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Lagrangian space consistency relation for large scale structure

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Abstract. Consistency relations, which relate the squeezed limit of an (N+1)-point correlation function to an N-point function, are non-perturbative symmetry statements that hold even if the associated high momentum modes are deep in the nonlinear regime and astrophysically complex. Recently, Kehagias & Riotto and Peloso & Pietroni discovered a consistency relation applicable to large scale structure. We show that this can be recast into a simple physical statement in Lagrangian space: that the squeezed correlation function (suitably normalized) vanishes. This holds regardless of whether the correlation observables are at the same time or not, and regardless of whether multiple-streaming is present. The simplicity of this statement suggests that an analytic understanding of large scale structure in the nonlinear regime may be particularly promising in Lagrangian space.

Keywords: particle physics - cosmology connection, cosmological parameters from LSS, cosmological perturbation theory

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1 Introduction

Consistency relations are statements which relate the squeezed limit of an (N+1)-point correlation function to an N-point function of cosmological perturbations; i.e., they take the following schematic form in momentum space:

$$\lim_{\mathbf{k}\to 0} \frac{\langle \pi_{\mathbf{k}} \mathcal{O}_{\mathbf{k_1}} \mathcal{O}_{\mathbf{k_2}} \ldots \mathcal{O}_{\mathbf{k_N}} \rangle^{c'}}{P_\pi(\mathrm{k})} \sim \langle \mathcal{O}_{\mathbf{k_1}} \mathcal{O}_{\mathbf{k_2}} \ldots \mathcal{O}_{\mathbf{k_N}} \rangle^{c'} , \tag{1.1}$$

where $\pi_{\mathbf{k}}$ represents a squeezed wavemode (long wavelength) of what turns out to be a Goldstone boson or pion, $P_\pi(\mathrm{k})$ is the power spectrum of the pion (k represents the magnitude of the vector $\mathbf{k}$), and $\mathcal{O}$ represents observables at high momenta $\mathbf{k_1}, \ldots, \mathbf{k_N}$. The symbol $\langle \ldots \rangle^{c'}$ denotes the connected correlation function with the overall delta function removed. Consistency relations can be understood as analogues of 'soft-pion' theorems in particle physics, which arise generally when a symmetry is spontaneously broken/nonlinearly realized. In the case of cosmology, the symmetries in question are diffeomorphisms (i.e. coordinate transformations), and consistency relations arise from a particular set of residual symmetries of a given gauge where the transformation does not fall off at infinity. The first example of a consistency relation was pointed out by Maldacena [3] in the context of a computation of the three-point correlation function from inflation. The utility of this as a test of single field/clock inflation was emphasized by Creminelli & Zaldarriaga [4]. Recent work pointed out new symmetries and therefore further consistency relations [5, 6], indeed an infinite tower of them [7], and explicated their non-perturbative nature [8–13].

These consistency relations are extremely robust: they remain valid when the high momentum modes ($\mathcal{O}$ in eq. (1.1)) are deep in the nonlinear regime, and even when the observables are astrophysically complex (such as galaxy density). This point might appear academic when applied to (small) perturbations in the early universe, such as are revealed in the cosmic microwave background. When applied to large scale structure (LSS) in the late universe, however, the robustness of the consistency relations becomes very interesting. It thus came as welcome news when Kehagias & Riotto [1] and Peloso & Pietroni [2] (KRPP)

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pointed out that non-trivial consistency relations exist even if all wavemodes (including the squeezed one) are within the Hubble radius, within the Newtonian regime which is the realm of LSS (see also [14–21]).

The KRPP consistency relation can be stated in the following form:

$$\lim_{\mathbf{k}\to 0} \frac{\langle v^j_{\mathbf{k}}(\eta)\, \mathcal{O}_{\mathbf{k}_1}(\eta_1)\, \ldots\, \mathcal{O}_{\mathbf{k}_N}(\eta_N)\,\rangle^{c'}}{P_v(\mathrm{k},\eta)} = i\mathrm{k}^j \sum_{a=1}^{N} \frac{D(\eta_a)}{D'(\eta)} \frac{\mathbf{k}\cdot\mathbf{k}_a}{\mathrm{k}^2} \langle \mathcal{O}_{\mathbf{k}_1}(\eta_1)\, \ldots .\, \mathcal{O}_{\mathbf{k}_N}(\eta_N)\,\rangle^{c'} , \quad (1.2)$$

where $v^j_{\mathbf{k}}$ is the j-th component of the peculiar velocity in momentum space, P_v is the velocity power spectrum defined by $\langle v^i_{\mathbf{k}}(\eta)\, v^j_{\mathbf{k}'}{}^*(\eta)\rangle = (2\pi)^3 \delta_D(\mathbf{k}-\mathbf{k}')(\mathrm{k}^i\mathrm{k}^j/\mathrm{k}^2) P_v(\mathrm{k},\eta)$,[1] the observables can be thought of as mass or galaxy overdensity at different momenta and times, and D and D' represent the linear growth factor and its conformal time derivative. The fluctuation variables will in general depend on time, although we will often suppress the time dependence to simplify the notation: $v^j_{\mathbf{k}}$ (and its power spectrum) is at conformal time η, $\mathcal{O}_{\mathbf{k}_1}$ is at time η_1, and so on. The times need not be equal. The symbol k^2 denotes $\mathbf{k}\cdot\mathbf{k}$.

We wish to show that the KRPP consistency relation takes a particularly simple form in Lagrangian space:

$$\lim_{\mathbf{p}\to 0} \frac{\langle \boldsymbol{v}_{\mathbf{p}}(\eta)\, \mathcal{O}_{\mathbf{p}_1}(\eta_1)\, \ldots\, \mathcal{O}_{\mathbf{p}_N}(\eta_N)\,\rangle^{c'}}{P_v(\mathrm{p},\eta)} = 0\,. \quad (1.3)$$

Unless otherwise stated, we use $\mathbf{p}$ to denote momentum in Lagrangian space and $\mathbf{k}$ to denote momentum in Eulerian space. In other words:

$$\mathcal{O}_{\mathbf{k}} = \int d^3\mathbf{x}\, \mathcal{O}(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}} \quad , \quad \mathcal{O}_{\mathbf{p}} = \int d^3\mathbf{q}\, \mathcal{O}(\mathbf{x}(\mathbf{q})) e^{i\mathbf{p}\cdot\mathbf{q}} \,, \quad (1.4)$$

where $\mathbf{x}$ and $\mathbf{q}$ are the Eulerian space and Lagrangian space coordinates respectively.[2] In both cases, we rely on context to distinguish between $\mathcal{O}$ in Fourier space and $\mathcal{O}$ in configuration space.

Since the velocity $\boldsymbol{v}$ (whose j-th component is v^j) is nothing other than the time derivative of the displacement $\boldsymbol{\Delta}$ in Lagrangian space, we can also rewrite the Lagrangian space consistency relation as:

$$\lim_{\mathbf{p}\to 0} \frac{\langle \boldsymbol{\Delta}_{\mathbf{p}}(\eta)\, \mathcal{O}_{\mathbf{p}_1}(\eta_1)\, \ldots\, \mathcal{O}_{\mathbf{p}_N}(\eta_N)\,\rangle^{c'}}{P_\Delta(\mathrm{p},\eta)} = 0\,, \quad (1.5)$$

where the power spectrum of displacement is defined by $\langle \Delta^i_{\mathbf{p}} \Delta^j_{\mathbf{p}'}{}^*\rangle = (2\pi)^3\delta_D(\mathbf{p}-\mathbf{p}')$ $(\mathrm{p}^i\mathrm{p}^j/\mathrm{p}^2)P_\Delta(\mathrm{p})$. It is important to emphasize that the Eulerian space consistency relation (eq. (1.2)) already yields a vanishing right hand side if $\eta_1 = \eta_2 \ldots = \eta_N$. The Lagrangian space consistency relation (eq. (1.3) or 1.5), on the other hand, has a vanishing right hand side *regardless* of what the times $\eta_1, \ldots, \eta_N$ happen to be. The consistency relation can also be viewed as a statement about how the squeezed correlation function (normalized by the soft power spectrum) scales with the soft momentum: the Eulerian space consistency relation states that such a squeezed correlation function goes like k^0 ($\mathbf{k}$ is the soft momentum); the

[1] This form of the velocity power spectrum assumes no vorticity. This is acceptable since $P_v(\mathrm{k})$ is used only for small k, or large scales, where the growing mode initial condition ensures gradient flow.

[2] The definitions given apply even in the presence of multiple streaming. See discussion in section 2.1.

Lagrangian space consistency relation states that there is no p^0 term, and at best there is a p^ϵ contribution with $\epsilon > 0$.

