Three Essays on Asset Pricing

by

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Abstract

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The first essay examines whether risk is explained based on cash flow (CF) or discount rate (DR). Realized returns comprise (ex-ante) expected returns plus (ex-post) innovations, and consequently both expected returns and returns innovations can be broken down into components reflecting fluctuations in CF and DR. I use a present-value model to identify the CF and DR risk factors which are latent from the time series and cross sections of price–dividend ratios. This setup accommodates models where CF risk dominates, like Bansal and Yaron (2004), and models where DR risk dominates, like Campbell and Cochrane (1999). I estimate the model on portfolios, which capture several of the most common cross-sectional anomalies, and decompose the expected and unexpected returns into CF and DR components along both time-series and cross-sectional dimensions. I find that (1) the DR risk is more likely to explain the variations of expected returns, (2) the CF risk drives the variations of unexpected returns, and (3)together they account for over 80% of the cross-sectional variance of the average stock returns.

The second essay develops a liability driven investment framework that incorporates downside risk penalties for not meeting liabilities. The shortfall between the asset and liabilities can be valued as an option which swaps the value of the endogenously determined optimal portfolio for the value of the liabilities. The optimal portfolio selection exhibits endogenous risk aversion and as the funding ratio deviates from the fully funded case in both directions, effective risk aversion decreases. When funding is low, the manager "swings for

the fences" to take on risk, betting on the chance that liabilities can be covered. Over-funded plans also can afford to take on more risk as liabilities are already well covered and so invest aggressively in risky securities.

The third essay introduces a methodology to estimate the historical time series of returns to investment in private equity. The approach is quite general, requires only an unbalanced panel of cash contributions and distributions accruing to limited partners, and is robust to sparse data. We decompose private equity returns into a component due to traded factors and a time-varying private equity premium. We find strong cyclicality in the premium component that differs according to fund type. The time-series estimates allow us to directly test theories about private equity cyclicality, and we find evidence in favor of the Kaplan and Strmberg (2009) hypothesis that capital market segmentation helps to determine the private equity premium.

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To my parents and Yun

Chapter 1

Cash-flow or Discount-rate Risk? Evidence from the Cross Section of Present Values

Prices are the sums of cash flows discounted by risk-adjusted discount rates. Thus, variations in price–dividend ratio and returns are due to change in cash flow (CF), or discount rate (DR), or both. I propose a present-value model to investigate how CF and DR risk drive the stock return variations from (ex-ante) expected and (ex-post) unexpected perspectives. This model treats the time-varying CF and DR risk factors as latent, following an exogenous time-series dynamics. Upon observations of the log of price–dividend ratio ($lnpd$) , I identify the latent factors and parameters in the model on portfolios capturing several of the most common cross-sectional anomalies. Then I decompose the realized returns into expected and unexpected CF and DR components based on the model estimates. I find the DR risk is more likely to explain the variations in expected returns, while CF risk drives the unexpected returns innovations.

Together, they can account for the common anomalies of cross-sectional stock returns.

In the model, the present-value and the latent-factor approaches function respectively in the analysis of expected and unexpected returns. On the expected return side, using *lnpd*, which reflects the ex-ante expectations in both of the future CF and DR, the present-value approach facilitates to bring the CF and DR fluctuations together and evaluate their likelihood of being priced as risks. Testing whether CF or DR risk dominates the expected return has a fundamental bearing on the theoretical modeling of asset prices. Theories characterized by either CF or DR risk quantitatively explain a wide range of asset-pricing phenomena. For example, the long-run risk literature argues that the CF risk explains stock returns in time series and cross sections,[1] while the models with changing risk aversion or sentiment emphasize DR risk as the main factor in pricing the stocks.[2]

In this paper, the present-value framework is general enough to accommodate both types of theories, featuring CF risk as in Bansal and Yaron (2004) and DR risk as in Campbell and Cochrane (1999). The model's building blocks include factors for the following states. Market-level dividend growth and expected return represent marketwise CF and DR. The stochastic volatilities of dividend growth and return characterize the marketwise prices of CF and DR risk. Moreover, for a piece of individual asset, the exposures of its CF and DR to the marketwise CF and DR factors capture its CF and DR risk. Either CF or

[1] See Bansal and Yaron (2004), Bansal, Dittmar and Lundblad (2005), Bansal, Dittmar and Kiku (2009), Bansal, Kiku, Shaliastovich and Yaron (2013), Drechsler and Yaron (2011), Hansen, Heaton and Li (2008), Lettau and Ludvigson (2009), Koijen, Lustig, Van Nieuwerburgh and Verdelhan (2010), Parker and Julliard (2005).

[2] Campbell and Cochrane (1999), Menzly, Santos and Veronesi (2004), Santos and Veronesi (2005, 2010), Wachter (2006)

DR risk channel imposes constraints on the pricing kernels of the corresponding nested model, and these constraints can be translated into hypothetical conditions on latent factors and parameters. This hierarchy of models enable the Bayes factor test to examine whether the proposed risk channel is likely to generate the cross-sectional data of *lnpd*. The results from this test emphasize the DR risk as the more likely risk channel.

Results beyond the Bayes factor test highlight the DR risk as having the main role in expected return. First, return and growth predictability is consistent with previous empirical findings under the DR risk model, but not under the CF risk model. When the DR risk dominates, the expected returns and growth constructed from DR and CF factors are significant predictors of realized returns and growth at both market and individual stock levels, with R^2 ranging from 1.9% to 8.5% for returns and 1.2% to 14% for dividend growth. The predictability of both return and dividend growth is much poorer under the model with CF risks. Second, the DR news leads CF news in contributing to the variations of *lnpd*. If the DR risk is more influential than the CF risk in pricing, the movement of prices should be driven more by DR changing, and vice versa. In the time-series variances of *lnpd* of all test portfolios, the innovation in market DR accounts for a fraction of 91.1% on average, compared with a fraction of 26.9% from the innovation in market CF. In explaining the cross-sectional variation of level of *lnpd*, the R^2 value attributed to the level of DR exposure is 81% and the R^2 value attributed to the level of CF exposure is -18% under the unconstrained model.

On the unexpected return side, the latent-factor approach highlights the importance of the difference between the ex-ante expected return and the ex-post realized return. In my model, the CF and DR state factors describe the ex-ante expected dividend growth and expected return. They are not observable directly and subject to exogenous dynamics. Discounting the future CF with time-varying DR, I develop an exact form of *lnpd* as a function of these latent factors. This observation equation together with the dynamics of the latent factors constitute a state-space model, and the latent CF and DR factors are readily estimated using Bayesian Gibbs sampling method. I therefore separate the ex-ante expected CF and DR from their ex-post unexpected shocks with the estimated latent factors. Under this approach, one considers *lnpd* as a proxy for both DR and CF and is able to estimate the latent factors without bias (Binsbergen and Koijen 2010). Without the latent-factor approach, one considers *lnpd* as a proxy only for DR (Fama and French 2002; Chen, Petkova and Zhang 2008), and it may result in bias in DR and CF. As a consequence of the latent approach, anomalies in cross-sectional returns may result from ex-ante conditional DR risk or ex-post unexpected realizations in CF and DR that are not priced in the expected returns. Given that the DR risk model is a version of the conditional CAPM, I find that the conditional CAPM explains well the cross section of the ex-ante expected return. The average of ex-ante expected return is totally explained by its level and stability of the time-varying market beta. To explain the ex-post realized return, the ex-ante expected return is in line with but not sufficient. Decomposing the time-series and cross-sectional variance of realized returns, one can see that the unexpected CF shocks mainly move the return. Intuitively, the value stocks, past winner stocks and etc, have more positive surprises than negative surprises in their dividends in the sample, which results in their high average realized returns. Together, the ex-ante DR risk and the ex-post CF shock can account for 81.2% of the cross-sectional variance of average returns.

This paper adds to the recent literature employing present-value structure to identify the DR and expected CF by using information of dividend yield and dividend growth. Ang and Liu (2004), among others,[3] provide expression for the price–dividend ratio as an infinite sum of exponentially quadratic forms of expected CF and DR. The information about DR of a portfolio is characterized by a one-factor model, and it is decomposed into information from marketwise DR and time-varying DR exposure (*beta*). To extend this framework, I integrate a one-factor structure on the CF side by bringing in marketwise CF and time-varying CF exposure to explain the CF of a portfolio. Adding this new ingredient, I endow the reduced-form model with the ability to embed CF risk in cross sections and conform to the motivations in long-run risk theories.

This paper is related to the literature focused on determining the main driving force between CF news and DR news in movement of prices and returns, including Vuolteenaho (2002), Campbell and Vuolteenaho (2004), Chen and Zhao (2009), and Chen, Da, and Zhao (2013).[4] These studies evaluate the importance of DR news and CF news in the movement of stock prices or returns. This paper focus on the validity of risk channels related to fluctuations in DR and CF, specifically, test the likelihood of two models emphasizing different risks. This is the first paper jointly test the DR risk model and CF risk model in time series and cross sections under the present-value framework. From the aspect of results, this paper provides answers unifying the seemingly conflicting

[3] See Kelly and Pruitt (2012), Lettau and van Nieuwerburgh (2008), Cochrane (2008), van Binsbergen and Koijen (2010), Pastor and Veronesi (2003, 2006), Pastor, Sinha and Swaminathan (2008), Ang and Bakaert (2007), and Goyal and Welch (2003).

[4] Also see Cohen, Polk and Vuolteenaho (2003) and Vuolteenaho (2002), among others.

results in previous literature. Emphasizing the DR risk, I argue that the DR news causes larger movement in prices or expected returns, consistent with Campbell and Vuolteenaho (2004). On the other hand, unexpected CF news, although not priced in expected returns, strongly moves realized returns. This is on the same wavelength as Chen, Da, and Zhao (2013).

1.1 Model

In this section, I construct a present-value model to analyze variations in expected and realized return. The cornerstone of the framework is the assumption that the price of an asset is equal to the present value of all the dividends discounted by the expected returns. I derive the closed-form expressions of the prices and returns in a general setup M_0 in subsection 1.1.1. Afterward, I show that both types of models characterized with CF and DR risks (M_{CF} and M_{DR}) can nest into the general framework in subsections 1.1.2. This approach facilitates a head-to-head comparison of the models featuring the two risks. In subsection 1.1.3, I briefly discuss the strategies of estimating the model.

1.1.1 General Present-Value Model: M_0

Let P_t be the price at time t; it is the discount value of all the future cash flows.

$$P_t = \mathrm{E}_t[\sum_{s=1}^{\infty}(\prod_{k=0}^{s-1}\exp(-\mu_{t+k}))D_{t+s}], \tag{1.1}$$

where D_t is the dividend distributed in the period $[t-1, t)$. In this paper, all the measures of price, return and dividend are in real terms, adjusted for inflation. The DR over the period $[t, t+1)$ is μ_t, which is defined as the log of expected return,

$$\exp(\mu_t) \equiv \mathrm{E}_t[\frac{P_{t+1} + D_{t+1}}{P_t}] \tag{1.2}$$

Let Δd_{t+1} denote the log of dividend growth,

$$\Delta d_{t+1} \equiv \log(\frac{D_{t+1}}{D_t}) \tag{1.3}$$

I can rewrite Equation (1.1) as

$$\frac{P_t}{D_t} = \mathrm{E}_t[\sum_{s=1}^{\infty}(\prod_{k=0}^{s-1}\exp(-\mu_{t+k} + \Delta d_{t+k+1}))] \tag{1.4}$$

As a result, the price–dividend ratio reflects variations in both CF (Δd_t) and DR (μ_t). I use conditional factor models to characterize CF and DR. This creates a hierarchy in the pricing of assets: the pricing of individual stocks depends on the pricing of the market portfolio.

1.1.1.1 Market Portfolio Price

I first focus on the market portfolio. I label the market-level variables with superscript M from here on. There are five factors in the market-level state vector to describe the information of aggregate CF and DR: $X_t^M = [r_t^f \; g_t^M \; (\sigma_{g,t}^M)^2 \; z_t \; (\sigma_{z,t}^M)^2]'$. The first is the risk-free rate. The next four are conditional expectations and variances of the log of market dividend growth and excess return, as defined in the following equations:

$$\Delta d_{t+1}^M = g_t^M + \sigma_{g,t}^M u_{t+1}^d \tag{1.5}$$

where $g_t^M = \mathrm{E}_t(\Delta d_{t+1}^M)$ and $(\sigma_{g,t}^M)^2 = \mathrm{var}_t(\Delta d_{t+1}^M)$; and

$$r_{t+1}^M \equiv \log \frac{P_{t+1} + D_{t+1}}{P_{t+1}} - r_t^f = z_t + \sigma_{z,t}^M u_{t+1}^r \tag{1.6}$$

where $z_t = \mathrm{E}_t(r_{t+1}^M)$ and $(\sigma_{z,t}^M)^2 = \mathrm{var}_t(r_{t+1}^M)$. These definitions are based on the assumption that the realized log of dividend growth and excess return are their conditional expectations with heteroskedastic shocks.

I include the state variables of volatility to address Jensen's terms, which are potentially important components in the risk premium. Combining equation (1.3) and equation (1.5), the log of expected dividend growth is written as

$$\log \mathrm{E}_t(\exp(\Delta d_{t+1}^M)) = g_t^M + \frac{1}{2}(\sigma_{g,t}^M)^2 \tag{1.7}$$

Combining equation (1.2) and equation (1.6), the DR μ_t^M, which is the log of expected return, is of the form as

$$\mu_t^M = r_t^f + z_t + \frac{1}{2}(\sigma_{z,t}^M)^2 \tag{1.8}$$

Among the state factors, the risk-free rate is exogenous, and the other four factors are treated as latent and endogenous. For simplicity of estimation, I use the state factors as the demeaned ones, and their unconditional means are treated as model parameters. The state vector evolves as a VAR(1) process.[5]

$$X_{t+1}^M = \Phi_M X_t^M + \Sigma_M^{\frac{1}{2}} \epsilon_{t+1} \tag{1.9}$$

[5]It has been argued that price–dividend rates have a persistent component; see, for example, Fama and French (1988), Campbell and Cochrane (1999), Ferson, Sarkissian, and Simin (2003), and Pastor and Stambaugh (2009). Further, many authors argue that the expected dividend growth is persistent, for instance, Bansal and Yaron (2004) and Lettau and Ludvigson (2005).

I specify Φ_M as diagonal for parsimonious reasons, but the variance is a set of full entries, which allows all possible correlations in the shocks to the state vector.

Under the VAR dynamic, the state vector of current CF and DR is able to capture the information of all future CF and DR. Therefore, I can link the latent state vector with the observable market price–dividend ratio in the following form.

$$\frac{P_t^M}{D_t^M} = \sum_{n=1}^{\infty} \exp(a_n^M + b_n^{M\prime} X_t^M) \tag{1.10}$$

The detail of the derivation is documented in Internet Appendix.

Equation (1.10) shows the price–dividend ratio is an infinite sum of exponential functions, a nonlinear form of the state vector. To estimate the state vector more efficiently, I apply the log-linearization[6] to equation (1.10). The observable market *lnpd* is then approximated by a linear function of the state vector.

$$lnpd_t^M = A^M + B^{M\prime} X_t^M + \sigma_v^M v_t^M \tag{1.11}$$

I approve the validity of this approximation method using simulations. There is little difference between *lnpd* calculated using the exact form and the linear proxy.[7]

1.1.1.2 Individual Stock Price

I next develop the individual level pricing based on the market-level pricing. I label all factors related to a specific stock P with superscript P. To capture the CF and DR of individual stocks, I each employ a conditional one-factor

[6] This methodology is proposed by King, Plosser and Rebelo (1988) and Campbell (1994)
[7] See Internet Appendix.

model:

$$g_t^P + \frac{1}{2}(\sigma_{g,t}^P)^2 = c^P + \gamma_t^P(g_t^M + \frac{1}{2}(\sigma_{g,t}^M)^2) \tag{1.12}$$

$$\mu_t^P - r_t^f = \alpha^P + \beta_t^P(\mu_t^M - r_t^f) \tag{1.13}$$

In equation (1.12), γ_t^P stands for the exposure of stock P's CF to the market CF, and c^P is the error in the level of CF under the conditional factor model. Similarly in equation (1.13), β_t^P is the sensitivity of stock P's DR to the aggregate DR and α^P is the conditional *alpha* (pricing error) in the level of DR.[8]

As a result, the state vector for individual stock P includes the time-varying CF and DR exposures in addition to the market state vector X_t^M: $X_t^P = [X_t^{M\prime}\ \gamma_t^P\ \beta_t^P]'$. Again, the dynamics of the state vector are subject to an exogenous VAR(1) process:

$$X_{t+1}^P = \Phi_P X_t^P + \Sigma_P^{\frac{1}{2}}\epsilon_{t+1} \tag{1.14}$$

Since the first five state factors of X_t^P are identical to X_t^M, the upper left 5×5 blocks of Φ_P and Σ_P are the same as Φ_M and Σ_M.

Following a similar procedure as for the market, one can represent the observable price–dividend ratio of an individual stock as a function of current state vector. Notice that the CF and DR of stock P are quadratic forms of the state vector X_t^P. The price–dividend ratio is an infinite sum of the exponential

[8] Ample evidence support time-varying expected market returns (Ferson and Harvey, 1991,1993; Lettau and Ludvigson, 2001; among others) and time-varying market beta (Ferson and Harvey, 1999; Ang and Chen, 2007; Ang and Kristensen, 2012; among others). The structure of the model is therefore originally motivated by these empirical findings.

quadratic forms of X_t^P.

$$\frac{P_t^P}{D_t^P} = \sum_{n=1}^{\infty} \exp(a_n^P + b_n^{P\prime} X_t^P + X_t^{P\prime} H_n^P X_t^P) \tag{1.15}$$

Applying log-linearization in equation (1.15), the linear observation equation for stock P is

$$lnpd_t^P = A_t^P + B_t^{P\prime} X_t^P + \sigma_v^P v_t^P \tag{1.16}$$

The details for deriving these equations can be found in Internet Appendix.

In summary, the present-value model is a state-space model. The state vector X_t^P is latent and subject to a VAR(1) process. The observation equations link the latent vector to the observable realized returns, dividend growth, and especially price–dividend ratios via the discounted cash flow formula for both market and individual stocks. This model structure enables me to estimate the latent factors given the observable information.

1.1.1.3 Returns and Expected Returns

In addition to the variation in *lnpd* revealed by equation (1.15), this paper also focuses on the variation in expected and realized returns. As shown by Campbell and Shiller (1988), the log of return relates to *lnpd* by

$$r_{t+1}^P = \kappa_0 + \kappa_1 lnpd_{t+1}^P - lnpd_t^P + \Delta d_{t+1}^P \tag{1.17}$$

Taking conditional expectations on both sides of equation (1.17), one can obtain the expected return as

$$\mathrm{E}_t(r_{t+1}^P) = \kappa_0 + \kappa_1 \mathrm{E}_t(lnpd_{t+1}^P) - lnpd_t^P + \mathrm{E}_t(\Delta d_{t+1}^P) \tag{1.18}$$

According to this equation, the expected return reflects the expected updates in price level, as well as the expected CF. Since the expected updates in price level include updates of information about DR and CF, the expected return is supposed to be a function of both DR and CF factors. Even though the DR factors in the state vector are designed to capture the expected returns, the expected returns may still have bearings on the CF factors because (1) the DR and CF factors are potentially correlated, and (2) the expected return is related to all future DR and CF.

In addition to the variations in expected returns, the variations in realized returns also consist of variations in unexpected components. Subtracting equation (1.18) from equation (1.17), one can divide the realized returns into expected and unexpected returns. The expected returns are driven by the state variables of expected CF and DR. The unexpected returns can be separate into DR innovation (I_{DR}), as the unexpected update in future DR, and CF innovation (I_{CF}), as the unexpected update in future CF and realization in distribution.

$$r_{t+1}^P = \mathrm{E}_t(r_{t+1}^P) + I_{DR,t+1}^P + I_{CF,t+1}^P \tag{1.19}$$

For a detailed explanation of the separation, I redirect the reader to Appendix A.4.

Equation (1.18) and (1.19) not only describe the time-series variations in expected and realized returns, but also decide the cross-sectional variations of their average. The average of the expected return is determined by the averages of DR and CF. The conditional one-factor model depicting DR with time-varying exposure is a version of conditional CAPM. As in Jagannathan and Wang (1996) and Lewellen and Nagel (2006), the average DR of stock P features an unconditional two-factor model.

$$\overline{\mu_t^P - r_t^f} = \alpha^P + \overline{\beta^P}(\bar{z} + \frac{1}{2}\overline{(\sigma_z^M)^2}) + \text{cov}(\beta_t^P, z_t + \frac{1}{2}(\sigma_{z,t}^M)^2) \tag{1.20}$$

The level of expected return results from conditional *alpha* (α^P), the level of DR exposure ($\overline{\beta^P}$), as well as the covariance terms describing the stability of DR exposure. Furthermore, the present-value model also allows influence on the mean of expected returns from the CF side. By the same token as DR, the average of CF is

$$\overline{g_t^P} + \frac{1}{2}\overline{(\sigma_g^P)^2} = c^P + \overline{\gamma^P}(\overline{g^M} + \frac{1}{2}\overline{(\sigma_g^M)^2}) + \text{cov}(\gamma_t^P, g_t^M + \frac{1}{2}(\sigma_{g,t}^M)^2) \tag{1.21}$$

The average of expected return may also depend on the level of CF exposure and covariance terms relevant to CF.

For unexpected return, I relax the rational expectation assumptions on CF and DR and summarize the average of realized returns of stock P as the sum of the mean of expected returns, DR innovations, and CF innovations.

$$\overline{r_{t+1}^P} = \overline{\text{E}_t(r_{t+1}^P)} + \overline{I_{DR}} + \overline{I_{CF}} \tag{1.22}$$

A high level of realized returns may result from a high average of expected return, as well as positive realizations of CF and DR innovations.

1.1.2 Models with CF and DR Risk

Notice that the model I present in the previous subsection is quite general, and the source of risks driving the expected return is not yet specified. I next show how the models featuring CF and DR risk can be nested in as special cases of the general model M_0. This hierarchy of model facilitates my choice of the more likely risk to explain expected return as choice of the more likely model to fit the data.

1.1.2.1 Constrained Model with CF risk: M_{CF}

Long-run risk in cash flows is a potential way of explaining the risk premium puzzle and cross-sectional anomalies.[9] In this subsection, I also present a long-run risk model following the models of Bansal and Yaron (2004), Bansal, Kiku, Shaliastovich and Yaron(2013) rather closely, as a special case of the present-value model featuring cash-flow risks. A crucial assumption in the long-run risk literature is the existence of a persistent long-run risk component in the dividend growth. The specification equation (1.5) of cash flow is in accordance with this assumption. The utility function the long-run risk models try to maximize is the Epstein and Zin (1989) utility function, and the log pricing kernel is therefore

$$m_{t+1} = \theta \log \delta - \frac{\theta}{\psi} \Delta d_{t+1}^{M} + (\theta - 1) r_{a,t+1} \tag{1.23}$$

[9]see Bansal and Yaron (2004), Bansal Dittmar and Lundblad (2005), Bansal Kiku and Yaron (2009), Bansal, Kiku, Shaliastovich and Yaron (2013).

Here δ is the discount factor, $\theta \equiv \frac{1-\nu}{1-\psi^{-1}}$, with ν as risk aversion and ψ as the intertemporal elasticity of substitution, and $r_{a,t}$ is the log return of the claim to aggregate consumption.

The following proposition provides necessary conditions on the market state variables in the specification of long-run risk.

Proposition 1.1 *Given the present-value model setup on market level as in equation (1.5), (1.6), (1.9), and (1.11), if the pricing kernel takes the form in equation (1.23), the conditional expectation and variance in excess return, z_t and $\sigma^M_{z,t}$, are linear in conditional variance in dividend growth. There exists parameters of χ_0, χ_1 and υ_0, υ_1, such that*

$$z_t = \chi_0 + \chi_1(\sigma_{g,t})^2 \tag{1.24}$$

$$(\sigma_{z,t})^2 = \upsilon_0 + \upsilon_1(\sigma_{g,t})^2 \tag{1.25}$$

As a result, the VAR coefficients of $(\sigma_{g,t})^2$, z_t and $(\sigma_{z,t})^2$ are the same, and the correlation between the shocks on the three state variables are ± 1.

$$\Phi_{\sigma_g} = \Phi_z = \Phi_{\sigma_z} \tag{1.26}$$

$$\rho_{\sigma_g z} = \rho_{\sigma_g \sigma_z} = \rho_{z\sigma_z} = \pm 1 \tag{1.27}$$

For the proof, I redirect the reader to the Appendix A.1.

At the market level, this proposition emphasizes the key implications from long-run risk models, that the market CF (g^M_t) and the CF volatility ($(\sigma^M_{g,t})^2$) are the only state factors. $(\sigma^M_{g,t})^2$ determines the market DR (z_t) and the DR volatility ($(\sigma^M_{z,t})^2$). This requires the persistence in the dynamics of $(\sigma^M_{g,t})^2$, z_t and $(\sigma^M_{z,t})^2$ are all the same, and the shocks to these state variables are perfectly correlated.

Regarding the portfolio-level implications, I state in the next proposition.

Proposition 1.2 *With the present-value model setup for Stock P as in equation (1.14), (1.16), and the pricing kernel of long-run risk model in equation (1.23), the risk exposure β_t^P linearly depends on the dividend growth's loading in cash-flow γ_t^P. There exists parameters of η_0 and η_1, such that*

$$\beta_t^P = \eta_0 + \eta_1 \gamma_t^P \tag{1.28}$$

As a result, the VAR coefficients of β_t^P and γ_t^P are the same, and the correlation between the shocks on these two state variables is 1.

$$\Phi_\beta = \Phi_\gamma \tag{1.29}$$

$$\rho_{\beta\gamma} = \rho_{\gamma\beta} = \pm 1 \tag{1.30}$$

The proof is also shown in Appendix A.1.

At an individual stock level, the constraints characterizing M_{CF}, require the DR exposure β_t^P to reflect the sensitivity of the individual's CF to aggregate CF. This proposition shows that the market risk in this setting is identical to the risks from CF exposure γ_t^P, and this risk is related to the fluctuation in CF. As a result, the dynamics of β_t^P and γ_t^P are supposed to have the same persistence and common shocks.

1.1.2.2 Constrained Model with DR Risk: M_{DR}

In this subsection, I embed the market risk of DR in the framework, specifically patterning it after the habit formation models as in Campbell and Cochrane (1999) and Santos and Veronesi (2004, 2010).

I follow the notation in Campbell and Cochrane (1999), and review the model in Appendix A.1. The habit formation model can be captured by two key features, the habit state variable and its dynamics. Here s_t is the log of consumption ratio surplus, reflecting the external habits, and it is subject to a heteroskedastic AR(1)

$$s_{t+1} = (1-\phi)\bar{s} + \phi s_t + \Lambda(s_t)\sigma_{c,t}v_{t+1}$$

where $\Lambda(s_t)$ is the sensitivity function characterizing the heteroskedastic innovation in s_t.

The following proposition shows how Campbell and Cochrane (1999) is linked to the present-value model as a special case.

Proposition 1.3 *Given the above notation of the habit formation model, there exists a specification of $\Lambda(s_t)$, such that the expected market excess returns is a function of s_t, $z_t = f(s_t)$; and the expected excess return of asset P is given as*

$$\mathrm{E}_t(r_{t+1}^P) + \frac{1}{2}var_t(r_{t+1}^P) - (r_t^f + \pi_t) = \beta_t^P(z_t + \frac{1}{2}(\sigma_{z,t}^M)^2) \tag{1.31}$$

where

$$\beta_t^P = cov_t(r_{t+1}^P, r_{t+1}^M)/var_t(r_{t+1}^M) \tag{1.32}$$

For the details of the proof, I redirect the reader to Appendix A.1.

At market level, this proposition does not impose any constraints to accommodate M_{DR}. Comparing the structures of model M_{DR} and M_0, the state variables s_t in M_{DR} can be mapped to the state variable z_t in M_0, showing that the DR factors may reflect the states of external habit. Because one can arbitrarily select $\Lambda(s_t)$, there must exist a choice of $\Lambda(s_t)$ to ensure that the state variable z_t, which s_t is mapped to, characterizes the DR. Since the empirical analysis of this paper focuses on the pricing in cross sections, the exact form of $\Lambda(s_t)$ is not of interest for studying. Given this choice of $\Lambda(s_t)$, one can see M_{DR} and M_0 as equivalent to price the market.

At an individual stock level, the special case of M_{DR} requires the time-varying *beta* to purely reflect conditional sensitivity of individual stock returns to market returns. In the general model M_0, the time-varying *beta* may reflect other information. This restriction, however, specially clarifies that *beta* is a risk channel associated with fluctuations in DR. Also it constitutes a crucial observation equation for the time-varying *beta*.

1.1.3 Estimation Strategy

In this section, I sketch the Gibbs sampling algorithm for estimating the present-value model, which is applicable to both general model M_0 and the two constrained models M_{DR} and M_{CF}. The estimates of the latent variables and relevant parameters are used to investigate the importance of CF and DR risks in expected returns and explain the realized returns.

Notice that present-value model is a state-space model, and the latent state variables and the parameters describing their dynamics are readily estimated by using the Gibbs sampling method. The evolution dynamics are described by equation (1.14), for both the market level and the individual level. The observation includes equations (1.5), (1.6), and (1.11) for the market level, as well as equation (1.12), (1.13), and (1.16) for the individual stocks.

Since the pricing is ordered from the market level to the individual stock level, the estimation strategy also adheres to this sequence. I first estimate the market-level state variables of $[g_t^M, (\sigma_{g,t}^M)^2, z_t, (\sigma_{z,t}^M)^2]$ using Kalman filtering with the evolution equations and observation equations on market level.[10] The set of market level parameters, $\Theta = (\bar{r}^f, \bar{g}^M, (\bar{\sigma}_g^M)^2, \bar{z},\ (\bar{\sigma}_z^M)^2, \Phi_M, \Sigma_M, (\sigma_v^M)^2)$ are estimated via iteration of the Gibbs sampling method. Provided with the uniform estimates of the market-level state vector and parameters, I then draw the posterior distributions of state vectors and parameters of individual stocks using the individual-level data. The state variables $[\gamma_t^P, \beta_t^P]$ are estimated using Kalman filtering, and the parameters $\Theta = (\bar{\gamma}^P, \bar{\beta}^P, \Phi^P, \Sigma^P, (\sigma_v^P)^2)$ are obtained using iteration of the Gibbs sampler.The details of the algorithm including specification of the prior distributions are documented in Internet Appendix.

1.2 Data

In this section, I first describe the data and test portfolios employed by the empirical study. I then summarize the observable variables from which the latent variables can be inferred.

1.2.1 Data Description

In the empirical work, I use all NYSE, Amex, and NASDAQ common stocks for the period 1964 to 2012 from Center for Research in Security Price (CRSP) monthly files, merged with accounting data from Compustat. To address time-series and cross-sectional returns, I estimate the model on the market and

[10] See Carter and Kohn (1994).

seven groups of quintile portfolios reflecting anomalies in averages of returns. The market portfolio is formed as the value-weighted portfolio containing all common stocks in the CRSP universe. The test portfolios are sorted by size, book-to-market ratios, past returns, past idiosyncratic volatility, accrual component in earnings, capital investment, and liquidity separately. I document the labels and the details of constructing the test portfolios in Appendix A.2.

For each test portfolio, the observable variables are calculated as follows. Assuming the dividends are reinvested in 3-month T-bills (nominal risk-free rate),[11] one is able to compute the observable dividend growth and *lnpd* from the value-weighted return with and without dividends.[12] All the dividend growth and returns are aggregated at an annual horizon to avoid seasonality, but sampled at quarterly frequency to include more observations. The data structure brings up an overlapping observation issue, as addressed by Hodrick (1992). My model can adapt to the overlapping data, and Internet Appendix address the technical details of specifications and model with overlapping data structure.

When the model M_{DR} is being estimated, there is one more observation equation about the time-varying *beta* itself. The observable benchmark of *beta* reflects the conditional covariance between market and portfolio returns, and is directly estimated from short-window regressions, following Lewellen and Nagel (2006). At the end of month t, one can simply run OLS with all the daily returns in the following quarter after the month t to get the benchmark

[11] I follow Chen (2009) argument that when stock returns enter the calculation of dividend, the CF effect on *lnpd* from dividend growth is contaminated.

[12] Given the surge in the share repurchase activity in this sample, I follow Bansal Dittmar and Lundblad (2005) to adjust capital gain and cash payment incorporating a candidate measure of repurchase.

of *beta*.

1.2.2 Data Summary

Table 1.1 summarizes the observable variables of the market and other test portfolios. For each sorting group, I choose the two extreme and the median portfolios for representative reporting. In columns 1-6, I report the mean and the standard error of the excess return, dividend growth, and the log price–dividend ratio. The last two columns report OLS estimates of the unconditional *alpha* ($\hat{\alpha}_{OLS}$) and the market *beta* ($\hat{\beta}_{OLS}$) under the unconditional CAPM.

The summary statistics reflect two features of the data. First, all three observables exhibit large time-series and cross-sectional variations. The standard deviations of the three observables are large for all portfolios, and the averages of the observables are dispersed. The valuation of CF and DR risk is based on how well the corresponding model fits these patterns. Second, the test portfolios demonstrate anomalies in the average of excess returns, which cannot be explained by unconditional CAPM.[13] For example, the portfolios sorted by momentum display a spread of $(9.4\% - 0.8\% =)$ 8.6% between average returns of "past winners" (MOM5) and "past losers" (MOM1). Taking CAPM market factor into account unconditionally, the spread between unconditional *alpha*'s become even higher as 9.1%. The model in this paper breaks through with a new approach from an unexpected perspective to explain these anomalies.

[13] An extensive literature documents the anomalies in return associated with these seven sorting: size (Fama and French 1992, 1993), book-to-market (Fama and French 1992,1993), momentum (Jagadeesh and Titman 1993), idiosyncratic volatility (Ang, Hodrick, Xing, and Zhang 2006) , accruals (Sloan 1996), capital investment (Titman, Sheridan, Wei, and Xie 2004; Liu, Whited, and Zhang 2009), illiquidity (Amihud 2002; Pastor and Stambaugh 2003).

