Columbia University

Department of Economics
Discussion Paper Series

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Massimiliano Amarante

Discussion Paper #:0102-50

Department of Economics
Columbia University
New York, NY 10027

April 2002

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Massimiliano Amarante*
Columbia University†

Revised: March 2001

Abstract

Crémer and McLean [1] and McAfee and Reny [3] showed that, in "nearly all auctions", the seller can offer a mechanism that obtains full rent extraction. The mechanism designed by Crémer and McLean differs from standard procedures in several respects. Notably, (a) bidders are required to buy "lotteries" in order to partecipate at the auction; (b) bidders are treated non-anonymously, in the sense that different bidders are given different strategic options; (c) the bidders' payments to the seller depend not only on the bids but also on some other "actions" taken by the bidders. In this paper, I keep (a) and (b), and impose that – like in the "standard model of auctions" – the bidders' payments to the seller depend on the bids alone. I find that the full-surplus extraction result no longer holds: There are "open sets" of auctions where the full extraction is not possible.

1 Introduction

Auctions occupy an important place within economic theory due not only to the unquestionable importance of this procedure in economic life ([8], [6]), but also because an auction mechanism can be considered an archetype of a general mechanism design problem. In fact, problems like optimal taxation, regulation, monopolistic price discrimination, trade under asymmetric information, public good provision display, essentially, the same structure as an auction problem.

In a typical auction an individual, the seller, auctions off a single object to N bidders. The seller maximizes his expected revenue from the auction by designing a set of rules according to which the object is awarded to one of the bidders. Each bidder privately knows how much the good is worth to him. Given such private information and the rules set by the seller, the strategic problem bidders face is modeled as a game with incomplete information. Hence, the seller's

*I am grateful to Jaques Crémer and Paolo Siconolfi for insightful comments and explanations.

†Department of Economics, 1019 IAB, 420W 118th St., NY, NY 10027. E-mail: ma734@columbia.edu.

problem can be viewed as that of choosing the game of incomplete information that the bidders will play. An extensive literature has studied how different mechanisms perform from the viewpoint of the seller (first-price, second-price, Dutch and English auction, etc.) as well as conditions under which all these mechanisms give the same expected revenue.

It is clear that the presence of bidders' private information (about their own valuations, their beliefs about the opponents' valuations, etc.) constitutes the main hurdle the sellers faces in pursuing his objective. In fact, it seems reasonable to conjecture that the bidders' private information would lead them to enjoy positive rents and, hence, limit the seller's ability to maximize his revenue. Yet, contrary to this intuition, work by Crémer and McLean [1] and McAfee and Reny [3] has shown that – in "nearly all auctions" – the seller can offer a mechanism that obtains full rent extraction.

Most likely, this result can be falsified in real life situations as the assumptions of the theoretical model are hardly met. Yet, it is unquestionable that, on a purely theoretical ground, the mechanism of Crémer and McLean does the job. This has generated a lot of skepticism, if not disbelief, to the point that McAfee and Reny conclude that "*the full rent extraction result* casts doubt on the value of the current mechanism design paradigm as a model of institutional design" ([3], p. 400). Because of this state of things, it seems worthwhile to investigate on the assumptions of Crémer and McLean's model as well as on the features of the mechanism that they proposed.

As it will be explained below, the mechanism that extracts full surplus differs from standard procedures in several respects. Notably, (a) bidders are required to buy "lotteries" in order to participate at the auction; (b) bidders are treated non-anonymously, in the sense that different bidders are given different strategic options; (c) the bidders' payments to the seller depend not only on the bids but also on some other "actions" taken by the bidders.

