Eigenvarieties and twisted eigenvarieties

Zhengyu Xiang

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Abstract

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For an arbitrary reductive group G, we construct the full eigenvariety $\mathfrak{E}$, which parameterizes all p-adic overconvergent cohomological eigenforms of G in the sense of Ash-Stevens and Urban. Further, given an algebraic automorphism ι of G, we construct the twisted eigenvariety $\mathfrak{E}^{\iota}$, a rigid subspace of $\mathfrak{E}$, which parameterizes all eigenforms that are invariant under ι. In particular, in the case $G = GL_n$, we prove that every self-dual automorphic representation can be deformed into a family of self-dual cuspidal forms containing a Zariski dense subset of classical points. This is the inverse of Ash-Pollack-Stevens conjecture. We also give some hint to this conjecture.

Contents

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Chapter 1

INTRODUCTION

The problem of eigenvarieties comes from the study of p-adic families of automorphic forms. For instance, the eigencurve constructed by Coleman-Mazur [6] gives a geometric and global interpretation of Hida and Coleman's earlier work on p-adic families of modular cusp eigenforms.

Generally, given a reductive group G, one approach to study congruences between Hecke eigenvalues is using cohomology of arithmetic subgroups. In this direction, to construct an eigenvariety parameterizing the Hecke eigensystems, a lot of works have been done for those reductive groups G whose archimedean part are compact modulo center [3],[7],[11]. More recently, Eric Urban constructed eigenvarieties for more general reductive groups whose archimedean part have discrete series [1]. More precisely, he constructed eigenvarieties associated to such reductive groups whose $\mathbb{Q}_p$-points are in bijection with p-adic overconvergent cohomological automorphic representation having non-trivial Euler-Poincare characteristic. So it is natural to ask that if we can construct for any reductive group an eigenvariety which parameterizes all the p-adic overconvergent cohomological automorphic representations.

To illustrate the situation, we say more here. For some undefined notations, we refer the chapter 2 below. Let G be a reductive group which is split over $\mathbb{Q}_p$, (B, T) a Borel pair, $\lambda \in X^*(T)$ a dominant algebraic weight which naturally lies in the weight space $\mathfrak{X}$, and $\mathbb{W}_\lambda$ be the irreducible algebraic representation of G with highest weight λ. We also fix $K = K_\infty K_f$, where $K_f = K^p I$ is some compact open subgroup of $G(\mathbb{A}_f)$ with I an Iwahori subgroup of $G(\mathbb{Q}_p)$. For any finite extension $L/\mathbb{Q}_p$, we can consider the cohomology over the locally symmetric space

$$H^*(S_G(K), \mathbb{W}_\lambda(L))$$

where the local symmetric space is defined as

$$S_G(K) := G(\mathbb{Q})\backslash G(\mathbb{A})/KZ_\infty$$

Let $\mathscr{U}_p$ be the local Hecke algebra at p, which is commutative as defined in chapater 2, and

$$\mathscr{H}_p(K^p) = C_c^\infty(K^p\backslash G(\mathbb{A}_f^p)/K^p)\hat{\otimes}\mathscr{U}_p$$

be the global Hecke algebra. Set $\mathscr{S}$ be the finite set of primes away from p and such that K_l^p is not local maximal compact for $l \in \mathscr{S}$, we consider the Hecke algebra

$$R_{\mathscr{S},p} := C_c^\infty(G(\mathbb{A}_f^{\mathscr{S}\cup\{p\}})//K^{\mathscr{S}\cup\{p\}}) \otimes \mathscr{U}_p$$

which lies in the center of $\mathscr{H}_p(K^p)$. It is well known that these Hecke algebras act on the cohomology.

Roughly speaking, a cohomological automorphic representation of weight λ is a representation of $G(\mathbb{A})$ occurring in $H^*(S_G(K), \mathbb{W}_\lambda)$ for some K. An (arithmetic) p-adic cohomological automorphic representation of arithmetic weight $\lambda = \lambda_0\epsilon$ is a representation of $(H)_p$ occurring in $H^*(S_G(K^pI), \mathbb{W}_{\lambda_0})(L)(\epsilon)$ with some p-adic field $L/\mathbb{Q}_p$. Where λ_0 is an algebraic weight and ϵ is a finite order character of I. Given an irreducible p-adic cohomological automorphic representation, its restriction on $R_{\mathscr{S},p}$ is a character, which is called a Hecke eigensystem. Given an irreducible cohomological automorphic representation of weight λ_0, its p-stabilization is an irreducible p-adic cohomological automorphic representation. This is the main examples of p-adic cohomological automorphic representations we are interested.