Table 1.1: Summary Statistics

I report summary statistics of the market portfolio, as well as the other test portfolios which are sorted on size, book-to-market, momentum, idiosyncratic volatility, accrual component in earnings, capital investment and liquidity. I choose the two extreme and the median portfolio within each quintile sorting group to report. Data are from CRSP and Compustat, spanning from Jan 1964 to Dec 2012. The variables are sampled at quarterly frequency but measured at annual horizon. The mean and standard error are annualized. The excess returns are defined as returns in excess of the annualized 3-month Treasury bill rate. In the last two columns, I report OLS estimates of the alphas and betas from unconditional CAPM, by regressing the quarterly portfolio annual excess return on the market excess returns.

							OLS Estimates	
	$\mu(R^e)$	$\sigma(R^e)$	$\mu(g)$	$\sigma(g)$	μ(lnpd)	σ(lnpd)	α^u	β^u
Market Portfolio	0.054	0.174	0.011	0.069	3.616	0.406	0.000	1.000
Size Portfolios								
SIZE1 (Large Cap)	0.051	0.169	0.006	0.069	3.575	0.442	0.000	0.948
SIZE3	0.074	0.204	0.026	0.121	3.932	0.546	0.016	1.082
SIZE5 (Small Cap)	0.078	0.259	0.036	0.176	4.372	0.516	0.014	1.186
B/M Portfolio								
BM1 (Growth)	0.046	0.188	0.010	0.125	4.083	0.387	-0.009	1.026
BM3	0.063	0.169	0.008	0.111	3.400	0.448	0.016	0.882
BM5 (Value)	0.096	0.193	0.031	0.265	3.436	0.476	0.046	0.928
Momentum Portoflios								
MOM1 (Losers)	0.008	0.249	-0.059	0.410	3.769	0.540	-0.057	1.216
MOM3	0.044	0.164	0.000	0.142	3.506	0.403	-0.004	0.886
MOM5 (Winners)	0.094	0.209	0.059	0.362	4.087	0.588	0.034	1.098
Idiosyncratic Portfolios								
VOL1 (Volatile)	-0.007	0.349	-0.073	0.712	5.428	1.070	-0.092	1.572
VOL3	0.067	0.246	0.011	0.247	4.379	0.799	-0.004	1.310
VOL5 (Stable)	0.058	0.151	0.012	0.067	3.469	0.410	0.013	0.829
Accural Portfolios								
ACC1 (High Accrual)	0.023	0.206	-0.012	0.214	4.201	0.580	-0.037	1.110
ACC3	0.057	0.169	0.002	0.166	3.579	0.508	0.008	0.912
ACC5 (Low Accrual)	0.071	0.199	0.041	0.198	3.798	0.528	0.013	1.068
Capital Investment Portfolios								
CI1 (High Cap Invest)	0.050	0.181	0.007	0.190	3.860	0.499	-0.004	0.999
CI3	0.054	0.160	0.009	0.163	3.542	0.413	0.006	0.887
CI5 (Low Cap Invest)	0.080	0.202	0.038	0.222	3.924	0.602	0.021	1.099
Liquidity Portfolio								
LIQ1 (Liquid)	0.050	0.162	0.007	0.094	3.464	0.391	0.002	0.905
LIQ3	0.083	0.200	0.025	0.183	3.704	0.466	0.029	0.986
LIQ5 (Illiquid)	0.090	0.238	0.043	0.319	4.002	0.456	0.031	1.095

1.3 Results

In this section, I report a comprehensive analysis of the variations in expected and realized returns using the proposed present-value model. First, in examining the variation in expected return, I test whether the fluctuation of CF or DR is likely to be priced as risk, with both market-level and individual-level stock data. Second, in examining the variation of realized return, I evaluate the power of risk from expected return, as well as that of unexpected CF and DR innovations, in explaining the time-series and cross-sectional realized return.

1.3.1 CF vs DR: Which is the more likely risk to model expected return?

To begin with, I argue that the DR risk is the more likely risk channel than the CF risk. One can draw this conclusion by testing the hypotheses as summarized in the propositions characterizing the CF and DR risk models. I list the hypothesized constraints associated with the risks at market and individual levels in the diagram below. Proposition 1.1 governs the CF risk constraints at market level, and Proposition 1.2 characterizes the CF risk at individual stock level. For DR risk, there are no market-level constraints, because M_0 is equivalent to a version of M_{DR} at market level, as shown in section 1.1.2.2. Proposition 1.3 specifies the constraints of DR risk at the individual stock level.

	M_{CF}	M_{DR}
Market Level	Proposition 1.1: $\Phi_{\sigma_g} = \Phi_z = \Phi_{\sigma_z}$ $\rho_{\sigma_g z} = \rho_{\sigma_g \sigma_z} = \rho_{z\sigma_z} = \pm 1$	-
Individual Stock Level	Proposition 1.2: $\Phi_\beta = \Phi_\gamma$ $\rho_{\gamma\beta} = \pm 1$	Proposition 1.3: $\beta_t^P = \frac{cov_t(r_{t+1}^P, r_{t+1}^M)}{var_t(r_{t+1}^M)}$

One can test these propositions with classical statistical tests using the posterior mean and posterior standard error of the factors and parameters from the Gibbs estimates. However, there is one major drawback to the classical tests using Bayesian estimates: the low statistical power. In Bayesian statistics, the Bayes factor test is the formal practice for quantifying the evidence in favor of a scientific theory (Jeffreys 1935, 1960; Kass and Raftery 1995). For a model M_1 under hypothesis constraints and the general model M_0 as the alternative, the Bayes factor (LR_1) is defined as twice the difference between posterior log-likelihood of M_0 and M_1 given the observable data Y: $LR_1 \equiv 2(L(M_0 \mid Y) - L(M_1 \mid Y))$. The higher the Bayes factor, the more likely the unconstrained model is true and the greater the evidence against the hypothesis. This test is positively against the hypothesis (model M_1) if the Bayes factor is greater than 2, and strongly against it if greater than 6 (Kass and Raftery 1995). I calculate the posterior likelihood of the models with the Gibbs outputs, following Chibs (1995). The details of the calculation can be found in Appendix A.3.

In addition, to make a choice of the more likely risk model for expected return, I examine whether the predictability of returns or dividend growth under each risk model is consistent with the existing literature. Furthermore, I evaluate the importance of CF and DR in driving *lnpd* by decomposing the time-series and cross-sectional variance of *lnpd*. The relative importance of DR and CF variations in *lnpd* has great relevance to modeling the expected return. For example, the crucial assumption of the long-run risk model is that *lnpd* is purely driven by the CF factors. As a result, under the general model M_0, the higher the portion of the variance of *lnpd* that is attributed to a factor, the more likely fluctuation in this factor is a risk.

1.3.1.1 Proposition 1.1: CF Risk at Market Level

To begin with, I test Proposition 1.1 to examine CF risk at the market level. Recall this proposition hypothesizes that the state variables of $(\sigma_{g,t})^2$, z_t, and $(\sigma_{z,t})^2$ have the same VAR coefficients and perfectly correlated shocks under M_0. In this subsection, I provide three reasons for rejecting these hypotheses.

First, I reject these hypotheses by using a classical test with the Bayes estimates. Table 1.2 reports the Bayes estimates of the market model parameters under M_0 and M_{CF}, separately in panel A and panel B. For each row, the posterior mean is reported as the upper number and the posterior standard error is reported as the lower one in parenthesis. The pertinent parameters in Proposition 1.1 are the last three estimates in the second row as VAR coefficients, and the last three in the bottom of correlation matrix of the shocks. Using these estimates, I jointly test this proposition with a χ^2_5 test and reject M_{CF} overall, since the p-value is less than 0.001.

Table 1.2: Estimates of Market Parameters under M_0 and M_{CF}

Panel A: Estimates under General Model M_0

	r^f	g^M	$(\sigma_g^M)^2$	z	$(\sigma_z^M)^2$
Average	0.009	0.011	0.004	0.057	0.032
	(0.002)	(0.005)	(0.000)	(0.013)	(0.000)
VAR coeff. Φ	0.938	0.958	0.671	0.984*	0.709
	(0.017)	(0.034)	(0.321)	(0.008)	(0.182)
Shock Variance Σ	0.084	0.041	0.007	0.011	0.461
	(0.009)	(0.047)	(0.006)	(0.006)	(0.240)

Correlation Between Shocks:ρ

	g_t	$(\sigma_{g,t}^M)^2$	z_t	$(\sigma_{z,t}^M)^2$
r_t^f	-0.244	-0.041	0.269	0.148
	(0.302)	(0.368)	(0.260)	(0.377)
g_t		-0.053	-0.369	-0.142
		(0.401)	(0.277)	(0.355)
$(\sigma_{g,t}^M)^2$			-0.027*	-0.032*
			(0.412)	(0.395)
z_t				0.199*
				(0.358)

Panel B: Estimates under CF Model M_{CF}.

	r^f	g^M	$(\sigma_g^M)^2$	z	$(\sigma_z^M)^2$
Average	0.009	0.012	0.004	0.056	0.033
	(0.002)	(0.005)	(0.000)	(0.012)	(0.000)
VAR coeff. Φ	0.939	0.975	0.684	0.684	0.684
	(0.017)	(0.009)	(0.184)	(0.184)	(0.184)
Shock Variance Σ	0.084	0.017	0.008	1.017	0.444
	(0.009)	(0.016)	(0.004)	(1.045)	(0.630)

Correlation Between Shocks:ρ

	g_t	$(\sigma_{g,t}^M)^2$	z_t	$(\sigma_{z,t}^M)^2$
r_t^f	-0.548	0.551	0.551	0.551
	(0.316)	(0.289)	(0.289)	(0.289)
g_t		-0.514	-0.514	-0.514
		(0.299)	(0.299)	(0.299)
$(\sigma_{g,t}^M)^2$			1.000	1.000
			(0.000)	(0.000)
z_t				1.000
				(0.000)

Note to Table 1.2 I report the Gibbs estimates of posterior mean and standard error of the parameters related to the market. For each parameter, the upper row is the posterior mean, and the posterior standard error is reported in the lower row with parenthesis. Panel A shows estimates under M_0 or M_{DR}, as they are equivalent at market level. Panel B shows estimates under M_{CF}. For each parameter, I perform a t-test on the equality of estimates under different models, and using asterisk showing the significance ("*": $p < 0.01$) in panel A.

To further examine the difference in model structure introduced by CF risk, I compare the corresponding parameters in panel A and panel B, and perform a t-test on each parameter to check whether the estimates under the two different models are equal. As shown in the second row of both panels, the market CF and DR factors are persistent under both M_0 and M_{CF}, consistent with findings in Ferson et al. (2003) and Campbell and Cochrane (1999). However, there are significant differences between the VAR coefficients in z_t in the two panels. Under the general model, the market DR factor is more persistent than when it is restricted to reflect CF volatility only. Furthermore, the correlation between shocks of z_t, $(\sigma_{z,t})^2$, and $(\sigma_{g,t})^2$ are significantly different from 1 as proposed in the hypothesis. In summary, the hypothesis is rejected due to the fact that variation in DR factor cannot be solely attributed to the change in CF volatility.

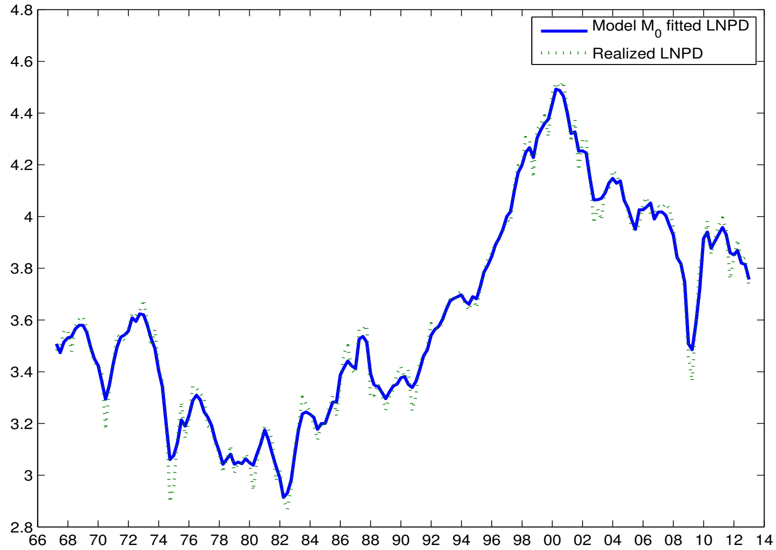
Second, the Bayes factor test rejects Proposition 1.1 and disproves CF risk as being plausible in modeling market prices. I calculate the Bayes factor statistics as 10.8, which can be interpreted as M_0 being about ($e^{10.8/2} =$) 221 times more likely to generate the market data than M_{CF}.

The inferior performance of the CF risk model also appears in fitting *lnpd*. Figure 3.1 compares the fitting of observable *lnpd* from estimates of M_0 and M_{CF}. The fitting R^2 values are 99% and 96%, respectively. Both models fit *lnpd* well overall; however, we can still observe that there is more deviation in the fitting by model M_{CF}.

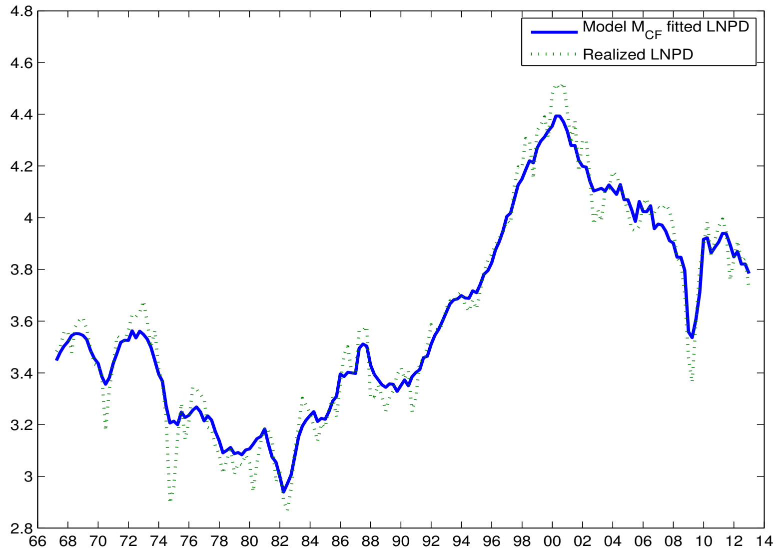
Third, the predictabilities of market return and dividend growth under M_{CF} are less consistent with previous findings than those under M_0. A large literature documents the existence of predictability in equity returns and dividend

Figure 1.1: Quality of Fitting Market Price-dividend Ratio under M_0 and M_{CF}

Model M_0 Fitted and Realized log of Price-dividend Ratio, $R^2 = 99\%$



Model M_{CF} Fitted and Realized log of Price-dividend Ratio, $R^2 = 96\%$



The plots compare the realized market log of price–dividend ratio and its fitted values from the models M_0 and M_{CF}. The R^2 is calculated by $1 - \text{var}(Real\ lnpd - Fitted\ lnpd)/\text{var}(Real\ lnpd)$.

growth.[14] As a result, a model characterized by an appropriate risk should produce expected returns and growth consistent with realizations. Since the predictability patterns for market return and dividend growth under M_{CF} contradicts former findings, I consider this as evidence for disqualifying CF risk.

Compared with the general model M_0, M_{CF} contradicts the existence of return and dividend growth predictability. I present estimates of the latent expected dividend growth g_t^M and expected excess returns z_t at market level in Figure 1.2. Under M_0, the predictive R^2 is 15% for the dividend growth, and 4.8% for the excess return.[15] My results are comparable to the predictability results shown by van Binsbergen and Koijen (2010),[16] and consistent with some other literature on predictability; for instance, Ang and Bakaert (2007), Lettau and van Nieuwerburgh (2008), and Cochrane (2008). All these papers report an R^2 of predictability in return is between 2% and 5%. In contrast, the expected dividend growth and excess return have negative R^2 under CF risk model M_{CF}, indicating the expectation is not a proper proxy for the realization.

The predictability literature better supports that *lnpd* predicts market returns but not dividend growth (van Binsbergen and Koijen 2010). When *lnpd* goes up, one expects a drop in return, not a rise in dividend growth. The estimates under M_0 are consistent with this argument, while the estimates under M_{CF} show the opposite. Considering the left two panels in Figure 1.2 for M_0, one can see that *lnpd* well predicts the market returns, as z_t negatively correlates with *lnpd*. When *lnpd* was low in the late 1970s and early 1980s, z_t

[14] See Fama and French (1988), Campbell and Shiller (1988), Cochrane (2008), Menzly, Santos, and Veronesi (2004), Goetzmann and Jorion (1995), among others.

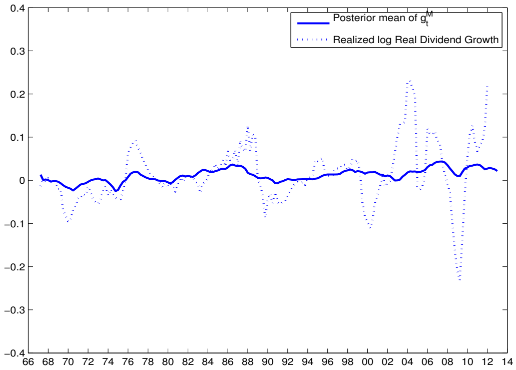
[15] R^2 is calculated as $1 - \frac{\mathrm{var}(Realized\ Value - Expected\ Value)}{\mathrm{var}(Realized\ Value)}$.

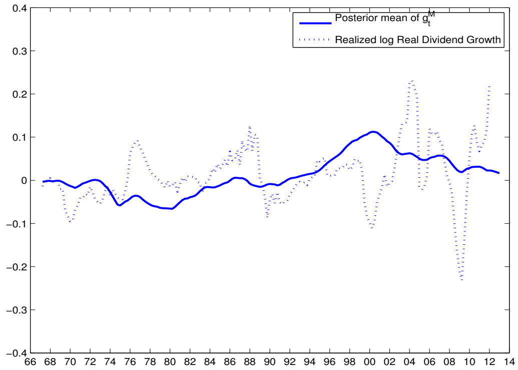
[16] van Binsbergen and Koijen (2010) estimate a reduced-form present value model. Using the data of dividend reinvested in cash, they show R^2 of predicting return is 8.2%, and R^2 of predicting cash flow is 13.9%.

Figure 1.2: CF and DR Predict Dividend Growth and Return under M_0, but not under M_{CF}

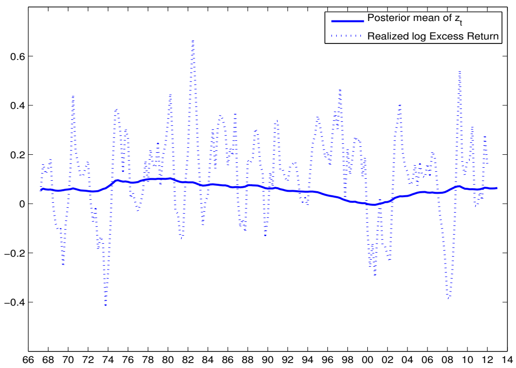
M_0 | M_{CF}

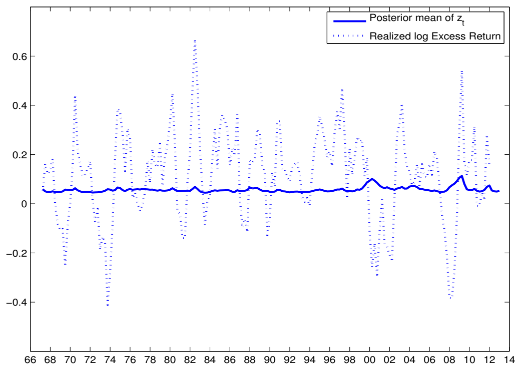
Dividend Growth ($R^2 = 15\%$) | Dividend Growth ($R^2 = -24\%$)





Excess Return ($R^2 = 4.8\%$) | Excess Return ($R^2 = 0.03\%$)





The plots compare the filtered series of expected dividend growths and excess returns (soild lines) with their realized values (dotted lines) under two specifications of risks. The upper two panels show the expected and realized log of dividend growth, while the lower two panels compare the expected and realized excess returns. The left two panels reflect estimations in the general model M_0, and the right panels represent the CF risk model M_{CF}. The R^2 is calculated by $1 - \frac{\text{var}(Realized\ Value - Expected\ Value)}{\text{var}(Realized\ Value)}$

reached its peak, but when *lnpd* was historically high around 2000, z_t was in the trough. According to nontabulated results, this correlation is -0.92. The CF factor g_t^M is less correlated with *lnpd*, and the correlation is about 0.44. In contrast, the two right panels tell a different story under M_{CF}. The estimates demonstrate predictability for dividend growth by *lnpd*, as the CF factor g_t^M becomes more correlated with *lnpd*. The correlation between g_t^M and *lnpd* is 0.96. And the expected return z_t covaries less with *lnpd*, with a correlation of 0.28. However, this is inconsistent with the established empirical findings.

1.3.1.2 Proposition 1.2: CF Risk at Individual Portfolio Level

In the following steps, I test the CF risk using data at the individual portfolio level. There are two reasons for employing the cross-sectional data. First, the risk is mostly reflected by cross-section of expected returns since different levels of risks the stocks bear result in different levels of awards in the expected returns. Second, bringing more information in relative pricing increases the sample size, therefore enhancing the power of the test.

Again, I begin with the classical statistical test. Proposition 1.2, associated with this scenario, hypothesizes that the DR and CF exposures have the same persistence and perfectly correlated shocks. Panel A of Table 1.3 reports the Gibbs estimates of Φ_β and Φ_γ and the correlation between their innovations $\rho_{\beta\gamma}$ under general model M_0. Overall, Proposition 1.2 is rejected by a joint χ^2_{70}-test, with a p-value of 0.001.

Taking a closer look at this proposition, I find the first part, $\Phi_\gamma = \Phi_\beta$, cannot be rejected using the t-test. Comparing the top left block with the middle left block in panel A of Table 1.3, one can see the means of Φ_γ and Φ_β

for the same test portfolio are almost identical. Moreover, the standard error of both estimates in the top and middle right blocks are large enough that this part of the proposition cannot be rejected with a t-test.

The second part of Proposition 1.2, $\rho_{\beta\gamma} = \pm 1$ gets rejected. In the bottom left block of panel A, I show that the correlations between the shocks of β and γ are significantly different from ± 1 for all portfolios, using a t-test. All $\rho_{\gamma\beta}$'s are around zero, among which the largest deviation from zero belongs to the second largest size portfolio (SIZE2), as 0.121. The standard errors of all the estimates in the bottom right block are below 0.4. As a result, ± 1 is outside of the two standard error boundary around the posterior mean.

Furthermore, the Bayes factor test also rejects Proposition 1.2. In panel B of Table 1.3, the LR statistics is 17.58 given all the test portfolios, indicating M_0 is more than 6000 times more likely to generate the cross-section of *lnpd* as observed. I also report the Bayes factors given the observation as portfolios within each sorting group and portfolios with different levels of returns; the model with CF risks can only weakly explain these data because the Bayes factors are larger than 6 in all cases.

At last, I show that CF risk is inconsistent with predictability of return and growth. Panel C of Table 1.3 compares the predictive R^2 for return and dividend growth under M_0 with R^2 under M_{CF}.[17] The predictive R^2 for returns of all test portfolios are around 2% under M_0 as shown in the top left block. This result is consistent with the predictability at market level. In contrast, as shown in the top right block, 21 out of the 35 test portfolios have a negative

[17] R^2 is calculated in the same way as for the case of the market portfolio. The expected return and dividend growth is calculated using CF and DR exposures and the corresponding market factors, as given by equation (1.12) and equation (1.13).

Table 1.3: Rejecting the CF Risk Model M_{CF} at Individual Level

Panel A: Classical Test

Joint χ^2 test of Proposition 1.2: $p_{\chi^2} = 0.001$

	$m(\Phi_\beta)$					$s(\Phi_\beta)$				
	Low Return	2	3	4	High Return	Low Return	2	3	4	High Return
SIZE	0.845	0.780	0.886	0.904	0.908	0.469	0.464	0.469	0.469	0.469
BM	0.876	0.832	0.914	0.903	0.865	0.473	0.467	0.475	0.474	0.468
MOM	0.831	0.830	0.716	0.776	0.789	0.454	0.456	0.453	0.455	0.459
VOL	0.868	0.858	0.809	0.842	0.867	0.448	0.458	0.460	0.469	0.474
ACC	0.763	0.700	0.683	0.671	0.725	0.452	0.447	0.450	0.448	0.446
CI	0.638	0.988	0.812	0.714	0.825	0.430	0.494	0.424	0.442	0.463
LIQ	0.895	0.903	0.912	0.930	0.906	0.479	0.469	0.470	0.475	0.464
	$m(\Phi_\gamma)$					$s(\Phi_\gamma)$				
	Low Return	2	3	4	High Return	Low Return	2	3	4	High Return
SIZE	0.851	0.764	0.866	0.895	0.901	0.474	0.461	0.476	0.474	0.468
BM	0.877	0.787	0.892	0.891	0.863	0.476	0.466	0.482	0.471	0.455
MOM	0.827	0.816	0.674	0.773	0.772	0.451	0.473	0.442	0.449	0.437
VOL	0.872	0.866	0.825	0.821	0.847	0.447	0.452	0.456	0.474	0.472
ACC	0.773	0.682	0.673	0.671	0.748	0.451	0.444	0.451	0.446	0.442
CI	0.643	0.929	0.909	0.721	0.833	0.435	0.464	0.455	0.449	0.459
LIQ	0.901	0.892	0.905	0.923	0.896	0.485	0.472	0.476	0.476	0.463
	$m(\rho_{\gamma\beta})$					$s(\rho_{\gamma\beta})$				
	Low Return	2	3	4	High Return	Low Return	2	3	4	High Return
SIZE	0.042*	0.140*	-0.062*	0.071*	0.119*	0.336	0.393	0.320	0.356	0.377
BM	0.121*	-0.038*	0.076*	0.058*	0.025*	0.351	0.280	0.384	0.368	0.358
MOM	0.060*	-0.038*	0.039*	0.063*	0.128*	0.362	0.287	0.359	0.387	0.368
VOL	0.080*	0.041*	0.008*	-0.010*	0.048*	0.354	0.357	0.355	0.306	0.381
ACC	0.109*	0.004*	-0.018*	0.123*	0.036*	0.390	0.320	0.309	0.382	0.375
CI	0.088*	-0.200*	0.036*	-0.012*	-0.010*	0.377	0.260	0.236	0.341	0.329
LIQ	0.002*	0.048*	0.104*	0.115*	-0.057*	0.322	0.354	0.367	0.396	0.308

Panel B: Bayes Factor Test

	ALL	Size	B/M	MOM	VOL	ACC
LR_{CF}	17.6	12.6	8.3	12.3	7.6	25.2
	CI	LIQ	High Ret	Med Ret	Low Ret	
LR_{CF}	21.7	10.5	14.0	7.3	33.2	

Panel C: Predictive R^2 in under M_0 and M_{CF}

R^2 for Return

	M_0					M_{CF}				
	Low Return	2	3	4	High Return	Low Return	2	3	4	High Return
SIZE	0.021	0.020	0.020	0.022	0.023	0.050	-0.025	-0.029	-0.003	-0.025
BM	0.017	0.020	0.024	0.026	0.019	0.046	-0.022	-0.023	-0.005	-0.022
MOM	0.017	0.022	0.022	0.023	0.022	0.049	-0.004	-0.008	0.061	-0.003
VOL	0.016	0.014	0.020	0.022	0.025	0.022	0.052	0.031	0.027	0.018
ACC	0.021	0.020	0.020	0.021	0.020	-0.025	0.050	-0.029	-0.033	-0.016
CI	0.020	0.022	0.026	0.020	0.022	-0.008	-0.025	-0.050	-0.020	-0.023
LIQ	0.023	0.021	0.021	0.030	0.029	0.045	-0.011	-0.016	0.050	0.050

R^2 for Dividend Growth

	M_0					M_{CF}				
	Low Return	2	3	4	High Return	Low Return	2	3	4	High Return
SIZE	0.137	0.051	0.066	0.068	0.032	-0.151	0.013	-0.044	-0.001	0.092
BM	0.040	0.041	0.039	0.062	0.048	0.010	0.074	0.048	0.022	0.060
MOM	0.018	0.015	0.055	0.031	0.015	0.058	-0.012	-0.079	-0.027	0.017
VOL	0.021	0.013	0.015	0.050	0.120	-0.009	-0.011	0.065	0.028	-0.071
ACC	0.037	0.034	0.022	0.057	0.006	0.012	-0.018	-0.095	-0.088	-0.079
CI	0.022	0.052	0.031	0.027	0.018	0.040	-0.002	-0.180	-0.059	0.049
LIQ	0.093	0.038	0.033	0.028	0.008	0.006	0.061	0.014	0.013	-0.023

Note to Table 1.3 I provide evidence that rejects M_{CF} using individual stock data from three aspects. The associated hypothesis is Proposition 1.2: the CF and DR exposure has the same VAR coefficients and their shocks are perfectly correlated.

$$\Phi_\gamma = \Phi_\beta; \quad \rho_{\beta\gamma} = \pm 1$$

Panel A reports the posterior mean $(m(\cdot))$ and standard error $(s(\cdot))$ of the pertinent parameters for the 35 test portfolios. For each parameter I also perform t-test on its specified hypothesis in the proposition and report the significance using asterisk ("*":$p < 0.01$).
Panel B reports Bayes factor as the log of the posterior likelihood ratio of the general model M_0 over the CF risk model M_{CF}, given various test data D.

$$LR_{CF} = 2\log\frac{Pr(M_0 \mid D)}{Pr(M_{CF} \mid D)}$$

Panel C compare the predictive R^2 under M_0 and M_{CF} at individual portfolio level, for return and growth. The R^2 for dividend growth of stock P is calculated by

$$R_g^2 = 1 - \frac{\text{var}(\Delta d_{t+1}^P - \mathbf{E}_t(\Delta d_{t+1}^P))}{\text{var}(\Delta d_{t+1}^P)}$$

The R^2 for excess return is calculated as

$$R_r^2 = 1 - \frac{\text{var}((R_t^{P,e}) - \mathbf{E}_t((R_t^{P,e})))}{\text{var}((R_t^{P,e}))}$$

predictive R^2 for returns under M_{CF}. For the dividend growth, the bottom left block shows predictive R^2 under M_0 ranges between 0.8% to 13.8%, while the R^2 values under M_{CF} in the bottom right block are mostly smaller and negative. The lack of predictability undermines the rationality of the expectations under the CF risks.

1.3.1.3 Proposition 1.3: DR risk at Individual Portfolio Level

Since there are no constraints associated with DR risk at market level, testing the DR risk with the data of individual portfolios is sufficient. The proposition characterizing DR risk requires the time-varying β to reflect the covariance between individual stock returns and the market returns. The results of classical test, Bayes factor test, and examination of predictability all support the DR risk.

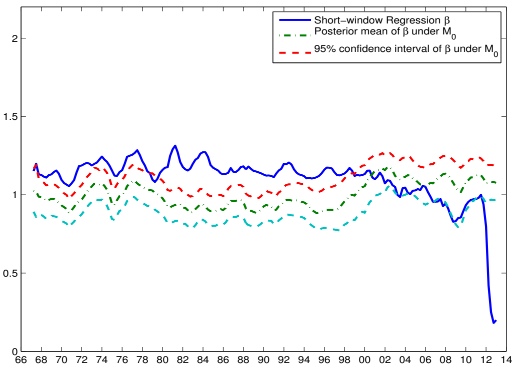
To test this proposition in the classical way, one needs to compare β estimated under M_0 with the time-varying benchmark of β. The benchmark is calculated using short-window regression of daily data as mentioned in section 1.1.3.

I find that this proposition cannot be rejected for all test portfolios. As representative illustrations, Figure 1.3 shows the comparison of the estimated β under M_0 and the benchmark for four test portfolios. The two left panels are for the cases of two extreme portfolios sorted by book-to-market ratio, and the two right panels are for the cases of two extreme portfolios sorted by past returns. The solid lines are the benchmark of conditional β calculated by rolling regression of daily returns. The dash-dotted lines represent the posterior mean of estimated β under M_0, and the dashed lines contour the boundary of

Figure 1.3: Estimates of Time-Varying *beta* under M_0 and Short-window Regression Benchmark under M_{DR}

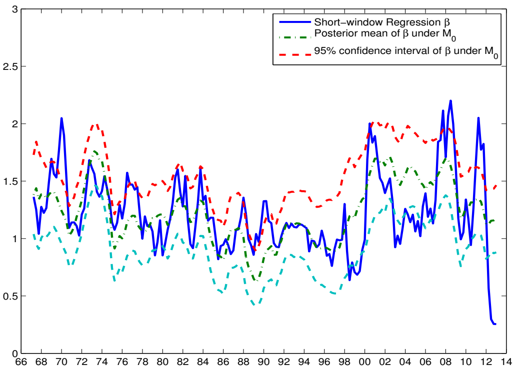
BM1 (Low Return)

$p_{\chi^2} = 0.99$



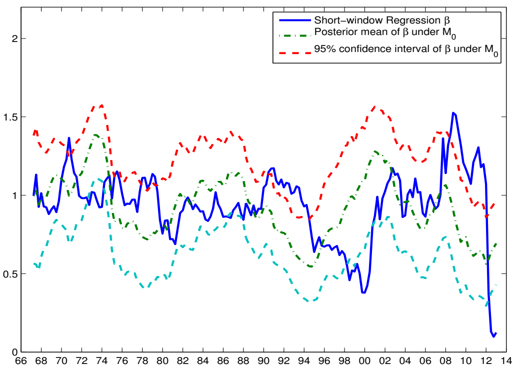
MOM1 (Low Return)

$p_{\chi^2} = 0.999$



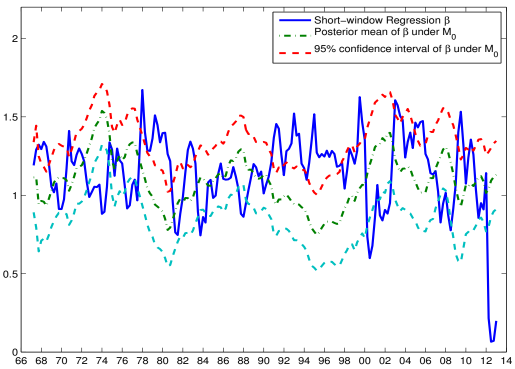
BM5(High Return)

$p_{\chi^2} = 0.999$



MOM5 (High Return)

$p_{\chi^2} = 0.475$



The plots compare the filtered series of *beta* under the general model M_0 with the Benchmark of *beta* under DR risk model M_{DR}. The Benchmark of M_{DR} (solid line) is calculated using short-window regression of daily individual stocks returns on daily market returns. The posterior mean of the time-varying *beta* (dash-dotted line) and its 95% confidence interval (dashed line) is estimated using Kalman filtering iterated in the Gibbs sampling method. The p-value is associated to the χ^2 test on the equality between point estimates of *beta* and the benchmark.

the 95% confidence interval of the posterior distribution of β. Although the correlations between posterior mean of β and the benchmark vary for different test portfolios, the 95% confidence intervals mostly enclose the benchmark of β for all panels. In addition, the p-values of a χ^2 test of the identity in the proposition are above 0.05 for all test portfolios.

