An Optimal Complexity Algorithm for Computing Topological Degree in Two Dimensions

by

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ABSTRACT

An algorithm is presented to compute the topological degree for any function from a class F. The class F consists of functions defined on a two dimensional unit square C, f, $C \rightarrow \mathbb{R}^2$, which satisfy Lipschitz condition with constant $K > 0$, and whose infinity norm on the boundary of C is at least $d > 0$. A worst case lower bound, $m^* = 4\lfloor K/4d \rfloor$, is established on the number of function evaluations necessary to compute the topological degree for any function f from the class F. The parallel information used by our algorithm permits the computation of the degree for every f in F with m^* function evaluations. The cost of our algorithm is shown to be almost equal to the complexity of the problem.

1. INTRODUCTION

The problem of computing the topological degree of a function has been studied in many recent papers, see [3,4,6,8–11]. From the topological degree one may ascertain whether there exists a zero of a function inside a domain. Namely, Kronecker's theorem, see [5], states that if the degree is not zero, then there exists at least one zero of a function inside the domain. By computing a sequence of domains with nonzero degrees and decreasing diameters one can obtain a region with arbitrarily small diameter which contains at least one zero of the function, see [3]. Algorithms proposed in these papers were tested by their authors on relatively easy examples. They concluded that the degree of arbitrary continuous function could be computed. It was observed, however, see [3,4,11], that the algorithms may require an arbitrarily large number of function evaluations.

In this paper we restrict the class of functions to be able to compute the degree for every element in the restricted class using an a priori bounded number of evaluations. We consider the class F of Lipschitz functions, with constant K, defined on the unit square $C \subset \mathbb{R}^2, f : C \rightarrow \mathbb{R}^2$ such that for every $f \in F, ||f(x)||_\infty \geq d > 0$ for all

$x \in \partial C$, the boundary of C, and also such that $K/4d \geq 1$. Note that if $K/2d < 1$ then the functions in F do not have zeros and therefore the degree is zero for every f. The case $1 \leq K/2d < 2$ is open. The information on $f, N_m(f)$, consists of m values of f on ∂C, which may be computed sequentially (adaptively). This form of information is assumed since the topological degree is uniquely determined by the values of f on ∂C, see [5]. The topological degree is computed by means of an algorithm φ which is a mapping depending on the information, $\varphi : N_n(F) \to I$, where I denotes the set of all integers.

In Sect. 1 we present basic definitions and give the formulation of the problem. In Sect. 2 we exhibit an algorithm φ^* using information N_{m^*} which allows us to compute the degree of any $f \in F$, using $m^* = 4 \cdot \lfloor K/4d \rfloor$ function evaluations. We then show, in Sect. 3, that m^* is a lower bound on the number of function evaluations necessary to compute the degree for every $f \in F$ using arbitrary information. In Sect. 4, we prove that the algorithm φ^* computes the degree for any $f \in F$. We conclude, in Sect. 5, with a calculation of the cost of φ^* and show that φ^* is an almost optimal complexity algorithm.

We remark that information N_{m^*} is *parallel (nonadaptive)*, i. e. the evaluation points are given a priori. Thus it can be easily implemented on a parallel computer yielding an almost optimal speed-up, see [14] for further discussion.

The reader interested in actual implementation details should consult [2], where an extension of the algorithm φ^* is detailed, and tested. The extensions presented therein include the ability to calculate the degree for arbitrary polygonal regions in 2 dimensions, and for the unit cube in N dimensions. We remark that while the complexity bounds in N dimensions are not as tight, [2] presents lower and upper bounds on the complexity of the computation of topological degree for Lipschitz functions in N dimensions.

1. Basic Definitions.

Let $C \equiv [0,1] \times [0,1]$ be the unit square in $\mathbb{R}^2, \|\cdot\| \equiv \|\cdot\|_\infty$ the infinity norm in $\mathbb{R}^2, \theta \equiv [0,0] \in \mathbb{R}^2$, and I be the set of all integers. For a given positive d and K define

$$F = \{f : C \to \mathbb{R}^2, f = (f_1, f_2) : \|f(x) - f(y)\| \leq K \cdot \|x - y\| \quad \forall\, x, y \in C \tag{1.1}$$
$$\|f(x)\| \geq d \quad \forall\, x \in \partial C \text{ and } K/4d \geq 1\}$$

Our problem is to find the topological degree, $\deg(f, C, \theta)$ of f relative to C at θ, see [5], for every f in F. To solve this problem we use *information* N_m and an algorithm φ using N_m. These are defined as in [14], to wit:

Let $f \in F$ and define

$$N_n(f) = [f(x_1), \ldots f(x_m)], \tag{1.2}$$

where $x_1 \in \partial C$ is given a priori, $x_j = \underline{\mathrm{x}}_j(f(x_1), \ldots, f(x_{j-1}))$ and $\underline{\mathrm{x}}_j$ is a transformation, $\underline{\mathrm{x}}_j : \mathbb{R}^{2\cdot(j-1)} \to \partial C, j = 2 \ldots m$. If $\underline{\mathrm{x}}_j$ are constant transformations then the information is called *parallel (nonadaptive)*, otherwise it is called *sequential (adaptive)*.

Knowing N_m we approximate $\deg(f, C, \theta)$ by an algorithm φ, which is an arbitrary mapping

$$\varphi : N_m(F) \to I. \tag{1.3}$$

Where I is the set of all integers.

