Probabilities

Studies in the Foundations of Bayesian Decision Theory

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Abstract

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One central issue in philosophy of probability concerns the interpretation of the very notion of probability. The fruitful tradition of modern Bayesian subjectivists seeks to ground the concept of probability in a normative theory of rational decision-making. The upshot is a representation theorem, by which the agent's preferences over actions are represented by derived subjective probabilities and utilities. As the development of Bayesian subjectivism becomes increasingly involved, the corresponding representation theorem has gained considerable complexity and has itself become a subject of philosophical scrutiny. This dissertation studies systematically various aspects of Bayesian decision theory, especially its foundational role in Bayesian subjective interpretation of probability. The first two chapters provide a detailed review of classical theories that are paradigmatic of such an approach with an emphasis on the works of Leonard J. Savage. As a technical interlude, Chapter III focuses on the additivity condition of the probabilities derived in Savage's theory of personal probability, where it is pointed out that Savage's arguments for not requiring probability measures derived in his system to be countable additive is inconclusive due to an oversight of set-theoretic details. Chapter IV treats the well-known problem of constant-acts in Savage's theory, where a simplification of the system is proposed which yields the representation theorem without the constant-act assumption. Chapter V addresses a series of issues in the epistemic foundations of game theory including the problem of asymmetry of viewpoints in multi-agent systems and that of self-prediction in a Bayesian setup. These issues are further analyzed in the context of epistemic games where a unification of different models that are based on different belief-representation structures is also proposed.

Contents

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Scope and Overview

This dissertation addresses a range of issues in the philosophical and mathematical foundations of Bayesian decision theory, broadly construed as covering topics in epistemology, philosophy of science, philosophical logic, and epistemic game theory. The Bayesian methodology emphasizes first and foremost the use of probability as the fundamental measure of uncertainty. Within this paradigm, the concept and problem of inductive inference—which have been a prominent issue in philosophy since Aristotle—are provided with a systematic account which can be ideally characterized in the following two steps:

PRIOR ESTIMATION: Given any scientific inquiry that involves uncertainty, establish *prior* probabilistic assessments over those quantities of ultimate interest (usually open hypotheses) as well as all other quantities that are relevant to the problem at hand (including background knowledge).

CONDITIONALIZATION: Update the probabilistic assessments over the quantities of interest in the presence of the incoming evidence, that is, calculate and interpret appropriate *posterior* probabilities of the quantities of interest *given* the observed data.

Great controversies have emerged surrounding each step of this methodological approach in both philosophical and scientific literatures ever since Thomas Bayes laid out the basic idea in the mid-eighteenth century. Yet great advancements in all these areas have also been made in the past fifty years, and many of which will be the starting points of the investigations in this dissertation.

At the heart of various controversies lies the philosophical question as to how the very notion of probability should be interpreted: whether probabilities are *objective* properties of the world that are independent of what we know or believe, or they are measures of our *subjective* attitudes as actual and justified degrees of beliefs when facing uncertainty. The difference between the two interpretations of probability goes beyond mere semantic disputes over the meaning of the term as it may lead to different opinions as to what constitutes the relevant knowledge of the underlying inquiry. For instance, an objectivist would take the probability of getting a head upon flipping a biased coin as limiting frequency of the occurrences of heads in long run experiments, and she would expect to find one outcome to appear more frequently than the other in a sequence of repeated trails on the same coin, from which an approximation of the probability of obtaining a head can be estimated. A Bayesian subjectivist, on the other hand, would probably see no reason for favoring one outcome over another given that the direction of bias is unknown, he however might update his initial probabilistic assessments when more experiments are conducted. As the number of trials increases the values of the two kinds of probabilities tend to converge, yet their respective interpretations remain distinct.

Although many Bayesian theorists embrace the subjective interpretation of probability, they differ, among other things, on the handling of statistical inferences as to whether Bayesian statistics requires further theoretical foundations, or that one should merely focus on the pragmatic aspect of Bayesian calculus, whose flexibility and generality, it is said, are sufficient for dealing with complex issues without the need to appeal to foundations.

This dissertation studies systematically the decision-theoretic foundations of Bayesian reasoning where statistical inference is viewed as a particular decision problem which is subject to further decision theoretic analyses. This approach seeks to ground the concept of probability in a normative theory of rational decision-making with the central aim of characterizing how decision makers, when facing uncertainty, may act rationally so that their respective best interests be maximally realized or that those undesirable consequences resulting from certain unwise choices of actions be on the whole avoided. The theory takes as basic assumptions that, in deliberating the best courses of action, the decision makers are primarily moved by their beliefs in those uncertain quantities of interest involved in the decision process as well as their concerns for the consequences of their actions. The goal is to establish a normative theory of decision making as to how agents' probabilistic beliefs fit in a coherent manner with value judgments in the light of changing evidence. This is usually carried out by prescribing various rationality principles and structural assumptions that govern the individual's preferences over potential actions in decision situations. The upshot is a representation theorem, by which the preferences over acts are represented by their expected utilities using derived subjective probabilities and personal utilities.

From the very beginning, with the prominent work of Frank P. Ramsey, and subsequently the works of Bruno de Finetti, Leonard J. Savage, among others, the decision-theoretic foundations of Bayesian subjectivism have been built with an important mathematical component. As the development of Bayesian theory becomes increasingly involved, the corresponding formalism has gained considerable complexity and has itself become a subject of philosophical scrutiny. Thus, a preliminary task of this dissertation is to provide a careful examination of the conceptual bases as well as the mathematics of some of the classical theories of subjective expected utility. The aim is to investigate at the foundational level the relationship between theoretical constructions and their interpretational characterizations in modern Bayesian theory. Modifications and generalizations based on various philosophical and mathematical considerations are then attempted.

In particular, as preparatory material for discussions in later parts of the dissertation and also as a show of continuous efforts that had been made in order to build a firm foundation for Bayesianism, we provide in Chapter I a condensed overview of some classical theories of expected utility, including the works of Ramsey (1926), von Neumann and Morgenstern (1944), Anscombe and Aumann (1963). The exposition highlights differences as well as interdependence among these theories. A unified mathematical reconstruction of all of these decision models is also given.

This will be followed by a detailed account of Savage's theory of subjective expected utility as presented in his seminal book *the Foundations of Statistics* (1954; 1972). Savage's theory is significant in that it provides a purely subjective interpretation of probability, where 'purely subjective' means that, unlike other competing accounts, the theory is characterized completely from the agent's personal perspective without appealing to any external chance mechanism. And, for this very reason, the theory has been seen as a paradigm of the decision theoretic foundations of Bayesian subjectivism. Ever since its first appearance in 1954, Savage's theory has been widely discussed in both economic and philosophical literatures and it was once celebrated as "the most brilliant axiomatic theory of utility ever developed" (Fishburn, 1970) and "the crowning glory of choice theory" (Kreps, 1988), due largely to its deep conceptual depth and high mathematical complexity. Chapter II will encompass Savage's system in its full scale. The philosophical and mathematical analyses made in this chapter will also serve as the basis for the rest of the dissertation. Our exposition retraces the methodological steps adopted by Savage in constructing his final representation theorem. This involves dissecting, evaluating, and reconstructing Savage's mathematical arguments for deriving numerical probabilities and utilities from the agent's preferences over actions. Along the way, we provide general discussions on various conceptual issues involved in Savage's system including the well-known "sure-thing" principle.

As a technical interlude, Chapter III is devoted to the discussion of a charged issue concerning the additivity condition of probability measures derived in Savage's theory of personal probability. The debate about finite versus countable additivity is a recurring issue in the foundations of probability theory. This is largely because de Finetti in his original theory of probability defended a finitistic account towards the additivity condition of personal probabilities which placed him in contrast with the countably additive measure tradition of Lebesgue and

Kolmogorov, a tradition that can retrospectively be traced back to the early writings of Pascal and Fermat (cf. Appendix A.7). Our focus in this chapter however is on the arguments against countable additivity given by Savage in his theory of personal probability. The discussion is divided into three main parts. First, we comment, by providing a brief historical review, on Savage's reasons for not requiring the subjective probabilities derived in his decision model to be countably additive. It is pointed out that Savage's argument for avoiding countable additivity is inconclusive due to an oversight of set-theoretic details. In the second part, we discuss some defects of employing merely finitely additive probability measures in Savage's system. A diagnosis is attempted which links the insufficiency of finite additivity to the failure of continuity in a rich background setting as employed in Savage's system. The analyses then lead, in the third part, to a proposal of introducing countable additivity as an added postulate to the theory, we then provide a discussion on various conditions under which utilities be extended from simple acts to acts in general. We will end with some general remarks which opt, more or less, for a dualistic view towards the probabilistic additivity condition.

Based on a joint work of Gaifman and Liu (2015), Chapter IV presents a theory of context-dependent decision making within Savage's framework. The development of this theory is motivated by solving the problem of the constant-act assumption. Briefly, a constant act defined in Savage's system is an act that has the same consequence with the same value in all states of the world. As an implicit structural assumption, Savage assumes that each possible consequence is a candidate for constructing a constant act. This implicit assumption is crucial for Savage as his final representation theorem hinges to a large extent on the structural properties it provides. The assumption, however, is highly problematic, in view of simple cases that Savage himself suggests at the beginning of his book as the kind of scenario that his system is supposed to handle. In this chapter we provide an in-depth analysis of the problems caused by the constant-act assumption and show that a certain weakened form of the representation theorem with *context-dependent* expected utilities can be derived without the implicit constant-acts assumption, a form that is sufficient for handling small-world scenarios of the kind proposed by Savage. The main innovation is that we provide a novel way of deriving utilities, a method that is different from Savage's original approach.

The decision-theoretic foundations of Bayesian subjectivist theory discussed in the preceding chapters have far-reaching influences in contemporary game theory, yet the former differ from the latter in important ways. Their differences sometimes can be seen as underlying various debates in the foundations of games which gives rise to many mind-boggling paradoxes or puzzles often discussed in the philosophical literature. Chapter V addresses a series of issues in the epistemic foundations of game theory including the problem of asymmetry of viewpoints in multi-agent systems and that of self-prediction in a Bayesian setup. Attempts are made to quantify various Bayesian probabilistic representations of beliefs in games under two decision-theoretic theses: (1) the players' beliefs be represented by imprecise probabilities in stead of sharp ones; (2) no self-predicting subjective probability should be assigned to the players' own acts. Unifying conditions compatible with these principles are proposed, and it is shown that under these conditions different belief structures can be coherently interrelated in the context of epistemic games.

Chapter I & II (and Appendix A) are expository in nature. The aim there is to, as the teacher used to say, "express in your own words." In our expositions we also provide a good amount of details that might be absent in the works where these theories were first presented. Original contributions are given in Chapter III, IV, & V. We are grateful to numerous anonymous reviewers and conference organizers and participants for their comments and criticisms, where early versions of various parts of these chapters were presented or published.

CHAPTER I

Subjective Expected Utility

> Let us now try to find a method of measuring beliefs as bases of possible actions .... The old-established way of measuring a person's belief is to propose a bet, and see what are the lowest odds which he will accept.
>
> —— Ramsey (1926)

1. Introduction

1.1. Degrees of belief and betting method. The opening quote is from Frank Ramsey's celebrated essay *Truth and Probability* (1926) where Ramsey proposed a theory of personal probability and utility. The theory contained basic ideas which were later developed or rediscovered in, most notably, the works of de Finetti, von Neumann and Morgenstern, Savage, Anscombe and Aumann, among others. Ramsey's subjectivism introduced a novel idea of measuring at the same time decision maker's subjective utilities and probabilities, where the agent's personal probability was portrayed as "the logic of partial beliefs" which was given an operational definition through coherent betting behavior. Let us highlight some basic ideas of Ramsey's theory.[1]

Let X be a set of prizes and, to simplify matters, let x^* and x_* represent respectively the best and the worst prizes considered by the decision maker.[2] In order to get a precise measure of her subjective valuations of the prizes in X, the decision

[1]Ramsey's paper was first read at the "Moral Sciences Club" at University of Cambridge in 1926 and published posthumously in *The Foundations of Mathematics and other Logical Essays* (1931), edited by Richard Braithwaite. It also appears in a collection of Ramsey's writings edited by Mellor, D. H., *Philosophical Papers*, Cambridge University Press, 1990.

[2]Note that Ramsey's original theory does not postulate the existence of these two distinguished prizes (in fact, any two distinctive prizes would do). We introduce x^* and x_* purely for illustrative purposes, which will be used in later expositions.

maker is presented with the following *betting* situation which involves an ethically neutral proposition p:

i. x^* if p; x_* if $\neg p$

ii. x^* if $\neg p$; x_* if p

where (i) and (ii) can be seen as two lottery tickets with the redeeming policy that if, say, ticket (i) is chosen and p is indeed true then the agent will be rewarded with x^*, x_* if p is false. According to Ramsey, a proposition p is said to be *ethically neutral* "if two possible worlds differing only in regard to the truth of p are always of equal value" (cf. Ramsey, 1926, p.73). In other words, the truth (or the falsity) of p itself has no added value in evaluating the value of a bet. As we shall see, this assumption is a forerunner of the state independent axiom adopted in various decision models which will be discussed later.

Now, under the above betting setup, if the decision maker is indifferent between (i) and (ii), then it is said that the agent has 1/2 *degree of belief* in p being true. Ramsey then postulates in the form of an axiom that there exists an ethically neutral proposition p believed to degree 1/2. This distinguished proposition p (with 1/2 degree of belief) can be further used to evaluate the values of other prizes in X. Consider the following bets

iii. x

iv. x^* if p; x_* if $\neg p$.

If the agent is indifferent between the two bets, then the value of x said to be equal to half of the total value of x^* and x_*. To represent numerically, let the utility of x^* be 1 and x_* be 0, in symbols $u(x^*) = 1$ and $u(x_*) = 0$. According to Ramsey, the fact that the decision maker's is indifferent between the two tickets implies that her evaluation of the utility of x is the midpoint of the utility scale from 0 to 1:

$$\overset{1}{u(x^*)} \text{ —— } \overset{1/2}{u(x)} \text{ —— } \overset{0}{u(x_*)}$$

It is further assumed that the above procedure can be repeated indefinitely, that is, for instance, there exists some prize x' whose utility halves the way from x_* to x with $u(x') = 1/4$, and so on. Hence, under this assumption, the utility scale between x^* and x_* can be calibrated to arbitrary precision. Then, for any $y \in X$, y can be assigned with a numerical utility representation $u(y)$ on the utility scale.

$$\overset{1}{u(x^*)} \text{ —— } \cdots \text{ —— } \overset{1/2}{u(x)} \text{ —— } \cdots \text{ —— } \overset{1/4}{u(x')} \text{ —— } \cdots \text{ —— } \overset{0}{u(x_*)}$$

With subjective utilities for all prizes in hand, Ramsey proceeds to define what it means to say that the agent believes in the truth of an arbitrary proposition q to certain *degree* using the following betting mechanism. For any q, if there exist prizes $x, y, z \in X$ with $u(y) \geq u(x) \geq u(z)$ such that the agent is indifferent between the following bets

v. x

vi. y if q; z if $\neg q$

then her *partial belief* in q, denoted by $\mu(q)$, is defined as

$$\mu(q) = \frac{u(x) - u(z)}{u(y) - u(z)}, \quad u(y) - u(z) > 0.$$

Using a "Dutch-book argument" Ramsey shows that if the agent's partial belief assignments are *coherent*, in the sense that no book can be made against her, then μ obeys the laws of probability calculus (cf. Ramsey, 1926, p.79).[3] We will not go further into the Dutch-book argument here which is a topic on its own, for further discussion see, for instance, Earman (1992, Ch.2), Hajek (2008). Our focus is rather to see how probabilities and utilities are derived in various formal systems.

1.2. Expected utility theory. Ramsey's essay marked the beginning of a series of extensive studies in utility theory. In this and the next chapter we explore three main representation theorems. Here is a quick preview. (The readers may ignore the technical details upon first reading.)

[3]For more detailed discussions/expositions on Ramsey's account, see Fishburn (1981, §5.1), Bradley (2001). .

vNM: Let X be a *finite* set of prizes/consequences, and $\mathcal{L}_X$ be the set of probability measures on X. Each $p \in \mathcal{L}_X$ is referred to as a *lottery* on X, the intended interpretation is that, for any prize $x \in X$, $p(x)$ is the probability of getting x. Let $\succsim$ be a preference relation on $\mathcal{L}_X$, the **von Neumann-Morgenstern** (vNM) expected utility theory states that if $\succsim$ satisfies certain postulated axioms then it can be presented by a utility function (EUF) $U : \mathcal{L}_X \mapsto \mathbb{R}$ such that

$$p \succsim q \iff U(p) \geq U(q),$$

where U can be expected utilities, that is, there exists a subjective utility function $u : X \mapsto \mathbb{R}$ for which

$$p \succsim q \iff \sum_{x \in X} p(x)u(x) \geq \sum_{x \in X} q(x)u(x),$$

and u is unique up to a positive linear transformation. We further extend this result to the case where X may contain *infinitely* many consequences and each $p \in \mathcal{L}_X$ is simple (i.e., has finite support).

A-A: Let S be a *finite* set of states of the world. Define a *horse-race lottery* to be a function h mapping from S to $\mathcal{L}_X$. Denote the space of horse race lotteries by $\mathcal{H}$. Then given any horse lottery h and state $s \in S$, $h(s)$ is a (vNM) lottery defined on X, we also write h_s for $h(s)$. Hence, for any prize $x \in X$, $h_s(x)$ is the probability that x is obtained in state s given the horse lottery h. An **Anscombe-Aumann** (A-A) representation of a preference relation $\succsim$ on $\mathcal{H}$ is that there exists a utility function $u : X \mapsto \mathbb{R}$ and a (subjective) probability measure μ on an algebra of S such that, for any $h, h' \in \mathcal{H}$,

$$h \succsim h' \iff \sum_{s \in S} \mu(s) \sum_{x \in X} h_s(x)u(x) \geq \sum_{s \in S} \mu(s) \sum_{x \in X} h'_s(x)u(x)$$

provided that $\succsim$ satisfies a set of postulated axioms.

SVG: Further, let S be an (uncountably) *infinite* set of states and $\mathcal{F}$ be some algebra equipped on S, X be a set of consequences, and let $\mathcal{A}$ be the set of functions mapping from $\mathcal{F}$ to X, each $f \in \mathcal{A}$ is referred to as an *act*. Then a **Savage** representation of the preference ordering $\succsim$ on $\mathcal{A}$ is that, under postulated axioms on $\succsim$, there exists a (subjective) probability measure μ on $(S, \mathcal{F})$ and a real-valued utility function u on X such that, for any $f, g \in \mathcal{A}$,

$$f \succsim g \iff \int_S u\big[f(s)\big]\, d\mu \geq \int_S u\big[g(s)\big]\, d\mu.$$

REMARK. 1. The decision-theoretic models listed here are *not* presented in chronicle order: the Anscombe-Aumann model appeared after the first edition of Savage's *Foundations of Statistics*. The materials presented here are organized based on the methodological approach they each adopts with increasing computational complexity.

2. In the three decision models above, the respective preference relations are defined on different sets of alternatives (see Table 1.1 for a comparison). To simplify notations, we adopt a systematic ambiguity and use, unless otherwise specified, the same notation "$\succsim$" for all preference relations and let the context determine on which set of alternatives a preference relation is defined.

TABLE 1.1. Models of Expected Utility.

	$\succsim$ defined on	subjective utility	subjective probability	objective probability
Ramsey	$\mathcal{A}^a$	✓	✓	–[b]
vNM	$\mathcal{L}_X$	✓	–	✓
A-A	$\mathcal{H}$	✓	✓	✓
SVG	$\mathcal{A}$	✓	✓	–

[a] Ramsey uses "propositions" instead of states and events and his preferences are defined for consequences, acts, and conditional acts.

[b] Strictly speaking there is no objective probability explicitly employed in Ramsey's model, yet it is easily seen that his notion of ethically neutral propositions with 1/2 degrees of belief, which is based on an apparent symmetry consideration, play a similar role as some chance mechanism.

1.3. Kinds of probability. In the discussions below, we pay close attention to different *kinds* of probability involved, by which we are referring to the (rough) distinction between *objective* and *subjective* probabilities. These probabilities may appear either as measures (subjective probability) of decision maker's personal probabilistic judgments over the occurrences of some events or in the form of some presupposed chance mechanism (objective probability).

> "Probability has often been visualized as a subjective concept more or less in the nature of an estimation. Since we propose to use it in constructing an individual, numerical estimation of utility, the above view of probability would not serve our purpose. The simplest procedure is, therefore, to insist upon the alternative, perfectly well founded interpretation of probability as frequency in the long run." (von Neumann and Morgenstern, 1944, p.19)

The quotation is from von Neumann and Morgenstern's well-known book "The Theory of Games and Economic Behavior," where they made a distinction between two different kinds of probabilities. As we shall see, the use of objective probabilities is crucial to the vNM model and the A-A model, where the decision maker's personal utilities and subjective probability (in the case of the A-A model) are essentially defined in terms of objective chances. Savage, on the other hand, adopted a purely subjective interpretation of probability upholding that subjective utility can be integrated with respect to subjective probability. The tradeoff is that Savage's theory is considerably more complicated than other models.

The plan for this chapter is as follows. In Section 2 and Section 3, we reconstruct the decision models developed in von Neumann and Morgenstern (1944) and Anscombe and Aumann (1963). The expositions owe much to Fishburn (1970, 1981, 1986, 1994); Hammond (1998a,b); Kreps (1988); Mehta (1998); Ok (2007, 2011); Rubinstein (2007), to name just a few. Full mathematical details of the theories discussed in this chapter can be found in the works just mentioned, our primary goal is to trace the methodological developments that are related to Savage's theory of subjective expected utility, which will be the main theme of the next chapter.

2. von Neumann-Morgenstern Utility Functions

2.1. Lotteries. As mentioned above, a *lottery* on a finite set X is a probability function p on X, we sometimes refer to p as a von Neumann-Morganstern (vNM) lottery. The intended interpretation is that X is a set of prizes and $p(x)$ is the *chance* that $x \in X$ obtains. Let $\mathcal{L}_X$ be the set of all probability functions on X.[4] In simple cases, define the *degenerate lottery* with respect to any given $x \in X$ to be the probability function $\delta_x \in \mathcal{L}_X$ such that, for any $y \in X$,

$$\delta_x(y) = \begin{cases} 1 & y = x \\ 0 & y \neq x \end{cases}. \tag{2.1}$$

That is, δ_x assigns probability 1 to x, 0 otherwise. Hence each prize $x \in X$ can be identified with a lottery δ_x that degenerates at x. Then, it is easily seen that, for any lottery $p \in \mathcal{L}_X$, p can be written as a combination of δ_x's,

$$p = \sum_{x \in X} p(x)\delta_x. \tag{2.2}$$

Example 2.1. Suppose that $X = \{x_1, x_2, x_3\}$. Write the degenerate lottery δ_{x_1} in the form of a triple $(1, 0, 0)$ which says that δ_{x_1} assigns probability 1 to prize x_1 and 0 to both x_2 and x_3. Then $\mathcal{L}_X$ can be represented geometrically by a 2-simplex in Figure 2.1 where each vertex of the equilateral triangle corresponds to a degenerate lottery.[5] Note that since in an equilateral triangle the sum of the perpendiculars from any internal point p to three sides equals its altitude (say, 1), write any point p in the triangle in the form of (p_1, p_2, p_3) where each coordinate p_i is the length of the perpendicular from p to the edge that is on the opposite

[4] $\mathcal{L}_X$ is often written as $\Delta(X)$, namely the space of probability functions defined on X.

[5] Strictly speaking, a standard n-simplex is a unit $n+1$-dimensional polygon in $\mathbb{R}^{n+1}$, Figure 2.1 is a special case where the 2-simplex is represented as a space of its own in $\mathbb{R}^2$.

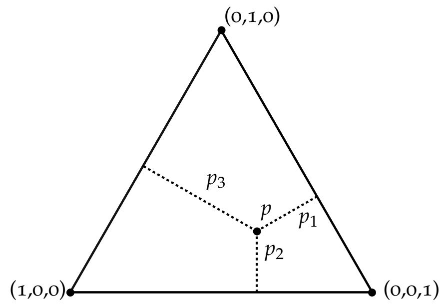

FIGURE 2.1. 2-Simplex with *unit* altitude.

side of vertex i, then $p_1 + p_2 + p_3 = 1$. Thus, Figure 2.1 in a representation of the lotteries in $\mathcal{L}_X$ where each point $p = (p_1, p_2, p_3)$ corresponds to a lottery in $\mathcal{L}_X$ with p_i being the probability that p assigns to x_i. $\triangleleft$

DEFINITION 2.2. Let $p, q \in \mathcal{L}_X$, a *compound lottery* of p and q with scalar $\lambda \in [0, 1]$ is a function r such that $r(x) = \lambda p(x) + (1 - \lambda)q(x)$ for all $x \in X$. Denote the compound lottery r by the following notation,

$$p \oplus_\lambda q := \lambda p(x) + (1 - \lambda)q(x). \tag{2.3}$$

Intuitively, given any $p, q \in \mathcal{L}_X$, a compound lottery $p \oplus_\lambda q$ can be considered as a (second-order) lottery ticket which has the payment policy that, with known chance λ, lottery p will transpire and, with probability $(1 - \lambda)$, lottery q obtains.

	λ	$1 - \lambda$
$p \oplus_\lambda q$	p	q

(2.4)

It is easy to see that $p \oplus_\lambda q \in \mathcal{L}_X$, that is, every (second-order) compound lottery is in effect equivalent to a (first-order) lottery in $\mathcal{L}_X$. To characterize this concept geometrically using the simplicial representation of Example 2.1, we have that the point that represents the compound lottery $p \oplus_\lambda q$ $(0 \leq \lambda \leq 1)$ in Figure 2.2 falls on the line segment that joins p and q.

2.1.1. *Preference over lotteries.* Presumably, the decision maker has preferences over the prizes. It is assumed that these preferences are reflected in her preferences

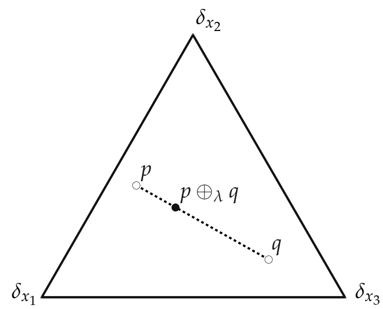

FIGURE 2.2. Compound lottery $p \oplus_\lambda q$ in 2-simplex.

over the lotteries with each lottery specifying the chances of getting the prizes.[6] For instance, in Example 2.1, suppose that the agent definitely prefers prize x_1 over other two prizes then it must be that she prefers δ_{x_1} over δ_{x_2} and δ_{x_3}, because the latter two lotteries assign probability 0 to the obtaining of x_1. Formally, let $\succsim$ be a preorder on $\mathcal{L}_X$ (see Appendix A.1), which represents the decision maker's preferences over all the lotteries. The following are the *von Neumann-Morgenstern postulates* on $\succsim$:[7]

vNM 1. $\succsim$ is a complete preference relation.

vNM 2. For all $p, q, r \in \mathcal{L}_X$ and any $\lambda \in (0, 1]$,

$$p \succ q \iff p \oplus_\lambda r \succ q \oplus_\lambda r.$$

vNM 3. For any $p, q, r \in \mathcal{L}_X$, there exist $a, b \in (0, 1)$, such that

$$p \succ r \succ q \implies p \oplus_a q \succ r \succ p \oplus_b q.$$

vNM 1 is often referred to as the *completeness* axiom which asserts that all lotteries are pair-wisely comparable. This axiom is often defended along the lines that the decision maker, if pressed, will eventually make a decision between a given pair of options regardless what her deliberation process might be. Note that

[6]See von Neumann and Morgenstern (1964, §3.3.1) for their discussions on the relation between probablistic reasonings and utility considerations.
[7]See Hammond (1998a, §3) for a discussion on different versions of the independence and the continuity axioms adopted in the literature. The current system (vNM 1-3) is provably equivalent to the theory presented there due to Jensen (1967) (see conditions (O), (I), (C), and Lemma 4.5(a), see also Fishburn (1977, 1981, 1982)).

given the agent's ordering among lotteries one can induce an ordering $\succsim^*$ over the prizes through degenerate lotteries as follows: for all $x, y \in X$,

$$x \succsim^* y \quad =_{\mathrm{Df}} \quad \delta_x \succsim \delta_y. \tag{2.5}$$

That is, prize x is said to be weakly preferred to prize y, if, under the initial ordering $\succsim$, the degenerate lottery δ_x is at least as good as the lottery that degenerates at y. It is easily seen that if $\succsim$ is totally (or partially) ordered so is $\succsim^*$. This preference relation among prizes induced through degenerate lotteries can be seen as a precursor of Savage's similar notion of preferences over consequences which is induced via the notion of constant acts defined in Definition 5.3. However, unlike degenerate lotteries the notion of constant acts is highly problematic, we shall address this issue in Section 14.1. vNM 2 is commonly known as the *independence* axiom. To explain in terms of compound lotteries, the axiom says that decision maker's (strict) preference between two lotteries remains the same when each is combined with the same lottery (with respect to the same scalar). To illustrate, observe that the compound lotteries in (2.6) are so arranged that they agree with one another on $(1-\lambda)$, then vNM 2 mandates that the preference between the two combined lotteries is solely determined on the part where they are different, i.e., on λ. This postulate is closely related to (or, perhaps, motivates) Savage's well known *sure-thing* principle which will be examined in §5.2.

	λ	$1-\lambda$
$p \oplus_\lambda r$	p	r
$q \oplus_\lambda r$	q	r

(2.6)

vNM 3 is sometimes called the *Archimedean* or *continuity* axiom. Intuitively, it says that no lottery p (q) is so good (bad) that, for any $r \succ q$ ($p \succ r$), the compound lottery of p and q is always better (worse) than r. Variants of vNM axioms are widely adopted in utility theory as they provide some basic characterization of the underlying preferential structure which mimics the behavior of the standard ordering $\geq$ on the real line. The latter paves the way for the eventual real-valued numerical utility representation of $\succsim$. The following properties can be derived from the axioms.

Lemma 2.3. For any $p, q, r \in \mathcal{L}_X$ and $\lambda \in (0,1]$,

(1) $p \sim q$ if and only if $p \sim p \oplus_\lambda q$;
(2) $p \succsim q$ if and only if $p \succsim p \oplus_\lambda q \succsim q$;
(3) for any $0 \leq \beta < \alpha \leq 1$, $p \succ q$ if and only if $p \oplus_\alpha q \succ p \oplus_\beta q$;
(4) if $p \succsim r \succsim q$ and $p \succ q$, then there exists a unique $\alpha \in [0,1]$ such that $r \sim p \oplus_\alpha q$.

Proof. (1) Suppose, to the contrary, that $p \succ p \oplus_\lambda q$. Write p as $p \oplus_\lambda p$, then we have $p \oplus_\lambda p \succ p \oplus_\lambda q$. The latter implies, via vNM 2, $p \succ q$, a contradiction. Hence $p \oplus_\lambda q \succsim p$ by vNM 1. Similarly, it can be shown $p \succsim p \oplus_\lambda q$. Thus $p \sim p \oplus_\lambda q$.

(2) Suppose, to the contrary, that $q \succ p \oplus_\lambda q$, that is, $q \oplus_\lambda q \succ p \oplus_\lambda q$. It follows, by vNM 2, that $q \succ p$, a contradiction. It can be similarly shown that it is not the case that $p \oplus_\lambda q \succ p$. Thus, by vNM 1, $p \succsim p \oplus_\lambda q \succsim q$.

(3) If $\beta = 0$, then, by vNM 2, $p \succ q$ implies $p \oplus_\alpha q \succ q \oplus_\alpha q = q = p \oplus_\beta q$. If $0 < \beta < \alpha \leq 1$, then $1 - \beta/\alpha \in (0,1)$, by vNM 2, $p \oplus_\alpha q \succ q$ implies that

$$
\begin{aligned}
p \oplus_\alpha q &= (p \oplus_\alpha q) \oplus_{1-\frac{\beta}{\alpha}} (p \oplus_\alpha q) \\
&\succ q \oplus_{1-\frac{\beta}{\alpha}} (p \oplus_\alpha q) \\
&= \Big(1 - \frac{\beta}{\alpha}\Big) q + \frac{\beta}{\alpha}[\alpha p + (1-\alpha) q] = \beta p + (1-\beta) q = p \oplus_\beta q.
\end{aligned}
$$

(4) The claim is trivially true if $r \sim p$ or $r \sim q$, in which cases $\alpha = 1$ or 0, respectively. We prove the case where $p \succ r \succ q$. Consider the sets

$$A := \{x \in [0,1] \mid p \oplus_x q \succsim r\};$$

$$B := \{x \in [0,1] \mid r \succsim p \oplus_x q\}.$$

Let $\alpha_* = \inf A$ and $\alpha^* = \sup B$. Note that, for any $a > \alpha_*$, there must exist some $a' \in A$ such that $a > a' \geq \alpha_*$ (for, otherwise, a is a lower bound of A that is greater than α_*, which contradicts the assumption $\alpha_* = \inf A$), hence, by claim (3) above, $p \oplus_a q \succ p \oplus_{a'} q \succsim r$. It follows, via vNM 1, that

$$a > \alpha_* \implies a \notin B. \tag{2.7}$$

The contrapositive of (2.7) says that, for any a, $a \in B$ implies that $\alpha_* \geq a$, in other words, α_* is an upper bound of B. and hence $\alpha_* \geq \alpha^*$. Similarly, one can show that, for any a,

$$\alpha^* > a \implies a \notin A \tag{2.8}$$

which leads to $\alpha^* \geq \alpha_*$. Now define $\alpha = \alpha_* = \alpha^*$. The proof is completed if we can show that $\alpha \in A \cap B$. Suppose, to the contrary, that $\alpha \notin B$, then, by vNM 1, $p \oplus_\alpha q \succ r$. It follows, by vNM 3 and the assumption $r \succ q$, that there exists some $a \in (0, 1)$ such that $(p \oplus_\alpha q) \oplus_a q \succ r$, that is, $p \oplus_{a \cdot \alpha} q \succ r$. This implies that $a \cdot \alpha \in A$. However, from $\alpha^* = \alpha > a \cdot \alpha$ we get, via (2.8), that $a \cdot \alpha \notin A$, a contradiction. Hence we have $\alpha \in B$. Similarly, one can show $\alpha \in A$. Uniqueness can be easily derived from (2.7) and (2.8). □

Remark 2.4. Note that vNM 3 can also be derived from Lemma 2.3(4) under vNM 1 and vNM 2. To see this, let p, q, r be such that $p \succ r \succ q$, we show that there exist $a, b \in (0, 1)$, such that $p \oplus_a q \succ r \succ p \oplus_b q$. By Lemma 2.3(4) there exists a unique $c \in (0, 1)$ for which $r \sim p \oplus_c q$. Then let a be any number in $(c, 1)$ and b be any number in $(0, c)$, then, by Lemma 2.3(3) (which is derivable from under vNM 1 and vNM 2), we are done. Thus vNM 3 is provably equivalent to Lemma 2.3(4) given vNM 1 and vNM 2. For this reason, we can use vNM 3 and Lemma 2.3(4) interchangeably as the continuity axiom of vNM theory.

2.2. Cardinal utility. Given the assumption that X is *finite*, it follows, by vNM 1, that the set of degenerate lotteries $\{\delta_x \mid x \in X\}$ has a $\succsim$-maximal and a $\succsim$-minimal element, that is, there exist a most desired prize x^* and a least desired prize x_* in X such that

$$\delta_{x^*} \succsim \delta_x \succsim \delta_{x_*}, \quad \text{for all } x \in X. \tag{2.9}$$

The following lemma shows that δ_{x^*} and δ_{x_*} are in fact extreme points for all lotteries in $\mathcal{L}_X$ under $\succsim$.

Lemma 2.5. There exist $x^*, x_* \in X$ such that $\delta_{x^*} \succsim p \succsim \delta_{x_*}$ for all $p \in \mathcal{L}_X$.

Proof. Let δ_{x^*} and δ_{x_*} be defined as in (2.9). Consider the non-trivial case where $\delta_{x^*} \succ \delta_{x_*}$. Note that, for any $p \in \mathcal{L}_X$, p can be rewritten as $\sum_{x \in X} p(x)\delta_x$ via (2.2). Then, by Lemma 2.3(4), for each δ_x, let $\lambda_x \in [0,1]$ be such that $\delta_x \sim \delta_{x^*} \oplus_{\lambda_x} \delta_{x_*}$, hence, by vNM 2,

$$\begin{aligned} p = \sum_{x \in X} p(x)\delta_x &\sim \sum_{x \in X} p(x)\big(\delta_{x^*} \oplus_{\lambda_x} \delta_{x_*}\big) \\ &= \sum_{x \in X} p(x)\lambda_x \delta_{x^*} + \Big[1 - \sum_{x \in X} p(x)\lambda_x\Big]\delta_{x_*}. \end{aligned} \tag{2.10}$$

Since $\delta_{x^*} \succ \delta_{x_*}$ and $0 \le \sum_{x \in X} p(x)\lambda_x \le 1$, then, by Lemma 2.3(2),

$$\delta_{x^*} \succsim \sum_{x \in X} p(x)\lambda_x \delta_{x^*} + \Big[1 - \sum_{x \in X} p(x)\lambda_x\Big]\delta_{x_*} \sim p.$$

Similarly, it can be shown that $p \succsim \delta_{x_*}$. □

Let us now proceed with the main theorem of this section.

Theorem 2.6 (von Neumann-Morgenstern). Let X be a nonempty finite set, and $\succsim$ be a preference relation on $\mathcal{L}_X$. Then $\succsim$ satisfies vNM 1-3 if and only if there exists a function $u \in \mathbb{R}^X$ such that

$$p \succsim q \;\text{ iff }\; \sum_{x \in X} p(x)u(x) \ge \sum_{x \in X} q(x)u(x), \tag{2.11}$$

where u is unique up to a positive linear transformation, that is, for any function $v \in \mathbb{R}^X$, v satisfies (2.11) if and only if, for some $a > 0$ and b,

$$u(x) = av(x) + b. \tag{2.12}$$

Proof. We only prove the non-trivial "only if" direction of the theorem in following steps:

(1) By Lemma 2.5, there exist $x^*, x_* \in X$ such that $\delta_{x^*} \succsim p \succsim \delta_{x_*}$ for all $p \in \mathcal{L}_X$. If $\delta_{x^*} \sim \delta_{x_*}$ then $p \sim q$ for all $p, q \in \mathcal{L}_X$. In this case, let u be any constant function. Otherwise, $\delta_{x^*} \succ \delta_{x_*}$, define function $U : \mathcal{L}_X \mapsto [0,1]$ as follows,

$$U(p) := \inf\left\{\alpha \in [0,1] \,\middle|\, \delta_{x^*} \oplus_\alpha \delta_{x_*} \succsim p\right\}.$$

By Lemma 2.3(3),

$$p \succsim q \text{ if and only if } U(p) \geq U(q) \text{ for all } p, q \in \mathcal{L}_X; \tag{2.13}$$

and by Lemma 2.3(4),

$$p \sim \delta_{x^*} \oplus_\lambda \delta_{x_*} \text{ if and only if } \lambda = U(p). \tag{2.14}$$

(2) We show that U is an affine function on $\mathcal{L}_X$, that is, for any sequences $\lambda_1, \ldots, \lambda_n \in [0,1]$ with $\sum_i \lambda_i = 1$ and for any $p_1, \ldots, p_n \in \mathcal{L}_X$, we have

$$U(\lambda_1 p_1 + \cdots + \lambda_n p_n) = \lambda_1 U(p_1) + \cdots + \lambda_n U(p_n). \tag{2.15}$$

It suffices to show, for any $p, q \in \mathcal{L}_X$ and $\lambda \in [0,1]$, that

$$U\Big(\lambda p + (1-\lambda) q\Big) = \lambda U(p) + (1-\lambda) U(q). \tag{2.16}$$

Note that, by (2.14), $p \sim \delta_{x^*} \oplus_{U(p)} \delta_{x_*}$ and $q \sim \delta_{x^*} \oplus_{U(q)} \delta_{x_*}$. Let $r = \lambda p + (1-\lambda)q$. Then, by vNM 2 (twice),

$$\begin{aligned}\delta_{x^*} \oplus_{U(r)} \delta_{x_*} \sim r &= \lambda p + (1-\lambda) q \\ &\sim \lambda\Big(\delta_{x^*} \oplus_{U(p)} \delta_{x_*}\Big) + (1-\lambda)\Big(\delta_{x^*} \oplus_{U(p)} \delta_{x_*}\Big) \\ &= \delta_{x^*} \oplus_{\lambda U(p) + (1-\lambda) U(q)} \delta_{x_*}.\end{aligned}$$

It follows that $U(r) = \lambda U(p) + (1-\lambda) U(q)$ via (2.14).

(3) Now, for any $p, q \in \mathcal{L}_X$, by (2.13) and (2.2),

$$p \succsim q \text{ if and only if } U\Big(\sum_{x\in X} p(x)\delta_x\Big) \geq U\Big(\sum_{y\in X} q(y)\delta_y\Big)$$

Then, by (2.15),

$$p \succsim q \text{ if and only if } \sum_{x\in X} p(x)U(\delta_x) \geq \sum_{y\in X} q(y)U(\delta_y).$$

Define $u : X \mapsto \mathbb{R}$ to be such that $u(x) = U(\delta_x)$, we have that

$$p \succsim q \text{ if and only if } \sum_{x\in X} p(x)u(x) \geq \sum_{y\in X} q(y)u(y).$$

(4) Finally, we show that u is unique up to a positive linear transformation, we prove only the nontrivial "only if" direction of the proof. As before, let $x^*, x_* \in X$ be such that $\delta_{x^*} \succsim \delta_x \succsim \delta_{x_*}$ for all $x \in X$. Further, let a, b be such that

$$u(x^*) = av(x^*) + b, \quad u(x_*) = av(x_*) + b,$$

where $a > 0$ (the existence of such a, b is guaranteed by the hypothesis $\delta_{x^*} \succ \delta_{x_*}$ in part (1)). By (2.14), for any $x \in X$, there exists a number λ for which $\delta_x \sim \delta_{x^*} \oplus_\lambda \delta_{x_*}$, then we have that

$$\begin{aligned} u(x) &= \lambda u(x^*) + (1-\lambda)u(x_*) \\ &= \lambda\big[av(x^*) + b\big] + (1-\lambda)\big[av(x_*) + b\big] \\ &= a\big[\lambda v(x^*) + (1-\lambda)v(x_*)\big] + b = av(x) + b. \end{aligned}$$

This completes the proof of the theorem. □

We refer to the derived function u above as an instance of *von Neumann-Morgenstern utility function* (vNMUF). A pair of vNMUFs u and v are said to be *cardinally equivalent* if (2.12) is satisfied. The following corollary is a generalization of Theorem 2.6, which will become handy later. The proof uses the same techniques as in the proof Theorem 2.6, and hence omitted.

Corollary 2.7. Let $X, \mathcal{L}_X$ be as above, and let C be any convex subset of $\mathcal{L}_X$. Suppose that $\succsim$ is a preference relation on C such that there is an $\succsim$-maximum and an $\succsim$-minimum in C. Then $\succsim$ satisfies vNM 1-3 if and only if there exists a (utility function) $u \in \mathbb{R}^X$ such that

$$p \succsim q \text{ iff } \sum_{x \in X} p(x)u(x) \geq \sum_{x \in X} q(x)u(x). \tag{2.17}$$

where u is unique up to a positive linear transformation.

2.3. Expected utility for simple lotteries. We now extend the von Neumann and Morgenstern Expected Utility Theorem 2.6 to a class of lotteries defined for some X that contains potentially *infinitely* many prizes.

Definition 2.8. Let X be an infinite set of prizes/consequences, a probability measure p on X is said to be *simple* if it has a *finite support*, that is, if

$$\left|\text{supp}(p)\right| = \left|\left\{x \in X \ : \ p(x) > 0\right\}\right| < \infty. \tag{2.18}$$

Denote by $\mathcal{L}^*_X$ the set of all simple probabilities on X, and we refer to $\mathcal{L}^*_X$ as an *extended space of lotteries*. The notational difference between $\mathcal{L}_X$ and $\mathcal{L}^*_X$ is that $\mathcal{L}_X$ contains all the probability measures defined on a *finite* X, whereas, for any $p \in \mathcal{L}^*_X$, p is defined on some *infinite* X but with finite support. Clearly, for any $\lambda \in [0, 1]$ and any simple probabilities p, q, the *mixture* of p and q, written $p \oplus_\lambda q$, is in $\mathcal{L}^*_X$. And for any $p \in \mathcal{L}^*_X$, p can be written as the sum of degenerate lotteries that support p, an analogue of (2.2):

$$p = \sum_{x \in \text{supp}(p)} p(x)\delta_x. \tag{2.19}$$

Then a similar argument for Lemma 2.5 leads to the following observation.