The simplest way to derive eq. (1.5) is to work out the implications of the KRPP symmetry entirely within Lagrangian space. This is done in section 2. We perform a perturbative check of this Lagrangian space consistency relation using Lagrangian perturbation theory in section 3.1. Because the Eulerian space and the Lagrangian space relations look so different, as a further check, we show how one can be obtained from the other in section 3.2. Since observations are performed in Eulerian, not Lagrangian, space, the fact that the consistency relation takes a particularly simple form in Lagrangian space is mainly of theoretical interest. The simplicity of the Lagrangian space consistency relation should not be interpreted as the lack of physical content, however — in the Lagrangian as well as in the Eulerian picture, the consistency relation can be viewed as a test of the single-field initial condition and of the equivalence principle. Rather, the simplicity suggests that an analytical understanding of nonlinear clustering might be most promising in Lagrangian space. This is discussed in section 4.

2 The Lagrangian space consistency relation: derivation

After a brief review of notation, we derive our main result — the Lagrangian space consistency relation — using the background wave argument phrased entirely in Lagrangian space.

2.1 Notation

We use $\mathbf{q}$ to denote the Lagrangian space coordinate of a particle, which coincides with its initial position, and $\mathbf{x}$ to denote the Eulerian space coordinate which is its position at a later time. To be definite, in cases where multiple components are present, the Lagrangian space coordinate $\mathbf{q}$ refers to that of the dark matter particle, which has only gravitational interactions.[3] Both coordinates are defined in comoving space where the expansion of the universe is scaled out. The (dark matter) displacement $\mathbf{\Delta}$ is the difference:

$$\mathbf{x}\left(\mathbf{q}, \eta\right) = \mathbf{q} + \mathbf{\Delta}\left(\mathbf{q}, \eta\right) . \tag{2.1}$$

The (dark matter) velocity is given by the conformal time derivative of $\mathbf{\Delta}$ at a fixed Lagrangian coordinate:

$$\boldsymbol{v}\left(\mathbf{q}, \eta\right) = \frac{\partial \mathbf{\Delta}}{\partial \eta}\bigg|_{\mathbf{q}} . \tag{2.2}$$

The (dark matter) overdensity δ can be obtained by mass conservation, assuming the initial overdensity is negligible:

$$1 + \delta(\mathbf{x}, \eta) = |J(\mathbf{q}, \eta)|^{-1} \tag{2.3}$$

with $J\left(\mathbf{q}, \eta\right)$ being the Jacobian relating the volume elements in Eulerian and Lagrangian space:

$$J(\mathbf{q}, \eta) \equiv \det\left[\frac{\partial x^i(\mathbf{q}, \eta)}{\partial q^j}\right] . \tag{2.4}$$

[3] Our derivation of the Lagrangian space consistency relation would go through even if we chose the Lagrangian coordinate to track other constituents of the universe.

The Jacobian J as a function of $\mathbf{q}$ is well-defined even in the presence of multiple-streaming — where a single $\mathbf{x}$ corresponds to multiple $\mathbf{q}$'s — but eq. (2.3) requires modification in that case:

$$1 + \delta\left(\mathbf{x}, \eta\right) = \sum_{\mathbf{x}=\mathbf{q}+\mathbf{\Delta}(\mathbf{q},\eta)} \left|J\left(\mathbf{q}, \eta\right)\right|^{-1}, \tag{2.5}$$

where the sum is over all $\mathbf{q}$'s that reach the same $\mathbf{x}$.

Suppose we have some LSS observable $\mathcal{O}$. This could represent many different quantities, such as mass overdensity or galaxy number overdensity.[4] What we typically observe is $\mathcal{O}$ as a function of $\mathbf{x}$ (and possibly time, which we suppress). Given this function $\mathcal{O}(\mathbf{x})$, one can define unambiguously a corresponding function of $\mathbf{q}$: $\mathcal{O}(\mathbf{x}(\mathbf{q}))$. In other words, suppose we are interested in the value of $\mathcal{O}$ at a Lagrangian location $\mathbf{q}$: we can define it by working out the $\mathbf{x}$ that $\mathbf{q}$ maps to, and then evaluating $\mathcal{O}(\mathbf{x})$. This procedure is well defined even if multiple $\mathbf{q}$'s map to the same $\mathbf{x}$, which is expected to happen for dark matter in the nonlinear regime.

Some quantities defined in Lagrangian space, on the other hand, might not have an unambiguous meaning in Eulerian space. For instance, the velocity $\boldsymbol{v}$ given in eq. (2.2) is defined for a dark matter particle labeled by the Lagrangian coordinate $\mathbf{q}$. At an Eulerian position $\mathbf{x}$ where multiple Lagrangian streams cross, additional inputs are required to define a velocity; a reasonable definition is:

$$\text{average } \boldsymbol{v} = \frac{1}{\mathcal{N}} \sum_{\mathbf{x}=\mathbf{q}+\mathbf{\Delta}} \boldsymbol{v}(\mathbf{q}), \tag{2.6}$$

where the sum is over all $\mathbf{q}$'s that map to the same $\mathbf{x}$, and $\mathcal{N}$ is the number of such $\mathbf{q}$'s. This gives a mass weighted velocity.

It is interesting to contrast the Fourier transform in Lagrangian versus Eulerian space, as described by eq. (1.4). In particular, the Eulerian space Fourier transform can be rewritten as (suppressing time dependence):

$$\mathcal{O}_{\mathbf{k}} = \int d^3\mathbf{x}\, \mathcal{O}(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}} = \int d^3\mathbf{q}\, J(\mathbf{q})\, \mathcal{O}(\mathbf{x}(\mathbf{q})) e^{i\mathbf{k}\cdot(\mathbf{q}+\mathbf{\Delta})}, \tag{2.7}$$

where J comes without absolute value; this expression remains valid in the presence of multiple streaming. Note how an Eulerian space Fourier transform of $\mathcal{O}(\mathbf{x})$ can be interpreted as a Lagrangian space Fourier transform of $J(\mathbf{q})\mathcal{O}(\mathbf{x}(\mathbf{q}))e^{i\mathbf{k}\cdot\mathbf{\Delta}}$.

2.2 Derivation from the displacement symmetry

We now deduce our main result, making use of a master formula derived in an earlier paper [21]. At the heart of the consistency relation is the existence of a nonlinearly realized symmetry, under which some field — the Goldstone boson or pion π — transforms as $\pi \to \pi + \Delta_{\rm lin.}\pi + \Delta_{\rm nl.}\pi$. Here, $\Delta_{\rm lin.}\pi$ is the part of the transformation that is linear in π, and $\Delta_{\rm nl.}\pi$ is the part of the transformation that is independent of π (i.e., nonlinear in π, though 'sub-linear' or 'inhomogeneous' would be a better description). The fact that $\Delta_{\rm nl.}\pi \neq 0$ is the sign of a nonlinearly realized, or spontaneously broken, symmetry. At the same time,

[4] Unless otherwise stated, whenever we discuss mass or galaxy density, we mean the mass or galaxy count per unit *Eulerian* space volume. Such a quantity can of course be expressed as a function of either Eulerian space coordinate $\mathbf{x}$ or Lagrangian space coordinate $\mathbf{q}$.

there are other fields or observables $\mathcal{O}$ that could have their own linear and/or nonlinear transformations. The master formula (in momentum space) reads [21]:

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{\langle \pi_{\mathbf{p}} \mathcal{O}_{\mathbf{p}_1} \cdots \mathcal{O}_{\mathbf{p}_N} \rangle^c}{P_\pi(\mathrm{p})} \Delta_{\mathrm{nl.}} \pi^*_{\mathbf{p}} = \Delta_{\mathrm{lin.}} \langle \mathcal{O}_{\mathbf{p}_1} \cdots \mathcal{O}_{\mathbf{p}_N} \rangle^c \,, \tag{2.8}$$

where $\langle \ldots \rangle^c$ refers to the connected correlation function *without* removing the overall delta function (as opposed to $\langle \ldots \rangle^{c'}$ which has the delta function removed). Note how it is the nonlinear transformation of π and the linear transformation of $\mathcal{O}$ that show up on the left and the right respectively. Note also that the $\mathcal{O}$'s need not even be the same observable. Nor do π and the $\mathcal{O}$'s need be at the same time: they can be at *arbitrary, potentially different, times.* The derivation of this master formula made no assumption about whether the quantities (or the Fourier transform thereof) are defined in Eulerian or Lagrangian space. We are thus free to use it in either. This master relation can be used to derive the large scale structure analog of Ward identities or soft-pion theorems in particle physics.

As a warm-up, let us first apply this formula to a simple system that involves the dark matter only. The dynamics is described by: (1) $\mathbf{x} = \mathbf{q} + \mathbf{\Delta}$ as in eq. (2.1); (2) the dark mater overdensity δ determined by the Jacobian as in eq. (2.5); (3) the displacement $\mathbf{\Delta}$ which evolves according to:

$$\frac{\partial^2 \mathbf{\Delta}}{\partial \eta^2}\bigg|_{\mathbf{q}} + \frac{a'}{a} \frac{\partial \mathbf{\Delta}}{\partial \eta}\bigg|_{\mathbf{q}} = -\boldsymbol{\nabla}_x \Phi \,, \tag{2.9}$$

where a is the scale factor, a' is its derivative with respect to conformal time η, Φ is the gravitational potential and $\boldsymbol{\nabla}_x$ is the partial derivative with respect to $\mathbf{x}$; lastly (4) the Poisson equation:

$$\nabla_x^2 \Phi = 4\pi G a^2 \bar{\rho} \delta \,, \tag{2.10}$$

where G is the Newton constant and $\bar{\rho}$ is the mean mass density.