By *minimal cardinality number* m^* we mean the minimal integer m for which there exists information N_m which uniquely determines the degree of every f in F, i.e., $N_m(f') = N_m(f) \Rightarrow \deg(f', C, \theta) = \deg(f, C, \theta) \quad \forall f', f \in F$. We define a segment $[x, y]$ to be the closed counter clockwise oriented portion of ∂C with end points x and y, and by (x, y) we mean the interior of $[x, y]$. We define a partition P of ∂C to be a set, $\{P_i\}_{i=1}^{q}$, of counter clockwise ordered points from ∂C such that

$$\partial C = \sum_{i=1}^{q} \quad [p_i, p_{i+1}] \text{ where } p_{q+1} = p_1 .$$

Definition 1.1.

The partition P forms a sufficient refinement of a boundary ∂C with respect to the sign of a function f iff $(p_i, p_{i+1}) \cap (p_j, p_{j+1}) = \emptyset \quad \forall i \neq j$, and on each $[p_i, p_{i+1}]$, there exists a component of f say f_{j_i}, that is of constant sign (i.e. $\neq 0$) on $[p_i, p_{i+1}]$, and the remaining component of f is non zero at p_i and p_{i+1}.

Stenger showed, see [8], that given a sufficient refinement of the boundary ∂C, then the topological degree can be computed by

$$\text{(1.4)} \qquad \deg(f, \partial C, \theta) = \frac{1}{4} \quad \sum_{i=1}^{q} (-1)^{j_i - 1} \cdot \deg(f_{j_{i+1}}, [p_i, p_{i+1}], 0) \cdot \operatorname{sgn}(f_{j_i}(p_i))$$

where j_i is the index of the component of f which has constant sign on $[p_i, p_{i+1}]$, $f_3 = f_1$, $\deg(f_j, [p_i, p_{i+1}], 0) = (\operatorname{sgn}(f_j(p_{i+1})) - \operatorname{sgn}(f_j(p_i)))/2$, and $\operatorname{sgn}(f_j(p_i)) = (1 \text{ if } f_j(p_i) > 0, -1 \text{ if } f_j(p_i) < 0)$.

2. Description of the Algorithm.

The algorithm calculates the degree using the equation (1.4). Using nonadaptive information $N_{m^*}(f)$ of cardinality $4 \cdot \lfloor K/4d \rfloor$ the algorithm constructs conceptually a partition P that forms a sufficient refinement of ∂C with respect to the sign of f, for every f in F. The algorithm calculates the sign of both components of f at each point of the partition P (the proof that this can be done using only $N_{m^*}(f)$ is given in Section 4), and calculates the degree using equation (1.4). The following is a flow chart of the major steps of the algorithm. Consult [1] for implementation details, as well as results of numerical tests.