We want to interpolate $H^\bullet(S_G(K), \mathbb{W}_\lambda)$ for λ in the weight space $\mathfrak{X}$. The idea (originally due to Hida, and generally to Ash-Stevens) is to replace the coefficients $\mathbb{W}_\lambda$ by $\mathscr{D}_\lambda$, a distribution space, in the cohomology group. We are going to realize the eigensystems with coefficients in classical cohomology among the eigensystems with coefficients in this "overconvergent cohomlogy", just like realizing the classical modular forms among the so-called "overconvergent modular forms". Just like the last paragraph, we call the representations (or Hecke eigensystems) occurring in the cohomology group $H^*(S_G(K), \mathscr{D}_\lambda)$ "overconvergent".

Let $\mathscr{U} \subset \mathfrak{X}$ be an affinoid subdomain, there is a universal distribution space $\mathscr{D}_{\mathscr{U}}$ interpolating $\mathscr{D}_\lambda$ for $\lambda \in \mathscr{U}$. Choose $f \in R_{\mathscr{S},p}$ which defines an compact operator on $H^*(S_G(K), \mathscr{D}_{\mathscr{U}})$, then a local piece of the full eigenvariety may be built over a Fredholm variety (which is called a spectral variety) in $\mathscr{U} \times \mathbb{A}_1^{rig}$ cut out by f acting on a finite $\mathscr{O}(\mathscr{U})$-torsion-free submodule of $H^*(S_G(K), \mathscr{D}_{\mathscr{U}})$.

Using the "overconvergent cohomology", Ash and Stevens have constructed local pieces of the full eigenvariety for Gl_n[3]. Basically, given a Hecke eigensystem θ of weight λ and slope h, they proved the existence of a neighborhood Ω such that there is an $\leq h$-slope decomposition of $H^*(S_G(K), \mathscr{D}_\Omega)$ and a control theorem over Ω which asserted that for any $\lambda' \in \Omega$ there is a morphism $H(S_G(K), \mathscr{D}_\Omega)^{\leq h} \to H(S_G(K), \mathscr{D}_{\lambda'})^{\leq h}$ sending $\mathscr{O}(\Omega)$ valued eigensystems occuring in $H(S_G(K), \mathscr{D}_\Omega)^{\leq h}$ to eigensystems occuring in $H(S_G(K), \mathscr{D}_{\lambda'})^{\leq h}$ via specialization. By this Ash-Stevens constructed the local eigenvarieties and, in particular, their local eigenvareities preserve all slope informations near the point (θ, λ).

However, since the choice of Ω depends on the weight λ and the slope h, it is not clear if those local pieces could be patched together. This gives us some hint that, to construct a global eigenvariety, we may have to loose the control of the slopes and consider some more general notions.

So the first goal of this work is to give a global construction of the full eigenvarieties for general reductive groups. We have proved in Chapter 4 that:

Theorem 1.1. *There is a rigid analytic space $\mathfrak{E}_{K^p}$ defined over $\mathbb{Q}_p$ of dimension at most dim$\mathfrak{X}$, and a projector $pr : \mathfrak{E}_K \to \mathfrak{X}$, such that every point $y \in \mathfrak{E}_{K^p}(\overline{\mathbb{Q}}_p)$ can be written into the form (λ_y, θ_y) with $\lambda_y = pr(y)$, and such a pair (λ, θ) is in $\mathfrak{E}_{K^p}(\overline{\mathbb{Q}}_p)$ if and only if θ is an overconvergent Hecke eigensystem of weight λ. Moreover, if y is contained in a component containing a Zariski dense subset of classical points, then the component is of codimension* 0 *if and only if θ_y has a non-trivial*

Euler-Poincare characteristic. So, in particular, the codimension zero components correspond to the eigenvariety constructed in [1] by Urban.

