Financial Innovation and Endogenous Uncertainty in Incomplete Asset Markets

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Abstract

We study endogenous uncertainty stemming from the introduction of new financial assets, so as to evaluate the risks as well as the welfare gains of financial innovation. The introduction of financial assets to hedge individual risk can lead to the risk of default, which is a collective risk. The possibility of default represents endogenous uncertainty, since it depends on economic variables. A proper allocation of risk in the face of new states of endogenous uncertainty requires the introduction of a large number of additional new securities, without which the market is incomplete. We prove the existence of a general equilibrium with default, in which the agents recontract trading positions and prices in the states of default (Theorem 1). We establish the existence of an open set of general equilibrium economies, called complex economies, in which the pattern of trade is highly interconnected so that default by one agent leads to default by an overwhelming majority of all other individuals (Theorem 2). We exhibit examples of complex economies in which the expected amount of default increases with the population (Proposition 3, Examples 3, and 4).

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Key words: Collective risk, default, endogenous uncertainty, individual risk, insurance, financial markets, financial innovation, incomplete markets.

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1. Introduction

The welfare gains from financial innovation should be evaluated together with the possible increases in financial instability, measured by the probability of widespread default. This is of interest in economies in which new financial instruments are introduced rapidly, and where the possibility of increasing the correlation of risks is therefore also significant. Most industrial economies fit this characteristic at present. Our purpose is to introduce a framework of analysis to describe rigourously the added uncertainty from financial innovation. For this we must define and explore a different type of uncertainty than is usually defined within general equilibrium models of markets, namely endogenous uncertainty: this refers to uncertainty which depends on economic behaviour along with nature's moves. In addition, it is useful to combine the notions of individual and collective risk; we shall argue that assets introduced to hedge individual risk often increase collective risk.

Risk and economic behaviour are closely linked. The complexity of the web of trades within a market can transmit individual risks and amplify them into collective risks. For example, a creditor in one market where payment fails to occur may be deprived of the means of delivery in another market where s/he is a debtor, thereby causing a further default in some other market, etc. Therefore default by one individual may lead, through a web of obligations, to default by a large number of individuals, namely to a collective risk. The risk of such widespread default can be called endogenous because it depends on economic behaviour\(^2\). Furthermore the transmission of default from one trader to another and from one market to another is intrinsically a general equilibrium phenomenon. We formalize here endogenous uncertainty stemming from financial innovation within a general equilibrium framework. Endogenous uncertainty is represented by new states where there is default on contracts, either for the trading of the new asset or for the trading of commodities. These states

\(^2\) The term endogenous uncertainty was introduced by Kurz (1974) in a very interesting discussion of models where uncertainty depends on economic behavior. Endogenous uncertainty is discussed further in Section 2.
of default emerge only after a new asset is introduced, and may involve a large number of individual agents. These states represent endogenous uncertainty because they depend on the diffusion of default throughout the economy and on the patterns of net trades, which cannot be predicted with certainty. These states represent collective risks because it affects a large number of agents.

In a world of rapid financial innovation it is important to analyse whether new financial assets can lead not only to welfare gains but also to endogenous uncertainty of default. We study a general equilibrium model with incomplete markets where default occurs as new assets are introduced, and prove existence of an market equilibrium with default. We examine the types of assets which lead to default, and the asymptotic behaviour of the economy as it increases in size. We show robust examples of economies where the expected level of default increases with the size of each (finite) economy, although at the limit, by the law of large numbers, there is no default. This suggests a "discontinuity at infinity" , which is formalized in Section 6, and which depends on the relationship between individual and collective risk and on the degree of trading complexity of the economy.

A first step is to show how default can emerge following the introduction of new assets. A natural framework is an economy where agents face individual risks, for example, illness or death. Default emerges naturally when assets are defined in terms of statistics rather than state by state. Such assets are typical of large economies, where it is difficult or costly to identify individual's characteristics and states. Default is therefore a typical problem in large economies with individual risks, since in such economies it is standard to use statistics to describe the characteristics of a group. For example, in Malinvaud's model (1972,1973), the statistic in question is the expected number of people in one state (such as illness), while the random variable itself is the actual number of sick people in each realization.

An Arrow - Debreu market has contracts contingent on each actual realization of the list of sick people. Such contracts, or the corresponding Arrow securities, are necessary for the efficiency of the competitive equilibrium. If, instead, we value insurance contracts at their
expected value (perhaps plus a premium), actions are taken contingent on a statistic, the expected value of the loss produced by sickness. As the population increases, the law of large numbers predicts that the random variable representing the number of sick people converges to a fixed number with certainty. Therefore in the limit, but only in the limit, selling insurance at its expected value matches premia precisely to the insurance payments to be made.

However close we are to the limit, if we are not actually there, the law of large numbers may not operate exactly. This means that insurance contracts designed to deal with an exact proportion of sick people will not be able to cope with the actual payments in those cases where the realized numbers exceed the limiting proportions. Therefore, with small but positive probabilities, contracts will default. In other words, insurance contracts offered at actuarially fair values promise payments that can exceed physical endowments.

Once default occurs, the complexity of the webs of trades within the economy will determine how widely it spreads, and the total amount defaulted. We analyze the effects of introducing certain assets, called Arrow - Lind - Malinvaud (ALM) assets, which are motivated by earlier work of Arrow and Lind (1970), and Malinvaud (1972, 1973). These are assets which are defined in terms of statistics, rather than contingent on each possible social state, and priced as a function of their expected value, such as actuarially fair insurance contracts for individual risks. These assets exist in practice because of the inherent difficulties of dealing with contracts contingent on exhaustive lists of all individual states. We show that the introduction of these assets leads to new states of collective endogenous risk, each described by a level of default of the asset. This default leads, in turn, to a chain reaction of defaults and recontracting of quantities and prices in all markets.

A widely known and interesting set of examples of Arrow - Lind - Malinvaud (ALM) assets is provided in Arrow and Lind (1970). They are provided by shareholding of firms

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3 See Duffie (1990) for a survey of financial innovation in other kinds of financial assets.

4 Another practical example is given by assets such as bonds which are evaluated on the basis of credit rankings, which provide expected default levels over a class of individuals. Any plan contingent on the value of such assets is prone to default, unless we are already in the limiting
which maximize expected profits. Arrow and Lind argue that expected profit is the preferred maximand in economies with individual risk, and point out that most risks in economic decisions are indeed of this type. When expected profits is the basis for decision making, we have statistical rather than contingent decisions. This leads again to default possibilities. The shareholders in a firm which maximizes expected profits must default their obligations in those states where the firm's profits (and thus the shareholders' wealth) fall short of what is expected. Our model can be applied to a more general framework than Malinvaud's model of individual risk. We consider more generally Arrow - Lind - Malinvaud (ALM) assets: these are assets which are defined as a function of their expected value (or of any other statistic) rather than on a state contingent basis.

Section 2 discusses the concept of endogenous uncertainty. The general equilibrium framework of individual and collective risk is formulated in Section 3. Theorem 1 in Section 4 proves the existence of a general equilibrium with default when agents recontract trades and prices in the default states. These default states are new, as they only exist following the introduction of the ALM asset. Each of the new states represents endogenous uncertainty. Therefore we arrive at a formalization of the concept of endogenous uncertainty within a general equilibrium model with incomplete asset markets.

It is well known that default arises for many reasons, and several explanations and formalizations have been proposed recently (e.g. Dubey, Shubik and Geanakoplos (1989), Dubey and Geanakoplos (1989), Zame (1990)). But however it is initiated, Theorem 2 in Section 5 shows that in an open set of economies called complex economies, it will lead to a collective state with a large expected value of default no matter how large the economy, i.e. no matter how close we are to the limiting economy. In other words, a robust set of large but finite complex economies, there exists a set of collective states with positive probability each,where the overwhelming majority of the households in the economy default (Theorem 2). economy and thus a proportion of individuals defaulting is equal to the expected value.
They define new endogenous states of collective uncertainty, and require in turn the introduction of new securities to complete the market and obtain efficient allocations.\textsuperscript{5} The diffusion of default in complex economies has an effect akin to a "multiplier", and has policy implications similar to those usually attributed to economies with multiplier effects.\textsuperscript{6} A policy which succeeds in stopping default by one agent will multiply its benefits, as it will prevent further default by a large number of other agents without additional costs.

Propositions 3, Examples 3 and 4 in Section 6 establish that the expected value of default may exceed any bound as the population increases, no matter how close the economy is to its limit, and that default per capita may increase as well with the population size. Since in the limit, with infinite populations, there is no default, this indicates a "discontinuity at infinity", which we formalize in Section 6. We provide examples of general equilibrium economies illustrating our conditions and results. A conclusion in Section 7 summarizes the results.

2. Endogenous Uncertainty and Default

A sharpening of the concept of risk is offered here to give a formal account of how endogenous uncertainty and default occur. Uncertainty is generally represented by random variables. Each realization of a random variable is called a \textbf{state}.

\textbf{Collective states} are realizations of random variables that affect all individuals of the

\textsuperscript{5} Cass, Chichilnisky and Wu (1991) explores how many assets, and of what type, are needed to reach Pareto efficient allocations under certain assumptions on the relation between individual and collective risk, namely when individual uncertainty is defined conditional on aggregate collective risks.

\textsuperscript{6} Our results share a commonly held view of the problem, exemplified in Geanakoplos (1990) who writes in another context: "Default is quintessentially a general equilibrium phenomenon. In a world in which promises can exceed physical endowments, each default can begin a chain reaction. A creditor in one market where payment does not occur is deprived of the means of delivery in another market where he/she is a debtor, thereby causing a further default in some other marke, etc. The indirect effects of default might be as important as the direct effects, but they are missed in partial equilibrium models. In general equilibrium models with incomplete markets, both the direct and the indirect effects of default can be captured, and the welfare implications of default studied."
economy, e.g. bad weather. An individual state is instead a realization of a random variable that affects one individual in the economy, e.g. the individual's sickness or death. In addition, we introduce here a second classification, between exogenous states, which describe nature's moves, and endogenous states, which derive from the operation of the economy. Kurz (1974) introduced the concept of endogenous uncertainty in an interesting discussion about economic models with uncertainty. Later Svensson (1981), Chichilnisky, Dutta and Heal (1991) and Henrotte (1992) studied endogenous price uncertainty in a sequential equilibrium framework. We introduce instead a general equilibrium approach to endogenous uncertainty induced by financial innovation.

**Exogenous collective states** describe moves by nature that affect all individuals in the economy. These are the only type of states considered by Arrow (1953) and Debreu (1959). Exogenous individual states are also nature's moves, but they affect only one individual, e.g. illness or death. These are the states considered in Arrow and Lind (1970) and Malinvaud (1972,1973).

We now turn to endogenous states. An endogenous state is an event which is described by a combination of values of exogenous and endogenous variables of the model, and which cannot be controlled or predicted with probability one by the agents of the economy. The definition of an endogenous state depends therefore on the model and on its informational structure. For example, with perfect foresight about prices, the set of all equilibrium prices of an economy is not an endogenous state, since it can be predicted. However, each of the equilibrium prices of an economy with more than one equilibrium and with perfect foresight is an endogenous state, if the agents of the economy are unable to predict precisely which equilibrium will be realized (see also Chichilnisky, Dutta and Heal (1991), Dasgupta and Heal (1979, p. 416 - 417), Hahn (1991) and Chichilnisky, Heal, Swinkles and Streufert (1991)).

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7 The definition of individual risk also requires that the proportion of all individuals in each state converges with certainty to a fixed proportion when the number of individuals goes to infinity; this is defined later in this section.
Another instance of endogenous risk is exemplified in the concept of "moral hazard".  