Lemma 2.9. If there exist a $\succsim$-maximal element x^* and a $\succsim$-minimal element x_* in X then, for each $p \in \mathcal{L}^*_X$, $\delta_{x^*} \succsim p \succsim \delta_{x_*}$.

THEOREM 2.10. Let $\succsim$ be a preference relation on $\mathcal{L}_X^*$. Then $\succsim$ satisfies vNM 1-3 if and only if there exists a vNMUF $u \in \mathbb{R}^X$ such that

$$p \succsim q \;\text{ iff }\; \sum_{x \in X} p(x)u(x) \geq \sum_{y \in X} q(y)u(y), \tag{2.20}$$

where u is unique up to a positive linear transformation.

PROOF OF THEOREM 2.10. We prove by modifying step (1)-(4) in the proof of Theorem 2.6 with (1*)-(4*) to account for the added assumption that X is infinite and that each $p \in \mathcal{L}_X^*$ is a simple probability measure. We only show the modified steps (1*) and (3*), namely the steps where the assumption of X being infinite plays a role. Steps (2*) and (4*) hold with obvious notational changes.

(1*) If for any $p, q \in \mathcal{L}_X^*$, $p \sim q$, then let u be any constant function, then we are done. Otherwise, fix any p, q satisfying $p \succ q$. By vNM 1, for any $r \in \mathcal{L}_X^*$, exactly one of the following cases holds:

(i) $p \succsim r \succsim q$, (ii) $r \succ p$, (iii) $q \succ r$.

For case (i), define function $U : \mathcal{L}_X^* \mapsto [0,1]$ as follows,

$$U(r) := \inf \left\{ \alpha \in [0,1] \;\middle|\; p \oplus_\alpha q \succsim r \right\}.$$

Then, by Lemma 2.3(4), $r \sim p \oplus_\lambda q$ if and only if $\lambda = U(r)$. It follows that $U(p) = 1$ and $U(q) = 0$. For any r in case (ii), by Lemma 2.3(4), let a be such that $p \sim r \oplus_a q$, define $U(r) = 1/a$. Similarly, for any r in case (iii), let a be such that $q \sim p \oplus_a r$, define $U(r) = a/(a-1)$. Thus, by Lemma 2.3(3), we have that U is a numerical representation of $\succsim$:

$$p \succsim q \;\text{ if and only if }\; U(p) \geq U(q) \text{ for all } p, q \in \mathcal{L}_X^*. \tag{2.21}$$

(3*) Define $u : X \to \mathbb{R}$ by

$$u(x) = U(\delta_x) \quad \text{for all } x \in X. \tag{2.22}$$

Then, for any $p, q \in \mathcal{L}_X^*$, a modified step (2*) together with (2.19)-(2.21) yield that

$$\begin{aligned} p \succsim q &\iff U\Bigg(\sum_{x \in \mathrm{supp}(p)} p(x)\delta_x\Bigg) \geq U\Bigg(\sum_{y \in \mathrm{supp}(q)} q(y)\delta_y\Bigg) \\ &\iff \sum_{x \in \mathrm{supp}(p)} p(x)U(\delta_x) \geq \sum_{y \in \mathrm{supp}(q)} q(y)U(\delta_y) \\ &\iff \sum_{x \in X} p(x)u(x) \geq \sum_{y \in X} q(y)u(y). \end{aligned}$$

This completes the proof of the theorem. □

Remark 2.11. Note that all the probability functions (either in $\mathcal{L}_X$ or in $\mathcal{L}_X^*$) considered in this section are simple (have finite support). The theorems proved above hold regardless of whether these probabilities are finitely or countably additive. This ceases to be true if probability functions defined over X are *not* simple: different constraints with different additivity conditions need to be added in order for the representation theorem to hold. See Fishburn (1970, Chapter 8) and Fishburn (1982, Chapter 3) for an extensive discussion for these cases.

3. Horse-race Lotteries

3.1. Risk versus uncertainty. In the von Neumann and Morgenstern expected utility model, the decision maker is uncertain as to which outcome/prize will transpire, where the uncertainty is associated with some objective chances attached to the outcomes. For instance, in a gambling situation, the gambler is uncertain about the outcome of the spin of a roulette wheel, where the betting on each possible outcome comes with a known risk (objective probability), and hence the vNM model is commonly referred to as a decision model under *risk*. Under the assumption of these known objective chances, the vNM expected theory provides a systematic way of retrieving decision makers' subjective utilities of the outcomes given their respective preferences among the probability distributions, i.e., among vNM lotteries. Now, it is conceivable that there are cases where these objective chances might be lacking. Consider, for instance, that in a horse race the gambler needs to choose between two gambles h and h' on the possible outcomes of the horse race: either horse H_1, or H_2, or H_3 will the winning horses and the payoffs of the two gambles are given in the matrix below.

$$\begin{array}{c|ccc} & H_1 & H_2 & H_3 \\ \hline h & \$100 & 0 & \$20 \\ h' & 0 & \$100 & \$20 \end{array} \tag{3.1}$$

That is, if gamble h is chosen and horse H_1 wins the race then the gambler will be paid with 100 dollars, 0 if H_2 is the winner, and so on. Here, the winning horses form a set of *possible states of the world*, denoted by S.[8]

As seen, in this example, there are no objective chances involved. In making a decision, the gambler needs to provide his own probabilistic estimation on the winning horse, based perhaps on his knowledge about the horses, past experiences with horse race, or some other considerations. A decision framework that treats this type of decision problems is often referred to as a decision model under *uncertainty*.

A complete treatment of the above case will have to wait until the next Chapter where we present Savage's theory of subjective expected utility. In this section we discuss an *intermediate* step where, instead of receiving direct cash reward, the gambler is paid with some other type of prizes, namely roulette lotteries p, q which indirectly lead to cash reward (e.g., if gamble h is chosen and H_1 wins the horse race then the gambler will be paid with roulette lottery p which in turn says that with 50-50 chance the gambler will get either \$100 or \$20 dollars).

$$\begin{array}{c|ccc} & H_1 & H_2 & H_3 \\ \hline h & p & 0 & q \\ h' & 0 & p & q \end{array} \quad \Rightarrow \quad \begin{array}{c|ccc} & \$100 & \$20 & 0 \\ \hline p & 1/2 & 1/2 & 0 \\ q & 0 & 1/2 & 1/2 \end{array}$$

The distinction between these two types of lotteries, namely horse-race lotteries/gambles and roulette lotteries, was made in Anscombe and Aumann (1963),

[8] We shall provide an analysis of the nature of the states in our discussion of Savage's decision model later. For the time being, a state of the world is taken as a specification of a possible way that the world may unfold that is relevant to the current decision situation.

where the roulette lotteries are just vNM lotteries. Their decision model is hence a mixture system containing both subjective and objective probabilities. The goal, as stated by the authors, is "to define the person's probabilities in terms of chances, by an extension of von Neumann-Morgenstern theory."

Let X be a finite set of prizes, $\mathcal{L}_X$ be the lottery space of X, and let S be a finite set of states of the world. A *horse-race lottery* (or *horse lottery* for short) is a function mapping from S to $\mathcal{L}_X$.[9] Denote the set of all horse lotteries by $\mathcal{H}$, that is, $\mathcal{H} = {\mathcal{L}_X}^S$. Given the definition of compound vNM lotteries in (2.3), define the operation of (convex) combination of horse lotteries as follows: a *compound horse lottery* of any $h, h' \in \mathcal{H}$ with scalar $\lambda \in [0,1]$, in symbols $h \oplus_\lambda h'$, is defined as

$$\big(h \oplus_\lambda h'\big)(s) =_{\mathrm{Df}} h(s) \oplus_\lambda h'(s) \quad \text{for all } s \in S. \tag{3.2}$$

Notation. By definition, $h(s)$ is itself a probability function in $\mathcal{L}_X$, we often write $h(s)(\cdot)$ as $h_s(\cdot)$ for short. Then, under our current notational convention, h denotes a horse lottery and h_s is a roulette lottery, i.e., a vNM lottery.

Note that, in (3.2), for any given state $s \in S$, it is clear that $h(s) \oplus_\lambda h'(s) \in \mathcal{L}_X$, and hence $h \oplus_\lambda h'$ is also a horse lottery in $\mathcal{H}$ by definition. The task for our decision maker is to choose among the horse lotteries the preferred one(s). The preferences are further represented by a preference relation $\succsim$, from which her subjective probability measure μ on the occurrences of the states and her subjective utility measure u on the prizes are to be deduced.

[9] In later chapters, we will be discussing the additivity condition of the subjective probability measures derived. This depends on the basic setups of the spaces in which various probability measures are defined. Anscombe and Aumann (1963) were not explicit about the cardinality of the set of prizes on which vNM lotteries are defined. They mentioned in passing that the restriction to finite space in Luce and Raiffa (1957)'s proof of the existence of a vNM utility representation is not necessary, however, the examples they used (i.e., roulette lotteries) and the details of their proofs involved are all finitary in nature. They also compared their system with that of Savage (1954), where the horse lotteries are just a special type of Savage acts (with vNM lotteries as consequences). They did not give further details on the structure of the state space on which their horse lotteries are defined. Here, again, their examples and the proposed axioms (Assumption 1 & 2) all use finite structures. To simplify matters, we discuss the case where all vNM lotteries are simple and the set of states is finite. That is, we consider $\mathcal{L}_X$ instead of $\mathcal{L}_X^*$.

3.2. State-dependent utility. Let $\succsim$ be a preference relation (a preorder) on the set of horse lotteries $\mathcal{H}$. In strict parallel to the von Neumann and Morgenstern postulates vNM 1-3, the first three Anscombe and Aumann (A-A) axioms on $\succsim$ take the following form.

A-A 1. $\succsim$ is complete.

A-A 2. For all $h, h', t \in \mathcal{H}$ and $\lambda \in (0, 1]$,

$$h \succ h' \iff h \oplus_\lambda t \succ h' \oplus_\lambda t.$$

A-A 3. For any $h, h', t \in \mathcal{H}$, there exist $a, b \in (0, 1)$, such that

$$h \succ t \succ h' \implies h \oplus_a h' \succ t \succ h \oplus_b h'.$$

These axioms are sufficient for deriving the following *state-dependent* representation theorem which can be seen as a direct consequence of Corollary 2.7.

Theorem 3.1. Let $\succsim$ be a preference relation on $\mathcal{H}_{S,X}$, then $\succsim$ satisfies A-A 1-3 if and only if there exist (state-dependent utility) functions $u : S \times X \mapsto \mathbb{R}$ such that, for any $h, h' \in \mathcal{H}_{S,X}$,

$$h \succsim h' \text{ iff } \sum_{s \in S} \sum_{x \in X} h_s(x) u(s, x) \geq \sum_{s \in S} \sum_{x \in X} h'_s(x) u(s, x). \tag{3.3}$$

The right hand side of (3.3) takes the full advantage of the fact that each h_s is itself a probability function, the task is then to show that there exists a utility function such that the comparison between horse lotteries can be represented by their expected utilities.

Proof. We give only the non-trivial "only if" direction of the proof. Let $\mathcal{L}_{S \times X}$ denote the set of probability functions defined on $S \times X$. Note that, for each horse lottery $h \in \mathcal{H} = \mathcal{L}_X{}^S$, there corresponds a $\hat{h} \in \mathcal{L}_{S \times X}$ such that for any $s \in S$ and any $x \in X$, $h_s(x) = |S| \cdot \hat{h}(s, x)$. Let $\hat{\mathcal{H}}$ be the set of all $\hat{h}$'s, that is,

$$\hat{\mathcal{H}} = \left\{ \hat{h} \mid h_s(x) = |S| \cdot \hat{h}(s, x) \text{ and } h \in \mathcal{H} \right\}. \tag{3.4}$$

Thence $\hat{\mathcal{H}}$ is subset of $\mathcal{L}_{S\times X}$ which is convex and compact (because $\mathcal{H}$ is). Further, define an ordering $\hat{\succsim}$ on $\hat{\mathcal{H}}$ such that, for any $\hat{h}, \hat{h}' \in \hat{\mathcal{H}}$,

$$\hat{h} \mathrel{\hat{\succsim}} \hat{h}' \iff h \succsim h'. \tag{3.5}$$

It is easy to see that $\succsim$ on $\mathcal{H}$ satisfies A-A 1-3 if and only if the induced ordering $\hat{\succsim}$ on $\hat{\mathcal{H}}$ satisfies vNM 1-3. By compactness and vNM 3 (i.e., continuity), there exists an $\hat{\succsim}$-maximum and an $\hat{\succsim}$-minimum in $\hat{\mathcal{H}}$. Hence, by Corollary 2.7 and (3.5), there exists a vNMUF $v \in \mathbb{R}^{S\times X}$ such that

$$\begin{aligned} h \succsim h' &\Leftrightarrow \sum_{(s,x)\in S\times X} \hat{h}(s,x)v(s,x) \geq \sum_{(s,x)\in S\times X} \hat{h}'(s,x)v(s,x) \\ &\Leftrightarrow \sum_{s\in S}\sum_{x\in X} \frac{1}{|S|} h_s(x)v(s,x) \geq \sum_{s\in S}\sum_{x\in X} \frac{1}{|S|} h'_s(x)v(s,x). \end{aligned}$$

The proof is completed once we define $u(s,x)$ to be $v(s,x)/|S|$. □

As seen, the derived two-place utility function u is *state-dependent* as the function value also depends on the state. For any $s \in S$, we write $u(s,\cdot)$ as $u_s(\cdot)$ and refer to the latter as the utility function with respect to state s. Theorem 3.1 then states that the agent's preference relation among horse lotteries can be represented using a series of state-dependent utility functions $\{u_s\}_{s\in S}$.

3.3. State-independent utility. Theorem 3.1 can be further strengthened by adding one more axiom so that the representation takes the form of a combination of agent's subjective probability on states and her subjective *state-independent* utility on the prizes. The strengthening relies on the following concept of "constant horse lotteries".[10]

Definition 3.2. A horse lottery is said to be *constant* with respect to $p \in \mathcal{L}_X$, written $\mathfrak{c}_p$ if $\mathfrak{c}_p(s) = p$ for all $s \in S$.

It is clear from the definition above that each constant horse lottery can be identified with a vNM lottery. This then enables us to define a preference ordering

[10]Constant horse lotteries are special cases of Savage's notion of "constant acts" in Definition 5.3 below. More discussion on Savage's constant acts will be given in Section 14.1.

$\succsim^*$ on $\mathcal{L}_X$ using the preference relation $\succsim$ over $\mathcal{H}$ as follows

$$p \succsim^* q \iff \mathfrak{c}_p \succsim \mathfrak{c}_q \quad \text{for all } p, q \in \mathcal{L}_X. \tag{3.6}$$

Call $\succsim^*$ the preference relation on $\mathcal{L}_X$ *induced by* $\succsim$.

Lemma 3.3. Let $\succsim$ be a preference on $\mathcal{H}$ and $\succsim^*$ the corresponding induced preference on $\mathcal{L}_X$, if $\succsim$ satisfies A-A 1-3 then $\succsim^*$ satisfies vNM 1-3.

Proof. We prove the lemma by direct verifications.

(1) vNM 1 can be easily verified using the definition of constant horse lottery.
(2) For vNM 2, suppose that $p \succ^* q$, then we have $\mathfrak{c}_p \succ \mathfrak{c}_q$ via (3.6). By A-A 2, for any $r \in \mathcal{L}_X$ and $\lambda \in (0, 1]$,

$$\mathfrak{c}_p \succ \mathfrak{c}_q \iff \mathfrak{c}_p \oplus_\lambda \mathfrak{c}_r \succ \mathfrak{c}_q \oplus_\lambda \mathfrak{c}_r.$$

By (3.2), $\mathfrak{c}_p \oplus_\lambda \mathfrak{c}_r(s) = \mathfrak{c}_p(s) \oplus_\lambda \mathfrak{c}_r(s)$ for all $s \in S$. Since $\mathfrak{c}_p, \mathfrak{c}_q, \mathfrak{c}_r$ are constant horse lotteries, we have that $\mathfrak{c}_p(s) \oplus_\lambda \mathfrak{c}_r(s) = p \oplus_\lambda r$ for all $s \in S$. Similarly, $\mathfrak{c}_q \oplus_\lambda \mathfrak{c}_r = q \oplus_\lambda r$. Hence $\mathfrak{c}_p \oplus_\lambda \mathfrak{c}_r \succ \mathfrak{c}_q \oplus_\lambda \mathfrak{c}_r$ if and only if $\mathfrak{c}_{p\oplus_\lambda r} \succ \mathfrak{c}_{q\oplus_\lambda r}$, thence $p \oplus_\lambda r \succ^* q \oplus_\lambda r$ via (3.6).
(3) Finally, suppose that $p \succ^* r \succ^* q$, then by (3.6) we have $\mathfrak{c}_p \succ \mathfrak{c}_r \succ \mathfrak{c}_q$. By A-A 3, there exists $a, b \in (0, 1]$ such that $\mathfrak{c}_p \oplus_a \mathfrak{c}_q \succ \mathfrak{c}_r \succ \mathfrak{c}_p \oplus_b \mathfrak{c}_q$. Using a similar argument as in (2), we get $\mathfrak{c}_{p\oplus_a q} \succ \mathfrak{c}_r \succ \mathfrak{c}_{p\oplus_b q}$. Therefore, $p \oplus_a q \succ^* r \succ^* p \oplus_b q$ via (3.6), and hence vNM 3. □

Further, a state $s \in S$ is said to be *null* if the agent is indifferent between any horse lotteries that differ only on s, s is *non-null* if it is not null. We are now in the position to state the fourth A-A axiom which facilitates state-independent utility representation.

A-A 4. For any $h, h' \in \mathcal{H}$,

(1) if $h_s \succsim^* h'_s$ for all $s \in S$ then $h \succsim h'$;
(2) if $h_s \succsim^* h'_s$ for all $s \in S$ and $h_s \succ^* h'_s$ for some non-null $s \in S$ then $h \succ h'$;

where $\succsim^*$ is the preference on $\mathcal{L}_X$ induced by $\succsim$.

Axiom A-A 4 is commonly known as the *Monotonicity* axiom (or sometimes the *Dominance* or *State-independent* axiom). It asserts that horse lottery h weakly dominates h' if, in each state s, the vNM lottery h_s weakly dominates h'_s (under the induced preference ordering through the notion of constant horse lottery); h strictly dominates h' if, for some state s, h_s strictly dominates h'_s. As shown the in following lemma, the axiom regulates in a very rigid way the two preferential systems ($\succsim$ and $\succsim^*$).

Lemma 3.4. Let $\succsim$ and $\succsim^*$ be as above. Suppose that $\succsim$ satisfies A-A 1-4 and that, for any $s \in S$, u_s be a utility function obtained in Theorem 3.1, then u_s is a vNMUF with respect to $\succsim^*$.

Proof. By Lemma 3.3, $\succsim^*$ satisfies vNM 1-3, then it suffices to show that, for any $p, q \in \mathcal{L}_X$,

$$p \succsim^* q \text{ if and only if } \sum_{x \in X} p(x) u_s(x) \geq \sum_{x \in X} q(x) u_s(x).$$

From $p \succsim^* q$ we have $\mathfrak{c}_p \succsim \mathfrak{c}_q$ by definition. Fix s, let h be the horse lottery that differs from $\mathfrak{c}_q$ in precisely the following way

$$h(v) = \begin{cases} p & \text{if } v = s \\ q & \text{if } v \neq s \end{cases}.$$

That is, h yields p at s but agrees with $\mathfrak{c}_q$ at all other states. Then, by A-A 4, we have $p \succsim^* q$ iff $h \succsim \mathfrak{c}_q$. It follows, by (3.3), that

$$\begin{aligned} h \succsim \mathfrak{c}_q \Leftrightarrow & \sum_{v \in S} \sum_{x \in X} h_v(x) u_v(x) \geq \sum_{v \in S} \sum_{x \in X} q(x) u_v(x) \\ \Leftrightarrow & \sum_{x \in X} p(x) u_s(x) + \sum_{v \in S \setminus \{s\}} \sum_{x \in X} q(x) u_v(x) \geq \sum_{v \in S} \sum_{x \in X} q(x) u_v(x) \\ \Leftrightarrow & \sum_{x \in X} p(x) u_s(x) \geq \sum_{x \in X} q(x) u_s(x). \end{aligned}$$

This completes the proof of the lemma. □

Theorem 3.5 (Anscombe and Aumann). Let $\succsim$ be a preference relation on $\mathcal{H}$. Then $\succsim$ satisfies A-A 1-4 if and only if there exists a utility function $u \in \mathbb{R}^X$ and a probability measure μ on S such that, for any $h, h' \in \mathcal{H}$,

$$h \succsim h' \text{ iff } \sum_{s\in S} \mu(s) \sum_{x\in X} h_s(x)u(x) \geq \sum_{s\in S} \mu(s) \sum_{x\in X} h'_s(x)u(x). \tag{3.7}$$

Proof. By Theorem 3.1, there exists a series of state-dependent functions $\{u_s\}_{s\in S}$ such that (3.3) holds. Further, Lemma 3.4 shows that the u_s's are vNM utility representations with respect to the same preference relation $\succsim^*$ over lotteries, and hence are unique up to positive affine transformations. That is, if we fix a state s' and let $u = u_{s'}$, then for any $s \in S$, there, by (2.12), exist a_s, b_s $(a_s > 0)$ such that $u_s = a_s u + b_s$. Then, from (3.3), we get

$$h \succsim h' \text{ iff } \sum_{s\in S} a_s \sum_{x\in X} h_s(x)u(x) \geq \sum_{s\in S} a_s \sum_{x\in X} h'_s(x)u(x). \tag{3.8}$$

Now, define $\mu : S \mapsto \mathbb{R}_+$ to be such that

$$\mu(s) = \frac{a_s}{\sum_{v\in S} a_v}. \tag{3.9}$$

This, together with (3.8), yield what we want. □

In the A-A system above, (3.9) is interpreted as the agent's subjective probability, which, as seen, is defined in terms of the coefficients of a series of vNM utility functions which, in turn, are defined through vNM lotteries. The model is hence a dualistic system featuring both subjective and objective probabilities. In the next chapter, we introduce Savage's theory of expected utility where probabilities are given a purely subjective and decision-theoretic interpretation. We shall compare the A-A system with Savage's theory in Section 16.

CHAPTER II

Savage's Subjectivism

> Personalistic views hold that probability measures the confidence that a particular individual has in the truth of a particular proposition, for example, the proposition that it will rain tomorrow. These views postulate that the individual concerned is in some ways "reasonable," but they do not deny the possibility that two reasonable individuals faced with the same evidence may have different degrees of confidence in the truth of the same proposition.
>
> —— Savage (1972)

4. Introduction

This chapter introduces Leonard J. Savage's theory of subjective expected utility as presented in his seminal book *the Foundations of Statistics*.[1] As indicated in the opening quote, one main objective of this project is to provide a subjective interpretation of the central notion employed in virtually all stages of statistical inferences, namely the notion of probability. Built on earlier works of Frank Ramsey, Bruno de Finetti, John von Neumann and Oskar Morgenstern, among others, Savage's theory seeks to ground a theory of personal probability in a normative theory of rational decision making of highly idealized reasonable agents, where by "reasonable agents" Savage means individuals who are capable of distinguishing "between coherent behavior and blunder, or demonstrable incoherence, in the face of uncertainty." This is achieved by prescribing various rationality principles and structural assumptions governing decision makers' behaviors in decision-making

[1]The first edition of Savage's book where the axiomatic theory was first introduced appeared in 1954 published by John Wiley & Sons. All citations in this dissertation refer to the second revised edition published by Dover Publications in 1972.

situations, by which the agents can police their own potential decisions against incoherency.

The methodological approach adopted by Savage begins with the decision maker's preferences over acts. A set of axioms is postulated on this preference relation. From the first five postulates a comparative notion of subjective probability is derived which reflects the agent's qualitative probabilistic judgments over possible circumstances under which these actions are taking place. With the sixth postulate, the derived qualitative probability is further represented by a numerical probability measure together with a personal utility function for simple acts (i.e., acts that may lead to finitely many potential consequences under different states). The last postulate is brought in so that the utility function for simple acts can be extended to all acts (cf. Table 4.1).

Table 4.1. Inferential order in Savage's system.

P1-5		**+ P6**		**+ P7**
Qualitative probability	$\Rightarrow$	Quantitative probability Utility for simple acts	$\Rightarrow$	Utility for all acts

Savage's approach differs from the methods used by Ramsey and Anscombe-Aumann in that, in the latter cases, the agents' subjective probabilities are derived from their personal utilities, which in turn are constructed based on some presupposed chance mechanisms (or, in the case of Ramsey, the notion of ethically neutral propositions, which plays a similar role as an unbiased coin receiving objective probability 1/2). This inferential order is reversed in Savage's subjectivism where the preference relation over acts is taken as the only primitive notion, from which the agent's personal probabilities and utilities are subsequently revealed. As a result of this reversal, Savage's approach may appear to have some computational disadvantages in the sense that the mathematical representation theorem given by Savage is considerably more involved than many of its alternatives, yet the theory is conceptually significant in that the system is maintained as a purely subjective framework with no direct reference to objective probabilities.

Our expositions will follow closely Savage's original approach. The plan is as follows. After an introduction of basic definitions and notations in Section 5.1, we provide an analysis of the well-known "sure-thing" principle (Section 5.2). This will be followed by a reconstruction of Savage's theory of qualitative probability (Section 6.1), quantitative probability (Section 6.2), and personal utility for simple acts (Section 7.1). In Section 7.2, we investigate the role of Savage's last postulate (i.e., P7) played in extending utility from simple acts to general acts.

5. Decision Matrix

5.1. States, consequences, and acts. The basic setup of Savage's decision model can be illustrated in the decision matrix in Table 5.1, where $S = \{s_1, s_2, \dots\}$ is an (infinite) set of *states of the world* specifying those possible circumstances that are relevant to the decision situation at hand,[2] $X = \{o_{1,1}, o_{1,2}, \dots\}$ is a (finite or infinite) set of *consequences* (or *outcomes*), and $f_1, f_2, \dots$ are commonly referred to as (Savage) *acts*, which are arbitrary functions mapping from S to X. The intended interpretation of an act, say f_m, is that the agent's choice of f_m will lead to consequence $o_{m,n}$ if s_n is the true state of the world. Denote the set of all acts by $\mathcal{A}$.

TABLE 5.1. Savage's decision matrix.

	s_1	s_2	$\cdots$	s_n	$\cdots$
f_1	$o_{1,1}$	$o_{1,2}$	$\cdots$	$o_{1,n}$	$\cdots$
f_2	$o_{2,1}$	$o_{2,2}$	$\cdots$	$o_{2,n}$	$\cdots$
$\vdots$			$\ddots$		
f_m	$o_{m,1}$	$o_{m,2}$	$\cdots$	$o_{m,n}$	$\cdots$

As a primitive assumption, the agent is assumed to have *preferences* over acts, which are modeled by a preorder $\succsim$ on $\mathcal{A}$. Thus, for any acts $f, g \in \mathcal{A}$, $f \succsim g$ is taken to mean that act f is *weakly preferred to* act g (or that g is not preferred to f) by the agent. We define $f \succ g \ =_{\text{Df}} f \succsim g$ and $g \not\succsim f$. This means that f is

[2] In fact, as a feature of Savage's theory, S must contain uncountably many states, we will return to this point later (cf. Remark 6.15 below).

strictly preferred to g. And define $f \sim g =_{\text{Df}} f \succsim g$ and $g \succsim f$, that is, f and g are *equi-preferable* (or that f is *indifferent* to g).

DEFINITION 5.1 (Combined acts). For any $f, g \in \mathcal{A}$, define the *combination* of f and g with respect to an event E (a set of states), written $f \oplus_E g$, to be such that:

$$(f \oplus_E g)(s) = \begin{cases} f(s) & \text{if } s \in E \\ g(s) & \text{if } s \in E^C, \end{cases} \tag{5.1}$$

where $E^C = S - E$ is the compliment of E.[3]

That is to say, $f \oplus_E g$ is the act which agrees with f on event E, with g on E^C, and it is easily seen that $f \oplus_E g \in \mathcal{A}$. Using the concept defined in (5.1), we can interpret $(f \oplus_E g) \oplus_F h$ as saying: do f if $E \cap F$ obtains, g if $F \cap E^C$ occurs, and h if F^C, and so on. The following is a list of simple properties of operation $\oplus_E$. The proof is immediate from the definition and hence omitted.

LEMMA 5.2. For any $E, F \in \mathcal{F}$, and for any acts $f, g \in \mathcal{A}$,

(1) $f \oplus_E g = g \oplus_{E^C} f$;
(2) $(f \oplus_E g) \oplus_F g = f \oplus_{E \cap F} g$;
(3) $f \oplus_E (f \oplus_F g) = f \oplus_{E \cup F} g$;
(4) $(f \oplus_E g) \oplus_{E^C} g = g$.

The following concept is a key structural component of Savage's theory, which will play an important role in the discussions that follows.

DEFINITION 5.3 (Constant acts). For any $a \in X$, an act is said to be *constant* with respect to consequence a, in symbols $\mathfrak{c}_a$, if

$$\mathfrak{c}_a(s) = a \quad \text{for all } s \in S. \tag{5.2}$$

In other words, act $\mathfrak{c}_a$ "constantly" outputs consequence a no matter which state $s \in S$ transpires. Now, given a preference ordering $\succsim$ on $\mathcal{A}$, an ordering $\succsim^*$

[3]Some writers use '(f, E, g)' or 'fEg' or '$f|E + g|E^C$' or '$[f$ on E, g on $E^C]$' for combined acts.

over consequences can be defined using constant acts by

$$a \succsim^* b \quad \Longleftrightarrow \quad \mathfrak{c}_a \succsim \mathfrak{c}_b \quad \text{for all } a, b \in X. \tag{5.3}$$

That is to say, consequence a is said to be weakly preferred to consequence b if the constant act $\mathfrak{c}_a$ is weakly preferred to $\mathfrak{c}_b$. Call $\succsim^*$ the preference relation on X *induced by* $\succsim$. For notational purpose, we often use the same symbol '$\succsim$' for both the preference relation over acts and the induced preference over consequences and let the context determine on which set of alternatives a given preference $\succsim$ is defined. With these two preference orderings, we proceed to define a (qualitative) relation among events.

Definition 5.4. For any events $E, F \in \mathcal{F}$, say that E is *weakly more probable* than F, written $E \succeq F$ (or $F \preceq E$), if, for any $a, b \in X$ with $a \succsim b$,

$$\mathfrak{c}_a \oplus_E \mathfrak{c}_b \succsim \mathfrak{c}_a \oplus_F \mathfrak{c}_b \tag{5.4}$$

(or equivalently if $\mathfrak{c}_b \oplus_F \mathfrak{c}_a \succsim \mathfrak{c}_b \oplus_E \mathfrak{c}_a$). E and F are said to be *equally probable*, in symbols $E \simeq F$, if both $E \succeq F$ and $F \succeq E$ hold.

The definition says that the agent's belief that E is more probable than F is manifested in her preference for the compound act $\mathfrak{c}_a \oplus_E \mathfrak{c}_b$ which, in turn, is determined by the agent's subjective estimation of the likelihood of obtaining the more favorable act $\mathfrak{c}_a$. (A postulate, i.e., SVG 4, is inserted in order to ensure that the notion of one event being more probable than another is well defined, that is, the definition in (5.4) does not depend on the choice of a, b.)

Remark. 1. Savage's "simple ordering" is, in our terminology, a total preorder. He uses 'F' for the set of consequences and he characterizes total preorders as "simple orderings". In particular, he uses boldface letters $\mathbf{f}$, $\mathbf{g}$, ... for acts and italics f, g, ... for values of "acts that are constant", writing $\mathbf{f} \equiv g$ when $\mathbf{f}(s) = g$ for all states s. He also uses 'f' for constant act whose value is f.

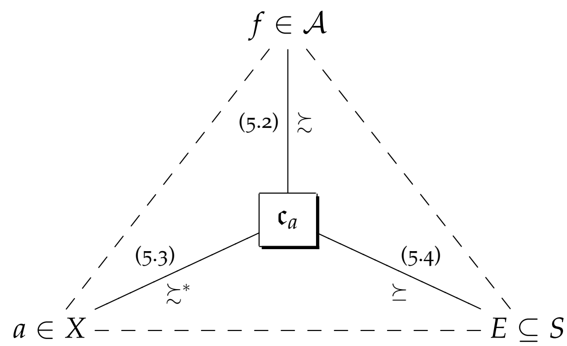

FIGURE 5.1. Constant act $\mathfrak{c}_a$ and other parameters in Savage's model.

Furthermore, he sometimes switches to italicized notation even when the function is not constant, as he does in the statement of P4 on p.31, where he writes $f_A(s)$ instead of $\mathbf{f}_A(s)$, or in Theorem 1 on page 70, where he writes $f(s) = f_i$ instead of $\mathbf{f}(s) = f_i$ as he should.

2. As seen from the definitions above, the construction of constant acts $\mathfrak{c}_a$ $(a \in X)$ plays a central role in associating various concepts in Savage's decision model, and it is through the notion of constant acts that different binary relations are interrelated (see Figure 5.1). This notion, however, is highly problematic. We shall address these issues brought by the assumption of the existence of constant acts (one for each consequence) in great detail in Chapter IV where we provide a simplification of Savage's theory without appealing to constant acts. The exposition in this chapter however will still use constant acts.

The goal is to show that the defined "more probable" relation $\succeq$ is a qualitative probability (will be made precise below) and that there exists a unique numerical probability measure μ on $(S, \mathcal{F})$ such that[4]

$$E \succeq F \iff \mu(E) \geq \mu(F), \quad \text{for all } E, F \in \mathcal{F}, \tag{5.5}$$

[4] Savage stated explicitly that in his theory probability is defined for all events which are taken to be *all* subsets of S, and hence $\mathcal{A} = X^S$ (Savage, 1972, p.40). Due to the reasons to be discussed in Chapter III, in our reconstruction, we restrict attention to "measurable acts." That is to say, given measurable spaces $(S, \mathcal{F})$ and $(X, \mathcal{G})$ where $\mathcal{F}$ and $\mathcal{G}$ are some σ-algebras equipped on S and X, respectively, we consider only those functions (acts) that are measurable $\mathcal{F}/\mathcal{G}$.

and that there is a real-valued function u on X for which

$$f \succsim g \iff \int_S u\big[f(s)\big]\,\mu(ds) \geq \int_S u\big[g(s)\big]\,\mu(ds), \tag{5.6}$$

for all $f, g \in \mathcal{A}$, where u is unique up to a positive transformation. This is Savage's the main *representation theorem* we seek to prove, which will be discussed in Section 6 and Section 7. But before proceeding to detailed reconstruction, let us first analyze Savage's well-known "sure-thing" principle and its formal configurations.

5.2. The sure-thing principle and postulate 2. The cornerstone of Savage's decision model is a postulated rationality principle known as the "sure-thing principle". The following is the example used by Savage to motivate this principle.

Example 5.5 (Businessman). A businessman contemplates buying a certain piece of property. He considers the outcome of the next presidential election relevant to the attractiveness the purchase. So, to clarify the matter for himself, he asks whether he would buy if he knew that the Republican candidate were going to win, and decides that he would do so. Similarly, he considers whether he would buy if he knew that the Democratic candidate were going to win, and again finds that he would do so. Seeing that the would buy in either event, he decides that the should buy, even though he does not know which event obtains, or will obtain, as we would ordinarily say. ◁

As illustrated in this example, the principle stems from an intuitive idea of reasoning by cases that if a decision maker takes certain course of action given the occurrence of some event and she will do the same if the event does *not* occur, then she shall proceed with *that* action without taking into account as to whether or not the event takes place, in other words, the implementation of the course of action is a "sure-thing". To state in terms of preferences over acts, the sure-thing principle says that

STP: If the decision maker prefers one act over another assuming either certain event obtains or that the compliment of the event obtains, then her preference over the two acts shall remain unchanged.

The principle is sometimes referred to as the *dominance principle*, which can be stated more generally as follows: if the state space is partitioned into n-many mutually exclusive cells, which represent n different decision situations, and if the consequence of one act weakly dominates that of another in each one of these possible situations, then the act is weakly preferred throughout. Savage takes this consideration to be fundamental to rational decision making: "I," he says, "know of no other extra-logical principle governing decisions that finds such ready acceptance" (*ibid.* p.21).

5.2.1. *Conditional preference.* Note that the statement of the sure-thing principle above employs explicitly a concept of conditional preference, that is, one act being preferred to another *given* the occurrence of certain event. Since the current formal setup is based entirely on *unconditional* preferences over acts, the notion of conditional preference is not directly expressible. Some alternative arrangements hence need to be made.

Definition 5.6 (Conditional preference). Let E be some event, then, given acts $f, g \in \mathcal{A}$, f is said to be weakly preferred to g *given* E, written $f \succsim_E g$, if for *all* pairs of acts $f', g' \in \mathcal{A}$ the following condition is satisfied

$$\left.\begin{array}{ll} f(s) = f'(s),\, g(s) = g'(s) & \text{if } s \in E \\ f'(s) = g'(s) & \text{if } s \in E^C \end{array}\right\} \implies f' \succsim g'. \tag{5.7}$$

That is to say, the conditional preference of f over g on the occurrence of event E is defined in terms of all unconditional preferences of f' over g' through the conditions that f' and g' agree, respectively, with f and g on event E and with each other on E^C, and that f' unconditionally weakly preferred to g'. Table 5.2a contains an illustration of conditional preference, where $\{E, E^C\}$ forms a simple partition of S for which $f(s) = a$ for all $s \in E$ and $f(s) = c$ for $s \in E^C$ (other

Table 5.2. Illustrations of

(a) Conditional preference

	E	E^C
f	a	c
g	b	d
f'	a	e
g'	b	e

(b) Savage's postulate 2

	E	E^C
f	a	c
g	b	c
f'	a	d
g'	b	d

acts defined similarly). Then the definition of conditional preference says that f is weakly preferred to g given E if $f' \succsim g'$ for all such f's and g's. Now, given the definition of conditional preference, **STP** can be translated into[5]

$$\left[f \succsim_E g, f \succsim_{E^C} g\right] \implies f \succsim g. \tag{STP}$$

Savage, however, was unwilling to incorporate (STP) directly into his system for the concern that the concept of conditional preference is based on knowledge of the possible occurrences of some events, the introduction of which may lead to, it is said, unsought philosophical complications.[6] Instead, he posited a different principle which is a technical approximation to **STP** known as the *formal* version of the sure-thing principle, and he left **STP** itself as an *informal* or, to use his phrase, a "loose" version of the sure-thing principle. This alternative principle contains no direct reference to conditional preferences and is officially stated as his second postulate (P2) for rational decision making, which says that, for any acts f, g, h, h' and for any event E,

$$f \oplus_E h \succsim g \oplus_E h \iff f \oplus_E h' \succsim g \oplus_E h', \tag{P2}$$

[5]In what follows, we use the boldface **STP** to refer to the informal statement of the principle and use (STP) to refer to its formulation in the formal model, same for **P2** and (P2) below.

[6] Savage (1972, p.22) explains: "The sure-thing principle [i.e., **STP** above] cannot appropriately be accepted as a postulate in the sense that P1 is, because it would introduce new undefined technical terms referring to knowledge and possibility that would refer it mathematically useless without still more postulates governing these terms." See Gaifman (2013, p.375) for a critique of this line of argument, where it is pointed out that Savage is guilty of confusing hypothetical reasoning with counterfactual knowledge: it is the former, not the latter, that is involved in formulating the sure-thing principle, which is conceptually transparent and non-problematic.

As the example in Table 5.2b illustrates, if f' and g' agree, respectively, with f and g on E and with each other on E^C, then (P2) mandates $f \succsim g$ iff $f' \succsim g'$. Here we remark that one technical motivation for imposing (P2) is to provide a provision to the definition of conditional preference in Definition 5.6 so that the notion is well defined. (Notice that, in the absence of (P2), an act f may fail to be conditionally preferred to another act g (i.e. $f \not\succsim_E g$) if there exist two pairs of acts (f', g') and (f'', g'') satisfying both conditions (1) and (2) for which $f' \succsim g'$ and $f'' \not\succsim g''$. This possibility for f to fail to be conditionally preferred to g is excluded by (P2), under which $f \not\succsim_E g$ if and only if, for all f' and g' satisfying (1) and (2), $f' \not\succsim g'$.) Beyond this technical reason for invoking (P2) as an additional constraint on the notion of conditional preferences, the rationale behind (P2) as a self-standing rationality principle may be characterized as follows

> **P2:** If the consequences of two acts differ on the occurrence of some event E but otherwise agree with each other, then their preferential comparison between these two acts shall be decided on those states in E and their corresponding consequences.

What underlies this principle seems to be the simple consideration that the difference between any two items is distinguished by the parts where they are actually different. Then **P2** implies that if two sets of decision problems have identical payoff structures on E but otherwise have respectively in-differentiable payoffs on E^C then if an option is preferred in the first set of decision problem it should also be favored in the second. Yet, we stress that, even with the presence of this compelling intuition, **P2** is after all a different and, in fact, more restrictive principle than **STP**. We illustrate this point by showing that (P2) is strictly stronger than (STP) in the current formal model.[7]

LEMMA 5.7. Let $\succsim$ be a preorder on $\mathcal{A}$, then (P2) implies (STP).

[7] Gaifman (2013, p.376) outlined a general method of distinguishing STP from P2 in a partial-act based system, where a partial-act is a (partial) function defined on some event and maybe undefined on other events. And it was shown that the counterpart of **P2** in a partial-act system is *independent of* that of **STP**. Here, we point out that, as far as Savage's total-act system is concerned, (P2) does imply (STP), but not vice versa.

Proof. Assuming (P2), it is easily seen that the definition of conditional preference can be equivalently stated as follows

$$f \succsim_E g \iff f \oplus_E h \succsim g \oplus_E h, \quad \text{for all } h \in \mathcal{A}. \tag{5.8}$$

Then the left-hand side of (STP) yields, via (5.8), that

$$f \oplus_E h \succsim g \oplus_E h \tag{5.9}$$

$$f \oplus_{E^C} h' \succsim g \oplus_{E^C} h'. \tag{5.10}$$

where h and h' are arbitrary acts in $\mathcal{A}$. Now, in (5.9), substitute h with $h \oplus_E f$, then, by (P2), we get $f = f \oplus_E (h \oplus_E f) \succsim g \oplus_E (h \oplus_E f) = g \oplus_E f$. Similarly, in (5.10), replace h' with $g \oplus_E h'$, then $f \oplus_{E^C} g = f \oplus_{E^C} (g \oplus_E h') \succsim g \oplus_{E^C} (g \oplus_E h') = g$. Together we have that $f \succsim g \oplus_E f$ and $f \oplus_{E^C} g \succsim g$. Note that, by Lemma 5.2(1), $g \oplus_E f = f \oplus_{E^C} g$, therefore, by transitivity of $\succsim$, we have $f \succsim g$. □

The converse, however, does not necessarily hold, that is, there are situations in which (STP) is satisfied but (P2) is violated as shown in the following example.

Example 5.8. Let $S = \{s_1, s_2\}$, $X = \{a, b\}$. Then there are four possible acts mapping from S to X as illustrated in the table below.[8]

	s_1	s_2
f_1	a	a
f_2	b	a
f_3	a	b
f_4	b	b

Consider the case where $f_1 \succsim f_2 \sim f_3 \prec f_4$. Then it is easy to see that P2 is violated but (STP) is trivially satisfied. (This is because our example is so arranged that, for any acts $f, g \in \{f_1, f_2, f_3, f_4\}$, (1) if f is different from g then, at least one of the conditional preferences $f \succsim_{s_1} g$ and $f \succsim_{s_2} g$ fails,[9] in which case the antecedent of

[8]Strictly speaking, the state space S in Savage system needs to contain uncountably many states (cf. Footnote 2). In writing $S = \{s_1, s_2\}$ we can assume that S is partitioned into two events s_1 and s_2.
[9]We write $f \succsim_{\{s_1\}} g$ as $f \succsim_{s_1} g$ for short, same below.

(STP) is false, and hence the conditional true; (2) if f is identical to g then (STP) is trivially true). ◁

Lemma 5.7 shows that Savage's proposed (P2) is deductively sufficient for enforcing (STP), however, as shown in Example 5.8, it is more demanding than what (STP) is intended for. Let us summarize the above discussion in the following theorem.

