdot}$ is a q-element **row** vector, the error process being a $MWN(\Sigma)$, i.e. a multivariate white noise process with mean zero and covariance matrix $\Sigma > 0$; normality is not necessary, as in the case of Johansen (1988), (1991).

"Dividing" $\Pi(L)$ by $(I-L)$, where L is the usual lag operator we find, after some rearrangement,

$$(I-L)X_{t\cdot} = -X_{t-1\cdot}\Pi(1) + x_{t\cdot}\Pi^* + \epsilon_{t\cdot}, \quad (2)$$

where

$$x_{t\cdot} = (\Delta X_{t-1\cdot}, \Delta X_{t-2\cdot}, \ldots, \Delta X_{t-n+1\cdot}), \quad t = 1, 2, \ldots T,$$

$$\Pi^* = (\Pi_1^{*\prime}, \ldots, \Pi_{n-1}^{*\prime})^{\prime}, \quad \Pi_j^* = \sum_{i=j+1}^{n} \Pi_i, \quad \Pi(1) = \sum_{j=0}^{n} \Pi_j.$$

If the process is cointegrated of rank r then $\Pi(1)$ is of (reduced) rank $r < q$. Hence, by the rank factorization theorem, see Dhrymes (1984) p. 23, there exist matrices Γ, B both of dimension $q \times r$ and rank r such that $\Pi(1) = B\Gamma^{\prime}$. A conformity test for cointegration consists of estimating $\Pi(1)$ without this restriction, say $\hat{\Pi}(1)$, and then forming the sample covariance matrix

$$\hat{M} = \frac{1}{T}\hat{\Pi}^{\prime}(1)P_{-1}^{\prime}P_{-1}\hat{\Pi}(1), \quad P_{-1} = (X_{t-1\cdot}), \quad t = 1, 2, \ldots T. \quad (3)$$

If the process is cointegrated of rank r then the rank of the limit of $\hat{M}$ must be r; if this hypothesis is rejected for any $r < q$ then the hypothesis of coitengration is rejected.

To implement this procedure obtain the least squares estimator

$$\hat{\Pi}(1) = \Pi(1) - (V'V)^{-1}V'U, \quad V = NP_{-1},$$

$$N = I_q - X(X'X)^{-1}X', \quad U = (\epsilon_{t\cdot}). \tag{4}$$

Define,

$$\hat{M} = \hat{\Pi}(1)'\left(\frac{P_{-1}'P_{-1}}{T}\right)\hat{\Pi}(1) = \left(\frac{\Pi(1)'P_{-1}'P_{-1}\Pi(1)}{T}\right)$$

$$+A_{12} + A_{12}' + A_{22}$$

$$A_{12} = -\frac{1}{T}\left(\frac{\Pi(1)'P_{-1}'P_{-1}}{T}\right)\left(\frac{U'NP_{-1}}{T}\right)\left(\frac{P_{-1}'P_{-1}}{T^2}\right)^{-1}, \tag{5}$$

$$A_{22} = \frac{1}{T}\left(\frac{P_{-1}'P_{-1}}{T^2}\right)^{-1}\left(\frac{P_{-1}'NU}{T}\right)\left(\frac{P_{-1}'P_{-1}}{T^2}\right)\left(\frac{U'NP_{-1}}{T}\right)\left(\frac{P_{-1}'P_{-1}}{T^2}\right)^{-1},$$

and note that

$$A_{11} = \left(\frac{\Pi(1)'P_{-1}'P_{-1}\Pi(1)}{T}\right) \xrightarrow{\text{a.c.}} M_{zz}, \quad A_{12} \xrightarrow{\text{d}} 0, \quad A_{22} \xrightarrow{\text{d}} 0, \quad \text{and}$$

$$\lambda I_q - \hat{M} \xrightarrow{\text{d}} \lambda I_q - M, \quad M = M_{zz}, \quad \frac{\Pi(1)'P_{-1}'P_{-1}\Pi(1)}{T} \xrightarrow{\text{a.c.}} M_{zz}. \tag{6}$$

Hence, the ordered characteristic roots of $\hat{M}$ converge to the (ordered) characteristic roots of $M = M_{zz}$, and it is evident that, as expected, the number of zero roots of $\Pi(1)$, and thus of M, corresponds to the number of unit roots of $|\Pi(z)| = 0$, and its rank corresponds to the cointegration rank.