The Bayes factor test on this proposition agrees that DR risk is an eligible risk channel given individual portfolio data. The test statistics of Bayes factors are reported in panel A in Table 1.4. Based on the data containing all test portfolios, the Bayes factor is -0.93, which slightly favors the model with DR risks. In the test using the seven portfolios with the highest/medium/lowest return in each sorting groups, the model with DR risks fits the data well, with negative Bayes factor in all three scenarios. When testing the model by sorting groups, four out of seven sorting groups have negative Bayes factors as evidence for the DR risk model M_{DR}. Moreover, the evidence against the DR risk model from the other three sorting groups is not strong, with all three LR statistics being less than 6.

Table 1.4: Supporting DR Risk Model M_{DR} at Individual Level

I provide evidence supporting M_{DR} with individual stock data. Panel A shows Bayes factor as the log of the posterior likelihood ratio of the general model M_0 over the DR risk model M_{DR}, given various test data D.

$$LR_{DR} = 2 \log \frac{Pr(M_0 \mid D)}{Pr(M_{DR} \mid D)}$$

Panel B reports individual portfolio level predictive R^2 for return and growth under M_{DR}. The R^2 for dividend growth of stock P is calculated by

$$R_g^2 = 1 - \frac{\text{var}(\Delta d_{t+1}^P - \mathbf{E}_t(\Delta d_{t+1}^P))}{\text{var}(\Delta d_{t+1}^P)}$$

The R^2 for excess return is calculated as

$$R_r^2 = 1 - \frac{\text{var}((R_t^{P,e}) - \mathbf{E}_t((R_t^{P,e})))}{\text{var}((R_t^{P,e}))}$$

Panel A: Bayes Factor Test

	ALL	Size	B/M	MOM	VOL	ACC
LR_{DR}	-0.93	2.0	0.3	3.0	5.6	0.1
	CI	LIQ	High Ret	Med Ret	Low Ret	
LR_{DR}	-1.2	-2.1	-2.8	-1.1	-1.2	

Panel B: Predictive R^2 under M_{DR}

	Predictive R^2 for Return					Predictive R^2 for Dividend Growth				
	Low Return	2	3	4	High Return	Low Return	2	3	4	High Return
SIZE	0.028	0.019	0.038	0.038	0.019	0.139	0.050	0.065	0.070	0.031
BM	0.030	0.027	0.022	0.025	0.025	0.041	0.041	0.038	0.062	0.047
MOM	0.025	0.025	0.029	0.049	0.083	0.016	0.015	0.055	0.030	0.014
VOL	0.014	0.017	0.017	0.023	0.028	0.019	0.013	0.015	0.051	0.120
ACC	0.022	0.023	0.024	0.027	0.024	0.037	0.034	0.022	0.057	0.006
CI	0.026	0.032	0.032	0.025	0.018	0.022	0.051	0.033	0.027	0.017
LIQ	0.037	0.021	0.020	0.020	0.025	0.098	0.037	0.033	0.030	0.007

In examining the predictability under M_{DR}, I find that the predictive R^2 for returns and dividend growth under M_{DR} is consistent with the existence of predictability. In panel B of Table 1.4, the R^2 values for returns range between 1.4% to 8.3%, and the R^2 values for growth range between 0.6% to 13.9% under M_{DR} across all test portfolios. They are similar to corresponding R^2 values under M_0, as reported in panel C of Table 1.3. Unlike CF risk, the DR risk can result in expectations that feature in predicting their realizations.

1.3.1.4 Variance Decomposition in *lnpd* under M_0

The result of this exercise demonstrates that DR dominates over CF in determining both dimensions of time-series and cross-sectional variations in *lnpd*, therefore it adds credit to the DR risk.

First, I decompose the time-series variance of $lnpd_t$ into fractions driven by CF and DR factors under the general model M_0, following a general approach in the literature.[18] I report the estimated fractions in $\text{var}(lnpd_t^P)$ of each state variable in Panel A of Table 1.5. The details of calculating the fractions are relegated to Appendix A.4.

[18] see Campbell (1991), Campbell and Vuolteenaho (2004), Chen and Zhao (2009), Binsbergen and Koijen (2010), and others.

Table 1.5: Variance Decomposition of Log Price-dividend Ratio

Panel A decomposes the time-series variance of *lnpd* and reports the percentage fractions of total variance driven by the state variables under M_0. AVG row presents the simple average of the fraction among all test portfolios. The "Min", "Med" and "Max" reports the minimum, maximum and median of the fractions across all the portfolios. The "High Ret", "Med Ret" and "Low Ret" report the median of the fraction among the portfolio with high, medium and low returns, respectively. Panel B decomposes the cross-sectional variance of *lnpd* and reports the fractions attributed to the influencing components,including average of CF and DR factors, conditional errors in CF and DR , and the covariance in CF and DR exposure with the aggregate factors. Column 1 is for the general model M_0, Column 2 is for the DR risk model M_{DR} and Column 3 is for the CF risk model M_{CF}.

Panel A: Decompose Time-series Variance of *lnpd*

	r^f	g	σ_g^2	z	σ_z^2	γ	β
AVG	6.4%	26.9%	0.0%	91.1%	0.1%	0.4%	1.0%
Max	10.1%	56.3%	0.0%	107.0%	0.1%	2.7%	6.5%
Median	6.2%	25.3%	0.0%	98.1%	0.1%	0.1%	0.3%
Min	2.4%	7.1%	0.0%	51.7%	0.0%	0.0%	0.0%
High Ret	5.9%	23.3%	0.0%	90.4%	0.1%	0.6%	1.0%
Med Ret	5.9%	27.4%	0.0%	91.6%	0.1%	0.5%	1.2%
Low Ret	6.1%	31.0%	0.0%	85.8%	0.1%	0.4%	1.4%

Panel B: Decompose Cross-sectional Variance of *lnpd*

	M_0	M_{DR}	M_{CF}
$\bar{\beta}$	81%	53%	31%
α	8%	6%	3%
$cov(\beta, z)$	22%	11%	11%
$cov(\beta, \sigma_z^2)$	-9%	-9%	-8%
$\bar{\gamma}$	-18%	-17%	-17%
c	-5%	-1%	-3%
$cov(\gamma, g)$	14%	10%	-36%
$cov(\gamma, \sigma_g^2)$	3%	2%	27%

The results take the DR factor as the more important driving force of movement in prices, and the market-level variation as more influential than the variation in portfolio specific exposures. In the first row, I show that, on average over all test portfolios, the market DR (z) explains 91.1% of the total variance of *lnpd*, dominating over the market CF (g^M), which accounts for a fraction of 26.9%. The fractions attributed to variation in CF and DR exposure (γ and β) are 0.4% and 1.0% in the total variance of *lnpd* on average. The exposure information is insignificant compared with market DR and CF factors, yet the DR exposure slightly outperforms the CF exposure. To show that the rank in explanatory power is not due to a special test portfolio, I report the maximum, median, and minimum of the estimated fractions over all test portfolios in row 2-4 of Panel A, as well as the decompositions using test portfolios with highest, medium, and lowest returns within each sorting group in row 5-7 in Panel A. The pattern of importance in attribution is consistent in all cases.

Second, one can cross-sectionally examine how the average of *lnpd* is affected by the average of factors. The average of *lnpd* depends on the average of CF and DR. Therefore, the cross-section variance is decomposed into variation in (1) the average of CF and DR exposures ($\bar{\gamma}$ and $\bar{\beta}$), (2) the conditional errors of CF and DR (c and α), and (3) the covariances describing the stability of the CF and DR exposures ($\text{cov}(\gamma, g^M)$, $\text{cov}(\gamma, \sigma_g^2)$, $\text{cov}(\beta, z)$, and $\text{cov}(\beta, \sigma_z^2)$). In Table 1.5, panel B reports the fraction of the cross-sectional variance of *lnpd* attributed to each component. The details of the calculation of the fraction are also relegated to Appendix A.4. As the covariances of these statistics depicting cross-sectional pricing are double calculated, the sum of these fractions is not necessarily equal to one.

In explaining the average of *lnpd*, the determinants on the DR side occupy the dominant roles. $\bar{\beta}$ accounts for 81% cross-sectional variation in *lnpd*. $\mathrm{cov}(\beta_t, z_t)$ explains a fraction of 22%, and α explains a fraction of 8%. On the other hand, the determinants on the CF side make a poor contribution to the variation in *lnpd*. The R^2 values explained by $\bar{\gamma}$ and c are negative. Yet $\mathrm{cov}(\gamma_t, \sigma_g^2)$ is an exception, contributing 14% to the cross-sectional variation in *lnpd*. These results are bases on the estimation from M_0, but they are robust and don't rely on the model used for estimation.

1.3.2 CF vs DR: Which is More Important in Determining Realized Returns

Given the DR risk as the more likely risk channel being priced in the expected return, it is natural to investigate the following ideas: (1) how the expected and unexpected components together drive the realized return in time series, and (2) whether the cross-section of average expected return can fully justify the cross-section of average realized return.

To answer these two questions, I decompose the time-series and cross-sectional variance in realized returns using the same method that was applied to *lnpd* in Section 1.3.1.4. Recall that the realized return is the sum of expected return and unexpected DR and CF innovations as expressed in equation (1.19). The variation in realized return is therefore due to the state factors affecting expected returns, as well as the DR and CF innovations.

In the time-series variance dimension, I argue that the unexpected CF plays the largest role in driving the realized returns. Panel A of Table 1.6 shows the fractions of time-series variance of returns attributed to the changes in expected state factors and unexpected DR and CF innovations under M_{DR}, as the DR risk is highlighted in the last section. The weights of expected DR and CF factors are dwarfed by the unexpected DR and CF innovations. According to the first row of panel A, the variation in z only accounts for 25.9% of the total variance of return on average, and g^M only accounts for 12.5%. In contrast, the CF innovation explains 59.1% on average, and the DR innovation explains 10.6% of the total variance in returns. To make sure these results are not due to only a few test portfolios, I provide a robustness check in the other rows of panel A.

Table 1.6: Variance Decomposition of Return

Panel A reports statistics of the percentage of variance in realized return driven by the state variables (covariances between state variables) over test portfolios under M_{DR}. AVG row presents the simple average of each fraction among all test portfolios. The "Min", "Med" and "Max" reports the minimum, maximum and median of each fraction across all the portfolios. The "High Ret", "Med Ret" and "Low Ret" report the median of each fraction among the portfolio with high, medium and low returns, respectively. Panel B reports the fractions of the cross-sectional variance of realized return attributed to the determinants, including average of CF and DR factors, conditional intercept of CF and DR , the covariance in CF and DR exposure with the aggregate factors, and average of DR and CF shocks. Column 1 is for the DR risk model M_{DR}, Column 2 is for the general model M_0 and Column 3 is for the CF risk model M_{CF}.

Panel A: Time-series Variance Decomposition

	r^f	g	σ_g^2	z	σ_z^2	γ	β	I_{DR}	I_{CF}
AVG	7.4%	12.7%	0.0%	25.7%	0.1%	0.2%	0.5%	10.6%	59.1%
Max	16.6%	41.0%	0.2%	40.3%	0.2%	1.3%	2.5%	17.1%	91.7%
Median	7.1%	11.7%	0.0%	25.8%	0.1%	0.0%	0.2%	10.3%	59.5%
Min	0.5%	3.3%	0.0%	2.5%	0.0%	0.0%	0.0%	2.3%	32.9%
High Ret	7.6%	9.8%	0.0%	23.4%	0.1%	0.3%	0.5%	9.4%	63.2%
Med Ret	9.0%	11.6%	0.0%	28.3%	0.2%	0.1%	0.2%	10.0%	59.5%
Low Ret	9.1%	18.6%	0.0%	26.4%	0.1%	0.1%	0.3%	12.6%	54.8%

Panel B: Cross-sectional Variance Decomposition

	M_{DR}	M_0	M_{CF}
$\bar{\beta}$	-7.5%	-5.4%	-6.6%
α	-1.4%	-1.9%	-1.01%
$cov(\beta, z)$	22.2%	24%	19.2%
$cov(\beta, \sigma_z^2)$	14.0%	8.6%	7.7%
$\bar{\gamma}$	-2.5%	-3.1%	-2.3%
c	10.0%	13.2%	4.9%
$cov(\gamma, g)$	5.6%	6.4%	8.5%
$cov(\gamma, \sigma_g^2)$	0.3%	0.3%	0.5%
$\overline{I_{DR}}$	0.2%	0.5%	0.3%
$\overline{I_{CF}}$	64.6%	56.2%	55.2%
Total	81.2%	80.8%	78.2%

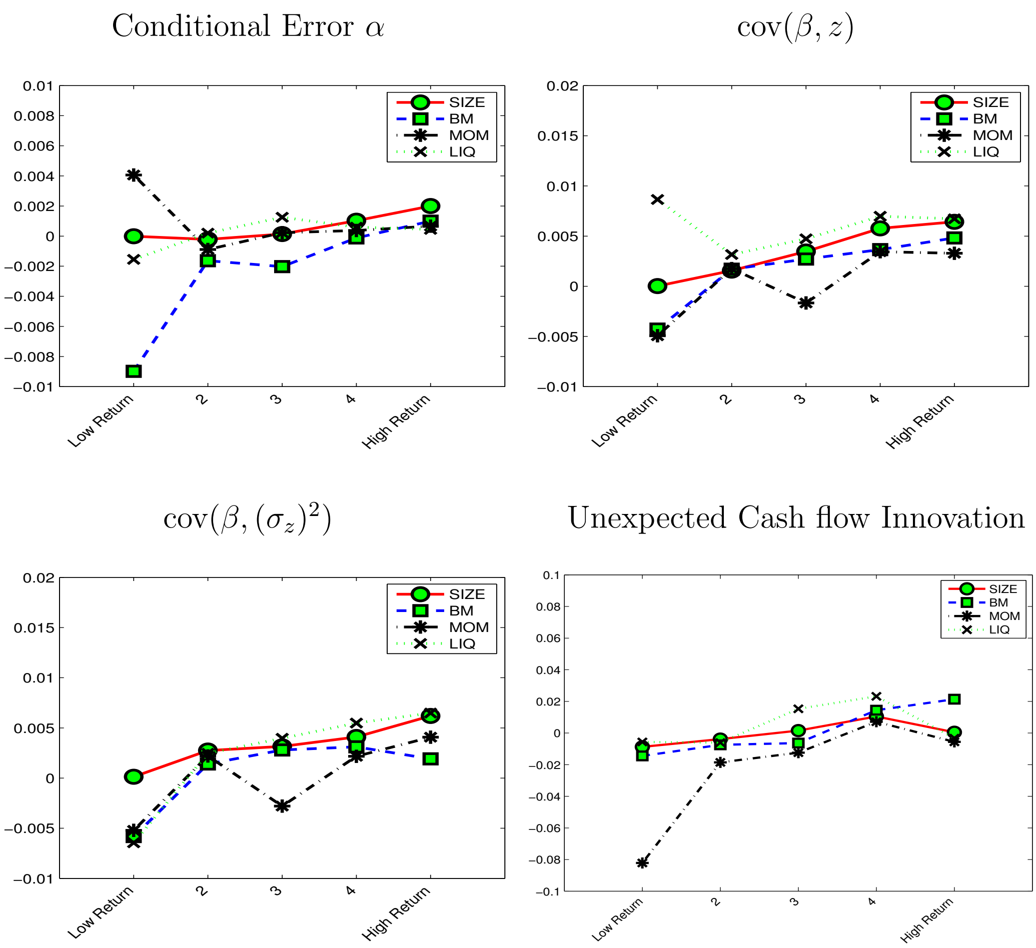
In the cross-sectional dimension, the results demonstrate that the cross-section of average expected returns, which are explained by the conditional risks, contribute to but are not sufficient to explain the cross-section of average realized returns. When estimating the model, I pin down the cross-sectional averages of DR and CF with the information about *lnpd* only. By doing so, I relax the assumption of rational expectation for returns and growth. Therefore, the averages of unexpected DR and CF innovations may not be zero, and a large fraction of average return spread is due to the average of unexpected CF. Bringing expected and unexpected components together, I can readily explain the cross-section of average realized returns.

Before jumping to the decomposition results in detail, I first exhibit the cross-sectional patterns of the determinants affecting the level of realized returns to intuit some meanings. In Figure 1.4, I display the cross-sections of these determinants under the DR risk model. The patterns are robust if one chooses M_0 or even M_{CF}.

The upper left panel plots the cross-section of the conditional DR error α. I choose four representative sorting groups as SIZE, B/M, MOM, and LIQ to show the patterns. For all portfolios, α is very small, mostly under 0.5% annually, and the cross-sectional pattern is flat, hardly explaining any dispersions in returns.

The upper right and the lower left panels in Figure 1.4 exhibit $\text{cov}(\beta, z^M)$ and $\text{cov}(\beta, (\sigma_z^M)^2)$. For all four sorting groups, the cross-sections of $\text{cov}(\beta, z^M)$ and $\text{cov}(\beta, (\sigma_z^M)^2)$ are in line with the patterns of cross-sectional returns. The portfolios with higher excess returns also have higher covariance terms describing the stability of β. This relation is consistent with the conditional CAPM

Figure 1.4: Determinants of Cross-sectional Returns under M_{DR}



The figure shows cross-sectional patterns of the statistics that affect the cross-sectional average realized returns. I choose four sorting groups as representatives. All the statistics are obtained under DR risk model M_{DR}. α is the conditional intercept as the pricing error in the expected return. $\text{cov}(\beta, z)$ and $\text{cov}(\beta, (\sigma_z)^2)$ characterize the stability of time-varying β. The unexpected cash flow is the average of the difference between realized CF and the expected CF.

literature, such as Jagannathan and Wang (1996). However, quantitatively, the covariance terms are still too small to fill the gap.

Lastly, the lower right panels show the average of unexpected CF innovations ($\overline{I_{CF}}$), which is calculated as the difference between the level of realized and expected dividend growth. One can see that the patterns of $\overline{I_{CF}}$ are in line with the realized returns for all sorting groups. For example, $\overline{I_{CF}}$ for the value portfolio (BM5) is higher than that for the growth portfolio (BM1) by 4 percentage points. For momentum sorted portfolios, the spread of $\overline{I_{CF}}$ between past winner (MOM5) and past loser (MOM1) is 8%. The quantity scale is large enough to potentially match the spreads in the unconditional *alpha* (α^u) shown in the summary Table 1.1.

The decomposition results substantiate the power of unexpected CF in explaining the spreads in realized returns. Panel B of Table 1.6 reports the fractions in the cross-sectional variation in realized returns explained by each determinant. The first column shows results under M_{DR}. The largest portion, a fraction of 64.6%, of the cross-sectional variance of returns is attributed to $\overline{I_{CF}}$. The covariance terms reflecting the stability of market beta determine the cross section of ex-ante expected return, hence realized return, as motivated by conditional CAPM: $\text{cov}(\beta, z)$ accounts for 22.2%, and $\text{cov}(\beta, \sigma_z^2)$ accounts for 14% of the cross-sectional variance of returns. The component influencing the expected CF c and $\text{cov}(\gamma, g^M)$ also contribute to the returns dispersion, but in a weaker way, accounting for 10% and 5.6% of the total cross-sectional variance of return, respectively. The levels of DR exposure $\bar{\beta}$ and CF exposure $\bar{\gamma}$, however, do not explain the return spreads. Taking all the factors into account, 81.2% of the cross-sectional average returns can be justified. These results are robust to the model I choose, and one can observe similar results in columns 2 and 3 in the table.

In summary, the realized returns reflect not only the variations in expectation, but the variations outside expectation as well. These unexpected variations are primarily attributed to the CF innovations. The unexpected CF innovations drive the realized returns' time-series variations. Furthermore, the overall level of the unexpected outcomes in dividend distribution is a key aspect affecting the average of realized returns.

1.4 Conclusion

This paper presents a general present-value model that treats CF and DR factors as time-varying and latent. Assuming the exogenous VAR(1) dynamic for these factors, I develop closed-form formulae linking the state factors with the observable price–dividend ratios, realized returns and dividend growths. Estimating the model using Gibbs sampling method on portfolios capturing the most common anomalies, I provides a comprehensive analysis on the time-series and cross-sectional variations of stock returns.

First, the model can make a judgment of the likelihood of CF risk and DR risk. Both the model characterized by CF risk, as in Bansal and Yaron (2004), and the model characterized by DR risk, as in Campbell and Cochrane (1999), can nest as a special case of the model by imposing constraints on the pricing kernel. I argue that the DR risk dominates according to the Bayes factor test constructed by the estimates. Moreover, under the model dominated by DR risk, the expected return and dividend growth display predictability in the realizations.

Second, since the DR risk model is a version of conditional CAPM, I find the conditional CAPM can explain well the cross section of ex-ante expected return. The dispersions in the ex-ante expected return are due to the level and stability of time-varying market beta.

Third, the ex-ante return is not sufficient to justify the ex-post realized return. The difference is mainly due to unexpected CF shocks. Integrating both expected and unexpected variations, the model can explain 80% of the dispersions in realized excess returns cross-sectionally.

A future possible extension from this paper is to apply its reduced-form model to evaluate liquidity effects by including time-varying and latent liquidity factors. The liquidity effects may be a breakout in explaining asset prices, especially those in an illiquid market.

Chapter 2

Liability Driven Investment under Downside Risk

2.1 Introduction

The fall in both interest rates and equity prices during the global financial crisis over 2007-2008 took a large toll on pension funds. Despite the rebound in asset values since 2009, funding ratios (asset values / projected benefit obligations) have not yet rebounded. In 2011, the funded status of the top 100 U.S. companies with the largest defined benefit pension assets was around 80% compared to above 100% in 2007 and was approximately 400 billion dollars lower than at year-end 2007.[1] The average funding ratios were close to 80% for these funds over 2009-2011. These downside risks are costly both for corporations, which need to pay higher insurance premiums, hold higher reserves, and transfer money to pension plans that could be used for other investments, and beneficiaries, who must bear higher default risk which is often highly correlated

[1] See Milliman 2012 Pension Funding Study.

with their main source of labor income.[2]

We present a liability-driven investment (LDI) approach to take into account downside risk. The approach is different from the surplus management approach developed by Sharpe and Tint (1990), Ezra (1991), and Leibowitz and Kogelman and Bader (1992) as we include a penalty term associated with not meeting liabilities.[3] Downside risk has a large effect on pension investments: optimal asset allocation is affected both by a current shortfall, when the value of the pension assets falls below the liabilities today, but also by the risk of a potential shortfall in the future. We show that the shortfall penalty can be valued as an option to exchange the optimal portfolio for the random value of the liabilities. The optimal portfolio, however, must be solved *simultaneously* with the value of the option. When the liabilities exceed the assets, the option is in the money. A cost factor parameter which multiplies the value of the exchange option in the manager's utility function can be interpreted as a downside risk aversion parameter. As the cost factor decreases to zero, the standard mean-variance framework holds.

The downside risk we address here is the failure of meeting liability. This is different from just taking into account liabilities in standard Sharpe-Tint surplus management. There are significant penalties in failing to meet liabilities in the real world. The 2006 Pension Protection Act requires that plan funding should equal 100% of the plan's liabilities. Sponsors of severely underfunded plans are required to fund their plans according to special rules that result in

[2]See, for example, Rauh (2006) for evidence that higher than expected contributions to pension plans reduces firm investment and Poterba (2003) on the excessive concentration of employer stock in pension plans.

[3] Some notable contributions in the area of optimal asset allocation of pension funds are Sundaresan and Zapatero (1997), Rudolf and Ziemba (2004), and van Binsbergen and Brandt (2009).

higher employer contributions to the plan. In addition, FAS 158, implemented in 2006, requires plan sponsors to "flow through" pension fund deficits into their financial statements. These have real impacts on earnings and stockholders' equity. A case study is AT&T whose funding status changed from $17 billion surplus in 2007 to a nearly $4 billion dollar deficit in 2008. This played a role in the decline of AT&T's equity from 2007 to 2008.

Taking into account downside risk leads to *endogenous* risk aversion. The funding ratio affects the likelihood of the assets being sufficient to cover the liabilities in the future, which affects the option value. There are pronounced non-linear effects of the funding ratio on risk taking. The fund manager's risk aversion peaks when the plan is close to fully funded. As the funding ratio deviates from the fully funded position, risk aversion decreases. An under-funded plan investment manager displays much lower risk aversion than the manager of a fully funded plan leading to a "swing for the fences" effect. If the fund is poorly funded, then only by taking on risk can the manager hope to avoid the shortfall. Managers of over-funded plans also act in a less risk averse manner because they can afford to take on more risk as the probability of the option being exercise falls as the funding ratio increases.

Our framework of LDI with downside risk is related to a portfolio choice literature that specifies drawdown constraints, such as Grossman and Zhou (1993) and Chekhlov, Uryasev and Zabarankin (2005). Constraints that capture shortfall risk have also been employed in surplus optimization problems by Leibowitz and Henriksson (1989), Jaeger and Zimmermann (1996), Amenc et al. (2010), and Berkelaar and Kouwenberg (2010). These approaches do not directly take into account the downside risk in the utility function of the manager, but instead specify a constraint that the surplus or the portfolio needs to satisfy. This constraint is usually that the surplus or portfolio return must be above a certain threshold with some probability. As an extension of the Sharpe and Tint (1990) to a dynamic setting, Detemple and Rindisbacher (2008) allow for a fund sponsor to exhibit aversion over a shortfall when a plan terminates. Their shortfall has a utility cost, whereas ours has an actual real-world value through an option calculation. Our shortfall cost is determined simultaneously with the optimal portfolio.

In addition, our research belongs to the literature studying the pension benefit guarantees as contingent claims. Bodie (1990) identifies the shortfall payoff be a put option payoff. Rudolf and Zimmerman (2001) evaluate this shortfall risk as a function of the underlying maturity mismatch. Steenkamp (1998) examines the liability of pensions based on this assumption of shortfall risk being valued as put option. Our work integrates this idea into the portfolio management with shortfall risk, and obtaining endogenous optimal strategies.

2.2 Model

LDI models treat fund liabilities as a state variable and specify an objective function of assets relative to liabilities. The investor takes into account the correlation between the liabilities and assets in determining the optimal portfolio allocation. We start by reviewing the simple LDI model of Sharpe and Tint (1990). Then we present our model with downside risk and show how to value the shortfall risk as an option.

2.2.1 Sharpe and Tint (1990)

Sharpe and Tint (1990) define surplus, S_t to be $S_t = A_t - L_t$, where A_t represents the plan's market value of assets and L_t is the value of the liabilities. Normalizing by the assets at the beginning of the period, we can define the surplus return over assets, z, as

$$z = \frac{S_1}{A_0} = \frac{A_1 - L_1}{A_0} = \left(1 - \frac{L_0}{A_0}\right) + \left(r_A - \frac{L_0}{A_0} r_L\right), \tag{2.1}$$

where $r_A = A_1/A_0 - 1$ is the return on assets, $r_L = L_1/L_0 - 1$ is the return on the liabilities, and L_0/A_0 is the inverse of the funding ratio.

The objective function is mean-variance over the surplus return:

$$\max_w \mathrm{E}(z) - \frac{\lambda}{2} \mathrm{var}(z), \tag{2.2}$$

where w is the portfolio of risky assets and λ is (standard) risk aversion in the mean-variance context. Sharpe and Tint show that this problem is equivalent to

$$\max_w \mathrm{E}(r_A) - \frac{\lambda}{2} \mathrm{var}(r_A) + \lambda \, \mathrm{cov}(r_A, r_L), \tag{2.3}$$

which emphasizes that the correlation of the liabilities with the asset returns influences the optimal portfolio holdings.

If the assets are uncorrelated with the liabilities, $\mathrm{cov}(r_A, r_L) = 0$, then the surplus problem in equation (2.3) is the standard mean-variance portfolio weight. Sharpe-Tint LDI does take into account downside covariance of assets and liabilities but it does this symmetrically with upside covariance through the $\mathrm{cov}(r_A, r_L)$ term. In our formulation, we will explicitly penalize *only* shortfall loss.

2.2.2 Liability Driven Investment with Downside Risk

We now introduce a LDI framework to include a penalty if the manager fails to meet the liability of the fund.

Asset Returns

We assume that the portfolio managers can only allocate wealth between two assets, risky equities (E) and risk-free bonds or cash (B). We analyze the case of risky bonds in the appendix and also investigate asset allocation over risky equities and bonds in our empirical calibration. We denote the liabilities by L.

We assume that equities are log normally distributed and and denote the risk-free rate as r_f:

$$\begin{aligned} B_1 &= B_0 \exp(r_f) \\ E_1 &= E_0 \exp\left((\mu - \frac{\sigma_E^2}{2}) + \sigma_E \varepsilon_1^E \right), \end{aligned} \tag{2.4}$$

where $\varepsilon_1^E \sim N(0,1)$. We also assume that liabilities are log normally distributed:

$$L_1 = L_0 \exp\left((\mu_L - \frac{\sigma_L^2}{2}) + \sigma_L \varepsilon_1^L \right), \tag{2.5}$$

where $\varepsilon_1^L \sim N(0,1)$. The correlation between the equity and liability shock is ρ.

Liability Shortfall

Following Sharpe and Tint (1990), we work in a one period setting and assume the asset weights are set at the beginning of the period, which we interpret as one year. The value of the assets at time 0 is denoted as A_0. The asset payoff at time 1 is a function of the equity weight, w:

$$A_1 = wA_0 \exp\left((\mu - \frac{\sigma_E^2}{2}) + \sigma_E \varepsilon_1^E\right) + (1-w)A_0 \exp(r_f). \tag{2.6}$$

Note that w is chosen at time 0.

The value of the shortfall is a put option on the terminal value of the assets at a strike price of L_1, which is unknown at time 0. The payoff of this option is

$$\max(L_1 - A_1, 0),$$

where L_1 and A_1 are given in equations (2.5) and (2.6), respectively. We denote the value of this option as $P(w, L_0, A_0)$. The notation emphasizes that the downside risk depends on the original funding level given by L_0 and A_0 and it also depends on the asset allocation policy, w, chosen by the fund manager.

Downside Liability Risk

We specify the objective function of the fund as mean-variance over asset returns plus a downside risk penalty on the liability shortfall:[4]

$$\max_{w} \mathrm{E}(r_A) - \frac{\lambda}{2}\mathrm{var}(r_A) - \frac{c}{A_0}P(w, L_0, A_0), \tag{2.7}$$

where c is a penalty cost associated with the downside risk. The parameter c can be interpreted as a downside risk aversion parameter in the context of shortfall loss. We scale the funding cost by assets to keep everything on a per dollar return metric. As the shortfall risk increases on the downside, the investor's utility is decreased. It is important to note that the option price, P, is the value of the shortfall risk at time 0; it is the value the fund manager would pay today to insure against the shortfall risk tomorrow.

The standard Sharpe-Tint (1990) LDI framework recognizes the fact that the correlation of assets with liabilities plays a role in driving optimal asset allocation. Our objective function (2.7) for LDI with downside risk replaces the Sharpe-Tint $\lambda\,\mathrm{cov}(r_A, r_L)$ term with a shortfall penalty term, cP/A_0. The value of the option is *endogenous* as the fund manager can reduce the value of this option by increasing the correlation of the optimal portfolio with the pension liabilities. Thus, the value of the insurance must be computed *simultaneously* with the optimal portfolio choice.

It is possible to extend equation (2.7) to include an additional term $\lambda\,\mathrm{cov}(r_A, r_L)$. This would then nest the traditional Sharpe-Tint (1990) surplus optimization

[4]This defines the downside risk limit as a funding ratio of 100%. The 2006 Pension Protection Act (PPA) aims at a minimum funding ratio of 100% and in cases of underfunding requires shortfalls be amortized and increases in contributions over certain horizons. Other countries, however, have other minimum levels of funding such as the Netherlands where the minimum funding level is 105%. Plan sponsors may consider other terminal funding levels for defining the strike of the option. It is also possible to extend the methodology to introduce a second option with an additional penalty at another strike. For example, under the PPA a fund is deemed "at-risk" and subject to onerous restrictions and steep contribution increases if the funding ratio falls below 80%.

(see equation (2.3)). We do not take this route because by including the $\text{cov}(r_A, r_L)$ term, we "double count" the effect of downside risk in both the covariance and the put option term. We purposely highlight the shortfall penalty in equation (2.7) to distinguish it from traditional Sharpe-Tint analysis in our calibrations below.

In their appendix, Amenc et al. (2010) in their appendix consider the related problem

$$\max_w \mathrm{E}\left[\frac{A_1}{L_1}\right] \quad \text{such that } A_t \geq kL_t \text{ for all } t,$$

which is similar to equation (2.7) in that the portfolio problem also involves an option. A major difference is that in our formulation the option valuation endogenously depends on the optimal portfolio strategy, and the optimal strategy simultaneously depends on the cost of the shortfall risk. In Amenc et al., the option value is stated *exogenously*.

2.2.3 Valuing the Shortfall Risk

The value of the shortfall risk is a put option. The shortfall process, however, does not follow a log-normal process and so we cannot use standard methods to value it. We can interpret the shortfall option as a spread option due to the stochastic evolution of both pension assets and pension liabilities. While the literature on spread options has not found a closed-form solution for valuation, there are very accurate approximations. The appendix shows how we can value the spread option following Alexander and Venkatramanan (2011) by representing the spread option as two compound exchange options.