This system has the following symmetry:

$$\mathbf{q} \to \mathbf{q} \quad , \quad \mathbf{\Delta} \to \mathbf{\Delta} + \mathbf{n}(\eta) \quad , \quad \Phi \to \Phi - \left(\mathbf{n}'' + \frac{a'}{a} \mathbf{n}' \right) \cdot \mathbf{x} \,, \tag{2.11}$$

where $\mathbf{n}(\eta)$ is a function of time alone. We will refer to this as the *displacement symmetry.* Note how $\mathbf{\Delta}$ shifts by a nonlinear (or sub-linear) amount and can be thought of as our Goldstone boson. The same is true for Φ. The interesting point is that the mass overdensity δ does not transform at all under this symmetry. Nor are $\mathbf{q}$ or η transformed. Applying the master formula, choosing the observable $\mathcal{O} = \delta$, we thus find:

$$\lim_{\mathbf{p} \to 0} \frac{\langle \mathbf{\Delta}_{\mathbf{p}} \delta_{\mathbf{p_1}} \ldots \delta_{\mathbf{p_N}} \rangle^{c'}}{P_\Delta(\mathrm{p})} = 0 \,. \tag{2.12}$$

Here, we have used the fact that the nonlinear transformation of $\mathbf{\Delta}$ in Fourier space is $\Delta_{\mathrm{nl.}} \mathbf{\Delta}_{\mathbf{p}} = \mathbf{n}(\eta)(2\pi)^3 \delta_D(\mathbf{p})$, where $\delta_D(\mathbf{p})$ is the Dirac delta function. We have also removed the overall momentum-conserving delta function. The power spectrum of displacement P_Δ is as defined in section 1.

Two comments are in order before we proceed to generalize this derivation to more realistic, astrophysically complex observables. First, while the Lagrangian coordinate $\mathbf{q}$ does not transform under the symmetry of interest, the Eulerian coordinate $\mathbf{x} = \mathbf{q} + \boldsymbol{\Delta}$ does, because the displacement $\boldsymbol{\Delta}$ shifts. This implies that an observable like δ, when expressed as a function of $\mathbf{x}$, transforms as: $\delta \to \delta + \Delta_{\rm lin.}\delta$ with $\Delta_{\rm lin.}\delta = \boldsymbol{\Delta}\cdot\boldsymbol{\nabla}\delta$. Plugging this into the master formula eq. (2.8), we see that there is a non-vanishing right hand side, unlike the situation in Lagrangian space where δ expressed as a function of $\mathbf{q}$ does not shift at all. The is the fundamental reason why the KRPP consistency relation takes a more complicated form in Eulerian space (eq. (1.2)) than in Lagrangian space (eq. (2.12)).

Second, the reader might wonder about the validity of our application of the master formula: on the one hand, the master relation is phrased in terms of a scalar pion; on the other, our application effectively uses the vector displacement $\boldsymbol{\Delta}$ as the pion. The short answer is that the master formula is applicable to any field π that shifts nonlinearly under the symmetry of interest; one can use it for each component of $\boldsymbol{\Delta}$ for instance. The long answer is: since $\boldsymbol{\Delta}$ is used in the consistency relation only as a soft (long wavelength) mode, one is justified in treating it as a gradient mode (assuming growing mode initial condition) with $\boldsymbol{\Delta} = \boldsymbol{\nabla}_q\pi$ with π playing the role of the displacement potential. The master formula can then be applied with the displacement potential as the pion. The resulting consistency relation can be shown to be equivalent to the one we have derived.[5]

Let us turn to the derivation of a stronger form of the Lagrangian space consistency relation. So far, we have focused on a simple system of dark matter particles that interact only gravitationally, as embodied in eqs. (2.9) and (2.10). Let us consider the addition of galaxies into the mix. They have their own number overdensity δ_g, displacement $\boldsymbol{\Delta}_g$ and velocity $\mathbf{v}_g = \boldsymbol{\Delta}'_g$. Their number density is not necessarily conserved by evolution, since galaxies can form and merge:

$$\delta'_g + (1+\delta_g)\boldsymbol{\nabla}_x\cdot\mathbf{v}_g = R_g\,, \tag{2.13}$$

where $'$ refers to conformal time derivative at a fixed Lagrangian coordinate and R_g is a source term that incorporates the formation and merger rates. The equation of motion for the galaxies is:

$$\boldsymbol{\Delta}''_g + \frac{a'}{a}\boldsymbol{\Delta}'_g = -\boldsymbol{\nabla}_x\Phi + \mathbf{F}_g\,, \tag{2.14}$$

where $\mathbf{F}_g$ encodes additional forces that might act on galaxies, such as gas pressure, dynamical friction et cetera. The gravitational potential Φ is determined of course by the Poisson equation (2.10) as before.

The displacement symmetry of eq. (2.11) can be extended to include also:

$$\boldsymbol{\Delta}_g \to \boldsymbol{\Delta}_g + \mathbf{n}(\eta)\,, \tag{2.15}$$

which also implies $\mathbf{v}_g \to \mathbf{v}_g + \mathbf{n}'$. The galaxy overdensity δ_g, like its dark matter counterpart, does not transform under this symmetry. Eqs. (2.11) and (2.15) represent the displacement symmetry of the combined dark-matter-galaxies system, as long as R_g and $\mathbf{F}_g$ depend only on (dark matter/galaxy) densities and gradients of (dark matter/galaxy) velocities — recall that neither shifts under our symmetry. What happens if R_g and/or $\mathbf{F}_g$ depends on velocities as opposed to gradients of velocities? In that case, shifting velocities by a spatially

[5] There are actually two different nonlinear realized symmetries associated with the displacement potential. One is shifting it by a constant or a function of time (but not space). The other is shifting it by a linear gradient, i.e., $\pi \to \pi + \mathbf{n}\cdot\mathbf{q}$ where $\mathbf{n}$ is the same as that in eq. (2.11). There are as a result two consistency relations which can be succinctly combined into one, eq. (2.12). See our earlier paper [21] for further discussions.

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constant amount would affect the galaxy formation and dynamics — this is a violation of the equivalence principle which states that local physical processes (such as galaxy formation, mergers and motion) should not be dependent on the absolute state of motion. Note that a dependence on the dark-matter-galaxy velocity *difference* $\mathbf{v} - \mathbf{v}_g$, on the other hand, is consistent with the equivalence principle, and the velocity difference is indeed unchanged under our symmetry. Thus, as long as the equivalence principle is respected, whether R_g and $\mathbf{F}_g$ depend on densities, gradients of velocities or velocity differences, the displacement symmetry holds. Furthermore, the same statement is expected to be valid in a system with many different species, such as baryons, galaxies or even dark matter of different kinds. The argument that leads to eq. (2.12) can be rerun to give the more general Lagrangian space consistency relation:

$$\lim_{\mathbf{p}\to 0} \frac{\langle \mathbf{\Delta}_{\mathbf{p}}(\eta)\, \mathcal{O}_{\mathbf{p}_1}(\eta_1) \ldots \mathcal{O}_{\mathbf{p}_N}(\eta_N) \rangle^{c'}}{P_\Delta(\mathrm{p}, \eta)} = 0\,, \tag{2.16}$$

where $\mathcal{O}$ is any observable that has no linear shift under the displacement symmetry — this includes for instance the densities, displacements and velocities of the galaxies and of dark matter.[6] Note that the $\mathcal{O}$'s need not be the same observables. We have restored the explicit time-dependence of each fluctuation variable to emphasize the fact that the times need not be equal. Note also that we have chosen the dark matter displacement to be the pion. We could have chosen the galaxy displacement instead. Assuming that gravity is the dominant interaction on large scales and adiabatic initial conditions, the two displacements are expected to coincide in any case in the soft limit.[7] Furthermore, we could have chosen the velocity instead of the displacement as the soft-pion, in which case eq. (1.3) follows.

3 The Lagrangian space consistency relation: checks

The above derivation of the Lagrangian space consistency relation is a bit terse, and the form the relation takes is surprisingly simple. It is thus worth performing some non-trivial checks of the relation. We will first do this using second order Lagrangian perturbation theory (section 3.1). Then, in section 3.2, we demonstrate how the Eulerian space consistency relation can be derived from its counterpart in Lagrangian space.