On a theoretical ground, the introduction of lotteries is not, by itself, particularly relevant. If the only effect of lotteries is to generate some additional payment from the bidders to the seller, then, in many instances, lotteries can be replaced by a simple participation fee. However, lotteries do come to play a big role when their introduction is combined with some other innovation, like (b) and/or (c) above. Because of this, it seems reasonable to mantain the use of the lotteries, and study the implications of (b) and (c) separately, by dropping them one at a time. Here I keep (a) and (b), and impose that the bidders' payments to the seller depend on the bids alone. This conforms to what is called the "standard model of auctions" (McAfee and McMillan [2]) and to the procedures commonly used in practice (first-price, second-price, Dutch and English auction, etc.). The finding is that the full-surplus extraction result no longer holds: There are "open sets" of auctions where the full extraction is not possible. More than the finding by itself, what is more important is the reason why it is so as, by contrast, this would clarify some aspects of Crémer and McLean's result as well. As the proof will make it clear (section 4), the "non-full surplus extraction" result is due to the following reason. Generally speaking, the emerging of rents does not depend on the fact that –at some level – the seller "does not know" what information the bidders have. For instance, the seller could picture a hypothetical individual with some valuation and some beliefs, and design a mechanism that extracts full rent from that individual. Since a seller can conceive the set of all possible bidders' valuations as well as the set of all possible bidders' beliefs, it is pretty clear that – for every single bidder's type – he can design a mechanism that targets that type, and extracts that type's rent. Then, from the viewpoint of the seller, the bidders' private information matters only if there is an inability to design a mechanism that simultaneously targets every single bidders' type. The proof shows that when the bidders' payments to the seller depend on the bids alone, the seller, so to speak, does not have enough "tools" to target all the bidders' types simultaneously.

The paper is organized as follows. Section 2 describes the model used in the paper, and Section 3 the mechanism of Crémer and McLean. Section 4 studied the case when payments depend on the bids alone. Section 5 concludes by discussing some related work.

2 The Model

Throughout the paper, the number of bidders is finite. The set of bidders is denoted by I, $I = \{1, 2, ..., N\}$. For simplicity, I deal with the independent private value case, only. All the reasonings can be effortlessly extended – but at the cost of tedious qualifications – to the common value case or to any common-private value mix. Since Crémer and McLean have remarked in several occasions how this can be done, there is no reason to do so here. The model is the same as Crémer and McLean's. It will be described by using the terminology of Mertens and Zamir [5] as this will make the main argument much more transparent.

Each bidders $i \in I$ has a valuation for the object $v \in V_i$, where V_i is a compact subset of the real line, which is denoted by R. The product $V^N = \overset{N}{\underset{1}{\times}} V_i$ is the basic domain of uncertainty for our problem. T_i denotes the set of types for bidder i, and $T^N = \underset{i}{\times} T_i$. Recall that a type can be identified to a probability measure on $V^N \times T^{N-1}$, and observe that – in our special case – any of such measures must be the unit mass, $\delta(\cdot)$, on that type's valuation. Hence, by means of the identification $v \longmapsto \delta(v)$, the universal belief space generated by V^N is $\Omega = T^N = \times T_i$.

I restrict to problems that display a common prior P on T. Then, each type $t^i \in T_i$ can be derived from P as a conditional probability. I do so since the mechanism of Crémer and McLean is designed for these type of situations. Following Crémer and McLean, we have

Definition 1 *An information structure with basic domain of uncertainty V^N is a pair (T^N, P), where P is a probability measure on T^N.*[1] *The set of all*

[1] Crémer and McLean call an information structure a triple (T^N, P, w_i), where w_i is a valuation function, i.e. a mapping $T_i \longrightarrow V_i$, which associate each type with his own valuation. I deleted the reference to w_i as a type's valuation is immediately derived from that type on

information structures with basic domain of uncertainty V^N is the set of all pairs (T^N, P) as P varies among all possible probability measures on T^N. Such a set is denoted by $\mathcal{I}(V^N)$.

Definition 2 *An information structure $(T^N, P) \in \mathcal{I}(V^N)$ is a finite information structure if P has finite support in T^N. The set of finite information structures in $\mathcal{I}(V^N)$ is denoted by $\mathcal{F}(V^N)$.*

Throughout the paper, the word generic refers to subsets of $\mathcal{I}(V^N)$, which is endowed with the weak-* topology.

The next definition is slightly different from Crémer and McLean's. It is so, because I need to distinguish between the case of auctions where payments depend on bids alone from the case dealt with by Crémer and McLean, where no such a restriction is imposed. For each $i \in I$, let A_i be a set, and let $a \in A = \times_i A_i$. One can interpret A_i as the set of possible "messages" that player i might send to the seller. Then,

Definition 3 *An auction with message space A is a collection of mappings $\{p_i, x_i\}_{i \in I}$, with $x_i : A \longrightarrow R$ and $p_i : A \longrightarrow R$ such that $p_i(a) \geq 0$ for all i and $a \in A$ and $\sum\limits_{i \in I} p_i(a) \leq 1$ for all a.*

The interpretation is that if players "announce" a, then bidder i pays an amount $x_i(a)$ to participate at the auction, and is awarded the object with probability $p_i(a)$. The seller's problem consists of choosing $\left(A, \{p_i, x_i\}_{i \in I}\right)$ so to maximize his expected revenue. Bidder i's utility is given by $v_i(t) - x_i(a(t))$ if he is awarded the object, and by $-x_i(a(t))$ if he is not.