The important economic concept is that the option value *endogenously* depends on the portfolio chosen by the pension plan. As the correlation between portfolio and the liabilities increases, the likelihood that there will be a shortfall decreases and the value of the option decreases. As the volatility increases, the option value increases. This property implies that when the penalty costs associated with downside risk increase, the optimal portfolio becomes more defensive. The correlation of the liabilities with the asset returns influences the portfolio weights, just as in a standard LDI problem, but the comovement of liabilities with assets also directly affects the risk of not meeting the liability schedules. As the penalty costs of downside risk increase (c increases), these effects dominate the standard Sharpe-Tint (1990) LDI effects.

2.2.4 Optimal Portfolios

Taking first order conditions of (2.7), we can solve for the optimal portfolio weight as

$$w^* = \frac{1}{\lambda}\sigma_E^{-2}\left[(\mu - r_f) - \frac{c}{A_0}P_w\right], \tag{2.8}$$

where $P_w = \partial P(w, L_0, A_0)/\partial w$. Note that the option value depends explicitly on the portfolio weight chosen by the pension fund manager. Thus, equation (2.8) implicitly defines both the optimal portfolio weight and the cost of the shortfall insurance.

Clearly if there is no downside penalty ($c = 0$), then the standard mean-variance efficient portfolio weight arises as a special case. If $c \to \infty$, the manager cares only about hedging the liability and does not care about mean-variance performance. As a result, we can define the liability hedging portfolio as

$$w^{LH} = \arg\max_{w} \; -\frac{1}{A_0}P(w, L_0, A_0). \tag{2.9}$$

Let us assume the time interval is small so that a log normal approximation holds. Then a Margrabe (1978) exchange option formula can be used to value the shortfall put option. This provides some intuition. In this case, the value of the exchange put option can be approximated by

$$P(w, L_0, A_0) = L_0 N(d_1(w)) - A_0 N(d_2(w)), \tag{2.10}$$

where $N(\cdot)$ represents the normal cumulative density function, the parameters d_1 and d_2 are given by

$$\begin{aligned} d_1(w) &= \frac{\ln(L_0/A_0) + \Omega^2(w)/2}{\Omega(w)} \\ d_2(w) &= \frac{\ln(L_0/A_0) - \Omega^2(w)/2}{\Omega(w)}, \end{aligned}$$

where

$$\Omega(w) = \sqrt{w^2\sigma_E^2 - 2w\rho\sigma_E\sigma_L + \sigma_L^2}$$

is the volatility of the portfolio relative to the liability.

Using the Margrabe (1978) approximation of the shortfall option in equation (2.10), we can write the optimal portfolio weight in equation (2.8) as a weighted average of the mean-variance efficient portfolio and the liability-hedging portfolio:

$$w^* = \frac{\lambda}{\lambda + \frac{cP_\Omega}{A_0\Omega}} w^{MV} + \frac{\frac{cP_\Omega}{A_0\Omega}}{\lambda + \frac{cP_\Omega}{A_0\Omega}} w^{LH}, \tag{2.11}$$

where

$$P_\Omega = A_0 n(d_2)$$

is the vega of the exchange option and $n(\cdot)$ is the normal probability density function. Note that from the chain rule, $P_w = P_\Omega \frac{\partial \Omega}{\partial w}$.

Equation (2.11) illustrates the trade-off in the optimal portfolio weight between the mean-variance performance seeking target and the liability hedge. When the downside risk penalty is very large (c is large), the liability hedging demand dominates in the optimal portfolio and the pension plan moves towards the liability hedging portfolio. When the shortfall penalty is close to zero, the optimal portfolio is just the mean-variance portfolio.

The weight placed on the liability-hedging portfolio is

$$\theta = \frac{cP_\Omega}{A_0 \Omega}.$$

The funding ratio, A_0/L_0, enters the vega of the exchange option through d_2 and therefore has a large influence on the liability hedging demand. It is easy to show that the weight on the liability-hedging portfolio increases with the downside risk penalty, c:

$$\frac{\partial \theta}{\partial c} = \frac{n(d_2)}{\Omega} > 0$$

. We can also show that

$$\frac{\partial \theta}{\partial (A_0/L_0)} = \frac{cn(d_2)}{A_0/L_0 \Omega^4}$$

and setting $\partial\theta/\partial(A_0/L_0) = 0$, we obtain a maximum value when $(A_0/L_0)^* = \exp(-\Omega^2/2)$. On either side of the maximum, θ declines with the funding ratio.

2.2.5 Endogenous Risk Aversion

In the mean-variance efficient portfolio, the risk aversion is λ. When the LDI objective function embodies downside risk ($c > 0$), there is an effective overall increase in risk aversion, which is seen in the denominator in the term $1/(\lambda + \frac{cP_\Omega}{A_0\Omega})$ in equation (2.11). With downside risk in equation (2.8), the risk aversion has increased by $\theta = (cP_\Omega)/(A_0\Omega)$. Effective risk aversion increases as the downside penalty, c, increases and there is a *nonlinear* relation between the funding ratio, A_0/L_0, and risk aversion.

Suppose the funding ratio is very high so that the option value is close to zero. Mathematically this makes $n(d_2) \approx 0$ in equation (2.11). The plan's assets are very far from liabilities, and so the pension fund manager effectively ignores the liabilities in setting the asset allocation policy – which is the mean-variance efficient portfolio, $w^* = w^{MV}$. Thus, for very high funding ratios, there will be little effect of downside risk even if the penalty c is large. On the other hand, when the funding ratio is around one, and the option value is most sensitive to the volatility, the investor wishes to avoid the shortfall risk. This pushes the investor to place a larger weight on the liability-hedging portfolio when the funding ratio is close to unity.

Interestingly, when the funding ratio is far below one, the liability hedging portfolio also becomes less important. In these regions, the option value is not sensitive to the volatility because it is deep in the money. This causes the optimal portfolio weight to move towards the mean-variance efficient portfolio. Intuitively, as the funding ratio decreases, the pension fund manager has an incentive to "swing for the fences" to avoid the shortfall.

2.3 Empirical Application

Our calibrations are intended to convey intuition; they are deliberately simple. We start with a cash and equity case because all the intuition can be conveyed in this benchmark case. We then consider a two-asset case of equities and bonds without a risk-free asset to show that the intuition carries over to the setting where there are portfolio constraints. This is also a more realistic case. We leave to further research the more complex portfolios held in industry which include alternative asset classes.

We present results for a range of shortfall costs represented by the parameter c. Just as the Sharpe-Tint (1990) setting does not micro-found the risk aversion in the objective function (the parameter λ), we do not discuss how the downside risk parameter, c, should be selected. How each fund selects its own set of parameters (c, λ) is beyond the scope of this article. The costs of downside risk, however, are real. Rauh (2006), for example, shows that higher than expected contributions to pension plans reduce firm investment in profitable business opportunities. These developments have dramatically increased the downside costs for sponsors of corporate pension plans.

2.3.1 Parameters

We take the S&P 500 total return index to represent equity, and the Ibbotson U.S. long-term corporate bond total return index to represent the bond fund. We sample these at a monthly frequency from January 1952 to December 2011. Over this sample, the mean bond and equity returns are 6.9% and 11.0%, respectively, with volatilities of 8.6% and 14.7%, respectively. We set the risk-free rate to be 4%. Following Leibowitz, Kogelman and Bader (1992)

Table 2.1: Data Summary Statistics

	Mean	Volatility	Correlations		
			Bond	Equity	Liability
Bond	6.92%	8.60%	1.00		
Equity	11.04%	14.69%	0.25	1.00	
Liability	6.92%	10.00%	0.98	0.35	1.00
Risk-Free	4.00%				

The table reports annualized expected returns, volatilities, and correlations of bonds and equities, which are total returns of the Ibbotson U.S. Long-Term Corporate Debt Index and the S&P 500 Index between January 1952 and December 2011. These are monthly frequency series and we annualize the means and volatilities by multiplying the monthly frequency mean and volatility by 12 and $\sqrt{12}$, respectively. Parameters for the liability and the risk-free rate are set by assumption and follow closely those set by Leibowitz, Kogelman and Bader (1992) and Jaeger and Zimmermann (1996).

and Jaeger and Zimmermann (1996), we assume that the liability has the same expected return as the bond fund and set the volatility of the liability to be slightly higher than the volatility of the bond fund at 10%. We also follow Leibowitz, Kogelman and Bader and set the correlation of the liabilities with bonds and equities to be 0.98 and 0.35, respectively.[5] We report these summary statistics, which we use in our calibrations, in Table 2.1. In all our calibrations we assume a horizon of one year.

2.3.2 Cash and Equities

We first assume that the pension plan allocates between a risk-free asset and equities.

[5] Leibowitz, Kogelman and Bader (1992) assume the correlation between the pension liability and bonds to be 1.00, but we set it at 0.98 as liabilities of pension funds are generally not tradeable and cannot be hedged perfectly. They also incorporate longevity and other risks as well as credit spread risk.

We set the risk aversion coefficient, λ, by taking the value of risk aversion such that the mean-variance efficient portfolio consists of a 60% equities/40% risk-free bond portfolio. This turns out to be $\lambda = 5.88$.

2.3.2.1 Comparison with Mean-Variance and Sharpe-Tint LDI

Table 2.2 reports the optimal portfolio weights held in equities in the mean-variance efficient portfolio, the Sharpe-Tint (1990) LDI portfolio, and our LDI portfolio that takes into account the downside shortfall risk. We assume a fully funded portfolio, $A_0/L_0 = 1$. By assumption of the value of λ, we start with a mean-variance efficient portfolio of 60% equities. The Sharpe-Tint portfolio places more weight on equities, at 84%, than the mean-variance portfolio because the liability is positively correlated with equities. Thus, equities serve to hedge the liabilities and the optimal Sharpe-Tint portfolio tilts towards the liability hedge portfolio. We report the effective risk aversion, which is the risk aversion required under mean-variance utility to produce the same portfolio as the optimal holding. An equity holding of 84% is produced by a risk aversion level of 4.21 in a mean-variance setting.

In the last column of Table 2.2, we report the optimal portfolio of the LDI with downside risk. We assume that $c = 1$. The LDI with downside risk portfolio undoes the higher equity position in the traditional Sharpe-Tint position. Taking into account the liability shortfall pushes the manager toward a lower holding in equities. The LDI position holds an equity proportion of 48%, which is lower than the mean-variance efficient portfolio. This is exactly the opposite of the Sharpe-Tint advice! The LDI with the downside penalty recognizes that although equities are positively correlated with the liabilities, there can

Table 2.2: Optimal Portfolio Choice Over Equities and Risk-Free Cash

	MV Efficient	Sharpe-Tint LDI	LDI with Downside Risk
Equity Portfolio Weight	0.60	0.84	0.48
Effective Risk Aversion	5.88	4.21	7.30

The table reports the portfolio weights for the mean-variance (MV) efficient, Sharpe-Tint (1990) LDI and the LDI with downside risk optimizations for a risk-free asset (cash) and equities. The "portfolio weight" row lists the proportion of the portfolio held in equities. We compute these using the expected returns, volatilities, and correlations given in Table 2.1. We use the parameters $\lambda = 5.88$, $c = 1$, and $A_0/L_0 = 1$ with a one-year horizon. The "effective risk aversion" is the risk aversion required in the mean-variance efficient portfolio weight to give the same weight in equities as the optimal portfolio weight.

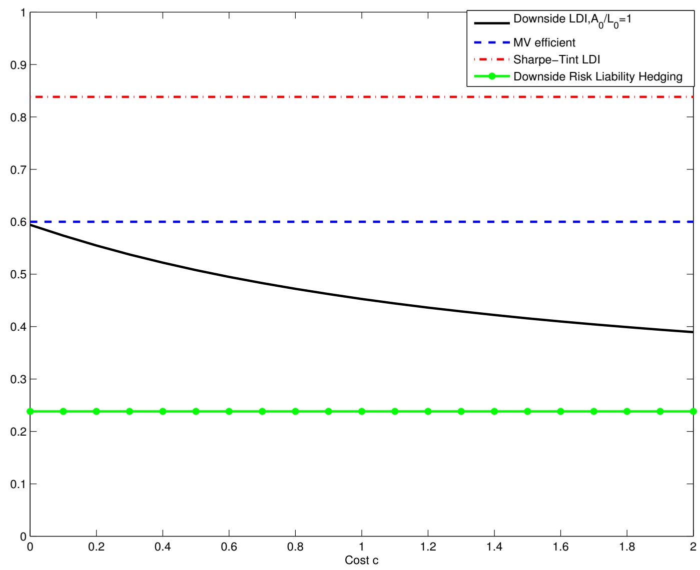
be instances of substantial underperformance when investing in equities. This is costly, and reflected in the value of the put option. Thus, the downside risk averse manager cuts back on equities.

In Figure 2.1, we show further the effect of the downside penalty, c, on the optimal portfolio weight in equities. The portfolio corresponding to $c = 0$ on the y-axis is the mean-variance efficient weight of 60% in Table 2.2. As c increases, the downside LDI strategy allocates less weight on equities. The dash dotted line draws the Sharpe-Tint LDI optimal weight on equity of 84%. As c increases, the optimal weight asymptotes to the liability-hedging portfolio, which holds 24% in equity (see equation (2.9)).

2.3.2.2 Funding Ratios and Endogenous Risk Aversion

In Figure 2.2, we compare the traditional Sharpe-Tint LDI with our LDI with downside risk. The top panel plots the optimal equity weight as a function of the initial funding ratio, A_0/L_0. We show four cases: the horizontal dotted line represents the case of no downside penalty, which is the regular mean-

Figure 2.1: Downside Risk Penalty: Stocks and Cash



The figure plots the optimal weight in equities as a function of the penalty cost, c, on downside shortfall risk for the LDI problem with downside risk with only a risk-free asset and equities. We use the expected returns, volatilities, and correlations given in Table 2.1 and the parameters $\lambda = 5.88$ and $A_0/L_0 = 1$ with a one-year horizon.

variance portfolio (60%); the dashed-dotted line shows the Sharpe-Tint LDI portfolio; the solid line plots the downside risk LDI with $c = 1$, and the dashed line plots the downside risk LDI with $c = 2$. The Sharpe-Tint portfolio holds more equities as the funding ratio decreases because the funding ratio decreases, the hedging demand increases. As equities are correlated with the liabilities, the fund holds more equities to hedge the liabilities as the funding drops. But this effect is a *linear* effect due to the liabilities entering surplus one-for-one. The downside risk induced by the put option ($c > 0$) is highly *nonlinear* and totally different from the Sharpe-Tint advice.

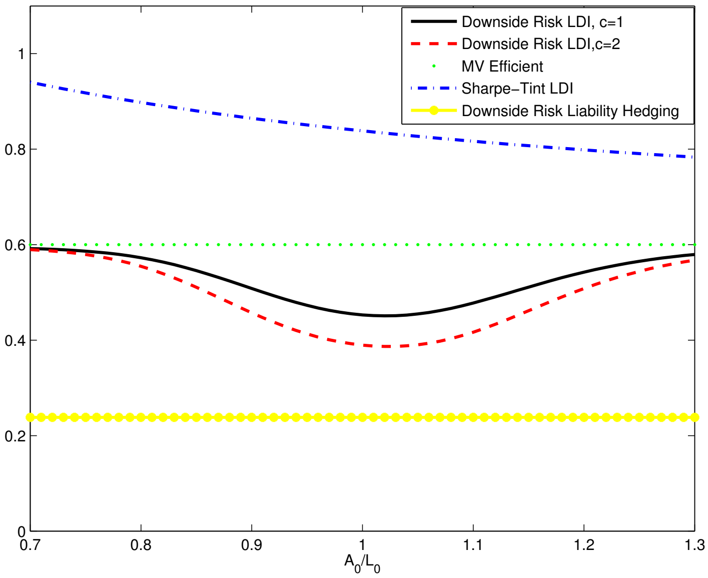
When downside risk is taken into account, the weight on equities reaches a minimum close to the fully funded case. At this point, the downside risk-averse manager places much less weight on the equities. In the case of $c = 1$, the equity weight reaches its minimum at 45% when the funding ratio is 1.03. The intuition is that at full funding, the fund value could easily just cover, or just be below, the liabilities at the end of the period. The manager dislikes this sensitivity and hedges by moving the portfolio towards lower holdings on equity.

As the initial funding ratio increases, it is less likely there will be a liability shortfall and the option value falls. For highly over-funded plans, the value of the option is negligible and the asset allocation problem is equivalent to mean-variance optimization.

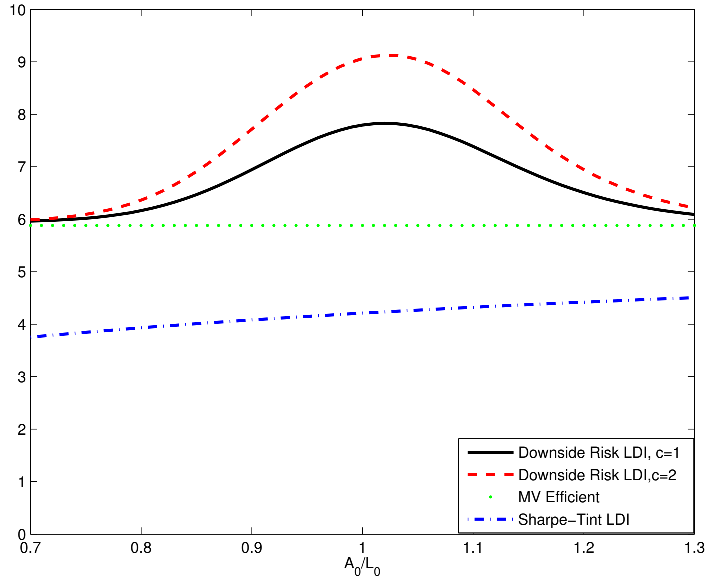
When the plan is very underfunded, the shortfall option is deep in the money. As a result, the objective function starts to put less weight on the shortfall risk because there is less ability of the manager to alter the portfolio choice to meet the liabilities. In extreme cases of chronic underfunding, the

Figure 2.2: Funding Ratios

Optimal Weight on Equities



Effective Risk Aversion



We consider a LDI problem with downside risk with only a risk-free asset and equities. Both plots are functions of the initial funding ratio, A_0/L_0. In the top panel, we plot the optimal weight in equities. The effective risk aversion in the bottom panel is the risk aversion required in the mean-variance efficient portfolio weight to give the same weight in equities as the optimal portfolio weight. We plot the portfolio in the bottom panel. We use the expected returns, volatilities, and correlations listed in Table 2.1 and the parameters $\lambda = 5.88$, $c = 1$, and $c = 2$ with a one-year horizon.

liabilities cannot be met in most states of the world and then the liabilities become irrelevant to the portfolio choice problem. In the limit as the funding ratio goes to zero, the fund just moves towards the mean-variance efficient portfolio to maximize performance.

Thus, there is an overall U-shaped equity weight as a function of the funding ratio, with a minimum weight on equities at a funding ratio close to one.

The bottom panel of Figure 2.2 plots the effective risk aversion, which is the risk aversion required in a standard mean-variance optimization over asset returns to yield the same portfolio weight in equities. By construction, as the equity weight decreases, effective risk aversion increases. The pension fund manager is most risk averse at the fully funded case where the option value reaches a maximum. For $c = 1$, the maximum of effective risk aversion is achieved when the funding ratio is 1.03. The corresponding effective risk aversion is 7.83. The manager is highly sensitive to the shortfall risk at this point and tilts the optimal portfolio resolutely towards holding more risk-free assets to minimize the cost of the shortfall.

In summary, both highly under-funded and over-funded plans are less risk averse than fully funded plans under LDI with downside risk and hold more equities. In particular, the manager "swings for the fences" as funding ratios decrease. There is some empirical evidence for this behavior, as Addoum, van Binsbergen and Brandt (2010) and Pennachi and Rastad (2011) find. Addoum, van Binsbergen and Brandt (2010) show that pension plans approaching a funding ratio of 80%, which subject plan sponsors to severe mandatory additional contributions, increase the risk of their portfolios. They also find similar, but weaker, results at a threshold of 100% where there are milder forms of required contributions. Pennachi and Rastad (2011) find that public pensions funds choose riskier portfolios following periods of relatively poor investment performance after their funding ratios have declined.

2.3.2.3 Funding Ratios and Option Values

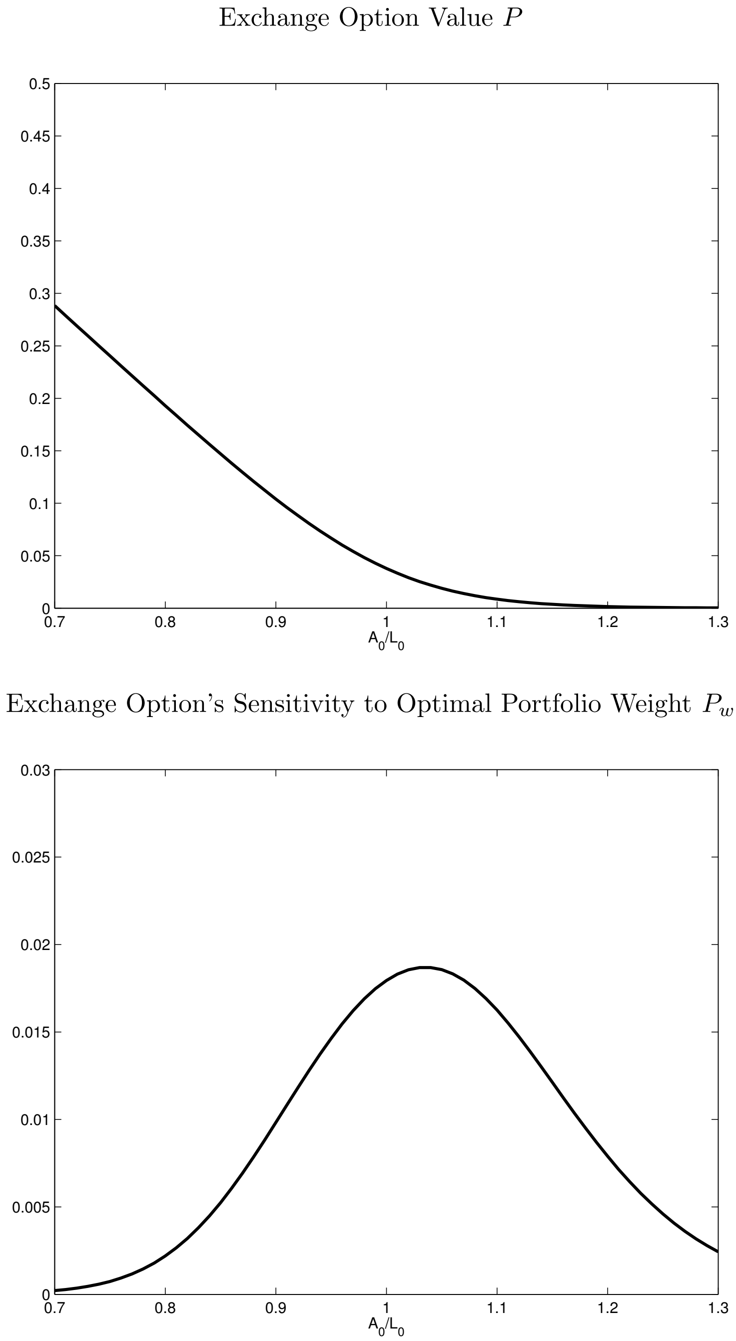
Figure 2.3 characterizes the shortfall option value as a function of the funding ratio. We take $c = 1$ for both plots. The top panel plots the option value. Not surprisingly, the option value decreases as the funding ratio increases as the higher the funding level, the lower the probability of a shortfall at the end of the period. In the bottom panel, we graph the shortfall option's sensitivity to the optimal weight on the equity, P_w, as a function of the funding ratio. The sensitivity is concave in A_0/L_0, and reaches a maximum when $A_0/L_0 = 1.04$. The option's sensitivity is highest when the pension plan is close to fully funded because the probability of the shortfall risk is highest at this point.

2.3.3 Equities and Bonds

Our second example is allocation over equities and bonds without a risk-free asset. In this case, we set the risk aversion coefficient, λ, by choosing its value so that it corresponds to a 60% equity/40% risky bond mean-variance efficient portfolio. This value is $\lambda = 4.37$.

Table 2.3 reports the optimal portfolios for the equities and bond allocation case. All three optimal portfolios are computed under the fully funded condition, $A_0/L_0 = 1$. The first column lists the mean-variance efficient portfolio of 60% in equities, and its effective risk aversion of $\lambda = 4.37$ by construction. The Sharpe-Tint (1990) LDI portfolio holds 45% equities, which corresponds to an

Figure 2.3: Shortfall Penalty Option Value and Option's Sensitivity to Optimal Portfolio Weight



We consider an LDI problem with downside risk with only a risk-free asset and equities. Both plots are functions of the initial funding ratio, A_0/L_0. In the top panel, we plot the option value, $P(w, A_0, L_0)$. We plot $\partial P(w, A_0, L_0)/\partial w$ in the bottom panel. We use the expected returns, volatilities, and correlations listed in Table 2.1 and the parameters $\lambda = 5.88$ and $c = 1$ with a one-year horizon.

Table 2.3: Optimal Portfolio Choice Over Stocks and Bonds

	MV Efficient	Sharpe-Tint LDI	LDI with Downside Risk
Equity Portfolio Weight	0.60	0.45	0.18
Effective Risk Aversion	4.37	6.73	–

The table reports the portfolio weights for mean-variance (MV) efficient, Sharpe-Tint (1990) LDI and the LDI with downside risk optimizations, respectively, for bonds and equities. The "portfolio weight" row lists the proportion of the portfolio held in equities. We compute these using the expected returns, volatilities, and correlations given in Table 2.1. We use the parameters $\lambda = 4.37$, $c = 1$, and $A_0/L_0 = 1$ with a one-year horizon. The "Effective risk aversion" is the risk aversion required in the mean-variance efficient portfolio weight to give the same weight in equities as the optimal portfolio weight. The LDI with downside risk portfolio does not have a corresponding effective mean-variance risk aversion coefficient because the mean-variance efficient portfolios are bounded below by 23%, which corresponds to a risk aversion of infinity.

effective mean-variance efficient risk aversion of 6.73. There is a decrease in the equity weight here compared to an increase in equities when only equities and cash were held (see Table 2.2) because now both equities and bonds are correlated with the liability, and the bond is a much better liability-hedging instrument than equities (bonds have a correlation of 0.98 with the liabilities).

The last column in Table 2.3 reports the optimal portfolio under LDI with downside risk, which has a weight of just 18% in equities. This does not have a corresponding effective mean-variance risk aversion coefficient. The mean-variance efficient portfolios are bounded below by 23%, which corresponds to a risk aversion of infinity. Taking into account downside risk tilts the optimal portfolio markedly towards bonds, rather than equities.

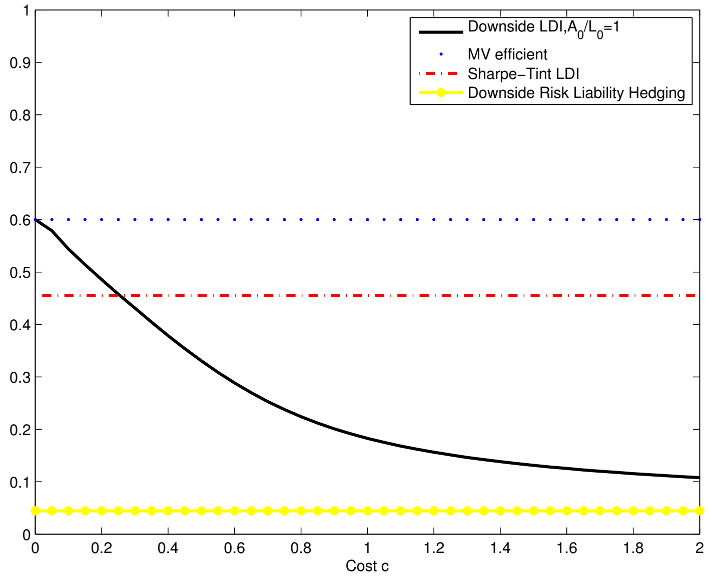
We plot the optimal equity portfolio weight in the top panel of Figure 2.4 as a function of the downside risk parameter, c. The Sharpe-Tint portfolio corresponds to the 45% horizontal dashed-dotted line. Like the equities-cash case in Figure 2.1, the optimal downside LDI portfolio weight decreases with c and asymptotes to the downside risk liability hedging portfolio of 4% equity/96% bond, as the weight on the shortfall risk increases. As the Sharpe-Tint portfolio holds less equity than the mean-variance efficient portfolio, when c is small ($c \leq 0.25$), the downside risk LDI put more weight on wealth on equities than the Sharpe-Tint portfolio. When meeting the liabilities is not so important (c is small), the downside risk LDI objective function seeks mean-variance performance. Even for modest c, there are marked reductions in the equity holdings.

In the bottom panel of Figure 2.4, we investigate the relation between the optimal portfolio weight and the initial funding ratio. The Sharpe-Tint LDI portfolio in the equity-bond case is upward sloping, and lies below the mean-variance portfolio. This is different from the equity-cash case because of the high correlation of the liability with bonds. The downside risk LDI portfolio weight is highly nonlinear and the manager holds the largest amount of equities at very low and high funding ratios.

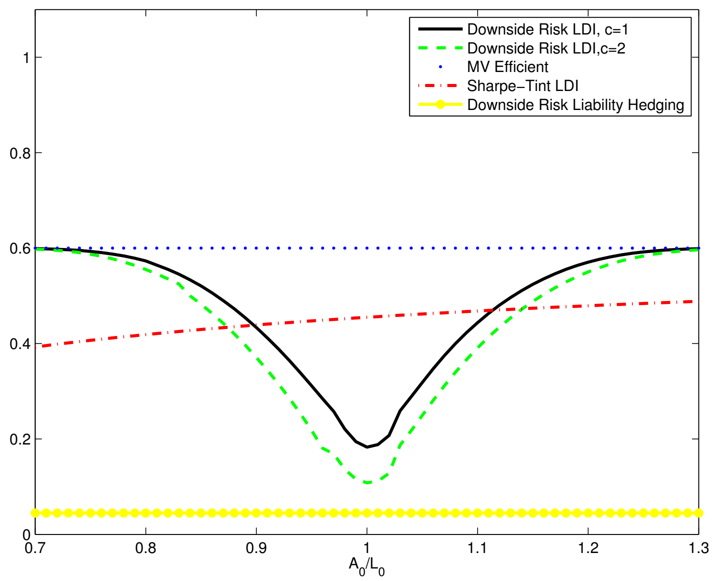
The manager is most risk averse around the fully funded case and holds the minimum amount of equities at this level. The minimum equity holdings are 18% for $c = 1$ and 11% for $c = 2$ and are both reached at $A_0/L_0 = 1.0$. As the funding ratio decreases, the manager "swings for the fences" and holds more equity because the manager's best option is to hold the mean-variance portfolio. When funding ratio is large, the manager can afford to take on risk because the option value is small. Only when the funding ratio is around one does the downside risk LDI optimally recommend a position heavily tilted towards risky bonds to hedge the liability.

Figure 2.4: Stocks and Bonds

Equity Weight as a Function of c



Equity Weight as a Function of A_0/L_0



We consider an LDI problem with downside risk, investing in equities and risky bonds. The top panel plots the optimal weight in equities as a function of the penalty cost, c, while fixing the parameter of funding ratio $A_0/L_0 = 1$. The bottom panel plots the optimal equity weight as a function of funding ratio A_0/L_0, the mean-variance efficient portfolio, the Sharpe-Tint (1990) LDI, and the LDI with downside risk. . We use the expected returns, volatilities, and correlations listed in Table 2.2 and the parameters $c = 1$, $c = 2$, and $\lambda = 4.37$ with a one-year horizon.

2.4 Conclusion

We extend the liability driven investment (LDI) framework to incorporate downside risk. We include a penalty term for the liability shortfall. This can reflect penalties on a plan sponsor, which could be imposed by a regulatory agency, or represent the opportunity cost of capital of a firm required to be diverted to the pension plan if the assets are not sufficient to meet the liabilities. It can also reflect the additional, asymmetric, risk borne by plan participants in shortfall situations.

We show the shortfall between assets and liabilities can be valued as an option. The option pays the difference between liability and asset value at maturity, if the liabilities are greater than the assets, and zero otherwise. The value of the option is determined *simultaneously* with the optimal portfolio, since an optimally chosen portfolio affects the probability of a liability shortfall in the future. The exposure to shortfall risk is controlled by a downside risk parameter. Optimal portfolio allocation with downside risk is very different from traditional Sharpe-Tint (1990) surplus optimization, which produces portfolio weights that are monotonic in funding ratios.

Under LDI with downside risk, the optimal portfolio exhibits *endogenous* risk aversion. Risk aversion peaks when the plan is approximately fully funded. At this point the optimal portfolio holds the *lowest* proportion in equities. The manager wishes to minimize the sensitivity of the portfolio to a shortfall event and the optimal portfolio is heavily tilted to the liability hedging portfolio, which has a low proportion of equities. As the funding ratio moves away from one in both directions, endogenous risk aversion decreases and the manager takes on more risk. Under-funded plans "swing for the fences" on the chance that the portfolio return may be sufficiently high to avoid a shortfall. When the plan is drastically under-funded, the shortfall option is way in the money and the manager has little ability to avoid the shortfall. In this case, the manager's best option is to hold the traditional mean-variance portfolio. For over-funded plans, the probability of a shortfall event is small and this allows the pension plan seek mean-variance performance and take on more risk.

We illustrate the LDI with downside risk framework with allocations only over equities and cash, and an equities and bond case. More practical application would require extension to many more asset classes. The same asymmetries for shortfall risk arise in many other asset management contexts like central bank reserves, sovereign wealth funds, and stabilization funds. These funds also bear downside risk. We also deliberately restrict our analysis to a simple two-date setting in order to compare the implications of our downside risk analysis with two well-known benchmarks in the pension fund industry that do not take into account downside risk: the mean-variance efficient portfolio and the Sharpe-Tint (1990) portfolio. Much of the economic intuition that we develop in this simplified static setting will carry over to an intertemporal setting, albeit with considerably more notational complexity. Developing our ideas into an intertemporal setting like Rudolf and Ziemba (2004) is a fruitful direction for future research.