3.1 Perturbative check

Let us perform an explicit check of eq. (2.16) using second-order Lagrangian space perturbation theory. For simplicity, we will focus on the case where the only species present is dark matter and the observable $\mathcal{O} = \delta$. We will confine the discussion to the squeezed three-point function; extension to a general (N+1)-point function is straightforward. Expanding eq. (2.3) to second order, we have

$$\delta(\mathbf{x}(\mathbf{q}, \eta), \eta) = -\nabla_{\mathbf{q}} \cdot \mathbf{\Delta} + \frac{1}{2}(\nabla_{\mathbf{q}} \cdot \mathbf{\Delta})^2 + \frac{1}{2}\nabla_{\mathbf{q}^i}\mathbf{\Delta}^j \nabla_{\mathbf{q}^j}\mathbf{\Delta}^i\,. \tag{3.1}$$

[6] The reader might wonder: given that the Lagrangian coordinate $\mathbf{q}$ does not get transformed at all under the displacement symmetry, is there any observable that has a linear shift? The answer is yes. For instance, the combination $\mathcal{O} = \mathbf{v}\delta$ transforms to $(\mathbf{v} + \mathbf{n}')\delta$ giving a shift that is linear in the fluctuation variable δ.

[7] The consistency relations will be violated in the presence of a primordial velocity bias between different species, and selection effects may introduce such a bias; see [21] for a more detailed discussion. See also e.g. [22] for an explicit calculation of the velocity and density of biased tracers in the soft limit.

Expanding out $\delta_{\mathbf{p}} = \delta^{(1)}_{\mathbf{p}} + \delta^{(2)}_{\mathbf{p}} + \cdots$, $\boldsymbol{\Delta}_{\mathbf{p}} = \boldsymbol{\Delta}^{(1)}_{\mathbf{p}} + \boldsymbol{\Delta}^{(2)}_{\mathbf{p}} + \cdots$, and plugging into eqs. (2.9) and (2.10), we have [23]:

$$
\begin{aligned}
\Delta^{j\,(1)}_{\mathbf{p}}(\eta) &= \frac{-i\,\mathrm{p}^j}{\mathrm{p}^2}\,\delta^{(1)}_{\mathbf{p}}(\eta)\,,\\
\Delta^{j\,(2)}_{\mathbf{p}}(\eta) &= \frac{1}{2}\frac{D_2(\eta)}{D(\eta)^2}\frac{i\,\mathrm{p}^j}{\mathrm{p}^2}\int\frac{d^3\mathbf{p}_A d^3\mathbf{p}_B}{(2\pi)^3}\delta_D(\mathbf{p}_A+\mathbf{p}_B-\mathbf{p})\\
&\qquad\times\left(1-\frac{(\mathbf{p}_A\cdot\mathbf{p}_B)^2}{\mathrm{p}_A^2\mathrm{p}_B^2}\right)\delta^{(1)}_{\mathbf{p}_A}(\eta)\delta^{(1)}_{\mathbf{p}_B}(\eta)\,,\\
\delta^{(2)}_{\mathbf{p}}(\eta) &= \frac{1}{2}\int\frac{d^3\mathbf{p}_A d^3\mathbf{p}_B}{(2\pi)^3}\delta_D(\mathbf{p}_A+\mathbf{p}_B-\mathbf{p})\\
&\qquad\times\left(1-\frac{D_2(\eta)}{D(\eta)^2}+\frac{(\mathbf{p}_A\cdot\mathbf{p}_B)^2}{\mathrm{p}_A^2\mathrm{p}_B^2}\left[1+\frac{D_2(\eta)}{D(\eta)^2}\right]\right)\delta^{(1)}_{\mathbf{p}_A}(\eta)\delta^{(1)}_{\mathbf{p}_B}(\eta)\,,
\end{aligned}
\tag{3.2}
$$

where D is the linear growth factor determining the time-dependence of the first order displacement (and density), and D_2 is the second order growth factor determining that of the second order displacement. They satisfy the equations:

$$
\begin{aligned}
D'' + \frac{a'}{a}D' - 4\pi G a^2\bar\rho D &= 0\,,\\
D_2'' + \frac{a'}{a}D_2' - 4\pi G a^2\bar\rho D_2 &= -4\pi G a^2\bar\rho D^2\,.
\end{aligned}
\tag{3.3}
$$

For instance, in a flat universe with $\Omega_m = 1$, $D_2 = -3D^2/7$. Using these expressions, we can work out the lowest order contributions to the relevant squeezed bispectrum:

$$
\begin{aligned}
\langle\Delta^j_{\mathbf{p}}(\eta)\delta_{\mathbf{p}_1}(\eta_1)\delta_{\mathbf{p}_2}(\eta_2)\rangle &= \langle\Delta^{j\,(2)}_{\mathbf{p}}(\eta)\delta^{(1)}_{\mathbf{p}_1}(\eta_1)\delta^{(1)}_{\mathbf{p}_2}(\eta_2)\rangle + \langle\Delta^{j\,(1)}_{\mathbf{p}}(\eta)\delta^{(2)}_{\mathbf{p}_1}(\eta_1)\delta^{(1)}_{\mathbf{p}_2}(\eta_2)\rangle\\
&\quad+\langle\Delta^{j\,(1)}_{\mathbf{p}}(\eta)\delta^{(1)}_{\mathbf{p}_1}(\eta_1)\delta^{(2)}_{\mathbf{p}_2}(\eta_2)\rangle\\
&= O(\mathrm{p}^j) + O(\mathrm{p}^j P_\Delta(\mathrm{p}))\,,
\end{aligned}
\tag{3.4}
$$

where we spell out the dependence on the soft momentum $\mathbf{p}$: the $O(\mathrm{p}^j)$ piece comes from the first term on the right in the first line, and the $O(\mathrm{p}^j P_\Delta(\mathrm{p}))$ piece comes from the other two terms. We have used the fact that $1-[(\mathbf{p}_A\cdot\mathbf{p}_B)^2/\mathrm{p}_A^2\mathrm{p}_B^2] = O(\mathrm{p}^2)$ for $\mathbf{p}_A+\mathbf{p}_B=\mathbf{p}$. The $O(\mathrm{p}^j P_\Delta(\mathrm{p}))$ piece is obviously compatible with the Lagrangian space consistency relation; i.e., it gives $\langle\Delta^j_{\mathbf{p}}\delta_{\mathbf{p}_1}\delta_{\mathbf{p}_2}\rangle^{c'}/P_\Delta(\mathrm{p}) = 0$ in the $\mathbf{p}\to 0$ limit. The $O(\mathrm{p}^j)$ piece does the same, provided the power spectrum $P_\Delta(\mathrm{p})$ is not too blue. Parameterizing the power spectrum $P_\Delta(\mathrm{p}) \propto \mathrm{p}^{n-2}$ in the low momentum limit, the consistency relation holds as long as $n < 3$. Exactly the same condition is needed for the Eulerian space consistency relation (see e.g. [21]).[8]

[8]The Eulerian space consistency relation is often given in a form where δ is used as the soft mode. One might be tempted to do the same for the Lagrangian space consistency relation. However, one can check using perturbation theory that such a consistency relation would have required $n < 1$, a condition considerably stronger than expected. This is related to the fact that $\delta^{(2)}_{\mathbf{p}}$ does not vanish in the $\mathbf{p}\to 0$ limit, unlike $\Delta^{j\,(2)}_{\mathbf{p}}$.

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3.2 Recovering the Eulerian space consistency relation from Lagrangian space

The consistency relation takes such a different form in Lagrangian versus Eulerian space that it is worth considering how one can be derived from the other. Let us compute the following:

$$
\begin{aligned}
E &\equiv E_L + E_R \\
E_L &\equiv \lim_{\mathbf{k}\to 0} \frac{\langle v^j_{\mathbf{k}}(\eta)\, \delta_{\mathbf{k}_1}(\eta_1)\, \delta_{\mathbf{k}_2}(\eta_2)\, \rangle^{c'}}{P_v(\mathrm{k},\eta)} \qquad , \\
E_R &\equiv \, - i\mathrm{k}^j \sum_{a=1}^{2} \frac{D(\eta_a)}{D'(\eta)} \frac{\mathbf{k}\cdot\mathbf{k}_a}{\mathrm{k}^2} \langle \delta_{\mathbf{k}_1}(\eta_1)\, \delta_{\mathbf{k}_2}(\eta_2)\, \rangle^{c'} .
\end{aligned}
\tag{3.5}
$$

The Eulerian space consistency condition is the statement that $E = 0$. We will content ourselves with deriving this — a special case of the more general Eulerian space consistency relation (1.2) — from the Lagrangian space consistency relation.