Theorem 5.9. Let $\succsim$ be a complete preorder on $\mathcal{A}$, then

(1) (P2) $\Longrightarrow$ (STP),

(2) (STP) $\not\Longrightarrow$ (P2).

To be sure, the reason that (P2) and (STP) are not deductively equivalent in Savage's system is largely due to the peculiar way how conditional preferences are formulated in his model, where the concept of conditional preference and (P2) are essentially interlocked. Gaifman (2013) suggested a way of defining conditional preference in a more straight forward manner so that **STP** can be formulated directly without going through Savage's roundabout way of using mutually dependent notions of conditional preference and (P2). Our discussions and generalizations in later sections will still be made within Savage's framework with total-acts, we, however, emphasize on a clear distinction between **STP** and **P2**, and their formalizations.

5.2.2. *Null events.* Further, an event $E \subseteq S$ is said to be a *null event* if, for any $f, g \in \mathcal{A}$, $f \succsim_E g$, that is, the agent is indifferent between any two acts *given* the occurrence of E. Intuitively, null events are those events whose occurrences take no effect in the agent's decision procedure as the individual believes that it is impossible that they obtain. As we shall soon see, in the current system null events corresponds to those events that receive probability zero. The following is a list of basic properties of null events.

Lemma 5.10. Let E be a null event, then given **P2**,

(1) $E \simeq \varnothing$;

(2) if $f(s) = f'(s)$ and $g(s) = g'(s)$ for all $s \in E^C$, then $f \succsim g$ iff $f' \succsim g'$;

(3) if $f(s) = g(s)$ for all $s \in E^C$, then $f \sim g$.

Proof. (1) Let $a, b \in X$ be such that $a \succsim b$. Since E is null, we have $\mathfrak{c}_a \succsim_E \mathfrak{c}_b$. This implies, by (P2) and (5.8), that

$$\mathfrak{c}_a \oplus_E \mathfrak{c}_b \succsim \mathfrak{c}_b \oplus_E \mathfrak{c}_b = \mathfrak{c}_b = \mathfrak{c}_b \oplus_\varnothing \mathfrak{c}_b.$$

By Definition 5.4, $E \succeq \varnothing$. Similarly, from E being null we get $\mathfrak{c}_b \succsim_E \mathfrak{c}_a$, thence

$$\mathfrak{c}_a \oplus_\varnothing \mathfrak{c}_b = \mathfrak{c}_b \oplus_E \mathfrak{c}_b \succsim \mathfrak{c}_a \oplus_E \mathfrak{c}_b.$$

By definition, $\varnothing \succeq E$. Together, we have $E \simeq \varnothing$.

(2) By symmetry, we show $f \succsim g$ implies $f' \succsim g'$. Note that, since E is null, we have $f' \succsim_E g'$. Then by (STP), we only need to show that $f' \succsim_{E^C} g'$. By the definition of conditional preference and (P2), it's sufficient to show that, there exists some $h \in \mathcal{A}$ such that

$$f' \oplus_{E^C} h \succsim g' \oplus_{E^C} h. \tag{5.11}$$

Since f' and g' agree respectively with f and g on E^C, (5.11) holds iff

$$f \oplus_{E^C} h \succsim g \oplus_{E^C} h.$$

Take h to be f, then the proof is completed if it can be shown that

$$f \succsim f \oplus_E g. \tag{5.12}$$

To this end, note that since E is null, we have $g \succsim_E f \oplus_E g$, it follows, through (5.8), that, for any $t \in \mathcal{A}$, $g \oplus_E t \succsim (f \oplus_E g) \oplus_E t$. Let $t = g$, then this together with the assumption $f \succsim g$ yield (5.12), which is what we want.

(3) This is an easy consequence of (2). □

Lemma 5.10(2) says that given any pairs of acts, if they differ pair-wisely only on events that are considered null then their relative preferences will remain the same (cf. the table below).

TABLE 5.3. $f \succsim g$ iff $f' \succsim g'$

	E (null)	E^C
f	a	e
g	b	f
f'	c	e
g'	d	f

As we shall see, this property plays an important role in deriving a utility function for consequences.

6. Subjective Probability

6.1. Qualitative probability. As the first step of our reconstruction of Savage's expected utility representation theory, we introduce the following concept of qualitative probability:

DEFINITION 6.1 (Qualitative probability). Let S be a nonempty set, a binary relation $\succcurlyeq$ on S is said to be a *qualitative probability* if, for any $A, B, C \in \mathcal{F}$,

i. $\succcurlyeq$ is a weak order (reflexive, transitive, and complete),
ii. $A \succcurlyeq \varnothing$,
iii. $S \succ \varnothing$,
iv. $A \succcurlyeq B$ if and only if $A \cup C \succcurlyeq B \cup C$, provided $A \cap C = B \cap C = \varnothing$.

where $\succ$ is the strict (i.e., the asymmetric) part of $\succcurlyeq$.

We show that if the preference relation $\succsim$ over acts satisfies the following list of axioms postulated by Savage then the binary relation $\succeq$ over events (sets of states) defined in (5.4) is a qualitative probability.[10]

SVG 1. $\succsim$ is a weak order (complete preorder).

[10]SVG 1-5 correspond respectively to P1-5 in Savage (1972), the only difference is that we present these postulates using the notations adopted here, same for SVG 6 and SVG 7 below.

SVG 2. For any $f, g \in \mathcal{A}$ and for any $E \subseteq S$, $f \succsim_E g$ or $g \succsim_E f$.

SVG 3. For any $a, b \in X$ and for any non-null event $E \subseteq S$, $\mathfrak{c}_a \succsim_E \mathfrak{c}_b$ if and only if $a \succsim b$.

SVG 4. For any $a, b, c, d \in X$ satisfying $a \succsim b$ and $c \succsim d$ and for any events $E, F \subseteq S$, $\mathfrak{c}_a \oplus_E \mathfrak{c}_b \succsim \mathfrak{c}_a \oplus_F \mathfrak{c}_b$ if and only if $\mathfrak{c}_c \oplus_E \mathfrak{c}_d \succsim \mathfrak{c}_c \oplus_F \mathfrak{c}_d$.

SVG 5. For some constant acts $\mathfrak{c}_a, \mathfrak{c}_b \in \mathcal{A}$, $\mathfrak{c}_b \succ \mathfrak{c}_a$.

Here, SVG 1 says that the preference relation is reflexive, transitive, and complete, in other words, it is assumed in Savage's system that all acts are pairwise comparable. SVG 2 can be easily derived from the completeness assumption and (P2), which says that the conditional preference relation over acts is definable for any given event and is complete. The next two postulates are commonly known as the "independence axioms" which place further assumptions that the agent's probabilistic estimations over events and value judgments on consequences are, generally speaking, mutually independent: SVG 3 says that the preference ranking of constant acts is solely dependent on the values of their respective consequences which are robust against all states and SVG 4 says that the agent's qualitative probability estimations are in independent of his value judgments over consequences (and that the relation "more probable" in Definition 5.4 is well defined). SVG 5 is brought in in order to rule out the trivial case where the agent is indifferent among all consequences (constant acts). With these postulates in hand, let's proceed to show the following preparatory results.

Lemma 6.2. For any consequences $a, b \in X$ and for any event $E \in \mathcal{F}$, if $a \succsim b$ then $\mathfrak{c}_a \succsim \mathfrak{c}_a \oplus_E \mathfrak{c}_b \succsim \mathfrak{c}_b$.

Proof. Given $a \succsim b$, we have that $\mathfrak{c}_a \succsim \mathfrak{c}_b$ by (5.3). Let E be any non-null event, then, by SVG 3, $\mathfrak{c}_a \succsim_E \mathfrak{c}_b$ (this holds trivially if E is a null event); from which we get, through (5.8), that for any $h \in \mathcal{A}$,

$$\mathfrak{c}_a \oplus_E h \succsim \mathfrak{c}_b \oplus_E h. \tag{6.1}$$

Let $h = \mathfrak{c}_b$, that is $\mathfrak{c}_a \oplus_E \mathfrak{c}_b \succsim \mathfrak{c}_b \oplus_E \mathfrak{c}_b = \mathfrak{c}_b$. This shows that $\mathfrak{c}_a \oplus_E \mathfrak{c}_b \succsim \mathfrak{c}_b$. Similarly, one can show $\mathfrak{c}_a \succsim \mathfrak{c}_b \oplus_{E^C} \mathfrak{c}_a$ by replacing h and E in (6.1) with $\mathfrak{c}_a$ and E^C, respectively. Then, by Lemma 5.2(1), $\mathfrak{c}_a \succsim \mathfrak{c}_a \oplus_E \mathfrak{c}_b$. □

LEMMA 6.3. For any $E, F \subseteq S$, if $F \subseteq E$ then $E \succeq F$.

PROOF. For any $a, b \in X$, assume that $a \succsim b$, and hence $\mathfrak{c}_a \succsim \mathfrak{c}_b$, then by Lemma 6.2, $\mathfrak{c}_a \succsim \mathfrak{c}_a \oplus_F \mathfrak{c}_b$. By SVG 2, for $\mathfrak{c}_a$ and $\mathfrak{c}_a \oplus_F \mathfrak{c}_b$, at least one of the following two conditions holds,

(i) $\mathfrak{c}_a \oplus_F \mathfrak{c}_b \succsim_E \mathfrak{c}_a$;

(ii) $\mathfrak{c}_a \succsim_E \mathfrak{c}_a \oplus_F \mathfrak{c}_b$.

Suppose that (i) is the case, then, by (5.8), for any $h \in \mathcal{A}$, $(\mathfrak{c}_a \oplus_F \mathfrak{c}_b) \oplus_E h \succsim \mathfrak{c}_a \oplus_E h$. Let $h = \mathfrak{c}_a$, we have $\mathfrak{c}_a \oplus_{E^C \cup F} \mathfrak{c}_b \succsim \mathfrak{c}_a$. On the other hand, given set $E^C \cup F$, Lemma 6.2 implies that $\mathfrak{c}_a \succsim \mathfrak{c}_b \oplus_{E^C \cup F} \mathfrak{c}_b$. Together, we have

$$\mathfrak{c}_a \oplus_{E^C \cup F} \mathfrak{c}_b \sim \mathfrak{c}_a. \tag{6.2}$$

Note that (6.2) holds for all $a, b \in X$ and all $E, F \subseteq S$ with $F \subseteq E$. Then let $E = S$ and $F = \varnothing$, from which we get $\mathfrak{c}_a \sim \mathfrak{c}_b$ for all $a, b \in \mathcal{A}$. But this is impossible if SVG 5 is in place.

The remaining possibility is (ii). In this case it follows, again by (5.8), that, for any $h \in \mathcal{A}$, $\mathfrak{c}_a \oplus_E h \succsim (\mathfrak{c}_a \oplus_F \mathfrak{c}_b) \oplus_E h$. Let $h = \mathfrak{c}_b$, we get $\mathfrak{c}_a \oplus_E \mathfrak{c}_b \succsim (\mathfrak{c}_a \oplus_F \mathfrak{c}_b) \oplus_E \mathfrak{c}_b$. Apply Lemma 5.2(2),

$$\mathfrak{c}_a \oplus_E \mathfrak{c}_b \succsim (\mathfrak{c}_a \oplus_F \mathfrak{c}_b) \oplus_E \mathfrak{c}_b = \mathfrak{c}_a \oplus_{F \cap E} \mathfrak{c}_b = \mathfrak{c}_a \oplus_F \mathfrak{c}_b.$$

This yields that $\mathfrak{c}_a \oplus_E \mathfrak{c}_b \succsim \mathfrak{c}_a \oplus_F \mathfrak{c}_b$, hence, by Definition 5.4, $E \succeq F$. □

THEOREM 6.4. If the preference relation $\succsim$ on $\mathcal{A}$ satisfies SVG 1-5 then the relation $\succeq$ over events is a qualitative probability.

PROOF. We prove by direct verifications that $\succeq$ as defined in Definition 5.4 satisfies conditions i-iv in Definition 6.1.

i. Suppose that $E \succeq E' \succeq E''$, we show $E \succeq E''$. By definition, for any $a, b \in Z$ with $a \succsim b$, we have that $\mathfrak{c}_a \oplus_E \mathfrak{c}_b \succsim \mathfrak{c}_a \oplus_{E'} \mathfrak{c}_b \succsim \mathfrak{c}_a \oplus_{E''} \mathfrak{c}_b$; then, by the transitivity of $\succsim$ (SVG 1), we get $\mathfrak{c}_a \oplus_E \mathfrak{c}_b \succsim \mathfrak{c}_a \oplus_{E''} \mathfrak{c}_b$, this yields that $E \succeq E''$. Hence, $\succeq$ is transitive. Completeness can be shown similarly.

ii. In Lemma 6.3 let $F = \varnothing$, we get $E \succeq \varnothing$ for all $E \subseteq S$.

iii. Let $a, b \in X$ be such that $\mathfrak{c}_a \succ \mathfrak{c}_b$ (i.e., $\mathfrak{c}_a \succsim \mathfrak{c}_b$ but $\mathfrak{c}_b \not\succsim \mathfrak{c}_a$, the existence of the pair is guaranteed by SVG 5). Suppose, to the contrary, that $\varnothing \succeq S$. Then, by (5.4), $\mathfrak{c}_a \oplus_\varnothing \mathfrak{c}_b \succsim \mathfrak{c}_a \oplus_S \mathfrak{c}_b$. On the other hand, note that $\mathfrak{c}_a \oplus_\varnothing \mathfrak{c}_b = \mathfrak{c}_b$ and $\mathfrak{c}_a \oplus_S \mathfrak{c}_b = \mathfrak{c}_a$, hence we have $\mathfrak{c}_b \succsim \mathfrak{c}_a$, a contradiction. Therefore, $S \succ \varnothing$.

iv. Suppose $E \succeq E'$ and let F be such that $E \cap F = E' \cap F = \varnothing$, we show $E \cup F \succeq E' \cup F$. By definition, for any $a, b \in X$ with $a \succsim b$, we have that $\mathfrak{c}_a \oplus_E \mathfrak{c}_b \succsim \mathfrak{c}_a \oplus_{E'} \mathfrak{c}_b$. Further, by SVG 2, one of the following is true,

(a) $\mathfrak{c}_a \oplus_{E'} \mathfrak{c}_b \succsim_{F^C} \mathfrak{c}_a \oplus_E \mathfrak{c}_b$;

(b) $\mathfrak{c}_a \oplus_E \mathfrak{c}_b \succsim_{F^C} \mathfrak{c}_a \oplus_{E'} \mathfrak{c}_b$.

Suppose that (a) is the case, this implies that, for any $h \in \mathcal{A}$,

$$\left(\mathfrak{c}_a \oplus_{E'} \mathfrak{c}_b\right) \oplus_{F^C} h \succsim \left(\mathfrak{c}_a \oplus_E \mathfrak{c}_b\right) \oplus_{F^C} h$$

Since $E \cap F = E' \cap F = \varnothing$, let $h = \mathfrak{c}_b$, we get, via Lemma 5.2(2),

$$\mathfrak{c}_a \oplus_{E'} \mathfrak{c}_b = \mathfrak{c}_a \oplus_{E' \cap F^C} \mathfrak{c}_b \succsim \mathfrak{c}_a \oplus_{E \cap F^C} \mathfrak{c}_b = \mathfrak{c}_a \oplus_E \mathfrak{c}_b$$

By definition, we have $E' \succeq E$. This, together with the assumption $E \succeq E'$, imply that for any $E, E' \subseteq S$, $E \succeq E'$ iff $E' \succeq E$, which contradicts (iii). The remaining possibility is (b), for which we have that, for any $h \in \mathcal{A}$,

$$\left(\mathfrak{c}_a \oplus_E \mathfrak{c}_b\right) \oplus_{F^C} h \succsim \left(\mathfrak{c}_a \oplus_{E'} \mathfrak{c}_b\right) \oplus_{F^C} h.$$

Let $h = \mathfrak{c}_a$. Then, by Lemma 5.2(1),

$$\mathfrak{c}_a \oplus_F \left(\mathfrak{c}_a \oplus_E \mathfrak{c}_b\right) \succsim \mathfrak{c}_a \oplus_F \left(\mathfrak{c}_a \oplus_{E'} \mathfrak{c}_b\right).$$

This yields, via Lemma 5.2(3), that $\mathfrak{c}_a \oplus_{E\cup F} \mathfrak{c}_b \succsim \mathfrak{c}_a \oplus_{E'\cup F} \mathfrak{c}_b$, and hence, by Definition 5.4, $E \cup F \succeq E' \cup F$. □

This completes the proof that the "more probable" relation $\succeq$ among events is indeed a qualitative probability. Before moving to show that there exists a unique probability measure that agrees with $\succeq$, let us explore some properties of qualitative probabilities which will become handy later.

Corollary 6.5. Let $\succeq$ be as in Definition 5.4, then for any $E, E', F, F' \subseteq S$ the following hold:

(1) if $F \succeq E$ and $F \cap F' = \varnothing$, then $F \cup F' \succeq E \cup F'$;
(2) if $\varnothing \succeq E$, then $E \cup F' \simeq F'$;
(3) if E is a null event, then $E \cup F \simeq F$;
(4) if $F \succeq E, F' \succeq E'$ and $F \cap F' = \varnothing$, then $F \cup F' \succeq E \cup E'$;
(5) if $F \cup F' \succeq E \cup E'$ and $E \cap E' = \varnothing$, then either $F \succeq E$ or $F' \succeq E'$;
(6) if $E^C \succeq E$ and $F \succeq F^C$, then $F \succeq E$.

Proof. By Theorem 6.4, $\succeq$ is a qualitative probability, and hence satisfies conditions (*a*)–(*d*) in Definition 6.1.

(1) Let $E_1 = E - F'$, then $E \cup F' = E_1 \cup F'$. From $E_1 \subseteq E$ and the assumption $F \succeq E$ it follows from Lemma 6.3 that $F \succeq E \succeq E_1$, hence $F \cup F' \succeq E_1 \cup F'$, that is, $F \cup F' \succeq E \cup F'$.
(2) In (1) let $F = \varnothing$, then $F' \succeq E \cup F'$. On the other hand $E \cup F' \succeq F'$ via Lemma 6.3. Hence $E \cup F' \simeq F'$.
(3) This is a direct consequence of (2) and Lemma 5.10(1).
(4) Let $F = A \cup Q, E' = B \cup Q$ where $A = F - E', B = E' - F$ and $Q = F \cap E'$, hence $F \cup B = E' \cup A = A \cup B \cup Q$. Since $B \cap F = \varnothing$, it follows from the assumption $F \succeq E$ and (1) that $E' \cup A = F \cup B \succeq E \cup B$. On the other hand, $A \subseteq F$ and $F \cap F' = \varnothing$ hence $A \cap F' = \varnothing$, then from $F' \succeq E'$ it follows that $F' \cup A \succeq E' \cup A$ via (1). Together we have $F' \cup A \succeq E \cup B$.

Finally, since $Q \cap (F' \cup A) = \varnothing$ we have, again by (1),

$$F \cup F' = F' \cup A \cup Q \succeq E \cup B \cup Q = E \cup E'.$$

(5) Otherwise, $E \succ F$ and $E' \succ F'$ which imply $E \succeq F$ and $E' \succeq F'$, then by (4), $E \cup E' \succeq F \cup F'$. It follows that $F \cup F' \simeq E \cup E'$ for all subsets E, E', F, F' of S with $E \cap E' = \varnothing$, which is absurd.

(6) Let $\{A, B, C, D\}$ be a partition of S such that $F = A \cup B, F^C = C \cup D, E = A \cup C$, and $E^C = B \cup D$. By assumption $E^C = B \cup D \succeq C \cup A = E$, this implies, through (5), that either $B \succeq C$ or $D \succeq A$:

i. If $B \succeq C$, it follows from the fact that A, B are disjoint that $B \cup A \succeq C \cup A$ via (1), and hence $F \succeq E$.

ii. If $D \succeq A$, also $F = A \cup B \succeq C \cup D = F^C$, then by (4) above, $A \cup B \cup D \succeq A \cup C \cup D$. It follows that $B \succeq C$ via (1), hence back to case (i).

Therefore, $F \succeq E$. □

It is easy to verify that Corollary 6.5 (1) and (4)-(6) continue to hold with '$\succ$' in place of '$\succeq$.' The following observations are easy consequences of Corollary 6.5 which will be useful in the proof of the existence of numerical probability representation below.

Corollary 6.6. Let $\{E_i\}_{i=1}^n$ and $\{F_j\}_{j=1}^n$ be partitions of S,

(1) if $\{E_i\}_{i=1}^n$ and $\{F_j\}_{j=1}^n$ are so indexed that $E_1 \preceq \cdots \preceq E_n$ and $F_1 \succeq \cdots \succeq F_n$, then for any $r = 1, \ldots, n$,

$$\bigcup_{j=1}^{r} F_j \succeq \bigcup_{i=1}^{r} E_i; \tag{6.3}$$

(2) if in addition $E_i \simeq E_j$ and $F_i \simeq F_j$ for all $i, j \in \{1, \ldots, n\}$, i.e., if $\{E_i\}_{i=1}^n$ and $\{F_j\}_{j=1}^n$ partition S into n equally probable events, then

$$\bigcup_{i=1}^{r} E_i \simeq \bigcup_{j=1}^{r} F_j. \tag{6.4}$$

PROOF. (1) We prove by induction on r. Note that for the case $r = 1$, it must be that $F_1 \succeq E_1$. For, otherwise $E_1 \succ F_1 \succeq \cdots \succeq F_n$. It follows that $E_i \succ F_i$ for all $i = 1, \ldots, n$, and hence $\bigcup_{i=1}^n E_i = S \succ \bigcup_{i=1}^n F_j = S$ (this is obtained by repeatedly applying the "$\succ$-version" of Corollary 6.5(4) since E_i's are mutually disjoint), a contradiction.

In the inductive step, assume that (6.3) holds for r, we show that it holds for the case $r+1$. Suppose, to the contrary, that $\bigcup_{i=1}^r E_i \cup E_{r+1} \succ \bigcup_{i=1}^r F_j \cup F_{r+1}$. Then, by the inductive hypothesis, $\bigcup_{i=1}^r F_j \succeq \bigcup_{i=1}^r E_i$, hence by Corollary 6.5(5) it must be that $E_{r+1} \succ F_{r+1}$. It follows that $E_i \succ F_i$ for all $i = r+1, \ldots, n$. This together with $\bigcup_{i=1}^{r+1} E_i \succ \bigcup_{i=1}^{r+1} F_j$ imply that $\bigcup_{i=1}^n E_i = S \succ \bigcup_{i=1}^n F_j = S$ via Corollary 6.5(4) , which is impossible.

(2) This is a direct consequence of (1) above. □

REMARK 6.7. Kraft et al. (1959) showed, through a counter example, that, contrary to what de Finetti (1951) had conjectured, the four conditions in Definition 6.1 are insufficient to bring about a numerical representation of $\succeq$ in the sense of (5.5) even when $|S|$ is finite. They then showed the extra condition that is needed in order that the probable relation be represented by a numerical probability in finite cases (see also Scott, 1964). We shall not pursue this direction here. In what follows, we study Savage's approach to the problem, which is more general, for it also treats infinite cases.

6.2. Quantitative probability. In this section we show that the qualitative probability relation derived from SVG 1-5 in Theorem 6.4 admits a unique numerical representation provided that an additional postulate is inserted. That is, we show that there is a unique probability measure μ on $(S, \mathcal{F})$ such that

$$E \succeq F \iff \mu(E) \geq \mu(F), \quad \text{for all } E, F \in \mathcal{F}, \tag{6.5}$$

In this case, we say that the probability measure μ *agrees* with the qualitative probability $\succeq$, and say that μ *almost agrees* with $\succeq$ if only the '$\Rightarrow$' direction of (6.5) holds. The representation rests on the following postulate.

SVG 6. For any $f, g \in \mathcal{A}$ and for any $a \in X$, if $f \succ g$ then there is a finite partition $\{P_i\}_{i=1}^n$ such that, for all i, $\mathfrak{c}_a \oplus_{P_i} f \succ g$ and $f \succ \mathfrak{c}_a \oplus_{P_i} g$.

The postulate says that if f is strictly preferred to g, then there exists a partition such that the preferential relation remains the same if f (g) is revised on a cell of the partition with a constant $\mathfrak{c}_a$. This postulate is a version of *continuity* axiom which is structurally similar to vNM 3 and A-A 3. It amounts to saying that the state space can be arbitrarily divided so that the revision of an act with respect to a constant-act on any member of the revision is considered as preferentially insignificant. As we shall soon see, SVG 6 imposes sufficient structural constraint on the system that facilities a numerical representation.

6.2.1. *Fineness and tightness.* As a show of the strength of SVG 6, let us first make the following observations.

LEMMA 6.8. Let $\succeq$ be a qualitative probability satisfying SVG 6, and E, F be any events. Suppose that $F \succ E$, then there exists a partition $\{P_i\}_{i=1}^n (n \leq \infty)$ of S such that $F \succ E \cup P_i$, for all $i = 1, \ldots, n$.

PROOF. By Definition 5.4, $F \succ E$ implies that, for any $a, b \in X$ with $a \succ b$, $\mathfrak{c}_a \oplus_F \mathfrak{c}_b \succ \mathfrak{c}_a \oplus_E \mathfrak{c}_b$. Now, in SVG 6 let $\mathfrak{c}_a \oplus_F \mathfrak{c}_b$ be in the place of f and let $\mathfrak{c}_a \oplus_E \mathfrak{c}_b$ be in that of g, then there exists a finite partition $\{P_i\}_{i=1}^n$ such that, for all i,

$$\mathfrak{c}_a \oplus_F \mathfrak{c}_b \succ \mathfrak{c}_a \oplus_{P_i} \big(\mathfrak{c}_a \oplus_E \mathfrak{c}_b\big).$$

By Lemma 5.2(3), it follows that

$$\mathfrak{c}_a \oplus_F \mathfrak{c}_b \succ \mathfrak{c}_a \oplus_{E \cup P_i} \mathfrak{c}_b.$$

Hence, by definition, $F \succ E \cup P_i$. ☐

Lemma 6.9. Given any two events E and F, if, for any non-null events G, H satisfying $E \cap G = F \cap H = \varnothing$, $E \cup G \succeq F$ and $F \cup H \succeq E$, then $E \simeq F$.

Proof. Suppose, to the contrary, that there exist E, F such that $E \succ F$ for all non-null G, H satisfying $E \cap G = F \cap H = \varnothing$, $E \cup G \succeq F$ and $F \cup H \succeq E$. Then by Lemma 6.8 there exists a partition $\{P_i\}_{i=1}^n (n \leq \infty)$ of S such that $E \succ F \cup P_i$ for all $i = 1, \ldots, n$. For each P_i, if $F \cap P_i \neq \varnothing$ then split it into two cells $F - P_i$ and $P_i - F$, then we can refine partition $\{P_i\}_{i=1}^n$ with a new partition $\{P'_j\}_{j=1}^m$ such that, for each new cell P'_j one of the following conditions holds

$$F \cap P'_j = \varnothing \text{ or } P'_j \subseteq F. \tag{6.6}$$

Since each P'_j is a subset of some P_i, by Lemma 6.3, $E \succ F \cup P_i$ implies that

$$E \succ F \cup P'_j \text{ for all } j = 1, \ldots, m.$$

Note that if $F \cap P'_j = \varnothing$ then P'_j must be null, for otherwise, by hypothesis, we have that $F \cup P'_j \succsim E$, a contradiction. By (6.6), it follows that the only non-null cells of $\{P'_j\}_{j=1}^m$ are the ones contained in F, then, by Lemma 6.5(3), we have

$$E \succ F \succeq F \cup \bigcup_j P'_j = S$$

which is impossible. Hence $E \not\succ F$. Similarly, it can be shown that $F \not\succ E$. □

Note that, in Lemma 6.8, let $E = \varnothing$, then we have that, for any $F \succ \varnothing$, there is a partition of S such that no element of which is as probable as F. In this case, we say that the qualitative probability $\succeq$ is *fine*. The property presented in Lemma 6.9 is often referred to as the *tightness* condition of $\succeq$. The above shows that both fineness and tightness are guaranteed if SVG 6 is in place.

6.2.2. *Savage's triples.* Next we show some further consequences of SVG 6. These properties reveal some fine structures the qualitative probability $\succeq$ under SVG 1-6.

Lemma 6.10. Let E, F, G, H, K be events, the following properties hold:

(1) if $E \succ \varnothing$ then E can be partitioned into G, H such that $G, H \succ \varnothing$;

(2) if E, K, F are pairwise disjoint with $E \cup K \succ F \succeq E$, then K can be partitioned into G, H such that $E \cup H \succ F \cup G$;

(3) if E, F are such that $E, F \succ \varnothing$ and $E \cap F = \varnothing$ then F can be partitioned into G, H for which $E \cup G \succeq H \succeq G$.

Proof. (1) By SVG 6, there exists some partition $\{P_i\}$ such that $E \succ P_i$, and hence, by Lemma 6.3, $E \succ E \cap P_i$ for all $i = 1, \dots, n$. Suppose that $E \cap P_i \simeq \varnothing$ for all i's, then by Corollary 6.5(2), $E \simeq \varnothing$, a contradiction. Suppose that there is only one P_i such that $E \cap P_i \not\simeq \varnothing$, then we have $E \simeq E \cap P_i$, again, a contradiction. Hence there are at least two cells P_i, P_j such that $E \cap P_i \not\simeq \varnothing$, $E \cap P_j \not\simeq \varnothing$, in which case let $G = E \cap P_i$ and $H = E - G$.

(2) If $E \succeq F$, by (1), F can be partitioned into $G, H \succ \varnothing$ with $H \succeq G$, in which case the claim follows trivially. Otherwise, $F \succ E \succ \varnothing$, then by SVG 6, there exists a n-partition $\{P_i\}$ such that $E \succ P_i$ and hence $E \succ P_i \cap F$ for $i = 1, \dots, n$. Rename $P_i \cap F$'s as Q_i's and let latter be arranged such that $Q_1 \preceq Q_2 \preceq \cdots \preceq Q_n$. Next, let m be such that

$$\bigcup_{i=1}^{m} Q_i \preceq \bigcup_{i=m+1}^{n} Q_i \preceq \bigcup_{i=1}^{m+1} Q_i \tag{6.7}$$

The existence of such an m is guaranteed by the assumption on $\{Q_i\}$ and the fact that $\succeq$ is a qualitative probability. Then let $G = \bigcup_{i=1}^{m} Q_i$ and $H = \bigcup_{i=m+1}^{n} Q_i$. Then (6.7) yields $G \preceq H \preceq G \cup Q_{m+1}$. Since $E \cap F = \varnothing$ and $E \succ Q_{m+1}$ we get $E \cup G \succeq Q_{m+1} \cup G \succeq H$.

(3) From the assumption $E \cup K \succ F \succeq E$ it is easy to see, via Corollary 6.5(2), that $K \succ \varnothing$. By SVG 6, there exists a n-partition $\{P_i\}$ such that $E \cup K \succ P_i \cup F$ for all i's and that there must be one cell, say P_i, of the partition such that $K \cap P_i \succ \varnothing$, then we have $E \cup K \succ (K \cap P_i) \cup F$. Next, by (1), $K \cap P_i$ can be partitioned into G, G' with $G' \succeq G$, then we have

$$E \cup \big[(K - G) \cup G\big] = E \cup K \succ (K \cap P_i) \cup F = G' \cup G \cup F.$$

This yields $E \cup (K - G) \succ G' \cup F \succeq G \cup F$ (because $G' \succeq G$). Let $H = K - G$, then we get what we want. □

The existence of a numerical probability over events depends on the following construction, which is sometimes referred to as *Savage triples*.

LEMMA 6.11. There exists a sequence of 3-fold partitions $\{C_n, G_n, D_n\}_{n=1}^{\infty}$ of the state space S satisfying

(1) $C_n \cup G_n \cup D_n = S$;
(2) $C_n \cup G_n \succeq D_n$ and $D_n \cup G_n \succeq C_n$;
(3) $C_n \subseteq C_{n+1}$, $D_n \subseteq D_{n+1}$, and $G_n \supseteq G_{n+1}$;
(4) $G_n - G_{n+1} \succeq G_{n+1}$.

PROOF. By Lemma 6.10(1), S can be partitioned into $E, F \succ \varnothing$. Assume, WLOG, that $F \succeq E$ (otherwise, relabel the two events), then, by Lemma 6.10(3), F can be further partitioned into H, G such that $E \cup G \succeq H \succeq G$. Let $C_1 = E$, $G_1 = G$, and $D_1 = H$. Then we have, for the case $n = 1$,

$$
\begin{aligned}
&C_1 \cup G_1 \cup D_1 = E \cup (G \cup H) = E \cup F = S, \\
&C_1 \cup G_1 = E \cup G \succeq H = D_1 \qquad (6.8) \\
&D_1 \cup G_1 = (G \cup H) = F \succeq E = C_1.
\end{aligned}
$$

Next, consider the following cases

a. If $G_1 \simeq \varnothing$ we have, via Corollary 6.5(2), $C_1 \simeq D_1$ then it is plain that the claim is proved if we let $C_n = C_1$ and $D_n = D_1$ for all n's.
b. If $G_1 \succ \varnothing$, we consider two subcases:
   i. If $C_1 \cup G_1 \preceq D_1$, then we have, via (6.8), that $C_1 \cup G_1 \simeq D_1$. Apply Lemma 6.10(3) to C_1 and G_1, we have that G_1 can be partitioned into H, G such that $C_1 \cup G \succeq H \succeq G$. In this case let $C_2 = C_1 \cup H$, $G_2 = G$, and $D_2 = D_1$,

then we have

$$\begin{aligned}
&C_2 \cup G_2 \cup D_2 = (C_1 \cup H) \cup G \cup D_1 = C_1 \cup G_1 \cup D_1 = S \\
&C_2 \supseteq C_2, G_2 \subseteq G_1, D_2 \supseteq D_1, \\
&C_2 \cup G_2 = (C_1 \cup H) \cup G = C_1 \cup G_1 \simeq D_1 = D_2, \\
&D_2 \cup G_2 = D_1 \cup G \simeq C_1 \cup G_1 \succeq C_1 \cup H = C_2, \\
&G_1 - G_2 = H \succeq G = G_2.
\end{aligned} \tag{6.9}$$

ii. Now suppose $C_1 \cup G_1 \succ D_1$, also, from (6.8), we have $D_1 \cup G_1 \succeq C_1$. For the latter, if $D_1 \cup G_1 \simeq C_1$ then we are back to the previous case, otherwise we have $C_1 \cup G_1 \succ D_1$ and $D_1 \cup G_1 \succ C_1$. WLOG, assume that $C_1 \succeq D_1$, apply Lemma 6.10(2), we have that G_1 can be partitioned into H', G' such that $D_1 \cup H' \succeq C_1 \cup G'$. Further, by Lemma 6.10(3), H' can be partitioned into H, G such that $G' \cup H \succeq G \succeq H$. In this case, let $C_2 = C_1 \cup G'$, $G_2 = G$, and $D_2 = D_1 \cup H$, then we have

$$\begin{aligned}
&C_2 \cup G_2 \cup D_2 = (C_1 \cup G') \cup G \cup (D_1 \cup H) = S \\
&C_2 \supseteq C_2, G_2 \subseteq G_1, D_2 \supseteq D_1, \\
&C_2 \cup G_2 = C_1 \cup G' \cup G \succeq D_1 \cup G \succeq D_1 \cup H = D_2, \\
&D_2 \cup G_2 = D_1 \cup H \cup G = D_1 \cup H' \succeq C_1 \cup G' = C_2, \\
&G_1 - G_2 = G' \cup H \succeq G = G_2.
\end{aligned} \tag{6.10}$$

Repeat the above procedure for all $n \geq 2$, then we get what we want. □

6.2.3. *Partition with equiprobable events.* One crucial step towards numerical probabilities is to show that, under SVG 1-6, the state space can be arbitrarily partitioned into equally probable events.

LEMMA 6.12. Let $\succeq$ be a qualitative probability satisfying SVG 6, then S can be partitioned into 2^n ($n < \infty$) many equiprobable events.

Proof. By Lemma 6.11, there exists a sequence of Savage-triples $\{C_n, G_n, D_n\}$. Then, for any event $E \succ \varnothing$, we have that $E \succeq G_n$ when n is large. For, otherwise, $G_n \succ E$ for all n. In this case let $\{P_i\}_{i=1}^m$ be an m-fold partition of S such that $E \succ P_i$ $(i = 1, \ldots, m)$ (the existence of such a partition is guaranteed by Lemma 6.8). We have $G_n \succ P_i$, for each i. Then, from conditions (3) and (4) above,

$$G_1 - G_2 \succeq \cdots \succeq G_{n-2} - G_{n-1} \succeq G_{n-1} \succeq G_{n-1} - G_n \succeq G_n \succ P_i$$

By the '$\succ$-version' of Corollary 6.5(4), it follows that $G_1 = (G_1 - G_2) \cup \cdots \cup (G_{n-1} - G_n) \cup G_n \succ \bigcup_i P_i = S$, which is impossible. Hence $E \succeq G_n$. Then from this we conclude, via Lemma 6.3, that

$$E \succeq \bigcap_n G_n \quad \text{for } any\ E \succ \varnothing. \tag{6.11}$$

Now suppose that $\bigcap_n G_n \succ \varnothing$, then there exists a partition $\{P_i\}_{i=1}^m$ of S such that $\bigcap_n G_n \succ P_i$ for all i. Further let some P_j in the partition be such that $P_j \succ \varnothing$ (such an P_j must exist, otherwise we have $S = \bigcup_i P_i \simeq \varnothing$, which is impossible). But observe that if in (6.11) we let $E = P_j$, then it follows that $P_j \succeq \bigcap_n G_n \succ P_j$, a contradiction. Hence $\bigcap_n G_n \preceq \varnothing$.

Now take $S_1 = \bigcup_n C_n$ and $S_2 = \bigcup_n D_n \cup \bigcap_n G_n$. By Lemma 6.5(2) and the conclusion that $\bigcap G_n \preceq \varnothing$, we have that $S_1 \simeq S_2$ via condition (2) above. Hence $\{S_1, S_2\}$ equally partitions S. Apply the above procedure to S_1 and S_2, and so on. Therefore, S can be partitioned into 2^n equivalent events for any n. □

Theorem 6.13. Let $\succsim$ and $\succeq$ be defined as above, then if $\succsim$ satisfies SVG 1-6, there exists a unique (finitely additive) probability measure μ that represents $\succeq$:

$$E \succeq F \iff \mu(E) \geq \mu(F). \tag{6.12}$$

Proof. We proceed in following steps.

(1) By Lemma 6.12, for any large $n \leq \infty$ in the form of 2^m for some m, there exists a partition $\{E_i\}_{i=1}^n$ of S such that $E_j \simeq E_k$ for all $j, k \in \{1, \ldots, n\}$. Let $\mu(\cdot)$ be a

real-valued function such that for each E_i

$$\mu(E_i) = \frac{1}{n} \quad i = 1, 2, \ldots, n. \tag{6.13}$$

Now fix an event B, let r be the largest integer such that the union of r-many E_i's is not more probable than B, that is,

$$\bigcup_{i=1}^{r+1} E_i \succ B \succeq \bigcup_{i=1}^{r} E_i. \tag{6.14}$$

Note that, for any fixed B, this integer r depends on n. However, as shown in Corollary 6.6(2), it is independent of the choice of n-fold partition of S. Let us denote r by a function $k(B,n)$, we show that $\left\{\frac{k(B,i)}{i}\right\}_{i=1}^{\infty}$ is a Cauchy sequence. To this end, suppose that $F_1, \ldots, F_m$ is an m-fold equal partition of S and t is the largest integer such that $\bigcup_{j=1}^{t+1} F_j \succ B \succeq \bigcup_{j=1}^{t} F_j$. Apply Lemma 6.12 again, we have that each $E_i (1 \leq i \leq n)$ and $F_j (1 \leq j \leq m)$ can be further partitioned, respectively, into m and n equally probable events, i.e., $E_i = \bigcup_{j=1}^{m} E_{ij}$ and $F_j = \bigcup_{i=1}^{n} F_{ij}$, where E_{ij} and F_{ji} are cells in the refined nm-fold equal partitions, then we have that

$$\bigcup_{j=1}^{t+1} \bigcup_{i=1}^{n} F_{ji} = \bigcup_{j=1}^{t+1} F_j \succ B \succeq \bigcup_{i=1}^{r} E_i \succeq \bigcup_{i=1}^{r-1} E_i \succeq \bigcup_{i=1}^{r-1} \bigcup_{j=1}^{m} E_{ij}.$$

Then, Corollary 6.6 implies that

$$(r-1)m \leq (t+1)n. \tag{6.15}$$

Here, by definition, $k(B,n) = r$ and $k(B,m) = t$, then (6.15) yields

$$\left| \frac{k(B,m)}{m} - \frac{k(B,n)}{n} \right| \leq \frac{1}{n} + \frac{1}{m} < \varepsilon,$$

where ε is an arbitrarily small number. The second inequality is met when m and n are sufficiently large. Hence, it is meaningful to define $\mu(B)$ by

$$\mu(B) =_{\mathrm{Df}} \lim_{n \to \infty} \frac{k(B,n)}{n}. \tag{6.16}$$

(2) We need to verify that $\mu(\cdot)$ defined in (6.13) is a (finitely additive) probability measure, that is, μ satisfies the following conditions: for any E, F,

(a) $\mu(E) \geq 0$;

(b) if $E \cap F = \varnothing$, then $\mu(E \cup F) = \mu(E) + \mu(F)$;

(c) $\mu(S) = 1$.

Condition (a) and (c) can be easily verified. To show condition (b), let $\{P_i\}_{i=1}^n$ be an n-fold equal partitions of S, and let $r = k(E, n)$, $t = k(F, n)$, and $u = k(E \cup F, n)$. Since $E \cap F = \varnothing$, by Corollary 6.5(4) and (6.14),

$$\bigcup_{i=1}^{u+1} P_i \succ E \cup F \succeq \bigcup_{i=1}^{r} P_i \cup \bigcup_{j=1}^{t} P_j \tag{6.17}$$

Note that (6.17) hold even when the P_i's and P_j's on the right hand side are disjoint, and hence $\bigcup_{i=1}^{u+1} P_i \succeq \bigcup_{j=1}^{r+t} P_j$. It follows that $r + t \leq u + 1$. On the other hand,

$$\bigcup_{i=1}^{r+t+2} P_i \succ E \cup F \succeq \bigcup_{i=1}^{u} P_i \tag{6.18}$$

This shows $u \leq r + t + 2$ (To see the first inequality, note that otherwise we have $E \cup F \succeq \bigcup_{i=1}^{r+1} P_i \cup \bigcup_{i=1}^{t+1} P_i$, then by Corollary 6.5(5), either $E \succeq \bigcup_{i=1}^{r+1} P_i$ or $F \succeq \bigcup_{i=1}^{t+1} P_i$. But neither case is possible). Hence

$$\frac{k(E,n)}{n} + \frac{k(F,n)}{n} - \frac{1}{n} \leq \frac{k(E \cup F, n)}{n} \leq \frac{k(E,n)}{n} + \frac{k(F,n)}{n} + \frac{2}{n}.$$

Let $n \to \infty$ we obtain that $\mu(E \cup F) = \mu(E) + \mu(F)$, which is what we want.

(3) Finally, we show that μ defined in (6.16) is unique. Consider otherwise, then let μ' be another probability measure on S such that (6.5) holds. It follows, via (6.14), that $\frac{k(B,n)}{n} \leq \mu'(B) \leq \frac{k(B,n)+1}{n}$. Now let $n \to \infty$, we get $\mu'(B) = \lim_{n\to\infty} \frac{k(B,n)}{n} = \mu(B)$. This shows uniqueness. □

One feature of the probability measure μ derived in the theorem above is that μ is *atomless*. That is, as the following corollary shows, it allows for partitions of the state space into sets of arbitrarily small probability.[11]

COROLLARY 6.14. Given the probability measure μ on S obtained above, for any $B \subseteq S$ and $0 \leq \rho \leq 1$, there exists $C \subseteq B$ such that $\mu(C) = \rho\mu(B)$.