Chapter 3

Estimating Private Equity Returns from Limited Partner Cash Flows

3.1 Introduction

Private equity is a major institutional asset class and represents a significant fraction of investments by colleges, foundations, pension funds and sovereign wealth funds, among others. [1]A major drawback of private equity is the lack of transactions-based performance measures. This greatly hampers the use of optimal portfolio allocation, which requires information about the risk, return, and covariance of asset classes. In liquid markets, these estimates are typically derived from statistical analysis of time-series returns. Most private equity time

[1]In 2011, institutional investors had over $2 trillion worth of investments in private equity funds worldwide, up from less than $0.4 trillion just ten years earlier. These funds are structured as private partnerships, invest in non-traded assets, and specialize in buyout, venture capital, real estate, etc. In these partnerships, investors commit capital ex ante and fund managers call this capital at their own discretion. The total amount of capital committed but uncalled in private equity funds stands at $1 trillion. This makes it, in a sense, a $3 trillion asset class. Source: https://www.preqin.com/docs/quarterly/PE/Private_Equity_Quarterly_Q3_2012.pdf

series are based on updated non-market estimates or on multi-year internal rates of return broken down by fund vintage years. We develop a methodology to estimate a time series of private equity returns based on cash flows accruing to limited partners. We analyze the dynamics of private equity over 1993 to 2011, as well as investigate private equity returns for different subclasses: venture capital, buyout, real estate, and debt funds. We decompose returns into a component due to exposure to traded factors and a time-varying private equity premium. The latter can be interpreted as the unique value-added by private equity which cannot be replicated by passive, liquid instruments. Given assumptions on the traded factors, the private equity premium can be interpreted as the time-varying private equity alpha.

Our methodology identifies private equity discount rates by using a net present value (NPV) framework. Under the null that the discount rates are correct and appropriate for the risk of the investment, the present value of the capital calls paid into the fund must equal the present value of the distributions from the fund. The NPV equation involving all limited partner (LP) cash flows should thus be zero in expected value both across time and across funds (cf. Driessen, Lin, and Phalippou (2012)). Using a Bayesian Markov Chain Monte Carlo (MCMC) procedure, we filter the time-varying private equity discount rates using the fund-level NPV equations as observation equations. As long as we have at least one fund in existence at a given time we can identify the private equity discount rate prevailing at that time, given additional assumptions about the data-generating process of the private equity returns. Intuitively, as the discount rate changes, the NPVs of those funds whose cash flows are affected by those discount rates also change. The estimation procedure can be interpreted as finding the set of discount rates which produce the smallest errors (statistically defined with respect to a distribution of those errors) in the NPV equations, given the dynamics of the discount rates. Through the appropriate use of priors, the procedure is robust to sparse data and can handle unbalanced panels of contributions and distributions. We find that the estimated time series of private equity returns are more volatile than standard industry indexes. For example, the volatility of our cash flow-based return time series for buyout funds is 25% per annum compared to 11% for the Cambridge Associates buyout index. Similarly, the NCREIF real estate index has a volatility of only 5%, while our estimated volatility of private real estate funds is 19%, which is close to the volatility of publicly traded REITS. There is a smaller difference in volatilities for venture capital, at 35% for our sample and 27% for the Venture Capital index produced by Cambridge Associates; but the volatility of the latter is largely due to a sharp spike in 1999. In addition, we find that our private equity return time series exhibit less serial dependence than industry indexes, even after allowing for a persistent component specific to private equity. This suggests that private equity return time series currently in use may be subject to smoothing biases due to the appraisal process or delayed and partial adjustment to market prices. The second major contribution of this paper is to introduce and apply a methodology for decomposing the time series of private equity returns into systematic and idiosyncratic components. The systematic component involves factor loadings on standard equity benchmarks including large-cap, small-cap, value, and liquidity factors. In this specification, we find that the most important systematic variable is the market factor, for which the private equity returns have a beta loading significantly greater than one. We estimate the market factor exposure for different types of private equity and find that they vary considerably, with venture funds having a high exposure and real estate funds having a low exposure. We term the remaining idiosyncratic portion of private equity returns the private equity premium. To the extent that the returns on traded factors can be elsewhere earned by investors, this private equity premium can be interpreted as a time-varying private equity alpha. We find that this premium is highly persistent and exhibits strong cyclicality. The cycles we uncover differ according to fund type and coincide with both anecdotal evidence and the time-series variation in private equity fundraising. For instance, we find that venture capital returns were high in the second half of the 1990s and low in the first half of 2000s, as was fundraising for this asset class. We also find that the buyout premium was low from 1998 to 2002 and then increased sharply from 2003 until 2007, which coincides with the well-known boom in buyout fundraising. Our broad finding about the private equity premium is that it contributed positively to total returns in the first half of the sample period and negatively in more recent years. This time-series variation allows us to identify macroeconomic variables which significantly comove with private equity returns, including the spread in the free-cash flow yield (EBITDA/Enterprise value) over the junk bond yield which was proposed and studied by Kaplan and Strmberg (2009), and behavioral variables proposed by Baker and Wurgler (2007) indicative of aggregate corporate mispricing. We find evidence consistent with the Kaplan and Strmberg hypothesis that capital market segmentation is a potential driver of the private equity premium, and that the private equity premium may be related to behavioral frictions. The rest of this paper is organized as follows. Section 3.2 describes the methodology.

Section 3.3 details the data. In Section 3.4 we present the empirical results, focusing on the estimated time-series of private equity returns and how they differ from industry benchmarks. We conclude in Section 3.5.

3.2 Methodology

The estimation procedure requires only the cash flows paid and received by investors (called Limited Partners; LPs) in different funds. The funds start and end at different periods in time, which allows us to identify the underlying unobservable discount rates. We present the model in Section 3.2.1 and the estimation procedure in Section 3.2.2. For readers unfamiliar with the structure of the data and the literature on non-traded asset risk evaluations, Appendix C conveys the intuition of our approach with a simple example.

3.2.1 Model

The key assumption of the model is that the cash flows associated with any investment market are generated by a time-varying portfolio of assets that have unobserved but continuous latent values. These assets are heterogeneous. However, we assume their returns are a linear function of an underlying systematic factor structure. Thus, if the latent asset values were observable, some portion of their return variance could be explained by common factors using standard regression methods. In addition, we allow (and test) for asset class-specific latent factors.

Let g_t denote the discount rate of private equity at time t, and g_t^e the excess discounted rate relative to the risk-free rate r_t^f.

$$g_t = g_t^e + r_t^f \tag{3.1}$$

The underlying return process, g_t , cannot be directly observed in the private equity data. We specify that private equity returns are driven by a set of J common tradable factors, $F_t = [F_{1,t}, \cdots, F_{J,t}]$, which are observable in public markets. We consider factors like the equity market, the Fama and French (1993) factors, and the liquidity factor of Pstor and Stambaugh (2003). In addition, we allow for an asset class-specific latent factor, f_t . This potentially makes private equity non-redundant in the space of tradable assets. Combining the two sources of return, we consider the following model for the private equity risk premium, g_t^e :

$$g_t^e = \alpha + \beta' F_t + f_t \tag{3.2}$$

where β are the loadings (betas) on the common factors, F_t .[2] We specify that the private equity return component, f_t , follows an AR(1) process:

$$f_t = \phi f_{t-1} + \sigma_f \epsilon_t \tag{3.3}$$

We specify that f_t is mean zero so that the α in equation (3.3) reflects the average level of private equity returns in excess of its systematic (and liquid) component of the private equity return. The error, , is drawn from an i.i.d. standard normal distribution. The latent factor process, f_t , can be viewed

[2]It is equivalent to model the total private equity return, g_t , as opposed to the private equity return in excess of the risk-free rate, g_t^e . We choose the latter because we are interested in the properties of the risk premium.

as the idiosyncratic component of private equity returns. Usually, traditional factor models for liquid asset returns specify that both systematic and idiosyncratic returns are i.i.d. This is driven by the assumption of market efficiency; predictable returns in a liquid market would be rapidly arbitraged away. In our specification, the f_t process is not exposed to the forces of arbitrage because, by design, it is not tradable and is orthogonal to factors in the public markets. Instead, it is intended to capture such features as persistent manager skill, the inter-temporal variation in good investment opportunities or the trends in performance due to non-constant returns to scale.[3] The specification allows us to test for trends in the private-equity-specific factor by testing whether $\phi = 0$ and also to more formally address the intuition that certain classes of private equity, like venture capital or buyouts, have different return premium properties after controlling for market effects.

The model nests the following special cases:

1. Constant expected returns, when $\beta = 0$, $\phi = 0$,and $\sigma_f = 0$;
2. CAPM, when $\alpha = 0$, $\phi = 0$, and $\sigma_f = 0$, and F_t contains only market excess returns as the systematic factor;
3. Constant excess returns above the CAPM model can be captured by $\alpha \neq 0$ when $\phi = 0$, and $\sigma_f = 0$, and F_t contains only market excess returns;
4. Private equity returns unrelated to public, systematic factors, when $\beta = 0$; and

[3]Imperfect information environments combined with the inability to immediately deploy capital can lead to large persistence in returns (see, for example, Abreu and Brunnermeier (2003), Brunnermeier (2005), Duffie (2010)).

5. The performance of private equity is explained entirely by liquid market returns, when $\sigma_f = 0$.

The full model allows for a rich set of dynamics for private equity returns. In the full model, private equity returns are related to systematic factors ($\beta \neq 0$) and they have characteristics unique to private equity, which may be persistent ($\phi \neq 0$, $\sigma_f \neq 0$). Private equity may offer risk-adjusted returns in excess of what is available in traded markets ($\alpha \neq 0$).

Our task is to estimate the latent factor, f_t , with dynamics given in equation (3.3). If the private equity returns were directly observable as would be the case for listed equity returns, then equations (3.2) and (3.3) would constitute a standard Kalman filter system (they actually represent a Kalman filter with exogenous variables.) The private equity returns are not directly observable; the non-observability can be thought of as a censoring process which renders the estimation a signal-extraction problem conditional on censoring. We discuss a Bayesian method of estimation to filter the returns.

3.2.2 Estimation

We observe cash flows to LPs across N private equity funds indexed by i . The cash flows include investments I_{it} paid into fund i at time t and distributions D_{it} received from fund i at time t. If the model is correctly specified, the cash flows satisfy a NPV condition of

$$\mathrm{E}[\sum_t I_{it}\delta_{it}] = \mathrm{E}[\sum_t D_{it}\delta_{it}] \tag{3.4}$$

where δ_{it} is the cumulative discount rate applicable to fund i at t, defined recursively as:

$$\delta_{it} = \delta_{i,t-1}(1 + g_t)^{-1} \tag{3.5}$$

with $\delta_{i,\tau} = 1$ at the inception of fund i when $t = \tau$ and g_t is the private equity return given in equation (3.1). We take each period to be one quarter in our estimation.

We specify that the ratio of the present value of investments to the Present Value (PV) of distributions is lognormally distributed, or

$$\ln \frac{\mathrm{E}[\sum_t I_{it}\delta_{it}]}{\mathrm{E}[\sum_t D_{it}\delta_{it}]} \sim N(-\frac{1}{2}\sigma^2, \sigma^2) \tag{3.6}$$

The mean of the log distribution is set at $-\frac{1}{2}\sigma^2$ so that the raw ratio PV of investments to the PV of distributions is centered at one. That is, this assumes that the log ratio has zero mean, and takes into account the Jensens inequality induced by taking the log transformation. We estimate the model using a Bayesian MCMC procedure described in Appendix C. We use equation (6) as the likelihood function and treat the unobserved discount rates as parameters to be estimated (which is called data augmentation), along with the other parameters of the data generating process, . In Appendix C we also report sensitivity analysis of the procedure to a range of assumptions, including robustness to different priors. We also show the small sample properties of the estimated parameters using Monte Carlo simulations. This estimation procedure is similar to that of Driessen, Lin, and Phalippou (2012), Franzoni, Nowak, and Phalippou (2012), and Korteweg and Nagel (2013). The key difference with respect to their work is that, in addition to estimating factor loadings, we estimate a quarterly time series of returns for private equityboth systematic and idiosyncraticfrom investor cash flows, while the previous papers only investigate average private equity returns and risk exposures. There are several caveats to our approach. First, a natural interpretation of the index is that it is the net return to investing in all of the private equity vehicles in the database in proportion to the aggregate inflows. This interpretation implicitly assumes that the returned capital D_t in any given period is immediately re-investable in all existing funds as opposed to only new funds. This is typically not the case. This assumption, however, only affects interpretation of the premium factor the latent factor series f_t component of the total return index. The passive component due to $\beta' F_t$ comprises only marketable factors, in which investors can re-invest or rebalance.

A more subtle point that is generally true in all manager performance studies which rely on estimated linear factor exposures is that, by presuming that the passive component is accessible to an investor, we are also implicitly assuming that leverage may be used to achieve a factor exposure greater than one. As we show below, a significant amount of the variation in the g_t^e series is explained by large exposures to public equity factors. Private equity may offer a means to relax borrowing constraints and this convenience may be priced (cf. Frazzini and Pedersen (2010)). We also use long-short factors, and implicit assume that short-selling is feasible and costless in replicating the performance of such factors.

Third, our procedure solves for the best fit of the private equity discount rates given fund cash flows. In computing present values, it may appear that we are implicitly assuming that these cash flows are independent of the discount rates. If cash flows are correlated with discount rates, then we are effectively solving for a time-varying series of discount rates that implicitly takes into account this covariance.[4]

Finally, as we infer private equity returns from LP cash flows, we require high quality data on cash flows. In theory, we would take funds that have terminated so that complete histories of cash flows are observable. In our empirical work, we relax this stringent constraint to take funds with a small portion of unrealized investments in a way we make more precise below. Part of our contribution is methodological, and the procedure can be used on any suitable dataset. An advantage of our estimation technique is that we can estimate private equity returns on data with very sparse cash flows, say a particular institutional investor track record, by using priors set from estimations on more extensive data sets which collate information across many investors.

3.3 Data

We use the cash flow dataset of Preqin purchased in March 2012; data are as of June 2011. Preqin collects the quarterly aggregated investments, distributions, and Net Asset Values (NAVs) made by private equity funds as recorded by U.S. pension funds. Preqin collects this data from public reports and routine Freedom of Information Act requests. The Preqin sample has some desirable characteristics and some limitations. Cash flows are likely to be accurately reported; pension funds would face serious regulatory issues if they deliberately

[4]Brennan (1997) and Ang and Liu (2004) consider the problem of discounting stochastic cash flows with time-varying discount rates and formulate a series of discount rates under these conditions. See also comments by Sorensen and Jagannathan (2013) and Korteweg and Nagel (2013) on how PME implicitly incorporates correlation between the cash flows and the discount rates.

misreport or only selectively report returns. In addition, data on a given fund can be cross-checked between the different pension funds which invest in it. One of the potential limitations is that, by conditioning on pension fund investments, we may not be picking up investments made by other institutional groups such as college endowments.[5] Preqin data have similar characteristics, including similar average and median returns, as those reported in other studies such as those of Robinson and Sensoy (2011) and Harris, Jenkinson, and Kaplan (2013) (cf. Phalippou (2013)).[6]

To assess the risk profile of funds, we need to observe the cash flows of a sufficient number of funds at any point in time. Because the number of funds in the dataset increases rapidly over the time period, we start in a year with at least five funds. This is 1992 for both venture capital and buyout funds. Ideally, we would include all funds from that point on. This approach would, however, assume that the reported NAVs are market values. Funds serving fiduciaries such as pension funds report their audited calculations of portfolio value (NAV) every year. In the U.S., FASB 157 requires fund assets to be fair market-valued, however the private nature of these investments and varying methodologies for evaluation leaves significant uncertainty. Ultimately reported fund NAVs represent the opinion of the fund manager about the assets. [7] It may therefore be problematic to take these NAVs at face value when trying to

[5] Lerner, Schoar, and Wongsunwai (2007) show that endowments have earned higher returns than other investors in private equity investments. Sensoy, Wang, and Weisbach (2013) show that the better performance of endowments is concentrated over the earlier part of the sample and in early stage venture capital.

[6] An additional and unique advantage of Preqin data is that they are publicly available.

[7] The process typically involves a valuation committee and for audited funds, the challenging of valuation assumptions by an auditing firm. Jenkinson, Sousa, and Stucke (2013) and Brown, Gredil, and Kaplan (2013) find that fund valuations are conservative except when follow-on funds are raised.

assess the underlying true returns.

One solution to this problem would be to include only funds that have passed their eight or tenth anniversary in order to both minimize the impact of NAVs and guarantee a representative sample for each of the included years. But doing so would result in the frequency of cash flows significantly decreasing in the later part of the sample. We thus include all (post-1992) funds as long as they have a relatively low NAV (50% of fund size or less). We exclude funds that do not have at least one distribution of at least 10% of fund size (which can be a cash distribution or the final NAV). These criteria mean that we keep a few funds each year in the sample all the way to 2008. This is essential in order to estimate the quarterly return of private equity. We label this sub-sample of funds the quasi-liquidated sub-sample. When reporting abnormal performance, we will show the results for both the quasi-liquidated sample of funds and the full sample of funds. The estimation of risk loadings, however, necessitates the use of the quasi-liquidated sample. This constraint has some potential effect if, for example, younger funds have different characteristics, or the management style and factor exposures for funds launched in the mid-2000s was not representative, then our estimates will be weighted away from these and towards more mature funds (cf. Barrot (2012)). Table 3.1 reports descriptive statistics for our data.[8] Panel A shows the number of funds entering the sample in each year. The Preqin sample appears to be similar to that of Harris, Jenkinson, and Kaplan (2013) in terms of size and years covered. Panel B compares the number of observations of the full sample and the quasi liquidated sample. It

[8] Selected funds are "closed" or "liquidated," and based in the United States. We exclude GCP California Fund, a partnership between Leonard Green and CalPERS to invest in California-related industries and underserved markets.

also breaks down the statistics of the quasi-liquidated sample per fund categories. The venture capital category includes funds classified as general venture capital, balanced, seeds, start-up, early stage, expansion and late stage. The buyout category includes funds classified as buyout and turnaround. The real estate and high yield debt categories include all funds classified as such. Note that the number of real estate and debt funds is relatively small.

Table 3.1: Descriptive Statistics

This table shows the number of observations in different samples. Panel A compares the Preqin dataset to that of proprietary dataset recently used in the literature. Panel B shows statistics on Preqin full sample and sub-samples. Venture Capital (VC) funds include funds classified (by Preqin) as either: expansion, late stage, general venture, balanced or growth. Buyout (BO) funds include funds classified as either turnaround or buyout. Debt funds include funds classified as either distressed debt or mezzanine. Private Equity (PE) funds refer to funds that are either VC, BO, Debt (D) or Real Estate (RE). Funds are from vintage years 1992 to 2008. The quasi liquidated sample is the sub-sample of funds with the latest NAV reported that is less or equal to 50% of fund size, with at least one cash distribution, and with the latest NAV reported (or the largest distribution) larger or equal to 10% of fund size.

Panel A: Number of observations Comparing Preqin and proprietary datasets

	Venture capital funds			Buyout funds		
	Harris, Jenkinson &Kaplan	Robinson &Sensoy	Preqin Full sample	Harris, Jenkinson &Kaplan	Robinson &Sensoy	Preqin Full sample
1992	17	4	10	5	4	7
1993	13	5	9	11	9	12
1994	20	7	12	13	24	15
1995	18	13	15	17	24	11
1996	20	13	16	9	41	18
1997	33	19	21	30	40	22
1998	46	36	31	38	59	39
1999	65	40	44	28	59	31
2000	80	55	79	39	68	38
2001	48	18	50	26	26	17
2002	18	7	28	21	5	19
2003	25		15	13	8	17
2004	32		28	46	3	29
2005	48	1	35	57	2	52
2006	62		49	67	8	52
2007	65	2	48	74	6	57
2008	45		26	68	12	42
Total	655	220	516	562	398	478

Panel B: Number of observations Preqin samples

	Full sample	Preqin equasi-liquidated sub-sample of funds				
	All PE funds (VC,BO,RE,D)	All PE funds (VC,BO,RE,D)	VC funds	BO funds	RE funds	Debt funds
1992	21	21	11	6	1	3
1993	22	22	9	12	0	1
1994	31	30	15	15	1	3
1995	29	28	15	10	1	2
1996	41	39	20	17	3	4
1997	52	50	27	22	1	7
1998	78	74	36	39	4	4
1999	81	67	60	26	1	4
2000	126	97	60	28	6	3
2001	77	52	28	14	2	8
2002	55	33	15	10	3	5
2003	42	12	2	2	4	4
2004	74	17	3	6	7	1
2005	114	11	3	4	4	0
2006	140	19	7	5	4	3
2007	146	27	3	12	8	4
2008	93	31	7	15	4	5
Total	1222	630	272	243	54	61

3.4 Empirical Results

3.4.1 Time-series estimates of private equity total returns and premiums

In this subsection, we discuss and plot the time series of our estimated private equity total returns and premiums. Recall that Appendix C offer further details on the methodology, the choice of priors, and robustness checks.

We apply the methodology to the sample of 630 quasi-liquidated funds described in Table 3.1. Figure 1 plots the cumulated total return index (g_t^e) obtained with a four-factor model for systematic risk. The four factors are the market portfolio, the Fama and French (1993) small-large and value-growth factors, and the Pstor and Stambaugh (2003) liquidity factor that goes long illiquid stocks with high returns and shorts more liquid stocks with relatively low returns. Figure 1 plots our estimated private equity returns expressed as an index, which starts with the value 1.0 in March 1993.

Figure 3.1: Private Equity Total Return Index v.s. US Index Funds

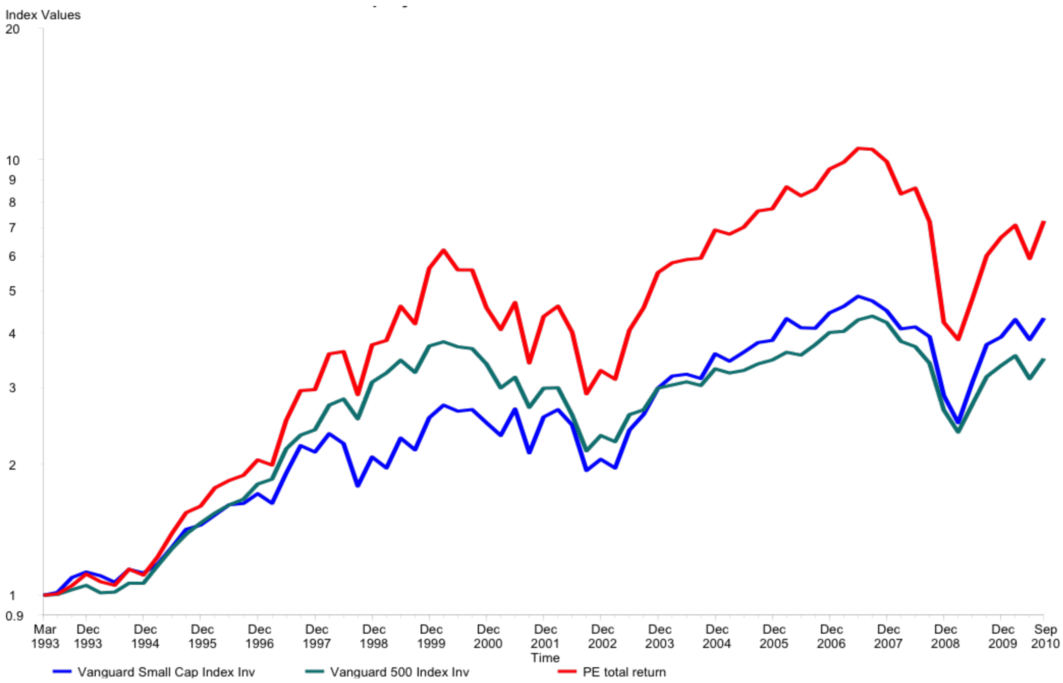


Figure 1 compares the return of our private equity index to the returns of two low-cost passive index portfolios offered by Vanguard: an S&P 500 index fund and a small-cap fund. Consistent with the findings of Harris, Jenkinson, and Kaplan (2013), and Robinson and Sensoy (2011), private equity beats the index portfolios over the time period 1993 to 2010. Part of the private equity performance, however, may be replicable using some passive factor exposures. Indeed, Figure 1 shows that there is significant co-movement between the private equity total returns and the Vanguard S&P 500 and small-cap index funds.

Figure 2 plots the total return, g_t^e , the return of the passive factor exposures, $\beta' F_t$, and the spread between the two which is the private equity return premium, f_t. The return f_t can also be interpreted as private equitys time-varying alpha. Over the sample, the cumulated private equity premium, f_t, is zero, so private equity has had an alpha of zero (when using the four factor asset pricing model). Nevertheless, over some time periods, there is a significant spread between the total return, g_t^e, and the systematic return, $\beta' F_t$, indicating that there is a non-negligible idiosyncratic component of private equity returns.

Figure 3.2: Decomposition of Private Equity Return Index into Passive and Premium Components

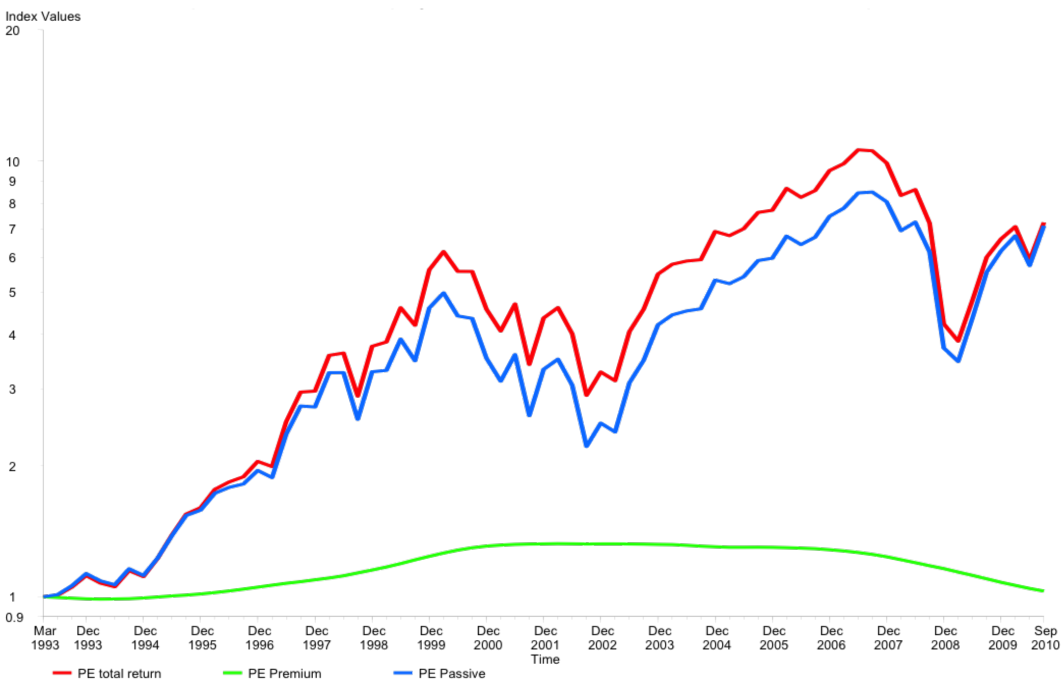
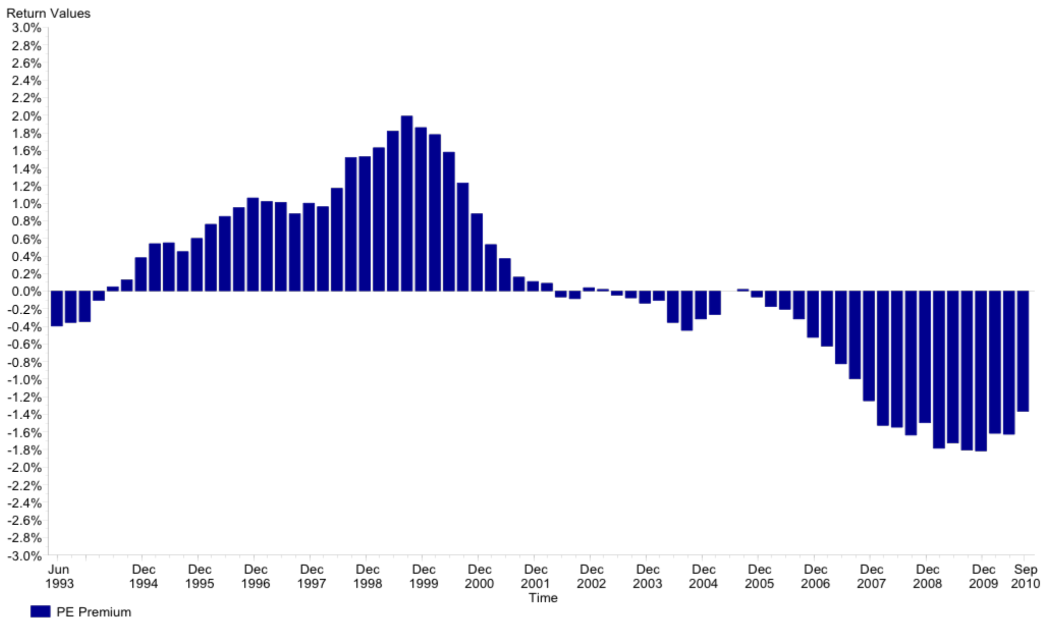


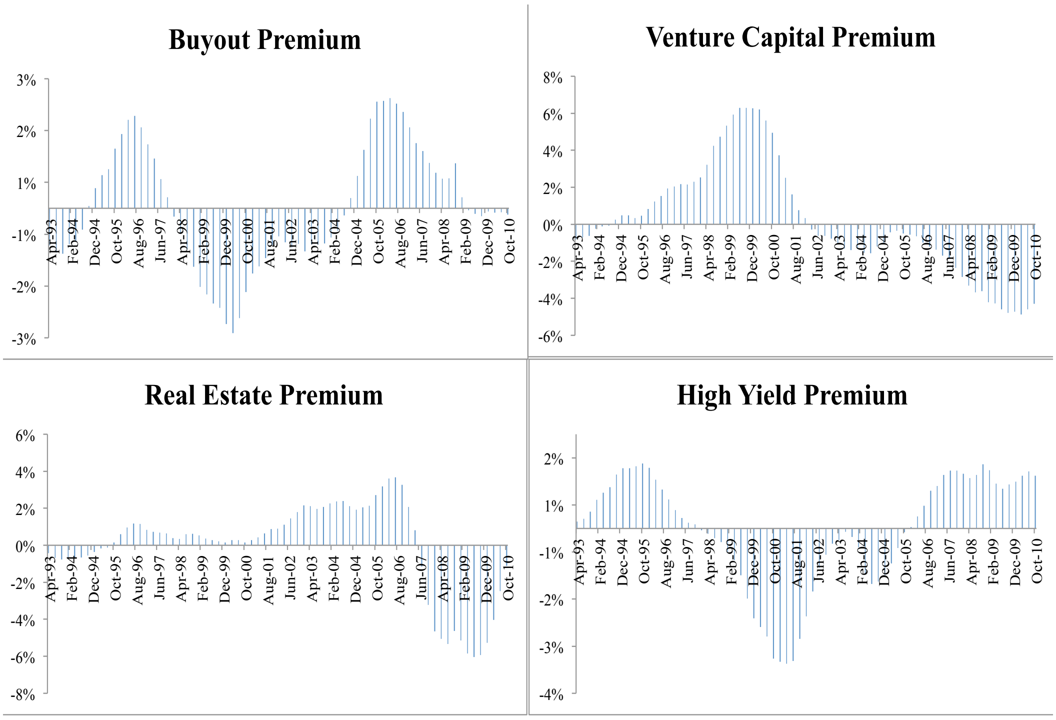
Figure 3 provides more detail about the timing of the premium, f_t , and shows the quarterly returns to the private equity premium, f_t , as bars in each period (so they are not cumulated like Figures 1 and 2). Although the overall average is zero, there is significant time variation. The premium is large and positive in the second half of the 1990s, approximately zero for the first half of the 2000s and then negative from 2006 including during the financial crisis. The pattern suggests that the private equity premium is cyclical, with as much as 10 years from peak to trough.

Figure 3.3: Private Equity Premium



The cyclical pattern of the private equity premium is most interesting when broken down into sub-asset classes. In Figure 4, we plot the premiums for our four subsets: buyouts, venture capital, real estate, and high yield (see Table 3.1, Panel B). The premiums for each asset types behave quite differently. Buyout funds experienced premiums of more than 1% in 1995-1996 and over 2005-2007, consistent with conventional beliefs as reflected in industry reports and press coverage. Venture capital funds had one very large peak of more than 5% in 1999-2000, coinciding with high valuations of internet companies during this time. Real estate peaked at more than 3% in 2006 (notably before available appraisal-based commercial property indexes captured a downturn). The premium to investing in high yield debt funds had two peaks of 2%–one in 1995 and another in 2009, co-moving only approximately with the premium to buyout funds.

Figure 3.4: Quarterly Private Equity Premium per Sub-classes



These imperfect co-movements suggest that the cycles to venture capital and real estate differ from those of buyout and high-yield funds, and that there are benefits to diversifying across private equity investment classesfor example high-yield premiums were positive when venture capital premiums were negative. More generally the evidence suggests that, even conditional on differing exposures for systematic factors, private equity premiums in different asset classes are exposed to different underlying factors unrelated to publicly traded securities.

3.4.2 Factor exposures and private equity premium

The private equity premium displayed in Figures 1-4 is computed using a four-factor asset pricing model with market, size, value, and liquidity factors. We find that the estimated overall private equity premiums, g_t^e, are relatively insensitive to the assumed model for systematic risk, but the systematic factors do affect the estimates of the private equity premium, f_t. In Table 3.2, we report parameter estimates of the factor loadings, β, the α coefficients, and the persistence of the private equity premiums, ϕ, with different asset pricing factor models. The table reports posterior means and standard deviations of the parameters.