To relate E to quantities in Lagrangian space, we will slightly abuse our notation. So far, we have been using $\mathbf{k}$ for the Eulerian space momentum and $\mathbf{p}$ for the Lagrangian space momentum. For instance, $\delta_{\mathbf{k}_1}$ is defined as

$$
\delta_{\mathbf{k}_1} = \int d^3\mathbf{x}\, \delta(\mathbf{x})\, e^{i\mathbf{k}_1\cdot\mathbf{x}} . \tag{3.6}
$$

Let us rewrite this as

$$
\begin{aligned}
\delta_{\mathbf{k}_1=\mathbf{p}_1} &= \int d^3\mathbf{q}\, J(\mathbf{q})\, \delta(\mathbf{x}(\mathbf{q}))\, e^{i\mathbf{p}_1\cdot(\mathbf{q}+\boldsymbol{\Delta}(\mathbf{q}))} \\
&= \int d^3\mathbf{q}\, J(\mathbf{q})\, \delta(\mathbf{x}(\mathbf{q}))\, e^{i\mathbf{p}_1\cdot\mathbf{q}} + i\mathrm{p}_1^m \int d^3\mathbf{q}\, J(\mathbf{q})\, \delta(\mathbf{x}(\mathbf{q}))\, \Delta^m(\mathbf{q})\, e^{i\mathbf{p}_1\cdot\mathbf{q}} \\
&\quad - \frac{1}{2}\mathrm{p}_1^m \mathrm{p}_1^n \int d^3\mathbf{q}\, J(\mathbf{q})\, \delta(\mathbf{x}(\mathbf{q}))\, \Delta^m(\mathbf{q})\Delta^n(\mathbf{q})\, e^{i\mathbf{p}_1\cdot\mathbf{q}} + \dots ,
\end{aligned}
\tag{3.7}
$$

where $J(\mathbf{q})$ is defined by eq. (2.4) (with no absolute value). Defining

$$
\tilde{\delta}(\mathbf{q}) \equiv J(\mathbf{q})\delta(\mathbf{x}(\mathbf{q})) , \tag{3.8}
$$

we see that the first term on the right of eq. (3.7) is $\tilde{\delta}_{\mathbf{p}_1}$, i.e. the Fourier transform of $\tilde{\delta}$ in *Lagrangian space*. The other terms on the right can likewise be thought of as the Fourier transform of some quantity in Lagrangian space. This is why we introduce $\mathbf{p}_1$ as the momentum label for these Fourier components. On the other hand, upon summation, they give the quantity on the left $\delta_{\mathbf{k}_1=\mathbf{p}_1}$ which is the Fourier transform of density in *Eulerian space* — this is why we use $\mathbf{k}_1$ as its momentum label; it just happens to take on the numerical value $\mathbf{p}_1$ which conveniently gives us the appropriate momentum label for quantities on the right. It is worth emphasizing that our definitions are general, in that they are valid even in the presence of multiple-streaming (see section 2.1).

The expansion in terms of Δ in eq. (3.7) is purely formal. In the nonlinear regime, there is no sense in which Δ is small. The expansion provides a convenient way to relate the Fourier transform in Eulerian space to the Fourier transform in Lagrangian space. We will argue $E = 0$ holds to arbitrary order in a power series expansion.

For the soft mode, we have

$$
v^j_{\mathbf{k}=\mathbf{p}} = v^j_{\mathbf{p}} + \dots , \tag{3.9}
$$

This is where our abuse of notation is the most egregious: on the left is the velocity Fourier transformed in Eulerian space; on the right is the velocity Fourier transformed in Lagrangian space. They agree only to lowest order in perturbations. For the soft-mode, ignoring the higher order corrections is permissible: the higher order corrections will give higher powers of the soft momentum $\mathbf{p}$ compared to what is kept in the consistency relation, provided that the soft power spectrum $P_v(\mathrm{p})$ or $P_\Delta(\mathrm{p})$ is not too blue (see section 3.1). Similarly, it can be shown that in the soft limit, there is no need to distinguish between P_v in Lagrangian versus Eulerian space.[9]

Let us substitute eq. (3.7) for the hard modes, and eq. (3.9) for the soft mode, into the expression for E in eq. (3.5). Consider first what contributes to E_L:

$$\begin{aligned}
&\langle v^j_{\mathbf{k}=\mathbf{p}} \delta_{\mathbf{k}_1=\mathbf{p}_1} \delta_{\mathbf{k}_2=\mathbf{p}_2}\rangle \\
&\quad = \langle v^j_{\mathbf{p}} \tilde{\delta}_{\mathbf{p}_1} \tilde{\delta}_{\mathbf{p}_2}\rangle + \left[i\mathrm{p}_1^m \int \frac{d^3\mathbf{p}_A}{(2\pi)^3} \langle v^j_{\mathbf{p}} \Delta^m_{\mathbf{p}_A}(\eta_1) \tilde{\delta}_{\mathbf{p}_1-\mathbf{p}_A} \tilde{\delta}_{\mathbf{p}_2}\rangle + 1 \leftrightarrow 2 \right] \\
&\qquad - \left[\mathrm{p}_1^m \mathrm{p}_2^n \int \frac{d^3\mathbf{p}_A}{(2\pi)^3} \frac{d^3\mathbf{p}_B}{(2\pi)^3} \langle v^j_{\mathbf{p}} \tilde{\delta}_{\mathbf{p}_1-\mathbf{p}_A} \Delta^m_{\mathbf{p}_A}(\eta_1) \tilde{\delta}_{\mathbf{p}_2-\mathbf{p}_B} \Delta^n_{\mathbf{p}_B}(\eta_2)\rangle \right. \\
&\qquad + \left. \left[\frac{1}{2} \mathrm{p}_1^m \mathrm{p}_1^n \int \frac{d^3\mathbf{p}_A}{(2\pi)^3} \frac{d^3\mathbf{p}_B}{(2\pi)^3} \langle v^j_{\mathbf{p}} \tilde{\delta}_{\mathbf{p}_1-\mathbf{p}_A-\mathbf{p}_B} \Delta^m_{\mathbf{p}_A}(\eta_1) \Delta^n_{\mathbf{p}_B}(\eta_1) \tilde{\delta}_{\mathbf{p}_2}\rangle + (1 \leftrightarrow 2) \right] \right] \\
&\qquad + O(\Delta^3) + \ldots
\end{aligned} \tag{3.10}$$

where we have largely suppressed the time-dependence to minimize clutter (η for the soft mode, and η_1 and η_2 respectively for the hard modes), except for variables with internal momenta. We emphasize that the expansion in Δ is purely formal, and comes entirely from expanding $e^{i\mathbf{p}_1\cdot\mathbf{\Delta}}$ or $e^{i\mathbf{p}_2\cdot\mathbf{\Delta}}$. The first term on the right can be set to zero by virtue of the Lagrangian space consistency condition (keeping in mind that this term is divided by $P_v(\mathrm{p})$ as part of the quantity E_L). We will be assuming the Lagrangian space consistency relation in its general form (eq. (1.3)):

$$\lim_{\mathbf{p}\to 0} \frac{\langle \boldsymbol{v}_{\mathbf{p}}(\eta)\, \mathcal{O}_{\mathbf{p}_1}(\eta_1) \ldots \mathcal{O}_{\mathbf{p}_N}(\eta_N)\, \rangle^{c'}}{P_v(\mathrm{p}, \eta)} = 0\,, \tag{3.11}$$

where the observables at hard momenta need not be the same observable. Finally, note that while we are interested in the connected part of the correlator on the left hand side of eq. (3.10), the correlators on the right hand side are the full correlators, minus the contributions where some proper subset of the original hard and soft momenta sum to zero. In particular, the correlators on the right hand side of eq. (3.10) contain both connected and disconnected pieces.

The second term on the right of eq. (3.10), formally $O(\Delta)$, equals

$$\begin{aligned}
i\mathrm{p}_1^m \int \frac{d^3\mathbf{p}_A}{(2\pi)^3} \Big[&\langle v^j_{\mathbf{p}} \Delta^m_{\mathbf{p}_A}(\eta_1)\rangle \langle \tilde{\delta}_{\mathbf{p}_1-\mathbf{p}_A} \tilde{\delta}_{\mathbf{p}_2}\rangle \\
&+ \langle v^j_{\mathbf{p}} \tilde{\delta}_{\mathbf{p}_1-\mathbf{p}_A}\rangle \langle \Delta^m_{\mathbf{p}_A}(\eta_1) \tilde{\delta}_{\mathbf{p}_2}\rangle + \langle v^j_{\mathbf{p}} \tilde{\delta}_{\mathbf{p}_2}\rangle \langle \Delta^m_{\mathbf{p}_A}(\eta_1) \tilde{\delta}_{\mathbf{p}_1-\mathbf{p}_A}\rangle \\
&+ \langle v^j_{\mathbf{p}} \Delta^m_{\mathbf{p}_A}(\eta_1) \tilde{\delta}_{\mathbf{p}_1-\mathbf{p}_A} \tilde{\delta}_{\mathbf{p}_2}\rangle^c \Big] + (1 \leftrightarrow 2)\,,
\end{aligned} \tag{3.12}$$

[9]It is also worth emphasizing that the notion of a well-defined velocity in Eulerian space is valid only when multiple-streaming is ignored. This is acceptable for the soft-mode. We do not assume single-streaming for the hard modes.

where the connected trispectrum term $\langle v^j_{\rm p} \ldots \rangle^c$ (anticipating division by $P_v({\rm p})$) can be set to zero using the Lagrangian space consistency relation, the terms involving $\langle v^j_{\bf p} \tilde\delta_{{\bf p}_1 - {\bf p}_A} \rangle$, $\langle v^j_{\bf p} \tilde\delta_{{\bf p}_2} \rangle$ and the like have one more power of the soft momentum $\mathbf{p}$ (and are thus subdominant) compared to terms involving $\langle v^j_{\rm p} \Delta^m_{{\bf p}_A} \rangle$ which give:

$$(2\pi)^3 \delta_D(\mathbf{p}_1 + \mathbf{p_2} + \mathbf{p})\, P_v({\rm p}, \eta)\, i{\rm p}^j \, \frac{D(\eta_1)}{D'(\eta)} \frac{\mathbf{p}_1 \cdot \mathbf{p}}{{\rm p}^2} \langle \tilde\delta_{{\bf p}_1 + {\bf p}}(\eta_1) \tilde\delta_{{\bf p}_2}(\eta_2) \rangle^{c'} + (1 \leftrightarrow 2)\,. \tag{3.13}$$