To examine the limiting distribution of such roots, we first note that A_{12}, A_{22} both converge to zero at the rate of T^{α}, for $\alpha \in [0,1)$. Thus, the limiting distribution of the roots depends entirely on the first term, and we obtain

$$\sqrt{T}\left(\hat{M} - M_{zz}\right) \sim \sqrt{T}\left[\left(\frac{1}{T}\Pi(1)'P_{-1}'P_{-1}\Pi(1) - M_{zz}\right)\right], \tag{7}$$

owing to the fact that

$$\sqrt{T}A_{12} \xrightarrow{\text{d}} 0, \quad \sqrt{T}A_{22} \xrightarrow{\text{d}} 0. \tag{8}$$

We note that

$$\frac{1}{\sqrt{T}}\Pi(1)'P_{-1}'P_{-1}\Pi(1) - \sqrt{T}M_{zz} = \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(z_{t-1\cdot}'z_{t-1\cdot} - M_{zz}\right), \qquad (9)$$

where $z_{t-1\cdot} = X_{t-1\cdot}\Pi(1)$, and

$$\left\{z_{t\cdot}'z_{t\cdot} - M_{zz} : t \in \mathcal{N}_+\right\}$$

is a zero mean strictly stationary process. If the $MWN(\Sigma)$ that defines the $VAR(n)$ model is assumed in this context to be normal, the entities of Eq. (9) obey

$$E \parallel z_{t\cdot}'z_{t\cdot} - M_{zz} \parallel^{2+\alpha} < \infty, \quad \alpha > 0. \qquad (10)$$

If normality is not assumed the moment condition above needs to be assumed explicitly. It follows, from Corollary 1a in Chapter 5, Dhrymes (1995) that

$$\begin{aligned}\psi_T &= \text{vec}\left(\frac{1}{\sqrt{T}}\Pi(1)'P_{-1}'P_{-1}\Pi(1) - \sqrt{T}M_{zz}\right)\\ &= \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\text{vec}\left(z_{t-1\cdot}'z_{t-1\cdot} - M_{zz}\right)\end{aligned}$$

obeys a CLT. Consequently, its limiting distribution is given by

$$\psi_T \sim N(0, \Psi^*), \quad \text{where} \quad \Psi^* = \int_0^1 [f(\lambda) \otimes f(\lambda)]\, d\lambda, \qquad (11)$$

and f denotes the **spectral matrix** of the cointegral vector $z_{t\cdot}$. The derivation of this particular covariance matrix may be found in Hannan (1970), p. 228.

We have therefore proved

Theorem 1. In the context of the discussion above, consider the **unrestricted** estimator $\hat{\Pi}(1)$ and the matrix

$$\hat{M} = \frac{1}{T}\hat{\Pi}(1)'P_{-1}'P_{-1}\hat{\Pi}(1).$$

The followinmg statements are true:

i. $\hat{M} \xrightarrow{\text{d}} M = M_{zz}$, and thus in probability as well, by Proposition 40, p. 263 in Dhrymes (1989);

ii. $\text{vec}[\sqrt{T}(\hat{M}-M)] \xrightarrow{d} N(0,\Psi^*)$, where Ψ^* is as defined in Eq. (11) and f is the spectral matrix of the cointegral vector.

We are now in a position to formulate a (conformity) test based on the result above. Evidently, we have cointegration if and only if rank $(M) < q$. But this means that some of the characteristic roots of M must be null. More precisely, if there is cointegration of rank r then $q-r$ of the characterstic roots of M must be zero. Therefore, to devise a test we need to obtain the limiting distribution of such roots, a result that is not available in the literature. To this end, we have

Theorem 2. In the context of Theorem 1, the (ordered) characteristic roots of $\hat{M}$, contained in the diagonal matrix $\tilde{\Lambda}$ obey

$$\sqrt{T}(\tilde{\Lambda}-\Lambda) \sim d^*[\sqrt{T}Q'(\hat{M}-M)Q],$$

where the notation $d^*[A]$ indicates the **diagonal elements of the matrix** A, and Q is the orthogonal matrix of the decomposition $M = Q\Lambda Q'$.