Consider a large economy with individual risks, as in Malinvaud (1972, 1973). Then the law of large numbers tells us that there is asymptotically no uncertainty about the proportion of individuals in each state: as the population grows, the proportion of people in each state converges to a known number, which equals the probability of one individual being in that state. This principle is utilized in pricing statistical assets such as insurance contracts. But convergence to the limit could be slow, and no matter how close the economy is to its limit, the economy could differ significantly from its limiting behaviour.

We observe that the law of large numbers may be defeated in practical terms by the complexity of the economy. A complex economy is an economy where failure to deliver on a contract by one individual leads to the same type of failure by all other individuals. An economy is $k$-complex, or has complexity $k$ when failure to deliver on a contract by any one agent leads to such failures by $k$ agents in the economy. This paper argues that the complexity of an economy amplifies and transmits risks, transforming exogenous individual asks into endogenous collective risks. When states of default are predicted, rational individuals will make decisions contingent on each such state, and recontract prices and quantities to stay within their new budgets in each state (taking into account the defaults). This leads to an incomplete market with as many new states and new budget constraints as states of default, an economy which requires as many new assets (Arrow securities) as states of default to reach efficiency.

Additionally, the large number of the states of default and the difficulty in collecting largely individual data, may defeat the exhaustive treatment of contingent markets called for by Arrow and Debreu to reach efficiency. Thus new ALM assets may be introduced, assets

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8 This is usually studied in a partial equilibrium framework (Arrow, 1963). Here again the risks depend jointly on exogenous data and also on rational actions taken by the agents (Pauly, 1968), except that the informational structure is assumed to be asymmetric. The agent knows his/her true state, the insurer may not. Examples where the introduction of an asset can increase the risk of moral hazard have been studied recently in Arnott and Stiglitz (1990).
which lead, in turn, to the same type of default. Thus in complex economies, the problem of collective default may emerge anew with every new set of ALM securities introduced to deal with the default of the previous set of securities. The complexity of the economy implies that, whenever default occurs, and for whatever reason, it is amplified as the population grows, because the trading patterns are highly interlinked.

3. Individual and Collective Risk

This section summarizes and extends Malinvaud's (1972, 1973) model of insurance for individual risk, and discusses its connection with the Arrow - Debreu model. Both models are used subsequently as benchmarks in the definition of our model, which contains both exogenous and endogenous uncertainty, and each of them has individual and collective aspects.

Arrow (1953) and Debreu (1959) consider economies with collective risks: these are risks shared by all agents in the economy, such as the weather. Efficient allocations are achieved when there exist competitive markets for all commodities contingent on all states of nature. Arrow and Lind (1970) introduce the concept of individual risk, which they consider the most common risk encountered in economic decision. These are risks which are peculiar to one individual only, such as sickness or death. Malinvaud (1972, 1973) formalizes individual risk in a general equilibrium model. In this context, he proves that in large economies the only assets needed for the asymptotic efficiency of markets are insurance contracts for individual risk. The saving in the number of transactions and in the information required is considerable. For example, with N goods and H individuals in two states (health or

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9 As an example of its wide application, the study of individual risk also underlies the theory of banking. In Diamond and Dybvig (1983) the banks, by offering demand deposit contracts, provide insurance against liquidity shocks to individuals and help achieve efficient risk allocations.

10 Malinvaud's results are asymptotic: they hold for the limit of a sequence of economies with increasingly large populations. In each finite economy the expected excess demand is zero, but excess demand need not be zero in all states. In Section 3.c we extend this model to a limiting economy with infinitely many individuals of several types, in which excess demand is always exactly zero.
sick) efficiency requires $N^2 H$ contingent markets, which increases exponentially with $H$.\textsuperscript{11} Instead, only $N$ spot markets and one insurance contract per individual type suffices for asymptotic efficiency.\textsuperscript{12}

Another practical advantage of insurance for individual risk is highlighted by the informational difficulty involved in trading Arrow–Debreu contingent contracts in a large economy. These contracts must be contingent on a large number of individual characteristics and states. For example, individuals must be able to purchase insurance contracts contingent on whether a long list of all others individuals in the economy are sick or well. Such would be the impractical nature of the Arrow–Debreu contracts or the Arrow securities needed for efficiency in an economy with individual risk. Insurance contracts, instead, depend solely on the individual's state. The efficiency of markets with individual insurance relies crucially on the limiting behavior of the distribution of states across the economy. As populations increase, the limiting proportion of people in each state (e.g. sick or healthy) is a fixed number about which there is no uncertainty. This is part of the definition of individual risk in Malinvaud (1973) who considers identical but not necessarily independent individual risk.

An extreme case of individual risk is provided by identically distributed and independent (iid) random variables.\textsuperscript{13} With iid's there is no connection between the individuals' risks; furthermore, the probability of a collective state (e.g. how many people are ill) is derived, indeed defined from, the probability of each individual's risks. Instead, Arrow–Debreu's approach to uncertainty described by "states of nature" is an extreme case of

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\textsuperscript{11} Or, following Arrow (1953, 1964) $N$ spot markets and $2^H$ Arrow securities, a number which still increases exponentially with the number of individuals, $H$.

\textsuperscript{12} See Section 3.b below. Malinvaud's (1972, 73) results hold for economies where all agents are identical. We extend his model to include several types of agents, in which case one insurance contract is needed for each type. Agents types are identified by their preferences, probabilities and endowments, which vary across individual states. All markets for collective risk are assumed to be described by the existing $N$ commodity markets; in this sense Malinvaud (1973) considers (exogenous) collective risks as well, but assumes that all markets for the optimal allocation of collective risks exist.

\textsuperscript{13} With i.i.d's, the limiting proportions of such individuals in one state equals the probability of one individual being in that state.
collective risk. In Arrow - Debreu models all risk is collective, and the probability distribution of risks for one individual is derived from, indeed is identical to the distribution for collective risks, because all individuals are exposed to the same risks simultaneously. However, between these two extreme cases there are many shades of risks which combine in different ways features of individual and collective risks. They are represented by random variables which have individual and collective components simultaneously. For example, a communicable disease is an individual risk which has also collective risks features. Another case where individual and collective risks are mixed appears in Cass, Chichilnisky and Wu (1991), which explores markets where individual risks, namely who will be affected, can be defined to be conditional on aggregate collective uncertainty, namely how many people will be affected. Chichilnisky and Heal (1992) explore another case where individual risks are unknown; even when there are no collective risks, ignorance about individual risks is shown to lead to collective uncertainty. The latter two papers study the markets which suffice to attain Pareto efficient allocations under specific conditions on the connection between collective and individual uncertainty. The definition of individual risk (see Malinvaud (1972, 1973), and below) implies that as the population increases the collective risk, defined by the proportion of people affected, disappears in the limit: with infinite populations, there is no such collective risk. In the limit there is no uncertainty about the distribution of risks across the population: this is known with certainty. However, with finite populations there is always potential mix of individual and collective risks.

In this paper we explore an important case where individual and collective risks are simultaneously determined, and the connection between individual and collective risks is mediated through economic behavior. We assume that the economy is large but finite. Therefore exhaustive lists of individual states are difficult to be included in contracts, so that assets promise payments contingent on statistics rather than state - by - state. This is shown to lead to new states of default. The complexity of trading in the economy transmits and enlarges this risk into states of collective default. Thus the introduction of assets to deal with individual
risks lead to more collective risks. Although the individual risks are exogenous, depending on states of nature, the new collective risks are endogenous, depending on economic behaviour such as how many agents default on their contracts. We may summarize this by the observation that assets introduced to deal with exogenous individual risks imply new endogenous collective risks. A representation is as shown in Figure 1 below:

A benchmark model for economies with individual risk is Malinvaud's (1972, 1973), where all agents are identical. Here we extend his model to include several types of agents, identified by their endowments, utilities and probabilities. First we establish the notation. Consider an exchange economy with $N$ consumption goods, with the $N$th good as the numeraire ($p_N = 1$). There are $H$ households, divided into types indexed by $i = 1,...,I$, and $H_i$ households of type $i$, so that $H = \sum_i H_i$. Each household faces the same set of $S$ individual states, indexed by $s = 1,...,S$. Let $\Omega$ denote the set of collective states, \( \Omega = \{ \sigma : \sigma \text{ is a function from } \{1,...,H\} \text{ into } \{1,...,S\} \} \). $\Omega$ consists of all possible lists of the individual states for the $H$ individuals, and it has $S^H$ elements. Let $s(h,\sigma)$ be the individual state given by the $h$-th component of the collective state $\sigma$, and $r_{is}(\sigma)$ be the proportion of all households of type $i$ for whom $s(h,\sigma) = s$. Then $\sum_{s=1}^S r_{is}(\sigma) = 1$. Let $r_1(\sigma) = (r_{11}(\sigma),...,r_{1S}(\sigma))$ be the vector of these proportions of households of type $i$ among $S$ individual states for any given collective state $\sigma$. Then $r_1(\sigma) \in \Delta$, the $S$-dimensional simplex. Similarly let $r(\sigma)$ be the proportion of households of all types for a given collective state, $r(\sigma) = (r_1(\sigma),...,r_{T}(\sigma)) \in \Delta^T$. Let $R_H$ be the set of vectors $r(\sigma)$ when $\sigma$ runs over $\Omega$; then $r(\sigma) \in R_H$ is called an aggregate collective state because it is defined only by the total number of individuals in each state for each type, and does not contain any information about the identities of the individuals themselves. $R_H$ is contained in $\Delta^T$ and has $A = \prod_1^{H_i+S-1}$ elements. Let $\Pi(\sigma)$ be the probability of the collective state $\sigma$. $\Pi(r)$ is the probability of the aggregate collective state $r$. The following anonymity
assumption is required for identical risk faced by all individuals:
\[ r(\sigma) = r(\sigma') \implies \Pi_\sigma = \Pi_{\sigma'} \]

Malinvaud (1973, p. 387) establishes that the probability that an aggregate collective state \( r \) obtains and that simultaneously, for a given household \( h \) of type \( i \), a particular state \( s \) also obtains is \( \Pi(r) r_{is} \). The probability \( \rho_{is} \) that, for a given \( h \) of type \( i \), a particular individual state \( s \) obtains is therefore given by

\[ \rho_{is} = \sum_{r \in R_H} \Pi(r) r_{is} \]

Note that the probability \( \Pi(r) \) on \( R_H \) is quite arbitrary. There is no assumption that individual risks are independent. However, we assume that as \( H \to \infty \) the probability distribution on \( R_H \) converges to a definitive limit supported at one point of \( \Delta^I \). If this is not satisfied, we may say that the risks involved have a collective component. Formally, denote now the probability distribution of \( R_H \) by \( \Pi_H(r) \) to indicate its dependence on the size of the population, \( H \). Then the concept of individual risk means that in the limit the proportion \( r_{is}^\infty \) of type \( i \) individuals in state \( s \), is fixed with probability one.

**Definition:** An economy is said to have individual risk if as \( H \to \infty \), \( \Pi_H(r) \to \Pi^\infty(r) \) where \( \Pi^\infty(r) \) is a point distribution on \( \Delta^I \) namely \( \Pi^\infty \) is a degenerate distribution concentrated on one point \( r^\infty \in \Delta^I \), \( \Pi^\infty(r^\infty) = 1 \).

The definition of individual risk provided here is from Malinvaud (1972, 1973); it does not require that the individual probabilities be identically and independently distributed (iid) random variables, although iid's certainly satisfy our definition of individual risk. This definition (the same as Malinvaud's (1973)) allows a wide class of populations of random variables in which correlations may exist between individual's random variables, provided that as the population increases in size all collective risk dissapears, or, in other words, in the limit the probability distribution over collective states is supported on a single point.