PROOF. The proof is trivial if B is null. Now assume that $\mu(B) = p > 0$. By Lemma 6.12 and Theorem 6.13, for any large n in the form of 2^m, there exists a partition $\{E_i\}_{i=1}^n$ of B and unique probability measure μ on S for which

$$\mu(E_i) = \frac{p}{n} \quad \text{for all } i = 1, \ldots, n$$

Now, let r be the largest number such that

$$\frac{(r+1)}{n} > \rho \geq \frac{r}{n}.$$

Define A_n, B_n by

$$A_n = \bigcup_{i=1}^{r} E_i, \quad B_n = \bigcup_{i=1}^{r+1} E_i.$$

Then we have

$$\mu(B_n) = \frac{p(r+1)}{n} > p\rho \geq \frac{pr}{n} = \mu(A_n).$$

By Theorem 6.13, $\mu(A_n) = \mu(B_n) = p\rho$ as $n \to \infty$. Define $C = \lim_{n\to\infty} A_n$. Then we have that $\mu(C) = \rho\mu(B)$. ☐

REMARK 6.15. Intuitively, Corollary 6.14 says that, for any event B receiving non-zero probability under μ, B can be infinitely and continuously divided. As a consequence of this feature, the state space S in Savage's decision model must contain *uncountably many states*. This, however, sets a limit to application of Savage's theory: it cannot be applied to cases with a finite or countable state space.

7. Personal Utility

[11]See also Savage (1972, p.34) and Fishburn (1970, p.199).

7.1. Utilities for simple acts. Next, we seek to construct a utility function for acts. This was approached by Savage in two steps. First, he considers a special set of simple acts, or *gambles* in his terminology, which are acts that potentially lead to only *finitely* many possible consequences, for which a von Neumann-Morgenstern utility function (vNMU) over consequences can be derived. The latter together with the derived subjective probability μ above give rise to a utility measure U_0 of simple acts. He then extends this utility for simple acts to general acts which can lead to potentially *infinitely* many consequences. The exposition here follows Savage's original approach, in Chapter IV we will provide an alternative method of deriving utilities without appealing to constant-acts.

Let us start with a close examination of relationship between gambles and the class of lotteries as introduced in §2.3.

Definition 7.1 (Gambles). An act $f \in X^S$ is said to be *simple* if there exist

(i) a n-partition $\{P_i\}_{i=1}^n$ of S, and
(ii) a finite sequence of consequences $x_1, x_2, \ldots, x_n$ such that $f(s) = x_i$ for all $s \in P_i$ $(i = 1, \ldots, n)$.

Denote the set of all simple acts by $\mathcal{A}_0$, we also refer to simple acts as *gambles*. It is plain that all constant acts $\mathfrak{c}_x (x \in X)$ are gambles/simple acts. Using our notation for compound acts, a gamble $f \in \mathcal{A}_0$ can be conveniently expressed by

$$f = \mathfrak{c}_{x_1} \oplus_{P_1} \big(\mathfrak{c}_{x_2} \oplus_{P_2} \big(\mathfrak{c}_{x_3} \oplus_{P_2} (\cdots \oplus_{P_{n-1}} \mathfrak{c}_{x_n}) \cdots \big)\big). \tag{7.1}$$

7.1.1. *Lotteries introduced by gambles.* Now, given the subjective probability μ on S derived from Theorem 6.13, each gamble $f \in \mathcal{A}_0$ defines a simple probability measure on X, written p_f, as follows

$$p_f(x_i) = \begin{cases} \mu[f(s) = x_i] & \text{if } x_i \in f(S), \\ 0 & \text{if } x_i \in X - f(S); \end{cases} \tag{7.2}$$

where $\mu[f(s) = x_i] = \mu\{s \in S \mid f(s) = x_i\}$ and $f(S)$ denotes the range of f (cf. §2.1). We refer to p_f as *the lottery on X introduced by gamble f.*

Recall that $\mathcal{L}_X^*$ is the set of simple probability measures defined on (an infinite) X (see Definition 2.8). Thus each $f \in \mathcal{A}_0$ corresponds to a simple probability measure in the extended lottery space $\mathcal{L}_X^*$. Observe that two different gambles may introduce the same lottery. Take, for instance, E, E^C be a partition of S for which $\mu(E) = \mu(E^C) = 1/2$ and let f, g be two acts defined in the table below.

	E	E^C
f	x_1	x_2
g	x_2	x_1

Then, we have an example where $f \neq g$, yet, by (7.2), $p_f = p_g$. That is, f and g induce the same lottery. We show in the following lemma that this is the case *only if* $f \sim g$. Intuitively, the lemma says that a pair of simple acts are considered equally preferable if the probabilities of getting each consequence under either one of the two acts are the same. As we shall soon see, this is a crucial step moving towards the full expected utility theory.

Lemma 7.2. For any gambles $f, g \in \mathcal{A}_0$, if $p_f = p_g$ then $f \sim g$.

Proof. We consider only the case where $f(S) = g(S)$. For if $f(S) \neq g(S)$, that is, if there is some $x_0 \in X$ such that, say, $x_0 \in f(S)$ but $x_0 \notin g(S)$, then, by the assumption that $p_f = p_g$ and (7.2), we have $\mu[f(s) = x_0] = 0$. In this case, we can construct an act f' which differs from f only on the null set $E_0 = \{s \mid f(s) = x_0\}$ (and hence $f' \sim f$ by Lemma 5.10(3)) such that $f'(S) = f(S) - \{x_0\}$. Repeat this process until we reach some f^* and g^* such that $f^* \sim f, g^* \sim g$ and $f^*(S) = g^*(S)$.

Now let $D = f(S) = g(S)$. The lemma is proved by induction on the size of D. Suppose that $|D| = 1$, then f, g are constant acts and $f = g$, and hence $f \sim g$. For the inductive step, assume that claim holds for $n - 1$, we show that it also holds when $|D| = n$. To this end, let $x_1, x_2, \ldots, x_n$ be an enumeration of the

TABLE 7.1

	P_1	P_2
Q_1	A	D
Q_2	C	B

f

x_1	x_2
x_1	x_2

g

x_1	x_1
x_2	x_2

consequences in D, and let $\{P_i\}_{i=1}^n$ and $\{Q_i\}_{i=1}^n$ be partitions of S such that

$$f(s) = g(t) = x_i \text{ for all } s \in P_i \text{ and } t \in Q_i \ (i = 1, \ldots, n). \tag{7.3}$$

We proceed with the following two possibilities:

(1) If for some j, P_j and Q_j are null events. It follows that $\mu(P_j) = \mu(Q_j) = 0$, and hence $\mu[f(s) = x_j] = \mu[g(t) = x_j] = 0$. In this case, let r be such that P_r, Q_r are non-null. Then construct new gambles f' and g' as follows

$$f'(s) = \begin{cases} x_i & \text{if } s \in P_i \text{ and } i \notin \{j, r\} \\ x_r & \text{if } s \in P_j \cup P_r \end{cases}; \text{ and}$$

$$g'(s) = \begin{cases} x_i & \text{if } s \in Q_i \text{ and } i \notin \{j, r\} \\ x_r & \text{if } s \in Q_j \cup Q_r \end{cases}.$$

That is to say, f' agrees with f on all cells of the partition $\{P_i\}_{i=1}^n$ except for the null cell P_j, in which $f(s) = x_j$ but $f'(s) = x_r$, same for g and g'. By Lemma 5.10(2), we have that,

$$f \succsim g \iff f' \succsim g'.$$

From the construction of f' and g' it is easily seen that they are gambles with $n-1$ partitions and that $f'(S) = g'(S) = D - \{x_j\}$. Then by the inductive hypothesis $f' \sim g'$, and hence $f \sim g$.

(2) The remaining case is that P_i, Q_i are not null for all $i = 1, \ldots, n$. We deal with this case in yet another two steps:

(a) As an illustration, consider the simple situation where $n = 2$. In this case we have that $X = \{x_1, x_2\}$ and that $\{P_1, P_2\}$ and $\{Q_1, Q_2\}$ are partitions of

S for which

$$f(s) = g(t) = x_i \text{ for all } s \in P_i, t \in Q_i \ (i = 1, 2) \tag{7.4}$$

$$\mu(P_1) = \mu(Q_1),\ \mu(P_2) = \mu(Q_2). \tag{7.5}$$

We want to show that $f \sim g$. To this end, let $A = P_1 \cap Q_1, B = P_2 \cap Q_2, C = P_1 \cap Q_2, D = P_2 \cap Q_1$, then (7.4) can be represented in Table 7.1. (for instance, $f(s) = x_1$ if $s \in A$ or $s \in C$). Next construct f' and g' which agree, respectively, with f and g on C and D and with each other on A and B. Then by the sure-thing principle (P2) $f \succsim g$ iff $f' \succsim g'$. It is hence sufficient to show that $f' \sim g'$.

	A	B	C	D
f	x_1	x_2	x_1	x_2
g	x_1	x_2	x_2	x_1
f'	x_2	x_2	x_1	x_2
g'	x_2	x_2	x_2	x_1

f'

x_2	x_2
x_1	x_2

g'

x_2	x_1
x_2	x_2

Note that (7.5) implies $\mu(C) = \mu(D)$, then, by Theorem 6.13, it must be

$$C \simeq D \tag{7.6}$$

One the other hand, f' and g' can be written as

$$f' = \mathfrak{c}_{x_1} \oplus_C \mathfrak{c}_{x_2}, \tag{7.7}$$

$$g' = \mathfrak{c}_{x_1} \oplus_D \mathfrak{c}_{x_2}. \tag{7.8}$$

By Definition 5.4, (7.6)-(7.8) imply that $f' \sim g'$.

(b) In general, let $\{P_i\}_{i=1}^n$ and $\{Q_i\}_{i=1}^n$ be the partitions of S with respect to f and g satisfying (7.3). Let $B = P_n \cap Q_n$, $C = Q_n - P_n$ and $D = P_n - Q_n$. By assumption, $\mu(Q_n) = \mu(P_n)$. This implies that $\mu(C) = \mu(D)$. We consider only the nontrivial case where $\mu(C) = \mu(D) > 0$. Further, C can be partitioned such that $C_i = Q_n \cap P_i$ $(i = 1, \ldots, n-1)$. And it is clear $f(s) = x_1$ for all $s \in C_i$ $(i = 1, \ldots, n-1)$. Next, let $\mu(C_1)/\mu(C) = \rho_1$, then by Corollary 6.14, there exists some $D_1 \subseteq D$ for which $\mu(D_1)/\mu(D) = \rho_1$, and hence $\mu(C_1) = \mu(D_1)$. It is easy to see that, by repeatedly applying Corollary 6.14, D can be partitioned into $D_1, \ldots, D_{n-1}$ for which

$$\mu(C_i) = \mu(D_i), \quad i = 1, \ldots, n-1. \tag{7.9}$$

TABLE 7.2

	P_1	P_2	$\cdots$	P_n
				D_1 D_2 $\vdots$
Q_n	C_1	C_2	$\cdots$	B

f				
				D_1 D_2 $\vdots$
Q_n	x_1	x_2	$\cdots$	x_n

Now construct an act h_1 such that it agrees with f on all parts of S except for C_1 and D_1 for which

$$h_1(s) = \begin{cases} x_n & \text{if } s \in C_1, \\ x_1 & \text{if } s \in D_1, \\ g(s) & \text{otherwise.} \end{cases}$$

Since $\mu(C_1) = \mu(D_1)$, using a similarly argument given in part (a) above, we conclude that $h_1 \sim f$. Repeat this process inductively we have that

$$h_{i+1}(s) = \begin{cases} x_n & \text{if } s \in C_2, \\ x_{i+1} & \text{if } s \in D_2, \\ h_i(s) & \text{otherwise,} \end{cases} \qquad (i < n-1).$$

From the construction h_i's we have that $h_{n-1}(s) = x_n$ for all $s \in C_i$ $(i = 1, \ldots, n-1)$ and $h_{n-1} \sim f$.

h_1				
				x_1 D_2 $\vdots$
Q_n	x_n	x_2	$\cdots$	x_n

$\Rightarrow \cdots \Rightarrow$

h_{n-1}				
				x_1 x_2 $\vdots$
Q_n	x_n	x_n	$\cdots$	x_n

The proof is completed if we show that $h_{n-1} \sim g$. To this end, note that h_{n-1} agrees with g on Q_n, and hence $h_{n-1} \sim_{Q_n} g$. In $S - Q_n$, there are only $n-1$ many elements, then by the construction of h_{n-1} and the inductive hypothesis we have that $h_{n-1} \sim_{S-Q_n} f \sim_{S-Q_n} g$. Together, we have $h_{n-1} \sim g$, which is what we want. □

7.1.2. *Gambles introduced by lotteries.* Conversely, each lottery $p \in \mathcal{L}_X^*$ can be associated with a gamble. To see this, let $x_1, x_2, \ldots, x_n$ be an enumeration of the members of X that are in the support of p, as defined in (2.18), and let $\{P_i\}_{i=0}^n$ be a partition of S such that

$$\mu(P_i) = \begin{cases} 0 & i = 0, \\ p(x_i) & i = 1, \ldots, n. \end{cases} \tag{7.10}$$

Note that the existence of such a partition is guaranteed by the fact that, by Lemma 6.12, S can be partitioned into arbitrarily fine equal-probable events and that μ is a well defined finitely additive probability measure on S for which Corollary 6.14 holds. Now, given p and the corresponding $\{P_i\}_{i=0}^n$, define f_p as follows

$$f_p(s) = \begin{cases} x & s \in P_0, \\ x_i & s \in P_i \quad (i = 1, \ldots, n), \end{cases} \tag{7.11}$$

where x is an arbitrary consequence that is not in the support of p. We refer to f_p as *a gamble introduced by lottery p*. The following observation says that, given any lottery q, let f_q be a gamble introduced by q as defined above, then the introduced lottery by f_q is equal to q. The proof is immediate from (7.2) and (7.11), and hence omitted.

LEMMA 7.3. For any $q \in \mathcal{L}_X^*$, $p_{f_q} = q$.

It shall be emphasized that, for any simple act $g \in \mathcal{A}_0$, it is in general *not* the case that $f_{p_g} = g$. As the the following example illustrates, this is due to the fact that, in general, more than one gambles can be associated with the same lottery.

EXAMPLE 7.4. Let $X = \{x_1, x_2, x_3\}$ and p be such that $p(x_1) = p(x_2) = 0$ and $p(x_3) = 1$. Construct f and g to be such that $\{P_1, P_2, P_3\}$ and $\{Q_1, Q_2, Q_3\}$ are their respective partitions of S for which $\mu(P_3) = \mu(Q_3) = 1$. By definition, both f and g are gambles introduced by p, but $f \neq g$.

	P_1	P_2	P_3
f	x_1	x_1	x_3

	Q_1	Q_2	Q_3
g	x_2	x_2	x_3

However, in the light of Lemma 7.2 and Lemma 7.3, we note that *all gambles introduced from the same lottery are equally preferable under* $\succsim$. It follows that each lottery $p \in \mathcal{L}_X^*$ can be identified with a class of equally preferable gambles introduced by p, which are ordered under the given preference $\succsim$ on $\mathcal{A}$.[12] For each p, let f_p be a representative of the associated equivalence class (under $\succsim$), then a preference relation $\mathcal{L}_X^*$ can be *induced* as follows: for any $p, q \in \mathcal{L}_X^*$,

$$p \succsim q \text{ if } f_p \succsim f_q. \tag{7.12}$$

We show that this induced preference on $\mathcal{L}_X^*$ satisfies von Neumann-Morgenstern axioms (cf. Remark 2.4).

LEMMA 7.5. If preference relation $\succsim$ on $\mathcal{A}$ satisfies SVG 1-6, then the induced ordering on $\mathcal{L}_X^*$ in (7.12) satisfies the following conditions:

(1) $\succsim$ is a complete preference relation;
(2) For all $p, q, r \in \mathcal{L}_X^*$ and $\lambda \in (0, 1]$, $p \succ q$ if and only if $p \oplus_\lambda r \succ q \oplus_\lambda r$;
(3) For any $p, q, r \in \mathcal{L}_X^*$, if $p \succsim r \succsim q$ and $p \succ q$, then there exists a unique $\alpha \in [0, 1]$ such that $r \sim p \oplus_\alpha q$.

PROOF. (1) This is immediate from (7.12) and SVG 1.

[12] Savage (1972, p.71) uses $\sum_i \rho_i f_i$ to denote the class of simple acts for which, to use his notations, there exist partitions B_i of S such that $P(B_i) = \rho_i$ and $f(s) = f_i$ for $s \in B_i$. He further remarks that if a simple act **f** is such that "the consequences f_i will befall the person in case B_i occurs, then the value of **f** is independent of how the partition B_i is chosen."

(2) By the definition of induced preference in (7.12), it is sufficient to show that the introduced gambles satisfy

$$f_p \succ f_q \text{ if and only if } f_{p\oplus_\lambda r} \succ f_{q\oplus_\lambda r}. \tag{7.13}$$

To this end, let $\{P_i\}_{i=0}^m$, $\{Q_j\}_{j=0}^r$, $\{R_k\}_{k=0}^n$ be partitions of S with respect to f_p, f_q, f_r, respectively, for which (7.10) and (7.11) are satisfied. By Corollary 6.14, construct $E_{ik} \subseteq P_i \cap R_k$ such that $\mu(E_{ik}) = \lambda\mu(P_i \cap R_k)$. Further, let $\overline{E}_{ik} = (P_i \cap R_k) - E_{ik}$, and hence $\mu(\overline{E}_{ik}) = (1-\lambda)\mu(P_i \cap R_k)$. It follows that

$$\mu\Big(\bigcup_k E_{ik}\Big) = \lambda\mu(P_i) \;\text{ and }\; \mu\Big(\bigcup_i \overline{E}_{ik}\Big) = (1-\lambda)\mu(R_k). \tag{7.14}$$

It is plain that $\{E_{ik}, \overline{E}_{ik}\}_{ik}$ forms a finer partition of S. Define a gamble f_1 to be such that

$$f_1(s) = \begin{cases} x_i & \text{if } s \in E_{ik} \\ x_k & \text{if } s \in \overline{E}_{ik} \end{cases},$$

where x_i is in the support of p and $f_p(s) = x_i$ for $s \in E_{ik} \subseteq P_i$ and similarly, x_k is in the support of r and $f_r(s) = x_k$ for $s \in \overline{E}_{ik} \subseteq R_k$. Now let p_{f_1} be the lottery introduced by f_1, then, by (7.14), for any $x_i \in X$,

$$\begin{aligned} p_{f_1}(x_i) = \mu[f_1(s) = x_i] &= \mu\Big[\bigcup_j E_{ij} \cup \bigcup_j \overline{E}_{ji}\Big] \\ &= \lambda\mu(P_i) + (1-\lambda)\mu(R_i) \\ &= \lambda p(x_i) + (1-\lambda)r(x_i) \\ &= (p \oplus_\lambda r)(x_i). \end{aligned} \tag{7.15}$$

By Lemma 7.3, $p_{f_{p\oplus_\lambda r}} = p \oplus_\lambda r$, it follows that $p_{f_1} = p_{f_{p\oplus_\lambda r}}$, hence $f_1 \sim f_{p\oplus_\lambda r}$ via Lemma 7.2.

Similarly, construct $F_{jk} \subseteq Q_j \cap R_k$ such that $\mu(F_{jk}) = \lambda\mu(Q_j \cap R_k)$, and let $\overline{F}_{jk} = (Q_j \cap R_k) - F_{jk}$. Then $\{F_{jk}, \overline{F}_{jk}\}_{jk}$ partitions S. Define a gamble f_2 to be such that

$$f_2(s) = \begin{cases} x_j & \text{if } s \in F_{jk} \\ x_k & \text{if } s \in \overline{F}_{jk} \end{cases}.$$

We have that $p_{f_2} = q \oplus_\lambda r = p_{f_{q \oplus_\lambda r}}$, and hence $f_2 \sim f_{q \oplus_\lambda r}$. Thus, by SVG 1, (7.13) is proved if it can be shown that $f_p \succ f_q$ if and only if $f_1 \succ f_2$.

Observe that, by (7.15), for any $x_i \in X$, $f_1(s) = x_i$ implies $s \in P_i \cup R_i$ and, similarly, $f_2(s) = x_i$ only if $s \in Q_i \cup R_i$. Further, since f_p, f_q, f_r satisfy (7.11), construct two sequences of gambles $h_1, \ldots, h_n$ and $h'_1, \ldots, h'_n$ as follows

$$h_1 = f_p \oplus_{R_1^C} f_r \text{ and } h'_1 = f_q \oplus_{R_1^C} f_r \tag{7.16}$$

$$h_{i+1} = h_i \oplus_{R_{i+1}^C} f_r \text{ and } h'_{i+1} = h'_i \oplus_{R_{i+1}^C} f_r \tag{7.17}$$

From the constructions of $f_1, f_2,$, it is easy to see that $f_1 = h_n$ and $f_2 = h'_n$. Finally, by the sure-thing principle, (7.16) and (7.17) imply that

$$h_1 \succ h'_1 \iff f_p \succ f_q$$

$$h_{i+1} \succ h'_{i+1} \iff h_i \succ h'_i$$

Therefore, $f_1 \succ f_2$ if and only if $f_p \succ f_q$, which is what we want.

(3) This claim can be similarly proved. □

By Theorem 2.10, if the induced preference $\succsim$ on $\mathcal{L}_X^*$ satisfies vNM axioms, then there exists a vNMUF u for all the consequences in X, and hence an *expected utility function* U_0 for gambles such that, for each $f \in \mathcal{A}_0$,

$$U_0[f] = \sum_{x \in X} \mu[f(s) = x] u(x) = \int_S u[f(s)] d\mu(s). \tag{7.18}$$

Thus, Lemma 7.5 and (7.18) lead to the following theorem.

Theorem 7.6. Let S be a set of states, X be a set of consequences, $\succsim$ be a preference over the set of acts $\mathcal{A} = X^S$, and let $\mathcal{A}_0 \subseteq \mathcal{A}$ be the set of gambles, then, if $\succsim$ satisfies SVG 1-6, there exists a utility function U_0 such that, for any

$f, g \in \mathcal{A}_0$,

$$f \succsim g \iff U_0[f] \geq U_0[g],$$

where $U_0[f] = \int u[f]\, d\mu$.

The next order of business is to extend the utility function obtained in Theorem 7.6 for simple acts to that for general acts, namely, to relax the restriction that acts being considered have only finitely many possible consequences, which will be the subject of the next subsection. Before moving on, we show that, for any general act g, if g is bounded by two simple acts then there exits a simple act/gamble that is equally preferable to g.

Corollary 7.7. Let $f_1, f_2 \in \mathcal{A}_0$ satisfying $f_1 \succ f_2$, and $g \in \mathcal{A}$. If $f_1 \succsim g \succsim f_2$, then there exists a $g_0 \in \mathcal{A}_0$ such that $g_0 \sim g$.

Proof. Our proof here parallels the proof of Lemma 2.3(4). In the notation of (7.2) and (7.11), let p_{f_1} and p_{f_2} be the lotteries induced by f_1, f_2, and $f_{p_{f_1} \oplus_\lambda p_{f_2}}$ is a gamble introduced by some mixer of p_{f_1} and p_{f_2}. Consider the following two sets

$$\begin{aligned} A := \{x \in [0,1] \mid f_{p_{f_1} \oplus_x p_{f_2}} \succsim g\}; \\ B := \{x \in [0,1] \mid g \succsim f_{p_{f_1} \oplus_x p_{f_2}}\}. \end{aligned} \tag{7.19}$$

Let $\alpha_* = \inf A$ and $\alpha^* = \sup B$. Note that, for any $a > \alpha_*$, there must exist some $a' \in A$ such that $a > a' \geq \alpha_*$. Then by Lemma 7.5(2) and Lemma 2.3(3), $f_{p_{f_1} \oplus_a p_{f_2}} \succ f_{p_{f_1} \oplus_{a'} p_{f_2}} \succsim g$. This means

$$a > \alpha_* \implies a \notin B. \tag{7.20}$$

The contrapositive of (7.20) says that, for any a, $a \in B$ implies that $\alpha_* \geq a$, in other words, α_* is an upper bound of B. and hence $\alpha_* \geq \alpha^*$. Similarly, one can show that, for any a,

$$\alpha^* > a \implies a \notin A \tag{7.21}$$

which leads to $\alpha^* \geq \alpha_*$. Now define $\alpha = \alpha_* = \alpha^*$. It can be similarly shown, by applying SVG 6, that it cannot be that $\alpha \notin A \cap B$. Finally, define $g_0 = f_{p_{f_1} \oplus_\alpha p_{f_2}}$, we have $g_0 \sim g$. □

7.2. Postulate 7 and utility extension to general acts. To extend the utility for simple acts to acts in general, Savage brought in one final postulate.

SVG 7. For any event $E \in \mathcal{F}$, if $f \succsim_E \mathfrak{c}_{g(s)}$ for all $s \in E$ then $f \succsim_E g$.

The postulate says that, for any event E, if the conditional preference of f given E is no less preferable to any of the constant acts constructed from the possible consequences of g under each state in E, then f is weakly preferred to g given E. As seen, this postulate uses constant acts in a systematic way, which, as we have discussed in Section 14.1, can be troublesome due to the issue of the applicability of the notion of constant acts. For the time being, let us focus on the following structural development of utility extension. Savage (1972, p.78) first demonstrated that SVG 7 is not derivable from the first six postulates. This was done by constructing a model which satisfies all of SVG 1-6 but fails SVG 7.

Example 7.8. Let $S = \mathbb{N}^+$ and $X = [0, 1)$ be the set of consequences, and λ be the finitely but not countably additive measure on positive integers given in Example A.4.7. For any act f, let $U[f] = \int_S u(f) d\lambda$ where $u(x) = x$ is a utility function on X and $V[f] = \lim_{\epsilon \to 0} \lambda[f(s) \geq 1 - \epsilon]$, and let

$$\begin{aligned} W[f] &= U[f] + V[f] \\ &= \int_S u(f(s)) d\lambda(s) + \lim_{\epsilon \to 0} \lambda[f(s) \geq 1 - \epsilon] \end{aligned} \tag{7.22}$$

Define $f \succsim^* g$ to mean that $W[f] \geq W[g]$. It is not difficulty to verify that the defined $\succsim^*$ satisfies SVG 1-6.[13] Note that for any act g with a finite range, i.e. a gamble, $V[g] = 0$, in this case $W[g] = U[g]$ is a utility function like the one given

[13]See Example 2.1 (and Lemma 1 & 2) in Seidenfeld and Schervish (1983).

in Theorem 7.6. To see SVG 7 is violated, let f, g be such that

$$f(x) = \begin{cases} 1 - 1/x & \text{if } x \text{ is even} \\ 0 & \text{if } x \text{ is odd} \end{cases} \qquad \text{and} \qquad g(x) = \max\{3/4, f(x)\}.$$

Then it is easy to calculate that

$$W[f] = \frac{1}{2} + \frac{1}{2} = 1, \quad \text{and} \quad W[g] = \Big(\frac{1}{2} + \frac{1}{2} \cdot \frac{3}{4}\Big) + \frac{1}{2} = \frac{11}{8}.$$

Hence $f \prec^* g$ by the definition of $\succsim^*$ in terms of $W[\cdot]$ above. On the other hand, for any $s \in S$, we have $g(x) < 1$. This means that, for the constant act $\mathfrak{c}_{g(s)}$, $W[\mathfrak{c}_{g(s)}] < 1$, and hence $f \succ^* \mathfrak{c}_{g(s)}$, from which we conclude that $f \succsim^* \mathfrak{c}_{g(s)}$ for all $s \in S$, but this contradicts SVG 7 (taking $E = S$). $\lhd$

Savage then showed that with SVG 7 the utility function U_0 for simple acts can be extended to a utility function U for general acts. To this end, we first prove the following lemmas.

Lemma 7.9. For any event E, if, for every consequence $a \in X$, $f \succsim \mathfrak{c}_a$ and $g \succsim \mathfrak{c}_a$, then $f \sim g$.

Proof. The lemma is proved by simple applications of SVG 7. □

Lemma 7.10. For any $f \in \mathcal{A}$, if there exists some $a \in X$ and $c < \infty$ such that $\mathfrak{c}_a \succsim f$ and $u\big(f(s)\big) \leq c$ for all $s \in S$, then there exists some gamble $g_0 \in \mathcal{A}_0$ for which

$$g_0 \succsim f \qquad \text{and} \qquad U_0[g_0] \leq c, \tag{7.23}$$

where u, U_0 are as in Theorem 7.6.

Proof. Suppose that $u(a) \leq c$, then we can define $g_0 = \mathfrak{c}_a$. Otherwise, $u(a) > c$, in this case, fix any $t \in S$ and let $f(t) = b \in X$, we have, by the hypothesis, $u[f(t)] = u\big(b\big) \leq c$. Let p^* be a probability mixer of a and b such that

$$p^*(a)u(a) + (1 - p^*(b))u(b) = c.$$

Let E, E^C be a partition of S such that $\mu(E) = p^*(a)$ and $\mu(E^C) = p^*(b)$, and define a gamble g_0 to be such that

$$g_0(s) = \begin{cases} a & \text{if } s \in E \\ b & \text{if } s \in E^C \end{cases}$$

From the construction, we have $U_0[g_0] = c$. Further, for any $s \in S$, we have $U_0[\mathfrak{c}_{f(s)}] \leq c$, and hence, by Theorem 7.6, $g_0 \geq \mathfrak{c}_{f(s)}$. That is, $g_0 \geq \mathfrak{c}_{f(s)}$ for all $s \in S$ then, by SVG 7, $g_0 \succsim f$. □

A small change of the proof above lead to the following corollary.

Corollary 7.11. For any $f \in \mathcal{A}$ and for any event E, if there exists some $a \in X$ and $c < \infty$ such that $\mathfrak{c}_a \succsim_E f$ and $u\big(f(s)\big) \leq c$ for all $s \in E$, then there exists some gamble $g_0 \in \mathcal{A}_0$ for which

$$g_0 \succsim_E f \qquad \text{and} \qquad U_0[g_0] \leq c. \tag{7.24}$$

Lemma 7.12. Let $\{P_i\}_{i=1}^n$ be a partition of S and $c_1, \dots, c_n < \infty$. Then, for any act $f \in \mathcal{A}$, if there is a gamble $h_0 \in \mathcal{A}_0$ such that $f \succsim h_0$ and $u\big(f(s)\big) \leq c_i$ for all $s \in P_i$, then

$$U_0[h_0] \leq \sum_{i=1}^n c_i \mu(P_i). \tag{7.25}$$

Proof. We consider the following two cases:

(1) If, for *each* P_i, there is some a_i such that $\mathfrak{c}_{a_i} \succsim_{P_i} f$. Then, by Corollary 7.11, there exists some g_i for which

$$g_i \succsim_{P_i} f \qquad \text{and} \qquad U_0[g_i] \leq c_i \qquad \text{for all } i = 1, \dots, n.$$

Define g_0 to be such that $g_0(s) = g_i(s)$ if $s \in P_i$. Then we have $g_0 \succsim f$, hence $g_0 \succsim h_0$. Since both h_0 and g_0 are gambles, by Theorem 7.6, we have

$$U_0[h_0] \leq U_0[g_0] \leq \sum_{i=1}^n c_i \mu(P_i).$$

In this case, (7.25) holds.

(2) Otherwise, for some P_i,

$$f \succ_{P_i} \mathfrak{c}_a \quad \text{for } \textit{all } a \in X. \tag{7.26}$$

We show that, in this case, f can be modified to some f' so that

1. for *each* P_i, there is some $b_i \in X$ such that $\mathfrak{c}_{b_i} \succsim_{P_i} f'$,
2. there exists some gamble h_0 such that $f \succsim f' \succsim h_0$, and
3. $u(f'(s)) \leq c_i$ for all $s \in P_i$.

If such a f' exists, this will take us back to case (1) for which (7.25) holds, then we are done. To this end, let $x^*, x_* \in X$ be such that $x^* \succ x_*$ and $u(x_*) < c_i$ (the existence of such a pair is guarantee by SVG 5 and the fact utility is unique up to some linear transformation). Fix any $a \in X$, then, by SVG 6, $f \succ_{P_i} \mathfrak{c}_a$ implies that that there is some non-null $A \subseteq P_i$ such that

$$\begin{aligned} \mathfrak{c}_{x^*} \oplus_A f \succ_{P_i} \mathfrak{c}_a, \\ \mathfrak{c}_{x_*} \oplus_A f \succ_{P_i} \mathfrak{c}_a. \end{aligned} \tag{7.27}$$

It is clear, by SVG 2, that

$$\mathfrak{c}_{x^*} \oplus_A f \succ_{P_i} \mathfrak{c}_{x_*} \oplus_A f. \tag{7.28}$$

Note that (7.26) implies, via SVG 7, that $f \succ_{P_i} \mathfrak{c}_{x^*} \oplus_A f$ and $f \succ_{P_i} \mathfrak{c}_{x_*} \oplus_A f$. Further, it cannot be the case that $\mathfrak{c}_{x_*} \oplus_A f \succ_{P_i} \mathfrak{c}_b$ for all $b \in X$, for, otherwise, by Lemma 7.9, $f \sim_{P_i} \mathfrak{c}_{x_*} \oplus_A f$, and hence $\mathfrak{c}_{x_*} \oplus_A f \succ_{P_i} \mathfrak{c}_{x^*} \oplus_A f$, which contradicts (7.28). This means that there is some $b_i \in X$ such that $\mathfrak{c}_{b_i} \succsim_{P_i} f \sim_{P_i} \mathfrak{c}_{x_*} \oplus_A f$. Let $f' = \mathfrak{c}_{x_*} \oplus_A f$, then this is what we need. □

Theorem 7.13 (Savage). If $\succsim$ satisfies SVG 1-7 then there exists a utility function u on X and and probability function μ on events such that for any $f, g \in \mathcal{A}$,

$$f \succsim g \iff \int u\big[f(s)\big]\, d\mu \geq \int u\big[f(s)\big]\, d\mu. \tag{7.29}$$

Proof. We prove the theorem in following steps:

(1) Under the derived utility u on X and μ on $\mathcal{F}$ from Theorem 7.6, we define the utility U of a general act f by[14]

$$U[f] = \int u[f]\, d\mu = \sup \sum_i \Big[\inf_{s \in P_i} u[f(s)] \Big] \mu(P_i), \tag{7.30}$$

where sup ranges over all possible finite partitions P_i of S intro $\mathcal{F}$-sets. The goal is then to show that such a utility U exists under SVG 1-7.

(2) Given any general act $f \in \mathcal{A}$, we consider the following possibilities:

(a) $\mathfrak{c}_a \succsim f \succsim \mathfrak{c}_b$ for some $a, b \in X$;

(b) $f \succ \mathfrak{c}_a$ for all $a \in X$;

(c) $\mathfrak{c}_a \succ f$ for all $a \in X$.

For case (a), partition S into $\{P_i\}_{i=1}^n$ and let P_i's be so arranged that, for any $s \in P_i$ ($i = 1, \dots, n$),

$$c_* + \frac{i-1}{n}(c^* - c_*) \le u[f(s)] \le c_* + \frac{i}{n}(c^* - c_*). \tag{7.31}$$

where c_* and c^* are respectively the greatest lower and least upper bounds of u.[15] Then from the definition of U in (7.30), it is easily seen that

$$\sum_{i=1}^n \Big[c_* + \frac{i-1}{n}(c^* - c_*) \Big] \mu(P_i) \le U[f] \le \sum_{i=1}^n \Big[c_* + \frac{i}{n}(c^* - c_*) \Big] \mu(P_i). \tag{7.32}$$

On the other hand, by Corollary 7.7, there exists some g_0 such that $g_0 \sim f$. Then from (7.32) we conclude via Lemma 7.11 (and an apparent symmetric argument) that

$$\sum_{i=1}^n \Big[c_* + \frac{i-1}{n}(c^* - c_*) \Big] \mu(P_i) \le U_0[g_0] \le \sum_{i=1}^n \Big[c_* + \frac{i}{n}(c^* - c_*) \Big] \mu(P_i). \tag{7.33}$$

Then (7.32) and (7.33) lead to

$$U[f] = U_0[g_0] \quad \text{as } n \to \infty. \tag{7.34}$$

[14] Cf. Section A.6.

[15] Theorem 1 on page 79 of Savage (1972) was proved under the assumption that both f, g are bounded. In fact, Theorem 14.5 of Fishburn (1970, p.206) shows that the utility function u derived in Theorem 7.6 is bounded under SVG 1-7. See the footnote on page 80 in Savage (1972).

If (b) is the case, then by Lemma 7.9, all acts that satisfy (b) are equally preferable. In this case, it is easy to show that $U[f] = c^*$. And similarly, for case (c), it can be shown that $U[f] = c_*$.

(3) Finally, observe that (7.29) holds if we consider a combination of cases where f and g are in situations (a)-(c) above. □

CHAPTER III

Interlude: Additivity Condition of Subjective Probability

8. Introduction

This chapter addresses the issue concerning the additivity condition of the subjective probability measures derived in Savage's system as presented in Chapter II. In a section titled "Some mathematical details," Savage (1972, §3.4) discusses the requirements of the additivity condition of the probability measures derived in his theory of subjective expected utility, he says,

> It is not usual to suppose, as has been done here, that *all* sets have a numerical probability, but rather a sufficiently rich class of sets do so, the remainder being considered unmeasurable . . . the theory being developed here does assume that probability is defined for all events, that is, for all sets of states, and it does *not* imply countable additivity, but only finite additivity . . . it is a little better not to assume countable additivity as a postulate, but rather as a special hypothesis in certain contexts. (*ibid.* p.40, emphases added)

One main mathematical reason provided by Savage for *not* requiring the derived probability measures in his theory to be countably additive (to be made precise shortly) is that, according to him, there does not exist a countably additive extension of the Lebesgue measure that is defined on the set of *all* subsets of the unit interval (or the real line), whereas in the case of finitely additive measures, such

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an extension does exist. Since events are taken to be "all sets of states" in his system (which can be interpreted as all subsets of the reals), countable additivity is ruled out because of this claimed defect.

Savage's remarks refer to the basic problem of measure theory posed by Henri Lebesgue at the turn of the twentieth century known as the *problem of measurability*. In what immediately follows, we point out that Savage's take on the countably additive extension of Lebesgue measure was in fact inaccurate due to an oversight of set-theoretic details. We illustrate this point by way of a brief historical review of Lebesgue's measure problem and some of its further developments. The goal is to situate our critical assessment of Savage's mathematical arguments for not requiring countable additivity in the historical development of the measure problem. In our discussion, we will also address some common misunderstandings concerning the interpretational values of sophisticated mathematical models used in practical sciences like decision theory. This will be followed in Section 10 by an extensive analysis of the imbalance between finite additivity and the rich mathematical structure employed in Savage's decision model, which, as we will see, echoes a well-known difficulty associated with finitely additive probability measures in the context of infinite state space with countable partition. We see these discussions as providing sufficient reasons for introducing countable additivity to Savage-style decision models. Thus, based on the work of Villegas (1964), we introduce in Section 11 a new postulate which brings in countable additivity, we then discuss the roles countable additivity and Savage postulate 7 played in extending utilities from simple acts to all acts.

9. Some Set-theoretic Details

9.1. The measure problem. In his 1902 thesis Lebesgue raised the following question about the real line: Does there exist a measure m such that

(a) m associates with each bounded subset X of $\mathbb{R}$ a real number $m(x)$;

(b) m is not identically zero, i.e., $m(X) \neq 0$ for some X;

(c) m is translation-invariant: for any $X \subseteq \mathbb{R}$ and any $r \in \mathbb{R}$ define $X + r := \{x + r \mid x \in X\}$, then $m(X) = m(X + r)$;

(d) m is countably additive (or σ-additive), that is, if $\{X_n\}_{n=1}^{\infty}$ is a collection of pairwise disjoint bounded subsets of $\mathbb{R}$, then $m\big(\bigcup_n X_n\big) = \sum_n m\big(X_n\big)$?

Lebesgue developed his measure towards the solution to this problem and, unlike other attempts made around the same period (see Bingham, 2000), the measure developed by him, later known as the *Lebesgue measure*, was constructed in accordance with certain algebraic structure of sets of the real numbers.[1] As seen, the measure problem would be solved if it could be shown that the Lebesgue measure satisfies all the measurability conditions (a)-(d). Lebesgue's question was soon answered in negative by Vitali (1905), who showed that, with the Axiom of Choice (AC), there exist sets of real numbers that are not (Lebesgue) measurable.[2] This means that, in the presence of AC, Lebesgue's measure is definable only for a proper class of subsets of the reals, the remainder being considered unmeasurable.

Then a natural question to ask is whether or not there exists an *extension* of Lebesgue measure which not only agrees with Lebesgue measure on all measurable sets, but is also definable for non-measurable sets. Let us refer to this question as the *revised measure problem*. The problem gives rise to a more general question as to whether there exists a real-valued measure on any infinite set. To anticipate our discussion on subjective probability measures, let us reformulate the question in terms of probabilistic measures for some general infinite set. Let S be a (countably or uncountably) infinite set, a (probability) *measure* on S is a non-negative real-valued function μ on $\mathcal{P}(S)$ such that

i. μ is defined for all subsets of S;

ii. $\mu(\varnothing) = 0$, $\mu(S) = 1$;

[1] Let S be a nonempty set, a collection $\mathcal{F}$ of subsets of S is called a σ-algebra if (1) $\varnothing \in \mathcal{F}$ and $S \in \mathcal{F}$; (2) $X \in \mathcal{F}$ implies $S - X \in \mathcal{F}$; (3) $X_1, X_2, \ldots \in \mathcal{F}$ implies $\bigcup_n X_n \in \mathcal{F}$ and $\bigcap_n X_n \in \mathcal{F}$. In the case of the reals where $S = \mathbb{R}$, let $\mathscr{B}$ be the (*Borel*) σ-algebra generated by (the smallest σ-algebra that contains) all the sets of the form $(a, b]$ where a, b are real numbers and $a < b$. Define a *Lebesgue measure* (over Borel sets) to be the real-valued function on $\mathscr{B}$ such that $\mu(\varnothing) = 0, \mu(\mathbb{R}) = +\infty$, and, for any $a, b \in \mathbb{R}$ with $a < b$, $\mu\big((a, b]\big) = b - a$.

[2] See Appendix A.2 for a Vitali-type construction.

iii. μ is countably additive (or σ-additive), that is, if $\{X_n\}_{n=1}^{\infty}$ is a collection of pairwise disjoint bounded subsets of $\mathbb{R}$, then

$$\mu\Big(\bigcup_{n=1}^{\infty} X_n\Big) = \sum_{n=1}^{\infty} \mu(X_n). \tag{9.1}$$

Here we distinguish two cases depending on the cardinality of S: If S contains uncountably many elements (e.g., $S = \mathbb{R}$), it is now known that an extension of Lebesgue measure exists if and only if there exists a measure on the continuum (or on any S with $|S| = 2^{\aleph_0}$) satisfying conditions (i)-(iii). Hence, the revised measure problem is solved if the latter can be answered. By referring to a result of Ulam (1930) on measures on infinite sets (cf. Footnote 6 below.), Savage gave a definitive answer that such an extension does *not* exist. One main aim of this section is to point out that, in fact, there is no straightforward answer to this question: the existence of a countably additive measure on $\mathcal{P}(S)$ that extends Lebesgue measure depends on the background set theory one chooses to work in. And this question is in close connection with the theory of large cardinals, we will return to this point presently in Section 9.2 below.

If, on the other hand, S contains only countably many elements (e.g., $S = \mathbb{N}$), then it is interesting to note that μ cannot be both countably additive and uniformly distributed (or, for that matter, μ cannot be a measure that assigns 0 to all singletons). Indeed, let $\{s_1, s_2, \ldots\}$ be an enumeration of all the elements in S. Suppose that μ is uniformly distributed on S, then it must be that $\mu(s_i) = 0$ for all $i \in \mathbb{N}$.[3] But, by countable additivity, $1 = \mu(S) = \mu\big(\bigcup_{i=1}^{\infty}\{s_i\}\big) = \sum_{i=1}^{\infty} \mu(s_i) = 0$, which is absurd. Hence there does not exist a σ-additive uniform distribution on a countable set. This, in part, is the reason why de Finetti opposed the employment of countable additivity across the board arguing that a rational agent should be ready to believe that the tickets in a countably infinite lottery be equally likely to win, a view shared, to a large extent, by Savage. This leads to the suggestion of weakening the additivity condition (iii), to which we now turn.

[3]We write $\mu(\{s_i\})$ as $\mu(s_i)$ for short.

9.2. Large cardinals and additivity conditions. An alternative option is to replace (iii) with the following condition.

iv. μ is finitely additive, that is, for any $X, Y \subseteq S$, if $X \cap Y = \varnothing$ then

$$\mu(X \cup Y) = \mu(X) + \mu(Y). \tag{9.2}$$

It is clear that (iii) implies (iv) but not vice versa, hence this condition amounts to weakening the additivity condition on subjective probabilities.