Instead of trying to prove a slope decomposition of $H^*(S_G(K), \mathscr{D}_{\mathscr{U}})$, we apply the "polynomial decomposition" to a perfect complex $R\Gamma^\bullet(\mathscr{D}_{\mathscr{U}})$ which computes the cohomology groups. (This perfect complex has been used by Ash-Stevens in [3] and Urhan in [1].) We then get a perfect finite torsion free subcomplex $N^\bullet_{\mathscr{U}}$. Using the method of Buzzard [8], we construct a rigid analytic space parameterizing the Hecke eigenvalues of $\oplus_i N^i_{\mathscr{U}}$. Then we determine a subspace which parameterize the Hecke eigenvalues of $h(N^\bullet_{\mathscr{U}})$, the cohomology groups of the complex $N^\bullet_{\mathscr{U}}$, by considering the support of $\mathscr{O}(\mathscr{U})[X]$ acting on $h(N^\bullet_{\mathscr{U}})$. This is our spectral variety. We then build our local pieces of eigenvariety over them. Polynomial decomposition is a more general notion than the slope decomposition. It allows us to handle "a piece of" variety at the same time, then to patch them easily. Moreover, since a slope decomposition is also a polynomial decomposition, that we could view the local pieces of eigenvariety constructed by Ash and Stevens as special pieces of ours.

Unfortunately, our full eigenvariety does not give much information of the classical points, and generally its components may not contain a Zariski dense subset of classical points. Moreover, the global geometry of $\mathfrak{E}$ may not be neat, for example, it is not equidimensinal. In general, however, we hope the full eigenvariety may still give a lot of information, especially globally. For example, trying to characterize its components may be interesting and important, as the last statement in the theorem above and Urban's conjectures stated in [1], which states some precise conjectures about the relation between the codimension of irreducible components of the eigenvariety and degrees of the cohomology. Also, it is interesting to see if the study of certain property of automorphic representations could be fit into the notion of eigenvariety. This generalizes the question that if we can deform an eigenform with certain property into a p-adic family with this property preserved. We will see later that the second work of this paper will also give such an example.

Another motivation of this paper is a conjecture made by Ash-Pollack-Stevens in [2] about p-adic deformation of an automorphic Hecke eigencharacter and its selfduality for Gl_3. Vaguely speaking, let θ be a finite slope cuspidal Hecke eigen character with cohomological weight λ, we say it is arithmetic if λ is arithmetic. then APS conjecture says:

Conjecture. *Let θ be a finite slope cuspidal Hecke eigen character of Gl_3, if θ can be deformed into a p-adically analytic family of finite slope arithmetic cuspidal Hecke eigensystems, then θ is essentially selfdual.*

This conjecture is interesting but could be very difficult to prove, however, in [1], Urban suggested that we can prove the inverse by constructing an twisted eigenvaritey parametrizing all self-dual representations using a method he developed there named finite slope character distibution. In this paper, we prove a stronger result, by a twisted version of Urban's method, as,

Theorem 1.2. *if θ is essentially selfdual, then it can be deformed into a p-adically analytic family of finite slope arithmetic cuspidal Hecke characters such that those characters are essentially selfdual.*

We actually deal with a more general situation. Let ι be a finite order algebraic automorphism of G and (B, T) is chosen to be stable under ι. Write $\mathfrak{X}^\iota$ the subspace of $\mathfrak{X}$ consisting of those weights invariant under ι. We say an automorphic representation π is self-dual under ι if $\pi^\iota \cong \pi$. In particular, if $G = Gl_n$ and ι is the Cartan involution given by transpose inverse, then we have the notion of self-dual as usual. We prove:

Theorem 1.3 (twisted eigenvarities)**.** *There is a rigid analytic space $\mathfrak{E}^\iota_{K^p}$ over $\mathbb{Q}_p$, satisfying:*

(*a*) *For any $(\lambda, \theta) \in \mathfrak{E}_{K^p}(\overline{\mathbb{Q}}_p)$, (λ, θ) is in $\mathfrak{E}^\iota_{K^p}(\overline{\mathbb{Q}}_p)$ if and only if θ has a non-trivial twisted Euler-Poincare characteristic and is self-dual with respect to ι.*

(*b*) *$\mathfrak{E}^\iota_{K^p}$ is equidimensional with the same dimension to $\mathfrak{X}^\iota$.*

(*c*) *Every irreducible component projects surjectively onto a Zariski subset of $\mathfrak{X}^\iota$.*

In particular, if $G = Gl_n$ and ι is given by transpose inverse, we can prove that the twisted Euler-Poincare characteristic of an (essentially) self-dual cuspidal automorphic representation is non-trivial. So we have an eigenvariety parameterizing all (essentially) self-dual cuspidal Hecke eigensystems in this case.