The von Neumann Morgenstern utility function in terms of collective states \( \sigma \) is

\[ W^i(z_h^i) = \sum_{\sigma} \Pi_{\sigma} U^i_{s(h,\sigma)}(z_h^i) \]
where \( z^i_{hs} \in \mathbb{R}^{N+} \). All households \( h \) of type \( i \) have the same endowment \( e^i_{sr} = e^i_s \) in any aggregate collective state \( r \) and individual state \( s \), the same probabilities \( \rho_{is} \) for each state \( s \), and their von - Neuman Morgenstem utility can also be written in terms of aggregate collective states \( r \) or individual states \( s \):

\[
W^i(z^i) = \sum_{r \in R^H} \Pi(r) \sum_{s=1}^S r^i_{is} U^i_s(z^i_{sr})
\]

or

\[
W^i(z^i) = \sum_{s=1}^S \rho_{is} U^i_s(z^i_s)
\]

where \( z^i_s \in \mathbb{R}^{N+} \) is the consumption of a household of type \( i \) in individual state \( s \), as explained in Appendix A.

3.a. The Arrow - Debreu model with contingent contracts for individual risk

Consider the set \( \Omega \) of all collective states \( \sigma \) consisting of a realization of one of the \( S \) states for each of the \( H \) households in the economy; \( \Omega \) has \( S^H \) elements. The endowments \( e_h \) of a household \( h \) is an \( NS^H \) dimensional vector. For each household \( h \) of type \( i \) the endowment is the same across all collective states in \( \Omega \) in which \( h \) is at the same individual state \( s \). A price vector \( p \) is an \( NS^H \) dimensional vector. An Arrow - Debreu equilibrium is a price vector \( p^* \) and \( H \) consumption plans \( z^*_h \) with \( NS^H \) components each, such that if individual \( h \) is of type \( i \), \( z^*_h \) maximizes \( W^i(z^i_h) \) subject to

\[
p^*(z^*_h - e^i_h) = 0
\]

and markets clear:

\[
\sum_{h=1}^H (z^*_h - e^i_h) = 0.
\]

and Arrow - Lind - Malinvaud assets.

Malinvaud notes that the number of contingent markets in the Arrow - Debreu model is impossibly high, indeed equal to $N^S$ (an exponential function of the number of individuals). He furthermore notes that as the population increases, then in the limit all contracts contingent on collective states become irrelevant. This is because with probability one, all collective states become equal, with probability one, to the single aggregate collective state $r^\infty$ having a fixed proportion $r^\infty_{is}$ of people of each type $i$ in each individual state $s$. Since the total initial endowments in the economy and total number of people with a given preference are fixed, this leads to a fixed set of prices for the $N$ commodities, $p \in \mathbb{R}^N$ (Malinvaud (1973), Proposition 5). For this reason, he suggests that, as the number of individuals goes to infinity: "The economy should be able to work properly with just $N$ markets, one for each good" (Malinvaud (1973), p.401). This requires, however, that individuals should be able to hedge appropriately their risk between bad and good individual states. For this purpose, Malinvaud introduces individual insurance: contingent commodities are substituted by an insurance system operating as a redistribution scheme. Suppose that the individual of type $i$ holds insurance contracts that will give him or her the net transfer $v^i_s$ of the numeraire good if he or she is in state $s$. Let now $z^i_{hs} \in \mathbb{R}^N$ be the consumption vector by individual $h$ of type $i$ of the $N$ goods in state $s$. Then if $p \in \mathbb{R}^N$ is the vector of (sure) commodity prices, the individual of type $i$ has a budget constraint:

\[
(6) \quad p(z^i_{s} - e^i_{s}) = v^i_{s}, \quad (s = 1, \ldots, S)
\]

Risk coverage means that $v^i_s$ will be positive in unfavorable states and correspondingly negative in favorable states. The individual chooses net transfers $v^i_s$, depending on the terms on which such insurance contracts are offered. Malinvaud assumes that a transfer vector $v^i_s$ is accessible to individual $h$ of type $i$ if and only if it is actuarially fair, i.e. its expected value is zero\(^{14}\):

\(^{14}\) This differs from the concept of "mutual insurance", see Cass, Chichilnisky and Wu (1991) in which the payoff is dependent on the individual state and also on the collective state. For
This is admittedly a strong assumption, but we note that nothing in what follows changes if instead the expected value is equal to a constant $c_i > 0$, where $c_i$ could be the return on investment across the economy in equilibrium, or a regulated level of profits. This assumption could be formalized as an equilibrium condition on the supply of insurance.\footnote{This could be formalized as a market clearing condition for Arrow - Lind - Malinvaud firms which maximize expected profits: assets with positive value would result in an infinite demand, while assets with negative expected value result in infinite supply.}

Arrow and Lind (1970) proposed that the expected profit should be the preferred maximand for public firms in economies with individual risks, and indeed share holding in such firms would also be assets valued as a function of their expected value. With these applications in mind, we consider more generally any asset which is offered at a price which is a function of its expected value, and we call this an \textbf{Arrow - Lind - Malinvaud (ALM asset)}. The introduction of such an asset has the effect of modifying the right hand side of equation (7) leading to

\begin{equation}
\sum_{s=1}^{S} \rho_{is} v^i_s = c_i.
\end{equation}

\textbf{Malinvaud's equilibrium $E_M$ with insurance for individual risk and several types of agents}\footnote{This is Malinvaud's (1973) market organization type C, extended here to include several type of individuals.} is defined as a vector of prices $p^* \in \mathbb{R}^N$, and for each household of type $i = 1, \ldots, I$, a plan consisting of $S$ vectors $z^i_s \in \mathbb{R}^N$ which maximize $W^i$, as defined in (3), subject to

\begin{equation}
\sum_{s=1}^{S} \rho_{is} p^*(z^i_s - c^i_s) = 0
\end{equation}

and expected total excess demand $\xi(p)$ is zero\footnote{Note that excess demand may not be zero at some aggregate collective state $r$.}:

\begin{equation}
\sum_{s=1}^{S} \rho_{is} v^i_s = c_i.
\end{equation}
When the right hand side of equation (9) is substituted by \( c_i \) as in (8), then we call this an equilibrium with individual risk and Arrow - Lind - Malinvaud (ALM) assets. Related concepts of equilibrium with zero expected excess demand have been studied by Hildenbrand (1971) and Wu (1988).

### 3.c. An infinite number of households of each type

Assume now that there are infinitely many individuals of each type \( i \). Then by the assumption of individual risk, \( \rho_{is} = r^\infty_{is} \), where \( r^\infty \in \Delta^I \) is the support of the limiting probability \( \Pi^\infty \) namely the (unique) limiting aggregate collective state giving a fixed proportion \( r^\infty_{is} \) is of individuals of type \( i \) which are in individual state \( s \). *Per capita expected excess demand* (\( 1/H \))\( \xi(p) \) is now used instead of excess demand because the latter may not be a finite number in the limit, as \( H \to \infty \). The equilibrium condition is now therefore written as

\[
(11) \quad (1/H)\xi(p) = 0 \text{ almost surely.}
\]

The assumption of individual risk and (11) imply that the limit of per capita excess demand as the number of individuals \( H \) increases, is zero almost surely. In the limit there is no default, since the proportion of people in an individual state \( s \) within an aggregated collective state \( r \) is exactly \( r^\infty_{is} = \rho_{is} \). This limiting economy is denoted \( E^\infty_M \). The Pareto optimality of the allocations in \( E^\infty_M \) is established in Malinvaud (1973).

### 4. Default with Incomplete Markets

This section formalizes a finite general equilibrium economy \( E \) with insurance and default using Malinvaud's model defined on Section 3 as a benchmark. As already noted, in a large but finite Malinvaud economy \( E_M \) extended to include several agent types, excess demand may fail to be zero at some aggregate collective state \( r \in R_H \) of positive probability,
because there are states \( r \) where promises exceed physical constraints. Similarly, there are states \( r \) where the insurer may fail to deliver on its contracts. The purpose of this section is to formalize such defaults within the equilibrium of a new (finite) economy \( E \) at which excess demand is zero in each state \( r \in R_E \).

Let \( A \) be the cardinality of the set of aggregate collective states \( R \) (Section 3). If \( r^\infty \) is the limiting aggregate collective state which exists by definition of individual risk, then we shall assume that the population size \( H \) is a multiple of \( 1/r^\infty \) for all \( i, s \) with \( r^\infty_{i s} \neq 0 \), so that there exists an aggregate collective state \( r \) in the finite economy \( E \) with \( r^\infty_{i s} = r^\infty_{i s} \) for all \( i, s \).

For each price vector \( p \in R^{NA} \), and \( r \in R_H \), the aggregate vector of (exact) excess demand per capita is:

\[
\chi_r(p) = \sum_{i, s} \left( H_i / H \right) \sum_{s=1}^S r^i_{i s} \xi_i(p),
\]

where \( r^i_{i s} \) is the proportion of individuals of type \( i \) in the individual state \( s \) within \( r \), and \( \xi_i(p) \) is the excess demand of the \( h \) household of type \( i \) in collective state \( r \) and individual state \( s \), derived from maximizing the utility function (2) over the set of \( y^{i}_{s r} \) satisfying \( p(y^{i}_{s r} - e^{i}_{s}) = v^{i}_{s} \). The difference between the expected per capita excess demand in (11), and the actual excess demand vector per capita in state \( r \) in (12) is

\[
\beta_r(p) = (1/H)\xi(p) - \chi_r(p),
\]

which may have positive or a negative coordinates. We say that there is individual default in state \( r \) at equilibrium prices \( p \) if some coordinate of the aggregate excess demand \( \sum_{i, s} H_i r_{i s} \xi_i(p) \) is strictly positive; \( r \) is then called a state of individual default. The set of all individual default states is denoted \( \Psi \).

Similarly for each \( r \) the difference between the receipts from insurance payments, and the total premia collected is:

\[
D(r) = \sum_{i, s} H_i r_{i s} v^i_s,
\]

where \( v^i_s \) is the transfer received by household of type \( i \) in individual state \( s \). Any \( r \) where \( D(r) \)
> 0 is a state of insurance default, and \(D(r)\) is the insurance default in state \(r\). The set of insurance default states is denoted \(\Gamma\). We also write \(\Gamma_H\) and \(D_H\) to indicate their dependence on the population size \(H\). The probability of the set \(\Gamma\) of states \(r\) on which \(D(r) > 0\), tends to zero as the population increases, i.e. \(\lim_{H \to \infty} (\Pi_H(\Gamma_H)) = 0\). Let \(\Psi\) be the union of the sets \(\Gamma\) and \(\Psi\). This set is called the set of default states.

We formalize below the assumption of limited liability on the part of the insurer: when default occurs, the actual insurance payments are made proportionally to what is owed. There are \(I\) insurance contracts, \(S\) - dimensional vectors of transfers \((v^i_s)\), one such vector for each type of individual, \(i\). We assume that there is no difficulty in identifying individual types. The insurer may have positive profits in some states \((D(r) < 0)\). The insurer is a (private or public) company; its owners share a utility function \(W(z_r) = \sum_{r \in R} \Pi_r U(z_r)\) which is risk neutral (or which is as in (2)), and an initial endowments \(e\), the same in all states.

We formalize below the assumption that at each default state \(r \in \Psi\), individuals recontract with each other, so that new net trades and market clearing prices emerge at each default state.

The informational structure of the model is as follows. Privacy is preserved, in the sense that individuals know their own endowments and preferences but not those of others. Individuals anticipate accurately that there is default in states \(r \in \Psi\), and also the extent of default and decrease in their insurance payments in such collective states. For any state of default \(r \in \Psi\), and for each individual state \(s\), define \(\delta^i_{sr}\) to be the actual change in the insurance payment (in the numeraire good) to the \(i\)th type of household in individual state \(s\) derived from this default policy. Individuals anticipate correctly each \(\delta^i_{sr}\) and that the value of

---

18 Note that if the asset sold is not actually fair insurance as defined in (7) above, but rather a mutual insurance policy, there will be no insurance default, since in that case the insurance premia always match the payments, see for example the definition of mutual insurance contracts in Cass, Chichilnisky and Wu (1991).