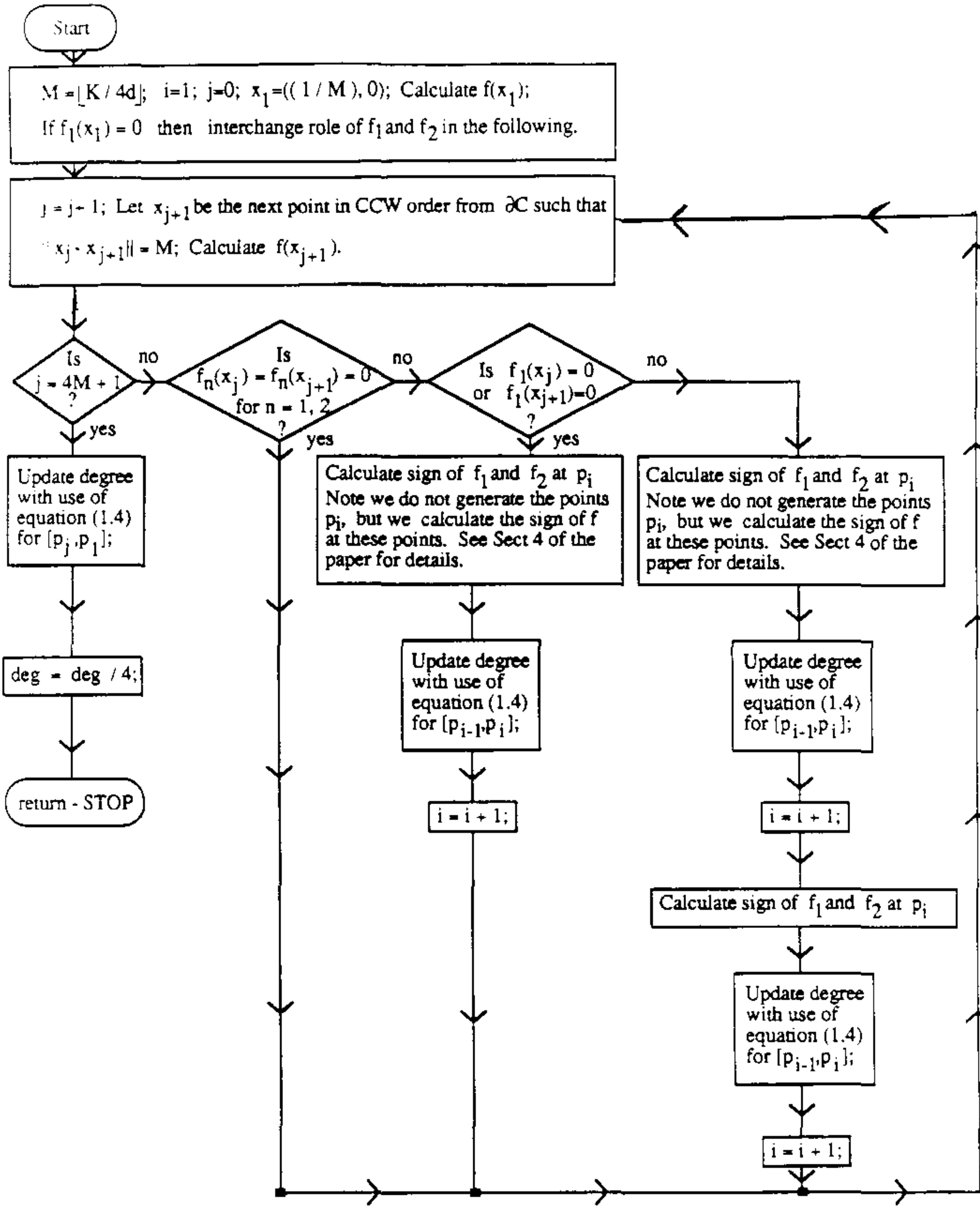

Figure 2.1: Flow Chart of Algorithm to Calculate Topological Degree

3. A Lower Bound.

In this section we show that the minimal cardinality number m^* is larger than $4 \cdot \lfloor K/4d \rfloor - 1$. Namely we prove

Lemma 3.1. *For any adaptive information operator N_m, of cardinality $m \leq 4 \cdot \lfloor K/4d \rfloor - 1$, there exist two functions f^*, f^{**} in F, such that*

$$N_m(f^*) = N_m(f^{**}), \ \deg(f^*, C, \theta) = 0, \text{ and } \deg(f^{**}, C, \theta) = \pm 1.$$

PROOF:

Let $M = \lfloor K/(4d) \rfloor$, and define

$$x_j = \begin{cases} (j/M \ , \ 0) & \text{for } j = 1, 2, \ldots, M \\ (1 \ , \ (j - M)/M) & \text{for } j = M+1, \ldots, 2M \\ (1 - (j - 2M)/M \ , \ 1) & \text{for } j = 2M+1, \ldots, 3M \\ (0 \ , \ 1 - (j - 3M)/M) & \text{for } j = 3M+1, \ldots, 4M \end{cases}$$

with the notation $x_{j \pm 4M} = x_j \quad \forall j$. Thus the x_j subdivides ∂C into $4M$ segments $[x_j, x_{j+1}]$ of size $\geq 4d/K$, i.e. $||x_j - x_{j+1}|| = 1/M \geq 4d/K$.

For a given point $C \in \mathbb{R}^2$ define the functions $h_i, h_i : \mathbb{R}^2 \to \mathbb{R}, i = 1, 2$, by

$$\begin{aligned} h_1(w; c) &= \min(d, -d + K||w - c||), \\ h_2(w; c) &= \min(d, \max(-d, -3d + K||w - c||)) \quad \forall w \in \mathbb{R}^2 . \end{aligned} \tag{3.1}$$

For given points $c_i \in R^2$ define the functions $g_i : \mathbb{R}^2 \to \mathbb{R}$, $i = 1, 2$, by

$$g_i(w; c_i) = \min(d, \max(-d, -2d + K||w - c_i||)) \forall w \in \mathbb{R}^2 . \tag{3.2}$$

Now we define the function $f^* : C \to \mathbb{R}^2$ by

$$\begin{aligned} f^*(w) = \Big(&\min(h_1(w, x_1), h_1(w, x_3), \ldots, h_1(w, x_{4M-3}), h_1(w, x_{4M-1})), \\ &\min(h_2(w, x_1), h_2(w, x_3), \ldots, h_2(w, x_{4M-3}), h_2(w, x_{4M-1})) \Big) \end{aligned}$$

see figure 3.2.

Note that f^* is Lipschitz with constant K, as the minimum of such functions, and $||f^*(w)|| = d$, $\forall\ w \in \partial C$. Thus f^* belongs to F. Note also that degree $(f^*, C, \theta) = 0$ since f^* has no zeros at all !

Next we generate the nonadaptive information $N_{m,f^*}(f) = [f(w_1), \ldots, f(w_m)]\ \forall f \in F$, where $w_i = \underline{w}_i(f^*(w_1), \ldots, f^*(w_{i-1}))$. Since the cardinality of this information is $= m \leq 4M - 1$, we know that there exists a segment on ∂C say $[x_{j-1}, x_j]$, that was not sampled by the information operator N_{m,f^*}, i.e. $w_i \notin [x_{j-1}, x_j]\ \forall i = 1..m$. Assume without loss of generality, that j is odd and define $f^{**} : C \to \mathbb{R}^2$ by:

$$\begin{aligned} f^{**}(w) = \Big(\ & \min(\ h_1(w, x_1), h_1(w, x_3), \ldots, h_1(w, x_{j-2}), g_1(w, x_j + \varsigma), \\ & \qquad h_1(w, x_{j+2}), \ldots h_1(w, x_{4M-1})), \\ & \min(\ h_2(w, x_1), h_2(w, x_3), \ldots, h_2(w, x_{j-2}), g_2(w, x_j + \xi), \\ & \qquad h_2(w, x_{j+2}), \ldots h_2(w, x_{4M-1})) \Big), \end{aligned}$$

where ς and ξ depend on j as shown in Table 3.1. To see that f^{**} is a member of the class F, note that f^{**} is a minimum of Lipschitz functions with constant K, and so too is Lipschitz with constant K. Obviously for every $w \in \partial C - [x_{j-1}, x_{j+1}]$ we have $||f^{**}(w)|| = d$. For every $w \in [x_{j-1}, x_{j+1}]$ the choice of the parameters ς and ξ assure that $||f^{**}(w)|| = d$, because either g_1 or g_2 is $= \pm d$ on that interval.