Due to Urban in [1], a finite slope character distribution is a morphism $I : \mathscr{H}_p(K^p) \to \overline{\mathbb{Q}}_p$ which could be expressed as a linear combination of traces of Hecke operators on irreducible finite slope representation automorphic representations. Urban proved that, if the coefficients of a finite slope character distribution (as a linear combination) are all positive (i.e. it is effective), then there is an eigenvariety associated to it, parameterizing those representations appearing in I.

Now we are going to interpolate those irreducible finite slope cuspidal self-dual (under ι) automorphic representations in $H^*(S_G(K), \mathscr{D}_\lambda)$. We need a twisted version of Urban's finite slope character distribution in which only those self-dual irreducible finite slope cuspidal automorphic representation will occur. Noticing that we can extend the action of ι to $H^*(S_G(K), \mathscr{D}_\lambda)$. If V_π is a non-self-dual irreducible finite slope automorphic representation realized in $H^*(S_G(K), \mathscr{D}_\lambda)$, the restriction of ι on V will not map V to itself, so $tr(\iota|V_\pi) = 0$. This observation tells us, if consider the twisted trace $tr(\iota \times f|H^\bullet(S_G(K), \mathbb{W}_\lambda^\vee(L)))$ with $f \in \mathscr{H}_p(K^p)$, then only the traces of those self-dual irreducible representatins will show up. Further, to isolate the cuspidal information from $tr(\iota \times f|H^\bullet(S_G(K), \mathbb{W}_\lambda^\vee(L)))$, we prove for $H^\bullet(S_G(K), \mathbb{W}_\lambda^\vee(\mathbb{C}))$ a twisted version of Franke's trace formula, which allows us to isolate cuspidal information over $\mathbb{C}$. Comparing $H^\bullet(S_G(K), \mathbb{W}_\lambda^\vee(L))$ with $H^\bullet(S_G(K), \mathbb{W}_\lambda^\vee(\mathbb{C}))$, we define by induction for the former a twisted finite slope character distribution I^ι, which is what we need. Moreover, an irreducible finite slope self-dual occurs in I^ι with a multiplicity which equals its twisted Euler-Poincare characteristic. Then, by a detailed study of the theory of Vogan-Zuckerman [20], we prove I^ι is actually effective. Finally, similar arguments as in [1] lead us to the twisted eigenvariety.

Chapter 2

PRELIMINARY

2.1 notation

Let G be a connected reductive group over $\mathbb{Q}$. To simplify the notation we assume that G is split over $\mathbb{Q}_p$. Let (B, T) be a fixed Borel pair of G, and $Z = Z_G$ the center of G. For any algebraic group H, we use the subscripts ∞ and f to indicate its archimedean and finite part respectively, we use the script $+$ to indicate the subgroup of $H(\mathbb{A})$ consisting of the elements whose infinite part belongs to the connected component of H_∞.

Now let $K = K_\infty K_f$ be an open compact subgroup of $G(\mathbb{A})$ with K_∞ the fixed maximal open compact subgroup of G_∞, and $S_G(K) = G(\mathbb{Q})\backslash G(\mathbb{A})/KZ_\infty$ the local symmetric space of G with respect to K. By the strong approximation,

$$G(\mathbb{A}) = \sqcup_i G(\mathbb{Q}) \times G_\infty^+ \times g_i K_f.$$

This enables us to decomposite

$$S_G(K) \cong \sqcup \Gamma_i \setminus \mathscr{H}_G$$

with

$$\Gamma_i = \Gamma(g_i, K) = \textit{the image of } g_i K g_i^{-1} \cap G(\mathbb{Q})^+ \textit{ in } G^{ad}(\mathbb{Q}),$$

and

$$\mathscr{H}_G = G_\infty^+ / K_\infty Z_\infty.$$

Throughout this paper, we assume that K is neat (that is, Γ_i contains no element of finite order). So $S_G(K)$ is a smooth real analytic variety of dimension, say, d.