19 Note that the sum of default in the economy may or not go to zero with \(H\) depending on how many other defaults are precipitated by the default in insurance payments and on the rate of convergence of \(\Pi_H\) as \(H \to \infty\), see Theorem 2 and Propositions 3, 4, and 5 below.
their consumption in such states does not exceed the value of their endowment plus the (reduced) insurance payments received. Then an individual’s income is now contingent on the aggregate collective state of the economy, r. Therefore A more states are now introduced (A is the cardinality of the set R) in the (extended) Malinvaud economy. With perfect foresight, therefore, we now have NA markets. A price \( p \) is now a vector in \( R^{NA} \). A consumption plan \( y^i \) for household \( h \) of type \( i \) consists of SA consumption vectors in \( R^N \) denoted \( y^i_{sr} \), one for each aggregate collective state \( r \in R \), and individual state \( s \). Recall that the insurance contracts provide transfers \( v^i_s \) with \( \sum_s \rho_{is} v^i_s = 0 \) in all no - default states. The data of the model, known to all the households, include now the probabilities \( \Pi_r \) of all aggregate collective states \( r \in R \), in particular those of the default states \( r \in V \), and also the shortfalls \( \delta^i_{sr} < 0 \), \( s=1,...,S \), on the payment to the \( h \) household of type \( i \) at the aggregate collective state \( r \). Note that with perfect foresight, the ALM asset with payoff vector \( (v^i_{1,...,i} \delta^i_{sr}) \), \( i = 1,...,I \), ceases to be a statistical asset and becomes a state contingent contract for the insured. This is because the insured is now aware of the contingent payment \( v^i_s + \delta^i_{sr} \) in all the collective states \( r \) in which there is default, \( r \in \Gamma \).

A general equilibrium with default is defined as follows. With \( H \) households of \( I \) types, the equilibrium consists of a vector \( p^* \) of prices in \( R^{NA} \), for each household \( h \) of type \( i \), SA consumption vectors \( y^i_{sr}^* \) in \( R^N \) representing consumption in the aggregate collective state \( r \) and individual state \( s \), and A vectors \( y^r \) in \( R^N \) representing consumption by the insurer in aggregate collective state \( r \), such that the vector \( y^i_{sr}^* = (y^i_{sr})^* \) maximizes the utility function\(^{20} \)

\[
W^i(y^i) = \sum_{r \in R_H} \Pi_r \sum_{s=1}^{S} r_{is} U^i_s (y^i_{sr})
\]

on the set of \( y^i = (y^i_{sr}) \) satisfying

\[
p^*(y^i_{sr} - e^i_s) = v^i_s, \quad \text{and} \quad \sum_s \rho_{is} v^i_s = 0, \quad s=1,...,S, \text{ if } r \in \Gamma
\]

(budget balance in states with no insurance default).

\(^{20}\) This is the utility function defined in (2) above, and discussed in Appendix A.
and

\[(6) \quad p_r^* (y_{sr}^i - e_s^i) = \delta_{sr} + \nu_i^s, \quad s = 1, \ldots, S \text{ if } r \in \Gamma\]

(budget balance for insurance default states)

The vector \( y^* = (y_r^*) \) maximizes the utility function

\[(17) \quad W(y) = \sum_{r \in \Gamma} \Pi_r U(y_r)\]

on the set of \( y = (y_r) \) satisfying

\[(18) \quad p_r^* (y_r - e) = \sum_{i,s} H_{rs} v_{is} \quad \text{if } r \in \Gamma\]

(budget of the insurer in non-default states)

or

\[(19) \quad p_r^* (y_r - e) = c, \quad c \leq 0^{21}, \quad \text{if } r \in \Gamma\]

(budget of the insurer in default states)

and, for each aggregate collective state \( r \in \Gamma \),

\[(20) \quad \sum_{i} H_{i} \sum_{s=1}^{S} (y_{sr}^i - e_{is}^i) + (y_r^* - e) = 0\]

(excess demand equals zero).

**The Asset Structure of the economy E**

In order to make explicit the asset structure of the model, we define the aggregate collective states as follows:

\( r \in \Gamma \) (insurance default states) = \{1, \ldots, V\}

and assume for simplicity in the exposition that there is one \( r \) without default,

\( r = 0 \) (the no-insurance-default state)

So we can write \( r = 0, 1, \ldots, V \), or \( r \in \{0, 1, \ldots, V\} \). The corresponding price vector is \( p_r, r = 0, \ldots, V \), \( p_r \in \mathbb{R}^{N(V+1)} \). The two budget constraints (15) and (16) can be written as one

\[ p_r^* (y_{sr}^i - e_{is}^i) = \delta_{sr} + \nu_i^s, \quad s = 1, \ldots, S \text{ if } r \in \Gamma\]

\[ p_r^* (y_r - e) = \sum_{i,s} H_{rs} v_{is} \quad \text{if } r \in \Gamma\]

\[ p_r^* (y_r - e) = c, \quad c \leq 0^{21}, \quad \text{if } r \in \Gamma\]

\[ \sum_{i} H_{i} \sum_{s=1}^{S} (y_{sr}^i - e_{is}^i) + (y_r^* - e) = 0\]

---

21 The parameter "c" could be considered a penalty levied on the insurer in default states. The structure of the equilibrium is the same for \( c = 0 \) or \( c \leq 0 \).
equation:

(21) \[ p_r(z_s^r - e_s^i) = \delta_s^i + v_s^i, \quad r = 0,1,\ldots,V, \quad s = 1,\ldots,S \]

where \( \delta_s^0 = 0 \) (i.e. when \( r = 0 \)),

\( z_s^r, e_s^i \in \mathbb{R}^N, \quad p_r \in \mathbb{R}^N, \quad \delta_s^i \) and \( v_s^i \) are scalars. The "\( a \)" operation is defined in a standard fashion:
for a given $s$, where the right hand side of (22) is an $(V+1) \times 1$ matrix. Then equation (21) can be written:

$$
\begin{bmatrix}
v_i^1 + 0 \\
\end{bmatrix}
$$

(23) $$p \circ (z_s^i - e_s^i) = Z_s^i = \begin{bmatrix} v_s^i + \delta_{sr}^i \\
v_s^i + \delta_{sv}^i \\
\end{bmatrix}$$

for $s = 1, \ldots, S$. We can also write (21) more compactly as follows. Let $Z^i = [Z_1^i, \ldots, Z_s^i, \ldots, Z_S^i]$, an $(V+1) \times S$ matrix, then (21) is

$$
[p \circ (z_1^i - e_1^i), \ldots, p \circ (z_s^i - e_s^i), \ldots, p \circ (z_S^i - e_S^i)] = Z^i
$$

(both $(V+1) \times S$ matrices), where $Z^i$ is the payoff matrix:

$$
Z^i = \begin{bmatrix}
v_1^i & \ldots & v_S^i \\
v_1^i + \delta_1^i & \ldots & v_S^i + \delta_S^i \\
v_1^i + \delta_1^i V & \ldots & v_S^i + \delta_S^i V \\
\end{bmatrix}
$$

Note that the asset structure of $E$ has restricted access: although there are $I$ insurance contracts, as many as individual types, a household of type $i$ can only purchase
insurance contract of type i, i.e. a vector of transfers \((v^i_s)\) with \(\sum_{s \in S} \rho_{ls} v^i_s = 0\). the market is *incomplete* because there are only I assets, while there are as many states as the cardinality of the set R, namely A (Section 3). At least A - I assets (or Arrow securities) are required to complete the market, allowing the households to transfer income in all states in R and thus reach Pareto efficient allocations.

**Theorem 1.** There exists a general equilibrium with default for the economy E of incomplete markets.

The proof of Theorem 1 is in Appendix B.

The economy E is also denoted \(E_H\) to indicate the number of its individuals. Note that the condition of individual risk implies that as the number of individuals H goes to infinity, the economy \(E_H\) converges to a limiting economy denoted \(E^{\infty}\), which is identical to the limit of the Malinvaud economies \(E_M\). Formally, \(\lim_{H \to \infty} E_H = E^{\infty} = E^{\infty}_M\) as defined in Section 3.

Alternatively, if the number of individuals H remains constant but we substitute the probability distribution \(\Pi\) on R by the singular probability \(\Pi^{\infty} = \lim_{H} \Pi_H\) supported on the single aggregate collective state \(r^{\infty}\), we obtain an economy \(E^*\) with the same finite number of individuals (H) but with a unique aggregate collective state \(r^{\infty}\) almost surely. Then \(\forall i,s \rho_{is} = r^{\infty}_{ls}\). There is no insurance default in \(E^*\) and excess demand is identical to expected excess demand almost surely. In per capita terms \(E^*\) is identical to \(E^{\infty}\). There is no collective risk in \(E^*\) because there is only one aggregate collective state \(r^{\infty}\) almost surely; all individual risk is covered by the insurance contracts \((v^i_s)\) which never default. \(E^*\) behaves as E is *expected to*, but sometimes doesn't. For this reason \(E^*\) is called a **benchmark** for E:

**Lemma 1.** The benchmark economy \(E^*\) is a complete market economy with no default almost surely.
5. Complex Economies

This section formalizes and examines the complexity of an economy, i.e. the total number of individuals who default following any one's default. We aim to determine the extent to which there is a chain reaction of defaults in an economy where the trade patterns are highly interlinked. The economy's complexity determines the extent to which there is a "multiplier" effect for policies designed to prevent financial default.

As already noted, there exist aggregate collective states $r$ in $\mathbb{R}$ where the insurer cannot meet promised payments to individuals, namely when $r \in \Gamma$. In the (finite) perfect foresight economy $E$, individuals take these defaults into account, and adjust their consumption in state $r$ appropriately (Section 4, (15)). In Section 4 we defined an abstract economy which we denoted the benchmark economy $E^*$, and which has no default; this economy is constructed for the purpose of comparing it with $E$, and measuring default and complexity. $E$ and $E^*$ are identical in all respects except on the structure of uncertainty. We could view $E^*$ as an idealized economy which functions always as $E$ is expected to function, but sometimes doesn't. The extent of default in an "unexpected" state of $E$ can then be measured by comparing the allocation in that state with the allocations at the economy $E^*$. We consider default in a state $r \in \Gamma$ where the insurer fails to honor its payment $v^i_s$ to an individual $h$ of type $i$ in individual state $s$ at a collective state $r$, and examine how many other individuals default as a consequence.

Formally, let $(z^i_{sr})$ denote an equilibrium allocation in $E$ of an individual $h$ of type $i$ in individual state $s$ at a collective state $r$. If its $m$-component $(z^i_{sr} - e^i_s)_m$ is negative, we say that $h$ is a net importer of $m$, or simply an importer of $m$, at this equilibrium. The same definition with the opposite sign applies to exporters of a good $m$. Similarly in $E^*$ individual $h$ of type $i$ is a net exporter of a good $m$ at the equilibrium allocation $(z^i^*)$, when $(z^i^*_s - e^i_s)_m > 0$, where $z^i^*_s$ is an equilibrium allocation for $E^*$. Let $r$ be a default state of the economy $E$ as defined in Section 4, and for $s,r$ let $h$ be an importer of good $m$ at an equilibrium allocation of $E$, $(z^i^*_{sr})$. Consider an equilibrium $(p^*, z^i^*_s)$ of $E^*$, and an equilibrium allocation $(z^i^*_{sr})$ of $E$.
in a state of insurance default \( r \in \Gamma \).