Now note that for every $w \in [x_j, x_{j+1}]$, we have $f^*(w) = f^{**}(w) = (\min(d, -d + K \cdot ||z - x_j||),\ \max(-d, -3d + K \cdot ||z - x_j||))$ (see Fig. 3.1 for the case with $0 < j < M$). Also note that $f^*(w) = f^{**}$, $\ \forall w$ from ∂C such that $w \notin [x_{j-1}, x_{j+1}]$. This implies that $N_m(f^*) = N_m(f^{**})$. However, by the choice of ς and ξ, f^{**} has exactly one zero in C, see Fig. 3.1, with the Jacobian of at that zero$=\pm K^2$, which implies that $\deg(f^{**}, C, \theta_2) = \pm 1$. This completes the proof of Lemma 3.1.

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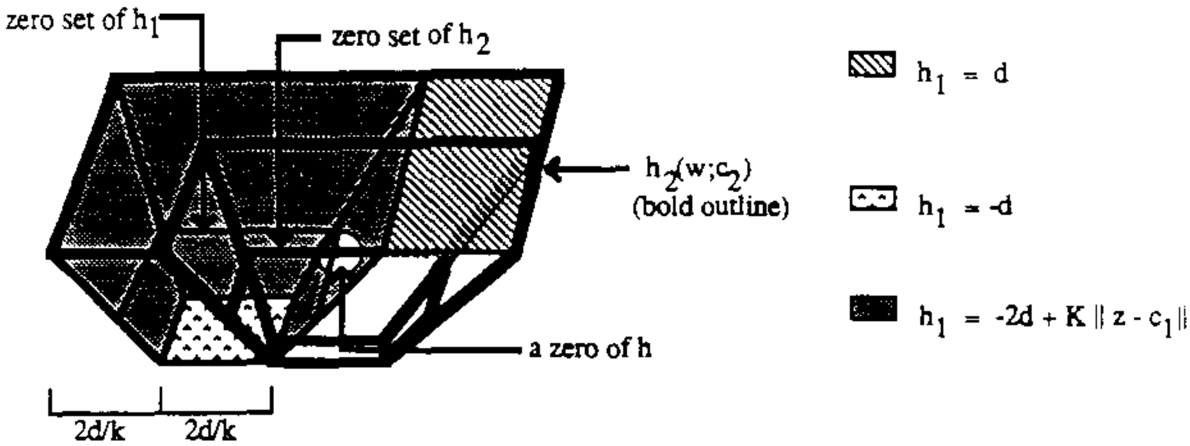
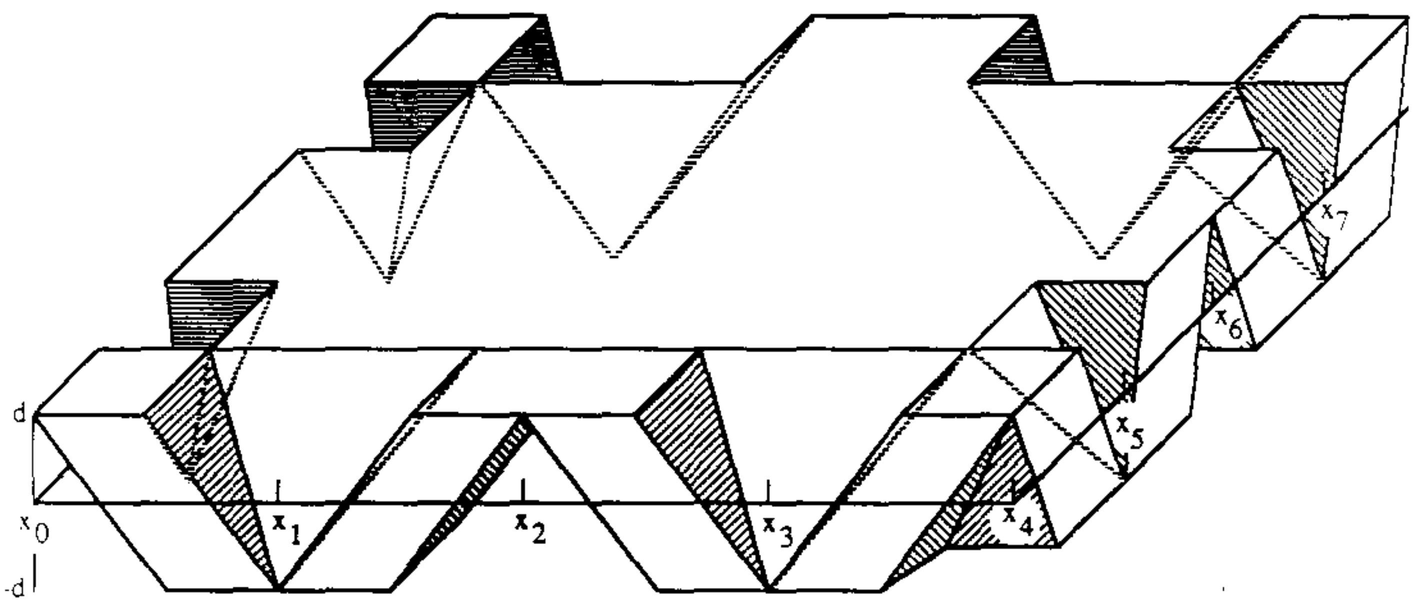