Admittedly, the employment of finitely additive probability measures has far-reaching merits. It can be shown that there does exist a finitely additive uniform distribution on S with at most countably many elements satisfying properties (i), (ii), and (iv).[4] In addition, in justifying his subjective interpretation of probability, de Finetti (1937b) showed that a rational player affords at least the possibility of avoiding exposure to a sure loss if and only if the set of betting quotients upon which the player accepts satisfies conditions (i), (ii), and (iv). More precisely, let S be a space of possible states, $\mathcal{F}$ be some algebra equipped on S, members of $\mathcal{F}$ are referred to as events. An event E *occurs* just in case $s \in E$ where s is the true state of the world. Let $\{E_1, \ldots, E_n\}$ be a finite partition of S where each $E_i \in \mathcal{F}$. Further, let $\mu(E_i)$ represent the decision maker's degree of belief on the occurrence of E_i, which is manifested, it is assumed, in her betting behavior that $\mu(E_i)$ is the rate at which the decision maker is willing to enter a bet dependent on the occurrence of E_i with a payoff of c_i for a cost of $c_i\mu(E_i)$, where c_i is decided by the opponent and is either positive or negative. The decision maker is said to be *coherent* if there is no selection of $\{c_i\}_{i=1}^n$ by the opponent such that $\sup_{s \in S} \sum_{i=1}^n c_i\big(\chi_{E_i}(s) - \mu(E_i)\big) < 0$, where χ_{E_i} is the characteristic function of E_i. In other words, the agent's subjective probability assignments are coherent if no sequence of bets can be arranged by the opponent that yields uniformly negative returns for the bettor regardless which state of the world actually obtains. Guided by this coherence principle, de Finetti showed that there exists at least one measure μ which admits that, for

[4]Appendix A.4 contains a set-theoretic construction of one such example, namely the existence of a finitely additive uniform distribution over the integers.

any selection of payoffs $\{c_i\}_{i=1}^n$, $\sup_{s\in S}\sum_{i=1}^n c_i\big(\chi_{E_i}(s)-\mu(E_i)\big)\ \geq\ 0$. In addition, it was shown by de Finetti (1930a) that μ can be extended to any collection of events containing $\mathcal{F}$, in particular, μ can be extended to $\mathcal{P}(S)$ so that condition (i) is satisfied; and that μ is a finitely additive probability measure, that is to say, μ satisfies (ii) and (iv). According to Regazzini (2013), these mathematical results developed by de Finetti in the 1920-30s played an important role in shaping his view on the issue of additivity, from which he concluded that there is *no need* to insist on countable additivity for probability measures on an algebra of events.[5]

Savage referred to the above conviction of de Finetti's (as well as a similar result given by Banach (1932)) as part of his mathematical reasons not to impose countable additivity. Before proceeding any further, let us recapitulate the main reasons that led Savage to take that "it is a little better not to assume countable additivity as a postulate":

1. There does not exist a countably additive uniform distribution over integers, whereas in the case of finite additivity such a distribution does exist.
2. According to Savage, there does not exist a countably additive extension of the Lebesgue measure to the set of all subsets of the reals, whereas in the case of finite additivity such an extension does exist.

Note that, for reason (1), Savage provided no further comments on the relationship between subjective probabilities derivable from his decision-theoretic model and uniform distribution over integers, he only briefly mentioned that many of us do have a strong intuitive tendency to regard such a distribution as necessary (p.43). We will not pursue any further as to under what circumstances this intuitive appeal to uniform distribution over integers may be grounded or refuted (which is based on an apparent symmetry consideration and it is an important topic of its own). Our contention however is that, granted that uniform distribution over

[5]The first chapter of de Finetti (1937b, English translation as de Finetti (1937a)) contains a non-technical summary of de Finetti (1930a, 1931, in Italian) on "the logic of the probable" where the aforementioned mathematical results were given (see Regazzini, 2013). Given that our current focus is on Savage's system, we shall not delve further into the discussion on de Finetti's reasons for rejecting countable additivity, for recent discussions, see Williamson (1999); Regazzini (2013).

integers is permissible in Savage's model, finite additivity is still insufficient in bringing about a coherent system with rich background settings: in Section 10 we will discuss a classical example, involving finitely additive uniform distribution over integers, which leads to the making of a money pump. More precisely, following the works of Adams (1962) and Wakker (1993), among others, we point out that the doctrine that (subjective) probability measure be merely finitely additive may stand in violations of various rationality constraints that are commonly adopted in decision-theoretic models.

Now, let us turn to reason (2). Savage (1972, p.41) cites the well-known result of Ulam (1930) testifying that any atomless σ-additive extension of Lebesgue measure to *all* subsets of the unit interval is incompatible with the continuum hypothesis (CH), from which he concludes that there is no extension that satisfies all of (i)-(iii). However, it is unclear as to why this constitutes a sufficient reason for rejecting countable additivity.[6]

As a matter of fact, in his article titled "a model of set theory in which every set of reals is Lebesgue measurable" Solovay (1970) showed that such a countably additive extension of Lebesgue measure to all sets of reals *does* exist if the existence of an inaccessible cardinal (I) and a weaker version of AC, i.e. the principle of dependent choice (DC), are assumed.[7] Thus, it seems that insofar as the possibility of obtaining a σ-additive extension of Lebesgue measure to all subsets of the reals

[6] Ulam (1930) proved that, for any uncountable set S with $|S| = \kappa$, it can be shown in ZFC that if κ is a successor (and hence a regular) cardinal (e.g., $\kappa = \aleph_1$), then there does not exist a measure on S satisfying all of (i)-(iii). It follows that if there is a σ-additive non-trivial extension of Lebesgue measure on $2^{\aleph_0}$ then CH must fail. (It is worth mentioning that, prior to Ulam, Banach and Kuratowski (1929) showed that if there is a measure on $2^{\aleph_0}$ then $2^{\aleph_0} > \aleph_1$.) Yet, even without the concern for CH, there is an aspect of Ulam's original results that was not addressed by Savage: it was shown by Ulam (1930) that, in ZFC, if there is a σ-additive non-trivial measure μ on any uncountable set S with $|S| = \kappa$ then μ is a measure on κ such that

(1) either κ is a measurable cardinal (and hence an inaccessible cardinal), on which a nontrivial σ-additive two-valued measure can be defined;
(2) or κ is a real-valued measurable cardinal (and hence a weakly inaccessible cardinal) such that $\kappa \leq 2^{\aleph_0}$, on which a nontrivial σ-additive atomless measure can be defined.

In the second case, it is plain that μ can be extended to a measure on $2^{\aleph_0}$: for any $X \subseteq 2^{\aleph_0}$, let $\mu(X) = \mu(X \cap \kappa)$. This leads to a general method of obtaining a countably additive measure on all subsets of the reals that extends Lebesgue measure (see Jech, 2003, p. 131).

[7] The relative consistency proof given by Solovay (1970) showed that if ZFC+I has a model then ZF+DC has a model in which every set of reals is Lebesgue measurable (see also Jech, 2003, p.50).

is concerned, Savage's set-theoretic argument, which calls for immediate exclusion of countable additivity, is inconclusive. This is because the existence of such an extension really depends on the background set theory: does not exist in ZFC+CH, but does exist in ZF+DC (assuming ZFC+I is consistent).

In view of the set-theoretic details above, a natural reaction one may have is that in order to determine whether or not to incorporate countable additivity in Savage's personal decision theory further investigation concerning the "appropriate" set theory that one should work with needs to be made. However, to provide a theoretic reason for choosing one set of set-theoretic axioms over another is no easy matter, it has become an exceedingly involved issue within the proper subject of set theory.[8] At this point, one might be puzzled that the task of isolating the "correct" axioms of large cardinals, although highly interesting as a subject of its own, might be of remote philosophical significance in revealing the role of additivity played within decision-theoretic models, especially when it comes to the development of a personal decision theory of a more "mundane kind," which was what Savage initially set forth to do. Hence, in defending finite additivity, one might be tempted to appeal to some practical reasons in matching up with the appropriate mathematical details contending that countable additivity should be ruled out precisely because the cost from set-theoretic complications is high, whereas in the case of finite additivity such complications can be largely minimized. Well, this suggested roundabout approach of circumventing set theory by suspending countable additivity would be celebrated if the strategy of restricting to merely finitely additive measures could serve the full purpose of keeping Savage's system in good health, which, as we will see, is not the case. We will return to this point in Section 10.

At this point, we would like to remark on a perhaps more important point: it would be a mistake to think that the issues associated with employing sophisticated mathematical constructions in modeling subjective probabilities are of

[8] See Gaifman (2012) and Koellner (2013) for pertinent discussions.

"merely internally mathematical interest" and have little conceptual importance in uncovering the philosophical understanding of the notion of probability and hence one should avoid such constructions when possible. This prejudice against the use of complicated formalism is based on a misconception about the role of mathematical models. One should bear in mind that there is an indisputable distinction between a mathematically sophisticated model of probability and the common-sense understanding of probability by "men in the street." But there is no conflict between having a good grasp of an everyday concept and forming a sophisticated theory about it. In fact, the latter often serve as an illuminating guide in achieving the former. As an example, we discuss in the next a few paragraphs how detailed set-theoretic considerations may shed light on our understanding of subjective probabilities and the spaces on which they are defined.

9.3. Logical omniscience and non-measurable sets. To be sure, Savage's set-theoretic argument for not requiring countable additivity was given in ZFC+CH, where it is known that, in the case of uncountable state space, there is no non-trivial measure satisfies simultaneously conditions (i) - (iii) above; and Savage's immediate reaction was to revise the third, i.e., the additivity condition, and restrict the attention to finitely additive measures.[9] The particular set-theoretic argument Savage relied on, namely the existence of Ulam matrix (which leads to the non-existence of a measure over $\aleph_1$, see Footnote 6), uses AC in an essential way. The latter allows for the construction of non-measurable sets in ZFC. Now if one insists on maintaining the first measurability condition of defining subjective probability measure for *all* sets of states, this amounts to introducing non-measurable sets into Savage's decision-theoretic model as representing certain events. Yet, it

[9]Savage is not alone in holding this view. Seidenfeld (2001) listed the non-existence of a non-trivial, σ-additive measure defined over the power set of an uncountable set as the first of his six reasons for considering theory of finitely additive probability. It is interesting to note that Seidenfeld also referred to the result of Solovay, however no further discussion on the significant of this result on additivity was given (see Seidenfeld, 2001, p.168). See Bingham (2010, §9) for a discussion and responses to each one of Seidenfeld's six reasons. Our set-theoretic argument presented here, in response to Seidenfeld's first, i.e., the measurability reason, is different from Bingham's "approximation" argument.

is unclear what one can benefit from making such high demand. Note that non-measurable sets are meaningful only insofar as we have a good understanding of the contexts in which they apply. These sets are highly interesting within certain branches of mathematics largely for the reason that their introduction reveals in a deep way the complex structures of the underlying mathematical systems. This however does not mean that these peculiar set-theoretic constructions should be carried over to a system that is primarily devised to guide rational decision making by rational agents.

Here, one might respond by pointing out that, as a normative theory, the current decision framework is designed only for *idealized* decision makers, not actual ones, hence it should be within the bounds of our super agents to conceive non-measurable sets on which subjective probabilities be defined. This line of response, we stress, misses the point of theoretical idealization. Indeed, in various highly idealized prescriptive decision-theoretic models, including Savage's original system and our analysis in later sections, it is assumed that the decision makers are equipped with extraordinary abilities. What we usually expect from these super agents are their supreme logical ability and exceptional computational capacity. However, it should be emphasized that this step of idealization is not based on a blind leap of faith, it is grounded in our understanding of the basic logical apparatus and computational processes involved. We acknowledge that, as actual reasoners, our inferential performances are bounded by various physical limitations, which prevent us from reaching too far. Yet a good grasp of the underlying logical machinery gives rise to the conceivable picture as to what it means for a logically omniscient agent to fulfill, *in principle*, the task of drawing all the logical consequences from a given proposition.[10] This justificatory picture, which is based

[10]With these being said, we should however also point out that the demand to have a deductively closed system remains as a challenge to any normative theory of beliefs. In his essay titled "difficulties in the theory of personal probability," Savage (1967, p.308) remarked that the postulates of his theory imply that one should behave in accordance with the logical implications of all that she knows, which can be very costly. In other words, conducting logical deductions is a very resource consuming activity, the merits it brings can sometimes be offset by its high costs. Hence some might say that the assumption of logical omniscience, a promise of the discharge of unlimited deductive resources, may at its best be seen as an unfeasible idealization. Here is not the place to open a new

on an apparent inductive reasoning, becomes increasingly blurry when we start asking our super agent to contemplate the intriguing nature of non-measurable sets in the context of rational decision making. The Banach-Tarski paradox sets just the example of how much we lack geometric and physical intuition when it comes to non-measurable sets. That means, unlike logical omniscience, we don't even have a clear idea of what to demand from our super agent: there is not a well-founded inductive basis for the jump to non-measurable sets beyond any specific set-theoretic context. Hence, it seems that if there is any set-theoretic oddity to be avoided in a personal decision theory it should be non-measurable sets.[11]

On this matter, we should add that Savage himself is fully aware that the set theoretical framework under which his personal decision model is being developed exceeds what one can expect from a rational decision maker who uses subjective probabilities to encompass the best courses of actions and fend against incoherency (Savage, 1967). He also cites the Banach-Tarski paradox as an example to show the extent to which highly abstract sophisticated set theory can contradict common sense intuitions. However, it seems that Savage's readiness to include as events "all sets of states" outshines his willingness to avoid this set-theoretic oddity. In practice, he takes the set of all subsets of the state space, the power set of the continuum in the case of the reals, as the background algebra, on which his subjective probabilities be defined. We however will go with a different approach.

In fact, the situation can be largely simplified if we choose to work, instead of with *all* subsets of the state space S, but with a sufficiently rich collection of subsets of S (for instance, the Borel sets $\mathscr{B}$ in the case where $S = \mathbb{R}$) where, as a

discussion on the legitimacy of logical omniscience. The point we are trying to make is rather that, unlike non-measurable sets, being logically all powerful, however unfeasible, is not something that is not conceptually entertainable.

[11] In an unpublished work, professor Haim Gaifman made a similarly point against the often cited analogy between finitely additive probabilities in countable partitions and countably additive probabilities in uncountable partitions in the literature (see, e.g., Schervish and Seidenfeld, 1986) that such an analogy plays a major heuristic role in set theory but provides no useful guideline in the case of subjective probabilities for the reason that certain mathematical structures required to make salient this analogy has no meaning in a personal decision theory.

well established theory, countable additivity is in perfectly good health. That is to say, instead of (i), we require that

v. μ is defined on (Lebesgue) measurable sets of S.

Note that this price of forfeiting the demand of defining probability measures on the set of all subsets of S is a rather small one to pay. It amounts to disregarding all those events that are defined by Lebesgue non-measurable sets. Indeed, even Savage himself conceded that

> All that has been, or is to be, formally deduced in this book concerning preferences among sets, could be modified, mutatis mutandis, so that the class of events would not be the class of all subsets of S, but rather a Borel field, that is, a σ-algebra, on S; the set of all consequences would be measurable space, that is, a set with a particular σ-algebra singled out; and an act would be a measurable function from the measurable space of events to the measurable space of consequences. (Savage, 1972, p.42)

It should be emphasized that this modification of definition of events from, say, the set of all subsets of $(0,1]$ to all the Borel set $\mathscr{B}$ of $(0,1]$ is not carried out at the expense of disregarding a large collection events that are otherwise representable. As noted by Billingsley (2012, p.23), "[i]n fact, $\mathscr{B}$ contains all the subsets of $(0,1]$ actually encountered in ordinary analysis and probability. It is large enough for all 'practical' purposes." Here, by "practical purposes" we take as meaning that all events and measurable functions considered in economic theories in particular are definable using only measurable sets, and, consequently, there is no need to appeal to non-measurable sets. Besides, it is easily seen that this way of restricting to Lebesgue measurable sets also has the advantage of circumventing the set-theoretic complications discussed above. We will proceed with our discussions of Savage's subjective utility representation theory in this spirit, where we require, unless otherwise specified (cf. Chapter IV), that all derived subjective probability measures be defined on the set of measurable sets of the state space and that they are countably additive, that is, they satisfy conditions (ii), (iii), and (v).

All in all, Savage chose to base his theory of subjective expected utility on finitely additive probability measures. This is because he took, for various mathematical considerations, that countable additivity is *insufficient* for the goals he set for his decision-theoretic model, whereas finite additivity, it is said, is sufficient for these purposes, which led him to think that it is *not necessary* to invoke the countable additivity condition, "I know," says Savage, "of no argument leading to the requirement of countable additivity" (*ibid.* p.43). The discussion we have made so far suggests that Savage's claim against countable additivity is inadequate due to an oversight of set-theoretic details. The main aim of the next section is then to provide an argument for the requirement of countable additivity in Savage's system: in Section 10, we point out that the restriction to merely finitely additive measures may render Savage's theory incoherent. These considerations will lead in Section 11 to the introduction of countable additivity as a formal component of the theory. We then discuss various attempts to extend utility function for simple acts to acts in general under Savage's P7 and countable additivity.

10. Finite and Countable Additivity in Savage's system

10.1. Strict finitism on additivity. Let us refer to the thesis that (subjective) probabilities be finitely but *not* countably additive as *strict finitism on additivity*. Note that this is not to be confused either with strict finitism (or ultrafinitism) in philosophy of mathematics which is a school of thoughts that casts doubt, in one form or another, on large finite sets[12] (let alone the use of infinite mathematical objects); or with finitism which allows the use of the infinite only in a controlled and reductively finitary manner. By strict finitism on additivity we mean the anti-pluralist position towards additivity that probability measures can only be finitely additive in all mathematical systems in which they apply. It is plain that probability measures are finitely additive in a finitary mathematical system, the dispute

[12]See Parikh (1971).

lies in whether subjective probabilities should be merely finitely additive in any given infinitary system. Clearly, Savage does not question the use of infinitary mathematics. In fact, many aspects of his theory (examples, proofs, etc.) use infinity in an essential way. Yet efforts were made by Savage to bring justification for finite additivity in his decision model leaving countable additivity "as a special hypothesis in certain contexts." In this section, we argue that in many ways Savage's own decision-theoretic framework is a context in which countable additivity should apply. We proceed by illustrating that strict finitism on additivity may lead to some discomforts in Savage's system.

As noted above, the derived probability measure in Savage's subjective expected utility theory features, among other things, the following properties:

Finitely Additive: The probability measure μ on the state space S is finitely additive.

All-inclusive: Events are defined as all subsets of S, not a collection of events measurable with respect to some σ-field of subsets of S smaller than $\mathcal{P}(S)$.

Atomless: The state space S is (uncountably) infinite and the probability μ is atomless.[13]

We point out an interesting property in systems that feature these three properties: it can be shown that for each disjoint sequence $\{A_i\}$ for which countable additivity fails there exists another sequence $\{C_i\}$ such that, for each i, the probability of A_i agrees with that of C_i while countable additivity holds for $\{C_i\}$.

Proposition 10.1. Let $\{A_i\}_{i=1}^{\infty}$ be any sequence of disjoint subsets of S, if μ is finitely additive, universal, and atomless then $\sum_{i=1}^{\infty} \mu(A_i) \neq \mu(\bigcup_{i=1}^{\infty} A_i)$ implies that there exists a partition $\{C_i\}_{i=0}^{\infty}$ of S satisfying

(1) $\mu(\bigcup_{i=1}^{\infty} C_i) = \sum_{i=1}^{\infty} \mu(A_i)$;

[13]More precisely, a set A is said to be an *atom* of a given measure μ if, for every $X \subseteq A$, either $\mu(X) = 0$ or $\mu(A - X) = 0$. The measure μ is *atomless* if there are no atoms. The atomlessness property is implied by the following stronger condition derivable in Savage's system: for any $A \subseteq S$ and any $0 \leq \rho \leq 1$, there exists $B \subseteq A$ such that $\mu(B) = \rho\mu(A)$, (cf. Savage, 1972, p.34, Theorem 2). In the following, it is this stronger condition we will be referring to as the atomlessness condition.

(2) $\mu(C_i) = \mu(A_i) \quad$ for all $i = 1, 2, \ldots;$

(3) $\sum_{i=1}^{\infty} \mu(C_i) = \mu(\bigcup_{i=1}^{\infty} C_i)$.

Proof. Suppose that $\{A_i\}_{i=1}^{\infty}$ is a sequence of pair-wise disjoint subsets of S such that $\sum_{i=1}^{\infty} \mu(A_i) \neq \mu(\bigcup_{i=1}^{\infty} A_i)$. The latter implies that exactly one of $<$ or $>$ holds. By finite additivity, $\sum_{i=1}^{n} \mu(A_i) = \mu\big(\bigcup_{i=1}^{n} A_i\big) \leq \mu\big(\bigcup_{i=1}^{\infty} A_i\big)$. So it can only be that

$$\sum_{i=1}^{\infty} \mu(A_i) < \mu\Big(\bigcup_{i=1}^{\infty} A_i\Big). \tag{10.1}$$

By the atomlessness property above, there exists some $B \subseteq S$ such that

$$\mu(B) = \sum_{i=1}^{\infty} \mu(A_i). \tag{10.2}$$

Similarly, there exists some $B_1 \subseteq B$ for which $\mu(B_1) = \mu(A_1)$. By finite additivity, we have $\mu(B - B_1) = \mu(\bigcup_{i=2}^{\infty} A_i)$. Using the same argument, there exist some $B_2 \subseteq B - B_1$ for which $\mu(B_2) = \mu(A_2)$. Repeat this process until we form a sequence $\{B_i\}_{i=1}^{\infty}$ of subsets of B such that $B_{i+1} \subseteq B - \bigcup_{j=1}^{i} B_j$ and $\mu(B_i) = \mu(A_i)$ for all $i = 1, 2, \ldots$. This together with (10.2) yield that

$$\mu(B) \geq \mu\Big(\bigcup_{i=1}^{\infty} B_i\Big) \geq \sum_{i=1}^{\infty} \mu(B_i) = \sum_{i=1}^{\infty} \mu(A_i) = \mu(B). \tag{10.3}$$

From which we conclude, via finite additivity, that $\mu(B - \bigcup_{i=1}^{\infty} B_i) = 0$. The proof is completed if we let $\{C_i\}_{i=0}^{\infty}$ be a partition of S such that $C_0 = (S - B) \cup (B - \bigcup_{i=1}^{\infty} B_i)$ and $C_i = B_i$ for all $i = 1, 2, \ldots$. □

In light of Proposition 10.1, it seems that, *probabilistically speaking*, there is little point in restricting to merely finitely additive measures in a system that features all three properties above. For, given any finitely (but not countably) additive measure satisfying all three properties, the proof above shows that, for every sequence of events that fails countable additivity, there exists another sequence of events which agrees with the original sequence on all local probabilistic details such that countable additivity holds under the same measure!

It may be argued that even though each C_i agrees probabilistically with A_i for all i's, this does not mean that $\{A_i\}$ can be replaced by $\{C_i\}$: they are after all different sequences of events and hence may lead to different configurations/partitions of the state space and different constructions of Savage acts. On this view, Proposition 10.1 can at most be seen as a peculiar feature of the system while maintaining that $\{A_i\}$ and $\{C_i\}$ remain as distinctive strings of events with different probabilistic properties. However, it should be pointed out that the admission of sequences of events with same local but different global probabilistic properties opens the door to various counter-intuitive examples, which can be constructed using precisely the imbalance between their different probabilistic characterizations. To name one such example, note that, in the proof of Proposition 10.1, if let A_0 denote $S - \bigcup_{i=1}^{\infty} A_i$, it is easily seen that the proposition implies that for any partition $\{A_i\}_{i=0}^{\infty}$ for which countable additivity fails there exists a partition $\{C_i\}_{i=0}^{\infty}$ such that the following holds: (1) $\mu(A_0) < \mu(C_0)$; (2) $\mu(A_i) = \mu(C_i)$, for all $i = 1, 2, \ldots$; (3) $\mu\big(\bigcup_{i=1}^{\infty} A_i\big) > \mu\big(\bigcup_{i=1}^{\infty} C_i\big)$. Using partitions like these, Wakker (1993) constructed two Savage acts that stand in clear violation of the (strict) *Stochastic dominance*.[14] The latter, however, is an intuitive rationality principle that is widely adopted in various decision-theoretic models. It would be scandalous if this rationality principle is violated in Savage's system.

To be sure, the issue we address here is closely related to what is called the *conglomerability* property for probability measures first defined by de Finetti (1930b). Without delving too far into this literature, we should briefly mention that Hill and Lane (1985) confirmed de Finetti's conjecture that a probability measure is conglomerable if and only if it is countably additive. Schervish and Seidenfeld

[14](Strict) Stochastic dominance mandates that act f is strictly preferred to act g if the probability of f yielding any consequence that is "at as valuable as" x is no less than that of g for all consequences $x \in X$ and with some strict inequality at some x:

$$\left.\begin{array}{l}\mu[f \succsim x] \geq \mu[g \succsim x] \text{ for all } x \in X \\ \mu[f \succsim x] > \mu[g \succsim x] \text{ for some } x \in X\end{array}\right\} \Longrightarrow f \succ g.$$

Assuming that consequences are linearly ordered with endpoints, Wakker (1993) showed that there are f and g constructed, respectively, from $\{A_i\}$ and $\{C_i\}$ such that f strictly dominates g even though they have the same expected utility.

(1986) showed that the failure of countable additivity (for the cases with countable partitions) may lead to the failure of dominance. That is, if μ is not countably additive, then there exist two acts f and g such that f is conditionally preferred to g for each cell of the partition $\mathcal{H} = \{H_1, H_2, \ldots\}$, yet f is not unconditionally preferred to g. This stands in clear violation of Savage's "loose" version of the sure-thing principle (cf. Savage (1972, p.22) and Section 5.2 above). So it seems that not requiring countable additivity can be a somewhat self-undermining strategy for Savage in setting up his decision model, for it may lead to violations of one of his most fundamental postulates for rational decision making.

10.2. Money pump. Prior to Wakker, Adams (1962) showed that there are scenarios in which the failure of countably additivity leads to a money pump. More precisely, Adams' example presents a betting situation where a (Bayes) rational gambler is justified in accepting, with a small fee, each bet of a sequences of bets, but the acceptance of all the bets leads to sure loss. For illustrative purpose, we reproduce this example here.[15]

Example 10.2. Let $S = \mathbb{N}$, $X = [-1, 1]$, and let the identity function $u(x) = x$ be the utility function on X. Let λ be the finitely but not countably additive measure on positive integers given in Example A.4.7 (in Appendix A.4) and let η be the countably additive probability measure on S given by

$$\eta(n) = \frac{1}{2^n} \quad \text{for all } n \in S.$$

Define subjective probability μ to be such that

$$\mu(n) = \frac{\lambda(n) + \eta(n)}{2} \quad \text{for all } n \in S. \tag{10.4}$$

The following is a list of simple properties of μ.

(1) μ is a finitely but not countably additive probability measure.

[15] The following example is altered from Stinchcombe (1997).

TABLE 10.1. Gamble g_n

	$B_n = \{n\}$	B_n^C
g_n	$1/2^{n+1} - r$	$1/2^{n+1}$

(2) For any $n < \infty$,

$$\mu(n) = \frac{0 + \eta(n)}{2} = \frac{1}{2^{n+1}}.$$

(3) $\mu(S) = 1$, whereas

$$\sum_{i=1}^{\infty} \mu(i) = \frac{\sum_{i=1}^{\infty} \lambda(i) + \sum_{i=1}^{\infty} \eta(i)}{2} = \frac{0+1}{2} = \frac{1}{2}.$$

(4) For any finite $E \subseteq S$,

$$\mu(E) = \frac{0 + \eta(E)}{2} = \sum_{i \in E} \frac{1}{2^{i+1}}.$$

Now, for each n, consider the gamble g_n with payoff described as in the matrix in Table 10.1. That is, g_n pays constantly $1/\left(2^{n+1}\right)$ no matter which state obtains, but will cost the gambler $r \in (\frac{1}{2}, 1)$ in the event of $B_n = \{n\}$ (see Table 10.1). Since $r < 1$, it is easy to calculate that, for each n, g_n has a positive expected utility:

$$U[g_n] = \frac{1}{2^{n+1}} \cdot \left(\frac{1}{2^{n+1}} - r\right) + \left(1 - \frac{1}{2^{n+1}}\right) \cdot \frac{1}{2^{n+1}} = \frac{1}{2^{n+1}} \cdot (1 - r).$$

Hence, a (Bayes) rational gambler should be willing to pay a small fee ($< U[g_n]$) to accept each gamble. However, the acceptance of all gambles leads to sure loss no matter which number eventually transpires. To see this, note that for any given m gamble g_n pays

$$g_n(m) = \frac{1}{2^{n+1}} - r \cdot \chi_{B_n}(m)$$

But, the joint of all g_n's yields

$$\sum_{i \in S} g_i(m) = \sum_{i \in S} \left[\frac{1}{2^{i+1}} - r \cdot \chi_{B_i}(m)\right] = \frac{1}{2} - r < 0.$$

That is to say, for each possible outcome $m \in S$, the expected value of getting m from accepting all the gambles g_n's is negative. ◁

10.3. Infinite bets. Adams' money pump is surprising, because it results in a series of *incoherent* choices made by the gambler which is precisely what Savage's subjective system is devised to prevent.

In dealing with this difficulty, advocates of finite additivity often argue that given that a subjective theory is a systematic characterization of coherent decision making by rational decision makers it is unclear as to what it means for a decision maker to fulfill the task of coherently choosing *infinitely* many times. Hence, on this view, the challenge from Example 10.2, which requires the gambler to accept infinitely many gambles, is really a non-starter, for it envisages a situation that is not conceptually admissible within a finitary framework with respect to coherent choosing which is best captured by finite additivity.[16]

We however differ from this reading of accepting infinitely many gambles. As noted before, there is a great deal of idealization built in the current decision-theoretic framework where we grant our decision maker with unlimited computing capacity. Again, this step of idealization is grounded in our understanding of the basic logical and computational apparatuses involved. The process of idealization is then the process of disregarding the physical limitations of our performances as actual reasoners. Admittedly, being deductively all-capable does not necessarily imply that our super agent has the ability of handling infinitely many gambles; besides, as we have discussed in the case of non-measurable sets, not all forms of idealization are meaningful for the purpose of developing a normative theory for personal decision making. However, it seems that the same inductive justification for the assumption of logical omniscience can be employed to provide a conceptual basis for extending the decision problem to include infinite gambles. Note that the procedure involved in deciding whether or not any given gamble g_n is acceptable is well understood (i.e., calculating its expected utility). The process of idealization is then the process of disregarding the sheer quantity of gambles a (Bayes) rational decision maker is obliged to accept, an ability our super agent

[16] We thank professor Isaac Levi for pointing out this line of objection, which echoes de Finetti's position that a rational agent is obliged to accept no more than finitely many fair bets at any time.

is expected to possess. Unlike non-measurable sets, this process is not something that we, *qua* actual reasoners, cannot conceive. It would be a double standard to insist that the decision maker should be infinitely capable when it comes to computational and inferential performances, but to appeal to physical/psychological realism when it comes to accepting bets.

11. Countable Additivity and Utility Extension

11.1. Quantitative and qualitative continuities. Note that, in Savage's representation theorem, it is crucial that the underlying algebra is closed under infinite unions and intersections. Indeed, Savage himself remarks that

> It may seem peculiar to insist on σ-algebras as opposed to finitely additive algebras even in a context where finitely additive measures are the central objects, but countable union do seems to be essential to some theorems of §3 ... (Savage, 1972, p.43)

This "peculiar" feature implies that, in a strictly finitely additive system, it is not always the case that the convergence of an infinite sequence of events at the limit point can be characterized in the corresponding probabilistic terms due to failure of continuity. To be more precise, let $(S, \mathcal{F}, \mu)$ be a measure space, μ is said to be *continuous from below*, if, for any sequence of events $\{A_n\}_{n=1}^{\infty}$ and event A in $\mathcal{F}$, $A_n \uparrow A$ implies that $\mu(A_n) \uparrow \mu(A)$; it is *continuous from above* if $A_n \downarrow A$ implies $\mu(A_n) \downarrow \mu(A)$, and it is *continuous* if it is continuous from both above and below.[17] It can be easily shown that continuity fails in general, if μ is merely finitely additive. As an illustration, it is interesting to note that the strictly finitely additive measure λ (Example A.4.7) used in constructing the probability measure in Adams' example is neither continuous from above nor from below. In fact, it can be proved that the aforementioned properties of continuity hold if and only if μ is countably additive.

[17] As notational conventions, $A_n \uparrow A$ means that $A_1 \subseteq A_2 \subseteq \cdots$ and $\bigcup_i A_i = A$, and $\mu(A_n) \uparrow \mu(A)$ means that $\mu(A_1) \leq \mu(A_2) \leq \cdots$ and $\mu(A_n) \to \mu(A)$ as $n \to \infty$. Similar for the other case.

Intuitively speaking, in order to establish continuity it is necessary that the set functions in question (in our case derived probability measures) be sensitive to the set operations of infinite union and intersection at limit points. This balance is undermined if we allow, on the one hand, infinite unions and intersections but do not require underlying measures to be countably additive on the other. This gives rise to the mismatch between accepting infinitely many gambles and the corresponding probability calculation, which ultimately is what fueled Adams' money pump.

One way to introduce countable additivity, and hence continuity, to Savage's system is to strengthen the underlying qualitative probability so that the aforementioned continuity conditions can be satisfied. Following Villegas (1964), let $\succeq$ be a qualitative probability (Definition 6.1) defined on a σ-algebra $\mathcal{F}$ of the state space S, $\succeq$ is said to be *monotonely continuous* if, given any sequence of events $A_n \uparrow A$ ($A_n \downarrow A$) and any event B,

$$A_n \preceq B \, (A_n \succeq B) \ \text{ for all } n \quad \Longrightarrow \quad A \preceq B \, (A \succeq B). \tag{11.1}$$

Moreover, Villegas showed that if a qualitative probability $\succeq$ is atomless and monotonely continuous then the numerical probability μ that agrees with $\succeq$ is unique and countably additive.[18]

Since the qualitative probability measures in Savage's system are non-atomic, it is sufficient to introduce the the property of monotone continuity in order to introduce countable additivity. We thus propose the following postulate, P8, to be added to Savage's P1-7 (cf. SVG 1-7 above), which is a reformulation of (11.1) in terms of preferences among Savage acts.

P8: For any $a, b \in X$ and for any event B and any sequence of events $\{A_n\}$,

[18] Villegas (1964) showed that monotone continuity is a necessary and sufficient condition for the agreeing numerical measure to be countably additive. It was further shown that any qualitative probability defined on a finitely additive algebra can be extended to a qualitative probability σ-algebra satisfying monotone continuity, fineness, tightness. Thanks largely to Savage's P6, the qualitative probabilities derived in the system are atomless, fine, and tight, then countable additivity obtains if the monotone continuity is in place.

(i) if $A_n \uparrow A$ and $\mathfrak{c}_a \oplus_{A_n} \mathfrak{c}_b \precsim \mathfrak{c}_a \oplus_B \mathfrak{c}_b$ for all n then $\mathfrak{c}_a \oplus_A \mathfrak{c}_b \precsim \mathfrak{c}_a \oplus_B \mathfrak{c}_b$;

(ii) if $A_n \downarrow A$ and $\mathfrak{c}_a \oplus_{A_n} \mathfrak{c}_b \succsim \mathfrak{c}_a \oplus_B \mathfrak{c}_b$ for all n then $\mathfrak{c}_a \oplus_A \mathfrak{c}_b \succsim \mathfrak{c}_a \oplus_B \mathfrak{c}_b$.

11.2. Countable additivity and P7. As seen, we have taken the step of introducing countable additivity as a formal assumption to Savage's decision model through an added postulate. Countable additivity, however, cannot replace P7. The latter, as seen in Section 7.2, is used by Savage to extend utility from simple acts to acts in general. In this section, we investigate the relations between countable additivity and P7 in extending utility extensions.

Recall that we have an example (Example 7.8) which satisfies the first six of Savage's seven postulates but not the last one. This shows that the seventh postulate (SVG 7) is independent from other postulates in Savage's original system. Upon showing the independence of P7, Savage (1972, p.78) remarked that "[f]inite, as opposed to countable, additivity seems to be essential to this example [i.e., Example 7.8]," and he conjectured that "perhaps, if the theory were worked out in a countably additive spirit from the start, little or no counterparts of P7 would be necessary." It turned out that this is inaccurate. Our aim is to provide a deeper analysis of Savage's remark on the relation between countable additivity and utility extension under various versions of P7. Let us start with the footnote Savage added to the remark above: "Fishburn (1970, Exercise 21, p.213) has suggested an appropriate wakening of P7." The following is Fishburn's suggestion (expressed using our notation).

P7b: For any event $E \in \mathcal{F}$ and $a \in X$, if $\mathfrak{c}_a \succsim_E \mathfrak{c}_{g(s)}$ for all $s \in E$ then $\mathfrak{c}_a \succsim_E g$; and if $\mathfrak{c}_a \precsim_E \mathfrak{c}_{g(s)}$ for all $s \in E$ then $\mathfrak{c}_a \precsim_E g$.

P7b is weaker than SVG 7 in that it compares act g with a constant act instead of another general act f. Note that Fishburn's P7b is derived from the following condition A4b occurred in his discussion on preferences axioms and bounded utilities (*ibid.* §10.4).

A4b: Let X be a set of prizes/consequences and $\Delta(X)$ be the set of all probability measure defined on X, then for any $P \in \Delta(X)$ and any $A \subseteq X$ if

$P(A) = 1$ and, for all $x \in A$, $\delta_x \succsim (\precsim)\delta_y$ for some $y \in X$ then $P \succsim (\precsim)\delta_y$, where, in the notation of Section 2.1, δ_x denotes the probability that degenerates at x.

A4b, together with other preference axioms discussed in the same section, are used to illustrate, among other things, the differences between measures that are countably additive and those that are not. It was proved by Fishburn that the expected utility hypothesis holds under A4b, that is,

$$P \succ Q \iff E(u, P) > E(u, Q), \quad \text{for all } P, Q \in \Delta(X) \tag{11.2}$$

if $\Delta(X)$ contains *only* countably additive measure. Fishburn then showed, by way of a counterexample, that the hypothesis fails if the set of probability measure contains also merely finitely additive ones. Because of its direct relevancy to our discussion on the additivity condition, let us reproduce this example (Fishburn, 1970, Theorem 10.2) here.

Example 11.1. Let $X = \mathbb{N}^+$ with $u(x) = x/(1 + x)$ for all $x \in X$. Let $\Delta(X)$ be the set of all probability measures on the set of all subsets of X and defined u on $\Delta(X)$ by

$$u(P) = E(u, P) + \inf\Big\{P\big[u(x) \geq 1 - \epsilon\big] : 0 < \epsilon \leq 1\Big\}. \tag{11.3}$$

Define $\succ$ on $\Delta(X)$ by $P \succ Q$ iff $u(P) > u(Q)$. It is easy to show that A4b holds under this definition. However if one takes P to be the measure in Example A.4.7, i.e., a finitely but not countably additivity probability measure, then we have $u(\lambda) = 1 + 1 = 2$ (cf. Example A.6.2). Hence $u(\lambda) \neq E(u, \lambda) = 1$. This shows the expected utility hypothesis fails under this example. $\triangleleft$

However, as pointed by Seidenfeld and Schervish (1983, Appendix), Fishburn's proof of (11.2) under A4b used the assumption that $\Delta(X)$ is closed under countable convex combination (condition S4 in Fishburn, 1970, p.137), which in fact is not derivable in Savage's system. They show through the following example (Example 2.3 in Seidenfeld and Schervish, 1983, p.404) that the expected hypothesis fails under the weakened P7b (together with SVG 1-SVG 6) and this is so even when the underlying probability be countably additive.

EXAMPLE 11.2. Let S be $[0,1)$ and X be the set of rational numbers in $[0,1)$. Let μ be uniform probability on measurable subsets of S and let all measurable function f from S to X satisfying $V[f] = \lim_{i\to\infty} \mu\big[f(s) \geq 1 - 2^{-i}\big]$ be acts. For any act f, let $U[f] = \int_S u(f)d\lambda$ where $u(x) = x$ is a utility function on X and define

$$\begin{aligned} W[f] &= U[f] + V[f] \\ &= \int_S u(f(s))d\mu(s) + \lim_{i\to\infty} \mu\Big[f(s) \geq 1 - \frac{1}{2^i}\Big]. \end{aligned} \tag{11.4}$$

Further, define $f \succ g$ if $W[f] > W[g]$. It is easy to see that SVG 1-SVG 6 are satisfied. To see that W satisfies P7b, note that if for any event E and any $a \in X$, $\mathfrak{c}_a \succsim_E \mathfrak{c}_{g(s)}$ for all $s \in E$, then by (11.4), we have $1 > u(a) \geq u(g(s))$ for any $s \in E$. Note that $1 > u(g(s))$ implies $V[g\chi_E] = 0$ where χ is the indicator function. Thus, $W[\mathfrak{c}_a\chi_E] = \int_E u(a)d\mu \geq \int_E u(g(s))d\mu(s) = W[g\chi_E]$. The case $\mathfrak{c}_a \precsim_E \mathfrak{c}_{g(s)}$ can be similarly shown. ◁

In other words, contrary to what Savage had thought, P7b is in fact insufficient in bringing about a full utility representation theorem even in the presence of countable additivity. This shows, *a fortiori*, that countable additivity alone is insufficient in carrying the utility function derived from SVG 1-SVG 6 from simple acts to general acts.[19] Seidenfeld and Schervish (1983, Example 2.2) also showed that this remains the case even the set of probabilities measure is taken to be closed under countable convex combination.

EXAMPLE 11.3. Let $S = X = [0,1]$ and let μ be uniform probability on measurable subsets of S. Let all measurable functions from S to X be acts and define

$$V[f] = \inf\Big\{\mu(E) : \ f \text{ takes only finitely many values on } E^C\Big\}. \tag{11.5}$$

[19] I thank Professor Teddy Seidenfeld for helping me see this point.

For any act f, let $U[f] = \int_S u(f)d\lambda$ where $u(x) = x$ is a utility function on X and define $W[f] = U[f] + V[f]$. As before, define $f \succ g$ if $W[f] > W[g]$. We have that the-defined preference relation $\succsim$ satisfies all of SVG 1-SVG 6 and the property of countable convex combination, but W is not an expected utility. In fact, W violates SVG 7. To see this, let f, g be such that

$$f(x) = \begin{cases} x & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases} \qquad \text{and} \qquad g(x) = 1 \text{ for all } x \in S.$$

Then $W[f] = .5 + 1 = 1.5$ and $W[g] = 1 + 0 = 1$, hence $f \succ g$ by the definition of $\succ$ by W. But for any fixed $s \in S$, $W[\mathfrak{c}_{f(s)}] = s$, and hence $\mathfrak{c}_{f(s)} \precsim g$ for all $s \in S$. But this contracts SVG 7 (taking $E = S$). $\lhd$

As we have seen, Savage's P7 which plays the role of extending the utilities from simple acts to acts in general cannot be easily weakened even in the presence of countable additivity. Yet, on the other hand, it is clear that countable additivity is a stronger condition than finite additivity originally adopted in Savage's theory. So, for future work, one might be interested in finding an appropriate weakening of P7 in a Savage-style system augmented with countable additivity.

12. Concluding Remarks

The debate about finite versus countable additivity often operates on two fronts. First, there is the concern for mathematical consequences as to whether or not the kind of additivity in use accords well with demanded mathematical details. Second, there is the concern for philosophical foundations as to whether or not the kind of additivity in use is conceptually justifiable (using our intuitions or rationality principles like the Dutch book argument etc.).

Regarding the first concern, Savage provided a set-theoretic argument against countable additivity arguing that countable additivity, it is said, is not in good conformity with demanded set-theoretic details. As we have seen in Section 9, this line of reasoning is unwarranted: his argument is misguided due to an overlook of some crucial technical details concerning large cardinals assumptions and non-measurable sets. Countable additivity *can* be coherently incorporated without serious set-theoretic complications.

As for the second concern with conceptual justification, we have seen above that there are cases in which finite additivity is insufficient in bringing about a coherent theory in Savage's framework due to the failure of continuity on the part of derived probability measures, which may lead to the construction of money pump. This is largely due to the imbalance between finite additivity and the rich background settings employed in his system, which naturally led us to enlist countable additivity as an additional postulate.

Indeed, as far as Savage's system is concerned, we see no sufficient reason as to why the decision model has to be restricted to just finitely additive measures. The general perception seems to be that there is no need to be dogmatic about this additivity condition, countable additivity can be employed as needed. Yet, on the other hand, given how widespread countably additive measures are used in modern probability theory, it seems it is advantageous to presuppose countable additivity and only to weaken this condition when it is called for.