We take models with one, three, and four systematic factors. The one factor model is the CAPM; the three-factor model is from Fama and French (1993) which adds SMB and HML factors, and the four-factor model is that of Pstor and Stambaugh (2003) which adds a liquidity factor. For robustness, we also report models for which we use the CRSP equally-weighted (EW) index instead of the CRSP value-weighted index as a measure of market returns. This is equivalent to the assumption that private equity funds acquire companies that are drawn from a pool resembling the CRSP sample; i.e. they are as likely to acquire a firm from the bottom decile as from the top decile of capitalization. This assumption is useful because the typical company purchased by a private equity fund is small compared to the firms in the S&P 500. The drawback is that the equal-weighted CRSP index is not investable. Table 3.2 shows that the CAPM estimate of the beta of private equity is 1.41, which is almost unchanged

Table 3.2: Private Equity Factor Exposures

This table shows the estimated risk loadings, abnormal returns and the persistence in abnormal returns using six different asset pricing factor models. All quasi-liquidated private equity funds are used in the analysis, irrespective of their type (venture capital, buyout, real estate, high yield). The quasi liquidated sample is the sub-sample of funds with the latest NAV reported that is less or equal to 50% of fund size, with at least one cash distribution, and with the latest NAV reported (or the largest distribution) larger or equal to 10% of fund size. The risk loadings are estimated using the quasi liquidated sample. The reported alpha is annualized (by compounding) and defined as the constant that makes the (equally weighted) average NPV equal to zero in either the full sample or the quasi-liquidated fund sample, given the estimated risk loadings. Underneath each coefficient, in italics, we report the posterior standard deviation of the estimated parameters. The factor models that we use are: the CAPM, the three factor model of Fama and French (1993), and the four factor model is that of Pstor and Stambaugh (2003). The equally weighted (EW) factor models are the same as the original model but with the CRSP equally-weighted index instead of the CRSP value-weighted index as a measure of market returns. The priors for the factor loadings are detailed in Appendix C.

Model	β_{market}	β_{size}	β_{value}	$\beta_{illiquidity}$	In-sample Alpha	Persistence of Alpha	Full sample Alpha	R-square
CAPM	1.41***				0.05***	0.4	0.04***	0.93
	0.24				*0.01*	*0.19*	*0.01*	
3 factors (FF)	1.49***	0.41	0.09		0.04***	0.43	0.03***	0.95
	0.23	*0.31*	*0.27*		*0.01*	*0.19*	*0.01*	
4 factors (PS)	1.41***	0.41	0.03	0.36	0	0.48	0	0.97
	0.21	*0.26*	*0.23*	*0.27*	*0.02*	*0.19*	*0.01*	
EW CAPM	1.42***				-0.04***	0.45	-0.04***	0.98
	0.18				*0.01*	*0.19*	*0.01*	
EW FF	1.47***	0.4	-0.11		-0.04***	0.47	-0.04***	0.98
	0.2	*0.25*	*0.21*		*0.01*	*0.19*	*0.01*	
EW PS	1.40***	0.33	-0.19	0.26	-0.05***	0.47	-0.05***	0.97
	0.22	*0.3*	*0.25*	*0.27*	*0.02*	*0.19*	*0.01*	

using an EW market index. The estimates for the four-factor loadings on market, size, value, and liquidity factors are 1.49 for the market excess return, 0.41 for SMB, 0.03 for HML, and 0.36 for the liquidity factor. In the four-factor model, the posterior means of the market and SMB loadings are more than two posterior standard deviations away from zero, but this is not the case for the value and liquidity factor loadings. Nevertheless, the economic magnitude of 0.36 for the liquidity factor beta is relatively large. We report two sets of alphas. The in-sample alpha is the premium computed, given the estimated set of risk loadings, using the sample of funds on which the model was estimated (i.e. the 630 quasi-liquidated sample). The full-sample alpha is the premium estimated using the full sample of 1,222 funds given the estimated set of risk loadings (from the quasi-liquidated sample). The two sets of alpha generally agree with each other, which indicates that our selection procedure does not lead to a bias towards better or worse performing funds. All alpha estimates in the table are annualized. Consistent with the literature (Robinson and Sensoy (2011) and Harris, Jenkinson, and Kaplan (2013)), we find that the alphas with respect to the S&P 500 and a three-factor model accounting for size and value effects are positive at 0.05 and 0.04, respectively, using in-sample estimates. Adding the liquidity factor drives the alpha to zero. Also consistent with the literature, substituting an index which weights small companies heavilythe equal-weighted CRSP indexreduces the alpha estimates dramatically. The alpha in the EW CAPM specification is negative at -0.04, and this is largely unchanged when the SMB, HML, and liquidity factors are added. In our model, the private equity premium is a non-arbitragable factor which is auto-correlated. Table 3.3 reports the persistence in the private equity premium measured at the quarterly horizon. Depending on the specification, this value ranges from 0.40 to 0.47. In all cases, the estimates are significantly different from zero. The autocorrelation estimates are a potentially useful measure because a variable with a significant autocorrelation coefficient is potentially forecastable. Our auto-correlation estimates indicate there is fairly strong persistence in the aggregate private equity premium. Value-enhancing shocks have a half-life of about one-quarter. A year later, the effect is 1/16 of its original intensity but still contributes to the net return. For example, Figure 3 shows an upward trend at the end of the time period, indicative of a potential reversion to a positive value in the future.

3.4.3 Factor exposures and private equity premium broken down by fund type

Table 3.3 reports estimations on the different private equity sub-classes. Venture capital funds have the highest estimated CAPM beta, followed by buyout, real estate, and high yield funds. The venture capital market beta is 1.67 in Panel A, which is a slight decrease from the previous estimates in the literature (see Appendix B and Appendix Table A.1). Venture capital has a significant negative loading on the Fama-French value factor, which is what we would expect from a strategy of buying high growth companies. The robustness specifications for venture capital use an equal-weighted portfolio of Nasdaq stocks instead of the equal-weighted CRSP. As expected, this change decreases the market beta and drives alpha to (close to) zero.

Two remarks are worth making on the venture capital alpha. First, there is a negative loading on the value premium. Venture capital strategies appear to be loading up on growth stocks, which have low average returns. Thus, the

Table 3.3: Risk Exposures Broken Down by Fund Type

This is the same table as Table 2. Instead of using all the funds we use (independently) sub-samples of funds based on their type: venture capital, buyout, real estate and high yield. Venture capital funds include funds classified (by Preqin) as either: expansion, late stage, general venture, balanced or growth. Buyout funds include funds classified as either turnaround or buyout. High yield funds include funds classified as either distressed debt or mezzanine. We report posterior means. The reported alpha is annualized (by compounding) and defined as the constant that makes the (equally weighted) average NPV equal to zero in either the full sample or the quasi-liquidated fund sample, given the estimated risk loadings. Underneath each coefficient, in italics, we report the posterior standard deviation of the estimated parameters. The bottom three models in each Panel are the same as the original models but with a different proxy used for market returns. The proxies used in each of the panels are, respectively: Equally-weighted Nasdaq index, Equally-weighted AMEX/NYSE index, FTSE REITS index, 10 years T-bonds returns.

Panel A: Venture Capital Funds

Model	β_{market}	β_{size}	β_{value}	$\beta_{illiquidity}$	In-sample Alpha	Persistence of Alpha	Full sample Alpha	R-square
CAPM	1.67***				0.05***	0.54***	0.04***	94.10%
	0.27				*0.01*	*0.19*	*0.01*	
3 factors (FF)	1.51***	0.45	-0.62*		0.08***	0.60***	0.06***	93.50%
	0.33	*0.42*	*0.38*		*0.02*	*0.17*	*0.02*	
4 factors (PS)	1.60***	0.53	-0.68*	0.16	0.06***	0.63***	0.05**	95.90%
	0.29	*0.42*	*0.37*	*0.36*	*0.02*	*0.17*	*0.02*	
EW CAPM	1.32***				-0.02	0.71***	-0.04***	97.30%
	0.19				*0.01*	*0.14*	*0.01*	
EW FF	1.11***	0.28	-0.59*		0.02	0.71***	0	96.00%
	0.25	*0.4*	*0.31*		*0.02*	*0.14*	*0.02*	
EW PS	1.15***	0.36	-0.57**	0	0.02	0.71***	0	97.00%
	0.23	*0.43*	*0.28*	*0.36*	*0.02*	*0.15*	*0.02*	

Panel B: Venture Capital Funds

Model	β_{market}	β_{size}	β_{value}	$\beta_{illiquidity}$	In-sample Alpha	Persistence of Alpha	Full sample Alpha	R-square
CAPM	1.31***				0.05***	0.42**	0.04***	88.70%
	0.25				*0.01*	*0.2*	*0.01*	
3 factors (FF)	1.39***	-0.07	0.74**		0.03***	0.42**	0.01	92.30%
	0.21	*0.3*	*0.29*		*0.01*	*0.19*	*0.01*	
4 factors (PS)	1.33***	-0.04	0.57**	0.59***	-0.02**	0.50***	-0.03***	96.90%
	0.14	*0.25*	*0.22*	*0.21*	*0.01*	*0.17*	*0.01*	
EW CAPM	1.30***				0	0.64***	-0.01	90.30%
	0.27				*0.01*	*0.16*	*0.01*	
EW FF	1.29***	-0.38	0.46		0	0.62***	-0.01	90.50%
	0.25	*0.37*	*0.33*		*0.01*	*0.18*	*0.01*	
EW PS	1.15***	-0.28	0.39	0.50*	-0.03**	0.70***	-0.04**	93.90%
	0.22	*0.36*	*0.32*	*0.3*	*0.02*	*0.15*	*0.02*	

Panel C: Venture Capital Funds

Model	β_{market}	β_{size}	β_{value}	$\beta_{illiquidity}$	In-sample Alpha	Persistence of Alpha	Full sample Alpha	R-square
CAPM	0.77***				0	0.72***	0	79.60%
	0.23				*0.01*	*0.11*	*0.01*	
3 factors (FF)	0.79***	0.21	0.76**		-0.03***	0.61***	-0.03**	87.10%
	0.22	*0.3*	*0.3*		*0.01*	*0.17*	*0.01*	
4 factors (PS)	0.74***	0.09	0.54	0.66*	-0.08***	0.54***	-0.07***	87.80%
	0.23	*0.37*	*0.37*	*0.38*	*0.02*	*0.19*	*0.02*	
EW CAPM	0.75***				-0.04***	0.61***	-0.04**	83.80%
	0.18				*0.01*	*0.16*	*0.02*	
EW FF	0.66***	0.1	0.49		-0.05***	0.58***	-0.04**	88.70%
	0.2	*0.29*	*0.3*		*0.01*	*0.18*	*0.02*	
EW PS	0.49**	0	0.26	0.59*	-0.07***	0.58***	-0.07***	83.40%
	0.24	*0.39*	*0.37*	*0.36*	*0.02*	*0.18*	*0.02*	

Panel D: Venture Capital Funds

Model	β_{market}	β_{size}	β_{value}	$\beta_{illiquidity}$	In-sample Alpha	Persistence of Alpha	Full sample Alpha	R-square
CAPM	0.62**				0.03	0.36*	0.03*	66.30%
	0.27				*0.02*	*0.19*	*0.02*	
3 factors (FF)	0.88***	1.18***	1.05***		-0.02**	0.49***	-0.03**	96.70%
	0.2	*0.26*	*0.27*		*0.01*	*0.18*	*0.01*	
4 factors (PS)	0.87***	1.14***	0.97***	0.29	-0.06***	0.49**	-0.06***	96.50%
	0.22	*0.26*	*0.28*	*0.24*	*0.02*	*0.19*	*0.02*	
EW CAPM	0.58***				-0.01	0.41**	-0.01	79.40%
	0.2				*0.02*	*0.18*	*0.02*	
EW FF	0.44	1.02***	0.73**		-0.03	0.58***	-0.02	88.90%
	0.28	*0.38*	*0.36*		*0.02*	*0.19*	*0.02*	
EW PS	0.36	0.99***	0.72*	0.16	-0.05*	0.57***	-0.04	89.60%
	0.24	*0.33*	*0.38*	*0.32*	*0.03*	*0.2*	*0.03*	

total returns of venture capital are relatively low, but the alpha is boosted up by the negative loading on the value factor. The second remark is that the value-weighted stock-market index used in the CAPM has low returns over our sample period, which sets a low bar in terms of performance. When the index is changed to the EW Nasdaq stocks, which have delivered better performance, then venture capital exhibits a negative alpha. Panel B of Table 3.3 reports results for the largest fund type in terms of asset under management, buyout funds. The buyout fund market beta for the standard specification is around 1.3, similar to previous estimates in the literature (see Appendix Table 1). The coefficients on value and liquidity factors are positive. The single-factor CAPM alpha is 0.05 in the estimation sample and 0.04 in the full Preqin sample. The alpha drops to 0.03, but remains significant in the estimation sample in the standard three-factor Fama-French specification. The inclusion of the Pstor-Stambaugh liquidity factor, however, changes the sign of the alpha. This can be interpreted as buyout funds harvesting a liquidity risk premium in the Pstor-Stambaugh sense (cf. Franzoni, Nowak, and Phalippou (2012)). Although we have relatively few real estate funds, results in Panel C show that the estimation procedure generates intuitively reasonable results. Real estate market betas vary from 0.74 to 0.79, consistent with previous estimates of a beta less than 1.0 for real property. The beta on the REIT index is less than one as well, ranging from 0.49 to 0.75. Most specifications show a negative alpha for real estate funds. For high yield funds in Panel D, we estimate a CAPM beta of 0.66, and find that all three of the factor loadings on the Fama-French model are significant: high yield has a beta greater than one for both size and value factors. As with real estate funds, most specifications show a negative alpha for high yield funds. Note that we have estimated the exposure of private equity investment to factors commonly used in the analysis of equity returns; and we have modeled a private equity premium as an auto-correlated latent factor. In several specifications we reject the null that private equity assets are redundant with respect to the standard Fama-French and Pstor-Stambaugh equity factors. Yet, these factors capture a large part of (and in some cases fully explain) the total returns to investing in private equity. This, however, does not necessarily imply that there is no value to private equity because none of these equity factors returns are available without incurring transaction costs. The next question is whether an investor can cheaply access the premiums of the tradable factors passively, or whether private equity investments are a more efficient way to access these factor premiums. This would involve an analysis of transactions costs (and investor size) that is beyond the scope of this paper. Finally, it is interesting to note that the persistence in the premium is strongest in venture capital and real estate, as reported in Table 3.3. It is less strong in buyout and even lower in high yield funds. These results are consistent with the idea that persistence is driven by non-scalability. Certainly venture capital and real estate are the most illiquid assets and are the most difficult investments to scale. The buyout and high yield debt strategies have more capacity.

3.4.4 Comparison to industry indices

One practical advantage of our cash flow-based index is that it seeks to attribute returns to the time period in which they occur. In practice, there are some industry indexes with the same objective but they use estimated asset values. These estimated values are potentially subject to inertiafor example anchoring on prior appraisal values. The econometrics of appraisal-based indexes have been well-studied for commercial real estate (cf. Geltner (1991)). Among other things, they have volatilities which under-estimate true volatilities and lag market values. In this section we examine the relationship of the cash-flow based index to industry indexes. In Table 3.4, we label our estimated index the CF PE index, which is produced using the four-factor model for systematic risk (see also Figures 1-3). The table shows the annualized mean, standard deviation, inter-quartile range and autocorrelation coefficient for some standard industry indexes and for our cash flow-based indexes. For buyout and venture capital we use the Cambridge Associates indexes; they are the most prominent ones in practice. For real estate we use the NCREIF index. This is the industry-standard appraisal-based index of unlevered property returns, which is computed using data reported by institutional investors to the National Council for Real Estate Investment Fiduciaries. All the mean and volatility estimates in Table 3.4 are annualized.

Table 3.4 shows that the cash flow-based indexes are more volatile than the industry indexes. The difference is particularly dramatic for real estate. We estimate a volatility of 19% per annum for real estate, compared to the NCREIF index volatility of 5%. The 19% is closer to the volatility of publicly traded real estate portfolios, REITS. This suggests that our estimated index may provide a more realistic estimate of real estate portfolio risk for investment managers. For buyout, the volatility of our cash flow-based return time series is more than twice as high as that of Cambridge Associates (25% compared to 11%). There is a smaller difference in volatilities for venture capital, at 35% for our sample and 27% for the venture capital index produced by Cambridge Associates; but

Table 3.4: Comparison of Private Equity Index with Industry Indices

This is the same table as Table 3.2. Instead of using all the funds we use (independently) sub-samples of funds based on their type: venture capital, buyout, real estate and high yield. Venture capital funds include funds classified (by Preqin) as either: expansion, late stage, general venture, balanced or growth. Buyout funds include funds classified as either turnaround or buyout. High yield funds include funds classified as either distressed debt or mezzanine. We report posterior means. The reported alpha is annualized (by compounding) and defined as the constant that makes the (equally weighted) average NPV equal to zero in either the full sample or the quasi-liquidated fund sample, given the estimated risk loadings. Underneath each coefficient, in italics, we report the posterior standard deviation of the estimated parameters. The bottom three models in each Panel are the same as the original models but with a different proxy used for market returns. The proxies used in each of the panels are, respectively: Equally-weighted Nasdaq index, Equally-weighted AMEX/NYSE index, FTSE REITS index, 10 years T-bonds returns.

	Mean	Volatility	Percentiles		Autocorrelation
			25th	75th	
CF buyout index	0.15	0.26	-0.12	0.53	0.06
Cambridge Associates buyout index	0.16	0.12	0.04	0.33	0.41
LPX listed buyout index	0.16	0.3	-0.05	0.45	0.22
CF venture capital index	0.18	0.34	-0.18	0.67	0.03
Cambridge Associates venture index	0.19	0.28	-0.03	0.35	0.61
LPX listed venture capital index	0.13	0.39	-0.33	0.64	0.14
CF real estate index	0.05	0.17	-0.12	0.31	0.24
NCREIF (Real Estate) index	0.09	0.05	0.07	0.15	0.82
CF private equity index	0.15	0.29	-0.15	0.58	0
LPX 50 index	0.13	0.35	-0.21	0.52	0.18

the latter is solely driven by a sharp spike in 1999. These results indicate that existing private equity return time series exhibit smoothing biases likely due to the appraisal process and the fact that valuations of illiquid assets may only partially adjust to market prices. In addition, we find that our private equity return time series exhibit much less serial dependence, if any, than industry indexes.

3.4.5 Vintage year comparisons

Industry participants and academic researchers have traditionally used vintage year IRRs, multiples of returned cash to investment, and PMEs to identify the cyclical behavior of private equity. Table 3.5 examines the relationship between these measures, our cash flow private equity indexes, and flows into private equity (changes in the number of funds and amount of capital entering the industry). Vintage year returns are computed by first aggregating all the cash flows of the funds from a given vintage year, and then computing the IRR, multiple and public market equivalent (PME) of that aggregated cash flow stream. We compute the PME with the discount rates derived from the factor loadings estimated with the four-factor model. In this respect it differs from a standard PME calculation that uses only S&P 500 returns to discount cash-flows. We do this to highlight the difference between our private equity returns with the PME on the basis of how the measures are computed, rather than having different discount rates in each measure. The sample is all the Preqin quasi-liquidated private equity funds.

The IRR, multiple, and PME measures display some common trends (Panel A of Table 3.5). They all start to decrease from 1994 and reach a low in 1999.

Table 3.5: Alternative Performance Measures and Capital Flows

This table compares different performance measures. The forward moving average of our private equity index, g_t, is the geometric yearly average return calculated from year t+1 to t+5. It is computed using the four-factor Pstor -Stambaugh (2003) model for systematic risk. The last year is 2009 and the forward moving average is not computed for 2008 as it is not meaningful (n.m.). Vintage year returns are computed by first aggregating all the cash flows of the funds from a given vintage year, and then computing the IRR, multiple and public market equivalent (PME) of that aggregated cash flow stream. The PME is calculated using the four-factor model cost of capital to discount cash flows. The sample is all Preqin quasi-liquidated private equity funds. The number of funds / capital allocated in a given year is taken from the full Preqin sample (see Table 3.1). Growth in year t refers to the growth rate in number of funds / capital raised from year t to year t+1. Panel D shows results from an OLS time series regression. The t-statistics are reported in italics and are based on Newest-West (1987) standard errors with four lags. Superscripts denote statistical significance at 1%, 5%, and 10% levels are denoted by a, b, and c, respectively. Cambridge Associates (CA) publishes one buyout quarterly return index and one venture capital quarterly return index. In regression analysis with either the full sample (PE) or the buyout (BO) sample we use the CA buyout index; when we use the sub-sample of venture capital funds we use the CA venture capital index.

Panel A: Yearly time-series of returns, flow, and yield spread

	Vintage year			CF PE annual	Forward moving	Growth	Growth
Year	IRR	Multiple	PME	PE index (g_t)	average of g_t	N-funds	Capital
1993	0.27	2.71	1.13	0.19	0.27	0.41	0.58
1994	0.36	2.77	1.38	-0.01	0.38	-0.06	0.19
1995	0.3	2.33	1.27	0.44	0.23	0.41	0.36
1996	0.17	1.77	1.06	0.28	0.16	0.27	0.84
1997	0.13	1.69	1.14	0.45	0.02	0.5	1.18
1998	0.09	1.5	1.05	0.27	0.08	0.04	-0.05
1999	0.07	1.36	0.89	0.5	0.04	0.56	0.95
2000	0.12	1.61	0.98	-0.19	0.11	-0.39	-0.41
2001	0.2	1.68	1.05	-0.05	0.17	-0.29	-0.33
2002	0.26	1.77	1.14	-0.25	0.25	-0.24	-0.09
2003	0.12	1.25	0.97	0.68	-0.05	0.76	0.47
2004	0.22	1.53	1.25	0.25	-0.01	0.54	1.56
2005	0.07	1.16	1.02	0.12	-0.04	0.23	0.79
2006	-0.21	0.67	0.62	0.23	-0.11	0.04	0.03
2007	-0.18	0.94	0.69	0.04	-0.18	-0.36	-0.32
2008	0.09	1.12	0.8	-0.57	n.m.	-0.51	-0.7

Panel B: Correlation matrix

	Vintage year			CF PE annual	Forward moving	Growth	
	IRR	Multiple	PME	PE index (g_t)	average of g_t	Nfunds	Capital
IRR	1	0.9	0.9	-0.05	0.87	0.16	0.21
Multiple	0.9	1	0.82	0.03	0.92	0.2	0.24
PME	0.9	0.82	1	0.17	0.75	0.37	0.46
Index (gt)	-0.05	0.03	0.17	1	-0.16	0.9	0.78
Forward gt	0.87	0.92	0.75	-0.16	1	0.02	0.06
Growth N-funds	0.16	0.2	0.37	0.9	0.02	1	0.92
Growth capital	0.21	0.24	0.46	0.78	0.06	0.92	1

Panel C:Venture capital funds and buyout funds

	Venture Capital funds				Buyout funds			
	CF index	Cambridge	Growth		CF index	Cambridge	Growth	
Year	($g_{vc,t}$)	Associates index	Nfunds	Capital	($g_{bo,t}$)	Associates index	Nfunds	Capital
1993	0.09	0.19	0.33	0.3	0.23	0.24	0.25	0.61
1994	0	0.17	0.25	0.93	0	0.13	-0.27	0.1
1995	0.52	0.47	0.07	0.18	0.4	0.24	0.64	0.21
1996	0.36	0.41	0.31	0.28	0.27	0.28	0.22	0.76
1997	0.46	0.34	0.48	0.71	0.52	0.31	0.77	1.61
1998	0.43	0.31	0.42	1.58	0.27	0.15	-0.21	-0.22
1999	1.14	2.93	0.8	1.43	0.06	0.44	0.23	0.79
2000	-0.29	0.2	-0.37	-0.39	-0.11	0.06	-0.55	-0.67
2001	-0.16	-0.4	-0.44	-0.68	0.05	-0.12	0.12	0.62
2002	-0.35	-0.34	-0.46	-0.42	-0.17	-0.08	-0.11	-0.14
2003	0.71	-0.04	0.87	0.77	0.5	0.22	0.71	0.36
2004	0.18	0.15	0.25	0.33	0.29	0.25	0.79	1.9
2005	0.06	0.07	0.4	1.48	0.26	0.28	0	0.71
2006	0.14	0.18	-0.02	-0.4	0.45	0.29	0.1	-0.13
2007	0.1	0.15	-0.46	-0.3	0.13	0.2	-0.26	-0.13
2008	-0.61	-0.16	-0.46	-0.3	-0.54	-0.22	-0.45	-0.71

Panel D: Regression analysis - capital flows and past performance

	Private equity		Venture capital		Buyout	
growth in	Nfunds	Capital	Nfunds	Capital	Nfunds	Capital
Constant	-0.19**	-0.09	-0.03	0.11	-0.27***	-0.43***
	-1.99	-0.38	-0.59	1.07	-2.6	-4.91
Our CF index, year t	1.11***	1.51***	1.01***	1.06***	1.52***	1.53***
	4.67	5.76	5.09	5.17	7.3	2.74
IRR, vintage year t	0.11	0.51	0.25	0.72	1.87***	3.80***
	0.77	1.48	0.85	1.02	6.11	5.54
Cambridge Associates index, year t	0.01	0.08*	-0.1	0.08	0.21	1.82***
	0.39	1.76	-1.56	0.89	0.53	2.93
IRR, vintage year t-1	-0.02	-0.39	-0.2	-0.51	-1.06***	-2.18***
	-0.12	-0.96	-0.52	-0.63	-5.16	-2.66
Adjusted R-square	75%	34%	59%	22%	64%	47%
Number of observations	16	16	16	16	16	16

Then, they start to increase. After the 2005 vintage year, the measures are also low. These patterns are counterintuitive because 1999 was anecdotally the best year ever for venture capital, as was the 2003-2007 time periods for buyout funds. In contrast, the returns in our cash flow-based index show 1999 and 2003 as high return years, while 2008 was the worst year. In other words, by unbundling vintage year returns we are able to more accurately identify good and bad years for private equity.

To further demonstrate that our contrasting result is mainly due to unbundling, we use our index to simulate vintage year returns by constructing a forward moving average of our index return. This forward moving average of the cash flow-based private equity index, , is the geometric yearly average return calculated from year $t + 1$ to $t + 5$. The four-year horizon reflects the typical duration of a private equity investment. [9] Our forward measure has a correlation of 0.87 with the vintage year IRR, which shows that our index is mainly an unbundled version of what is done in practice and in the literature.

[9]See Lopez-de-Silanes, Phalippou, and Gottschalg (2013). Note: The last year we use is 2009 and the forward moving average is not computed for 2008 as it is not meaningful.

This unbundling is important because it allows the identification of the performance cycles. These cycles cannot be identified with vintage years IRR (e.g. our yearly index exhibits a correlation of -0.05 with the vintage year IRR).

To assess whether our index captures actual performance cycles, we study the correlation between capital flows in the private equity fund industry and different past performance measures. Results in Panel B of Table 3.5 show that our index correlates highly with capital flows. [10] The correlation is as high as 90% with year-on-year growth in the number of funds. In contrast, the vintage-based performance measures have correlations close to zero with industry growth.

Panel C of Table 3.5 breaks down results for venture capital and buyout fund sub-samples and compares them with existing industry annual returns. The Cambridge Associate indexes track our indices fairly closely although significant differences occur for the venture capital series in 1999 and 2000. Our cash flow-based venture capital index is 114% in 1999 (while the Cambridge Associates index reaches 293%) and is -29% in 2000, while the downturn manifested itself in the Cambridge Associates indexes only in the following year.

Panel D documents the relationship between capital flows and different return series. We regress the measures of industry growth on the various industry return measures as well as our cash flow-based index. With only 16 years of data, the regression should be carefully interpreted and is only suggestive evidence of relative significance. With this caveat, we note that the coefficient on our index based private equity index and capital flows is positive and

[10] Growth in the number of funds and in capital raised have a 92% correlation with one another. The metric using number of funds is less sensitive to one large fund missing. Data source is the full Preqin sample (Table 3.1).

strongly statistically significant in all cases. In contrast the coefficient on both the vintage-based return measures and Cambridge Associates return series are only significant for the buyout sub-sample.

3.4.6 Private equity return cycles

Table 3.6 uses our cash flow-based private equity premium indexesthe alphas, to identify peaks and troughs in returns specific to private equity. The time-varying alpha is the pure private equity return component. Table 3.6 shows the start and end of private equity cycles broken down by fund types. In Panels A and B, a boom period is one that has more than two quarters in a row with time-varying alpha above one standard deviation above the mean. A bust period is one that has more than two quarters in a row with the time-varying alpha below one standard deviation below the mean. We thus identify cycles in a similar way various economic institutions, like the NBER, define economic cycles.

Table 3.6: Private Equity Premium Return Cycles

This table shows the start and end of private equity cycles broken down by fund types. In Panels A and B, a boom period is one that has more than two quarters in a row with the private equity return premium, , more than one standard deviation above the mean. A bust period is one that has more than two quarters in a row with more than one standard deviation below the mean. In panel A, alpha is derived from the four-factor model of Pstor and Stambaugh (2003). In Panel B, alpha is derived from the CAPM model. In Panel C, the definition of a boom/bust is the same except that we use the Cambridge Associates NAV-based quarterly return series. The time series starts at 1993:Q1 and ends at 2010:Q4.

Panel A: Alpha cycle in private equity using the Pstor -Stambaugh four-factor model

	Boom		Bust		Boom		Bust	
	Starts	Ends	Starts	Ends	Starts	Ends	Starts	Ends
All private equity funds	Q1-1997	Q3-2000					Q4-2007	-
Venture capital funds	Q1-1998	Q1-2001					Q2-2008	-
Buyout funds	Q4-1995	Q1-1997	Q4-1998	Q2-2001	Q2-2005	Q3-2007		
Real estate funds					Q4-2005	Q4-2006	Q4-2007	Q3-2010
High yield debt funds	Q1-1995	Q2-1996	Q4-1999	Q2-2002	Q3-2007	Q1-2008		

Panel B: Alpha cycle in private equity using the CAPM

	Boom		Bust		Boom		Bust	
	Starts	Ends	Starts	Ends	Starts	Ends	Starts	Ends
All private equity funds	Q3-1998	Q4-2000					Q3-2007	-
Venture capital funds	Q1-1998	Q4-2000					Q3-2008	-
Buyout funds	Q2-1996	Q3-1996	Q3-1998	Q3-2000	Q1-2005	Q1-2007		
Real estate funds					Q4-2002	Q4-2006	Q4-2007	Q2-2010
High yield debt funds			Q1-1997	Q2-2001	Q2-2002	Q1-2004		

Panel C: Return cycles in private equity according to the Cambridge Associates NAV-based index

	Boom		Bust		Boom		Bust	
	Starts	Ends	Starts	Ends	Starts	Ends	Starts	Ends
Venture capital funds	Q1-1999	Q1-2000	Q2-2002	Q4-2002				
Buyout funds							Q3-2008	Q1-2009
Real estate funds							Q4-2008	Q4-2009

In Panel A of Table 3.6, the alpha is derived from the four-factor model of Pstor and Stambaugh (2003). In Panel B, alpha is computed using the CAPM model. In Panel C, the definition of a boom or bust is the same as the other panels, except that we use the Cambridge Associates NAV-based quarterly return series. Table 3.6 clearly identifies the venture capital boom of late 1990s, along with the buyout boom of the mid-2000s. The real estate boom in the mid-2000s coincides with the buyout boom. The real estate bust around the crisis can also be seen in the data. These results are similar if we use a single-factor CAPM model to derive alpha or a four-factor model (Panels B and C). In contrast, cycles identified from the Cambridge Associates returns do not exhibit much boom-bust dynamics, if at all.

3.4.7 Test of the market segmentation hypothesis

The cyclicality of private equity represents a challenge to private equity investors who are faced with the decision to time their investments, or to maintain a continuous commitment to the asset class and manage expectations about short-term performance. This pattern is also difficult to explain in a standard economic framework. Kaplan and Strmberg (2009) introduce a novel theory of boom and bust cycles in private equity. They propose that buyout funds exploit segmentation between the debt and equity markets. [11] Kaplan and Strmberg (2009) extend the insights of the behavioral corporate finance literature to explain this correlation. In particular, Baker, Greenwood, and Wurgler (2003), and Baker and Wurgler (2000) present evidence that corporations

[11]Prior researchers have noted the connection between low interest rates and buyout fundraising, such as Ljungqvist, Richardson, and Wolfenzon (2008), Demiroglu and James (2010), Ivashina and Kovner (2011), Axelson et al. (2013).

choose financing channels based on the relative capital market demand for equity vs. debt. Kaplan and Strmberg (2009) argue that the ultimate source of the variation in relative demand for debt vs. equity is market sentiment, and they report suggestive evidence of this by charting a variable defined as the EBITDA/enterprise value minus the high yield spread. When this variable is high, private equity buyouts should be relatively profitable because the cost of debt financing in low compared to the return on asset.

Our cash flow-based private equity indexes allow us to empirically test the behavioral market segmentation hypothesis. In particular, we test whether private equity is profitable when the Kaplan-Stromberg asset-debt yield spread is higher. Table 3.7 reports the results of regressions in which our private equity cash flow returns are dependent variables and the independent variables include the asset-debt yield spread, the Baker-Wurgler sentiment index and a set of macro-economic variables that capture credit conditions (the default spread, which is the difference in yields on AAA and BAA AAA rated debt, and a survey of loan officers) and the health of the economy (growth in industrial production, inflation, and the change in the VIX index). [12]

Our specification jointly tests the theory that market sentiment provides the opportunity for private equity managers to create value, and that the source of that value is the asset-debt yield spread. If market sentiment is a significant determinant of the private equity return premium, we expect a positive sign on the sentiment index and a negative sign on the change in the VIX. In our specification the sign on the default spread may go either way since, by construction, it is negatively correlated to the asset-debt spread. Chen, Roll, and

[12] We use Newey and West (1987) standard errors and have 72 quarters of observations.

Ross (1986) argue that the default spread captures investor confidence about the economy. Another measure of confidence is the survey of loan officers. This and industrial production growth should be positively associated with the aggregate cash flow private equity indexes since buyout funds are, in effect, a levered exposure to the corporate sector of the economy. Innovations in these macroeconomic variables are rapidly priced in public capital markets, but not necessarily incorporated in private capital markets. Likewise, inflation is likely to have negative effects on nominal cash flow measures. The key prediction is that the asset-debt spread should be a positive determinant of the private equity return premium.

Panel A of Table 3.7 reports results for three specifications using both the aggregate cash flow- based private equity indices (g_t) and the private equity premium series (f_t). The specifications include either the asset-debt spread, industrial production or both. The first three columns of coefficients show that the index is significantly positively related to the asset-debt spread and the sentiment index, consistent with the Kaplan-Stromberg hypothesis. It is negatively related to the VIX inflation and the default spread and positively correlated to production and the survey of loan officers. The coefficients on production and inflation are insignificantly different from zero. These results are consistent with the hypothesis that private equity does well when the economy does well and when sentiment about the economy is positive. The second set of regressions repeats the estimate using the private equity premium. In this specification, only two variables are significant: growth in industrial production and the VIX.