Cross Section of $h_1(w,c_1)$ and $h_2(w,c_2)$ with $\| c_1 - c_2 \| = 2d/K$

Figure 3.1



Function f^* for $d / K = 1 / 16$

Figure 3.2

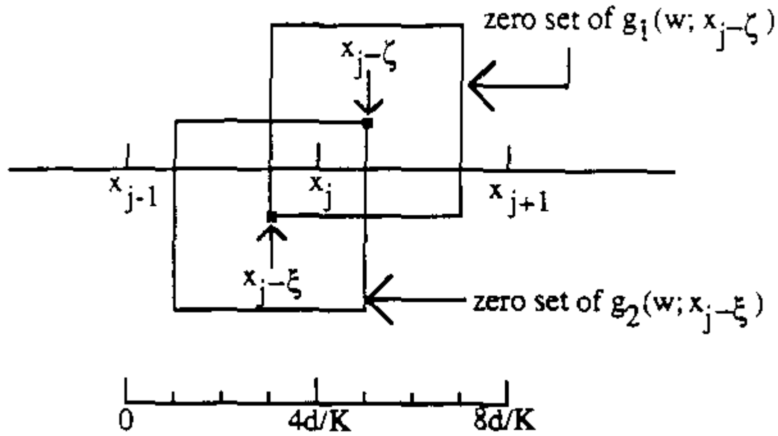


Figure 3.3

Range for j	Value for ζ	Value for ξ
$0 < j \leq M$	$(-d/K, -d/K)$	$(d/K, d/K)$
$M < j \leq 2M$	$(d/K, -d/K)$	$(-d/K, d/K)$
$2M < j \leq 3M$	$(d/K, d/K)$	$(-d/K, -d/K)$
$3M < j \leq 4M$	$(-d/K, d/K)$	$(d/K, -d/K)$

Table 3.1

4. Upper Bound

Lemma 3.1 implies that the minimal cardinality number m^*, must be greater than $4 \cdot \lfloor K/4d \rfloor - 1$. In this section (in Lemma 4.1) we prove that m^* is equal to $4 \cdot \lfloor K/4d \rfloor$, by exhibiting a nonadaptive information $N_q, q = 4 \cdot \lfloor K/4d \rfloor$, such that it generates a sufficient refinement of ∂C with respect to the sign of f, and therefore allows us to calculate the degree for any f in F.

In Sect. 2 of this paper we outlined the idea of an algorithm using N_q to compute the degree. Here (in Lemma 4.2) we exhibit the details of the algorithm and prove that it calculates the degree of any $f \in F$.

Let $f \equiv (f_1, f_2) \in F$, let $M \equiv 4 \cdot d/K - \varepsilon$, for arbitrarily small $\varepsilon > 0$, and let

$q = 4 \cdot \lfloor 1/M \rfloor$. Observe that $q = 4 \cdot \lfloor K/4d \rfloor$.

Let $x_0 = (0,0)$ and, $x_j, j = 1, \ldots, q$ be counter clockwise (CCW) ordered points along ∂C such that $||x_j - x_{j+1}|| = M,\ j = 0, \ldots, q$, where $x_{q+1} = x_1$.

Define the information operator $N_q(f) \equiv [f(x_1), \ldots, f(x_q)]$.

Lemma 4.1. *For every f in F, the information $N_q(f)$ uniquely determines the degree of f, i.e. $\forall\ \bar{f} \in F : N_q(\bar{f}) = N_q(f) \Rightarrow \deg(\bar{f}) = \deg(f)$, by generating a partition P of ∂C which forms a sufficient refinement of ∂C with respect to the sign of f.*

∎

Proof:

For every function f in F we build a partition $P = \{p_j\}_{j=1}^{Q}$ of ∂C, (where $Q < 2 \cdot q$, is the number of points in the partition, and Q depends on f) using only x_i and $f(x_i), i = 1, ..., q$, such that P forms a sufficient refinement of ∂C with respect to the sign of f.

First we construct the partition P. Assume that $f_1(x_1) \neq 0$, if not then interchange the roles of f_1 and f_2 in the definition below.

Definition 4.1 Let $i = j = 0$, and $P = \emptyset$. Then while $i \leq q + 1$ iteratively define the partition, P, as follows:

Let $i = j$ and redefine j as the smallest index $> i$ such that $f_1(x_j) \neq 0$. Then let

$$
P := \begin{cases}
P & \text{if } f_2(x_i) = f_2(x_j) = 0 \\
P \cup \{\beta_L\} \cup \{\beta_R\} & \text{if} \exists z \in [x_i, x_j] \text{ such that } f_1(z) = 0 \\
\quad \text{where } \beta_L, \beta_R \text{ are such that} & \\
\qquad [\beta_L, \beta_R] \subset (x_i, x_j),\ ||\beta_L - x_i|| < ||\beta_R - x_i|| & \\
\quad \text{and } \beta_L \in (x_i, x_{i+1}) \text{ is such that } f_1 \neq 0 \text{ on } [x_i, \beta_L], \text{ and} & (4.1) \\
\qquad \exists z_1 \in [x_i, x_{i+1}] \text{ such that } f_1(z_1) = 0 \text{ and } ||z_1 - \beta_L|| \leq M/2, & \\
\quad \text{and } \beta_R \in (x_{j-1}, x_j) \text{ is such that } f_1 \neq 0 \text{ on } [\beta_R, x_j], \text{ and} & (4.2) \\
\qquad \exists z_2 \in [x_{j-1}, x_j] \text{ such that } f_1(z_2) = 0 \text{ and } ||z_2 - \beta_R|| \leq M/2. & \\
P \cup \{\gamma_L\} \cup \{\gamma_R\} & \text{Otherwise, (i.e. } j = i + 1\text{),} \\
\quad \text{where } \gamma_L, \gamma_R \text{ are such that} & \\
\qquad [\gamma_L, \gamma_R] \subset (x_i, x_{i+1}), ||\gamma_L - x_i|| < ||\gamma_R - x_i|| \text{ and} & \\
||\gamma_R - x_i|| = \min(M/2, |f_2(x_i)|/(2K)) \text{ if } p = i & (4.3) \\
||\gamma_L - x_{i+1}|| = \min(M/2, |f_2(x_{i+1})|/(2K)) \text{ if } p = i + 1 & (4.4)
\end{cases}
$$