The third term on the right of eq. (3.10), formally $O(\Delta^2)$, can be treated in a similar way: some can be ignored by assuming the Lagrangian space consistency relation, some are subdominant in the soft-limit (i.e., they vanish upon division by $P_v(p)$ and sending $\mathbf{p} \to 0$), and the dominant terms are those that involve $\langle v^j_{\bf p} \Delta \rangle$ which give:

$$\begin{aligned} & -(2\pi)^3 \delta_D(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}) P_v({\rm p}, \eta) \\ & \quad \times \int \frac{d^3 \mathbf{p}_A}{(2\pi)^3} \Bigg[{\rm p}^j {\rm p}_2^m \frac{D(\eta_1)}{D'(\eta)} \frac{\mathbf{p}_1 \cdot \mathbf{p}}{{\rm p}^2} \langle \tilde\delta_{{\bf p}_1 + {\bf p}}(\eta_1) \tilde\delta_{{\bf p}_2 - {\bf p}_A}(\eta_2) \Delta^m_{{\bf p}_A}(\eta_2) \rangle^{c'} \\ & \qquad\qquad + {\rm p}^j {\rm p}_1^m \frac{D(\eta_1)}{D'(\eta)} \frac{\mathbf{p}_1 \cdot \mathbf{p}}{{\rm p}^2} \langle \tilde\delta_{{\bf p}_1 + {\bf p} - {\bf p}_A}(\eta_1) \tilde\delta_{{\bf p}_2}(\eta_2) \Delta^m_{{\bf p}_A}(\eta_1) \rangle^{c'} + (1 \leftrightarrow 2) \Bigg]. \end{aligned} \tag{3.14}$$

Thus, combining eqs. (3.13) and (3.14), E_L of eq. (3.5) can be rewritten as:

$$\begin{aligned} E_L \equiv & \lim_{\mathbf{p} \to 0} \frac{\langle v^j_{\mathbf{k}=\mathbf{p}}(\eta)\, \delta_{\mathbf{k}_1 = \mathbf{p}_1}(\eta_1)\, \delta_{\mathbf{k}_2 = \mathbf{p}_2}(\eta_2) \, \rangle^{c'}}{P_v({\rm p}, \eta)} \\ = & \, i{\rm p}^j \frac{D(\eta_1)}{D'(\eta)} \frac{\mathbf{p}_1 \cdot \mathbf{p}}{{\rm p}^2} \langle \tilde\delta_{{\bf p}_1}(\eta_1) \tilde\delta_{{\bf p}_2}(\eta_2) \rangle^{c'} \\ & - \int \frac{d^3 \mathbf{p}_A}{(2\pi)^3} \Bigg[{\rm p}^j {\rm p}_2^m \frac{D(\eta_1)}{D'(\eta)} \frac{\mathbf{p}_1 \cdot \mathbf{p}}{{\rm p}^2} \langle \tilde\delta_{{\bf p}_1}(\eta_1) \tilde\delta_{{\bf p}_2 - {\bf p}_A}(\eta_2) \Delta^m_{{\bf p}_A}(\eta_2) \rangle^{c'} \\ & \qquad + {\rm p}^j {\rm p}_1^m \frac{D(\eta_1)}{D'(\eta)} \frac{\mathbf{p}_1 \cdot \mathbf{p}}{{\rm p}^2} \langle \tilde\delta_{{\bf p}_1 - {\bf p}_A}(\eta_1) \tilde\delta_{{\bf p}_2}(\eta_2) \Delta^m_{{\bf p}_A}(\eta_1) \rangle^{c'} \Bigg] + (1 \leftrightarrow 2) + \ldots\,. \end{aligned} \tag{3.15}$$

Next, let us rewrite E_R using the same strategy:

$$\begin{aligned} E_R \equiv & - i{\rm p}^j \sum_{a=1}^{2} \frac{D(\eta_a)}{D'(\eta)} \frac{\mathbf{p} \cdot \mathbf{p}_a}{{\rm p}^2} \langle \delta_{\mathbf{k}_1 = \mathbf{p}_1}(\eta_1)\, \delta_{\mathbf{k}_2 = \mathbf{p}_2}(\eta_2) \, \rangle^{c'} \\ = & - i{\rm p}^j \frac{D(\eta_1)}{D'(\eta)} \frac{\mathbf{p} \cdot \mathbf{p}_1}{{\rm p}^2} \langle \tilde\delta_{{\bf p}_1}(\eta_1)\, \tilde\delta_{{\bf p}_2}(\eta_2) \, \rangle^{c'} \\ & + \int \frac{d^3 \mathbf{p}_A}{(2\pi)^3} \Bigg[{\rm p}^j {\rm p}_2^m \frac{D(\eta_1)}{D'(\eta)} \frac{\mathbf{p}_1 \cdot \mathbf{p}}{{\rm p}^2} \langle \tilde\delta_{{\bf p}_1}(\eta_1) \tilde\delta_{{\bf p}_2 - {\bf p}_A}(\eta_2) \Delta^m_{{\bf p}_A}(\eta_2) \rangle^{c'} \\ & \qquad + {\rm p}^j {\rm p}_1^m \frac{D(\eta_1)}{D'(\eta)} \frac{\mathbf{p}_1 \cdot \mathbf{p}}{{\rm p}^2} \langle \tilde\delta_{{\bf p}_1 - {\bf p}_A}(\eta_1) \tilde\delta_{{\bf p}_2}(\eta_2) \Delta^m_{{\bf p}_A}(\eta_1) \rangle^{c'} \Bigg] + (1 \leftrightarrow 2) + \ldots\,. \end{aligned} \tag{3.16}$$

Thus, we see that $E \equiv E_L + E_R = 0$, at least to the two lowest non-trivial orders in Δ. The cancelation works like this: expanding $e^{i\mathbf{p}_1 \cdot \mathbf{\Delta}}$ and $e^{i\mathbf{p}_2 \cdot \mathbf{\Delta}}$ as a formal power series in Δ, a given order for E_L is canceled by one lower order for E_R. It can be shown that this pattern continues to arbitrarily high orders. The proof is given in the appendix. This completes our derivation of the Eulerian space consistency relation, embodied in the statement $E = 0$ (eq. (3.5)), from the Lagrangian space consistency relation (eq. (3.11)).

4 Discussion

We have shown that the consistency relation takes a particularly simple form in Lagrangian space: the squeezed correlation function, suitably normalized, vanishes (eqs. (1.5)):

$$\lim_{\mathbf{p}\to 0} \frac{\langle \mathbf{\Delta}_{\mathbf{p}}(\eta)\, \mathcal{O}_{\mathbf{p}_1}(\eta_1) \ldots \mathcal{O}_{\mathbf{p}_N}(\eta_N) \rangle^{c'}}{P_\Delta(\mathrm{p}, \eta)} = 0\,, \tag{4.1}$$

where Δ is the displacement, and $\mathcal{O}$ can be many different observables such as mass or galaxy density; the quantities can be at different times, and $\mathbf{p}, \mathbf{p}_1, \mathbf{p}_2, \ldots$ label the momenta with $\mathbf{p}$ being the soft one.[10] The derivation given in section 2.2 is fully non-perturbative and is valid even in the presence of multiple-streaming. It makes use of a master formula that was derived in an earlier paper [21], which relates an (N+1)-point function to the linear transformation of an N-point function, for a general nonlinearly-realized symmetry (eq. (2.8)). The key realization is that the nonlinearly-realized symmetry of interest — the displacement symmetry — does not require transforming the Lagrangian coordinate $\mathbf{q}$ (eq. (2.11)):

$$\mathbf{q} \to \mathbf{q} \quad , \quad \mathbf{\Delta} \to \mathbf{\Delta} + \mathbf{n}(\eta) \quad , \quad \Phi \to \Phi - \left(\mathbf{n}'' + \frac{a'}{a}\mathbf{n}'\right) \cdot \mathbf{x}\,, \tag{4.2}$$

where $\mathbf{\Delta}$ is the displacement, $\mathbf{n}$ is some function of time, Φ is the gravitational potential, a is the scale factor and $\mathbf{x}$ is the Eulerian coordinate.[11] Thus, many observables $\mathcal{O}$ such as the mass density or the galaxy density,[12] when expressed as functions of the Lagrangian coordinate $\mathbf{q}$, do not receive linear transformations, and so the right hand side of the master formula vanishes.[13] This contrasts with what happens when these observables are thought of as functions of the Eulerian coordinate $\mathbf{x}$; they receive linear transformations because under the same symmetry

$$\mathbf{x} \to \mathbf{x} + \mathbf{n}(\eta)\,. \tag{4.3}$$

It is worth mentioning that the observables $\mathcal{O}$ can even be quantities in redshift space. For instance, the redshift space mass density $\delta_s(\mathbf{s})$, where $\mathbf{s}$ is the redshift space coordinate, can always be written as a function of $\mathbf{q}$, just like what we have done for the real (Eulerian) space mass density. This works even in the presence of (real or redshift-space) multiple-streaming.