CHAPTER IV

Context-dependent Decision Making

13. Introduction

As we have seen in Chapter II, Savage's framework of subjective preference among acts provides a paradigmatic derivation of rational subjective probabilities within a more general theory of rational decisions. The system is based on a set of possible states of the world, and on acts, which are functions that assign to each state a "consequence." The representation theorem states that the given preference between acts can be represented by their expected utilities, based on uniquely determined finitely additive probabilities (assigned to sets of states), and numeric utilities (assigned to consequences). Savage's derivation, however, is based on a highly problematic assumption not included among his postulates: for any consequence of an act in some state, there is a "constant act" which has that consequence in all states. This chapter is devoted to addressing and solving the problems caused by this implicit assumption. As the analysis in the next section shows, this ability to transfer consequences from state to state is, in many cases, miraculous – including simple scenarios suggested by Savage as natural cases for applying his theory. In Section 15, we propose a simplification of the system, which yields the representation theorem without the constant act assumption. This is done at the cost of reducing the richness of the set of acts included in the setup. The reduction excludes certain theoretical infinitary scenarios, but includes the scenarios that should be handled by a system that models human decisions.

This chapter is slightly adapted from a joint work of Gaifman and Liu (2015). A version of this chapter was presented at the 5th International Workshop on Logic, Rationality and Interaction (LORI-V), Taipei, Taiwan.

14. The Problem of the Constant-act Assumption

14.1. Miraculous acts. One crucial element of the system is the notion of *constant acts* or, in Savage's phrasing, "acts that are constant" (p.25). The idea is that a constant act has the same consequence in all states. To be precise, being a constant act is not a property of a single act, but is subject to an axiom that applies to a bunch of acts: the preference between two constant acts, given some event, does not depend on the event. The fifth postulate (P5) posits the existence of two non-equivalent constant acts.

Savage's representation theorem claims that a preference relation that satisfies the postulates determines a unique (finitely additive) probability on $\mathcal{B}$ and a utility function (unique up to a linear transformation) which assigns numeric utilities to consequences, such that $f \succsim g$ iff the expected utility of f is greater or equal to that of g. As seen in Chapter II the derivation of a probability and a utility is carried out in two stages. In the first stage a finitely additive probability is derived from a preference relation, which satisfies the postulates P1–P6. As far as constant acts are concerned, this derivation does not require more than P5 (the existence of two non-equivalent constant acts is sufficient). But in the second stage—the derivation of a utility in chapter 5—Savage tacitly assumes the following:

> **CAA (constant-acts assumption):** For every consequence $a \in C$ there exists a constant act $\mathfrak{c}_a$, such that $\mathfrak{c}_a(s) = a$, for all $s \in S$.

Note that after introducing "acts that are constant" Savage hardly uses the term anymore and one has to infer that such and such acts are constant only from the notation, which is not always consistent. Fishburn (1970) who observed that CAA is required for the proof of the representation theorem, has also pointed out the problematic nature of CAA (cf. Footnote 1 below). Among others who have also emphasized the need for CAA in Savage's system are Pratt (1974); Seidenfeld and Schervish (1983); Shafer (1986). This assumption, we shall argue, does not sit well with certain simple scenarios of decision making, which Savage considers as the kind of situations that his system is supposed to handle.

Table 14.1. Savage omelet example.

Act	State	
	Good	Rotten
break into bowl	six-egg omelet	no omelet and all five eggs destroyed
break into saucer	six-egg omelet and a saucer to wash	five-egg omelet and a saucer to wash
throw away	five-egg omelet and one good egg wasted	five-egg omelet

The difficulty is the fact that the very possibility of some consequence may depend on the world being in a certain state: *the consequence could not exist in a different state of the world.* At the beginning of his book Savage (1972, p.14) proposes the following omelet-making problem to illustrate the way his system works. The agent, call him John (in the book it is 'you'), has to finish making an omelet, which was begun by his wife. She broke into a bowl five good eggs and John finds a sixth egg, which can be added to the bowl or thrown away (we assume that there is no option of keeping it for future use). John does not know if the egg is good or rotten and has to decide between three acts: (1) Break it into the bowl (2) break it into a saucer to see if it is good or rotten (3) throw it away. There are two possible states of the world *good* and *rotten*, which are determined by the state of the sixth egg. The consequences of each act are given in Table 14.1, as it appears in the book.

John's ranking of the acts (that is, his preference relation, $\succsim$) reflects both his probabilistic estimates regarding the likeliness of each state, as well as the utility values of the consequences; for example, if he is sufficiently confident that the egg is good and if washing the saucer is, for him, of considerable nuisance, he will prefer "break into bowl" to "break into saucer". His preferences for these three acts cannot, of course, determine the probabilities and utilities, but if the set of acts over which the preference relation is defined is sufficiently rich (where "sufficiently rich" is determined by the postulates), then we get probabilities and utilities. Obviously the consequence "six-egg omelet" means an omelet made of the six eggs of the story, in the case where the sixth egg is good. Yet CAA requires that there should be a constant act that yields that consequence also in the state in which the sixth egg is rotten. It would involve a miraculous production of a good six-egg omelet out of five good eggs and a rotten one.[1]

The problem arises also in the second scenario, which Savage proposes for the very purpose of clarifying what is implied by a constant act (*ibid.* p.25). A person, call her Jane, plans to go with friends on a picnic, and she has to choose between buying a tennis racquet and buying a bathing suit (assume that buying both is ruled out for financial reasons). The bathing suit would be handier if the picnic is held near water where one can swim; the racquet would be better, if the picnic is not held near water but near a tennis court. One might consider the possession of a bathing suit and the possession of a tennis racquet as constant, state-independent consequences. But Savage makes it clear that this would not do, since the preference order of possessing a racquet and possessing a bathing suit depends on the state of the world, where the state of the world includes the picnic-location. Savage argues that the payoffs should be entities such as: "a refreshing swim with friends, or sitting on a shadeless beach twiddling a brand-new tennis racquet while one's friends swim". That, however, does not make the constant-acts problem easier. To get a constant act, we have to appeal to the theoretical possibility that while Jane sits on a shadeless beach twiddling a brand new tennis racket, she has somehow the enjoyment of a refreshing swim with her friends.

Perhaps the constant-acts problem is not so difficult if we consider getting sums of money, or some other quantitative goods, as being of equivalent value to the consequences in question. In the omelet scenario, John may consider getting $\$k$ as being equivalent to a six-egg omelet and this can serve also as a payoff

[1] In passing, Fishburn (1970, p.166-7) also voiced this unsatisfactory feature of CAA. He pointed out that, for any states $s, s' \in S$, if $W(s)$ and $W(s')$ are respectively the sets of consequences that may occur under s and s', then it might well be that $W(s) \neq W(s')$ (or even that $W(s) \cap W(s') = \emptyset$), in which case the CAA fails. He remarked that he is not aware of any axiomatic system that does not make the assumption that $W(s) = W(s') = C$ for all $s, s' \in S$, and he left this line of research as an open question (see also Fishburn, 1981, p.162).

in the state "rotten". But it is not clear what the equivalence of $\$k$ with a six-egg omelet means in the given context where John has to finish making the omelet. We may consider replacing Table 14.1 by the following table, in which the entries are dollar amounts; this would turn the problem into a problem of choosing between gambles. (Obviously, k is assumed to be the largest payoff, l is the smallest, $m > n$

Act	State	
	Good	Rotten
Gamble 1	$\$k$	$\$l$
Gamble 2	$\$m$	$\$n$
Gamble 3	$\$p$	$\$q$

and $q > n$.) And we may consider offering John the choice of not completing the task – throwing out all eggs – and getting in return to choose a gamble from the table above. But this artificial dubious device undermines the big attraction of Savage's system: its ability to evaluate consequences that do not consist in winning or loosing sums of money or goods. If all consequences are to be replaced by dollar sums before the system is applied, the main point of the system is lost.

One objective of this chapter is to show that CAA is not required for applying Savage's system to any finitistic problem, that is to say, *a problem that is stated in terms of finitely many evants, finitely many acts and finitely many possible consequences.* All that we need is the existence of two distinguished constant acts.

14.2. The Significance of the Set of Acts and the Boolean Algebra. The weaker the postulates and the presuppositions which are needed to get the representation theorem, the stronger the theorem is. The basic presupposition of Savages system is that the preference relation is defined over some very rich set of acts. In some places Savage even considers every function from states to consequences to be an act, in situations in which the set of states, as well as the set of consequences, has the cardinality of the continuum. This is exorbitant. Of course the set of acts should be sufficient for handling the kind of problems that the system is designed for. As a rule, these problems are stated in terms of finitely many simple acts, where a simple act is an act, f, which has finitely many values, such that, $f^{-1}(x)$ is an event (a member of the Boolean algebra $\mathcal{B}$) for each consequence x that is a value of f. Such acts are called by Savage gambles.

In the initial scenario the agent is supposed to decide between given options that belong to some finite set of simple acts. P6 implies however that the preference is to be defined over richer sets that involve more refined events (cf. Theorem 15.3 below). But, as we shall show, we never need more than simple acts. (In Section 15.2.2, we comment on how our model can be generalized to treat certain infinitary cases.)

Now the richness of the set of acts is also determined by the richness of the Boolean algebra $\mathcal{B}$ of events, namely the collection of subsets that constitute events. As noted, Savage considers possibilities in which this Boolean algebra consists of all subsets of real numbers (cf. Chapter III). But his proof of the representation theorem requires only that it be a σ-algebra, that is, closed under unions of countable many sets. Our results can be now stated as follows:

i. While we assume that the Boolean algebra is a σ-algebra, we can derive the representation theorem if we consider only a preference defined over simple acts, which include two non-equivalent constant ones.
ii. Moreover, we can also give up the assumption that the algebra is a σ-algebra and get the representation theorem, nonetheless. In fact, we need only a countable Boolean algebra so that the simple acts defined over it satisfy P6.

(i) is proved by using Savage's derivation of probabilities from two constant acts. We deviate from him in the derivation of expected utilities for simple acts (where the set of consequences is arbitrary). In the next section, we lay out the basic ideas behind our construction. (ii) is a more difficult result that is based on a more difficult derivation of probabilities, we hope to address in future work. We will highlight in Section 16 difference between our model and Savage's system together some remarks on related literature.

15. Context-dependent Decision-making

We seek to develop a theory of context-dependent decision making where the relevancy and the value of a consequence depend on the contexts under which it arises. We assume that the set of acts consists of those acts that are either included in the given initial decision problem (like f_1, f_2, f_3 in the omelet example) or generated by applying one or more of the axioms (under the assumption of the background algebra). We show that a variant of Savage's representation theorem can be derived without appealing to SVG 7 or CAA.

15.1. Deriving probability and ordinal utility without CAA. Recall from Section 6 that to derive subjective probability from preferences Savage uses SVG 1-6. The construction starts with a derivation of qualitative probabilities.

Definition 15.1. For any events E, F, say that E is *weakly more probable* than F, written $E \succeq F$ (or $F \preceq E$), if, for any constant acts $\mathfrak{c}_a$ and $\mathfrak{c}_b$ such that $\mathfrak{c}_a \succsim \mathfrak{c}_b$,

$$\mathfrak{c}_a \oplus_E \mathfrak{c}_b \succsim \mathfrak{c}_a \oplus_F \mathfrak{c}_b. \tag{15.1}$$

Savage's SVG 4 guarantees that (15.1) does not depend on the choice of the pair of constant acts. Obviously, this concept does not rely on CAA. It is also not difficult to show that $\succsim$ is a qualitative probability (Theorem 6.4 above). The task is to show that this qualitative probability admits a numerical *representation*: there exists a real-valued probability measure μ defined on an algebra of events satisfying:

$$E \succeq F \iff \mu(E) \geq \mu(F). \tag{15.2}$$

15.1.1. *Numerical probability.* As noted in Chapter III, Savage takes as background algebra $\mathcal{F}$ the power set of the state space. In both Chapter II and this chapter, we deviate from Savage's approach by considering a σ-algebra of S over which μ is defined. (That one can do without the assumption of a σ-algebra but with a countable algebra is, as noted above, beyond the scope of discussion in this dissertation.) So far only two non-equivalent constant acts are required.[2] This then leads us to the following theorem.

THEOREM 15.2 (Savage). Let $\succsim$ be a preference relation among acts. Suppose that $\succsim$ satisfies SVG 1-6 and that the Boolean algebra $\mathcal{F}$ of events is a σ-algebra, then there exists a unique (finitely additive) probability measure μ for which (15.2) holds.

The proof of the theorem establishes also the following result (i.e., Corollary 6.14 above), which holds under the assumption that the algebra of events is a σ-algebra. (Note that, unlike Theorem 15.2, Theorem 15.3 fails if the assumption that the Boolean algebra is a σ-algebra is omitted. A weaker version of it holds: The set of all ρ for which the equality holds is dense in $(0, 1)$.)

THEOREM 15.3. Given the probability measure μ obtained above, for any event E and any $0 \leq \rho \leq 1$, there exists some $F \subseteq E$ such that $\mu(F) = \rho\mu(E)$.

15.1.2. *Utilities for acts.* In deriving numerical utilities for acts, we need neither SVG 7 nor CAA. All we need are two distinct constant acts, say $\mathfrak{c}_0$ and $\mathfrak{c}_1$, which will serve as reference points for which an ordinal utility function can be defined. The following are some simple properties of the two distinguished constant acts, which are immediate from the definitions and Theorem 15.2 above.

LEMMA 15.4. For any events E, F,

(1) $\mu(E) > \mu(F)$ iff $\mathfrak{c}_1 \oplus_E \mathfrak{c}_0 \succ \mathfrak{c}_1 \oplus_F \mathfrak{c}_0$,
(2) $\mu(E) = \mu(F)$ iff $\mathfrak{c}_1 \oplus_E \mathfrak{c}_0 \sim \mathfrak{c}_1 \oplus_F \mathfrak{c}_0$.

We show that, under SVG 1-6 and the assumption that there exist two constant acts $\mathfrak{c}_0$ and $\mathfrak{c}_1$, the agent's preferences can be represented by an ordinal utility function in Savage's system without appealing to CAA. To this end, we first observe

[2]This observation is also noted in Fishburn (1981, p.161) where he remarked that "[as far as obtaining a unique probability measure is concerned] Savage's $\mathscr{C}$ [i.e., the set of consequences] can contain as few as two consequences." See Section 6 or Fishburn (1970, §14.1-3) for an exposition of Savage's proof of (15.2), and see especially Fishburn's §14.3 for an illustration of the role of Savage's SVG 1-6 played in deriving numerical probability.

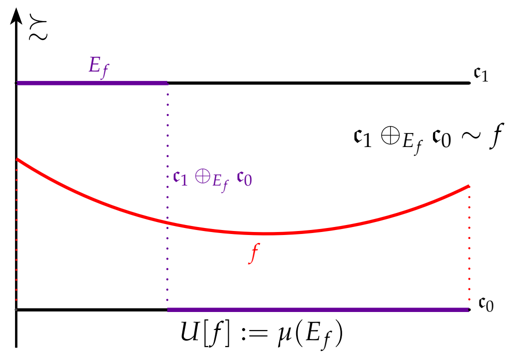

FIGURE 15.1. The case where $\mathfrak{c}_1 \succsim f \succsim \mathfrak{c}_0$

that to each act $f \in \mathcal{A}$ satisfying $\mathfrak{c}_1 \succsim f \succsim \mathfrak{c}_0$ there corresponds a combined act using the two distinguished constant acts which is indifferent to f under $\succsim$.

LEMMA 15.5. For and $f \in \mathcal{A}$, if $\mathfrak{c}_1 \succsim f \succsim \mathfrak{c}_0$, there exists an event E_f such that

$$\mathfrak{c}_1 \oplus_{E_f} \mathfrak{c}_0 \sim f. \tag{15.3}$$

In proving this lemma, we make full use of the derived personal probability μ from Theorem 15.2, the proof given here is somewhat standard in utility theory. Figure 15.1 provides an illustration of the general method involved in the proof, where $\mathfrak{c}_1 \oplus_{E_f} \mathfrak{c}_0$ is the act that yields $\mathfrak{c}_1$ if E_f occurs, status quo otherwise. The aim is to find the appropriate E_f so that the given event f is indifferent to this combined act.

PROOF OF LEMMA 15.5. Let us consider the following two sets of events.

$$\mathcal{B} := \left\{ E \;\middle|\; \mathfrak{c}_1 \oplus_E \mathfrak{c}_0 \succsim f \right\};$$
$$\mathcal{C} := \left\{ E \;\middle|\; \mathfrak{c}_1 \oplus_E \mathfrak{c}_0 \precsim f \right\}.$$

It is easily seen that $\mathcal{B}$ and $\mathcal{C}$ are nonempty, for at least we have $S \in \mathcal{B}$ and $\emptyset \in \mathcal{C}$. Let μ be the probability measure derived from Theorem 15.2, Next, consider the following sets defined in terms of $\mathcal{B}, \mathcal{C}$ and μ:

$$\begin{aligned} \mathcal{B}_\mu &:= \left\{ \mu(E) \;\middle|\; E \in \mathcal{B} \right\}; \\ \mathcal{C}_\mu &:= \left\{ \mu(E) \;\middle|\; E \in \mathcal{C} \right\}. \end{aligned} \tag{15.4}$$

Let $\alpha_* = \inf \mathcal{B}_\mu$ and $\alpha^* = \sup \mathcal{C}_\mu$. Note that, for any $a > \alpha_*$, there must exist some $a' \in \mathcal{B}_\mu$ such that $a > a' \geq \alpha_*$ (for, otherwise, a is a lower bound of $\mathcal{B}_\mu$ strictly greater than α_*, which contradicts the assumption $\alpha_* = \inf \mathcal{B}_\mu$). Since $a' \in \mathcal{B}_\mu$ then, by the definition of $\mathcal{B}_\mu$ in (15.4), there is some event, say, $F' \in \mathcal{B}$ such that $\mu(F') = a'$. Further, let F be an event such that $\mu(F) = a$ (the existence of F is guaranteed by Theorem 15.3). Then, by Lemma 15.4, $\mu(F) = a > \mu(F') = a' \geq \alpha_*$ implies $\mathfrak{c}_1 \oplus_F \mathfrak{c}_0 \succ \mathfrak{c}_1 \oplus_{F'} \mathfrak{c}_0 \succsim f$. It follows, via SVG 1, that, for any F,

$$\mu(F) > \alpha_* \implies F \notin \mathcal{C}. \tag{15.5}$$

The contrapositive of (15.5) says that, for any F, $F \in \mathcal{C}$ implies that $\mu(F) \leq \alpha_*$. In other words, α_* is an upper bound of $\mathcal{C}_\mu$, and hence $\alpha^* = \sup \mathcal{C}_\mu \leq \alpha_*$. Using a symmetric argument one can show that $\alpha^* \geq \alpha_*$. Hence $\alpha^* = \alpha_*$.

Next, let E_f be such that $\mu(E_f) = \alpha^* = \alpha_*$ (again, the existence of E_f is guaranteed by Theorem 15.3). The proof is completed if we can show that $E_f \in \mathcal{B} \cap \mathcal{C}$. Suppose, to the contrary, $E_f \notin \mathcal{B}$, then, by P1, $f \succ \mathfrak{c}_1 \oplus_{E_f} \mathfrak{c}_0$. The latter implies, via P6, there exists a partition $\{P_i\}_{i=1}^n$ such that, for all $i = 1, \ldots, n$, $f \succ \mathfrak{c}_1 \oplus_{P_i} \left[\mathfrak{c}_1 \oplus_{E_f} \mathfrak{c}_0\right]$, that is,

$$f \succ \mathfrak{c}_1 \oplus_{E_f \cup P_i} \mathfrak{c}_0 \quad (i = 1, \ldots, n). \tag{15.6}$$

Then, it follows that $E_f \cup P_i \in \mathcal{C}$ for *all* $i = 1, \ldots, n$. On the other hand, note that P_i's form a partition of S, we consider two cases:

(1) If for *some* P_j in the partition we have $\mu(E_f \cup P_j) > \mu(E_f) = \alpha_*$, then, by (15.5), $E_f \cup P_j \notin \mathcal{C}$, a contradiction.

(2) If $\mu(E_f \cup P_j) \leq \mu(E_f) = \alpha_*$ for *all* $j = 1, \dots, n$, then it is easily seen that $\mu(E_f) = 1$. By Corollary 15.4(2), it follows that $\mathfrak{c}_1 \oplus_{E_f} \mathfrak{c}_0 \sim \mathfrak{c}_1 \oplus_S \mathfrak{c}_0 = \mathfrak{c}_1$, and hence $E_f \in \mathcal{B}$, but this contradicts the hypothesis $E_f \notin \mathcal{B}$.

Hence, E_f must be in $\mathcal{B}$. Similarly, it can be shown that $E_f \in \mathcal{C}$. Then we have $E_f \in \mathcal{B} \cap \mathcal{C}$. This completes the proof of the lemma. □

REMARK. 1. In light of the lemma, for any $f \in \mathcal{A}$ satisfying $\mathfrak{c}_1 \succsim f \succsim \mathfrak{c}_0$, let E_f be such that (15.3) holds, we define the *utility* of f to be

$$U[f] := \mu(E_f), \tag{15.7}$$

where μ is obtained through Theorem 15.2 and E_f is from (15.3).

2. Notice that, if there exists another event E'_f for which (15.3) holds, then we have $\mathfrak{c}_1 \oplus_{E_f} \mathfrak{c}_0 \sim \mathfrak{c}_1 \oplus_{E'_f} \mathfrak{c}_0$. It follows, via Lemma 15.4(2), that $\mu(E'_f) = \mu(E_f)$, hence $U[f]$ is well defined.
3. For the two distinguished constant acts $\mathfrak{c}_1$ and $\mathfrak{c}_0$, trivially we have $E_{\mathfrak{c}_1} = S$ and $E_{\mathfrak{c}_0} = \emptyset$, then (15.7) yields that $U[\mathfrak{c}_1] = 1$ and $U[\mathfrak{c}_0] = 0$.
4. It is plain that U does not need to be uniquely defined by (15.7): if h is any monotonically increasing function on the reals (or any order preserving function), then U can also be defined by $h \circ \mu$.
5. If $f \succ \mathfrak{c}_1$ (or $\mathfrak{c}_0 \succ f$), it is easy to see that Lemma 15.5 can be adjusted to show that there exists some E_f such that $f \oplus_{E_f} \mathfrak{c}_0 \sim \mathfrak{c}_1$ (or $\mathfrak{c}_1 \oplus_{E_f} f \sim \mathfrak{c}_0$), in which case U can be defined standardly as in (15.9) below.

THEOREM 15.6. Let $\succsim$ be a preference relation over acts, if $\succsim$ satisfies SVG 1-6, then there exists a real-valued function U on $\mathcal{A}$ satisfying, for all $f, g \in \mathcal{A}$,

$$f \succsim g \iff U[f] \geq U[g], \tag{15.8}$$

where

$$U[f] := \begin{cases} \frac{1}{\mu(E_f)} & \text{if } f \succ \mathfrak{c}_1, \\ \mu(E_f) & \text{if } \mathfrak{c}_1 \succsim f \succsim \mathfrak{c}_0, \\ \frac{\mu(E_f)}{\mu(E_f)-1} & \text{if } \mathfrak{c}_0 \succ f. \end{cases} \tag{15.9}$$

15.2. Context-dependent expected utility for simple acts. The strength of SVG 1-6 does not end with Theorem 15.6, we show that under these postulates the utility of a simple act can be further decomposed into (context-dependent) expected utilities. After proving this claim, we shall comment on how this method can be extended to certain infinitary cases.

15.2.1. *Finitary cases.* Let $f \in \mathcal{A}_0$ be a simple act. Then there are finitely many consequences, say, $x_1, \ldots, x_n$, and a partition $\{P_1, \ldots, P_n\}$ of the state space S where each P_i is the set of states under which x_i obtains, that is,

$$\begin{aligned} &P_i = f^{-1}(x_i) \quad (i = 1, \ldots, n), \\ &P_i \cap P_j = \emptyset \quad (i \neq j) \quad \text{and} \quad \bigcup_{i=1}^{n} P_i = S. \end{aligned} \tag{15.10}$$

We seek to define a utility function u over consequences such that the utility of a simple act $U[f]$ can be represented by its expected utility:

$$U[f] = \sum_{i=1}^{n} \mu(P_i) u(P_i, x_i), \tag{15.11}$$

where $u(P_i, x_i)$ is the utility of consequences x_i *given* that P_i occurs. We thus speak of *context-dependent* utilities where the relevancy of value of a consequence depend on the context P_i it arises. To this end, let us first adopt the following notation:

$$\mathfrak{c}_x^*(s) := \begin{cases} x & \text{if } s \in E, \\ 0 & \text{if } s \notin E, \end{cases} \quad \text{for some } E \in \mathcal{B}. \tag{15.12}$$

We refer to $\mathfrak{c}_x^*$ as a *locally constant act* which yields x in all states in E, 0 (status quo) otherwise. It is obvious that $\mathfrak{c}_x^*$ is a generalization of Savage's notion of constant act. But, unlike Savage, we do not impose any structural assumptions on

$\mathfrak{c}_x^*$, in particular E needs not to be the universal set S. With (15.12), a simple act f satisfying (15.10) can be conveniently expressed by

$$f = \mathfrak{c}_{x_1}^* \oplus_{P_1} \left(\mathfrak{c}_{x_2}^* \oplus_{P_2} \left(\mathfrak{c}_{x_3}^* \oplus_{P_3} \left(\cdots \oplus_{P_{n-1}} \mathfrak{c}_{x_n}^* \right) \cdots \right) \right). \tag{15.13}$$

The goal is to represent simple acts in the form of (15.13) by expected utilities. Observe that, if $\mu(P_i) = 0$ for some P_i, then the term $\mu(P_i)u(P_i, x_i)$ in (15.11) is 0, in which case x_i can be seen as having no contribution to the total utility calculation. As a rule, one can assign in this situation an arbitrary finite value to the consequence $f(s)$ where $s \in P_i$. If, on the other hand, $\mu(P_i) \neq 0$, consider act $\mathfrak{c}_{x_i}^* \oplus_{P_i} \mathfrak{c}_0$. Then in light of Theorem 15.6, define the utility of x_i in P_i in terms of the utility of $\mathfrak{c}_{x_i}^* \oplus_{P_i} \mathfrak{c}_0$ as follows

$$u(P_i, x_i) := \begin{cases} c & \text{if } \mu(P_i) = 0, \\ \frac{U[\mathfrak{c}_{x_i}^* \oplus_{P_i} \mathfrak{c}_0]}{\mu(P_i)} & \text{if } \mu(P_i) \neq 0, \end{cases} \tag{15.14}$$

where c can be any number in $[0, 1]$. Finally, it remains to verify that $\succsim$ among simple acts indeed admits an expected utility representation using the probability measure μ and utility function u given above. We put this claim in the form of the following theorem.

THEOREM 15.7. Let $\succsim$ be a preference relation over acts, if $\succsim$ satisfies SVG 1-6, then there exist a probability measure μ on events and a utility function u on events and the set of consequences such that, for any $f, g \in \mathcal{A}_0$,

$$f \succsim g \iff \sum_{x \in f(S)} \mu[f(s) = x] u(f^{-1}(x), x) \geq \sum_{x \in g(S)} \mu[g(s) = x] u(g^{-1}(x), x).$$

PROOF. Let μ be the subjective provability measure derived in Theorem 15.2. By Theorem 15.6, there exists a utility function U for which (15.8) holds. We show that if U takes the form of (15.9) then it can be further decomposed into (15.11) using the utility function defined in (15.14). To simplify matters, assume that $f = \mathfrak{c}_{x_1}^* \oplus_{P_1} \mathfrak{c}_{x_2}^*$ and that $\mathfrak{c}_1 \succsim \mathfrak{c}_{x_i}^* \oplus_{P_i} \mathfrak{c}_0 \succsim \mathfrak{c}_0$ ($i = 1, 2$). (The more general case of (15.13) can be similarly shown). Let us consider the non-trivial case where $\mu(P_1), \mu(P_2) \neq 0$. By (15.9) and (15.14), there exist some E_{x_1} and E_{x_2} such that

$$\begin{aligned} u(P_1, x_1) &= \frac{U[\mathfrak{c}^*_{x_1} \oplus_{P_1} \mathfrak{c}_0]}{\mu(P_1)} = \frac{\mu(E_{x_1})}{\mu(P_1)}, \\ u(P_2, x_2) &= \frac{U[\mathfrak{c}^*_{x_2} \oplus_{P_2} \mathfrak{c}_0]}{\mu(P_2)} = \frac{\mu(E_{x_2})}{\mu(P_2)}. \end{aligned} \tag{15.15}$$

It is not difficulty to see that $u(P_1, x_1), u(P_2, x_2) \leq 1$. Apply Theorem 15.3 again, we get $F_1 \subseteq P_1$ and $F_2 \subseteq P_2$ such that $\mu(F_1) = \mu(E_{x_1})$ and $\mu(F_2) = \mu(E_{x_2})$, then, by Lemma 15.4 we have

$$\begin{aligned} \mathfrak{c}_1 \oplus_{F_1} \mathfrak{c}_0 &\equiv \mathfrak{c}_1 \oplus_{E_{x_1}} \mathfrak{c}_0 \sim \mathfrak{c}^*_{x_1} \oplus_{P_1} \mathfrak{c}_0, \\ \mathfrak{c}_1 \oplus_{F_2} \mathfrak{c}_0 &\equiv \mathfrak{c}_1 \oplus_{E_{x_2}} \mathfrak{c}_0 \sim \mathfrak{c}^*_{x_2} \oplus_{P_2} \mathfrak{c}_0; \end{aligned} \tag{15.16}$$

and

$$\frac{\mu(F_1)}{\mu(P_1)} = u(P_1, x_1) \quad \text{and} \quad \frac{\mu(F_2)}{\mu(P_2)} = u(P_2, x_2). \tag{15.17}$$

Note that $\{P_1, P_2\}$ partitions S and $F_1 \subseteq P_1, F_2 \subseteq P_2$, it follows, via SVG 2 and that

$$\begin{aligned} f &= \mathfrak{c}^*_{x_1} \oplus_{P_1} \mathfrak{c}^*_{x_2} \\ &\sim \big(\mathfrak{c}_1 \oplus_{F_1} \mathfrak{c}_0\big) \oplus_{P_1} \big(\mathfrak{c}_1 \oplus_{F_2} \mathfrak{c}_0\big) \\ &= \mathfrak{c}_1 \oplus_{(P_1 \cap F_1) \cup (P_2 \cap F_2)} \mathfrak{c}_0 \\ &= \mathfrak{c}_1 \oplus_{F_1 \cup F_2} \mathfrak{c}_0. \end{aligned} \tag{15.18}$$

Finally, by the definition of U in (15.7), (15.17) and (15.18) imply

$$U[f] = \mu\big(F_1 \cup F_2\big) = \mu(F_1) + \mu(F_2) = \mu(P_1)u(P_1, x_1) + \mu(P_2)u(P_2, x_2).$$

This shows, via Theorem 15.6, that if f is the form of $\mathfrak{c}^*_{x_1} \oplus_{P_1} \mathfrak{c}^*_{x_2}$, it indeed admits a context-dependent expected utility representation. □

15.2.2. *Infinitary Cases.* Our method can be generalized to treat certain infinitary case. There are acts, f, in which there are countably many consequences, say $x_1, x_2, \ldots, x_n, \ldots$ such that $f^{-1}(x_n)$ is a non-null set for every n. In other words, we allow the number of cells of the partition in (15.10) to be unbounded. Then (15.14) and Theorem 15.7 also apply to this case, where the expected utility of f can be defined by

$$\sum_{i=1}^{\infty} \mu\big[f(s) = x_i\big]\, u\big(f^{-1}(x_i), x_i\big) \tag{15.19}$$

provided that $\sum_{i=1}^{\infty} \mu\big[f(s) = x_i\big] \cdot \big|u\big(f^{-1}(x_i), x_i\big)\big|$ converges. It is defined as the sum of the positive values minus the sum of the negative ones. Note that μ does not need to be countably additive. The expectation in that case is defined for discrete random variables, for which the sum absolutely converges.

Finally, we point out that Savage needed the CAA because he wanted to extend utilities to continuous random variables, that is, he wanted to define the integral:

$$\int X(s)\, d\mu(s) \tag{15.20}$$

where X is a measurable function, which is interpreted in his system as a general act with potentially uncountably many consequences, and μ is a finitely additive probability. This can be seen as an attempt to integrate modern measure theory (albeit with finitely additive probability measures) into the theory of expected utilities. Mathematically this is interesting. But it is unclear whether this is required for applying his system to decision scenarios which a rational human agent is expected to face.

16. Concluding Remarks

Savage's method of deriving expected utilities for simple acts is different from what we have just shown. As detailed in Section 7, his approach relies on defining a von Neumann-Morgenstern (vNM) lottery space in terms of the derived subjective probability measure μ and the set of all simple acts in order that a vNM utility function u for consequences can be derived. More precisely, the method adopted by Savage in deriving the utility function u for consequences includes the following logical steps:

(1) Each simple act f induces a simple probability function (a vNM lottery), written p_f, on the set C of consequences as follows

$$p_f(x_i) = \begin{cases} \mu[f(s) = x_i] & \text{if } x_i \in f(S), \\ 0 & \text{if } x_i \in C - f(S). \end{cases}$$

(2) To each vNM lottery p (a simple probability measure defined on C) there corresponds a simple act f_p. To define f_p, let $x_1, x_2, \ldots, x_n$ be an enumeration of the members of C that are in the support of p and let $\{P_i\}_{i=0}^n$ be a partition of S such that

$$\mu(P_i) = \begin{cases} 0 & i = 0, \\ p(x_i) & i = 1, \ldots, n. \end{cases}$$

Then define f_p as follows

$$f_p(s) = \begin{cases} x_0 & s \in P_0, \\ x_i & s \in P_i \quad (i = 1, \ldots, n), \end{cases}$$

where x_0 is an arbitrary consequence that is not in the support of p.

(3) Show that, under SVG 1-6, if two simple acts f, g induce the same probability measure on C then it must be that $f \sim g$, that is,

$$p_f = p_g \implies f \sim g.$$

(4) Under the above constructions, each vNM lottery can be identified with a equivalent class of Savage simple acts under $\succsim$.

(5) Induce a preference relation $\succsim^*$ among vNM lotteries:

$$p \succsim^* q \quad \text{if} \quad f_p \succsim f_q.$$

(6) Finally, show that if $\succsim$ satisfies SVG 1-6 then $\succsim^*$ satisfies all the vNM axioms. The latter then yields a utility function u for all consequences.

Note that in order for this method of deriving utility to fully function it is crucial that CAA is in place, this is because, under the above constructions, each constant $\mathfrak{c}_a$ ($a \in C$) corresponds to the vNM lottery that degenerates at a, and the latter is needed in order to apply the vNM expected utility theorem. In contrast, our approach does not need to go through such a construction.

A different approach to state-dependent utility representation has been studied by Karni et al. (1983), Schervish et al. (1990), among others. These works are based on the analytic framework of Anscombe and Aumann (1963). The axiomatic system developed by Anscombe and Aumann (A-A) was intended to be a "simplification" of Savage's system (Savage, 1954), and the objective, as stated by the authors, was "to define the person's probabilities in terms of chances, by an extension of the von Neumann-Morgenstern theory." More precisely, the A-A theory starts by enriching Savage's system with an auxiliary randomization mechanism which introduces vNM lotteries over consequences, then, under the standard vNM axioms, a utility function for consequences can be arrived at. The person's subjective probability over states is further separated from the derived utility function by imposing an additional, i.e., the dominance axiom. The result is a state-independent representation theorem. Based on a proposal of Karni et al. (1983), Schervish et al. (1990) generalizes the A-A theory by allowing the vNM lotteries to be defined on a product space of both consequences and states, from which a vNM utility function for each consequence-state lottery can be extracted (and hence the utilities are taken to be state-dependent). The agent's subjective probability is separated from this derived state-dependent utility by a *consistency* axiom (see Schervish et al. (1990, p.864), see also Karni (1993)).

Observe that the A-A theory reverses the inferential order originally adopted by Savage in that the subjective probability in the A-A system is derived from personal utility which in turn is defined in terms of some extraneous chance mechanism (cf. Figure 19.1). The resulting theory is hence a dualistic system featuring both objective and subjective probabilities. Admittedly, this approach has certain

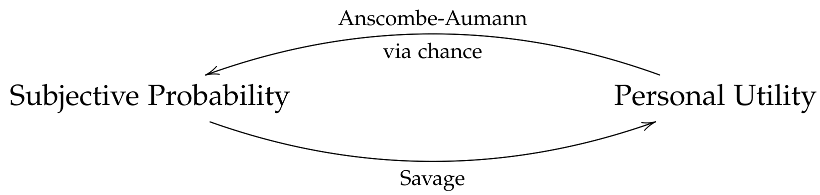

Figure 16.1. Inferential orders in various theories of expected utility.

computational advantages in the sense that the mathematical representation theorem proved in A-A is considerably simpler than that in Savage's original theory. There is, however, an undue conceptual complication. As known, one primary goal of Savage's project was to provide a *subjective interpretation* of probability for Bayesian inferences in general. The A-A system impoverishes this goal by essentially defining subjective probabilities in terms of objective ones. Our proof, on the other hand, follows Savage's original approach where we make no reference to any chance mechanism, our model remains as a purely subjective framework.

CHAPTER V

Epistemic Limitations and Bayesian Priors in Games

17. Introduction

The epistemic approach to non-cooperative game theory is largely directed at refining various game-theoretic "solution concepts" with explicitly articulated epistemic profiles of different parties of a given game, where the players' choices of the best courses of actions are determined not only by the structure of the underlying game (including the set of players, their possible actions, payoffs, and preferences, etc.) but also by their beliefs about the structure of the game, their beliefs about other players' beliefs about the game, and so on. An adequate game-theoretic model hence seeks to include, as a constituent component of the model, systematic representations of players' beliefs. Two main types of formal representation of beliefs are widely adopted in the philosophical literature, namely the *Kripke doxastic structure* $\mathcal{K}$ and the *Bayesian probabilistic structure* $\mathcal{P}$. Both structures are constructed on some possible world framework where an agent's belief in a given proposition (or belief about the occurrence of an event in the probabilistic case) is modeled in terms of various qualitative and quantitative properties among sets of possible worlds (or sets of possible states of the world).

In a game-theoretic setting, each possible world corresponds to a complete description of an alternative way a given game may evolve, and it is customary in game theory that players' beliefs be modeled using *Aumann information structures*

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$\mathcal{I}$, a game-theoretic counterpart of Kripke structure, where different representations of beliefs are seen as different measures of information distributions among players. The Bayesian probabilistic model of beliefs is echoed in game-theoretic analyses by various *Harsanyi type structures* $\mathcal{T}$. Briefly, depending on the particular modeling, a type is a probability function that measures a player's uncertainties about other players' payoffs (or actions, preferences, beliefs, etc.), which is considered as an integral part of overall doxastic assessments of the player.

Discussions on various representations of knowledge/beliefs in games and their interrelationships are readily available in Aumann (1999a,b), Battigalli and Bonanno (1999), Halpern (2003), Samuelson (2004), Pacuit and Roy (2015), to name a few. The main objectives of the present work are to (1) address various conceptual difficulties involved in these multi-agent systems, and (2) attempt a synthesis of the aforementioned belief structures ($\mathcal{K}, \mathcal{I}; \mathcal{P}, \mathcal{T}$) under two epistemic restrictions on probability assignments in all probabilistic models of beliefs:

Imprecise Probabilities (IP): The agent's beliefs are measured by *imprecise probabilities* (a set of subjective probabilities).

No Self-referential Probabilities for Acts (NSPA): No subjective probabilities should be assigned to the players' *own* actions that are under their current deliberations.

To limit the scope of our discussion, in this work, we do not engage in the philosophical debate about the first principle as to whether or not rational agents' beliefs should be represented by sharp probabilities. Instead, we take for granted that they are represented by imprecise ones. Our focus is rather to see how different belief structures fit into one another with IP in sight.[1]

[1]The literature on imprecise probabilities is vast: Early works on intervals/sets of probabilities include Koopman (1940), Smith (1961), Good (1962), Dempster (1967). These were followed by a systematic philosophical account defended by Levi (1974, 1986) that the credal state of a rational agent should be represented by indeterminate probability (see also Levi, 1985, for his distinction between indeterminate and imprecise probabilities). The current term 'imprecise probaiblities' was coined by Walley in his influential book 'Statistical Reasoning with Imprecise Probabilities' (1991). Systems that use or motivate IP include Gilboa and Schmeidler (1989), Seidenfeld et al. (1995). Gilboa et al. (2008) discusses a range of models in economic literature that use IP as representations

Different versions of the second, i.e., the NSPA principle, have been defended by Spohn (1977), Levi (1989, 1996), Gaifman (1999), among others. We stress that this principle, which places a strict restriction on assigning subjective probability to agents' self-actions, is incompatible with some of the existing game-theoretic models. In particular, according to Aumann (1987, p.2) "subjective probabilities should be assignable to every prospect, including that of players choosing certain strategies in certain games," where "every prospect" includes assigning probabilities to the players own actions, and hence contradicts the NSPA principle. The aim of the next section is to provide a detailed conceptual analysis of this conflict.

18. Epistemic Limitations

Note that by "subjective probabilities" Aumann means that each player is assumed to "conform to the Savage theory." In what follows, we point out that this take on Savage's conception of personal probability is misguided: there is no evidence that Savage's theory suggests that the agent's subjective probability can be assigned to just "every prospect" including the agent's own actions. This discussion will then be followed by an analysis of the notion of all-inclusive states, over which Aumann's subjective probabilities (player's priors) are defined. We show, by advancing a similar argument given by Spohn (1977) and Levi (1996), that in the presence of NSPA there is a serious conceptual difficulty in allowing Bayesian priors to be assigned to states that are all-inclusive.

18.1. Self-prediction and subjective probability. The so-called no subjective probability for acts thesis was already hinted at in Savage's discussion on the "small world" semantics where the discussion seems to suggest that probabilities for acts play no role in individual decision making. To illustrate, let us revisit the example provided by Savage. Suppose that John is torn between either (i) buying a sedan, (ii) buying a convertible, or (iii) keeping the money without buying any car. In a simple decision scenario, it is conceivable that the decision maker's

of decision makers' beliefs (multi-prior models). For recent discussions/surveys of philosophical motivations for employing IP, see Joyce (2010), Hájek and Smithson (2012), Bradley (2014).

choice of action is solely determined by her preferences over the consequences to which each action leads: in our example, if John prefers the convertible the most, then he should just go ahead and buy a convertible, "[c]hance and uncertainty are considered to have nothing to do with the situation" (Savage, 1972, p.83).

At this point, one might object by suggesting that if, say, John likes equally the sedan, he might come to make a decision on his purchasing either a convertible or a sedan by, say, flipping a coin or by utilizing some internal randomizing mechanism (whim, impulse, etc.). Then, in this case, there seems to be a sense in which one could say that John will take such-and-such action with so-and-so probability. However, the very formulation of the objection makes it clear that the probability in question is essentially about the randomizing device in use, which can hardly qualify for the agent's genuine subjective probability assignment for his actions.

In a more complicated scenario where more careful deliberation is required, the agent's actions are further evaluated in terms of different consequences obtained under different contingencies, where different contingencies are seen as different decision situations under which the actions are to take place. According to Savage, it is these possible decision situations (i.e., states of the world) that are subject to the agent's probabilistic estimations. Say that John has finally made a decision that he will buy a convertible and this is because he realized that he will be taking a vacation in Monterey, California next month, in which case the enjoyment of driving a convertible by the seaside in warm spring breeze will be maximally materialized. In other words, the agent's choices of actions are not deliberated in isolation: they are always placed in various *decision contexts* in which the acts are being evaluated, where these contexts come with descriptions of various decision situations and different consequences as results of implementing different actions. These considerations led Savage to his *belief-act-consequence* model, where acts are taken as functions mapping from (act-independent) states to consequences, and it is the states over which the acts are defined that are the subject of uncertainty but not the acts themselves. Hence, Aumann's universal probability assignment is not completely in line with Savage's theory.

Note that even if we grant that Aumann's notion of subjective probabilities be understood through some general Bayesian betting interpretation, it is not difficult to see that no non-trivial subjective probabilities can be meaningfully assigned to one's own acts.[2] To wit, suppose that John is faced with choices of either going to an Italian restaurant or a French restaurant for dinner. In an attempt to elicit his subjective probabilities assigned to his two possible actions, John is offered a bet with payoffs as follows

(1) win $\$X$ if John goes to an Italian restaurant,
(2) nothing if John goes to a French restaurant.