Where p is given by :

$$
p = \begin{cases}
i & \text{if}((|f_2(x_i)| < d \text{ or } |f_2(x_{i+1})| < d)) \text{ and } |f_2(x_{i+1})| \leq |f_2(x_i)|) \\
i + 1 & \text{if}((|f_2(x_i)| < d \text{ or } |f_2(x_{i+1})| < d)) \text{ and } |f_2(x_{i+1})| > |f_2(x_i)|) \\
i & \text{if}((|f_2(x_i)| \geq d \text{ and } |f_2(x_{i+1})| \geq d) \text{ and } |f_1(x_i)| \leq |f_1(x_i + 1)|) \\
i + 1 & \textit{Otherwise.}
\end{cases}
$$

∎

Note that because we do not always add the same number of points to the partition, it is not possible to know in advance, how many points will be in the partition. We let Q be the size of the partition, i.e., $P = \bigcup_{i=1}^{Q} \{p_i\}$. In the above definition of the partition P, we define 2 types of segments, $\beta = [\beta_L, \beta_R]$ segments where f_1 has a zero, and $\gamma = [\gamma_L, \gamma_R]$ segments, where f_1 does not have a zero. The end points of the β and γ segments form our partition. For clarity in the following discussion, let us index these segments (each type independently) using the CCW order, i.e. we have $\beta_j = [\beta_{j,L}, \beta_{j,R}], j = 1..Q_1, \quad \gamma_j = [\gamma_{j,L}, \gamma_{j,R}], j = 1..Q_2$, where $Q_1 + Q_2 = Q/2$. Observe that the above definition results in the following properties:

$$\forall j \text{ and } \forall x \in \beta_j, \exists z \in \beta_j \text{ such that } f_1(z) = 0 \text{ and } ||x - z|| \leq M/2. \tag{4.5}$$

(This follows from (4.1–4.2) and $||x_j - x_{j+1}|| = M$).

$$\forall i, f_1(x_i) = 0 \Leftrightarrow \exists j \text{ such that } x_i \in \beta_j. \tag{4.6}$$

$$\text{If } \beta_i \subset [x_j, x_{j+1}] \text{ then } \forall z \in [x_j, x_{j+1}], f_1(z) = 0 \Rightarrow z \in \text{Int}(\beta_j.) \tag{4.7}$$

$$\forall z \in \partial C \text{ such that } f_1(z) = 0, \quad \exists j \text{ such that } z \in \text{Int}(\beta_j). \tag{4.8}$$

$$\forall i, j, (\beta_i \cap \gamma_j) = \emptyset. \tag{4.9}$$

$$\forall i, j, \quad x_i \notin \gamma_j. \tag{4.10}$$

(4.11) $$\forall_j, f_1 \neq 0 \text{ on } \gamma_j \text{ and } f_2 \neq 0 \text{ on } \gamma_j.$$

.

(4.12) $$\forall i \quad x_i \notin P.$$

.

A segment $[x_j, x_{j+1}]$ contains no points of P iff

(4.13) $$f_1(x_j) = f_1(x_{j+1}) = 0 \text{ or } f_2(x_j) = f_2(x_{j+1}) = 0.$$

A segment $[x_j, x_{j+1}]$ contains 1 point of P iff

(4.14) $$f_1(x_j) = 0 \text{ or } f_1(x_{j+1}) = 0 \text{ but not both.}$$

(4.15) A segment $[x_j, x_{j+1}]$ never contains more than 2 points of P.

Now we prove a less obvious property.

(4.16)
Let z be a point such that $f_1(z) = 0$. By definition of the class F we have $|f_2(z)| \geq d$. Since it would take a distance greater or equal d/K from z for $|f_1|$ to be $\geq d$, and at least d/K more for f_2 to reach zero, there exists no point $y \in \partial C$ such that $f_2(y) = 0$ and $||y - z|| \leq M/2 < 2d/K$.

Next we show that P forms a sufficient refinement of ∂C with respect to the sign of f for arbitrary f in F. Namely for each segment $[L, R] L, R \in P$ such that $(L, R) \cap P = \emptyset$ we show:

$$f_1 \neq 0 \text{ on } [L, R] \text{ or } f_2 \neq 0 \text{ on } [L, R] \tag{4.17}$$

and

$$f_1 \text{ and } f_2 \text{ are non-zero at both } L \text{ and } R. \tag{4.18}$$

To wit, by property (4.8) we know that f_1 is of constant sign (and $\neq 0$) on all segments except when $\exists i$ such that $[L, R] = [\beta_{i,L}, \beta_{i,R}]$. By properties (4.5) and (4.16) we have $f_2 \neq 0$ on $[\beta_{i,L}, \beta_{i,R}]$ $\forall i$ which yields (4.17). By definition, we know that for all j, $f_1(\beta_{j,L}) \neq 0$ and $f_1(\beta_{j,R}) \neq 0$ which yields that $f_1 \neq 0$ at all points of P. That $f_2 \neq 0$ at all points of P, follows from (4.11), and the above argument. Thus (4.18) holds, which completes the proof of Lemma 4.1. ∎

LEMMA 4.2.