It is worth reviewing the assumptions behind the master formula: it assumes (1) single-field/clock initial condition — that which follows from single-field inflation — and (2) the adiabatic mode condition (first emphasized in [24]) — that the (displacement) mode generated by the symmetry of interest is the long wavelength limit of an actual physical mode. In particular, the latter condition requires that $\mathbf{n}(\eta)$ have the same time dependence as the linear growth factor $D(\eta)$.[14] This is the reason why the growth factor shows up on the right hand

[10] See also eq. (1.3) with velocity $\boldsymbol{v}$ as the soft mode.

[11] If there are multiple species present such as dark matter and galaxies, the same transformation applies to the displacement of all species. See eq. (2.15).

[12] See footnote 4.

[13] Note that even quantities such as Δ or Φ have no linear shift (a shift that is linear in fluctuation variables). More complicated observables could have a linear shift; see footnote 6.

[14] In the presence of multiple species, such as dark matter and galaxies, the fact that the displacements for all species have to be transformed by the same $\mathbf{n}(\eta) \propto D(\eta)$ is a manifestation of the equivalence principle: all species fall at the same rate on large scales.

side of the Eulerian space consistency relation (eq. (1.2)). It is interesting that because the Lagrangian space consistency relation has a vanishing right hand side, the time-dependence of $\mathbf{n}(\eta)$ has no direct bearing on the form it takes.

An important point: the fact that the consistency relation can be written in such a simple, even trivial, form in Lagrangian space should not be taken to imply the lack of physical meaning. Indeed, the consistency relation can be violated if the initial conditions were not of the single-field/clock type. Rather, the simplicity suggests that an analytic understanding of nonlinear clustering is perhaps more promising in Lagrangian space. This view has a long history, starting from Zeldovich [25]. What is interesting is that the consistency relation, by virtue of its being a symmetry statement, is non-perturbative, and thus goes beyond perturbative treatments such as the Zeldovich approximation.

Ultimately, observations are performed in Eulerian space, not Lagrangian space. At the nonlinear level, the relation between the two descriptions is complex. Our derivation of the consistency relation in Eulerian space from its counterpart in Lagrangian space is a case in point (section 3.2). It requires a formal series expansion in the displacement Δ. In relating the two descriptions, the expansion in Δ is done in an uneven manner: only phase factors such as $e^{i\mathbf{p}_1\cdot\boldsymbol{\Delta}}$ are expanded even though other variables, such as the density δ, also depend on the displacement. This is not unexpected in relations that are purported to be non-perturbative — *partial* resummation of perturbations is often a useful technique. Can our example point to a useful, new resummation scheme?

A natural question is whether there are relativistic generalizations of statements like eq. (4.1) — consistency relations with a vanishing right hand side. The Lagrangian coordinate (attached to dark matter particles) is essentially the freely-falling coordinate.[15] Indeed [27] showed that using the freely-falling coordinate, the dilation consistency relation [3] can be rewritten in a similarly simple form (see also [28]). Their derivation is perturbative. It should be possible to extend their proof using the non-perturbative arguments presented here. More generally, it would be interesting to see if further general relativistic consistency relations, such as those found by [7], can also be recast in this fashion.

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A Recovering the Eulerian space consistency relation from Lagrangian space — To arbitrary orders in displacement

In section 3.2, we argue that the Eulerian space consistency relation follows from its Lagrangian space counter-part, at least to the two lowest non-trivial orders in a formal expansion in displacement. In this appendix, we show that this works to arbitrary orders.

We begin by expanding eq. (3.7) to all orders in Δ:

$$\begin{aligned}\delta_{\mathbf{k}_1=\mathbf{p}_1} &= \int d^3\mathbf{q}\, J(\mathbf{q})\,\delta(\mathbf{x}(\mathbf{q}))\, e^{i\mathbf{p}_1\cdot(\mathbf{q}+\boldsymbol{\Delta}(\mathbf{q}))} \\ &= \sum_{n=0}^{\infty}\frac{i^n}{n!}\mathrm{p}_1^{i_1}\cdots\mathrm{p}_1^{i_n}\int_{\mathbf{p}_{j_1},\cdots,\mathbf{p}_{j_n}}\tilde{\delta}_{\mathbf{p}_1-\mathbf{p}_{j_1}-\cdots-\mathbf{p}_{j_n}}\Delta^{i_1}_{\mathbf{p}_{j_1}}\cdots\Delta^{i_n}_{\mathbf{p}_{j_n}}\end{aligned} \tag{A.1}$$

[15] See [26] for a recent discussion of freely falling coordinates in cosmological settings.

where as before, $\tilde{\delta}(\mathbf{q}) = J(\mathbf{q})\delta(\mathbf{x}(\mathbf{q}))$. Collecting together the terms of order Δ^n in the three-point correlator $\langle v^j_{\mathbf{p}}(\eta)\delta_{\mathbf{p}_1}(\eta_1)\delta_{\mathbf{p}_2}(\eta_2)\rangle$ in E_L, we have

$$\sum_{m=0}^{n} \frac{i^m}{m!}\frac{i^{n-m}}{(n-m)!}\mathrm{p}_1^{i_1}\cdots\mathrm{p}_1^{i_m}\mathrm{p}_2^{i_{m+1}}\cdots\mathrm{p}_2^{i_n} \\ \times\int_{\mathbf{p}_{j_1},\cdots,\mathbf{p}_{j_n}} \langle v^j_{\mathbf{p}}(\eta)\tilde{\delta}_{\mathbf{p}_1-\mathbf{p}_{j_1}-\cdots-\mathbf{p}_{j_m}}(\eta_1)\Delta^{i_1}_{\mathbf{p}_{j_1}}(\eta_1)\cdots\Delta^{i_m}_{\mathbf{p}_{j_m}}(\eta_1) \\ \cdot\tilde{\delta}_{\mathbf{p}_2-\mathbf{p}_{j_{m+1}}-\cdots-\mathbf{p}_{j_n}}(\eta_2)\Delta^{i_{m+1}}_{\mathbf{p}_{j_{m+1}}}(\eta_2)\cdots\Delta^{i_n}_{\mathbf{p}_{j_n}}(\eta_2)\rangle \tag{A.2}$$

where the correlator in the integral is the full correlator containing both connected and disconnected pieces, but where no proper subset of the original momenta $\mathbf{p},\mathbf{p}_1,\mathbf{p}_2$ sums to zero, since it is the connected correlator that appears in E_L. The correlator can be split into a sum over products of connected blocks. Anticipating division by $P_v(\mathbf{p})$, we see that the Lagrangian space consistency relation implies that all contributions where the soft velocity $v^j(\mathbf{p})$ is part of a connected correlator with two or more other fields will vanish. The remaining terms contain a factor of either

$$\langle v^j(\mathbf{p})\tilde{\delta}_{\mathbf{p}_1-\cdots}\rangle\,,\ \langle v^j(\mathbf{p})\tilde{\delta}_{\mathbf{p}_2-\cdots}\rangle\,,\ \langle v^j(\mathbf{p})\Delta_{\mathbf{p}_j}(\eta_1)\rangle\,,\ \langle v^j(\mathbf{p})\Delta_{\mathbf{p}_j}(\eta_2)\rangle\,. \tag{A.3}$$

The first two types of terms will be suppressed by an additional power of the soft momentum, and are subdominant in the squeezed limit. The final set of terms are

$$\sum_{m=0}^{n} \frac{i^m}{m!}\frac{i^{n-m}}{(n-m)!}\mathrm{p}_1^{i_1}\cdots\mathrm{p}_1^{i_m}\mathrm{p}_2^{i_{m+1}}\cdots\mathrm{p}_2^{i_n}\int_{\mathbf{p}_{j_1},\cdots,\mathbf{p}_{j_n}}\Big[m\langle v^j_{\mathbf{p}}(\eta)\Delta^{i_1}_{\mathbf{p}_{j_1}}(\eta_1)\rangle\langle\tilde{\delta}_{\mathbf{p}_1-\mathbf{p}_{j_1}-\cdots-\mathbf{p}_{j_m}}(\eta_1) \\ \cdot\Delta^{i_1}_{\mathbf{p}_{j_1}}(\eta_1)\cdots\Delta^{i_m}_{\mathbf{p}_{j_m}}(\eta_1)\tilde{\delta}_{\mathbf{p}_2-\mathbf{p}_{j_{m+1}}-\cdots-\mathbf{p}_{j_n}}(\eta_2)\Delta^{i_{m+1}}_{\mathbf{p}_{j_{m+1}}}(\eta_2)\cdots\Delta^{i_n}_{\mathbf{p}_{j_n}}(\eta_2)\rangle\Big] \\ +(1\leftrightarrow 2) \tag{A.4}$$