Now suppose that John's subjective probability for his going to an Italian restaurant is p, then he should be willing to pay a fee of pX to accept the bet in exchange of a reward of X on the event that he indeed is going to have Italian food for dinner. So far, the example accords well with the standard Bayesian betting interpretation of subjective probabilities. But situation changes once we notice that the mere fact that John is willing to accept the bet of his going to an Italian restaurant at a cost of $pX > 0$ implies that he will be going to an Italian restaurant *for sure*! For, otherwise, it would be extremely unwise for John to *knowingly* pay a fee of pX but actually go to a French restaurant while gaining nothing from the bet he paid for, where the lost can easily be avoided by simply rejecting the bet. And this is true for any $0 < p \leq 1$. Furthermore, if $p = 0$ then this means that John will be going to a French restaurant *for sure*. Then it follows that the betting rate upon which John is willing to pay a fair price for his acts collapses into 1 or 0. In other words, personal probabilities tend to be "gappy" when it comes to the agents' own actions.

[2]See Spohn (1977) and Levi (1989) for further discussion on the argument presented here. Rabinowicz (2002) and Levi (2007) contain a pair of exchange on Levi's well-known thesis "deliberation crowds out prediction."

To be sure, there shall be no mystery as to why the above betting interpretation falls apart: unlike the car example where John has no control over the occurrences of the events in the decision problem he is facing, namely the weather condition in Monterey next month (although he does seem to have a strong belief that it will be good), the "events" of his going to either restaurant in the betting situation above are under his immediate control, in which case the values of his actions of going to Italian or French restaurant are solely determined by the values of the consequences they lead to: Italian dinner together with $\$X$ reward versus French food. Then he should act upon whichever way he favors. Again, "chance and uncertainty are considered to have nothing to do with the situation."

Thus it seems that, when understood either in the sense of Savage's original theory of personal probability or under standard Bayesian betting interpretation, no subjective probabilities can be meaningfully assigned by an agent to his or her own actions that are being deliberated, or, more generally, to events whose occurrences depend directly on the agent's actions.[3] This restriction on Bayesian prior assignments however poses a serious challenge to many classical game-theoretic models. This is because the states of the world in these systems usually include *complete* descriptions of players' actions/strategies and the players are assumed to have prior probabilistic assessments over the occurrences of these states. Then, if

[3]Spohn (1977) remarked that the decision model developed in Jeffrey (1965) is problematic in that the subjective probabilities in Jeffrey's system are assignable to *act-dependent* events. More precisely, unlike Savage's system where there is a strict separation of states, consequences and acts with events comprising solely states, Jeffrey's model has a single algebra of events consisting of descriptions of all aspects of decision making processes, then a probability assignment to these events unavoidably includes an assignment to acts, and hence contradict the NSPA principle. Spohn then mounted a similar criticism against Luce and Krantz (1971) arguing that the occurrence of certain event A in their model implies that f_A (a Savage-style act partially defined on A) is performed, and hence, according to Spohn, their systems "contain hidden probabilities for acts." Here, we point out that there seems to be a misunderstanding on the part of Spohn. Note that one main motivation for Luce and Krantz to define acts as partial, instead of total, functions mapping from states to consequences is that they want to avoid a known interpretational difficulty in Savage's system concerning the meaningfulness of some structurally constructed acts (see also Fishburn, 1981; Shafer, 1986). In Luce and Krantz's system, a (conditional) act f_A is taken to be an act that is meaningful under A (it needs not to be definable in states outside A). Hence in Luce and Krantz (1971) it is not that the events are act-dependent, it is rather that the meaningfulness of acts relies on the events over which they are defined. This is an important issue in conditional/partial-acts based systems, we hope to address it more fully elsewhere.

probability assignments to self-actions are to be avoided then no Bayesian priors should be assigned *by the players* to those states which contain descriptions of their own actions. To substantiate this remark, let us turn to a more detailed analysis of the nature of states. We will start with a related and recurring issue concerning different epistemic viewpoints in modeling games.

18.2. The problem of asymmetric epistemic viewpoints. In a series of exchanges with Harsanyi (1982a,b) on the foundations of Bayesian games, Kadane and Larkey (1982a,b) argued that many of the core assumptions that are the bases of various "solution concepts" in classical game theory, assumptions like the common knowledge of rationality on the behaviors of the participants in a game and that of their prior probabilistic judgments on the incomplete information about the underlying game etc., are formed from the perspective of an external observer; yet, from the point of view of an individual player, other players' behaviors and beliefs (including their higher order beliefs about yet other players' behaviors and beliefs, etc.) are just as much parts of the setting of the game as everything else. Hence, they argued, it seems that no principle of rationality can mandate that one should, or even could, take the stance of an external observer and accept solution recommendations prescribed from "the above and beyond," because the implementation of which has far exceeded the individual player's epistemic capacity. Let us refer to the asymmetry between different levels of viewpoints in modeling a game, namely the point of view of an individual player and that of an external observer, as the first-person/third-person distinction in the epistemic standpoints of games; and refer to the criticism Kadane and Larkey made against the conventional equilibrium approach in game theory as the *problem of asymmetric epistemic viewpoints*.

Harsanyi (1982b), in response, formulated this well-known difficulty for multiagent systems in terms of the tension between normative (or prescriptive) and descriptive (or predictive) accounts of games. According to him, the problem of asymmetric viewpoints can be resolved within a normative theory by stipulating various rationality postulates on the part of the players. A descriptive model, on the other hand, theorizes "actual—often error-prone—human behavior in game situations" where individual players' epistemic expectations may deviate from what is prescribed in a normative theory. The two ways of modeling a game belong to different intellectual enterprises: one theoretical and one practical, yet they are related in that the former provides helpful guidelines for the latter. To illustrate this point, Harsanyi (1982a,b) drew an analogy to arithmetic arguing that arithmetic, being identified with a normative theory for "correct" computation, serves also an explanatory purpose for understanding actual computations performed by human agents: "[n]o psychologist studying how people perform arithmetic computations can develop a realistic descriptive theory of computing behavior without knowing arithmetic."

The last point on arithmetic is very true, however it should be pointed out that there is an important difference between arithmetic being instructive in theorizing human computation on the one hand and a normative theory of games being instrumental in guiding the study of the actual play of a game on the other.

Notice that, in actual computations, the human agents are subject to the same basic principles of arithmetic as described in a normative theory (simple definitions and properties of the natural numbers, rules of calculations, etc.), their failure to conform to the correct computation is largely due to their limited physical capacities or perhaps to some other psychological factors that prevent them from fulfilling the task of computing, which nonetheless can in principle be enhanced or in some cases be even eliminated (by, for instance, using a calculator or being permitted with more time to compute in an environment with less pressure or stress, etc.); whereas in the case of games, many of the fundamental assumptions made in a normative theory are different from that of a descriptive theory (which, if I understood correctly, was precisely the point Kadane and Larkey were trying to make), and the players' failure to conform to the paradigm prescribed in a normative account is often due to the first-person standpoint they are in, which cannot be lifted into a third-person stance by mere stipulation. In other words, what is being criticized is not the *strength* of the player's epistemic capacity but its very *possibility*. There is hence a gap created by the epistemic limitations on the part of participating agents.

18.3. All-inclusive states. Aumann (1987) made considerable attempts to fill this gap, where he reformulated the asymmetry of different epistemic viewpoints in terms of the tension between the "Bayesian" and the "game-theoretic" views of the world maintaining that the two accounts can be coherently integrated through the notion of *subjective correlated equilibrium* and he claims that the choices made by Bayes rational players will form a correlated equilibrium. The cornerstone of Aumann's construction is the use of "all-inclusive" states of the world, an assumption that is widely adopted in game-theoretic analysis.[4] Here is a characterization of the nature of an all-inclusive state summarized by Geanakoplos (1992):

> A "state of the world" is very detailed. It specifies the physical universe, past, present, and future; it describes what every agent knows, and what every agent knows about what every agent knows, and so on; it specifies what every agent does, and what every agent thinks about what every agent does, and what every agent thinks about what every agent thinks about what every agent does, and so on; it specifies the utility to every agent of every action, not only of those that are taken in that state of nature, but also those that *hypothetically* might have been taken, and it specifies what everybody thinks about the utility to everybody else of every possible action, and so on; it specifies not only what agents know,

[4] Aumann and Brandenburger (1995) adopted a more refined approach to the problem where each player's belief about other players' actions and beliefs are explicitly represented through the notions of *conjectures* and *theories* respectively. Both concepts are constituent components of the player's *types*, an idea originated by Harsanyi (1968), which essentially plays the same role as all-inclusive states with perhaps less informative contents about the physical world. But, at any rate, each type profile (a state) includes a description of actions of all players and it is further assumed that there is a common prior defined over all states. This implies that the players have prior probabilistic judgments over their own actions. We will raise further concerns in regard to this feature of the model in the next section.

> but what probability they assign to every event, and what probability they assign to every other agent assigning some probability to each event, and so on. (*ibid.* p.57, emphasis added)

In other words, a state is assumed to be a *complete* description of the world, which includes not only the information about the actions the players may carry out and their mutual beliefs about each other's actions and beliefs, which are usually direct targets of game-theoretic modeling (call information of this type information at the *theoretic level*); it contains also *meta-theoretic* information such as players' global probabilities judgments over *all* the states, their criteria for rational decision making (i.e., to what extend the other players are being rational), and their information structures over the states (to be made precise below). The slogan is "conditional on one particular state, everybody knows everything!"

Now, it is easily seen that the aforementioned first-person/third-person distinction is really a nonstarter in a framework that embraces all-inclusive states, for the latter encode not only information that is obtainable by each individual player (first-person) but also meta-theoretic information which is usually accessible only to the theorist (third-person). As a result of this universal encoding, the problem of asymmetric viewpoints seems to have disappeared. This is because, according to Aumann, in a given game situation the players are uncertain as to which state is the true state of the world, each player is however "informed of" a set of states which constitutes their information sets. Since each member of an information set (i.e., an all-inclusive state) is assumed to carry meta-theoretic information, it is then tautologically true, it is said, that the players will be automatically equipped with information of higher orders, which includes Bayesian rationality, common knowledge of information partitions of all players, and common prior over the states. Consequently, for Bayes rational players, the convergence to equilibrium points will proceed in the same way as in classical theory (cf. Aumann, 1987).

However, the immediate epistemological question to ask is: How can any individual player come across a state of this kind in the first place? Recall that our initial task was to explicate the intriguing question concerning the epistemic asymmetry between different viewpoints and its consequences in game-theoretic analyses, that is, how is it possible that, by mere stipulation, players can acquire information that is beyond their epistemic capabilities. Aumann approaches the question by invoking, as a primitive notion in his theory, the concept of all-inclusive states, and, as discussed above, in each one of these states the distinction between the first-person and the third-person perspectives is removed *as a result* of the very assumption. Hence, it seems that, instead of explicating as to how it is possible to alternate between internal and external viewpoints, the distinction between the two is suppressed and subsumed under the assumption of all-inclusive states. We will return to this with more technical details in Section 19.3.

Before moving on, it should be mentioned that Savage (1954) was among the first to advocate the distinction between, what he calls, normative and empirical interpretations of a formal theory. He held that, by prescribing various rationality and structural principles governing decision makers' behaviors during decision-making processes, a normative theory provides healthy guidance by which the agents can police their own potential decisions against incoherency. In his theory he also took that the description of a state of the world can be arbitrarily refined to include as much detailed information as needed (in order that the consequences under each state be precisely stated). Admittedly, Savage's theory has its own difficulties (see, e.g., Fishburn, 1981; Savage, 1967; Shafer, 1986), yet his views on issues concerning the informativeness of a state and probability assignments over acts are quite different from what we have discussed above in important ways. First, the states in Savage's theory are taken to be act-independent: they do not include descriptions of the agent's actions nor do they encode any meta-theoretic information. Moreover, Savage's theory is a single-agent decision theory which theorizes decisions made by a highly idealized self-policing Bayesian agent. The criticisms mounted against Savage's theory from the empirical side are usually that the various postulates made in his system are computationally or psychologically too demanding for actual decision makers. But Savage's theory, as a single-agent decision system, does not suffers from the difficulty of asymmetric epistemic viewpoints. The latter requires the agents to transcend their epistemic limitations, which might appear as a tall order.

In what follows, we re-examine different representations $(\mathcal{K}, \mathcal{I}; \mathcal{P}, \mathcal{T})$ of beliefs that are widely adopted in game-theoretic models, the goal is to adjust these models so that various epistemic considerations and quantifications discussed above be distinguished within the formalism.

The plan for the rest of the chapter is as follows: since information takes the center stage in various game-theoretic analyses, we first provide an extended discussion on non-probabilistic information structures and its relation to Kripke structures, where, as a case study, a purely information-theoretic version of Aumann's (1976) impossibility theorem is analyzed. The proof relies on the notion of doxastic blindspots and a generalized Savage's sure-thing principle (GSTP) which will be made clear shortly. We then discuss how these formal results should be interpreted given the clear epistemic limitation of the model. This will then be followed by an introduction of different probabilistic representations of beliefs. Conditions (A1, A2, B1, C1, and C2) are given in an attempt to unify different belief structures in the context of epistemic games.

19. Information Structure and Asymmetric Viewpoints

One mysterious feature of the notion of states of the world in game models seems to be that there is an imbalance between the conceptual presupposition of their being all-inclusive on the one hand and their precise theoretical implementations in a concrete model on the other, by which we mean that, far from being all-inclusive, the actual formulations of states in a game-theoretic model are always highly restrictive and precise: they are usually points in a Cartesian product space of players' actions and beliefs (about other players' actions and/or beliefs, represented by some probability measures) or some other variation of this joint space. Meta-theoretic information is often not modeled directly but inserted informally as further constraints as to how those precisely formulated states are structured. Hence there seems to be two different notions of states tacitly employed in a quantitative model: one, being all-inclusive, is the conceptual presupposition which plays an important interpretational role in constructing the theory; the other, being highly restrictive and precise, is the mathematical underpinning of the conceptual idealization. Considerable information however may get lost or smuggled in when one moves from one notion of states to another.

The situation echoes the tacit shifts between first-person and third-person viewpoints in a game-theoretic model discussed above. And this, we argue, is the source of many puzzlements in multi-agent systems, especially when it comes to the interpretations of various formal results proved in a representational system of knowledge or beliefs. In what follows, we introduce a notion of *doxastic blindspots* to our game model.[5] As we will see, as a component of the underlying formal model the concept of doxastic blindspots makes sense only from a third-person (the external theorist's) point of view. This allows us to separate the interpretational values of a formal system viewed from different epistemic standpoints. Our goal in this section is to tackle the issue of asymmetric viewpoints in concrete game models, some formalism is hence needed.

19.1. Information structures and blindspots. We start with the well known Kripke and Aumann information structures. Along the way we will list various formal properties of these structures, the proofs of some of the claims are standard and hence omitted, our focus is rather on their epistemic implications.

Let Ω be a (finite) set which is referred to as the state space. A Kripke model (of beliefs) over Ω is a relational structure that is distinguished by a binary relation $\rightsquigarrow \subseteq \Omega \times \Omega$, where $\rightsquigarrow$ is often referred to as a *doxastic accessibility relation* among possible states. Intuitively, $\omega \rightsquigarrow^i \omega'$ says that, from the perspective of player i,

[5] Our concept of doxastic blindspot is altered from Collins (1996), who in turn introduced the term from Sorensen (1988).

ω' is considered doxastically possible in state ω. We also say that state ω' is $\leadsto^i$-*accessible* from ω. The following is a list of properties of $\leadsto$ that are commonly adopted in a doxastic model: for any $\omega, \omega', \omega'' \in \Omega$,

Seriality: for each ω there exists an ω' such that $\omega \leadsto \omega'$.

Transitivity: if $\omega \leadsto \omega''$and $\omega'' \leadsto \omega'$ then $\omega \leadsto \omega'$.

Euclid: if $\omega \leadsto \omega'$ and $\omega \leadsto \omega''$ then $\omega' \leadsto \omega''$.

19.1.1. *Information structures.* Given $\leadsto^i$ above, define function $\mathcal{I}^i : \Omega \to 2^\Omega$ by

$$\mathcal{I}^i(\omega) = \Big\{\omega' \in \Omega \mid \omega \leadsto^i \omega'\Big\}. \tag{A1}$$

Then $\mathcal{I}^i(\omega)$ is the set of states that are $\leadsto^i$-accessible from ω, call $\mathcal{I}^i(\omega)$ the *information set* of i in state ω, the intended interpretation is that $\mathcal{I}^i(\omega)$ contains all the relevant information that is accessible by i at ω. Player i is said to be *informed of* certain *event* E (a set of states) in state ω if $\mathcal{I}^i(\omega) \subseteq E$ (i.e., if the information i possesses at ω is contained in E). For any $E \subseteq \Omega$, denote by $\mathcal{I}^i(E)$ the set of all states that are $\leadsto^i$-accessible from the states in E, i.e., $\mathcal{I}^i(E) = \bigcup_{\omega \in E} \mathcal{I}^i(\omega)$. Further, let $\mathcal{I}^i = \{\mathcal{I}^i(\omega) \mid \omega \in \Omega\}$ be the *information structure* of player i. Hence $\mathcal{I}^i$ provides a full description of what information player i has in each state.

Here we adopt a systematic ambiguity using $\mathcal{I}^i$ to denote the information structure of player i and using $\mathcal{I}^i(\cdot)$ to denote the information function of i. There should be no danger of confusion: $\mathcal{I}^i(\omega)$ is the information set of player i at ω, which is also an element of $\mathcal{I}^i$.

Alternatively, one can take the information function $\mathcal{I}^i : \Omega \to 2^\Omega$ of player i as primitive and define i's accessibility relation $\leadsto^i$ over Ω by

$$\leadsto^i := \Big\{(\omega, \omega') \in \Omega \times \Omega \mid \omega' \in \mathcal{I}^i(\omega)\Big\}. \tag{A2}$$

Consider the following properties of an information structure $\mathcal{I}$: for any $\omega \in \Omega$,

Viability: $\mathcal{I}(\omega) \neq \emptyset$.

Inclusion: if $\omega' \in \mathcal{I}(\omega)$ then $\mathcal{I}(\omega') \subseteq \mathcal{I}(\omega)$.

Mutuality: if $\omega', \omega'' \in \mathcal{I}(\omega)$ then $\omega'' \in \mathcal{I}(\omega')$ and $\omega' \in \mathcal{I}(\omega'')$.

Proposition 19.1. Let $\mathcal{I}$ be an information structure and $\rightsquigarrow$ be the corresponding accessibility relation for which (A1) and (A2) are satisfied, then

(1) $\rightsquigarrow$ is serial if and only if $\mathcal{I}$ is viable,

(2) $\rightsquigarrow$ is transitive if and only if $\mathcal{I}$ is inclusive,

(3) $\rightsquigarrow$ is Euclidean if and only if $\mathcal{I}$ is mutual.

Definition 19.2. An information structure $\mathcal{I}$ is said to be *divisible* if it is (a) viable, (b) inclusive, and (c) mutual; $\mathcal{I}$ is *partitional* if it is divisible and $\mathcal{I}(\Omega) = \Omega$.

Proposition 19.3. If $\mathcal{I}$ is divisible then, for any $\omega, \omega' \in \Omega$, either $\mathcal{I}(\omega) \cap \mathcal{I}(\omega') = \emptyset$ or $\mathcal{I}(\omega) = \mathcal{I}(\omega')$.

Proof. Let $\omega, \omega' \in \Omega$, then, by viability, $\mathcal{I}(\omega)$ and $\mathcal{I}(\omega')$ are non-empty. If $\mathcal{I}(\omega) \cap \mathcal{I}(\omega') = \emptyset$ then we are done. Otherwise, let $\nu' \in \mathcal{I}(\omega) \cap \mathcal{I}(\omega')$ and $\nu \in \mathcal{I}(\omega)$, from the former we get $\nu' \in \mathcal{I}(\omega)$ and hence $\mathcal{I}(\nu') \subseteq \mathcal{I}(\omega')$ by inclusion. Then $\nu \in \mathcal{I}(\omega)$ and $\nu' \in \mathcal{I}(\omega)$ implies $\nu \in \mathcal{I}(\nu')$ via mutuality, hence $\nu \in \mathcal{I}(\omega')$. This is shows that $\mathcal{I}(\omega) \subseteq \mathcal{I}(\omega')$. Similarly, $\mathcal{I}(\omega') \subseteq \mathcal{I}(\omega)$. Together, we have $\mathcal{I}(\omega) = \mathcal{I}(\omega')$. □

It is easy to see that if $\mathcal{I}$ is divisible then, by Proposition 19.1, the corresponding accessibility relation $\rightsquigarrow$ forms an equivalence relation over Ω, in this case we have that $\mathcal{I}(\omega) = [\omega]_{\rightsquigarrow}$ for all $\omega \in \Omega$.

19.1.2. *Doxastic blindspot.* The above construction enables a formal classification of states in Ω. Note that if $\mathcal{I}(\Omega)$ is a proper subset of Ω then it follows that there are some states that do not belong to any information set. In other words, given (A1) and (A2), members of $\Omega - \mathcal{I}(\Omega)$ are states that are *not* $\rightsquigarrow^i$-accessible for i from *any* state in Ω. We refer to such states as player i's doxastic blindspots. Formally, for any $\omega \in \Omega$, ω is said to be a *doxastic blindspot* (or *blindspot* for short) of player i if there is no $\nu \in \Omega$ such that $\omega \in \mathcal{I}^i(\nu)$. Denote the set of all blindspots of i by $\mathcal{B}^i(\Omega)$.[6]

[6] Collins (1996) defines a doxastic blindspot for player i to a singleton event $\{\omega\}$ such that $\omega \notin \mathcal{I}^i(\omega)$ ($\omega \notin P_i(\omega)$ in his notation). Our definition is equivalent to his in a KD45 system.

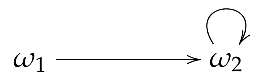

FIGURE 19.1. Player i's information structure.

EXAMPLE 19.4. Let $\Omega = \{\omega_1, \omega_2\}$ and $\mathcal{I}^i(\omega_1) = \mathcal{I}^i(\omega_2) = \{\omega_2\}$. Then ω_1 is a doxastic blindspot for i, see Figure 19.1. $\lhd$

A doxastic blindspot ω can also be interpreted as saying that the agent may falsely believe $\mathcal{I}^i(\omega)$ as she does not consider ω as an epistemic possibility. Then, from the definition, it is clear that the concept of blindspots is only intelligible if it is modeled from the third person point of view, a lesson we learn from G. E. Moore. The following are some simple formal properties of doxastic blindspots.

PROPOSITION 19.5. $\mathcal{I}^i(\Omega) \cap \mathcal{B}^i(\Omega) = \emptyset$ and $\mathcal{I}^i(\Omega) \cup \mathcal{B}^i(\Omega) = \Omega$.

PROPOSITION 19.6. Suppose that, for any $i, j \in N$, the information structures $\mathcal{I}^i$ and $\mathcal{I}^j$ are divisible, then

(1) $\omega \notin \mathcal{I}^i(\omega)$ if and only if $\omega \in \mathcal{B}^i(\Omega)$;
(2) if $\nu \in \mathcal{I}^i(\omega)$ then $\nu \in \mathcal{I}^i(\nu) = \mathcal{I}^i(\omega)$;
(3) if $\mathcal{I}^i(\Omega) = \mathcal{I}^j(\Omega)$ then, for any $\nu \in \mathcal{I}^i(\Omega)$, $\nu \in \mathcal{I}^j(\nu)$.

PROOF. By definition, $\mathcal{I}^i$ and $\mathcal{I}^j$ are divisible if and only if they are viable, inclusive, and mutual.

(1) If ω is a blindspot it is trivially true that $\omega \notin \mathcal{I}^i(\omega)$ by definition. It remains to show the "only if" direction. Suppose, for contradiction, that $\omega \notin \mathcal{B}^i(\Omega)$, then by definition this implies, there is some $\nu \in \Omega$ for which $\omega \in \mathcal{I}^i(\nu)$, from which we get $\mathcal{I}^i(\omega) \subseteq \mathcal{I}^i(\nu)$ via inclusion. On the other other hand, by viability, $\mathcal{I}^i(\omega)$ is non-empty, then let $\omega' \in \mathcal{I}^i(\omega)$, hence $\omega' \in \mathcal{I}^i(\nu)$. From $\omega \in \mathcal{I}^i(\nu)$ and $\omega' \in \mathcal{I}^i(\nu)$ we conclude that $\omega \in \mathcal{I}^i(\omega')$ and $\omega' \in \mathcal{I}^i(\omega)$ via mutuality. The latter also implies that $\mathcal{I}^i(\omega') \subseteq \mathcal{I}^i(\omega)$, and hence $\omega \in \mathcal{I}^i(\omega)$, a contradiction. Thus, if $\mathcal{I}^i$ is divisible, then $\omega \notin \mathcal{I}^i(\omega)$ implies that $\omega \in \mathcal{B}^i(\Omega)$.

(2) Suppose, to the contrary, that $\nu \notin \mathcal{I}^i(\nu)$, then, by (1), $\nu \in \mathcal{B}^i(\Omega)$, but this contradicts the assumption that $\nu \in \mathcal{I}^i(\omega)$, and hence $\nu \in \mathcal{I}^i(\nu)$. This yields $\nu \in \mathcal{I}^i(\nu) \cap \mathcal{I}^i(\omega)$, then, by Proposition 19.3, $\mathcal{I}^i(\nu) = \mathcal{I}^i(\omega)$.

(3) For any $\nu \in \mathcal{I}^i(\Omega)$, we have $\nu \in \mathcal{I}^j(\Omega)$. The latter implies that there is some $\omega' \in \Omega$ for which $\nu \in \mathcal{I}^j(\omega')$. Then, by (2), $\nu \in \mathcal{I}^j(\nu)$. □

Further, note that the existence of blindspots differentiates a doxastic information model from an epistemic one, the latter is widely used in representing knowledge where it is assumed that $\omega \in \mathcal{I}^i(\omega)$ for all $\omega \in \Omega$, that is, it is assumed that it is impossible that the players' information sets exclude "the true state of the world" (and hence there is no blindspot). The assumption is often referred to as the "truth condition" of information sets which is connected to the assumption that knowledge is infallible.

This assumption however is not uncontroversial, especially when observing from the agent's point of view, it is unclear as to how blindspots can be eliminated by stipulating that players' information be always truthful. Yet, contrary to what many had thought, the infallibility condition is in fact not essential to the agreeing-to-disagree type arguments. In the next section, we provide a purely information-theoretic analysis of Aumann (1976), where we present a scenario without the infallibility assumption under which a variant of Aumann's impossibility result holds under weakened conditions. The aim is to uncover the logic behind the formal derivation of the agreement theorem and, most importantly, the epistemic presuppositions needed for the interpretation of the underlying model. The analysis relies on a generalized Savage's sure-thing principle and a notion of *common information*, which is analogous to the concept of common knowledge (or common beliefs), to which we now turn.

19.2. Agreeing to disagree and different types of common knowledge. In the interactive situation, let $\rightsquigarrow^N$ be the smallest transitive relation that contains all the $\rightsquigarrow^i$ relations, that is,

$$\rightsquigarrow^N := \mathrm{TC}\Big(\bigcup_{i\in N} \rightsquigarrow^i\Big), \tag{19.1}$$

where 'TC' stands for the transitive closure operator. Then, relation $\rightsquigarrow^N$ represents the maximum reachability of all $\rightsquigarrow^i$'s (cf. Proposition 19.7(1) below). Call $\rightsquigarrow^N$ the *group accessibility relation* of N. For any $(\omega, \omega') \in \rightsquigarrow^N$, we also say that ω' is $\rightsquigarrow^N$-*accessible from* ω. From the group accessibility relation $\rightsquigarrow^N$ a corresponding notion of *group information function* $\mathcal{I}^N : \Omega \to 2^\Omega$ can be defined by

$$\mathcal{I}^N(\omega) = \{\omega' \in \Omega \mid \omega \rightsquigarrow^N \omega'\}. \tag{19.2}$$

And let $\mathcal{I}^N$ be the *group information structure* such that $\mathcal{I}^N = \{\mathcal{I}^N(\omega) \mid \omega \in \Omega\}$. Alternatively, one can take players' information structures $\mathcal{I}^i, \ldots, \mathcal{I}^n$ as primitive and define group information structure $\mathcal{I}^N$ as the *meet* of the $\mathcal{I}^i$s, i.e., $\mathcal{I}^N = \bigwedge_{i\in N} \mathcal{I}^i$, and define group accessibility relation $\rightsquigarrow^N$ by

$$\rightsquigarrow^N := \{(\omega, \omega') \in \Omega \times \Omega \mid \omega' \in \mathcal{I}^N(\omega)\}. \tag{19.1$'$}$$

19.2.1. *Common information.* Let E be any event, say that E is *common information* among members of group N at ω, if $\mathcal{I}^N(\omega) \subseteq E$. The following is a list of basic properties of the group accessibility relation $\rightsquigarrow^N$ and the group information structure $\mathcal{I}^N$.

Proposition 19.7. Let $\mathcal{I}^i, \rightsquigarrow^N$, and $\mathcal{I}^N$ be defined as above, then we have

(1) For any $(\omega, \omega') \in \rightsquigarrow^N$, there corresponds a sequence $i_1, i_2, \ldots, i_k \in N$ and a sequence of states $\omega_0, \omega_1, \ldots, \omega_k \in \{v \mid \omega \rightsquigarrow^N v\}$ with $\omega_0 = \omega$ and $\omega_k = \omega'$ such that $\omega_0 \rightsquigarrow^{i_1} \omega_1 \rightsquigarrow^{i_2} \cdots \rightsquigarrow^{i_k} \omega_k$, where $0 \le k < \infty$.
(2) For any $\omega \in \Omega$ we have $\mathcal{I}^i(\omega) \subseteq \mathcal{I}^N(\omega)$.
(3) For any $\omega \in \Omega$ and for any $i \in N$, we have $\mathcal{I}^i\big(\mathcal{I}^N(\omega)\big) \subseteq \mathcal{I}^N(\omega)$.
(4) Given any $\omega \in \Omega$, if, for any $i, j \in N$, $\mathcal{I}^i$ and $\mathcal{I}^j$ are divisible and $\mathcal{I}^i(\Omega) = \mathcal{I}^j(\Omega)$, then $\mathcal{I}^N(\omega) = \mathcal{I}^i\big(\mathcal{I}^N(\omega)\big)$.

PROOF. (1) and (2) are immediate from the definitions. To see (3), note that

$$\mathcal{I}^i\big(\mathcal{I}^N(\omega)\big) = \bigcup_{\nu \in \mathcal{I}^N(\omega)} \mathcal{I}^i(\nu). \tag{19.3}$$

Now suppose, to the contrary, that there is some δ such that $\delta \in \bigcup_{\nu \in \mathcal{I}^N(\omega)} \mathcal{I}^i(\nu)$ but $\delta \notin \mathcal{I}^N(\omega)$. From the latter we conclude that δ is not $\rightsquigarrow^N$-accessible from ω; the former, on the other hand, implies that there is some $\nu \in \mathcal{I}^N(\omega)$ such that $\delta \in \mathcal{I}^i(\nu)$, from which we get that δ can be reached from ω (first to ν via $\rightsquigarrow^N$ and then to δ through $\rightsquigarrow^i$), and hence is $\rightsquigarrow^N$-accessible from ω, a contradiction.

For (4), it is sufficient to show that $\mathcal{I}^N(\omega) \subseteq \mathcal{I}^i(\mathcal{I}^N(\omega))$ under the assumption that $\mathcal{I}^i(\Omega) = \mathcal{I}^j(\Omega)$ for all $i, j \in N$. Let $\nu \in \mathcal{I}^N(\omega)$, then, by (1), there should be a sequence $i_1, i_2, \dots, i_k \in N$ and a sequence $\omega_0, \dots, \omega_{k-1}, \omega_k \in \mathcal{I}^N(\omega)$ with $\omega_0 = \omega$ and $\omega_k = \nu$ such that $\nu \in \mathcal{I}^{i_k}(\omega_{k-1})$. The latter implies that $\nu \in \mathcal{I}^{i_k}(\Omega)$, then, by Proposition 19.6(3), $\nu \in \mathcal{I}^i(\nu)$, and hence $\nu \in \bigcup_{\nu \in \mathcal{I}^N(\omega)} \mathcal{I}^i(\nu)$ in (19.3). This completes the proof of the lemma. □

19.2.2. *The sure-thing principle and agreeing-to-disagree.* By Proposition 19.7(2), if, for some event E, $\mathcal{I}^N(\omega) \subseteq E$, that is, if E is common information shared among group N in state ω, then player i must be informed of E at ω. These properties together with the following generalized notion of Savage's sure-thing principle will lead to an information-theoretic version of the Agreement theorem of Aumann (1976).

> **Generalized Sure-thing Principle:** If a decision maker makes the same decision conditional on the information she has in all possible decision situations, then she should make the same decision unconditionally.[7]

[7] Recall that Savage's second postulate (P2) is derived from, what he calls, a "loose" version of the sure-thing principle (STP) which says that if the decision maker prefers one act over another assuming either certain event obtains or the compliment of the event obtains, then her preference over the two acts should remain unchanged. This "loose" version of STP is sometimes referred to as the *dominance principle* which captures an intuitive idea of reasoning by cases: if one act is weakly preferred to another in all cases than it should be weakly preferred throughout. Our notion of informational decision function is defined in this spirit.

Formally, let D be a nonempty set with unspecified domain, call $f : 2^{\Omega} \to D$ an *informational decision function for* i if for any $S \subseteq \Omega$ the following condition holds:

$$f\big(\mathcal{I}^i(\nu)\big) = d \ \text{ for all } \nu \in S \quad \Longrightarrow \quad f\Big(\bigcup_{\nu \in S} \mathcal{I}^i(\nu)\Big) = d, \tag{GSTP}$$

where $f\big(\mathcal{I}^i(\nu)\big) = d$ is a decision made by player i in state ν based on her information set $\mathcal{I}^i(\nu)$, and S is a set of possible situations. Then (GSTP) says that if player i makes the same decision d in all possible situations (states) in S then she should decide on d without differentiating the information generated in S (i.e., $\mathcal{I}^i(\nu)$'s where $\nu \in S$). We may now prove the following:

Theorem 19.8. Let $\Omega, N, \mathcal{I}^i$ be defined as above and ω be the actual state of the world. Suppose that, for any $i, j \in N$,

(1) $\mathcal{I}^i, \mathcal{I}^j$ are divisible,
(2) $\mathcal{B}^i(\Omega) = \mathcal{B}^j(\Omega)$;
(3) f is an informational decision function for N; and
(4) i's decision d^i is common information shared among members of N at ω.

Then, $d^i = d^j$ for all $i, j \in N$.

Proof. For any $i \in N$, consider the event

$$E^i = \Big\{\nu \in \Omega \,\Big|\, f\big(\mathcal{I}^i(\nu)\big) = d^i\Big\}, \quad d^i \in D \tag{19.4}$$

where E^i is the set of possible states in which f yield d^i given player i's information $\mathcal{I}^i(\nu)$ at ν. It is plain that $\omega \in E^i$ for all $i \in N$. By definition, the assumption that d^i's are common information at ω amounts to

$$\mathcal{I}^N(\omega) \subseteq \bigcap_{i \in N} E^i. \tag{19.5}$$

Then, $\nu \in \mathcal{I}^N(\omega)$ implies $\nu \in E^i$ via (19.5), and hence, by (19.4),

$$f\big(\mathcal{I}^i(\nu)\big) = d^i. \tag{19.6}$$

Note that each $\mathcal{I}^i$ is assumed to be divisible, this implies, by Proposition 19.7(4), that, for each $i \in N$,

$$\mathcal{I}^N(\omega) = \bigcup_{\nu \in \mathcal{I}^N(\omega)} \mathcal{I}^i(\nu). \tag{19.7}$$

Finally, since f is an informational decision function for each i, apply (GSTP) to (19.6) and (19.7), we get

$$f\big(\mathcal{I}^N(\omega)\big) = d^i. \tag{19.8}$$

Therefore, $d^i = d^j$ for all $i, j \in N$. □

REMARK 19.9. If f in the theorem above is intended to be a conditional probability of some event A (i.e., if $f(\cdot) = \Pr(A \mid \cdot)$) and $\mathcal{I}^i(\Omega) = \Omega$ (and hence $\mathcal{B}^i(\Omega) = \emptyset$ by Proposition 19.5), then we have the impossibility result of Aumann (1976) as a special case of the theorem above.

19.3. Different types of common knowledge? Intuitively, Theorem 19.8 says that if the players (1) have appropriate information distributions, (2) are "epistemically ignorant" in an identical way, (3) follow the same decision-making protocol, and (4) share, as common information, their decisions within the group, then it is impossible for them to make different decisions.

As with Aumann's original impossibility theorem, this statement sounds striking. Indeed, Geanakoplos (1992, 1994) gave various interesting real world "applications" of the agreement theorem which show how counter-intuitive these theorems may seem. Our aim here, however, is to see how, epistemically speaking, the frameworks under which these theorems are derived be interpreted.

Recall that Aumann's original proof uses a knowledge model that is provably equivalent to a S5 multi-agent system in epistemic logic. In other words, the players in Aumann's theory are assumed to be fully introspective agents: their knowledge is infallible, they know what they know, and they know what they don't know. In addition, it is assumed that the players commonly know what everyone, in principle, could have known in different situations, namely they are assumed to *commonly know everyone's knowledge structure*. According to Aumann (1987, p.9-10, where his information model of knowledge is more fully articulated), the latter is a different type of knowledge which is not explicitly expressible within the knowledge model yet it is "not an assumption, but a "theorem", a tautology; it is implicit in the model itself." This is because, we are told, the same meta-theoretic information about all players' information partitions is already coded in each possible state of the world which can be read off by each player regardless of what immediate epistemic situation he or she is in.

Now, in view of Theorem 19.8, we argue that Aumann's second type common knowledge of player's knowledge/information structures is indeed a substantive presupposition instead of an innocuous internal property of the system.

Observe that Theorem 19.8 not only reproduces the mathematics behind Aumann's original agreement theorem to the effect that the logical steps we have taken from (19.4) to (19.8) almost parallel the ones in Aumann's proof, it also allows us to separate the formal derivation from the assumptions made on players' epistemic capacities in the agreement argument. The interpretational difference between Theorem 19.8 and Aumann's theorem is that the players in the former need not be S5 agents and, more importantly, that they are not assumed to have access to any meta-theoretic information about their information structures. In fact, given the notion of doxastic blindspots incorporated in the current model we *cannot* assume that the players in Theorem 19.8 have access to meta-theoretic information about their information structures: the theorem proved here is intelligible only from an external point of view.

With these said, we should add that this is not to say that the players can never acquire meta-theoretic information about the information partitions. Chess game is an example where the players themselves can come to theorize the entire play *as if* they are external observers. But, in these cases, the acquisition of the theorist's external point view is rather *accidental*: it is the particular game situation with perfect information that gives rise to the merge of the internal and the external points of view. Yet this is still far removed from saying that the meta-theoretic information is a built-in feature and can be "known" by all the players as tautological truth of the underlying model.

Hence it seems that the notion of all-inclusive states is convenient only as a theoretic presupposition yet it is unclear how it can be fully realized in a concrete theory. And the asymmetry of different epistemic viewpoints is not something to be ignored in game analyses, and certainly not something that can eliminated simply by stipulation.

20. Bayesian Epistemic Games

In this section we discuss different formal probabilistic representations of beliefs and their relations to the model discussed in the last section. Efforts will be made in order that various epistemic limitations discussed above are accommodated in the constructions of the formal systems. In particular, guided by the IP principle, in all Bayesian probabilistic models, the players' beliefs will be represented by imprecise probabilities instead of sharp ones. In addition, a distinction will be made between *global* and *local* probabilities, the former are taken to be assignable only by an external theorist, the later by the players of the underlying game. This is to avoid the conceptual difficulty encountered when players assign subjective probabilities to act-dependent events as mandated by the NSPA principle. The aim is to see the interrelationships between different structures of beliefs in modeling games, for which a series of unifying conditions are provided so that various concepts can be correlated in the context of epistemic games.

20.1. Belief operator and Bayesian priors. To simplify matters, we assume that the state space Ω is finite and the algebra $\mathcal{F}$ equipped on Ω is 2^{Ω}. In a standard information model of beliefs, to say that player i *believes* certain event $E \in \mathcal{F}$, in symbols $B^i E$, is for the player to have the information about E. Given the characterization of the information structure of player i above, we say that player i *believes* certain event E in state ω if $\mathcal{I}^i(\omega) \subseteq E$. Formally, a *belief operator* B

is a set-valued function on 2^{Ω} satisfying

$$BE = \{\omega \in \Omega \mid \mathcal{I}(\omega) \subseteq E\}; \tag{20.1}$$

or, equivalently, by (A1),

$$BE = \big\{\omega \in \Omega \mid \text{for any } \nu \in \Omega, \omega \rightsquigarrow \nu \text{ implies } \nu \in E\big\}. \tag{20.2}$$

The following is a list of properties that are commonly associate with a belief operator B: for any $E, F \subseteq \Omega$,

N: $B\Omega = \Omega$

K: $B(E \cup F) \cap B\neg E \subseteq BF$

D: $BE \subseteq \neg B \neg E$

4: $BE \subseteq BBE$

5: $\neg BE \subseteq B\neg BE$

where $\neg E$ stand for $\Omega - E$. The labels of these properties echoes the corresponding rule of necessitation and axioms in modal logic. It is easily seen that if B is a belief operator, then B satisfies **N** and **K**.

Proposition 20.1. Let B be a belief operator defined in (20.1), then

(1) B satisfies **D** if and only if the underlying information structure $\mathcal{I}$ is viable;
(2) B satisfies **4** if and only if $\mathcal{I}$ is inclusive;
(3) B satisfies **5** if and only if $\mathcal{I}$ is mutual.

The proof of above proposition is straightforward and hence omitted. Now, in light of Proposition 19.1 and Proposition 20.1, let us summarize the relations between properties of a belief operator B and that of its defining information structure $\mathcal{I}$ and doxastic accessibility relation $\rightsquigarrow$ in Table 20.1.

20.1.1. *Beliefs and partial beliefs.* In defining a belief operator in (20.1), an event E is said to be believed by player i if the information i possesses in state ω is *completely* included in E. Now let us consider the case where the inclusion is not

Table 20.1

B	$\rightsquigarrow$	$\mathcal{I}$
D	serial	viable
4	transitive	inclusive
5	Euclidean	mutual

complete, that is, the information i has at ω is only partially contained in E. In this case, the player is said to be uncertain about the occurrence of E. We hence need a more fine-grained measure of beliefs. This is usually fulfilled by using probability measures to represent players' partial beliefs.

Formally, let player i's credal states be represented by a set $\mathcal{P}^i$ of probability functions on $\mathcal{F}$ (imprecise probabilities). We refer to $\mathcal{P}^i$ as *the global probability assignments for player i*. As we will see later, the description of a state may contain the player's own actions, hence global probabilities are prescribed purely from an external observer's point of view.

We stress that here is where we depart from classical models where $p \in \mathcal{P}^i$ is often taken to be player i's prior probability defined over the state space Ω, which, as we have argued in length, is highly problematic given the NSPA principle.

Presumably, in a given state space Ω, the doxastic accessibility relation $\rightsquigarrow^i$ among members of Ω and the probability functions in $\mathcal{P}^i$ over Ω are different modelings of the same epistemic capability of player i, namely his beliefs. Hence, there shall be some compatibility requirement that correlates these different measures. Recall that $\mathcal{B}^i(\Omega)$ stands for the set of blindspots of i in Ω. That is to say, for any $\omega \in \mathcal{B}^i(\Omega)$, ω is not considered by the agent to be accessible from any state. It is then conceivable that, in the corresponding probabilistic model, doxastic blindspots be identified with those states with zero probability. This leads to the following consistency requirement,[8]

$$\omega \in \mathcal{B}^i(\Omega) \text{ if and only if } \rho^i(\omega) = 0, \text{ for any } \rho^i \in \mathcal{P}^i. \tag{B1}$$

[8]For notational convenience, we write $\rho^i(\omega)$ for $\rho^i(\{\omega\})$.