**Default:** An importer \( h \) of good \( m \) in economy \( E \) is said to default at \( s, r \) when the value of \( h \)'s net purchases of good \( m \) at the default state \( r \in \Gamma \) is lower than what \( h \) contracted to purchase at the individual state \( s \) in \( E^* \), i.e., \( |p^*_m(z^*_{sr}(m) - e^i_s(m))| < |p^*_m(z^*_{s}(m) - e^i_s(m))| \).

A similar definition holds for exporters. Note that the values are given by the equilibrium price \( p^* \) of the benchmark economy \( E^* \). Default therefore identifies the decrease in the value of a contract resulting from a failure in payments by the insurer. The complexity of the economy measures how many other individuals default in state \( r \) as a consequence of the default by the insurer on payments to individuals of type \( i \).

**Complexity:** The economy \( E \) is said to have complexity \( k \) at a default state \( r \), when there are \( k \) defaults following any one default, at any equilibrium allocation of \( E^* \) and \( E \). \( E \) is called complex when its complexity is \( H \).

Note that the concept of default defined here is not related to bilateral trade between the individuals, nor to sequential trading of any form; default compares two simultaneous situations, and refers to contracts (net trades) for delivery to the "market exchange" or to an "auctioneer", as in the Arrow - Debreu market formulation. In other words, no information is given here about who trades with whom or how much. Such information is explicitly forbidden by the assumption of preserving privacy. In our context, default involves simply an equilibrium net trade vector at a default state which differs from that contracted at a no-default equilibrium.

The concept of complexity and the analysis of complex economies is useful for two reasons. The first is for gauging the collective nature of the states of default in the economy \( E \) defined in Section 4. For example, in an economy with complexity \( k \geq H/2 \), a default state involves defaults by a large number of individuals. The default states in \( E \) are always
collective states, because they depend on the number of people in a bad state across the population. However, the collective nature of these states is emphasized further when the complexity of the economy is high. In such economies the states of default not only depend on a collective state (the number of people in a given individual state) but also affect a large number of individuals as well. This is formalized in Theorem 2.

Theorem 2 shows that there exists an open set of economies \( E \) with complexity \( k \), for any \( k \geq 2 \). This establishes that in an open set of economies the states of default affect at least \( k \) individuals. Therefore when \( k \) is large, the risk of default cannot be hedged properly by insurance.

Consider now an economy \( E \) with \( N \) goods, \( H \) individuals of \( I \) different types. There are \( S \) individual states. Each individual of type \( i \) has an endowment \( e^i_s \) and a utility as in (2) and \( W \) (Section 4, (17)). In addition, an \( S \) vector of transfers \( (v^i_s) \) is available to each individual of type \( i \) provided it has zero expected value, as in (7). The economy can be described therefore as \( E = (N, H, I, e^i_s, W^i, W) \). Insurance contracts do not appear in this parameterization, since they are chosen as an optimal set of transfers \( (v^i_s) \), \( s=1,...,S \), by the individuals of type \( i \), among all possible transfers of zero expected value. The space \( E \) of all such economies can be parameterized by the endowments \( \{e^i_s\} \in \mathbb{R}^{NSI} \) and by the utilities \( W^i, i=1,...,I \), and \( W \), and topologized by the product topology \( \tau \) defined by the Euclidean metric on endowments and the Whitney \( C^2 \) topology on the space of \( C^2 \) utility functions. The Whitney topology on the space of \( C^2 \) functions \( U^h: \mathbb{R}^N \rightarrow \mathbb{R} \) is defined by specifying neighborhoods of zero, since this is a linear space (Smale (1974), Peixoto (1967)). Such a neighborhood \( M_\epsilon \) is defined by each strictly positive continuous function \( f: \mathbb{R}^N \rightarrow \mathbb{R} \) as follows: the function \( U \in V_\epsilon \) iff \( \|U(x)\| < f(x), \|DU(x)\| < f(x), \|D^2(U(x))\| < f(x) \) for all \( x \in \mathbb{R}^N \) where \( DU \) and \( D^2U \) are the first and second derivatives of \( U \), and \( \|\cdot\| \) is the Euclidean norm in finite dimensional spaces. An open set of economies in the space \( E \) means an open set in the topology \( \tau \), as endowments and utilities vary.
Theorem 2. Let \( N > H \), and \( k \geq 2 \). There exists in \( E \) an open set of economies of complexity \( k \).

Proof of Theorem 2 is in Appendix B.

The following corollary refers to small variations of the endowments of the economy, leaving utilities invariant. The set of economies is now parameterized by its endowments and these are endowed with any bounded measure which is positive on open sets:

**Corollary 1:** Let \( N \geq H \), and \( k < H \). Then there exists a set of positive measure of Arrow - Debreu economies which are \( k \) - complex, for any default level \( \delta > 0 \).

**Proof:** This follows from Theorems 1 and 2 and the fact that open sets have positive measure.

**Example 1: A complex economy**

We now construct a general equilibrium economy \( E \) as defined in Section 4. There are \( I = 3 \) types of households, \( H \) of each, three goods, \( N = 3 \), and two states of individual risk for each household, \( S = \{ 1, 2 \} \). Assume that for each \( i \) and all \( s \), individual risk is defined by \( \rho_{is} = 0.5 \), and that individual risks are identical and independently distributed random variables. \( H \) is assumed to be even so that \( H/2 \) is an integer and there exist collective states with \( r_{is} = 0.5 \). The utilities of the agents are:

\[
W^i_s(x^i_s) = \sum_{s=1}^{2} \rho_{is} U^i(x^i_{s})
\]

where \( U^i \) is state independent and Cobb - Douglas (the same for all \( s \)), \( x^i_{sn} \) is consumption of good \( n \) by the \( i \)th type of households in state \( s \) and \( x^i_s = (x^i_{sn}) \). Endowments of households of type 1 and 3 are state independent; they are denoted \( e^1 = (0,1,0) \) and \( e^3 = (1,0,0) \). Type 2 households have different endowments in state 2, the unfavorable state, than in state 1, the favorable state: \( e^2 = (0,0,0) \) and \( e^2_1 = (0,0,2) \). This implies that there are \( 2^H \) collective states
in \( \Omega \) (rather than \( 2^{-3H} \)), because only agents of type 2 face individual risk. There are \( A = H+1 \) aggregate collective states \( r \) in \( R \), each identified by the number of agents of type 2 who are in state 2. When consumption of good \( m \) is state independent it is denoted \( x^m_i \). The utilities of type \( i \) households \( U^i: R^3 \to R \) are:

\[
U^1 = 4 \ln x^1_1 + \ln x^2_2 \\
U^2 = 1/2(4 \ln x^2_{12} + \ln x^2_{13}) + 1/2(4 \ln x^2_{22} + \ln x^2_{23}) \\
U^3 = 4 \ln x^3_3 + \ln x^1_1
\]

The market is incomplete so far because there are no assets to deal with the risk faced by the second type of household.

An ALM asset is now introduced, i.e. a set of transfers across different states with expected value equal to zero. We now compute an equilibrium of this economy, as defined in Section 4. Let \( p = (p_1, p_2, p_2) \) and assume that good 3 is the numeraire, \( p_3 = 1 \). Type 2 wants to purchase insurance offered by the insurer, called agent type 0: type 2 pays \( q \) units of the numeraire (good 3) in both states and receives 1 unit of good 3 in the unfavourable state \( s = 2 \), \( q \) is the insurance premium for each unit of this contract. \( \mu \) is the amount of insurance contract purchased by type 2. Both \( q \) and \( \mu \) are determined endogenously. Type 0 is risk neutral, it only has utility for good 3, and it offers an insurance contract that is actuarially fair, i.e. \( q = \rho_{2s} = 1/2 \), with zero expected value (either due to competition or regulatory constraint). Type 2 agents maximize their utility \( U^2 \) subject to

\[
\begin{align*}
p_2 x^2_{12} + x^2_{13} &= 2 - q \mu, \quad s = 1 \\
p_2 x^2_{22} + x^2_{23} &= (1-q)\mu, \quad s = 2
\end{align*}
\]

obtaining

\[
\begin{align*}
x^2_{12} &= (4/5)(2 - q \mu)/p_2, \quad x^2_{13} = (1/5)(2 - q \mu) \\
x^2_{22} &= (4/5)(1 - q) \mu/p_2, \quad x^2_{23} = (1/5)(1 - q)\mu
\end{align*}
\]

and

\( \mu = 1/q. \)

Substituting \( q = 1/2 \), we obtain the demand of type 2 household:

\[
x^2_{12} = x^2_{22} = x^2_2 = (4/5)(1/p_2)
\]
\[ x_{13}^2 = x_{23}^2 = x_3 = 1/5 \]
\[ x_1^2 = 0. \]

Type 1 household maximizes

\[ U^1 = 4 \ln x_1^1 + \ln x_2^1 \]

subject to

\[ p_1 x_1^1 + p_2 x_2^1 + p_3 x_3^1 = p_2. \]

So the demand of type 1 household is:

\[ x_1^1 = 4p_2/5p_1, \quad x_2^1 = 1/5, \quad x_3^1 = 0. \]

Similarly, one obtains

\[ x_3^1 = 1/5, \quad x_2^3 = 0, \quad x_3^3 = 4p_1/5p_3 \]

Since type 0 (insurer) has no need to trade, there is a unique price equilibrium \( p^* \):

\[ p_1^* = p_2^* = p_3^* = 1. \]

The equilibrium consumption vectors are:

\[ x_1^* = (4/5, 1/5, 0), \quad x_2^* = (0, 4/5, 1/5), \quad x_3^* = (1/5, 0, 4/5) \]

with net trade equilibrium vectors:

\[ z_1^* = x_1^* - e^1 = (4/5, -4/5, 0), \quad z_2^* = (0, 4/5, -4/5), \quad z_3^* = (-4/5, 0, 4/5). \]

Now we define the benchmark economy \( E^* \) as in Section 4: for all \( i \) and \( s \), \( r_{is} = \rho_{is} = 1/2, \quad \Pi^\infty(r^\infty) = 1. \) \( E^* \) has only one aggregate collective state with non-zero probability, \( r^\infty \); in this state 50% of the individuals of type 2 are in state 1, and 50% in state 2. Note that \( E^* \) has the same unique equilibrium as computed above, there is no default in \( E^* \) and the equilibrium excess demand is zero almost surely.

To see that \( E \) is complex we refer to the proof of Theorem 2 in Appendix B. The endowments and preference in this economy correspond to those of the economy in the proof of Theorem 2: for each good there is only one type of household who is a net importer and only one type of household who is a net exporter. This economy with an ALM insurance contract is thus complex.
Example 2 studies default in the economy of Example 1, and analyses its complexity.

**Example 2: Default equilibrium in the complex economy E**

After the insurance contract is introduced in the economy of Example 1, and overpayment or underpayment to the insurer occurs, new state contingent prices and allocations emerge. We compute these now.

Recall that the total number of aggregate states in the perfect foresight equilibrium is $V+1$, one for the no-default state and $V$ default states with $V = H/2$ if $H$ is even and $V = (H+1)/2$ if $H$ is odd. In terms of the notation of Example 1, $\rho_2 = 1/2$ ($s=1,2$), $v^i_s = -1$ if $s = 1$, and $v^i_s = 1$ if $s = 2$ and type $i = 2$. $r \in \Gamma$ (insurance default state) if $r = \varphi H$ and $\varphi > 1/2$. For $r \in \Gamma$, $p^*_r$ satisfies $p^*_1 = p^*_2 = 2(1-\varphi)$, $p^*_3 = 1$. Recall that default is denoted $\delta$: $\delta_s = 0$ if $s = 1$, $\delta_s = -(1-\varphi)/\varphi < 0$ if $s = 2$.

Type 2 agents purchase full insurance as in Example 1. After the realization of individual state ($s = 1,2$) for type 2, there are $2^H$ collective states and $A = H + I$ aggregate collective states, indexed by $r = 0,...,\varphi H,...,H$, i.e. by the number of type 2 agents in the unfavourable state 2.