Crémer and McLean's result (see below) can be rephrased by saying that the seller can choose $A = T^N$ and mappings $\{p_i, x_i\}_{i \in I}$ so that full-surplus extraction obtains.

Let B_i be the set of possible bids for player i, and let $B = \times_i B_i$.

Definition 4 *We say that payments depend on bids alone if $A = B$. In such a case, we will say that the auction is a "standard auction".*

3 The Mechanism of Crémer and McLean

The mechanism can be described as follows. Each bidder i has to pick an element in a set $\mathcal{L}_i$. A typical element $l_i \in \mathcal{L}_i$ is a mapping $l_i : \underset{j \neq i}{\times} (\mathcal{L}_j \times B_j) \longrightarrow R$, specifying bidder i's payment to the seller as a function of the other bidders' choices as well as their bids. I will refer to elements in $\mathcal{L}_i$ as lotteries as they are such from the viewpoint of bidder i. The object is awarded to the highest bid (with the addition of a tie-breaking rule, which is irrelevant for our purposes).

the basis of the observation made in the previous paragraph.

These rules define payoff functions $u_i : T_i \times \left(\underset{j}{\times} (\mathcal{L}_j \times B_j) \right) \rightarrow R$ for each bidder i. A strategy for bidder i is a mapping $\sigma_i : T_i \rightarrow \Delta (\mathcal{L}_i \times B_i)$. This completes the description of the game with incomplete information chosen by the seller. Then, when T^N is a finite set, we have

Theorem 5 ***[Crémer and McLean [1]]*** *Generically (in the space of all the P's on T^N), the seller can choose the sets $\mathcal{L}_i$'s so that there exists an equilibrium of the corresponding game such that (I) Players play dominant strategies; (II) Full surplus extraction obtains.*

A similar result holds for the uncountable compact case (McAfee and Reny [3]).

To see how full surplus extraction obtains, let us begin by observing that since for each bidder the payment is determined by means of an $l_i \in \mathcal{L}_i$, bidder i's payment does not depend on his own bid. Hence, the rule that the object is awarded to the highest bid implies that it is a dominant strategy for bidder i to bid his own valuation.

Denote by t_k^i a type for bidder i who has valuation v_k^i for the object, and by g_k^i the expected gain gross of the payment that type t_k^i obtains *at this dominant-strategy profile.*

Under these circumstances, the seller's problem is solved if, for each bidder i, the seller can find a set of lotteries, $\mathcal{L}_i$, such that (at the above dominant strategies) the following is true for each i's type

$$\begin{aligned} E(l_k^i \quad &| \quad t_k^i) = g_k^i \\ E(l_z^i \quad &| \quad t_k^i) > E(l_k^i \mid t_k^i), \qquad z \neq k \end{aligned} \tag{1}$$

In words, when offered a choice among the lotteries in the set $\mathcal{L}_i$, type t_k^i picks lottery l_k^i and his (total) expected gain is zero.

For simplicity, I illustrate the argument in the two-bidder case. Let $T = \{t_1, t_2, ..., t_m\}$ be the set of bidder 1's types, and let $\Theta = \{\theta_1, \theta_2, ..., \theta_m\}$ be that of player 2.[2] In such a case, the common prior P is a matrix, $P = (p_{ij})$, on $T \times \Theta$. Denote by $p_k = P(\cdot \mid t_k)$ and $q_k = P(\cdot \mid \theta_k)$ the conditionals computed from P (the types).

Now, suppose that bidder 2 is offered a set of lotteries $\Lambda = \{\lambda_1, ..., \lambda_m\}$, and that type θ_k picks lottery λ_k. Then, we must offer bidder 1 a set of lotteries $\left\{\tilde{l}_1, ..., \tilde{l}_m\right\}$ such that

$$\begin{aligned} p_k \cdot \tilde{l}_k &= g_k^1 \qquad && k = 1, ..., m \\ p_k \cdot \tilde{l}_j &> g_k^1 \qquad && k \neq j \end{aligned}$$

or,

$$\begin{aligned} p_k \cdot \tilde{l}_k &= g_k^1 \qquad && k = 1, ..., m \\ p_j \cdot \tilde{l}_k &> g_j^1 \qquad && k \neq j \end{aligned}$$

[2] To shorten the exposition, I have assumed that T and Θ have the same cardinality. Obviously, the Crémer and McLean argument does not require such a condition.