For every function f in F the following formulas hold for the sign of f_1 and f_2 at all points of the partition P from Lemma 4.1, contained in $[x_i - x_{i+1}]$, $\forall i = 1, \ldots, q$:

If $[x_i, x_{i+1}]$ contains exactly one point T of P then

$$\begin{cases} \text{sign } f_1(T) = \text{sign } f_1(x_{i+1}) & \text{if } f_1(x_i) = 0 \text{ and } f_1(x_{i+1}) \neq 0; \\ \text{sign } f_2(T) = \text{sign } f_2(x_i) & \end{cases} \tag{4.19}$$

and

$$\begin{cases} \text{sign } f_1(T) = \text{sign } f_1(x_i) & \text{if } f_1(x_i) \neq 0 \text{ and } f_1(x_{i+1}) = 0; \\ \text{sign } f_2(T) = \text{sign } f_2(x_{i+1}) & \end{cases} \tag{4.20}$$

If $[x_i, x_{i+1}]$ *contains two points* L *and* R *of* P *then for* $j = 1, 2$

$$(4.21)\qquad \begin{cases} \text{sign } f_j(L) = \text{sign } f_j(x_i) & \text{if sign } f_2(x_i) = \text{sign } f_2(x_{i+1}), \\ \text{sign } f_j(R) = \text{sign } f_j(x_{i+1}) & \end{cases}$$

and

$$(4.22)\qquad \begin{cases} \text{sign } f_1(L) = \text{sign } f_1(x_i), & \text{if sign } f_2(x_i) \neq \text{sign } f_2(x_{i+1}), \\ \text{sign } f_1(R) = \text{sign } f_1(x_{i+1}), & \\ \text{sign } f_2(L) = \text{sign } f_2(R) = \text{sign } f_2(x_p), & \end{cases}$$

where p is given in Definition 4.1.

∎

COROLLARY

Observe that (4.13), (4.14) and (4.15) imply that for every f in F we can calculate the number of points of P in each $[x_i, x_{i+1}]$. Thus using the formulas (4.19)–(4.22) we can calculate the degree of every f in F using equation (1.4). ∎

Proof of Lemma 4.2

For $[x_i, x_{i+1}]$ containing exactly one point of P we have two cases:

If $f_1(x_i) = 0$ and $f_1(x_{i+1}) \neq 0$ then for some j, $T = \beta_{j,r}$ and (4.2) implies sign $f_1(T)$ = sign $f_1(x_{i+1})$. Moreover (4.5) and (4.6) directly yield sign $f_2(T)$ = sign $f_2(x_i)$.

If $f_1(x_i) \neq 0$ and $f_1(x_{i+1}) = 0$ then for some j, $T = \beta_{j,L}$ and (4.20) follows by similar arguments as (4.19).

Now we consider the case of $[x_i, x_{i+1}]$ containing exactly two points of P. Note that L and R in this case are the end points of a β_j or γ_j for some j. Moreover (4.13) and (4.14) imply that $f_1(x_i) \neq 0 \neq f_1(x_{i+1})$ and $(f_2(x_i) \neq 0$ or $f_2(x_{i+1}) \neq 0)$.

For the formula (4.21) observe that if f_1 is not zero in $[x_i, x_{i+1}]$ then for some j, $L = \gamma_{j,L}$ and $R = \gamma_{j,R}$. The formula (4.21) for f_1 in this case is obvious and for f_2

follows directly from (4.4) and (4.11). If f_1 has a zero in $[x_i, x_{i+1}]$ then for some j, $L = \beta_{j,L}$ and $R = \beta_{j,R}$. Thus (4.7) yields the formula for f_1. As in (4.16) it follows that f_2 is of constant sign within $M/2 < 2d/K$ from a zero of f_1, which combined with (4.1), (4.2) and $||x_i - x_{i+1}|| = M$ yields the formula for f_2.

Now we prove formula (4.22). If f_i is not zero in $[x_i, x_{i+1}]$ then (4.22) obviously holds for f_1, and since in this case $L = \gamma_{j,L}$ and $R = \gamma_{j,R}$ for some j, then (4.22) for f_2 follows from (4.3), (4.4) and the definition of the index p. Suppose now that f_1 has some zero in $[x_i, x_{i+1}]$. Then as in (4.20) for some $j, L = \beta_{j,L}$ and $R = \beta_{i,R}$. Thus (4.7) yields (4.22) for f_1. Since L and R are both within $M/2$ from some zero of f_1 then all zeros of f_2 must be in $(x_i, L) \cup (R, x_{i+1})$. Thus sign $f_2(L) =$ sign $(R) =$ sign $f_2(\alpha_1)$, where α_1 is an arbitrary zero of f_1 from $[x_i, x_{i+1}]$. To show (4.22) for f_2 we prove that sign $f_2(x_p) =$ sign $f_2(\alpha_1)$.

Assume first that $|f_2(x_i)| < d$ or $|f_2(x_{i+1})| < d$ and that $p = i$, i.e. $|f_2(x_{i+1})| \leq |f_2(x_i)|$, and suppose by contrary that sign $f_2(x_p) \neq$ sign $f_2(\alpha_1)$ where α_1 any zero of f_1. Thus there exists α_2, a zero of f_2, in $[x_i, L] \subset [x_i, \alpha_1]$. Then $||\alpha_2 - x_i|| \geq |f_2(x_i)|/K$ and (4.16) yields $||\alpha_2 - \alpha_1|| \geq 2d/K$. It takes a distance d/K from α_1 for $|f_1|$ to become at least d, say at z, and a distance $d/K - |f_2(x_{i+1})|/K$ from z for $|f_2|$ to become less than d at x_{i+1}. The sum of these distances is obviously equal to $||x_i - x_{i+1}||$, thus $||x_i - x_{i+1}|| = ||\alpha_2 - x_i|| + ||\alpha_2 - \alpha_1|| + ||\alpha_1 - z|| + ||z - x_{i+1}|| \geq |f_2(x_i)|/K + 2d/K + d/K + d/K - |f_2(x_{i+1})|/K \geq 4d/K > ||x_i - x_{i+1}||$ a contradiction. The proof for $p = i + 1$ is essentially the same.