When $r = \varphi H$, $\varphi > 1/2$, and $s = 2$, the payment from the insurer is only $(1-\varphi)/\varphi < 1$ (due to the limited liability provision). This is a default state, $r \in \Gamma$. If $s = 1$, type 2's endowment is $(0,0,1)$ for $(1-\varphi)H$ of them; for the same case but when $s = 2$, type 2's endowment is $(0,0,(1-\varphi)/\varphi) \varphi H$ of them. Given these endowments, type 2 agent's maximize utility $U^2$ and we derive type 2's demand:

$(1-\varphi)H$ of type 2's demand is $x^2_1 = 0$, $x^2_2 = (4/5)(1/p_2)$, $x^2_3 = 1/5$

and excess demand is $(0,4/5p_2, -4/5)$

$\varphi H$ of type 2's demand is $x^2_1 = 0$, $x^2_2 = (4/5p_2)(1-\varphi)/\varphi$, $x^2_3 = (1/5)(1-\varphi)/\varphi$

and excess demand is $(0,(4/5p_2)(1-\varphi)/\varphi, (-4/5)(1-\varphi)/\varphi)$
Type 1's excess demand is \(((4p_2/5p_1),(-4/5),0), H\) of them

Type 3's excess demand is \(((4/5),0,(4/5)p_1), H\) of them

Market clearing conditions are:

Good 1: \((4p_2/5p_1) H - 4H/5 = 0\) \(\rightarrow p_1 = p_2\)

Good 2: \((1-\varphi)4/5p_2 + (4H/5p2)(1-\varphi) - 4H/5 = 0\) \(\rightarrow p_2 = 2(1-\varphi) < 1\)

Good 3: \((-4/5)(1-\varphi)H - (4/5)(1-\varphi)H + (4/5)p_1 = 0\) \(\rightarrow p_1 = 2(1-\varphi) < 1\)

The renegotiated equilibrium price vector for the aggregate collective state with default \((r = \varphi H, \varphi > 1/2)\) is \(p_1^* = p_2^* = 2(1-\varphi), p_3 = 1\).

Equilibrium consumption vectors are

\[x^1 = (4/5, 1/5, 0)\] (same as in Example 1, not affected by default)

\[x^2 = (0, 4/10\varphi, (1/5)(1-\varphi)\varphi/\varphi)\] for \(\varphi H\) of them (less consumption)

and \(x^3 = (0, 4/10(1-\varphi), 1/5)\) for \((1-\varphi)H\) of them (more consumption)

Note that the market clears: sum of all demand equals \((H, H, 2(1-\varphi)H)\) which is the total supply of commodities and \(x^0 = (0, 0, 0)\) for the insurer. The equilibrium prices for aggregate collective state without default \((r = \varphi H, \varphi \leq 1/2, \epsilon^2 = (0,0,1)\) always

Type 1's demand is \(x^1_1 = 4p_2/5p_1, x^1_2 = 1/5, x^1_3 = 0\)

Type 2's demand is \(x^2_3 = 0, x^2_2 = 4/5p_2, x^2_3 = 1/5\)

Type 3's demand is \(x^3_1 = 1/5, x^3_2 = 0, x^3_3 = 4p_1/5\)

Now type 0 receives \((1-\varphi)H\) of good 3 and pays out \(\varphi H\) only, so it has a surplus of \((1-2\varphi)H > 0\) of good 3. Since by assumption, agent 0's utility is a function of good 3 only, all the surplus \((1 -2\varphi)H\) is consumed directly in good 3, and this agent's excess demand is the zero vector.
Market clearing conditions are:

Good 1: $4p_2H/5p_1 - 4H/5 = 0 \implies p_1 = p_2$

Good 2: $-4H/5 + 4H/5p_2 = 0 \implies p_2 = 1$

Good 3: $-4H/5 + 4p_1H/5 \implies p_1 = 1$

if $r = \phi H$, $\phi \leq 1/2$, then $p^*_1 = p^*_2 = p^*_3 = 1$ (prices stay constant)

Consumption vectors are

$$x^1 = (4/5, 1/5, 0)$$
$$x^2 = (0, 4/5, 1/5)$$
$$x^3 = (1/5, 0, 4/5)$$

and

$$x^0 = (0, 0, (1-2\phi)H)$$

Note that markets clear: the sum of all demand vectors equals the total supply of commodities.

6. Financial Reserves and Asymptotic Risk

Consider now the economy $E$ defined in Section 4, which we now shall denote also $E_H$ in order to highlight the number of its individuals, $H$. In order to study the asymptotic properties of this economy, we shall analyse a sequence of economies $E_H$ as the number of individuals tends to infinity, i.e. $H \to \infty$. In view of the assumption of individual risk, the probabilities over collective states satisfy $\Pi_H \to \Pi^\infty$, where $\Pi^\infty$ is a distribution supported on a single point, a known proportion of each type in each state. Recall that at the limit, i.e for $H = \infty$, $E^\infty$ coincides with $E^\infty_M$, the Malinvaud economy with infinite agents defined in 3.c. In particular, when actuarially fair insurance contracts exists for each type $i=1,\ldots,I$, the economy $E^\infty$ is always a complete economy with no default.

In order to study the asymptotic behavior of economies where assets are valued in
terms of statistics, (i.e. ALM assets) we now need a few definitions.

Recall that the set $V_H$ represents all states of default in the economy $E_H$, and $\Pi_H(V_H)$ its measure, and that convergence of probabilities means weak convergence (Billingsley, 1968). Consider an economic statistic on an endogenous risk, such for example the expected value of total default of the economy $E_H$, or the expected value of per-capital default of $E_H$. Formally, an endogenous economic statistic is defined as a continuous function from the space $(F_H \times E_H)$ of all probability distributions on the space of collective states $R_H$ times the space of economies $E_H$, into the real numbers, $R$, i.e. $s(p, E_H) \in R$ for $p, E_H \in F_H \times E_H$. Notice that an endogenous economic statistic is defined continuously for probability distributions and economies with a population of a fixed size $H$.

Now consider as in Section 3 the product of the $(S-1)$-dimensional unit simplex with itself $I$ times, denoted $\Delta^I$. A point in $\Delta^I$ represents a proportion of each type $i=1,...,I$ in each of the $S$ individual states. Then a probability measure $\Pi_H$ on $R_H$ defines a measure on $\Delta^I$, supported on a finite subset of $\Delta^I$, given say by a density function $\Pi_H(r)$ for $r \in R_H$, as represented in Figure 2 below, see also Malinvaud (1973):

As already mentioned, the assumption of individual risk in Section 3, implies $\lim_{H \to \infty} (\Pi_H) = \Pi^\infty$, a degenerate measure on $\Delta^I$ supported in one point only, denoted $r^\infty$. Note that even if the endogenous economic statistic $s(p, E_H)$ is a continuous function of $p \in F_H$ for a fixed population size $H$, convergence in measure of the probabilities $\Pi_H$ need not imply that $\lim_{H \to \infty} s(\Pi_H, E_H) = s(\Pi^\infty, E^\infty)$, since the value of the endogenous statistic $s$ depends on the behavior of the economy $E_H$ as well as on the probabilities $\Pi_H$.

**Definition:** When $\lim_{H \to \infty} s(\Pi_H) \neq s(\Pi^\infty)$, we say that there exists a "discontinuity at infinity" for the endogenous economic statistic $s$.

Proposition 3, and examples 4, and 5 below provide robust examples of economies with individual risk in which the various statistics related to the expected level of default have a
discontinuity at infinity. In other words, although \( s(\Pi_{r_0}^\infty, E^\infty) = 0 \), \( \lim_{R \to \infty} s(\Pi_{H^r}^E_{H^r}) \neq 0 \). For example the expected default per capita statistic \( s(\Pi_{H^r}^E_{H^r}) \) may grow with \( H \) even though once at the limit \( s(\Pi_{r_0}^\infty, E^\infty) = 0 \).

A natural question is whether default may be avoided by requiring that the agents hold their positive profits from favourable collective states in reserve and use these reserves to satisfy claims in unfavourable collective states. However, unless reserves which equal the maximum exposure are required, the problem of default emerges all the same and leads to the same consequences. The only difference is in the probability of default which is typically decreased. This provides support to policies requiring some forms of reserves in the insurance and banking industry and in financial markets, which can enhance the financial stability of the economy.

Yet asymptotically, even with these decreased probabilities the "discontinuity at infinity" may emerge, as shown in the following propositions. We shall argue that since requiring reserves equal to the maximum possible exposure is not a realistic assumption, the problem of default still emerges when reserves are required.

In order to formalize the problem with reserves within the same framework as above, the commodity space \( R^N \) in the definition of the economy \( E \) is now assumed to include time dependent commodities, and we may assume that the excess (positive) returns on insurance premia over payments are saved as "reserves", and used to cover the (negative) shortfalls in the unfavorable collective states. With time indexed commodities, borrowing and lending on the reserves is possible in successive years, with the maximum permitted length of such a contract being, for example, \( Y \) years. If the distribution of collective states \( \sigma \) in \( \Gamma \) is perfectly symmetric on aggregate collective states, then this will yield good coverage against default in the limit, i.e. as \( Y = \infty \). However, for any finite period \( Y \) there will still exist collective states with small probability in which the insurance contracts must default. All such collective states constitute the default states with reserves. Therefore our analysis can still be applied if we use
the set of default states with reserves instead of the set $\mathcal{V}$ of default states, except that the probabilities of the default states decrease. Yet, depending on the complexity of the economy and on the implied correlation of risks, it is possible, as we show in the following results, that the per capita level of default will increase without bound as the population increases.

In the following, if $\{b_n\}$ is a sequence of real numbers we say that $\{b_n\} = O(\ln H)$ if $\lim_{H \to \infty} \frac{|b_n|}{\ln H} = \infty$. A class of economies is called robust when it is an open set in the topology on the space of all economies defined in Section 5. If the class of economies has variable population $H = 1, 2, \ldots$, then we say that the class is robust when it is an open set under the topology defined in Section 5 for each population size $H$.

The following result considers a class of economies of population size $H$ and of complexity $H$, such as the complex economies illustrated in Figure 3 below. As $H$ increases, the number of agents and of goods increases with $H$:

**Proposition 3**: Consider a sequence of increasingly large and complex economies $E^*_H$ with a population of $H$ individuals facing individual risk, $H \to \infty$ and satisfying $|\mathcal{V}_H| = O(\ln H)$. Then the expected value of default increases with the size of the population $H$. The result is robust as it holds for an open set of economies.

**Proof**: In Appendix B.

The following result considers a class of economies $E^*_H$ in which the number of agents increases, the number of goods remains constant or increases; the per capita endowment of each agent need not increase.

**Example 3**: There exists a robust class of economies in which autarky occurs in states of default, for all population levels.