and they give (relabeling m as $m+1$)

$$\Big[i\mathrm{p}^j\frac{\mathbf{p}\cdot\mathbf{p}_1}{\mathbf{p}^2}\frac{D(\eta_1)}{D'(\eta)}\sum_{m=0}^{n-1}\frac{i^{n-1}}{m!(n-1-m)!}\mathrm{p}_1^{i_1}\cdots\mathrm{p}_1^{i_m}\mathrm{p}_2^{i_{m+1}}\cdots\mathrm{p}_2^{i_n}\int_{\mathbf{p}_{j_1},\cdots,\mathbf{p}_{j_n}}\langle\tilde{\delta}_{\mathbf{p}_1+\mathbf{p}-\mathbf{p}_{j_1}-\cdots-\mathbf{p}_{j_m}}(\eta_1) \\ \cdot\Delta^{i_1}_{\mathbf{p}_{j_1}}(\eta_1)\cdots\Delta^{i_m}_{\mathbf{p}_{j_m}}(\eta_1)\tilde{\delta}_{\mathbf{p}_2-\mathbf{p}_{j_{m+1}}-\cdots-\mathbf{p}_{j_n}}(\eta_2)\Delta^{i_{m+1}}_{\mathbf{p}_{j_{m+1}}}(\eta_2)\cdots\Delta^{i_n}_{\mathbf{p}_{j_n}}(\eta_2)\rangle\Big] \\ +(1\leftrightarrow 2) \tag{A.5}$$

Comparing this to the order Δ^{n-1} terms in the expansion of E_R,

$$\Big[-i\mathrm{p}^j\frac{\mathbf{p}\cdot\mathbf{p}_1}{\mathbf{p}^2}\frac{D(\eta_1)}{D'(\eta)}\sum_{m=0}^{n-1}\frac{i^{n-1}}{m!(n-1-m)!}\mathrm{p}_1^{i_1}\cdots\mathrm{p}_1^{i_m}\mathrm{p}_2^{i_{m+1}}\cdots\mathrm{p}_2^{i_n}\int_{\mathbf{p}_{j_1},\cdots,\mathbf{p}_{j_n}}\langle\tilde{\delta}_{\mathbf{p}_1-\mathbf{p}_{j_1}-\cdots-\mathbf{p}_{j_m}}(\eta_1) \\ \cdot\Delta^{i_1}_{\mathbf{p}_{j_1}}(\eta_1)\cdots\Delta^{i_m}_{\mathbf{p}_{j_m}}(\eta_1)\tilde{\delta}_{\mathbf{p}_2-\mathbf{p}_{j_{m+1}}-\cdots-\mathbf{p}_{j_n}}(\eta_2)\Delta^{i_{m+1}}_{\mathbf{p}_{j_{m+1}}}(\eta_2)\cdots\Delta^{i_n}_{\mathbf{p}_{j_n}}(\eta_2)\rangle\Big] \\ +(1\leftrightarrow 2) \tag{A.6}$$

the terms cancel in the $\mathbf{p}\to 0$ limit to give $E_L+E_R=0$ order by order in the formal expansion in Δ, Q.E.D.

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References

[1] A. Kehagias and A. Riotto, *Symmetries and consistency relations in the large scale structure of the universe*, Nucl. Phys. **B 873** (2013) 514 [arXiv:1302.0130] [INSPIRE].

[2] M. Peloso and M. Pietroni, *Galilean invariance and the consistency relation for the nonlinear squeezed bispectrum of large scale structure*, JCAP **05** (2013) 031 [arXiv:1302.0223] [INSPIRE].

[3] J.M. Maldacena, *Non-gaussian features of primordial fluctuations in single field inflationary models*, JHEP **05** (2003) 013 [astro-ph/0210603] [INSPIRE].

[4] P. Creminelli and M. Zaldarriaga, *Single field consistency relation for the* 3*-point function*, JCAP **10** (2004) 006 [astro-ph/0407059] [INSPIRE].

[5] P. Creminelli, J. Noreña and M. Simonović, *Conformal consistency relations for single-field inflation*, JCAP **07** (2012) 052 [arXiv:1203.4595] [INSPIRE].

[6] K. Hinterbichler, L. Hui and J. Khoury, *Conformal symmetries of adiabatic modes in cosmology*, JCAP **08** (2012) 017 [arXiv:1203.6351] [INSPIRE].

[7] K. Hinterbichler, L. Hui and J. Khoury, *An infinite set of Ward identities for adiabatic modes in cosmology*, JCAP **01** (2014) 039 [arXiv:1304.5527] [INSPIRE].

[8] V. Assassi, D. Baumann and D. Green, *On soft limits of inflationary correlation functions*, JCAP **11** (2012) 047 [arXiv:1204.4207] [INSPIRE].

[9] V. Assassi, D. Baumann and D. Green, *Symmetries and loops in inflation*, JHEP **02** (2013) 151 [arXiv:1210.7792] [INSPIRE].

[10] A. Kehagias and A. Riotto, *Operator product expansion of inflationary correlators and conformal symmetry of de Sitter*, Nucl. Phys. **B 864** (2012) 492 [arXiv:1205.1523] [INSPIRE].

[11] W.D. Goldberger, L. Hui and A. Nicolis, *One-particle-irreducible consistency relations for cosmological perturbations*, Phys. Rev. **D 87** (2013) 103520 [arXiv:1303.1193] [INSPIRE].

[12] G.L. Pimentel, *Inflationary consistency conditions from a wavefunctional perspective*, JHEP **02** (2014) 124 [arXiv:1309.1793] [INSPIRE].

[13] L. Berezhiani and J. Khoury, *Slavnov-Taylor identities for primordial perturbations*, JCAP **02** (2014) 003 [arXiv:1309.4461] [INSPIRE].

[14] P. Creminelli, J. Noreña, M. Simonović and F. Vernizzi, *Single-field consistency relations of large scale structure*, JCAP **12** (2013) 025 [arXiv:1309.3557] [INSPIRE].

[15] M. Peloso and M. Pietroni, *Ward identities and consistency relations for the large scale structure with multiple species*, JCAP **04** (2014) 011 [arXiv:1310.7915] [INSPIRE].

[16] A. Kehagias, J. Noreña, H. Perrier and A. Riotto, *Consequences of symmetries and consistency relations in the large-scale structure of the universe for non-local bias and modified gravity*, Nucl. Phys. **B 883** (2014) 83 [arXiv:1311.0786] [INSPIRE].

[17] P. Berger, A. Kehagias and A. Riotto, *Testing the origin of cosmological magnetic fields through the large-scale structure consistency relations*, JCAP **05** (2014) 025 [arXiv:1402.1044] [INSPIRE].

[18] P. Creminelli, J. Gleyzes, M. Simonović and F. Vernizzi, *Single-field consistency relations of large scale structure. Part II: resummation and redshift space*, JCAP **02** (2014) 051 [arXiv:1311.0290] [INSPIRE].

[19] P. Valageas, *Kinematic consistency relations of large-scale structures*, Phys. Rev. **D 89** (2014) 083534 [arXiv:1311.1236] [INSPIRE].

[20] P. Creminelli, J. Gleyzes, L. Hui, M. Simonović and F. Vernizzi, *Single-field consistency relations of large scale structure. Part III: test of the equivalence principle*, *JCAP* **06** (2014) 009 [arXiv:1312.6074] [INSPIRE].

[21] B. Horn, L. Hui and X. Xiao, *Soft-pion theorems for large scale structure*, *JCAP* **09** (2014) 044 [arXiv:1406.0842] [INSPIRE].

[22] M. Mirbabayi, F. Schmidt and M. Zaldarriaga, *Biased tracers and time evolution*, *JCAP* **07** (2015) 030 [arXiv:1412.5169] [INSPIRE].

[23] F. Bernardeau, S. Colombi, E. Gaztanaga and R. Scoccimarro, *Large scale structure of the universe and cosmological perturbation theory*, *Phys. Rept.* **367** (2002) 1 [astro-ph/0112551] [INSPIRE].

[24] S. Weinberg, *Adiabatic modes in cosmology*, *Phys. Rev.* **D 67** (2003) 123504 [astro-ph/0302326] [INSPIRE].

[25] Ya.B. Zeldovich, *Gravitational instability: an approximate theory for large density perturbations*, *Astron. Astrophys.* **5** (1970) 84 [INSPIRE].

[26] L. Dai, E. Pajer and F. Schmidt, *Conformal Fermi coordinates*, arXiv:1502.02011 [INSPIRE].

[27] E. Pajer, F. Schmidt and M. Zaldarriaga, *The observed squeezed limit of cosmological three-point functions*, *Phys. Rev.* **D 88** (2013) 083502 [arXiv:1305.0824] [INSPIRE].

[28] T. Tanaka and Y. Urakawa, *Dominance of gauge artifact in the consistency relation for the primordial bispectrum*, *JCAP* **05** (2011) 014 [arXiv:1103.1251] [INSPIRE].