Equivalently, we must find m linear functionals, $\{l_1, ..., l_m\}$, such that

$$\begin{array}{rclr} p_k \cdot l_k & = & 0 & k = 1, ..., m \\ p_j \cdot l_k & > & 0 & k \neq j \end{array}$$

In other words, the linear functional l_k must separate p_k and $co\{p_1, ..., p_{k-1}, p_{k+1}, ...p_m\}$ [co denotes the convex hull], and, for each k, we must find a linear functional that does so.

It's clear that we can fulfill such a request as long as

$$p_k \cap co\{p_1, ..., p_{k-1}, p_{k+1}, ...p_m\} = \emptyset, \qquad \forall k \tag{2}$$

Let P_1 be the matrix of conditional probabilities of bidder 1. Since $(\det P_1 \neq 0) \Longrightarrow$ condition (2), generically we can do so. Hence, given bidder 2's choices, $\{\lambda_1, ..., \lambda_m\}$, and valuations, $\{v_1, ..., v_m\}$, we can use the components of the vector l_k to define a function

$$l_k(\lambda_j, v_j) \longmapsto R$$

which is our desired lottery.

Finally, we can proceed in a similar way for bidder 2.

It is worth remarking that, in this way, Crémer and McLean constructively provide a mechanism that leads to full surplus extraction. One of the mechanism's virtues is that the actual state of the world the bidders' payment is contingent upon is verifiable not only by the bidders and the seller but also by an outside observer. This is so because the payment depends on the lotteries and the bids chosen by the other bidders, both of which are observable, and not on the other bidders' types (which are unobservable).

4 The Case when Payments Depend on the Bids Alone

Throughout this section, I consider mechanisms that have the same structure as the Crémer and McLean's mechanism above, but with the additional restrictions that payments depend on the bids alone. I will refer to these mechanisms as BA (for bids alone) mechanisms. BA mechanisms imply a restriction on the domain of the lotteries: elements of $\mathcal{L}_i$ are now mappings $l^i : \underset{j \neq i}{\times} B_j \longrightarrow R$. Just like before, bidder i's payment does not depend on his own bid, and, once again, the rule that the object is awarded to the highest bid implies that it is a dominant strategy for bidder i to bid his own valuation. It follows that (at such a dominant-strategy profile) the only beliefs that are relevant to determine player i's expected payment are the first-order beliefs, that is i's beliefs on other players' valuations. I am going to show that

Theorem 6 *Generically in $\mathcal{I}(V^N)$, BA mechanisms do not obtain full surplus extraction.*

The proof will be given first for the case of a finite set of possible valuations, and then extended to the case of uncountable but compact set of possible valuations.

4.1 Finite Set of Valuations

Recall that the space of possible types for a player can be identified to the space of probability distributions on $V^N \times T^{N-1}$. Here, I assume that V is a finite set, $V = \{v_1, ..., v_k\}$. To avoid trivialities, I also assume that $k > 1$.

To ease the exposition, I present the proof for the two-bidder case since the argument goes unchanged for the N-bidder case.

Let $(\Theta_1 \times \Theta_2, P)$ be a finite 2-bidder information structure with common prior P. Here, $\Theta_1 \times \Theta_2$ denotes the support of P in $\Omega = T_1 \times T_2$, the universal beliefs space. To avoid tedious qualifications, I treat players symmetrically by assuming that $\Theta_1 = \Theta_2 = \Theta$. The assumption does not entail any loss in generality since, in all of my reasoning, it will be sufficient to consider the player with the larger number of types. In the 2-bidder case, the common prior P can be represented by a matrix on $\Theta \times \Theta$. Notice that since P is a common prior and $T_1 \times T_2$ is constructed as a projective limit, for any j we have a map $P \longmapsto P_i^j$, which associates the common prior with the beliefs of order j of player i. In particular, i's first-order beliefs, P_i^1, are univoquely determined. In what follows, I will suppress the reference to player i, since all the reasonings refer to one bidder only.

Let us begin by giving necessary an sufficient conditions for a BA mechanism to obtain full-surplus extraction.

Lemma 7 *Let $\varphi \in \mathcal{F}(V^N)$ be a finite information structure, and let $m = card(\Theta)$. Let P^1 be the matrix of first-order beliefs. Then, there exists a BA mechanism which obtains full surplus extraction iff no row in P^1 is a convex combination of the other rows.*

Obviously, this is the same argument that we saw for Crémer and McLean, but now restricted to first-order beliefs. The proof is included only for completeness.