Consider an economy $E^*_H$ with $H_i$ households of type $i$, $i=1, \ldots, I$, a total population of $\sum H_i = H$. 
There are $N \geq 1$ goods. One type, type 1, faces uncertainty: the households of type i can be in one of two individual states $S = \{0,1\}$. The unfavorable state is 1, and 0 is the favorable state. The endowments of households of type 1 are $(0,...,0) \in \mathbb{R}^N$ in the unfavourable state and $(1,0,...,0)$ in the favourable state where the first good is the numeraire. Assume that the distribution of risks over aggregate collective states for households of type 1 is given by a probability distribution $\Pi^n_H$ defined by $\Pi^n_H(1,...,1) = 1/(2 \ln H)$; $\Pi^n_H(1,...,1,0,...,0) = \Pi^n_H(1/2 H_i$ in state 1 and $1/2 H_i$ in state 0) = $1/(2 \ln H)$; and $\Pi^n_H(0,...,0) = 1 - (1/\ln H)$. For all other aggregate collective states $r$, $\Pi^n_H(r) = 0$. The individual probabilities for households of type 1 are $p_Q = 1/(4 \ln H) + (1 - 1/\ln H) = (4 \ln H - 3)/(4 \ln H)$, and $p_i = 1/(2 \ln H) + 1/(4 \ln H) = 3/(4 \ln H)$, so that $p_Q + p_i = 1$. Then the probabilities $\Pi^n_H \to \Pi^\infty$ weakly, where $\Pi^\infty$ is the probability measure which assigns measure 1 to the aggregate collective state $r^\infty$ where everyone of type i is in the favorable state $s = 0$, i.e. $r^\infty_{is} = 1$ if $s = 0$, $r^\infty_{is} = 0$ if $s = 1$ ($i = 1$). Therefore the conditions for individual risk are satisfied. Individuals of type 1 purchase actuarially fair insurance to equate consumption in the two states: they contract to pay a premium $\rho_1$ in state 0 and receive a payment $\rho_0$ in state 1. For any finite $H$, there is one aggregate collective state of insurance default, denoted $r^n_H$, which occurs with probability $1/(2 \ln H)$: when all households of type 1 are in state 1. The probability of $r^n_H$ goes to 0 with $H$. Assume now that the economy $E_H$ is complex, as in the example of Section 5 and as illustrated in Figure 3; it has a unique equilibrium in which there is one type which is a net importer of each good, and one type which is a net exporter of each good. At the equilibrium prices of $E_H$ all the households of type 1 have zero endowments and trade nothing in the default state; by construction, no - one else will trade in this state either, leading to autarky. Therefore as $H$ increases, the economy is always in autarky in the default state $r^n_H$. This example is robust for small changes in initial endowments and preferences, since the net trades at equilibrium are a continuous fuction of endowments and preferences in the chosen topology.

The following result considers economies in which the number of agents and the their initial endowments increases:
Example 4: There exists a robust class of economies with individual risk where the expected value of per capita default increases with the population

Consider an economy $E_H$ with $H_i$ households of type $i$, $i=1,...,I$, a total population of $\sum_i H_i = H$. There are $N \geq 1$ goods. One type, type 1 faces uncertainty: the households of type $i$ can be in one of two individual states $S = \{0,1\}$. The unfavorable state is 1, and 0 is the favorable state. The endowments of households of type $i$ are $(0,...,0) \in \mathbb{R}^N$ in the unfavorable state and $(H,0,...,0)$ in the favorable state where the first good is the numeraire. Assume that the distribution of risks over aggregate collective states for households of type 1 is given by a probability distribution $\Pi_H$ defined by $\Pi_H(1,...,1) = 1/(2 \ln H)$; $\Pi_H(1,...,1,0,...,0)= \Pi_H \{1/2 H_i$ in state 1 and $1/2 H_i$ in state 0$\} = 1/(2.1n H)$; and $\Pi_H(0,...,0) = 1 - (1/\ln H)$. For all other aggregate collective states $r$, $\Pi_H(r) = 0$. The individual probabilities for households of type 1 are $\rho_0 = 1/(4 \ln H) + (1 - 1/\ln H) = (4 \ln H - 3)/(4 \ln H)$, and $\rho_1 = 1/(2\ln H) + 1/(4 \ln H) = 3/(4 \ln H)$, so that $\rho_0 + \rho_1 = 1$. Then the probabilities $\Pi_H \rightarrow \Pi^\infty$ weakly, where $\Pi^\infty$ is the probability measure which assigns measure 1 to the aggregate collective state $r^\infty$ where everyone of type $i$ is in the favorable state $s= 0$, i.e. $r^\infty_{is} = 1$ if $s = 0$, $r^\infty_{is} = 0$ if $s =1$ ($i = 1$). Therefore the conditions for individual risk are satisfied. Individuals of type 1 purchase actuarially fair insurance to equate consumption in the two states: they contract to pay a premium $\rho_1 H$ in state 0 and receive a payment $\rho_0 H$ in state 1. For any finite $H$, there is one aggregate collective state of insurance default, denoted $r_H$, which occurs with probability $1/(2.\ln H)$: when all households of type 1 are in state 1. This aggregate collective state will be used to derive a lower bound for the per-capita default of this economy as $H$ increases. The probability of $r_H$ goes to 0 with $H$. Assume now that the economy $E_H$ is complex, as in the example of Section 5 and as indicated in Figure 3, having a unique equilibrium in which there is one type which is a net importer of each good, and one type which is a net exporter of each good. At the equilibrium prices of $E_H$ the default of the insurer in collective state equals $\rho_0 H_1 H$ units of the numeraire. units of the numeraire defaulted in collective state $r_H$, and, in per-capital terms, to $\rho_0 H_1$ units of the numeraire. Therefore as $H$ increases, the expected level of per-capita
default is $p_0 H_1 (1/2 \ln H)$ which goes to infinity since $H_1$ is a fixed fraction of $H$. Note that this example is robust for small changes in initial endowments and preferences, since the net trades at equilibrium are a continuous function of endowments and preferences in the chosen topology.

7. Conclusion

We have analyzed the effect of introducing ALM assets in a large but finite economy to hedge against individual risk. Examples of ALM assets are actuarially fair insurance contracts, or shares in a firm which maximizes expected profits. An economy $E$ with incomplete markets was defined in which states of default emerge to account for the fact that ALM assets promise deliveries that can sometimes exceed physical endowments. Theorem 1 proved the existence of an equilibrium with incomplete markets and endogenous uncertainty on the contract's default. The new states of default in $E$ are collective states because they depend on collective events, such as the proportion of the population in each individual state. In complex economies, furthermore, many individuals default at once at each state of default, emphasizing the collective nature of default. Theorem 2 shows that there is an open set of complex economies of complexity $k$, for any $k \geq 2$. In such economies default by one individual leads to default by $k$ individuals, due to the pattern of trading. The introduction of ALM assets to hedge against individual risk therefore may increase collective risk (of default).

We studied the asymptotic properties of risk, namely the level of expected default as the population size increases towards infinity. At the limit, and only at the limit all the economies considered here have no default. This is because the distribution of risks across the population converges with certainty to a known quantity. However, matters are quite different before the limit is reached. Indeed, there might be a "discontinuity at infinity" measured by the fact that an economic statistic depending on endogenous risk (of default) increases beyond limit as the population goes to infinity even though at the limit there is no default.

Propositions 3, Examples 3 and 4 show that there are robust classes of economies where there is a "discontinuity at infinity", in the sense that although in the limiting case
(infinite populations) there is no default, yet the expected default increases with the population. Furthermore we show that the economy goes into autarky in states of default at all population levels no matter how large, and the per capital level of default increases with the population as well.

Our results illustrate a familiar concern about financial innovation. The concern is that the introduction of new financial assets can lead to more instability, namely to new states of collective default. The results we presented formalize this concern. They could also offer a way to measure the benefits of financial innovation, as well as its drawbacks.

The key is to understand the two circumstances under which collective risk increases following the introduction of new financial assets. These circumstances are: the assets are ALM, and the economies are complex. The first feature of the problem, that the assets be ALM, is almost inevitable in large economies with individual risk, because of the difficulty inherent to assets which depend on long list of individual characteristics, most of which are difficult to observe. So this first condition cannot be easily avoided. The second feature refers to the complexity of the economy. Here our results could be used to study the impact of financial innovation. The introduction of an asset will typically increase the web of trading in an economy and thus can be generally expected to increase its complexity. However, certain assets will increase the collective risk of default, and the total expected default, more than others. They create "correlated" risks which cannot be properly insured. The computation of the collective risk of default and the total expected default from different assets, as defined in Section 3, could help to determine the extent of collective risk introduced by the asset.

One generally expects that a new asset will increase the efficiency of the economy by improving the allocation of risks across favourable and unfavourable states, and thus lead to Pareto improvements. An interesting area of research would include the computation of the costs and benefits from the introduction of new securities. The benefits can be measured in terms of Pareto improvements in welfare, and the costs could be measured in terms of the increase is collective risks.
One implication of our results is that they formalize a "multiplier effect" for policy. In a complex economy, financial policies which succeed in preventing default by one agent also prevent, by a chain reaction, a large number of other defaults at no additional cost. Therefore the benefits have a "multiplier effect". Our results provide support for the policy of requiring reserves to enhance financial stability.

It should be noted that the relevant measurement of financial instability could in some cases be the total expected default, rather than the quantity of defaults following a given default. The complexity of the economy is not a problem in itself, unless it leads to large correlated risks.

If after the introduction of an ALM asset a second layer of securities is introduced to deal with the endogenous risk created by the first, and the latter securities are also of the ALM type, then the process is replicated. More endogenous uncertainty is created, piling up the risk of default of an asset which was introduced to hedge against the risk of default of another. More securities are needed at each point in this sequence to complete the market. These results also suggest that large and complex market economies with individual risks are likely to be incomplete. If as economies grow in size they develop more trading complexity, they can become more vulnerable to the collective risks arising from financial innovation.

8. Appendix

Part A. The Utility Function of Households

The von - Neuman Morgenstem utility function can be written as:

\[ W^i(z^i_h) = \sum_\sigma \Pi_\sigma U^i_s(h,\sigma)(z^i_{h\sigma}) \]

\[ z^i_{h\sigma} \in \mathbb{R}^{N^+} \], indicating that the h household of type i has preferences on consumption which may be represented by a "state separated" utility function \( U^i \) defined from S elementary utility functions \( U^i_s \). The functions \( U^i_s \) are assumed to be \( C^2 \), strictly increasing, strictly quasi
concave, and the closure of the indifference surfaces \((U^i_s)^{-1}(x) \subset \text{int}(R^N)\) for all \(x \in R^+\).

We consider, like Malinvaud (1973), an important class of cases in which the activity of the household \(h\) depends on \(\sigma\) only through the aggregate collective state \(r(\sigma)\). If household \(h\) takes into account first what happens to her or him i.e. \(s = s(h, \sigma)\), and second which frequency distribution \(r(\sigma)\) happens to appear, but nothing else, then the consumption plan \(z^i_s = \sum y^i_s(h, \sigma)(r(\sigma))\). The summation with respect to collective states \(\sigma\) can now be made first with respect to each aggregate collective state. To a particular \(r\) and \(s\) for which \(r_s \neq 0\), there usually corresponds a number of \(\sigma\) leading to \(r(\sigma) = r\) and \(s(h, \sigma) = s\), hence to the same \(U^i_s(h\sigma)(z^i_h\sigma) = U^i_s(y^i_s(r))\). Hence \(W\) may also be written as

\[
W^i(y^i) = \sum_{r \in R_H^H} \Pi(r) \sum_{s=1}^S r_s U^i_s(y^i_s(r)),
\]

which depends on the type \(i\) but not on the household \(h\). But if we make the further assumption that the household \(h\) only takes into account what happens to him/her, then the utility function of the \(i\)-th type of household in (A.1) can be rewritten as

\[
W^i(z^i) = \sum_{s=1}^S r_s U^i_s(z^i_s)
\]

where \(z^i_s \in R^N\) is the consumption of a household of type \(i\) in individual state \(s\) (Malinvaud (1973), p 390, (12)). Clearly \(z^i_h\sigma = z^i_s\) if individual \(h\) is of type \(i\), and is in individual state \(s\) at the collective state \(\sigma\).