One problem with interpreting the results based on the aggregate indices is that the Kaplan- Strmberg theory is actually about buyout funds. Our aggregate cash flow-based private equity indices are comprised of the returns for all four types of funds. To the extent that all asset types are similarly exposed to macro-economic conditions, this improves the power for estimate the relationship of private equity to the general economy, but it adds noise to the estimate of the co-variates of the premium series, f_t . We have seen that the premium cycles for buyout, venture capital, private equity and high yield funds differ significantly. Panel B uses only the cash flow-based buyout return indices. It also reports results for the Cambridge Associates buyout index.

The first two specifications show the results for our private equity total return index. As with the broad index results, the asset-debt yield variable is positive and significant, the survey of loan officers is positive and significant, the default spread and VIX coefficients are negative and significant. Sentiment and production lose significance for the buyout sub-index. Turning to the buyout premium index results, we see that the asset-debt coefficient is positive and significant as predicted, while the signs on the default spread and the VIX change. The significant, positive coefficient on the buyout premium represents a rejection of the null hypothesis that cheap relative financing terms are not a source of value-creation by private equity managers.[13]

One qualification of these findings is that we are measuring the contemporaneous effects of the asset-debt yield spread. The proposed channel by which this adds value is via the purchase of a higher yielding asset financed by issuing cheap debt. The fund cash flows we observe are deployment or realization of

[13] In the last two columns of Panel B we report regressions for the Cambridge buyout index. The results are consistent with theory and with the estimates from our aggregate cash flow private equity index.

capital and are thus conditional on such a transaction occurring. Nevertheless, our premium index assumes that all firms in operation at a given date experience the same shocks. If we could separate transacting firms from firms that were not exploiting the spread, we may find a larger effect.

3.5 Conclusion

Researchers and practitioners interested in understanding private equity investment have been limited by the structure and nature of the data. This has made it particularly difficult to evaluate its time-series characteristics. We present a methodology for extracting a latent performance measure from non-periodic cash flow information, and demonstrate how it may be further decomposed into passive and active components. We find that private equity returns are only partially spanned by investable passive indices. Our estimate suggests that private equity is, to a first approximation, a levered investment in small and mid-cap equities. We model the residual component of private equity returns which cannot be replicated in traded, public markets as an orthogonal variable with cyclical characteristics. We find that in the first part of our sample period the private equity premium contributed positively to returns and in the second period it detracted from returns. Our estimated autocorrelation coefficient is consistent with long-horizon cyclical behavior. We estimate the private equity premium for separate classes and show that their cycles are not highly correlated. This suggests that a diversified strategy across sub-asset classes of private equity may be beneficial. Our cash flow-based private equity indexes allow us to test current theories about the cyclical nature of private equity returns. In particular, we test the Kaplan and Strmberg (2009) hypothesis that relative yields on corporate asset compared to high-yield debt explain returns of buyout investment. We find evidence that the buyout premium is higher in quarters for which the asset-debt yield spread is higher. Consistent with the conjecture that this investment opportunity is related to behavioral frictions, we also find that the Baker-Wurgler sentiment variable is correlated to total private equity returns.

Table 3.7: Private Equity Premium Return Cycles

This Table shows how our cash flow-based total private equity return indexes and those of industry relate to macroeconomic variables. There are three different dependent variables: i) Time varying alpha (f_t) for private equity funds or only buyout funds; ii) the total private equity return index (g_t), either for all private equity funds or only buyout funds; iii) Cambridge Associates quarterly NAV-based buyout returns. The factor model used to derive and is the four-factor model of Pstor and Stambaugh (2003). We compute t-statistics using Newey-West (1987) standard errors with four lags, which are shown underneath each coefficient in italics. Time period is from the first quarter of 1993 to the last quarter of 2010.

Panel A:Cash flow-based private equity indices

	CF PE index (g_t)			Premium (f_t)		
Constant	0	0.05***	0	0	0	0
	-0.05	*4.36*	*0.09*	*-0.13*	*-0.88*	*0.71*
Ebitda/EV - High Yield spread	2.82***		2.55**	0.02		-0.24
	3.49		*2.46*	*0.09*		*-1.55*
Industrial Production growth		2.01***	0.47		0.30***	0.44***
		2.67	*0.48*		*2.72*	*3.38*
Default Spread (BAA-AAA)	-1.64***	-2.18***	-1.61***	0.02	0.1	0.04
	-2.73	*-4.53*	*-2.71*	*0.24*	*1.6*	*0.66*
Inflation	-1.72	-1.79	-1.69	0.1	0.14	0.13
	-1.19	*-1.32*	*-1.22*	*0.55*	*0.8*	*0.76*
Sentiment index	0.73***	0.67***	0.70***	0.04	0.01	0.01
	3.5	*3.02*	*3.15*	*1.3*	*0.34*	*0.27*
Survey of Loan Officer	0.24**	0.26**	0.24**	0.02	0.01	0.02
	2.09	*2.4*	*2.13*	*1.19*	*1.05*	*1.16*
Return VIX	-0.22***	-0.26***	-0.22***	0	0	0
	-6.04	*-5.92*	*-4.85*	*0.3*	*-0.51*	*-1.32*
Adjusted R-square	67%	64%	67%	-2%	12%	17%
Number of observations	72	72	72	72	72	72

Panel B: Cash flow-based buyout indices

Dependent Variable:	CF BO index ($g_{BO,t}$)		Premium ($f_{BO,t}$)		CA index	
Constant	0.05***	-0.01	0	-0.01**	0.04***	0.01
	5.41	*-0.49*	*-0.21*	*-2.48*	*5.07*	*0.46*
Ebitda/EV - High yield spread		3.01a		0.51***		1.45***
		3.06		*2.78*		*3.41*
Industrial production growth	2.02***	0.21	0.04	-0.27*	1.51***	0.63*
	2.74	*0.2*	*0.35*	*-1.75*	*5.41*	*1.86*
Default spread (BAA-AAA)	-2.55***	-1.88***	0.06	0.17**	-0.37	-0.04
	-5.72	*-3.8*	*1.11*	*2.54*	*-1.45*	*-0.17*
Inflation	-0.85	-0.73	0.04	0.06	-0.22	-0.16
	-0.61	*-0.54*	*0.38*	*0.7*	*-0.46*	*-0.32*
Sentiment index	0.03	0.06	0.01	0.01	0.36***	0.38***
	0.2	*0.43*	*0.2*	*0.43*	*3.09*	*3.17*
Survey of loan officer	0.23***	0.21***	0.01	0	0.05	0.04
	3.12	*2.6*	*0.38*	*0.11*	*1.63*	*1.44*
Return VIX	-0.22***	-0.18***	0	0.01**	-0.06***	-0.04**
	-5.25	*-4.07*	*0.75*	*2.34*	*-3.55*	*-2.34*
Adjusted R-square	58%	64%	-7%	14%	57%	63%
Number of observations	72	72	72	72	72	72

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Appendix **A**

Appendix to Chapter 1

A.1 Proof of the Propositions

In this section, I present the proofs of the Propositions 1.3, 1.1, 1.2 in the main body of the text.

Proof of Proposition 1.1

In following Bansal, Kiku, Shaliastovich and Yaron (2013), I assume that the market portfolio gives the aggregate consumption.[1] The total wealth portfolio's return, i.e. market return, is of the form

$$r_{M,t+1} = \kappa_0 + \kappa_1 lnpd_{t+1}^M - lnpd_t^M + \Delta d_{t+1}^M$$

According to the assumption of long-run risk model, the log price-dividend ratio only depend on the long-run expectation and conditional volatility of Δd_{t+1}^M.

$$lnpd_{t+1}^M = A_0 + A_1 g_t^M + A_2 \sigma_{g,t}^2$$

In the equation above, A_0, A_1 and A_2 are parameters, whose exact forms are ready to pinned down, but not important.

The log of pricing kernel in the Bansal Yaron (2004) model is $m_{t+1} = \theta \log \delta - \frac{\theta}{\psi} \Delta d_{t+1}^M + (\theta - 1) r_{a,t+1}$. Without loss of generality, I identify $r_{a,t+1}$ with the market return. By log-linearization in Campbell and Shiller (1991), the return is approximated by $r_{a,t+1} = \kappa_0 + \kappa_1 lnpd_{t+1}^M - lnpd_{t+1}^M + \Delta d_{t+1}^M$. Here the log of price-dividend ratio depends only on the long-run component in dividend growth and its conditional volatility by the assumption of long-run risk

[1] It is a traditional practice of substituting the return to aggregate wealth with the return on the stock marke in the static CAPM literature. It has also been practiced in empirical work based on recursive preferences (e.g., Epstein and Zin (1991) among others). The main reason of my choice of this assumption is that the empirical analysis this paper focus is the relative pricing in cross sections, this expedition in modeling has no influence on the covariances of the cash-flows and the discount rates.

model.
$$lnpd_t^M = A_0 + A_1 g_t^M + A_2(\sigma_{g,t}^M)^2$$
Plug it into the equation of aggregate returns, I can show that the innovation in the aggregate return is
$$r_{a,t+1} - \mathrm{E}_t(r_{a,t+1}) = \kappa_1 A_1 \Sigma_{gg}^{\frac{1}{2}} \epsilon_{t+1}^g + \kappa_1 A_2 \Sigma_{\sigma_g\sigma_g}^{\frac{1}{2}} \epsilon_{t+1}^{\sigma_g} + \sigma_{g,t}^M u_{t+1}^d$$
As a result the conditional variance in the return is linear in $(\sigma_{g,t}^M)^2$.
$$(\sigma_{z,t}^M)^2 \equiv \mathbf{Var}_t(r_{a,t+1}) = (\sigma_{g,t}^M)^2 + const$$
Plug the innovation in aggregate returns in the log of pricing kernel, I get the innovation in the pricing kernel as
$$m_{t+1} - \mathrm{E}_t(m_{t+1}) = (\theta - 1 - \frac{\theta}{\psi})\sigma_{g,t}^M u_{t+1}^d + (\theta - 1)[\kappa_1 A_1 \Sigma_{gg}^{\frac{1}{2}} \epsilon_{t+1}^g + \kappa_1 A_2 \Sigma_{\sigma_g\sigma_g}^{\frac{1}{2}} \epsilon_{t+1}^{\sigma_g}]$$
The risk premium is therefore given by
$$\begin{aligned} z_t \equiv \mathrm{E}_t(r_{a,t+1} - r_t^f) &= -\mathbf{Cov}_t(m_{t+1}, r_{a,t+1}) - \frac{1}{2}\mathbf{Var}_t(r_{a,t+1}) \\ &= -(\theta - 1 - \frac{\theta}{\psi})(\sigma_{g,t}^M)^2 - (\theta-1)\kappa^2 A_1^2 \Sigma_{gg} \\ &\quad -(\theta-1)\kappa_1^2 A_2^2 \Sigma_{\sigma_g\sigma_g} - \frac{1}{2}(\sigma_{z,t}^M)^2 \end{aligned}$$
I show that z_t is linear in $(\sigma_{g,t}^M)^2$ and $(\sigma_{g,t}^M)^2$, and since $(\sigma_{z,t}^M)^2$ is linear in $(\sigma_{g,t}^M)^2$, z_t is only a linear function of $(\sigma_{g,t}^M)^2$. The restrictions in the parameters are necessary given the linear relationships.

Proof of Proposition 1.2

It suffices to show that the market risk β is linear in the cash-flow exposure γ. For a specific Portfolio P, the pricing equation is
$$\begin{aligned} \mathrm{E}_t(r_{t+1}^P) + \frac{1}{2}\mathbf{Var}_t(r_{t+1}^P) - r_t^f &= -\mathbf{Cov}_t(r_{t+1}^P, m_{t+1}) \\ &= \frac{\mathbf{Cov}_t(r_{t+1}^P, m_{t+1})}{\mathbf{Cov}_t(r_{a,t+1}, m_{t+1})}[-\mathbf{Cov}_t(r_{a,t+1}, m_{t+1})] \end{aligned}$$
The term in the bracket is the price of risk, and the ratio of covariances is the market risk β. Given the assumption that the only resource is the long-run risk from the cash-flow. The β is therefore linear in the cash-flow exposure γ.

Proof of Proposition 1.3

In reviewing Campbell and Cochrane (1999), the utility function maximized by identical agents

$$\mathrm{E}\sum_{t=0}^{\infty}\delta^t\frac{(C_t - Hb_t)^{-\nu}-1}{\nu-1}$$

Here δ is the time discount factor, C_t is the level of consumption and Hb_t is the level of habit. Without loss of generality, I follow the original paper by setting the consumption growth as i.i.d.

$$\Delta c_{t+1} \equiv \log\frac{C_{t+1}}{C_t} = g_t^c + \sigma_{c,t}v_{t+1}$$

The log of pricing kernel is therefore

$$m_t = \log\delta - \nu(s_{t+1}-s_t) - \nu\Delta c_{t+1}$$

Here s_t is the log of surplus consumption ratio $S_t \equiv \frac{C_t - Hb_t}{C_t}$, and it subjects to a heteroskedastic AR(1)

$$s_{t+1} = (1-\phi)\bar{s} + \phi s_t + \Lambda(s_t)\sigma_{c,t}v_{t+1}$$

where $\Lambda(s_t)$ is the sensitivity function characterizing the heteroskedastic innovation in s_t. Denote λ_t as $z_t + \frac{1}{2}(\sigma_{z,t}^M)^2$. If the pricing kernel can be put in the form of

$$m_{t+1} = -r_t^f - \frac{1}{2}\frac{\lambda_t^2}{(\sigma_{z,t}^M)^2} - \frac{\lambda_t}{\sigma_{z,t}^M}v_{t+1}$$

, then equations (1.31) and (1.32) are satisfied. Comparing the above form of pricing kernel with the Campbell and Cochrane (1999) pricing kernel, as long as $\Lambda(s_t)$ and g_t^M satisfying

$$\lambda_t = \nu(1+\Lambda(s_t))\sigma_{g,t}\sigma_{z,t}$$

$$r_t^f = -\log\delta + \nu g_t^M + \nu[(\phi-1)(s_t-\bar{s})] - \frac{1}{2}\nu^2(1+\Lambda(s_t))^2(\sigma_{g,t}^M)^2,$$

then the Campbell and Cochrane (1999) model is embedded in the present-value model as a special case automatically.

A.2 Detail of Portfolio Sorting

I employ 35 test portfolios, which are seven groups of quintile portfolios sorted on different stock characteristics. These portfolios are common portfolios reflecting the anomali in the cross-sectional returns. For each sorting, I label the portfolios according to the *alpha* relative to the unconditional CAPM. The portfolios with the lowest *alpha* are labeled by number "1", while the portfolios with highest *alpha* are labled as number "5".

The size portfolios (SIZE1-SIZE5) are formed at the end of each December based on the market capitalization at the end of year.The book-to-market portfolios (BM1-BM5) are rebalanced at the end of June in every year, based on the book-to-market ratio of each stock. The practice of forming both sets of portfolios follow the instruction in Fama and French (1992). We form and rebalance the momentum portfolios (MOM1-MOM5) by the rank of past returns between month $t-12$ and month $t-1$, at the end of each quarter of a year, following Jegadeesh and Titman (1993). The set of idiosyncratic volatility portfolios (VOL1-VOL5) are constructed as described in Ang et al (2006). In accordance with the notation in their paper, we construct a 3/0/3 portfolio, which means the portfolios are rebalanced at the end of a quarter and held for three months, according to the idiosyncratic volatility in the past quarter. We form the the quintile Accruals portfolios (ACC1-ACC5) at the end of each March in year t based on the Accrual component in the earnings at the end of the year $t-1$. The definition of the accrual is from Sloan (1996). We sort tertile portfolios (CI1-CI5) at the end of each June based on the capital investment of the previous year, following Titman, et al (2004) and Liu, et al (2009). The last sets of tertile portfolios (LIQ1-LIQ5) are formed by sorting on the liquidity of the stocks, using the liquidity measure constructed in Amihud (2002).

A.3 Details of Calculating Bayes Factor

Denote M_1 as the model characterized by the hypothesis I want to test and M_0 is the general model without hypothetical constraints. The Bayes factor to test the hypothesis characterizing model M_1 is calculated as $2(L(M_0 \mid Y) - L(M_1 \mid Y))$, which is twice of the difference between marginal likelihood of the the two models given observable data Y. It suffices to show how to calculate these two marginal likelihood.

The methods of calculating these two marginal likelihoods are the same, following Chibs (1995). The marginal likelihood is obtained using output from Gibbs sampling. Without loss of generality, I show this algorithm of $L(M_0 \mid Y)$ only.

Suppose $p(Y|\Theta, X)$ is the sampling density (likelihood function) and $\pi(\Theta)$ and $\pi(X)$ are the prior density. Then, the marginal likelihood can be written as

$$p(M_0 \mid Y) = \frac{p(Y|\Theta, X)\pi(X)\pi(\Theta)}{p(X, \Theta \mid Y)}$$

owing to normalized by posterior density of latent variables and parameters $p(X, \Theta \mid Y)$. This identity holds for any Θ and X_t. For any value of Θ^* and X_t^*, the proposed estimates of log of marginal density is therefore

$$L(M_0 \mid Y) = \log p(Y \mid \Theta^*, X^*) + \log \pi(\Theta^*) + \log \pi(X^*) - \log p(\Theta^*, X^* \mid Y)$$

The likelihood function and priors density are ready given any specification of X^* and Θ^*. I only need to estimate the posterior density. As the Gibbs sampler is defined through iterations of conditional density of X and Θ, the log of posterior density of latent variables and parameters can be written as

$$\log p(\Theta^*, X^* \mid Y) = \log p(\Theta^* \mid X^*, Y) \log p(X^* \mid Y)$$

The first term is the marginal coordinate, which can be estimated by initial draw of Gibbs sampler. The second term is given by

$$p(X^* \mid Y) = \int p(X^* \mid \Theta, Y) p(\Theta \mid Y) d\Theta$$

This can be estimated by draws from the reduced complete conditional Gibbs run.

$$\hat{p}(X^* \mid Y) = N^{-1} \sum_{j=1}^{N} p(X^* \mid \Theta^{(j)}, Y)$$

A.4 Details of Calculating Fractions of Variation in Variance Decomposition

In this section, I present how to calculate the fractions accounted by state variables, in decomposition of the time-series and cross-sectional variance of *lnpd*, expected and realized returns.

In general, the state vector with I variables can be written as $X_t = (x_t^{(i)})_{i=1}^{I}$. Notate the state vector muting the $i-th$ variable as $X_t^{(-i)}$.

A.4.1 Time-series Variance Decomposition

I first address the time-series variance decomposition. As aforementioned in Equation (**??**), the *lnpd* is a function of these state variables.

$$lnpd_t = lnpd(X_t) \equiv A^P + B^{P\prime} X_t + X_t' Q^P X_t' \tag{A.4.1}$$

The the fraction of time-series variance of *lnpd* explained by the $i-th$ $(i \leq I)$ state variable is therefore,

$$R_i^2 = 1 - \frac{var(lnpd(X_t^{-i}))}{var(lnpd(X_t))}$$

The expected and realized return can also be written as function of state vector X_t, by virtue of *lnpd*. Using Campbell and Shiller (1988) log-linearization, the expected return relates to the expected and current *lnpd*.

$$\mathrm{E}_t(r_{t+1}^P) = \kappa_0 + \kappa_1 \mathrm{E}_t(lnpd_{t+1}^P) - lnpd_t^P + \mathrm{E}_t(\Delta d_{t+1}^P) \tag{A.4.2}$$

From Equation (A.4.1), one can derive the expected *lnpd* as function of state vector X_t.

$$\mathrm{E}_t(lnpd_{t+1}) = A^P + B^{P\prime}\Phi_P X_t + X_t'\Phi_P' Q^P \Phi_P X_t \quad \text{(A.4.3)}$$

Therefore one can see $\mathrm{E}_t(r_{t+1}) = Er(X_t)$ is a function of the state vector X_t.

Then the fraction of time-series variance of expected return explained by the $i-th$ $(i \leq I)$ state variable is,

$$R_i^2 = 1 - \frac{var(Er(X_t^{-i}))}{var(Er(X_t))}$$

For the realized return, Campbell and Shiller (1988) shows it has three components, expected return, unexpected shock in *lnpd* and unexpected distribution in CF.

$$r_{t+1} = \mathrm{E}_t(r_{t+1}) + \kappa_1(lnpd_{t+1} - \mathrm{E}_t(lnpd_{t+1})) + (\Delta d_{t+1} - \mathrm{E}_t(\Delta d_{t+1})) \quad \text{(A.4.4)}$$

As the shock in *lnpd* has both DR and CF information, I further decompose it into two shocks: one is purely due to change in DR (I_{DR}) and the other is purely due to change in CF (I_{CF}).

$$\kappa_1(lnpd_{t+1} - \mathrm{E}_t(lnpd_{t+1})) = I_{DR,t+1} + I'_{CF,t+1} \quad \text{(A.4.5)}$$

where

$$I_{DR,t+1} = \frac{1}{2}\kappa_1[(lnpd(\mu_{t+1},\omega_{t+1}) - lnpd(\mu_t,\omega_{t+1})) + (lnpd(\mu_{t+1},\omega_t) - \mathrm{E}_t(lnpd_{t+1}))] \quad \text{(A.4.6)}$$

$$I'_{CF,t+1} = \frac{1}{2}\kappa_1[(lnpd(\mu_{t+1},\omega_{t+1}) - lnpd(\mu_{t+1},\omega_t)) + (lnpd(\mu_t,\omega_{t+1}) - \mathrm{E}_t(lnpd_{t+1}))] \quad \text{(A.4.7)}$$

I then combine the shocks due to update in CF and unexpected realization of CF together to get $I_{CF,t+1} = I'_{CF,t+1} + (\Delta d_{t+1} - \mathrm{E}_t(\Delta d_{t+1}))$.

As a result, besides the state vector X_t, the realized return is also function of these two unexpected components, $r_{t+1} = r(X_t, I_{DR,t+1}, I_{CF,t+1})$. For convenience, I notate $I_{DR,t+1}$ as x_t^{I+1}, and $I_{CF,t+1}$ as x_t^{I+2}.

The fraction of time-series variance of realized return explained by the $i-th$ $(i \leq I+2)$state variable is

$$R_i^2 = 1 - \frac{var(r(X_t^{-i}))}{var(r(X_t))}$$

A.4.2 Cross-sectional Variance Decomposition

I first illustrate the cross-section variation decomposition in a general notation. Suppose one is interested in analyzing the cross-sectional variation in the average of f, and the average of f is of the function form of K cross-sectional statistics $s_1, ..., s_K$.

$$\bar{f}^P = \bar{f}(s_1^P, ..., s_K^P)$$

Therefore, the fraction of the cross-sectional variance attributed to the $k-th$ cross-sectional statistics is calculated as the remaining portion that cannot be explained by the other arguments.

$$R_k^2 = 1 - \frac{var(\bar{f}(s_1^P, ..., s_{k-1}^P, 0, s_{k+1}^P, ..., s_K^P)}{var(\bar{f}(s_1^P, ..., s_K^P))} \tag{A.4.8}$$

To analyze the cross-sectional variance of *lnpd*, expected and realized returns, one need to find the function forms of their mean. Taking unconditional expectation on both sides of Equation (A.4.1), (A.4.2) and (A.4.4), one can see the mean of *lnpd* and returns as function of cross-sectional statistics characterizing mean of DR and CF of individual stocks. I use over-line to notate for the average.

$$\overline{lnpd_t^P} = \overline{lnpd}(\bar{\beta}^P, \alpha^P, \text{cov}(\beta^P, z), \text{cov}(\beta^P, (\sigma_z)^2), \gamma^P, c^P, \text{cov}(\gamma^P, g), \text{cov}(\gamma^P, (\sigma_g)^2)) \tag{A.4.9}$$

$$\overline{\text{E}_t(r_{t+1}^P)} = \overline{Er}(\bar{\beta}^P, \alpha^P, \text{cov}(\beta^P, z), \text{cov}(\beta^P, (\sigma_z)^2), \gamma^P, c^P, \text{cov}(\gamma^P, g), \text{cov}(\gamma^P, (\sigma_g)^2)) \tag{A.4.10}$$

$$\overline{r_{t+1}^P} = \overline{r}(\bar{\beta}^P, \alpha^P, \text{cov}(\beta^P, z), \text{cov}(\beta^P, (\sigma_z)^2), \gamma^P, c^P, \text{cov}(\gamma^P, g), \text{cov}(\gamma^P, (\sigma_g)^2), \overline{I_{DR}}, \overline{I_{CF}}) \tag{A.4.11}$$

Plugging these functions forms in Equation (A.4.8), one can obtain the fraction of the total cross-section variance of various observables attributed to each explaining components.

Appendix **B**

Appendix to Chapter 2

In this appendix, we value the shortfall option. Our approach follows the analytical approximation in Alexander and Venkatramanan (2011). The time to maturity of the option is assumed to be one year.

B.1 Spread Option Interpretation

In the first place, we study the asset allocation between risky equity and risk-free cash, as a benchmark case. The case of allocation between equities and risky bonds will be addressed in the next section. We denote the market value of liability and the asset portfolio by L_t and A_t respectively, and the payoff of the option by $\max\{L_1 - A_1, 0\}$.

The market value of the liability at the end of the year is

$$L_1 = L_0 \exp\left((\mu_L - \frac{\sigma_L^2}{2}) + \sigma_L W_1^L\right),$$

where W_t^L is a Brownian motion process for the liabilities. We assume that the weight on equity and cash are chosen at the beginning of the period and not rebalanced during the year. The market value of the portfolio managed by the fund is

$$A_1 = wA_0 \exp\left((\mu - \frac{\sigma_E^2}{2}) + \sigma_E W_1^E\right) + (1-w)A_0 \exp(r_f),$$

where W_t^E is a Brownian motion process for equities. Note that A_1 does not satisfy the assumption of a log-normal diffusion. Thus, the exchange option pricing formulas of Fisher (1978) and Margrabe (1978) do not apply for valuing $P(w, A_0, L_0) = \mathbf{E}^{\mathbb{Q}}[\max(L_1 - A_1, 0)]$, where $\mathbb{Q}$ is the risk-neutral measure. The exchange options are good approximations when option maturities are very short, as Alexander and Venkatramanan (2011) comment.

Let us define

$$\begin{aligned} S_{1,t} &= L_t \\ S_{2,t} &= wA_0 \exp\left((\mu - \frac{\sigma_E^2}{2})t + \sigma W_t^E\right) \\ K &= (1-w)A_0 \exp(r_f). \end{aligned} \tag{B.1}$$

As both S_1 and S_2 are log-normally distributed, we can transform the problem into pricing a spread option with the underlying assets being S_1 and S_2, and the strike spread being K:

$$P(w, A_0, L_0) = \mathbf{E}^{\mathbb{Q}}[\max(L_1 - A_1, 0)] = \mathbf{E}^{\mathbb{Q}}[\max(S_{1,1} - S_{2,1} - K, 0)]. \tag{B.2}$$

We employ the analytical approximation of spread options as compound exchange options following Alexander and Venkaramanan (2011). The compound exchange option representation appears to provide the most precise estimate of the value of spread options.

B.2 Valuation of the Shortfall Option

Define m to be a real number such that $m \geq 1$. Let us define the regions:

$$\begin{aligned} \mathbf{L} &= \{S_{1,1} - S_{2,1} - K \geq 0\} \\ \mathbf{A} &= \{S_{1,1} - mK \geq 0\} \\ \mathbf{B} &= \{S_{2,1} - (m-1)K \geq 0\}. \end{aligned} \tag{B.3}$$

Then the spread option's payoff of strike K at year end can be written as:

$$\begin{aligned} \mathbf{1_L}[S_{1,1} - S_{2,1} - K] &= \mathbf{1_L}(\mathbf{1_A}[S_{1,1} - mK] - \mathbf{1_B}[S_{2,1} - (m-1)K] \\ &\quad +(1 - \mathbf{1_A})[S_{1,1} - mK] - (1 - \mathbf{1_B})[S_{2,1} - (m-1)K]). \end{aligned} \tag{B.4}$$

With some algebraic manipulation, Alexander and Venaramanan (2011) show that

$$P(w, L_0, A_0) = e^{-r_f}(\mathbf{E}^{\mathbb{Q}}\{[(U_{1,1} - U_{2,1})^+\} + \mathbf{E}^{\mathbb{Q}}\{[(V_{2,1} - V_{1,1})^+]\}) \tag{B.5}$$

where $U_{1,1}, V_{1,1}$ are payoffs to European call and put options on S_1 with strike mK, respectively. Likewise, $U_{2,1}, V_{2,1}$ are European call and put options on S_2 with strike $(m-1)K$, respectively. The spread option is thus equivalent to compound exchange options on two calls and two puts. The parameter m is chosen such that the single-asset call options are deep-in-the-money. For our calibrations we choose $m = 5$ which satisfies the approximation conditions in Alexander and Venaramanan (2011).

The calls and puts can be described as:

$$\begin{aligned} dU_{i,t} &= r_f U_{i,t} dt + \xi_i U_{i,t} dW^{\mathbb{Q}}_{i,t} \\ dV_{i,t} &= r_f V_{i,t} dt + \eta_i V_{i,t} dW^{\mathbb{Q}}_{i,t}. \end{aligned} \quad \text{(B.6)}$$

Note that $U_{1,0}$ and $V_{1,0}$ are the Black-Scholes prices at $t = 0$. By applying Ito's theorem on the calls and the puts, the parameters ξ and η are:

$$\begin{aligned} \xi_i &= \sigma_i \frac{S_{i,t}}{U_{i,t}} \frac{\partial U_{i,t}}{\partial S_{i,t}} \\ \eta_i &= \sigma_i \frac{S_{i,t}}{V_{i,t}} \left| \frac{\partial V_{i,t}}{\partial S_{i,t}} \right| \end{aligned} \quad \text{(B.7)}$$

Under our assumption that S_1 and S_2 are driven by geometrical Brownian motion processes, the calls and puts in equation (B.5) can be approximated as log normal even though the spread option is not log normal by suitable choice of m.

We can now apply the Margrabe's (1978) formula for exchange options on equation (B.5):

$$P(w, L_0, A_0) = e^{-r_f}[U_{1,0}N(d_{1U}) - U_{2,0}N(d_{2U}) - (V_{1,0}N(-d_{1V}) - V_{2,0}N(-d_{2V})] \quad \text{(B.8)}$$

where $N(\cdot)$ represents the normal cumulative density function, and the parameters d_{1X} and d_{2X} for $X \in \{U, V\}$ are given by

$$\begin{aligned} d_{1X} &= \frac{\ln\left(\frac{X_{10}}{X_{20}}\right) + \frac{1}{2}\sigma_X^2}{\sigma_X} \\ d_{2X} &= d_{1X} - \sigma_X \end{aligned} \quad \text{(B.9)}$$

The volatilities of the call and put are given by

$$\begin{aligned} \sigma_U &= \sqrt{\xi_1^2 + \xi_2^2 - 2\rho\xi_1\xi_2} \\ \sigma_V &= \sqrt{\eta_1^2 + \eta_2^2 - 2\rho\eta_1\eta_2} \end{aligned} \quad \text{(B.10)}$$

The correlation used to compute the exchange option volatility is the implied correlation between two vanilla calls or puts. They are the same as the correlation between the underlying prices of the two assets, as the options and the underlying prices are driven by the same Brownian motions. Note also that as the underlying asset 1 is the liability, $S_{1,0} = L_0$ and $\sigma_1 = \sigma_L$. Similarly as the underlying asset 2 is the equity portion of the portfolio, $S_{2,0} = wA_0$ and $\sigma_2 = \sigma_E$.

B.3 Equity and Risky Bond Case

In this section, we value the shortfall option when the bond is risky. The risky bond has log normal price process:

$$\frac{dB}{B} = (\mu_B - \frac{\sigma_B^2}{2})dt + \sigma_B dW_{B,t} \tag{B.11}$$

We choose the risky bond as the numeraire, and the equivalent martingale measure associated with this numeraire is denoted as $\mathbb{R}$. The liability and equities have the following normalized price processes:

$$\begin{aligned} \frac{d(L/B)}{L/B} &= \sigma_L dW^{\mathbb{R}}_{L,t} - \sigma_B dW^{\mathbb{R}}_{B,t} \\ \frac{d(E/B)}{E/B} &= \sigma_E dW^{\mathbb{R}}_{E,t} - \sigma_B dW^{\mathbb{R}}_{B,t} \end{aligned} \tag{B.12}$$

The contingent claim $[L_1 - A_1]^+$ can be priced as

$$P(w, A_0, L_0, B_0) = B_0 \mathbf{E}^{\mathbb{R}}\{[L_1 - A_1]^+ / B_1\} \tag{B.13}$$

We can then write $[L_1 - A_1]^+/B_1$ in the form of $(S_{1,1} - S_{2,1} - K)^+$, where

$$\begin{aligned} S_{1,t} &= L_t / B_t \\ S_{2,t} &= w \frac{A_0}{B_0} \exp\left(-\frac{\sigma_E^2 - 2\rho_{EB}\sigma_E\sigma_B - \sigma_B^2}{2} t + \sigma_E W^{\mathbb{R}}_{E,t} - \sigma_B W^{\mathbb{R}}_{B,t} \right) \\ K &= (1-w) A_0 / B_0 \end{aligned} \tag{B.14}$$

The rest of the pricing approach is identical to what we outlined in the previous section.

Appendix C

Appendix to Chapter 3

C.1 Identification of Private Equity Returns

Consider the following four funds, which live between times $t = 0$ and $t = 4$:

Times	PE return (g)	Fund 1	Fund 2	Fund 3	Fund 4
0		-100	-100		
1	5.90%	**53.0**	0	-100	-100
2	17.50%	**31.1**	**124.4**	**117.5**	0
3	-4.80%	**29.6**	0		111.9
4	31.70%			**131.7**	

All the cash flows represent money paid or received by Limited Partners (LPs). The contributions into the funds are denoted by negative signs and have all been normalized to 100. Distributions from the funds are marked in bold and are represented by positive numbers. Each of the four funds begins with an initial investment of 100. Funds 1 and 2 start at time 0, and funds 3 and 4 start at time 1. Fund 1 pays intermediary dividends and pays half of the fund value out each year, except in the last year where it pays out the remainder. Fund 3 invests in two projects sequentially and the other funds dissolve after only one project.