Proof. Denote by p_j the j-th row of P^1, $j = 1, 2, ..., m$. The k-component vector p_j is a probability distribution on $V = \{v_1, ..., v_k\}$.

As explained in the above, in order to show that full surplus extraction obtains we need to show that there exist m linear functionals – $\tilde{l}_1, \tilde{l}_2, ..., \tilde{l}_m$ – such that

$$\begin{aligned} p_j\tilde{l}_j &= E(u \mid \theta_j) \\ p_j\tilde{l}_k &> E(u \mid \theta_j) \quad , \quad k \neq j \end{aligned} \tag{3}$$

or equivalently that there exist m linear functionals – $l_1, l_2, ..., l_m$ – such that

$$\begin{aligned} p_j l_j &= 0 \\ p_k l_j &> 0 \quad , \quad k \neq j \end{aligned} \tag{4}$$

Just like before, m linear functional, $l_1, l_2, ..., l_m$, with the desired properties exist if and only if

$$p_j \cap co\{p_1, ..., p_{j-1}, p_{j+1}, ..., p_m\} = \varnothing \qquad , \qquad j = 1, ..., m \tag{5}$$

In fact, if condition (5) is satisfied for p_j, then the desired linear functional l_j exists by any elementary separation theorem. The converse (necessity) is obvious. ■

From the lemma, it emerges that we need to inquire into the conditions under which no row in P^1 is a convex combination of the other rows, which is what the next lemma addresses.

Recall that the common prior $P = (p_{ij})$ is a matrix on $\Theta \times \Theta$. The element p_{ij} gives the probability that bidder 1 is of type θ_i and bidder 2 is of type θ_j. Recall also that a type is a conditional probability on $V^2 \times \Theta$, and that his first order belief is a probability on V, computed as a marginal from his type. Therefore, bidder 1's first-order beliefs give the probability that bidder 2 has valuation v_j given that bidder 1 is of type θ_i. Denote by $\mathbf{1}_{v_l}$ the map on Θ that takes value 1 if type θ has valuation v_l, and 0 otherwise. Clearly, player 1's first-order beliefs (the elements of the matrix P^1) are computed from P according to the formula

$$\text{Prob}(v_l \mid \theta_i) = \frac{\sum_{j=1}^{k} \mathbf{1}_{v_l} p_{ij}}{\sum_{j=1}^{k} p_{ij}} \tag{6}$$

Let $\{J_l\}_1^k$ be a partition of the set of types, Θ, so that two types, θ and θ', are in the same class if they have the same valuation ($v = v' = v_l$, for some l). Then,

Lemma 8 *A row in P^1 is a convex combination of the other rows if $\exists \lambda \in R^m$, $\lambda \neq 0$, such that*

$$\lambda \left(\sum_{j \in J_l} c_j \right) = 0 \qquad , \qquad l = 1, ..., k$$

where c_j is the j-th column in P (the common prior).

If the n-th row is one of such rows, then λ is given by

$$\lambda = \left(-\beta_1 \frac{p_{\cdot n}}{p_{\cdot 1}}, -\beta_2 \frac{p_{\cdot n}}{p_{\cdot 2}}, ..., \underset{n\text{-}th\ pos}{1}, ..., -\beta_m \frac{p_{\cdot n}}{p_{\cdot m}} \right) \tag{7}$$

where $p_{\cdot j}$ is the sum of the elements of the j-th row in P, and $\sum_{j \neq n} \beta_j = 1$.

Conversely, if $\exists \lambda$ satisfying (7), then the n-th row in P^1 is a convex combination of the other rows.

Proof. We saw that a generic element, p^1_{il}, in P^1 is of the form

$$p^1_{il} = \frac{\sum_{j=1}^{k} \mathbf{1}_{v_l} p_{ij}}{\sum_{j=1}^{k} p_{ij}} = \frac{\sum_{j \in J_l} p_{ij}}{\sum_{j=1}^{k} p_{ij}} \tag{8}$$