**Appendix Part B: Proofs of results**

The economy in Theorem 1 has incomplete asset markets. Each asset has exogenously determined and fixed yields denominated in terms of the numeraire good. The economy has restricted access, because each type of agent can only purchase one of the assets available in the economy (i.e. the insurance contract of type \(i\)). The proof of existence of equilibrium in such an economy (e.g. Geanakoplos and Polemarchakis, 1986, Chichilnisky and Heal, 1991) is
standard. The first part of the proof consists of verifying that Walras Law is satisfied in the model of Theorem 1; it suffices to prove this for the case where the penalty for the insurer \( c = 0 \); the same proof follows for any given \( c \leq 0 \):

**Lemma 1.** Walras Law is satisfied in the economy of Theorem 1, i.e.

\[
P_r \cdot \sum_{i=0}^{S} \sum_{s=1}^{L} H_{i} r_{is} (y_{sr}^i - e_{s}^i) = 0, \text{ for } r = 0, 1, \ldots, V.
\]

**Proof:**

Let \( \Delta_r = \{ p_r \in \mathbb{R}^N : \sum_{n=1}^{N} p_{rn} = 1 \} \) be the price simplex for the aggregate collective state \( r = 0, \ldots, V \). After we find the equilibrium price \( p^* \in \Delta_r \) with \( p^* > 0 \), we can renormalize to make \( p^*_r = 1 \) for the numeraire good. Let the price simplex be denoted \( \Delta = \Delta_0 \times \ldots \times \Delta_V \). In the aggregate collective state \( r \), the total insurance payments to all individual agents of type \( i = 1, \ldots, I \) is

\[
T_r = \sum_{i=1}^{I} \sum_{s=1}^{S} H_{i} r_{is} v_{sr}^i, r = 0, 1, \ldots, V.
\]

where \( r_{is} = \rho_{is} \) for \( i = 1, \ldots, I \) and \( s = 1, \ldots, S \), then we know that \( T_r = 0 \) from \( \sum_{s=1}^{S} \rho_{is} v_{s}^i = 0 \) (actuarially fair insurance). Otherwise \( T_r \neq 0 \). If \( T_r \leq 0 \) there is no default and the insurer \( (i = 0) \) has net income \( -T_0 \) to be spent on consumption goods in state \( r = 0 \). If \( T_r > 0 \), the insurer has to default and consume nothing (the insurer has zero endowment, \( e_0 = 0 \)) with the provision of limited liability which corresponds to the default state \( r = 1, \ldots, V \). The shortfalls \( \delta_{sr}^i \) are chosen then to satisfy the provision of limited liability:

\[
\sum_{i=1}^{I} \sum_{s=1}^{S} H_{i} r_{is} (v_{sr}^i + \delta_{sr}^i) = 0 \quad \text{for } r = 1, \ldots, V,
\]

\[
\delta_{sr}^i = 0 \quad \text{if } v_{sr}^i \leq 0,
\]

\[
(B.3a) \quad (a)
\]
(B.3b) (b) If \( v^i_s > 0 \), then \( \delta^i_{sr} \leq 0 \), and \( \delta^i_{sr} + v^i_s \geq 0 \).

(B.3a) means that there is no shortfall when the insurer does not pay in state \( s \). (B3b) means that shortfalls happen only when the insurer is supposed to pay according to the actuarially fair insurance contract, and the shortfalls cannot exceed the originally promised payment. Now we can state and prove the Walras' Law in our economy.

For \( r = 0 \), and \( i = 1, \ldots, I \), \( p^r_r (y^{i}_{sr} - e^{i}_{s}) = v^i_s \), from the budget constraint and \( \delta^i_{sr} = 0 \).

For \( r = 0 \) and \( i = 0 \) (insurer), \( p^r_r (y^{O}_r - e^{O}) = p^r_r y^r_r = -T^r_o \). Hence

\[
p^r_r \sum_{i=0}^{I} \sum_{s=1}^{S} H^r_{is} (y^{i}_{sr} - e^{i}_{s}) = \sum_{i=1}^{I} \sum_{s=1}^{S} H^r_{is} v^{i}_{s} - T^r_o = 0 \quad \text{from (B.1) and } H^o_o = 1, r^o_o = 1.
\]

For \( r = 1, \ldots, V \), \( p^r_r \cdot (y^{i}_{sr} - e^{i}_{s}) = v^{i}_{s} + \delta^i_{sr} \) from the budget constraint. Summing over \( i \) and \( s \) we have

\[
p^r_r \cdot \sum_{i=0}^{I} \sum_{s=1}^{S} H^r_{is} (y^{i}_{sr} - e^{i}_{s}) = \sum_{i=1}^{I} \sum_{s=1}^{S} H^r_{is} (v^{i}_{s} + \delta^i_{sr}) = 0
\]

from (B.2), for \( r = 1, \ldots, V \). *

**Proof of Theorem 1**

Since \( W^i(y^i) \) is additively separable across \( r \) and \( s \), an equilibrium can be represented as a pair \((p, y)\) with \( p = (p^r_r) \), \( r = 0, \ldots, V \), \( p^r_r \in A^r_r \), and \( y \in R^{INS(V+1)} \), such that:

(B4) \( y = (y^i_{sr}) \), \( y^i_{sr} \) is in the demand correspondence \( D^i_{sr}(p^r_r) = \{y^i_{sr} \in R^N: y^i_{sr} \text{ maximizes } U^i_{sr}(y^i_{sr}) \text{ within } B^i_{sr}(p^r_r)\} \),

where

(B5) \( B^i_{sr}(p^r_r) \) is the budget set \( \{y^i_{sr} \in R^N: p^r_r (y^i_{sr} - e^i_s) = v^i_s + \delta^i_{sr}\} \) for \( s = 1, \ldots, S \), \( r = 0, \ldots, V \), and markets clear:

(B6) \( 0 \in \phi^i_{r}(p^r_r) \) the excess demand correspondence \( \sum_{i=0}^{I} \sum_{s=1}^{S} H^i_{is} (D^i_{sr}(p^r_r) - \{e^i_s\}) \) for \( r = 0, \ldots, V \).

For each \( r \) the budget set \( B^i_{sr}(p^r_r) \) defines a non empty, compact, convex valued and
continuous correspondence, so that \( \phi_r(p) \) is a non empty, convex-valued and upper hemicontinuous correspondence (Berge's theorem). Furthermore, the excess demand correspondence \( \phi_r \) satisfies Walras' Law for each \( r \) (Lemma 1), and the Boundary condition of Debreu (1982, p. 722). Therefore all the conditions of Theorem 8 of Debreu (1982, p. 722) are satisfied so that there exists a \( p^* \) such that \( 0 \in \phi_r(p^*) \). The vector \( p^* = (p^*_r) \) and the corresponding allocation \( y^*(p) \) is an equilibrium, completing the proof.

**Proof of Theorem 2:**

This proof has two steps: in the first step we construct an example of a complex economy, \( E \). In the second step we show that small variations of endowments and utilities from those of \( E \) will remain within the class of complex economies. This latter step uses Smale's (1974) results which establish that for a generic (open and dense) set of economies, the equilibrium allocations and prices depend locally continuously on the initial endowments and preferences. We construct a complex economy in steps.

First consider the case \( N = H \). Let \( E_0 \) be an economy with \( N \) goods, \( H \) households, and equilibrium prices \( p^* \), where at the equilibrium allocation corresponding to \( p^* \), household 1 is the only net exporter of good 1, and the only net importer of good 2; household 2 is the only net exporter of good 2 and the only net importer of good 3, etc., finally household \( H \) is the only net exporter of good \( N \), and the only net importer of good 1. Note that \( E_0 \) is complex, for any default of any amount initiated by any of the households in any commodity \( n = 1, \ldots, H \).

The argument is now extended to \( N = H + b \), \( b \leq H \). It suffices to modify \( E_0 \) as follows. Assume that household 1 is the only net exporter of goods 1 and \( H+1 \), and the only net importer of goods 2 and \( H+2 \); household 2 is the only net exporter of goods 2 and \( H+2 \); etc., household \( b \) is the only net exporter of goods \( b \) and \( H+b \) and the only net importer of goods \( b+1 \) and 1; and finally household \( H \) is the only exporter of good \( H \) and importer of good \( H+1 \). The economy \( E_1 \) thus defined is clearly complex. Finally consider the general case of an economy \( E \), where \( N = aH+b \), \( b < N \), for some \( a \geq 0 \). Define \( i \text{ mod } H \) as the set of all natural
numbers $n > H$ such that $i$ is the remainder from dividing $n$ into $H$, i.e. such that $i$ satisfies $n = aH + i$, for some $a > 0$. Then define the economy $E$ so that household 1 is the only net importer of all goods $1 \mod H$ and the only net exporter of all goods $2 \mod H$; household 2 is the only net exporter of all goods $2 \mod H$ and importer of all goods $3 \mod H$; etc.; finally $H$ is the only net exporter of all goods $0 \mod H$ and net importer of all goods $1 \mod H$. $E$ is clearly complex, see Figure 3.

Figure 3 here

Such an economy $E$ arises by assigning to household $i$ an endowment consisting exclusively of goods $i \mod H$, a utility function with no utility for good $i \mod H$, and the sum of Cobb-Douglas utilities for goods $i+1 \mod H$. Example 1 in Section 5 illustrates such an economy. Finally note that the complexity of $E$ survives small variations in net trades at the equilibrium. To see that the economy remains complex for small variations in net trades consider the following modification of the economy $E$. Consider first the case $N = H$. Assume that $p^* = (1,\ldots,1)$ - this can always be assumed without loss of generality by changing the commodities' units of measurement. Now consumer 1 is no longer the only net importer of good 1, but the main one: consumer 1 is a net importer of one unit of good 1 (in terms of the numeraire), while all other consumers together import less than $E$ at the equilibrium, $E < 1/2H$. Similarly consumer 1 is a net exporter of one unit of good 2, while all other consumers together export less than $1/2H$ of good 2, etc., until consumer $H$ who is a net importer of 1 unit of good 1, while the rest of all consumers import at most $1/2H$ units of good $H$; and finally $H$ exports one unit of good 1, while all other consumers together export at most $1/2H$ units of good 1. Then any default of at least one unit of good $n$, $n = 1,\ldots,H$, leads to default by all $H$ individuals. A small enough modification of endowments and preferences of produces another complex economy. The same argument can be employed to show that for small variations of endowments and utilities the economy $E$ with $N = aH + b$, $a > 0$, $b < H$, remains complex. Therefore, $E$ may be modified so as to be within Smale's (1974) open and sense class of
economies in which net trades vary (locally) continuously with initial endowments and utilities, and remain complex. Any further small modification will still remain within the class of complex economies, proving that there exists an open class of complex economies.

To prove that \( k \) - complexity is an open property when \( N > H \), construct an economy \( E \) where \( k \) individuals replicate economy \( E_0 \) of Theorem 1, for \( k \) goods. Assume all other individuals have zero net trades at the equilibrium \( p^* \). Then default can only be initiated by one of \( k \) individuals, and leads to \( k - 1 \) other defaults. This example is open since for any level of default \( \delta > 0 \), we can modify the original economy so that the net trades of the households \( h = k + 1, \ldots, H \) add up to less than \( \delta \).

**Proof of Proposition 3**

This follows from Theorem 1 and the results of Section 5. In complex economies such as \( E_0 \) in Theorem 1, default always increases at least as a linear function of the population \( H \), while the definition of individual risk is consistent with an arbitrary rate of convergence of the probabilities \( \Pi_H \) as \( H \to \infty \), as it requires only weak convergence of the probability measures \( \Pi_H \to \Pi^\infty \), see e.g. Malinvaud (1973), p. 387, para 4.
Classification of Risk

Assets introduced to hedge individual risk increase collective risk.

Figure 1
The shaded area contains the support of the distribution over collective states in the finite economy; the point inside it indicates the support of the limiting distribution which is a point because of the assumption of individual risk.
The complex economy $F$ of Theorem 2. Household 1 is the only net importer of all goods $1 \text{ mod } H$, and the only net exporter of all goods $2 \text{ mod } H$. There are $H$ households and $N$ goods, $N \geq H$. 
8. References