In other words, ω is a doxastic blindspot for i if and only if any global probabilities of i assign 0 to ω.[9] By Proposition 19.5, (B1) implies that, for any $\omega \in \mathcal{I}^i(\Omega)$, $\rho^i(\omega) \neq 0$ for all $\rho^i \in \mathcal{P}^i$, that is, ω is $\rightsquigarrow^i$-accessible from some state if and only if $\rho^i(\omega) > 0$ for all $\rho^i \in \mathcal{P}^i$. Condition (B1) characterizes a global relationship between $\mathcal{P}^i$ and $\mathcal{I}^i(\Omega)$ (and $\mathcal{B}^i(\Omega)$); the next question is, at the local level, what is the relationship between $\mathcal{P}^i$ and each $\mathcal{I}^i(\omega)$, for all $\omega \in \Omega$. We discuss this in the context of epistemic games.

20.2. Belief structures in epistemic games. We consider models of games in strategic form augmented by type structure. Formally, let $\Gamma = \langle N, \{A^i\}, \{u^i\} \rangle$ be a (finite) strategic game, where N is a finite group of players and, for each player $i \in N$, A^i is a nonempty set of *actions* from which player i choose to act. Let $\mathcal{A} = \prod_{i \in N} A^i$ be the product space of all action spaces, each member of which forms an *action profile*, that is, for any $a \in \mathcal{A}$, $a = (a_1, \ldots, a_n)$ where $a_i \in A^i$. We refer to a real function $u^i : \mathcal{A} \to \mathbb{R}$ as the *payoff* function of i. The interpretation is that if the players in N adopt respectively the actions described in the profile $a \in \mathcal{A}$ then $u^i(a)$ is the *payoff* (or the *outcome*) for i. Following the convention in game theory, we often write an action profile $a = (a_1, \ldots, a_n) \in \mathcal{A}$ as (a^i, a^{-i}) where a^i is the ith coordinate of a, which is also denoted by $(a)_i$, i.e., the action adopted by player i in profile a, and a^{-i} is the action profile of a of the players other than i. Write A^{-i} for $\prod_{j \neq i} A^j$, hence, for any $a \in \mathcal{A}$, we have $a^i \in A^i$ and $a^{-i} \in A^{-i}$. Further, let $T^i = \Delta(A^{-i} \times T^{-i})$ denote the set of the probability measures defined on the product space of A^{-i} and T^{-i}. We refer to a member t^i of T^i as a *type* of player i, which characterizes i's belief about *other players'* actions and beliefs. Now, let $\mathcal{T} = \prod_{i \in N} T^i$ be the type space of N, and hence, for any $t = (t_1, \ldots, t_n) \in \mathcal{T}$, t provides a *type profile* of group N. Similarly, write T^{-i} for $\prod_{j \neq i} T^j$, then, for any $t = (t^i, t^{-i}) \in \mathcal{T}$, $t^i \in T^i$ and $t^{-i} \in T^{-i}$. An *epistemic extension* Γ' of strategic game

[9] This essentially amounts to identifying doxastically possible states with probabilistically non-zero states and doxastically impossible states with probabilistically zero ones. This is not problematic for a space with finitely many states. But see Hájek (2013) for discussion on the issue concerning regularity for infinite space.

Γ is a structure of the form $\Gamma' = \langle N, \{A^i\}, \{u^i\}, \{T^i\}\rangle$. Now, let the state space Ω be $\mathcal{A} \times \mathcal{T}$, then, for each $\omega \in \Omega$, ω is in the form of

$$\omega = (a, t) = (a^i, t^i; a^{-i}, t^{-i}).$$

We refer to each ω as a *state-description* of the given game situation which includes a profile of actions and a profile of types of all players. For notational purpose, we use $[\![a^{-i}, t^{-i}]\!]$ to denote the set of all states in Ω that contain profile (a^{-i}, t^{-i}):

$$[\![a^{-i}, t^{-i}]\!] = \left\{\omega \in \Omega \;\middle|\; \omega = (b^i, s^i; b^{-i}, s^{-i}) \text{ and } b^{-i} = a^{-i} \text{ and } s^{-i} = t^{-i}\right\}.$$

As before, let $\mathcal{P}^i$ be the set of global probability functions of player i defined over Ω. Then player i's types should be consistent with the globally probabilistic assessments, that is, for each $t^i \in T^i$, there is some $\rho^i \in \mathcal{P}^i$ such that,

$$t^i(a^{-i}, t^{-i}) = \rho^i\big([\![a^{-i}, t^{-i}]\!]\big) \quad \text{for all } (a^{-i}, t^{-i}) \in A^{-i} \times T^{-i}. \tag{C1}$$

Further, to relate to player i's doxastic accessibility relation $\rightsquigarrow^i$ in Ω, define that, for any $\omega = (a^i, t^i; a^{-i}, t^{-i})$ and $\omega' = (b^i, s^i; b^{-i}, s^{-i})$,

$$\omega \rightsquigarrow^i \omega' \quad \text{if and only if} \quad t^i\big(b^{-i}, s^{-i}\big) > 0. \tag{C2}$$

That is to say, state ω' is considered by player i as a doxastic possibility in state ω if i assigns a non-zero probability to (b^{-i}, s^{-i}) contained in ω'.

Theorem 20.2. Given (B1) and (C1), the accessibility relation $\rightsquigarrow^i$ $(i \in N)$ defined in (C2) is (1) serial, (2) transitive, and (3) Euclidean.

Proof. Let $\omega = (a^i, t^i; a^{-i}, t^{-i})$, $\omega' = (b^i, s^i; b^{-i}, s^{-i})$ and $\omega'' = (c^i, u^i; c^{-i}, u^{-i})$.

(1) Suppose that, for ω, there is no ω' such that $\omega \rightsquigarrow^i \omega'$, this implies, by (C2), that, for any $(b^{-i}, s^{-i}) \in A^{-i} \times T^{-i}$, $t^i(b^{-i}, s^{-i}) = 0$, and hence

$$\sum_{(b^{-i}, s^{-i}) \in A^{-i} \times T^{-i}} t^i(b^{-i}, s^{-i}) = 0.$$

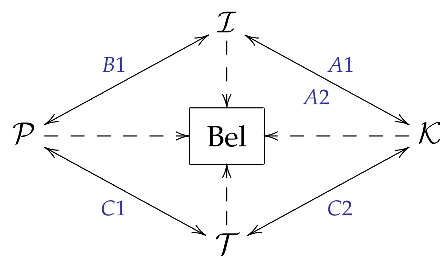

FIGURE 20.1. Unifying belief structures in epistemic games

But this contradicts the assumption that t^i is probability function on $A^{-i} \times T^{-i}$. Thus, for any $\omega \in \Omega$, there exists some ω' such that $\omega \rightsquigarrow^i \omega'$.

(2) Assume that $\omega \rightsquigarrow^i \omega'$ and $\omega' \rightsquigarrow^i \omega''$, then, by (C2),

$$t^i(b^{-i}, s^{-i}) > 0; \quad \text{and} \quad s^i(c^{-i}, u^{-i}) > 0.$$

We show $t^i(c^{-i}, u^{-i}) > 0$, i.e., $\omega \rightsquigarrow^i \omega''$. Suppose that this is not the case, then $t^i(c^{-i}, u^{-i}) = 0$, hence, by (C1), there is some $\rho^i \in \mathcal{P}^i$ for which $t^i(c^{-i}, u^{-i}) = \rho^i([\![c^{-i}, u^{-i}]\!]) = 0$. This implies that $[\![c^{-i}, u^{-i}]\!]$ is a set of blindspots via (B1). On the other hand, from $s^i(c^{-i}, u^{-i}) > 0$ we conclude that there is some $\sigma^i \in \mathcal{P}^i$ such that $s^i(c^{-i}, u^{-i}) = \sigma^i\big([\![c^{-i}, u^{-i}]\!]\big) > 0$. But if $[\![c^{-i}, u^{-i}]\!]$ is a set of blindspots then, by (B1), $\sigma^i\big([\![c^{-i}, u^{-i}]\!]\big) = 0$, a contradiction. Hence, $\rightsquigarrow^i$ is transitive.

(3) Finally, assume that $\omega \rightsquigarrow^i \omega'$ and $\omega \rightsquigarrow^i \omega''$, we show that $\omega' \rightsquigarrow^i \omega''$. The assumption implies that $t^i(b^{-i}, s^{-i}) > 0$ and $t^i(c^{-i}, u^{-i}) > 0$, and hence, for some $\rho^i \in \mathcal{P}^i$, $t^i(c^{-i}, u^{-i}) = \rho^i([\![c^{-i}, u^{-i}]\!]) > 0$. Now suppose, to the contrary, that $\omega' \not\rightsquigarrow^i \omega''$, that is, $s^i(c^{-i}, u^{-i}) = 0$. The latter implies, via (C1) and (B1), that $[\![c^{-i}, u^{-i}]\!]$ is a set of blindspots, and hence $\rho^i([\![c^{-i}, u^{-i}]\!]) = 0$, a contradiction. Therefore, $\rightsquigarrow^i$ is Euclidean. □

REMARK 20.3. As a direct consequence of Proposition 19.1, Proposition 20.1 and Theorem 20.2, we have that, for any $i \in N$, let $\mathcal{I}^i$ and B^i be defined in terms of $\rightsquigarrow^i$ given in (C2), then each $\mathcal{I}^i$ is divisible and B^i satisfies **KD45** (cf. Figure 20.1).

21. Concluding Remarks

To sum up, we addressed in this chapter various issues concerning epistemic limitations in modeling games including the problem of asymmetric viewpoints, the IP principle, and the NSPA principle. The discussion then led us to reconstruct various models of beliefs in accordance with these principles, for which we proposed a series of unifying conditions A1, A2, B1, C1, and C2 in the context of epistemic games. Figure 20.1 is an illustration of the interrelationships of these conditions. As seen, the Kripke structure can be related to the probabilistic structure through the concept of types, where a type $t \in \mathcal{T}$ is taken to be a player local probabilistic assessments of other players' actions and beliefs, which in turn is a constituent part of the global probabilistic assessments $\mathcal{P}$. Local probabilities also provide a refined measurement of doxastic accessibility, for, by (C2), we can now say that ω' is $\rightsquigarrow^i$-accessible from ω *to degree* c if $t^i(b^{-i}, s^{-i}) = c$, which accords well with the intuition that some states are "more" $\rightsquigarrow^i$-accessible than others.

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APPENDIX A

Some Mathematical Details

Gathered here are some of the definitions and results used or referred to in the main texts. They deliver some more details that complement discussions above. References of the sources are given from time to time, but all mistakes are mine.

A.1. Binary relations. Let X be a nonempty set, a *binary relation* R on X is a set of ordered pairs of elements of X. Following a notational convention, we sometimes write $(x, y) \in R$ in the form of xRy. The following is a list of properties of R: for any $x, y, z \in X$,

reflexivity: xRx
irreflexivity: $\neg xRx$
symmetry: $xRy \Rightarrow yRx$
asymmetry: $xRy \Rightarrow \neg yRx$
antisymmetry: $(xRy \wedge xRy) \Rightarrow x = y$
transitivity: $(xRy \wedge yRz) \Rightarrow xRz$
negatively transitivity: $(\neg xRz \wedge \neg zRy) \Rightarrow \neg xRy$ or $xRy \Rightarrow (xRz \vee zRy)$
completeness: xRy or yRx.

Definition A.1.1. Let R be a binary relation on X, R is

(1) a *preorder* if it is reflexive and transitive;
(2) a *weak order* (or *total order*) if it a complete preorder;
(3) a *partial order* if it is an antisymmetric preorder;
(4) a *linear order* if it is a complete partial order.

For any given preorder $\succsim$, by the *symmetric part* of $\succsim$, denoted by $\sim$, we mean $\sim = \{(x,y) \in \succsim \mid x \succsim y \text{ and } y \succsim x\}$, and by the *asymmetric part* (i.e., the *strict* part) of $\succsim$, denoted by $\succ$, we mean $\succ = \{(x,y) \in \succsim \mid x \succsim y \text{ and } y \not\succsim x\}$.

Definition A.1.2. A *preordered set* is a structure $(X, \succsim)$ where X is a nonempty set and $\succsim$ is a preorder on X. A preordered set is said to be a *poset* $(X, \succeq)$ if $\succeq$ is a partial order on X; it is a *loset* $(X, \geq)$ if $\geq$ is a linear order on X.

A binary relation E on X is said to be an *equivalence relation* if it is reflexive, symmetric and transitive. For any $x \in X$, the *equivalence class* of x with respect to E is the set

$$[x]_E = \{v \in X \mid xEv\}.$$

The collection of all equivalence classes of X with respect to E, denoted by $X/_E$ is the *quotient set* of X with respect to E, that is, $X/_E = \{[x]_E \mid x \in X\}$. It is plain that, for any given preorder set $(X, \succsim)$, $\succsim$ induces a partial order $\succeq$ on the quotient set $X/_\sim$ of X such that

$$[x]_\sim \succ [y]_\sim \text{ if and only if } x \succ y$$

$$[x]_\sim = [y]_\sim \text{ if and only if } x \sim y.$$

Definition A.1.3. Let $(X, \succsim)$ be a preordered set, any x, y of X are said to be $\succsim$*-comparable* if either $x \succsim y$ or $y \succsim x$, and they are $\succsim$*-incomparable* if they are not $\succsim$-comparable, that is, if $x \not\succsim y$ and $y \not\succsim x$, denoted by $x \bowtie y$ (some writers also use '$x \parallel y$' for incomparability).

A.2. Non-measurable sets. The following example is due to Vitali (1905). It shows that there exist sets of real numbers that are not Lebesgue measurable.

Example A.2.1 (Vitali). Define an equivalence relation $\sim$ on $\mathbb{R}$ by:

$$x \sim y \text{ if and only if } x - y \in \mathbb{Q}.$$

By the Axiom of Choice, there exists a set V of representatives from each equivalent class. Now consider the set $\{V + r \mid r \in \mathbb{Q}\}$, it has following two properties:

(1) For any distinct rational numbers r_1, r_2,

$$V + r_1 \cap V + r_2 = \varnothing.$$

(Otherwise, $V + r_1$ and $V + r_2$ share some point $h_1 + r_1 = h_2 + r_2$, then $h_1 \sim h_2$. Since h_1, h_2 are representatives it follows $h_1 = h_2$, and hence $r_1 = r_2$, a contradiction.)

(2) For any $x \in \mathbb{R}$, $x \in V + r$ for some $r \in \mathbb{Q}$, that is,

$$\mathbb{R} = \bigcup \{V + r \mid r \in \mathbb{Q}\}. \tag{A.2.1}$$

(For, x must lie in some equivalence class with a representative, say, h. Then, by definition, $x - h = r'$ for some $r' \in \mathbb{Q}$, hence $x \in V + r'$.)

We show that it cannot be the case that $V \in \mathscr{B}$. Note that if $V \in \mathscr{B}$ then it must be that $\mu(V) > 0$. For, otherwise, $\mu(V) = 0$, then $\mu(V + r) = 0$ for all $r \in \mathbb{Q}$, since μ is translation-invariant. But, by (A.2.1) and countable additivity,

$$\mu(\mathbb{R}) = \mu\Big(\bigcup \{V + r \mid r \in \mathbb{Q}\}\Big) = \sum_{r \in \mathbb{Q}} \mu(V + r) = 0$$

which is impossible. Further, if $\mu(V) > 0$ then there must be some $(a, b]$ for which $\mu(V \cap (a, b]) = c$ for some $c > 0$. Again, by translation-invariance,

$$\mu\Big(V \cap (a, b]\Big) = \mu\Big(V \cap (a, b] + r\Big) = c \quad \text{for all } r \in \mathbb{Q}. \tag{A.2.2}$$

On the other hand, consider all the rationals in $[0, 1]$, we have

$$\bigcup_{r \in \mathbb{Q} \cap [0,1]} (V \cap (a, b]) + r \subseteq (a, b + 1].$$

It follows that

$$\sum_{r \in \mathbb{Q} \cap [0,1]} \mu\Big(V \cap (a, b] + r\Big) \leq \mu(a, b + 1] = b + 1 - a. \tag{A.2.3}$$

However, by (A.2.2), the left hand of (A.2.3) is the sum of countable many c's which add to $+\infty$, a contradiction. $\lhd$

A.3. Szpilrajn extension theorem. The following result is due to Szpilrajn (1930). It shows that every partial ordering can be extended to a linear ordering.

THEOREM A.3.1. Let $\succ$ be a strict partial order on a set X. Then there exists a strict total order $>$ on X that extends $\succ$.

PROOF. Let $\mathcal{P}$ be the set of all the strict partial orders on X that extend $\succ$. Then it is plain that $\mathcal{P}$ is partially ordered under $\subseteq$. Let $\mathcal{C}$ be any chain in the poset $(\mathcal{P}, \subseteq)$, then $\bigcup \mathcal{C}$ is an upper bound of $\mathcal{C}$. To see this, we show that $\bigcup \mathcal{C}$ is irreflexive and transitive, and hence $\bigcup \mathcal{C} \in \mathcal{P}$. Suppose, to the contrary, that there is an $x \in X$ such that $(x, x) \in \bigcup \mathcal{C}$, this implies that there exists some $C \in \mathcal{C}$ for which $(x, x) \in C$, which contradicts the assumption that C is a strict partial order. As for transitivity, suppose that $(x, y), (y, z) \in \bigcup \mathcal{C}$, then there exist $C_1, C_2 \in \mathcal{C}$ such that $(x, y) \in C_1$ and $(y, z) \in C_2$. Since $\mathcal{C}$ is totally ordered under $\subseteq$, assume, without loss of generality, that $C_1 \subseteq C_2$, we get that $(x, z) \in C_2$, and hence $(x, z) \in \mathcal{C}$.

By Zorn's lemma, $\mathcal{P}$ contains a maximal element $\overline{P}$, that is, for any $P \in \mathcal{P}$, $\overline{P} \subseteq P$ implies that $P = \overline{P}$. We claim that $\overline{P}$ must be a complete relation on X. For, otherwise, there exist some $x, y \in X$ such that neither $x\overline{P}y$ nor $y\overline{P}x$ hold. In this case, define $\overline{P}' = \overline{P} \cup A$ where $A = \{x\} \cup \{z \mid z\overline{P}x\} \times \{y\} \cup \{z \mid y\overline{P}z\}$. Then it is clear that $\overline{P}'$ is a strict partial order on X that properly extends $\overline{P}$, which contradicts the maximality of $\overline{P}$. Thus, $\overline{P}$ is irreflexive, transitive, and complete. Finally, denote $\overline{P}$ by $>$, we have that $>$ is a strict total order that extends $\succ$. $\square$

A.4. Existence of uniform distribution over natural numbers. A uniformly distributed probabilistic measure on natural numbers $\mathbb{N}$ is of particular interest because (1) it serves a good purpose of delineating the difference between finite additivity and countable additivity; (2) its use is often tied to the notion of randomness: it amounts to saying that choose a number "at random." The latter is commonly understood in the following relative frequentist interpretation of uniformity of natural numbers.

A.4.1. *Density function.* Let A be any subset of $\mathbb{N}$. For each number $n < \infty$, denote the number of elements in A that are less or equal to n by $A(n)$, that is,

$$A(n) = \big|A \cap \{1, \ldots, n\}\big|. \tag{A.4.1}$$

Define the *density* of A by the limit (if exists)

$$d(A) = \lim_{n\to\infty} \frac{A(n)}{n}. \tag{A.4.2}$$

Let $\mathcal{C}_d$ be the collection of all sets of natural numbers that have densities. The following properties of the density function are easy to verify.

Proposition A.4.1. (1) $d(\varnothing) = 0$ and $d(\mathbb{N}) = 1$.

(2) For each natural number n, $d(\{n\}) = 0$.
(3) For any finite $A \in \mathcal{C}_d$, $d(A) = 0$.
(4) If $A, B, A \cup B \in \mathcal{C}_d$ and $A \cap B = \varnothing$, then $d(A \cup B) = d(A) + d(B)$.
(5) If $A \in \mathcal{C}_d$, then, for any number n, $A + n \in \mathcal{C}_d$ and $d(A) = d(A + n)$, where $A + n = \{x + n \mid x \in A\}$.
(6) The set of even numbers has density $1/2$, or more generally, the set of numbers that are divisible by $m < \infty$ has density $1/m$.

Thus, the density function (A.4.2) satisfies conditions (ii) and (iv) listed in the preceding section. However, d is not defined for *all* subsets of $\mathbb{N}$ ($\mathcal{C}_d$ is not a field of natural numbers). We hence seek to extend d to a finitely additive probability measure μ so that μ is defined for all subsets of the natural numbers and that μ agrees with d on $\mathcal{C}_d$ (Theorem A.4.6 below). One version of the extension theorem has been given by Rao and Rao (1983, Theorem 3.2.10).[1] The set-theoretic approach explicated in the next subsection is adapted from Hrbacek and Jech (1999, Ch.

[1] Kadane and O'Hagan (1995, Theorem 1) show that the monotonicity condition given by Rao and Rao (1983) in their extension theorem is also necessary, see also Schirokauer and Kadane (2007).

11). We include this construction for completion, readers may proceed directly to Example A.4.7 below without losing much on the flow of the main argument.

A.4.2. *Filter and ultrafilter.* A *filter* on a nonempty set S is a collection $\mathcal{F}$ of subsets of S such that

(1) $S \in \mathcal{F}$ and $\varnothing \notin \mathcal{F}$,
(2) if $X, Y \in \mathcal{F}$, then $X \cap Y \in \mathcal{F}$,
(3) if $X, Y \subseteq S$ and $X \in \mathcal{F}$, then $Y \in \mathcal{F}$.

Example A.4.2. (1) A trivial filter $\mathcal{F} = \{S\}$.

(2) Let $A \subseteq S$, a *principal filter* generate by A is the collection $\{X \subseteq S | A \subseteq X\}$. In the case of natural numbers where $S = \mathbb{N}$, a principal filter generated by $n_0 < \infty$ is the collection $\mathcal{F}_{n_0}$ of sets of numbers such that $X \in \mathcal{F}_{n_0}$ if and only if $n_0 \in X$.
(3) As for an example of a *nonprinciple filter*, let S an infinite set, the *Fréchet filter* on S is the collection

$$\mathcal{F} = \{X \subseteq S \mid S - X \text{ is finite}\}. \tag{A.4.3}$$

That is, $\mathcal{F}$ is the filter of all *cofinite* subsets of S. ◁

A filter $\mathcal{U}$ is said to be an *ultrafilter* if, for each $X \subseteq S$, either $X \in \mathcal{U}$ or $S - X \in \mathcal{U}$. The following extension theorem (due to Tarski, 1930) is crucial to our construction of an finitely additive probability measure on $\mathcal{P}(\mathbb{N})$. The proof uses Zorn's lemma and is widely available (see, for instance, Jech, 2003, §7).

Theorem A.4.3 (Tarski). Every filter can be extended to an ultrafilter.

Recall that our main concern in the last subsection is that the density function $d(\cdot)$ is not defined for all the subset of natural numbers, in other words, there exists some $A \subseteq \mathbb{N}$ such that the sequence $\{A(n)/n\}_{n=1}^{\infty}$ does not converge. The goal is to extend d to some measure so that (A.4.2) holds for all A's. To this end, we define a general notion of *convergence in an ultrafilter*, which has the property that, given an ultrafilter of natural numbers, *every* bounded sequences converges. As we shall see, this leads to the extension of d to $\mathcal{P}(\mathbb{N})$ as required.

Definition A.4.4. Let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers and let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$. For some $a \in \mathbb{R}$, $\{a_n\}_{n=1}^{\infty}$ is said to be *convergent in* $\mathcal{U}$ to a (or a is a $\mathcal{U}$*-limit* of the sequence), written $a = \lim_{\mathcal{U}} a_n$, if for every small $\epsilon > 0$,

$$\left\{n \,\middle|\, |a_n - a| < \epsilon\right\} \in \mathcal{U}. \tag{A.4.4}$$

Lemma A.4.5. Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$, then, for any bounded real sequence $\{a_n\}$, there exists a unique $\mathcal{U}$-limit.

Proof. Since $\{a_n\}$ is bounded, for every $x < \infty$, let

$$A_x = \{n \mid a_n < x\}.$$

Further, let

$$a = \sup\{x \mid A_x \notin \mathcal{U}\}.$$

We show that $\lim_{\mathcal{U}} a_n = a$, that is, we show that, for any $\epsilon > 0$, (A.4.4) holds. Note that, for any $x < y$, $A_x \subseteq A_y$, hence if $A_x \in \mathcal{U}$ then $A_y \in \mathcal{U}$. Since a is the least upper bound of x for which $A_x \notin \mathcal{U}$, we have $A_{a+\epsilon} \in \mathcal{U}$ but $A_{a-\epsilon/2} \notin \mathcal{U}$. Given that $\mathcal{U}$ is an ultrafilter, the latter implies that $S - A_{a-\epsilon/2} \in \mathcal{U}$, that is,

$$S - A_{a-2\epsilon} = \left\{n \,\middle|\, a - \frac{\epsilon}{2} \leq a_n\right\} \in \mathcal{U}.$$

Since $A_{a+\epsilon} = \{n \mid a_n < a + \epsilon\} \in \mathcal{U}$ and $\{n \mid a - \epsilon/2 \leq a_n\} \subseteq \{n \mid a - \epsilon < a_n\}$, we have that $\left\{n \,\middle|\, |a_n - a| < \epsilon\right\} = \{n \mid a_n < a + \epsilon\} \cap \{n \mid a - \epsilon < a_n\} \in \mathcal{U}$, and hence (A.4.4). To show uniqueness, note that if there is some $b \neq a$ such that $b = \lim_{\mathcal{U}} a_n$. Let $\epsilon = |a - b|$, then, by (A.4.4), both $A = \{n \mid |a_n - a| < \epsilon/2\}$ and $B = \{n \mid |a_n - b| < \epsilon/2\}$ are in $\mathcal{U}$. Clearly, $A \cap B = \varnothing$, and hence $B \subseteq S - A$. But this implies, from $B \in \mathcal{U}$ and the fact that $\mathcal{U}$ is an ultrafilter, that $S - A$ is also in $\mathcal{U}$, which is impossible. □

THEOREM A.4.6. There exists a finitely additive probability measure on all subsets of $\mathbb{N}$ that extends the density function d.

PROOF. Let $\mathcal{U}$ be a Fréchet ultrafilter on $\mathbb{N}$ (the existence of $\mathcal{U}$ is guaranteed by Example A.4.2 (3) and Theorem A.4.3). Define a measure μ on $\mathcal{P}(\mathbb{N})$ to be such that

$$\mu(A) = \lim_{\mathcal{U}} \frac{A(n)}{n}, \tag{A.4.5}$$

where $A(n)$ is defined as in (A.4.1). By Lemma A.4.5, μ is well defined for all $A \in \mathcal{P}(\mathbb{N})$. Note that, for any A, if $d(A)$ exists, say $d(A) = a$, then $a = \mu(A)$. For, by definition, if for any small ϵ there exists some N such that, for all $n > N$, $|A(n)/n - a| < \epsilon$, then, given that $\mathcal{U}$ is the ultrafilter of all cofinite subsets of $\mathbb{N}$, it follows that $\{n \mid |A(n)/n - a| < \epsilon\} \in \mathcal{U}$, and hence $\mu(A) = a$.

It remains to show that μ is indeed a finitely additive probability measure. Clearly, $\mu(\varnothing) = 0$ and $\mu(\mathbb{N}) = 1$. We show μ is finitely additive. To this end, let A, B be any disjoint subsets of $\mathbb{N}$. By (A.4.5) and the fact that $A \cap B = \varnothing$,

$$\begin{aligned}
\mu(A \cup B) &= \lim_{\mathcal{U}} \frac{(A \cup B)(n)}{n} \\
&= \lim_{\mathcal{U}} \frac{A(n) + B(n)}{n} \\
&= \lim_{\mathcal{U}} \frac{A(n)}{n} + \lim_{\mathcal{U}} \frac{B(n)}{n} = \mu(A) + \mu(B).
\end{aligned}$$

(Actually, it can also be easily seen that μ is also translation-invariant.) Therefore, μ is a measure defined for all subsets of $\mathbb{N}$ that extends the density function d. □

The following is a classical example of finitely but not countably additive probability measure on the natural numbers which is a simple form of the density function d introduced above.

Example A.4.7. Let $\{\lambda_n\}$ be a sequence of functions defined on $\mathbb{N}$ such that[2]

$$\lambda_n(i) = \begin{cases} 1/n & \text{if } 1 \leq i \leq n, \\ 0 & \text{if } i > n. \end{cases} \tag{A.4.6}$$

Clearly, each $\lambda_n(i)$ takes the form of $A(n)/n$ in (A.4.2) where $A = \{i\}$, and $\{\lambda_n\}$ converges point-wisely to the density function d (on singletons). By Theorem A.4.6, there exists a function λ defined for all subsets of $\mathbb{N}$ that extends d. Further, by Proposition A.4.1, λ satisfies the following properties:

(1) λ is defined for all subsets of $\mathbb{N}$.
(2) $\lambda(\varnothing) = 0$ and $\lambda(\mathbb{N}) = 1$.
(3) λ is finitely additive.
(4) λ is *not* countably additive.
(5) For any $i < \infty$, $\lambda(\{i\}) = 0$.
(6) For any $A \subseteq \mathbb{N}$, if A is finite then $\lambda(A) = 0$; if A is cofinite (i.e. if $\mathbb{N} - A$ if finite) then $\lambda(A) = 1$.
(7) $\lambda(\{2n \mid n \in \mathbb{N}\}) = 1/2$, i.e., the set of even numbers has measure $1/2$.
(8) In general, the set of numbers that are divisible by $m < \infty$ has measure $1/m$, that is, $\lambda(\{1m, 2m, 3m, \ldots\}) = 1/m$. As a result of this property, we have that the assignment of μ can be arbitrarily small: for any $\lambda > 0$, there exists some n such that the set of numbers that are divisible by n has measure $1/n < \epsilon$. $\lhd$

A.5. Convergences. Let $\{f_n\}$, f be measurable functions on the measure space $(\Omega, \mathcal{F}, \mu)$,

(1) f_n is said to *converges point-wisely* to f, in symbols $f_n \to f$, if

$$\lim_{n\to\infty} f_n(\omega) = f(\omega), \quad \text{for all } \omega \in \Omega. \tag{A.5.1}$$

[2] Dubins and Savage (1965) call probability measure of this type *diffuse*.

(2) f_n is said to *converges uniformly* to f if, for any $\epsilon > 0$, there is some large N such that

$$\Big| f_n(\omega) - f(\omega) \Big| < \epsilon, \quad \text{for all } \omega \in \Omega, n \geq N. \tag{A.5.2}$$

(3) f_n is said to converge to f *almost everywhere* (*a.e.*) if there exists a measurable set $E \subseteq \Omega$ satisfying

$$\mu(E) = 0 \quad \text{and} \quad \lim_n f_n(\omega) = f(\omega) \text{ for all } \omega \in \Omega - E. \tag{A.5.3}$$

(4) f_n is said to converge to f *in measure* if

$$\lim_{n \to \infty} \mu\Big[|f_n - f| \geq \epsilon \Big] = 0 \quad \text{for all } \epsilon > 0. \tag{A.5.4}$$

Lemma A.5.1. Given any finitely additive measure μ on measurable space $(\Omega, \mathcal{F})$, if f_n converges to 0 almost everywhere implies that f_n converges to 0 in measure, then the measure is also countably additive.

Proof. Assume that $\{B_i\}$ is any sequence of pairwise disjoint sets in the measurable space, define

$$B = \bigcup_i B_i = \bigcup_{i \leq n} B_i \cup \bigcup_{i > n} B_i$$

Let $A_n = \bigcup_{i>n} B_i$, hence $A_n \downarrow \varnothing$. We show that $\mu(A_n) \to 0$ as $n \to \infty$. To this end, let χ_{A_n} be the characteristic function of A_n, it is plain that $\mu(A_n) \to 0$ if and only if $\chi_{A_n} \to 0$ in measure. By assumption, it is enough to ask $\chi_{A_n} \to 0$ *a.e.* but this follows trivially from the fact that $\bigcap A_n = \varnothing$. Next, note that, by finite additivity,

$$\mu(A_n) = \mu\big(B - \bigcup_{i \leq n} B_i\big) = \mu\big(B\big) - \sum_{i=1}^{n} \mu(B_i)$$

Hence, from $\mu(A_n) \to 0$, we get $\mu\big(B\big) = \sum_{i=1}^{\infty} \mu(B_i)$. This shows countable additivity. □

A.6. Expectations. Let $\{f_n\}$, $\{g_n\}$ f, g be real-valued measurable functions on the measure space $(\Omega, \mathcal{F}, \mu)$. f is said to be *simple* if there n-many distinct values $c_1, \ldots, c_n$ and a partition $\{P_i\}_{i=1}^n$ of Ω such that $f(x) = c_i$ for all $x \in P_i$ $(i = 1, \ldots, n)$. Define the expectation of f with respect to μ to be

$$E(f, \mu) = \sum_{i=1}^{n} c_i \mu(P_i). \tag{A.6.1}$$

Definition A.6.1 (Expection). If f is bounded and $\{f_n\}$ is a sequence of simple measurable functions converges uniformly to f then

$$E(f, \mu) = \sup \{E(f_n, \mu) : n = 1, 2, \ldots \}. \tag{A.6.2}$$

It can be shown that the above definition does not depend on the selection of the sequences of simple functions converging to f. As shown below, any bounded measurable function f can be approximated by a particular sequence of *simple* functions, and hence in (A.6.2) we can use this sequence of simple functions to calculate the expectation of f through (A.6.1). Suppose that $c_* \leq f \leq c^*$, for each $n < \infty$, define a n-partition $\{P_i\}_{i=1}^n$ of Ω to be such that

$$P_i = \left\{ x \,\middle|\, c_* + \frac{(i-1)(c^* - c_*)}{n} \leq f(x) \leq c_* + \frac{i(c^* - c_*)}{n} \right\}, \tag{A.6.3}$$

and define f_n by

$$f_n(x) = c_* + \frac{(i-1)(c^* - c_*)}{n} \quad \text{for all } x \in P_i. \tag{A.6.4}$$

For each n, f_n is a simple function by definition, then we have that for all $x \in \Omega$,

$$|f(x) - f_n(x)| \leq \frac{c_* - c_*}{n}. \tag{A.6.5}$$

Hence $\{f_n\}$ uniformly convergences to f, in which case we have

$$E(f, \mu) = \sup \left\{ \sum_{i=1}^{n} \big[\inf_{x \in P_i} f(x) \big] \mu(P_i) : n = 1, 2, \ldots \right\}. \tag{A.6.6}$$

Note that the requirement of uniform convergence is crucial for those measure spaces with mere finitely additive probabilities. The following is the example commonly used in the literature to illustrate this point.

Example A.6.2. Let $\Omega = \{0, 1, 2, \ldots\}$ and λ be a diffuse (Example A.4.7) defined on $(\Omega, \mathcal{F})$ and $f(x) = x/(1+x)$ for all $x \in \Omega$. Using the construction from (A.6.3) to (A.6.6), we can define a sequences $\{f_n\}$ of functions uniformly converging to f such that

$$f_n(x) = \frac{i-1}{n} \quad \text{for all } x \in P_i = \Big\{x \,\Big|\, \frac{i-1}{n} \leq f(x) \leq \frac{i}{n}\Big\} (i = 1, \ldots, n).$$

Since, for each $i < n$, P_i is finite and hence $\lambda(P_i) = 0$, then we have

$$\begin{aligned} E(f, \lambda) &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i-1}{n} \lambda(P_i) \\ &= \lim_{n\to\infty} \Big[\sum_{i=1}^{n-1} \frac{i-1}{n} \lambda(P_i) + \frac{n-1}{n} \lambda(P_n) \Big] \\ &= \lim_{n\to\infty} \frac{n-1}{n} = 1. \end{aligned}$$

Now consider another sequence g_n of functions constructed as follows. Let Q_i be a $n-1$-partition of Ω such that

$$\begin{aligned} Q_1 &= \Big\{x \,\Big|\, 0 \leq f(x) \leq \frac{1}{n}\Big\} \cup \Big\{x \,\Big|\, \frac{n-1}{n} \leq f(x) \leq 1\Big\} \\ Q_i &= \Big\{x \,\Big|\, \frac{i-1}{n} \leq f(x) \leq \frac{i}{n}\Big\} \quad (i = 2, \ldots, n-1). \end{aligned}$$

Define g_n by

$$g_n(x) = \inf\Big\{f(y) \,\Big|\, y \in Q_i\Big\} \quad \text{for all } x \in Q_i (i = 1, \ldots, n-1).$$

We have that, for each n, g_n is a simple function and $E(g_n, \lambda) \equiv 0$, and $\sup\{E(g_n, \lambda) : n = 1, 2, \ldots\} = 0 \neq E(f, \lambda) = 1$. Note that the difference between $\{f_n\}$ and $\{g_n\}$ is that the latter does not converge uniformly to f. ◁

A.7. Gambler's Ruin and Countable Additivity. The story we are about to tell points to one important source of the *countable additivity* condition for probability measures. The issue is closely related to the modern philosophical debate about finite versus countable additivity. As we shall see, countable additivity is needed even at very early stage of the development of probability theory.

The Gambler's Ruin. The original Gambler's Ruin is the problem posed by Pascal to Fermat through a letter from Carcavi to Huygens on 28 September 1656 (cf. Hald, 2003, p.76). The problem goes like this: A and B are playing a game which involves the rolling of three fair dice. Each player is given 12 counters as his initial capital. The rule of the game is that if 11 points are shown, A gives a counter to B and if 14 points are shown, B gives a counter to A, then whoever first collects all the counters wins the game. The question is which one of the two players is more prone to win the game.[3]

Solution. Let us modernize the story: suppose that a gambler enters a game with capital a and adopts the strategy of continuing to bet at unit stakes with chance p of winning each bet (and chance $q = 1 - p$ of losing a bet) until his fortune increases to c or his funds are exhausted. Then the question is what is the probability of his achieving goal?

Let $X_1, X_2, \dots, X_n, \dots$ be a sequence of random variables taking on $+1$ or -1 as values with probabilities $\Pr[X_n = +1] = p$ and $\Pr[X_n = -1] = q$. Define

$$S_0 = 0;$$

$$S_n = X_1 + \cdots + X_n.$$

Intuitively, S_n counts the wins of the gambler in the first n bets, and his fortune after the n's bet is $a + S_n$. The event that the gambler achieved his goal after n's round of betting can be described as

$$A_{a,n} = \left[a + S_n = c\right] \cap \bigcap_{k=1}^{n-1} \left[0 < a + S_k < c\right], \tag{A.7.1}$$

where $\left[0 < a + S_n < c\right]$ represent the set of sequences of rolling such that the gambler's goal is not reached in the first k tries. It is easy to see that $m \neq n$ implies $A_{a,n} \cap A_{a,m} = \emptyset$. Then the probability of the gambler winning the game

[3] The mathematial detail below is pulled mainly from Billingsley (2012, §2 and §7).

with capital a and goal c, denoted by $s_c(a)$, is

$$\begin{aligned} s_c(a) &= \sum_{n=1}^{\infty} \Pr(A_{a,n}) = \Pr\Big(\bigcup_{n=1}^{\infty} A_{a,n}\Big); \\ s_c(0) &= 0, \quad s_c(a) = 1. \end{aligned} \tag{A.7.2}$$

Now apply Huyens' idea of shifting the betting sequence one step to the right, that is, from $X_1, X_2, \ldots$ to $X_2, X_3, \ldots$. Then, the initial game is equivalent to a game where the gambler has either probability p to start betting with a capital of $a+1$ or probability q with capital $a-1$. This generates the following recursive function,

$$s_c(a) = ps_c(a+1) + qs_c(a-1). \tag{A.7.3}$$

Assuming $0 \le a \le c$, let $r = q/p$, then the above equation can be solved as[4]

$$s_c(a) = \begin{cases} \frac{r^a-1}{r^c-1} & \text{if } r \neq 1 \\ a/c & \text{if } r = 1 \end{cases}. \tag{A.7.4}$$

Note. The construction above involves

(1) infinite sequences of observable results: in (A.7.1), $A_{a,n}$ (as $n \to \infty$) is the set of infinite sequence of rollings;

(2) countably additivity probability: in (A.7.2), the probability of winning the game with initial capital a and goal c is the *sum* of probabilities of winning the game after n rollings ($n = 1, 2, \ldots$).

Hence the solution in (A.7.4) is justifiable only if it can be shown the the underlying probability measure is (1) definable for infinite sequences and (2) is countable additive, to which we now turn.

Sequence space. Let S be a (finite) set of possible outcomes and ρ be a (simple) probability function defined on S. In the example above, $S = \{1,2,3,4,5,6\}$ and $\rho(i) = 1/6$, for all $i \in S$. Let $\Omega = S^\infty$ and, for any $\omega \in \Omega$, let $z_k(\omega) : S^\infty \to S$ be the kth coordinate projection function for all $k \ge 1$. Define a cylinder of rank n to

[4]See also DeGroot and Schervish (2012, p.87).

be a set of the form

$$A = [\omega : \big(z_1(\omega), \ldots, z_n(\omega)\big) \in H],$$

where $H \subseteq S^n$. Let $\mathscr{C}_0$ be the class of cylinders of all ranks, then it is easy to verify that $\mathscr{C}_0$ is a field. The goal is to define a probability measure on measurable space $(\Omega, \mathscr{C}_0)$. Now consider the following set function $\Pr(\cdot)$ on $\mathscr{C}_0$ defined by

$$\Pr(A) = \sum_H \rho\big(z_1(\omega)\big) \times \cdots \times \rho\big(z_n(\omega)\big). \tag{A.7.5}$$

We show that Pr is a well defined probability measure on $(\Omega, \mathscr{C}_0)$. For this, we only show that $\Pr(\cdot)$ is *finitely additive*: let A be as above and

$$B = [\omega : \big(z_1(\omega), \ldots, z_m(\omega)\big) \in I]$$

for some $I \subseteq S^m$. WLOG, assume that $n \leq m$, then let $H' \subseteq S^m$ be such that, for each $\omega \in \Omega$, $\big(z_1(\omega), \ldots, z_n(\omega), \ldots, z_m(\omega)\big) \in H'$ iff $\big(z_1(\omega), \ldots, z_n(\omega)\big) \in H$, and hence

$$A = [\omega : \big(z_1(\omega), \ldots, z_m(\omega)\big) \in H']$$

Now suppose that $A \cap B = \emptyset$, then by (A.7.5)

$$\begin{aligned}\Pr(A \cup B) &= \sum_{H' \cup I} \rho\big(z_1(\omega)\big) \cdots \rho\big(z_m(\omega)\big) \\ &= \sum_{H'} \rho\big(z_1(\omega)\big) \cdots \rho\big(z_m(\omega)\big) + \sum_I \rho\big(z_1(\omega)\big) \cdots \rho\big(z_m(\omega)\big) \\ &= \Pr(A) + \Pr(B).\end{aligned}$$

$\Pr(\cdot)$ is referred to as a (finitely additive) *product measure* on $(\Omega, \mathscr{C}_0)$. We point out that Pr is at the same time countably additive. To this end, we first turn to the following observations

LEMMA A.7.1. If $\Pr(\cdot)$ is a finitely additive probability measure on the field $\mathscr{F}$, and if $A_n \downarrow \emptyset$ for sets A_n in $\mathscr{F}$ implies $\Pr(A_n) \downarrow 0$, then $\Pr(\cdot)$ is countably additive.

Proof. Assume that $\{B_n\}$ is a sequence of pairwise disjoint sets in $\mathscr{F}$, define

$$B = \bigcup_i B_i = \bigcup_{i \le n} B_i \cup \bigcup_{i>n} B_i$$

Let $A_n = \bigcup_{i>n} B_i$, then $A_n \downarrow \emptyset$ as $n \to \infty$. Note that, by finite additivity,

$$\Pr(A_n) = \Pr\left(B - \bigcup_{i \le n} B_i\right) = \Pr\left(B\right) - \sum_{i=1}^{n} \Pr(B_i)$$

Hence, from $\Pr(A_n) \to 0$, we get $\Pr\left(B\right) = \sum_{i=1}^{\infty} \Pr(B_i)$. □

Lemma A.7.2. If $A_n \downarrow A$, where A_n are nonempty cylinders, then $A \neq \emptyset$.

Proof. See Billingsley (2012, p.30) □

Theorem A.7.3. Every finitely additive product measure on $\mathscr{C}_0$ is countably additive.

Proof. Assume, to the contrary, that $\Pr(\cdot)$ is not countably additive, then apply Lemma A.7.1: there is some sequence $\{A_n\}$ in $\mathscr{C}_0$ such that $A_n \downarrow \emptyset$ and $\Pr(A_n)$ does not converge to 0, that is, there is some $\epsilon > 0$ for which $\Pr(A_n) > \epsilon$ as $n \to \infty$. This implies, by Lemma A.7.2, $\emptyset = A = \bigcap_n A_n \neq \emptyset$, which is absurd. □