Suppose that row n in P^1 is a convex combination of the other rows. This means that there are $m-1$ numbers, $\{\beta_1, ..., \beta_{n-11}, \beta_{n+1}, ..., \beta_m\}$, such that $\beta_j \geq 0$ and $\sum_{j \neq n} \beta_j = 1$ such that

$$p^1_{nz} = \sum_{j \neq n} \beta_j p^1_{jz} \qquad , \qquad z = 1, ..., k$$

or, using (8)

$$\frac{\sum_{j \in J_z} p_{nj}}{\sum_{j=1}^{k} p_{nj}} = \sum_{j \neq n} \beta_j \frac{\sum_{l \in J_z} p_{jl}}{\sum_{j=1}^{k} p_{jl}} \qquad , \qquad z = 1, ..., k$$

To ease the notation, denote by $p_{\cdot j}$ the sum of the elements of the j-th row in P, and set $p_{n,J_z} = \sum_{j \in J_z} p_{nj}$. With this notation the above expression becomes

$$\frac{p_{n,J_z}}{p_{\cdot n}} = \sum_{j \neq n} \beta_j \frac{p_{j,J_z}}{p_{\cdot j}} \qquad , \qquad z = 1, ..., k$$

Rearranging, the preceding becomes

$$\lambda \left(\sum_{j \in J_z} c_j \right) = 0 \qquad , \qquad z = 1, ..., k$$

where

$$\lambda = \left(-\beta_1 \frac{p_{\cdot n}}{p_{\cdot 1}}, -\beta_2 \frac{p_{\cdot n}}{p_{\cdot 2}}, ..., \underset{n\text{-th pos}}{1}, ..., -\beta_m \frac{p_{\cdot n}}{p_{\cdot m}} \right)$$

The converse is obvious. ■

The proof of the theorem rests on the following simple

Lemma 9 *Let $d_1, ..., d_k \in R^m$, $k < m$. Then, $\exists \lambda \in R^m$, $\lambda \neq 0$, such that*

$$\lambda d_l = 0 \qquad , \qquad l = 1, ..., k$$

Moreover, either λ is proportional to an element of the canonical basis of R^m or λ can be taken of the form

$$\lambda = \left(\frac{\alpha_j}{\sum_l d_{jl}} \right)$$

with $\sum_{j=1}^{m} \alpha_j = 0.$

Proof. Let $d_1, ..., d_k \in R^m$. By the dimension theorem and the orthogonal decomposition of Euclidean spaces, $k < m$ implies $\exists \lambda \in R^m$, $\lambda \neq 0$, such that $\lambda d_l = 0$, $l = 1, ..., k$.

First, suppose that $d_{jl} \neq 0$, $j = 1, ..., m$ and $l = 1, ..., k$. Let $\{\alpha_j\}_{j=1}^m$ be a set of numbers such that $\sum_{j=1}^{m} \alpha_j = 0$. Then, clearly the vector

$$\beta^l = (\beta_j^l) = \left(\frac{\alpha_j}{d_{jl}} \right)$$

solves the l-th equation, $l = 1, ..., k$.

Now, define a vector $\lambda \in R^m$ by

$$\lambda = (\lambda_j) = \left(\frac{1}{\sum_l d_{jl}} \sum_l d_{jl} \beta_j^l \right)$$

Then λ is of the form $\lambda = \left(\frac{\alpha_j}{\sum_l d_{jl}} \right)$, and for any l, $l = 1, ..., k$

$$\begin{aligned} \sum_j \lambda_j d_{jl} &= \sum_j \left[\frac{1}{\sum_l d_{jl}} \sum_l d_{jl} \beta_j^l \right] d_{jl} \\ &= \sum_j \left[\frac{1}{\sum_l d_{jl}} \sum_l d_{jl} \frac{\alpha_j}{d_{jl}} \right] d_{jl} \\ &= \sum_j \left[\frac{1}{\sum_l d_{jl}} \alpha_j \sum_l d_{jl} \right] \\ &= \sum_j \alpha_j = 0 \end{aligned}$$

If $d_{jl} = 0$ for some j and l, then it is clear – by repeating the same reasoning or by simple geometric inspection – that a λ of the same form exists as long as $k < m - 1$ or at least one of the $d_1, ..., d_k$ has at least two components different from zero (this is obtained by simply setting in the above reasoning $\alpha_j = 0$ when $d_{jl} = 0$).