Proof: Since $\hat{M}$ and M are at least positive semidefinite by Proposition 52 in Dhrymes (1984), pp. 61-62, they have the (orthogonal) decomposition

$$\hat{M} = \tilde{Q}\tilde{\Lambda}\tilde{Q}', \quad M = Q\Lambda Q'. \tag{12}$$

Moreover, by the results of Theorem 1, Proposition 28, Corollary 5, pp. 242-244 in Dhrymes (1989), $\tilde{Q}$, $\tilde{\Lambda}$ converge, respectively, to Q and Λ; in addition, $\sqrt{T}(\tilde{Q}-Q)$, and $\sqrt{T}(\tilde{\Lambda}-\Lambda)$ have well defined limiting distributions. Next, consider

$$\sqrt{T}[Q'(\hat{M}-M)Q] = \sqrt{T}(\tilde{M}^*-\Lambda), \quad \tilde{M}^* = Q'\tilde{Q}\tilde{\Lambda}\tilde{Q}'Q, \tag{13}$$

and put

$$\sqrt{T}(\tilde{Q}'Q-I_q) = C, \quad \sqrt{T}(\tilde{\Lambda}-\Lambda) = D, \quad \sqrt{T}(\tilde{M}^*-\Lambda) = G. \tag{14}$$

Note that by Theorem 1, C, D, G, are all a.c. finite random variables, in the sense that they have well defined limiting distributions. It follows immediately that

$$\sqrt{T}(\tilde{M}^*-\Lambda) = G \sim D + \Lambda C - C\Lambda. \tag{15}$$

Since

$$g_{ii} = d_{ii}, \quad g_{ij} = (\lambda_i-\lambda_j)c_{ij}, \quad \text{or} \quad c_{ij} = g_{ij}/(\lambda_i-\lambda_j), \quad i \neq j, \tag{16}$$

this concludes the proof of the theorem, for the case where all characteristic roots are different so that the elements c_{ij} are well defined. We now examine the case where

$$\Lambda = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{bmatrix},$$

where Λ_1 is a diagonal matrix containing the (r) positive roots under the null of cointegration. Partitioning the other matrices conformably we determine, in this case

$$\sqrt{T}(\tilde{M}^* - \Lambda) = G \sim \begin{bmatrix} D_{11} & 0 \\ 0 & D_{22} \end{bmatrix} + \begin{bmatrix} \Lambda_1 C_{11} - C_{11}\Lambda_1 & \Lambda_1 C_{12} \\ -C_{21}\Lambda_1 & 0 \end{bmatrix}. \quad (17)$$

Consequently, we have again

$$\begin{aligned} g_{ii} &= d_{ii}, \quad i = 1, 2, \ldots, q, \quad g_{ij} = (\lambda_i - \lambda_j)c_{ij}, \quad i, j = 1, 2, \ldots, r, \quad i \neq j; \\ g_{ij} &= \lambda_i c_{ij}, \quad i = 1, 2, \ldots, r, \quad j = r+1, r+2, \ldots, q; \\ &= \lambda_j c_{ij}, \quad i = r+1, r+2, \ldots, q, \quad i = 1, 2, \ldots, r; \\ &= 0, \quad i, j = r+1, r+2, \ldots, q, \quad i \neq j. \end{aligned} \quad (18)$$

q.e.d.

Corollary 1. The characteristic roots of $\hat{M}$, and hence their limiting distributions, are exactly those of $\tilde{M}^*$.

Proof: Obvious since Q is a fixed orthogonal matrix.

Corollary 2. The distribution of the (associated) characteristic vectors is given by the distribution of QC'.

Corollary 3. A test on the rank of cointegration may be carried out as follows: let $\hat{\lambda}_{(i)}$, $i = 1, 2, \ldots, q$ be the characteristic roots of

$$|\lambda I_q - \hat{M}| = 0,$$

arranged in **decreasing order**, and let it be desired to test the hypothesis

H_0: the rank of cointegration is r
as against the alternative
H_1: the rank of cointegration is $r + s \leq q$.

Consider the entity

$$\tau'^{*} = (\hat{\lambda}_{(r+1)}, \hat{\lambda}_{(r+2)}, \ldots, \hat{\lambda}_{(r+s)});$$

the characteristic roots contained therein are, under H_0, $N(0, \Psi_s^*)$, where Ψ_s^* is the submatrix of Ψ^* corresponding to the roots in τ^*. Thus,

$$\tau'^{*} \Psi_s^{*-1} \tau^{*} \sim \chi_s^2. \tag{19}$$

Corollary 4. For $s = 1$, we have the simple rank test $H_0 : \text{rank}(M) = r$, as against the alternative $H_1 : \text{rank}(M) = r + 1$. For $s = q - r$ we have the test for the existence of cointegration, i.e. $H_0 : \text{rank}(M) = r$, as against the alternative $H_1 : \text{rank}(M) = q$.

Remark 1. The major computational burden entailed by this procedure is the estimation of the spectral matrix of the (estimated) cointegral vector $\hat{z}_{t\cdot} = X_{t-1\cdot}\hat{\Pi}(1)$.

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