We do not observe the private equity return, g_t . If the funds are correctly priced, then the fund investments must satisfy a NPV condition, which is

$$PV(I) = PV(D) \tag{C.1}$$

where PV denotes present value, I represents the investments made, and D the distributions received. The NPV conditions for the four funds are:

$$
\begin{aligned}
Fund1 :100 &= \frac{53}{1+g_1} + \frac{31.1}{(1+g_1)(1+g_2)} + \frac{29.6}{(1+g_1)(1+g_2)(1+g_3)}, \\
Fund2 :100 &= \frac{124.4}{(1+g_1)(1+g_2)}, \\
Fund3 :100 + \frac{100}{(1+g_2)(1+g_3)} &= \frac{117.5}{1+g_1} + \frac{131.7}{(1+g_2)(1+g_3)(1+g_4)}, \\
Fund4 :100 &= \frac{111.9}{(1+g_2)(1+g_3)}.
\end{aligned}
\tag{C.2}
$$

This constitutes a non-linear system of four equations with four unknowns, g_1, g_2, g_3, and g_4 . Treating each of these discount rates as separate parameters, we can solve this system (with a non-linear root solver). This yields the solution of the private equity returns listed in the table. Hence, using the NPV conditions allows us to estimate the private equity returns each period using LP cash flows.

The NPVs of all the funds do not involve the complete set of discount rates. Since only funds 1 and 2 are alive at time 1, only those funds identify g_1. All funds are alive at time 2, so all their NPVs involve the time 2 discount rate, g_2. As only fund 4 is alive at time 4, g_4 enters only the NPV equation of fund 4. Intuitively, identification is obtained because when we change a particular discount rate, like g_4, only certain NPVs are affected by that change. At a given time, all funds that are alive at that time are subject to the same discount rate. If the discount rate at that time changes, the NPVs of the funds alive at that time are affected.

Now suppose that the same discount rate applies at all periods, so $g_t = \bar{g}$ for $t = 1, \cdots, 4$. Then, there are four NPV equations but only one discount rate. Thus, the system is over-identified. We can estimate the constant discount rate by assuming an orthogonality condition or distribution for the NPV equations. In our empirical work, we work with the ratio

$$\frac{PV(I)}{PV(D)} = 1 \tag{C.3}$$

which is equivalent to equation (C.1). [1] If we use the NPV itself in equation (C.1), the error in fitting the NPV condition may be large simply because the fund size is large. In equation (C.3), the ratio of the present value of investments to the present value of distributions does not have this problem as the size of the cash flows roughly cancels out in both the numerator and denominator.

There is an intermediate case between assuming that each discount rate for each period is a free parameter and the case of a constant discount rate for all periods. We parameterize the private equity return to be a persistent process, so that we require fewer funds than discount rates for identification. In fact, it is precisely over-identification which makes our procedure robust. The many funds in existence at a point in time provide over-identifying restrictions to estimate the latent discount rate. Since there are different funds that start and end across time, we can identify discount rates across time.

It is useful to contrast the returns with the IRR. Our returns apply to all funds. In contrast, the internal rate of return (IRR) commonly used as a return heuristic by private equity industry participants is usually computed at the fund level. Funds are often grouped into separate vintages, and the IRRs associated with funds in different vintages are taken as performance measures. Our approach differs in two ways from the IRR. First, we estimate the same set of discount rates across funds, rather than inferring one discount rate, the IRR, from each fund. Second, by using many simultaneous funds with different cash flows in different periods, we can identify a time series of returns which are common to all private equity projects. The only time variation that can be achieved by fund-level IRRs is to examine ten-years overlapping IRRs of funds in different vintage years.

The literature has used various estimation procedures when the system is over-identified. In the real estate literature, estimation has typically involved (generalized) least-squares procedures. These techniques have been applied to residential real estate (Bailey, Muth, and Nourse (1963), Case and Shiller (1987)) and commercial property (Geltner and Goetzmann (2000)). Similar procedures have been used by Peng (2001) and Hwang, Quigley, and Woodward (2005) to estimate returns to venture capital. In the private equity literature, Driessen, Lin, and Phalippou (2012) employ a generalized method of moments

[1] Equation (C.3) is similar to the ratios introduced by Ljungqvist and Richardson (2003) and Kaplan and Schoar (2005). In the public market equivalent (PME) ratio of Kaplan and Schoar (2005), the present values of the investments and distributions are computed with the market return, or equivalently it is assumed that the private equitys discount rate is the same as the aggregate equity market. They interpret private equity as out-performing the market if the PME is greater than one. In equation (C.3), we compute the present values using discount rates which are endogenously determined. Nevertheless, the same intuition as Kaplan and Schoar holds in the sense that when private equity investments are fairly priced with appropriate discount rates, the ratio of the PV of investments to the PV of distributions should equal one.

estimator to a set of counstant discount rates, similar to the assumption that $g_t = \bar{g}$ for all t.

Our innovation is to introduce a way to extract multiple latent factorsand factor loadingfrom infrequent transactions data when the latent factor can be persistent, and there are also observable factors. The estimation is detailed in the following section.

C.2 Estimation of the Model

We re-state the model here for convenience. We can merge equations (3.2) and (3.3) into one equation containing only the latent state variable,g_t^e, which is the state equation:

$$g_t^2 = (1 - \phi)\alpha + \phi g_{t-1}^e + \beta'(F_t - \phi F_{t-1}) + \sigma_g \epsilon_t, \tag{C.4}$$

where the systematic factors, F_t, are observable. We assume that the zero NPV condition in equation (3.4) holds, and we specify that the log ratio of the PV of the distributions to the PV of investments is normally distributed:

$$\ln \frac{PV_i(D)}{PV_i(I)} \sim N(\mu, \sigma^2), \tag{C.5}$$

Equation (C.2) which repeats equation (3.6) represents the likelihood function of the cash flows. To ensure that the ratio of the present value of distributions and the present value of investments are centered at one, we set $\mu = -\frac{1}{2}\sigma^2$. This is equivalent to assuming that the errors of the log ratio of the PV of distributions to the PV of investments have zero mean.

Equations (C.1) and (C.2) constitute a state equation and a non-linear observation equation. The following algorithm filters the latent state variable g_t^e given the observation equations. Once g_t^e is estimated, we can infer the private equity-specific return, f_t , using

$$f_t = g_t^e - (\alpha + \beta' F_t), \tag{C.6}$$

We denote the parameters $\theta = (\alpha, \beta, \phi, \sigma_g, \sigma)$ and let θ denote the full set of parameters less the parameter that is being estimated in each conditional draw. We collect the exogenous private equity cash flow data and the common tradable factors F_t as $Y_t = \{\{I_{it}\}, \{D_{it}\}, \{F_t\}\}$. We estimate the model described by MCMC and Gibbs sampling. Other similar models are estimated by Jacquier, Polson, and Rossi (2004), Jacquier, Polson, and Rossi (1994), Ang and Chen (2007), which involve latent state variables. These papers are able to directly use observable returns. In contrast, we use non-linear NPV equations to infer returns. This makes our estimation more similar to Chen (2013), who infers latent discount rate and cash flow factors from price-dividend ratios.

A textbook exposition of Gibbs sampling is provided by Robert and Casella (1999).

In each of our estimations, we use a burn-in period of 20,000 draws and sample for 80,000 draws to produce the posterior distributions of latent state variables and parameters. With this large number of sampling, our estimation converges in a sense of passing the Geweke (1992) convergence test.

The Gibbs sampler iterates over the following sets of states and parameters conditioned on other parameters and states variables, to converge to the posterior distribution of $p(\{g_t^e\}, \theta|Y)$:

1. Private equity returns: $p(\{g_t^e\}|\theta, Y)$,
2. Parameters of the private equity-specific return: $p(\beta, \phi, \alpha|\theta_-, \{g_t^e\}, Y)$,
3. Standard deviation of the private equity return shocks: $p(\sigma_g|\theta_-, \{g_t^e\}, Y)$, and
4. Standard deviation of likelihood errors$p(\sigma|\theta_-, \{g_t^e\}, Y)$:

We discuss each one in turn.

Private equity returns, $p(\{g_t^e\}|\theta, Y)$

We draw g_t^e using single-state updating Metropolis-Hasting algorithm (see Jacquier, Polson, and Rossi (1994), Jacquier, Polson, and Rossi (2004)). For a single state update, the joint posterior is:

$$\begin{aligned} p(g_t^e|\{g_i^e\}_{i\neq t}\theta, Y) &\propto p(Y|\{g_t^e\}_{t=1}^T, \theta)p(\{g_t^e\}_{t=1}^T, \theta, Y) \\ &\propto p(Y|\{g_t^e\}_{t=1}^T, \theta)p(g_t^e|g_{t-1}^e, g_{t+1}^e, \theta, Y) \\ &\propto p(Y|\{g_t^e\}_{t=1}^T, \theta)p(g_t^e|g_{t-1}^e, \theta, Y)p(g_{t+1}^e|g_t^e, \theta, Y)p(g_t^e) \end{aligned} \tag{C.7}$$

We can go from the second to third line in equation (C.7) because g_t^e is Markov. In equation (C.7), the distribution $p(Y|\{g_t^e\}_{t=1}^T, \theta)$ is the likelihood function in equation (C.5). The distribution of $p(g_t^e|g_{t-1}^e, \theta, Y)$ and $p(g_{t+1}^e|g_t^e, \theta, Y)$ are implied by the dynamics of g_t^e in equation (C.4).

They can be expressed as:

$$\begin{aligned} p(g_t^e|g_{t-1}^e, \theta, Y) &\propto \exp(-\frac{1}{2\sigma_g^2}(g_t^e - (1-\phi)\alpha - \phi g_{t-1}^e - \beta'(F_t - \phi F_{t-1}))^2) \\ p(g_{t+1}^e|g_t^e, \theta, Y) &\propto \exp(-\frac{1}{2\sigma_g^2}(g_{t+1}^e - (1-\phi)\alpha - \phi g_t^e - \beta'(F_{t+1} - \phi F_t))^2) \end{aligned} \tag{C.8}$$

Collecting terms and completing the squares, we obtain

$$p(g_t^e|\{g_i^e\}_{i\neq t}, \theta, Y) \propto p(Y|\{g_t^e\}_{t=1}^T, \theta)\exp(-\frac{(g_t^e - \mu)^2}{2\sigma_g^2}(1+\sigma^2))p(g_t^e) \tag{C.9}$$

where

$$\mu_t = \frac{\phi(g^e_{t-1} + g^e_{t+1} + (1-\phi)\alpha + \beta'((1+\phi^2)F_t - \phi(F_{t+1} + F_{t-1}))}{1+\phi^2} \tag{C.10}$$

For the prior of g^e_t, we impose an uninformative prior, $p(g^e_t) \propto 1$. We use a Metropolis-Hasting draw with the proposal density

$$q(g^e_t) \propto \exp(-\frac{(g^e_t - \mu_t)^2}{2\sigma^2_g}(1+\phi^2)) \tag{C.11}$$

The acceptance probability for the $(k+1) - th$ draw, $g^{e,(k+1)}_t$, is

$$\min(\frac{p(Y|g^{e,(k+1)}_t, \{g^e_i\}_{i\neq t}, \theta)}{p(Y|g^{e,(k)}_t, \{g^e_i\}_{i\neq t}, \theta)}, 1). \tag{C.12}$$

When drawing g^e_t at the beginning or the end of the sample, we integrate out the initial and end values drawing from the process in equation (C.4).

Parameters of the private equity-specific return: $p(\beta, \phi, \alpha|\theta_-, \{g^e_t\}, Y)$

Consider the factor loadings, β. We can write the posterior

$$\begin{aligned} p(\beta|\theta_-, \{g^e_t\}, Y) &\propto p(Y|\beta, \theta_-, \{g^e_t\})p(\{g^e_t\}|\beta, \theta_-)p(\beta) \\ &\propto p(\{g^e_t\}|\beta, \theta_-)p(\beta) \end{aligned} \tag{C.13}$$

because β does not enter the dynamics of the private equity returns, g^e_t. We can rewrite equation (C.4) as

$$g^e_t - (1-\phi)\alpha - \phi g^e_{t-1} = \beta'(F_t - \phi F_{t-1}) + \sigma_g \epsilon_t, \tag{C.14}$$

which implies a standard regression draw for β . We use a normal conjugate prior. The draws of ϕ and α are similar. Although they could be drawn directly in a multivariate conjugate regression draw, we separate them. This allows us to place separate priors on each parameter.

Standard deviation of the private equity return shocks: $p(\sigma_g|\theta_-, \{g^e_t\}, Y)$

We draw σ^2_g using a conjugate Inverse Gamma draw. We select a truncated conjugate prior by confining the range of between 0.1% and 100% per quarter. We assume the prior

$$p(\sigma^2_g) \sim IG(\frac{a_0}{2}, \frac{b_0}{2})\mathbf{1}_{[10^{-6},1]} \tag{C.15}$$

where $a_0 = 2$ and $b_0 = 10^{-6}$. The peak of this prior is far left to the lower bound of our range; therefore, the truncated prior is approximately a uniform distribution on the range.

We draw the posterior distribution of σ_g^2 from its truncated conjugate posterior:

$$p(\sigma_g^2|\theta_-, Y) \sim IG(\frac{a_1}{2}, \frac{b_1}{2})\mathbf{1}_{[10^{-6},1]} \quad \text{(C.16)}$$

where $a_1 = a_0 + T - 1$, and $b_1 = b_0 + u$, and u is given by

$$u = \sum (g_t^e - (1-\phi)\alpha - \phi g_t^e - \beta'(F_t - F_{t-1}))^2 \quad \text{(C.17)}$$

Standard deviation of likelihood errors$p(\sigma|\theta_-, \{g_t^e\}, Y)$

We draw σ^2 using a conjugate truncated Inverse Gamma distribution. This follows a similar method to the draw for σ^2 . We assume the prior

$$p(\sigma^2) \sim IG(\frac{A_0}{2}, \frac{B_0}{2})\mathbf{1}_{[10^{-6},1]} \quad \text{(C.18)}$$

with $A_0 = 10^{-6}$ and $B_0 = 10^{-6}$. Denote

$$s = \sum_i (\ln \frac{PV(D_i)}{PV(I_i)})^2.$$

Then the posterior distribution is

$$p(\sigma^2|\theta_-, Y) \sim IG(\frac{A_0 + N}{2}, \frac{B_0 + s}{2})\mathbf{1}_{[10^{-6},1]} \quad \text{(C.19)}$$

Priors

Like any Bayesian procedure, the estimation requires assumptions on the prior distributions of parameters. The prior on betas are taken from the current literature on private equity as listed in Appendix Table C.1 (Brav and Gompers (1997), Driessen, Lin, and Phalippou (2012), Derwall et al. (2009), Ewens, Jones, and Rhodes-Kropf (2013), Korteweg and Sorensen (2010), Cao and Lerner (2009), Franzoni, Nowak, and Phalippou (2012), Jegadeesh, Krussl, and Pollet (2009), Chiang, Lee, and Wisen (2005), Lin and Yung (2004), Elton et al. (2001)). These studies estimate a three factor Fama-French model for venture capital, buyout, real estate or high yield bonds. [2] Real estate estimates are derived from REITs, and high yield debt estimates are derived from Industrial BBB-rated bonds of 10-year maturities. The weighted average across sub-classes takes the four sub-classes averages and weights them by the number of funds in each sub-classes. The loadings are rounded at 0.05. The average loading in each category is used as priors.

[2] Note that Jegadeesh, Krussl, and Pollet (2009) use a dataset that contains predominantly but not exclusively buyout related vehicles (the rest of their sample is venture capital related).

Table C.1: Literature Estimates of the Risk Exposures of Private Equity Funds

This table shows the factor loading estimates shown in the literature. Selected papers are those that estimated a three factor model for venture capital, buyout, real estate or high yield bonds. Jegadeesh, Kraussl and Pollet (2010) use a dataset that contains predominantly but not exclusively buyout related vehicles (the rest of their sample is venture capital related). Real estate estimates are derived from real estate investment trusts, and high yield debt estimates are derived from Industrial BBB-rated bonds of 10-year maturities. The weighted average across sub-classes takes the four sub-classes averages and weights them by the number of funds in each sub-class. The loadings are rounded to increments of 0.05. The average loading in each category is used as priors in our Bayesian estimations.

		Venture capital funds		
Authors	Year	β_{mkt}	β_{smb}	β_{hml}
Brav, and Gompers	1997	1.1	1.3	-0.7
Driessen, Lin and Phalippou	2012	2.4	0.9	-0.25
Ewens, Jones and Rhodes-Kropf	2013	1.05	-0.1	-0.9
Korteweg, and Sorensen	2009	2.3	1	-1.55
Average venture capital funds		**1.7**	**0.8**	**-0.85**
		Buyout funds		
Authors	Year	β_{mkt}	β_{smb}	β_{hml}
Cao, and Lerner	2007	1.3	0.75	0.2
Driessen, Lin and Phalippou	2012	1.7	-0.9	1.4
Ewens, Jones and Rhodes-Kropf	2013	0.8	0.1	0.25
Franzoni, Nowak and Phalippou	2012	1.4	-0.1	0.7
Jegadeesh, Krussl and Pollet	2010	1.05	0.6	0.35
Average buyout funds		**1.25**	**0.1**	**0.6**
		Real estate		
Authors	Year	β_{mkt}	β_{smb}	β_{hml}
Chiang, Lee and Wisen	2005	0.55	0.4	0.5
Derwall, Huij, Brounen, and Marquering	2009	0.65	0.4	0.6
Lin and Yung	2004	0.55	0.4	0.7
Average real estate		**0.6**	**0.4**	**0.6**
		High yield debt		
Authors	Year	β_{mkt}	β_{smb}	β_{hml}
Elton, Gruber, Agrawal, and Mann	2001	0.7	1.3	1.45
Average high yield debt		0.7	1.3	1.45
Weighted average across sub-classes		1.3	0.55	0.05

For the loading on the liquidity factor, there is only the Franzoni, Nowak, and Phalippou (2012) estimate available in the literature and it is available only for buyout. They find a beta of 0.7, which we then use as a prior for buyout funds. For the other sub-samples and the pooled private equity fund samples we use a prior of 0.5. From the estimates in Table C.1, we set prior means for market, size, value, and liquidity factors at 1.30, 0.55, 0.05, and 0.50, respectively, for the private equity sample. When we estimate factor loadings per fund category (venture capital, buyout etc.) we use the average estimate in the literature in the corresponding category. The prior for alpha is set at zero for simplicity. Since are computed at the end of each iteration by setting the NPV to zero from the factor loadings in that iteration, the estimation is insensitive to the alpha priors.

To compute the prior for the premium persistence, we use the individual buyout investment return database of Lopez-de-Silanes, Phalippou, and Gottschalg (2013). We compute the correlation between successive investment IRRs of the same private equity firm (IRR of investment i and IRR of investment $i+1$) and find it to be 0.25. The average spread in starting dates is around six months and investments last for four years. If the process is assumed to be AR(1), this means an autocorrelation coefficient of 0.5 at yearly frequency, which is what we use as a prior.

We set bounds of ± 1 around the prior mean for betas. There are no bounds on alpha. The autocorrelation parameter, ϕ , is restricted to lie between 0 and 0.9 and the latent return is restricted to lie in between -0.50 and 1.00 per quarter.

The standard deviation of the prior captures how diffuse the priors are. We choose a large standard deviation for the priors equal to 10. This would represent an extremely diffuse prior in most contexts. We find, however, that the posterior distributions depend on the volatility of the latent factors more than the priors of the parameters. The volatility of the latent factor is equivalent to determining the R^2 , that is how much of the private equity return is attributable to systematic factors. Since f_t is latent, we could exactly match any private equity return process if there were no restrictions. This can be clearly seen in a traditional linear context, but also occurs in the likelihood for our private equity cashflows. Thus, the volatility of the latent process effectively influences the informativeness of the priors.

We cap the volatility of the latent returns at a large 100% per quarter. For the prior for the volatility of the latent return, we use a mean of 20% per quarter. We examine robustness to these choice and the sample selection choices in next section.

C.3 Robustness Checks And Small Sample Properties

In this appendix, we show that our estimation method is robust. First, we report the sensitivity of the estimation procedure to different choices of the priors of the parameters, their informativeness, and sample selection; we find no substantial influence on the estimation results. Second, we run Monte Carlo simulations with known (pre-set) parameters and state factors. We find that our Bayesian Gibbs sampling method and identification strategy generates little small-sample bias.

Sensitivity to priors and assumptions

Since we use a Bayesian framework, it is useful to understand how the priors and their informativeness affect the results. In addition, an important decision in our framework is the threshold to use as a NAV cut-off for definition of a quasi-liquidated fund. In Table C.2, we examine robustness of our estimations to different priors and NAV thresholds. The first line shows the results with the default specification; the estimates coincide with those reported in the main tables. Each line then shows the results of implementing one change. Panel A shows results for the CAPM for the sample of private equity funds. We first examine the effects of changing the priors about the likelihood standard error, σ , in equation (9). Raising the maximum prior for σ slightly increases the mean of the estimated betas to 1.43. The alpha estimates are unchanged. Note also that persistence estimates drop when the maximum on σ increases.

Priors on beta obviously make a difference. The prior mean used in our estimation is 1.30. Increasing the prior mean to 1.80 results in a posterior mean estimate of 1.65, roughly one standard error above the default estimated mean value. Decreasing the prior by 0.5, i.e. from 1.3 to 0.8 yields a beta of 1.15. Thus, priors on betas matter to estimate the systematic risk exposure, although we cannot statistically reject the null that these are equal to the baseline estimation. Variation of the beta priors has little effect on the in-sample alpha. The fact that the posterior always move in the direction of our original prior can be seen as additional support for our original set of priors.

Turning to the NAV cutoff assumption we find that beta estimates are relatively unchanged if we lower the threshold for the percentage of the fund liquidated from 50% to 33%. The CAPM alpha increases to 0.07. Raising the threshold to 66% and 75% has little effect. In other words, restricting the sample to funds that have more liquidated investments makes private equity look more attractive (in that sample). This is consistent with private equity funds holding on to losers, making better performing funds liquidating faster (Lopez-de-Silanes, Phalippou, and Gottschalg (2013)). This may reflect an upward

Table C.2: Risk Exposures of Private Equity Funds – Robustness Tests

The risk loading estimates are re-estimated separately with one change in our estimation methodology at a time. The first line shows the results with the default specification; the estimates coincide with those reported in the main tables. Each line then specifies the change made and the corresponding results. Panel A shows results for the CAPM for the sample of private equity funds. Panels B, C, and D show results for the four-factor model of Pstor and Stambaugh (2003) for the sample of private equity funds, venture capital funds, and buyout funds, respectively.

Panel A: CAPM

Change	β_{mkt}	In-sample Alpha	Persistence of Alpha	R-square	Nobs
Default	1.41***	0.05***	0.4	93.00%	630
	0.24	*0.01*	*0.19*		
Sigma max from 50% to 75%	1.43***	0.05***	0.27	81.10%	630
	0.4	*0.01*	*0.18*		
Sigma priors from 10 to 5	1.35***	0.05***	0.42***	90.10%	630
	0.18	*0.01*	*0.16*		
Sigma priors from 10 to 2	1.31***	0.05***	0.46***	86.60%	630
	0.1	*0.01*	*0.1*		
Beta priors increase by 0.5	1.65***	0.04***	0.40**	95.50%	630
	0.22	*0.00*	*0.19*		
Beta priors decrease by 0.5	1.15***	0.05***	0.40**	87.80%	630
	0.26	*0.01*	*0.19*		
NAV threshold from 50% to 33%	1.39***	0.07***	0.41**	92.60%	484
	0.26	*0.01*	*0.2*		
NAV threshold from 50% to 66%	1.37***	0.04***	0.34*	93.20%	790
	0.23	*0.00*	*0.18*		
NAV threshold from 50% to 75%	1.34***	0.04***	0.32*	93.10%	868
	0.23	*0.00*	*0.17*		

Panel B: Four-factor model

Change	β_{mkt}	β_{size}	β_{value}	$\beta_{illiquidity}$	In-sample Alpha	Persistence of Alpha	R-square	Nobs
Default	1.41***	0.41	0.03	0.36	0	0.48	97.00%	630
	0.21	*0.26*	*0.23*	*0.27*	*0.02*	*0.19*		
Sigma max from 50% to 75%	1.46***	0.4	-0.13	0.22	0.02	0.28	88.20%	630
	0.38	*0.52*	*0.47*	*0.53*	*0.02*	*0.18*		
Sigma priors from 10 to 5	1.36***	0.50***	0.03	0.46***	-0.01**	0.49***	96.30%	630
	0.14	*0.15*	*0.15*	*0.15*	*0.01*	*0.13*		
Sigma priors from 10 to 2	1.31***	0.54***	0.04	0.49***	-0.02***	0.50***	95.10%	630
	0.07	*0.07*	*0.07*	*0.07*	*0.01*	*0.07*		
Beta priors increase by 0.5	1.53***	0.78***	0.25	0.50*	-0.04***	0.51***	98.50%	630
	0.18	*0.28*	*0.24*	*0.26*	*0.01*	*0.19*		
Beta priors decrease by 0.5	1.13***	-0.01	-0.29	-0.01	0.07***	0.41**	88.50%	630
	0.32	*0.39*	*0.34*	*0.42*	*0.01*	*0.19*		
NAV threshold from 50% to 33%	1.41***	0.42	0.03	0.36	0.02*	0.48***	96.70%	484
	0.22	*0.28*	*0.25*	*0.29*	*0.01*	*0.19*		
NAV threshold from 50% to 66%	1.40***	0.41	-0.02	0.35	-0.01	0.44**	97.00%	790
	0.2	*0.26*	*0.23*	*0.28*	*0.01*	*0.17*		
NAV threshold from 50% to 75%	1.37***	0.39	-0.05	0.31	-0.01	0.41**	96.50%	868
	0.22	*0.27*	*0.24*	*0.29*	*0.01*	*0.18*		

Panel C: Four-factor for VC

Change	Alpha Full sample	Alpha Est. sample	Persistence	β_{mkt}	R^2	Nobs
Default	1.60%	3.90%	0.56***	1.62***	0.94	272
		0.60%	*0.18*	*0.26*		
Sigma max from 50% to 75%	1.60%	3.80%	0.38*	1.66***	0.82	272
		0.80%	*0.2*	*0.41*		
Sigma priors from 10 to 5	1.60%	3.80%	0.52***	1.66***	0.93	272
		0.60%	*0.16*	*0.19*		
Sigma priors from 10 to 2	1.60%	3.80%	0.49***	1.69***	0.9	272
		0.60%	*0.1*	*0.11*		
Beta priors increase by 0.5	1.70%	3.70%	0.54***	1.84***	0.96	272
		0.60%	*0.19*	*0.22*		
Beta priors decrease by 0.5	1.90%	4.40%	0.58***	1.35***	0.9	272
		0.60%	*0.18*	*0.3*		
NAV threshold from 50% to 33%	1.60%	7.30%	0.58***	1.63***	0.94	203
		0.60%	*0.19*	*0.26*		
NAV threshold from 50% to 66%	1.60%	2.60%	0.51***	1.62***	0.94	344
		0.60%	*0.19*	*0.27*		
NAV threshold from 50% to 75%	1.60%	2.40%	0.50**	1.60***	0.93	376
		0.60%	*0.19*	*0.27*		

Panel D: Four-factor for BO

Change	Alpha Full sample	Alpha Est. sample	Persistence	β_{mkt}	R^2	Nobs
Default	4.70%	5.60%	0.41**	1.22***	0.87	243
		0.50%	*0.19*	*0.27*		
Sigma max from 50% to 75%	4.70%	5.60%	0.38*	1.20***	0.82	243
		0.50%	*0.2*	*0.29*		
Sigma priors from 10 to 5	4.70%	5.60%	0.42**	1.23***	0.85	243
		0.50%	*0.17*	*0.2*		
Sigma priors from 10 to 2	4.70%	5.60%	0.46***	1.24***	0.81	243
		0.50%	*0.11*	*0.12*		
Beta priors increase by 0.5	4.70%	5.50%	0.42**	1.42***	0.91	243
		0.50%	*0.2*	*0.26*		
Beta priors decrease by 0.5	4.90%	5.80%	0.42**	1.01***	0.79	243
		0.50%	*0.2*	*0.3*		
NAV threshold from 50% to 33%	4.70%	6.00%	0.36*	1.15***	0.86	193
		0.60%	*0.2*	*0.29*		
NAV threshold from 50% to 66%	4.70%	5.10%	0.40**	1.16***	0.88	306
		0.40%	*0.19*	*0.23*		
NAV threshold from 50% to 75%	4.80%	5.10%	0.42**	1.12***	0.86	336
		0.40%	*0.19*	*0.23*		

bias in estimated NAVs. More comforting is the evidence from increasing the threshold. Including more funds leaves the estimate unchanged. This suggests that higher alpha based on looser censoring is upward-biased but that results using the 50% threshold are representative. Persistence estimates decrease with a greater threshold. This may be due to the decreasing sample size. [3]

Panels B, C, and D show results for the four-factor Pstor and Stambaugh (2003) model for the sample of private equity funds, venture capital funds, and buyout funds, respectively. In general, changes in priors about have similar effects to those noted for the CAPM specification. Raising the maximum increases the posterior mean of the market beta, while decreasing the prior lowers the posterior mean of beta. Interestingly, widening the prior increases the in-sample alpha (even though beta goes up), while tightening the sigma prior decreases the in-sample alpha. This latter result appears to be due to an increase in the size and illiquidity betas. As with the single factor model, varying the priors on beta changes the estimated betas, although does not push them beyond approximately one posterior standard deviation from the estimates under the default assumptions. The beta priors do, however, significantly affect the estimated in-sample alphas. Naturally, decreasing betas by 0.5 raises the posterior alpha mean estimate to 0.07 and raising betas by 0.5 lowers the posterior in-sample mean alpha to -0.04. The posterior standard deviations of the

[3] Goetzmann (1992) shows that, in a similar estimation procedure, negative autocorrelation is induced by thin data.

alphas are about 0.01, so this difference is large. The effect of varying the NAV threshold is similar to that observed for the CAPM model, raising the threshold lowers the alpha. It is likely that the higher NAV threshold reduces the likelihood of upward bias due to stale marks.

Simulations with known alphas and betas

In order to test the precision and potential bias in the estimation procedure we conduct a simulation that constructs hypothetical funds for which we assume what the true parameters are. The procedure uses the timing of actual cash flows from funds sampled from the Preqin data. The time span is set to match our sample, at 20 years. For each simulated fund, we randomly choose a fund from our sample and randomly choose a date to start our simulated fund. We match the timing of cash flows of the simulated fund with the selected fund. We simulate the return over the investment period using a true alpha set at 1.25% per quarter (5% per year), a true beta as 1.5 and true phi as 0.5. These are applied to the actual S&P 500 returns over the sample interval. This exercise is repeated 100 times for different sample sizes of simulated funds. Table C.3 shows the results. The mean, standard deviation and the quartile threshold of the 100 estimations are displayed in Panel A. The mean of the alpha distribution appears to be downward biased in small samples but the bias is modest at about 1% per year. The autocorrelation coefficient is also slightly downward biased in small sample, which is consistent with the well-known bias of autocorrelation parameters from Kendall (1952). Beta and sigma estimates are less affected by small sample bias.

Panel B shows the summary statistics for the simulated cash flow private equity total return index, g_t , and the time-varying private equity component, f_t, measured at the quarterly horizon. The mean of the simulated is slightly higher than the true value for all sample sizes, as is the median of the time-varying private equity component, f_t. The most important requirement of our index is that it captures the true dynamics of the private equity index. Since we know the actual returns of the true index by construction, we can measure its average correlation to the estimated indices. Panel C reports the average correlation between the true return g_t and the Gibbs sampler estimates. Even in small sample the correlation is greater than 50%. This may explain why our real estate index, measured with relatively few funds, appears to capture the dynamics of a broad-based commercial property return index. As sample size increases, the correlation increases rapidly towards one. The correlation of the true return g_t to other performance measures (IRR, multiple, and PME) is around zero.

Table C.3: Monte Carlo Simulations

The time span in the simulation is set at 80 quarters. For each simulated fund, we randomly choose a fund in the buyout funds collection as a matching fund. After randomly choose a time to start our simulated fund, we match the timing of cash flows of the simulated fund with the matching fund. Our market return is the same as the S&P 500. This exercise is repeated with 100 times with the true set at 1.25% per quarter (5% per year), true of 1.5, and true of 0.5. The mean, standard deviation, and the component, . The frequency is quarterly. Panel C reports the correlations between the true return of and our Gibbs sampling estimates, and other performance measures (IRR, multiple, and PME). quartile threshold of the 100 estimations are reported in Panel A. Panel B shows the summary statistics of the latent total returns, g_t , and the private equity premium f_t

Panel A: Factor parameters

	True Value	N=200	N=400	N=1000
Mean α	0.0125	0.007	0.01	0.011
Std α		0.008	0.006	0.005
Mean ϕ	0.5	0.423	0.444	0.478
Std ϕ		0.083	0.076	0.074
Mean β	1.5	1.486	1.533	1.498
Std β		0.287	0.257	0.188
Mean σ_f	0.05	0.046	0.052	0.048
Std σ_f		0.008	0.007	0.005
Mean σ	0.1	0.11	0.129	0.12
Std σ		0.035	0.016	0.014
Mean $RMSE(g_t)$		0.051	0.05	0.045
Std σ		0.008	0.007	0.004
Lower quartile α		0.002	0.004	0.004
Upper quartile α		0.011	0.014	0.015
Lower quartile β		1.363	1.344	1.402
Upper quartile β		1.636	1.702	1.675

Panel B: Index estimates

	N=200		N=400		N=1000	
	True value	Estimation	True value	Estimation	True value	Estimation
Mean g_t	3.76%	3.90%	3.72%	4.08%	3.91%	4.09%
Std g_t	14.32%	13.29%	14.44%	13.81%	14.09%	13.95%
Lower quartile f_t	-3.58%	-0.73%	-3.82%	-0.83%	-3.83%	-2.13%
Median f_t	0.20%	0.57%	0.03%	0.72%	0.06%	0.86%
Upper quartile f_t	3.87%	1.93%	3.82%	2.33%	3.98%	2.87%

Panel C: Correlation between true and other performance measures

	N=200	N=400	N=1000
Estimated	54.97%	94.55%	95.24%
IRR	1.91%	3.28%	3.77%
Multiple	-2.45%	-2.36%	-4.58%
PME	-1.15%	-0.89%	-3.00%