The only case where the above reasoning does not apply is when $k = m - 1$ and each d_j is proportional to a vector of the canonical basis of R^m. In such a case, our λ must be itself proportional to a vector of the canonical basis of R^m.
■

We can now prove a useful intermediate result

Theorem 10 *Given a finite information structure $\varphi \in \mathcal{F}(V^N)$, there exists a BA mechanisms which obtains full surplus extraction if and only if both the following conditions are satisfied:*

(a) P has full rank

(b) $v \neq v' \Longrightarrow \theta \neq \theta'$

Proof. Let $card(\Theta) = m$. The matrix P^1 of a bidder's first-order beliefs is an $m \times k$ matrix. Hence, its k columns are vectors in R^m. Let $m > k$. Then, it cannot be that all the vectors are vectors of the canonical basis in R^m (this would imply that at least one row in the $m \times m$ matrix P is 0, contradicting $\text{supp}(P) = \Theta \times \Theta$). Hence, by Lemmata 9 and 8, $\exists \lambda \in R^m$, $\lambda \neq 0$, $\lambda = \left(\frac{\alpha_j}{\sum_l d_{jl}} \right)$, such that $\lambda \left(\sum_{j \in J_l} c_j \right) = 0$, $l = 1, ..., k$. Hence, at least one row in P^1 is a convex combination of the others, and by Lemma 7 full surplus extraction does not obtain.

It follows that a necessary condition for full surplus extraction is $m = k$, that is different valuations are associated to different types[3]. In such a case, P^1 is a $k \times k$ matrix, and by Lemma 7 a necessary and sufficient condition for full surplus extraction is that P^1 has full rank. In turn, this is the case if and only if the common prior P is itself a $k \times k$ matrix with full rank. ■

We can now complete the proof of the theorem for the finite case.

Proof. Recall that the set $\mathcal{F}(V^N)$ of finite information structures is the set of all pairs (Ω, P), where $\Omega = T^N$ is the universal space of beliefs and P has finite support in Ω. It follows from theorem 10 that the set of all auctions for which there exists a BA mechanism which obtains full surplus extraction can be identified to a subset of the probability distribution whose support is a set of $k \times k$ points. Hence, it is a subset of a linear space of dimension $(k \times k - 1)$, and is not dense in $\mathcal{F}(V^N)$, which has infinite dimension. From this, by observing that the set of probability distribution with finite support in Ω is dense in the set of all probability distributions on Ω ([4]), we have that the full surplus extraction property of BA mechanisms in not generic in $\mathcal{I}(V^N)$. ■

Remark 11 *By observing that a subspace of finite dimension h is closed with empty interior in a space of dimension $z > h$, the proof shows a stronger result: Generically, a BA mechanism does not obtain full surplus extraction.*

4.2 Compact set of valuations

The extension of the argument to the case where V is an uncountable though compact subset of the real line does not respond to a mere technical need. In fact, the explanatory power of the model with a finite set of valuations is severely limited by the presumption that the seller as well as all the bidders have an exact

[3] Recall that one of the requirements in the construction of the type space is that a type knows his own valuation.

knowledge of the possible valuations. As soon as we admit that such a knowledge might not be exact, as it seems to be the case in most circumstances, we are naturally led to consider models with an uncountable set of possible valuations. This circumstance is clearly reflected by the widespread use of such a model in the economic literature. The remaining of this section proves Theorem 6 for such a case.

Proof. Suppose that generically in $\mathcal{I}(V^N)$ there exists a BA mechanism which obtains full surplus extraction (FSE). It follows at once from this assumption that there exists one information structure (Ω, P^*) such that P^* has full support and FSE obtains.

The set $\{(\Omega, P) \mid P \text{ has finite support}\}$ is dense in $\mathcal{I}(V^N)$. Hence, there is a sequence $\{(\Omega, P_j)\}_0^\infty$ such that

(i) $(\Omega, P_j) \longrightarrow (\Omega, P^*)$

(ii) for any integer j, P_j has finite support.

The assumption of genericity of FSE implies that we can construct the sequence $\{(\Omega, P_j)\}_0^\infty$ so that FSE obtains for (Ω, P_j), for any integer j. Moreover, without loss, we can assume that, along $\{(\Omega, P_j)\}_0^\infty$, $\text{Supp}(P_j) \subseteq \text{Supp}(P_{j+1})$ for each j [we can obtain this by simply constructing a new sequence with the desired property from $\{(\Omega, P_j)\}_0^\infty$].

With the sequence $\{(\Omega, P_j)\}_0^\infty$, there is associated the sequence $\left\{(P_{i,j}^1)\right\}_0^\infty$ of first-order beliefs of player i, $i = 1, ..., N$.