Fix a prime number p, and at the place p, denote the Iwahori subgroup of depth m by I_m, that is, $I_m = \{g \in G(\mathbb{Z}_p) | g \in B(\mathbb{Z}/p^m\mathbb{Z}) \ mod\ p^m\}$. To simplify the notation, we set $I = I_1$. By Iwahori decomposition,

$$I_m = (I_m \cap N^-(\mathbb{Q}_p))T(\mathbb{Z}_p)N(\mathbb{Z}_p)$$

Moreover, we denote

$$T^+ = \{t \in T(\mathbb{Q}_p) | tN(\mathbb{Z}_p)t^{-1} \subset N(\mathbb{Z}_p)\}$$

$$T^{++} = \{t \in T^+ | \bigcap_{i \geq 1} t^i N(\mathbb{Z}_p)t^{-i} = \{1\}\},$$

We set $\Delta_m^+ = I_m T^+ I_m$ and $\Delta_m^{++} = I_m T^{++} I_m$ and consider the p-adic cell

$$\Omega_m = I_m \cap N^-(\mathbb{Q}_p) \backslash I_m \subseteq N^-(\mathbb{Q}_p) \backslash G(\mathbb{Q}_p).$$

Now I_m acts on Ω_m by right translation, it is known that we can extend this action to an $*$-action of I and Δ^+ on the p-adic cells (see [1] and [3]). Concretely speaking, fix a splitting ξ of the exact sequence

$$1 \to T(\mathbb{Z}_p) \to T(\mathbb{Q}_p) \to T(\mathbb{Q}_p)/T(\mathbb{Z}_p) \to 1$$

for $g \in \Delta$, by Iwahori decomposition, write $g = n_g^- t_g n_g^+$, then for any $[x] \in \Omega$, we define

$$[x] * g = [\xi(t_g)^{-1} x g]$$

We denote by $\mathscr{U}_p$ the local Hecke algebra at p, that is,

$$\mathscr{U}_p = C_c^\infty(\Delta^+ // I, \mathbb{Z}_p) \simeq \mathbb{Z}_p[T^+/T(\mathbb{Z}_p)]$$

and the global Hecke algebras

$$\mathscr{H}_p := \mathscr{H}_p(G) = C_c^\infty(G(\mathbb{A}_f^p))\hat{\otimes}\mathscr{U}_p$$

$$\mathscr{H}_p(K^p) = C_c^\infty(K^p\backslash G(\mathbb{A}_f^p)/K^p)\hat{\otimes}\mathscr{U}_p$$

Now set $\mathscr{S}$ be the finite set of primes where K is not local maximal compact, we consider the Hecke algebra

$$R_{\mathscr{S},p} := C_c^\infty(G(\mathbb{A}_f^{\mathscr{S}\cup\{p\}})//K^{\mathscr{S}\cup\{p\}}) \otimes \mathscr{U}_p$$

which can be identified in the center of $\mathscr{H}_p(K^p)$. It is well known that the Hecke algebras acting on the cohomology.

We denote by ι an finite order algebraic automorphism of G, it sends $g \in G$ to g^ι. For given ι, we can find a minimal parabolic subgroup P_0 fixed by ι. Throughout this paper, we always choose such P_0 and a standard Borel pair (B, T) fixed by ι, in particular, ι will fix the corresponding Iwahori subgroups I_m. We also choose K_f^p fixed by ι. With these notation, we can consider ι acting on the Hecke algebras as sending f to f^ι.

2.2 weight spaces

2.2.1

As a finitely generated $\mathbb{Z}_p$-module, $T(\mathbb{Z}_p)$ can be written as $\mathbb{Z}_p^r \times \Pi$, with a finite group Π. There is a rigid space $\mathfrak{X}_{T(\mathbb{Z}_p)}$, such that for any finite extension L of $\mathbb{Q}_p$,

$$\mathfrak{X}_{T(\mathbb{Z}_p)}(L) = Hom_{cont}(T(\mathbb{Z}_p), L^\times)$$

and

$$\mathfrak{X}_{T(\mathbb{Z}_p)}(\overline{\mathbb{Q}}_p^\times) = Hom_{qp}(\Pi, \overline{\mathbb{Q}}_p^\times) \times (B_{1,1}(\overline{\mathbb{Q}}_p)^\circ)^r.$$

$\mathfrak{X}_{T(\mathbb{Z}_p)}$ is called the weight space associated to $T(\mathbb{Z}_p)$, and its elements are (p-adically) continuous weights.