Assume finally that $|f_2(x_1)| \geq d$ and $|f_2(x_{i+1})| \geq d$. Then $|f_1(x_i)| < d$ or $|f_1(x_{i+1})| < d$ since it is impossible for both $|f_1|$ and $|f_2|$ to be at least d at both x_i, x_{i+1} and have zeros in $[x_i, x_{i+1}]$.

As before we show that the sign$f_2(x_p)$= sign$f_2(\alpha_1)$ for some zero α_1 of f_1. Suppose the contrary. and also assume that $p = i$. Then there would exist a zero of f_2, α_2, in $[x_i, \alpha_1]$. Since in this case $|f_1(x_i)| < d$, then $||x_i - x_{i+1}|| = ||x_i - \alpha_2|| + ||\alpha_2 - \alpha_1|| +$

$$||\alpha_1 - x_{i+1}|| = d/K - |f_1(x_i)|/K + d/K + 2d/K + |f_1(x_{i+1})|/K \geq 4d/K > ||x_i - x_{i+1}||,$$

a contradiction. The proof for $p = i + 1$ is essentially the same. This finally completes the proof of Lemma 4.2. ∎

5. Complexity of the Problem and of Our Algorithm

In this section we examine the complexity, i.e. minimal cost, of the problem of the calculation of topological degree for functions from our class. We then give the complexity of our algorithm, φ^*, and conclude that it is an almost optimal complexity algorithm.

We assume that each arithmetic operation $(+, -, *, \div, abs())$, each logical operation (and, or, not) or comparison $(>, <, \geq, \leq, =, \neq)$ costs unity and that each function evaluation costs c.

Lemma 3.1 implies that any algorithm that solves the problem must use at least $m^* = 4 * \lfloor K/4d \rfloor$ function evaluations. Thus a lower bound on the computational complexity is $m^* \times (c+2) - 1$, since there are $2m^*$ numbers that must go into the calculation and combining them takes at least $2m^* - 1$ operations.

The complexity (cost) of our algorithm, φ^*, is equal to the cost of computing the information, and the algebraic complexity of the algorithm. The location of the evaluation points can, for known K and d, be calculated once (i.e. precomputed), in which case the cost of the information would be $m^* \times c$. However, the algorithm as implemented in [1] calculates the location of the next information point based on the location of the current one. This approach results in a information cost of $m^* \times (c + 17)$. The algebraic complexity for φ^* is $31 \times m^* + 12$. Thus the computational complexity of φ^* as implemented is $m^* \times (c + 48) + 12$. However, with precomputation of the information points,and removal of the bells and whistles we can reduce the algorithmic cost down to $m^* \times (c + 20) + 5$. Therefore φ^* is an almost optimal complexity algorithm.

6.Numerical Results

We report on some of the numerical results of the algorithm implemented in [1]. The algorithm presented therein is an extension of the algorithm described here to the case where the domain of interest is an arbitrary polygon in $\mathbb{R}^2$. The algorithm was implemented in FORTRAN IV and the timing results are from a Dec-20 (Digital Equipment Corporation). In table 6.1 we present the timing results for a number of test functions. The CPU seconds were obtained by averaging the total CPU time of 10 successive calls to the algorithm to reduce the effect of extraneous CPU time that is counted in the system timing figures (e.g. loading the algorithm into memory, page swaps, user timeouts, etc.).

Test function 1: $F_1 = ((\sin(11.5 \times \pi \times x) + .01), \cos(11 \times \pi \times y))$. This function has 121 zeros within the unit square, the minimum value of its infinity norm on the boundary is .01, and it is Lipschitz with constant $11.5 \times \pi < 40.0$. The correct degree is -1.

Test function 2: $F_2 = ((x^2 + y^2 - .5), (2xy - .5))$. This function has 1 zero (of multiplicity 2) within the unit square, the minimum value of its infinity norm on the boundary is .5, and it is Lipschitz with constant 2. The correct degree is 0.

Test function 3: $F_3 = ((x^2 - 4y), (y^2 - 2x + 4y))$.with the domain as the rectangle $-2 \leq x \leq 2$ and $-.25 \leq y \leq .25$. This function has 1 zero within the domain the minimum value of its infinity norm on the boundary is $\geq .4375$, and it is Lipschitz with constant 4. The correct degree for this function is 1.

Test function 4: $F_4 = ((\sin(1.5 \times \pi \times x) + .01), -\cos(\pi \times y))$. This function has 1 zero within the unit square, the minimum value of its infinity norm on the boundary is .01, and it is Lipschitz with constant $1.5 \times \pi < 5.0$. The true degree of this function in this domain is 1.

Function #	degree	K	D	# Evaluations	CPU seconds
1	− 1	40	.01	4001	1.086
1	− 1	40	.005	8001	2.115
2	0	2	.5	5	.040
2	0	2	.05	41	.044
2	0	2	.001	2001	.37
2	0	20	.0003	66678	11.194
3	1	4	.25	18	.045
3	1	4	.1	46	.055
3	1	4	.01	451	.196
4	1	5	.01	501	.168
4	1	5	.005	1001	.303

Table 6.1

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