By construction, FSE obtains on $\{(\Omega, P_j)\}_0^\infty$, $\forall j$. From the proof for the finite case, we know that for this to be true it must be true that for each j there is a bijection

$$b_j : V_j \longrightarrow T_j$$

where V_j is a finite subset of V and T_j is a finite subset of T (the set of types for player i) [From the proof for the finite case, we know that for finite subsets of valuations, FSE obtains iff the mapping from valuations to first-order beliefs is a bijection. Then, with each first-order belief associate that type with that first order belief]. Recall that the consistency condition that a type knows his own valuation implies valuation v is associated to a type who has valuation v

$$b_j(v) = t_v \tag{9}$$

Now, each b_j satisfies (9), and since $T_j \subseteq T_{j+1}$ and b_j is a bijection

$$b_j \leq b_{j+1}$$

in the ordering of partial functions.

Let $\mathcal{B}$ be the set of partial functions defined on closed subsets of V which satisfy property (9). By Zorn's Lemma, $\mathcal{B}$ has a maximal element.

As $(\Omega, P_j) \longrightarrow (\Omega, P^*)$, $T_j \longrightarrow T$ and, along such a sequence, b_j can be extended to a maximal element in $\mathcal{B}$, which has necessarily V as domain.

Summarizing, along the sequence $(\Omega, P_j) \longrightarrow (\Omega, P^*)$ we can construct a bijection $b : V \longrightarrow T$ which satisfies property (9).

Because of property (9) and the construction of Ω, such a b is defined by

$$\begin{array}{ccc} \Omega = V^N \times T^N & \xrightarrow{t} & T \\ i \uparrow & \overset{b}{\nearrow} & \\ V & & \end{array}$$

where i is the canonical injection. Hence, b can be taken to be continuous.

Now, let b be as defined above. Then, the following diagram commutes

$$\begin{array}{ccc} & & T \\ & & \downarrow \\ & & \Pi(V^N) \\ & & \wr \\ & \overset{b}{\nearrow} & v \times \Pi(V^{N-1}) \\ & & \downarrow \pi_1 \\ V & \xleftarrow{id} & V \end{array}$$

where $\sim$ denotes a homeomorphism. It follows that the mapping $T \longrightarrow V$ constructed on the right hand part of the diagram is the inverse of b, and it is continuous.

Therefore, along the sequence $(\Omega, P_j) \longrightarrow (\Omega, P^*)$, the mapping $b : V \longrightarrow T$ can be taken to be a homeomorphism.

But, Ω is V^N-based. Hence, T can be taken to be $\Pi(V^N \times T^{N-1})$, and our construction implies that there is a homeomorphism $V \sim \Pi(V^N \times T^{N-1})$. Such a homeomorphism requires both V and T to be one-point spaces, contrary to our hypothesis. ■

5 Related work

In this paper, we have established that, if the bidders' payments to the seller depend on the bids alone, full extraction of the surplus is, generally speaking, no longer possible. As we saw in Section 4, the reason is that there is no bijection between the set of states of the world and the set of bidders' incentive-compatible choices associated to full surplus extraction.

Recently, the work on full surplus extraction has been criticized by Neeman [7]. According to Neeman [7], the result is driven by the assumption, implicitly embedded into the model, of a one-to-one relation between a player's beliefs (over the other players' types) and his willingness to pay for the object.

While this is true for a large part of the applied work on auctions, in my view this critique does not apply to the work of Crémer and McLean. To see the point, the reader should observe that in most of the applied work a bidder's type is identified to his valuation for the object. As a consequence, the requirement that the matrix of a bidder's beliefs (over the other types) have full-rank immediately produces a one-to-one relation between beliefs and valuations. On the contrary, no such identification (between types and valuations) is made in Crémer and

McLean. Moreover, from the model described in Section 2, it follows at once that no one-to-one relation between types and beliefs can be postulated. This is immediately seen in the case of a finite set of valuations. Since no restriction is imposed on the cardinality of the set of types, this latter can be much larger (and it is so in all the relevant cases) than the set of possible valuations (this is the only case to consider as two different valuations produce two different first-order beliefs – see Section 2 – and, hence, two different types).

As explained in Section 3, the reason why the mechanism still works is that, by making a player's payment depend not only on the other players' bids but also on the lotteries chosen by them, one can still establish the existence of a bijection between incentive-compatible choices and states of the world. This is equivalent to the bidders declaring their types, i.e. their valuations, their first-order beliefs, second-order beliefs, etc.. Given this, since the bidders' expected payments are linear in the probabilities, the seller can set the payments contingent on the various states of the world so that full surplus extraction obtains.

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