Given K as in the last section, we denote $Z_K = Z(\mathbb{Q}) \bigcap K$ and $\mathfrak{X} \subseteq \mathfrak{X}_{T(\mathbb{Z}_p)}$ the Zariski closure of those weight λ which is trivial on Z_K. Indeed, if we set Z_p be the p-adic closure of Z_K in $T(\mathbb{Z}_p)$, then $\mathfrak{X} = \mathfrak{X}_{T(\mathbb{Z}_p)/Z_p}$. $\mathfrak{X}$ will be the weight space we work on later.

Moreover, ι induces an algebraic automorphism on $\mathfrak{X}$ which sents λ to λ^ι, such that $\lambda^\iota(t) = \lambda(t^{\iota^{-1}})$. A p-adic weight is called self-dual under ι if $\lambda^\iota = \lambda$ and $\mathfrak{X}^\iota$ is defined to be the subspace of all self-dual weights.

2.2.2

For any n, we can associate a rigid space $T(\mathbb{Z}_p)_n^{rig}$ to $T(\mathbb{Z}_p)$, such that for any finite extension L of $\mathbb{Q}_p$,

$$\mathscr{O}(T(\mathbb{Z}_p)_n^{rig}{}_{/L}) = \mathscr{A}_n(T(\mathbb{Z}_p), L),$$

where $\mathscr{A}_n(T(\mathbb{Z}_p), L)$ is the space of locally n-analytic L-valued functions on $T(\mathbb{Z}_p)$.

For any L as above, we have a natural pairing

$$\mathfrak{X}_{T(\mathbb{Z}_p)}(L) \times T(\mathbb{Z}_p) \to L^\times, (\lambda, t) \mapsto \lambda(t).$$

This deduces a continuous injective homomorphism $T(\mathbb{Z}_p) \hookrightarrow \mathscr{O}(\mathfrak{X}_{T(\mathbb{Z}_p)})^\times$.

More important, we have following lemma [1, lemma 3.4.6]

Lemma 2.1. *For any affinoid subdomain $\mathfrak{U} \subseteq \mathfrak{X}_{T(\mathbb{Z}_p)}$ or $\mathfrak{X}^\iota$, there exist a smallest integer $n(\mathfrak{U})$, such that for any finite extension L of $\mathbb{Q}_p$, every element $\lambda \in \mathfrak{U}(L)$ is $n(\mathfrak{U})$-locally analytic. Moreover, there is a rigid analytic map $\mathfrak{U} \times T(\mathbb{Z}_p)_{n(\mathfrak{U})}^{rig} \to B_{1,1}$, such that for any L, its realization in L is given by the pairing above.*

2.3 analytic induced modules and distribution spaces

2.3.1

We work on the p-adic cell Ω_1 defined above. For $\lambda^{alg} \in X^*(T)$, we denote by $W_{\lambda^{alg}}$ the finite dimensional algebraic representation of G with highest weight λ^{alg}. So

$$W_{\lambda^{alg}} = ind_B^G(\lambda^{alg})^{alg}$$

If $\lambda = \lambda^{alg}\epsilon$ with ϵ of order m, we also write the m-locally analytic induction of λ as

$$\mathscr{W}_\lambda = ind_B^G(\lambda)^{m-an},$$

so

$$W_{\lambda^{alg}}(L)(\epsilon) \hookrightarrow \mathscr{W}_\lambda(L)$$

The $*$-action of Δ^+ induces an action on $\mathscr{W}_\lambda(L)$, via the right $*$-translation. Moreover, if $\lambda \in \mathfrak{X}^\iota$, then ι gives actions on $W_{\lambda^{alg}}$ and $\mathscr{W}_\lambda$. Concretely speaking, for $f \in W_{\lambda^{alg}}$, ι sends f to f^ι such that $f^\iota(g) = f(g^{\iota^{-1}})$.

Given L a finite extension of $\mathbb{Q}_p$, and $\lambda \in \mathfrak{X}(L)$, we denote $\mathscr{A}_\lambda(L)$ the space of locally L-analytic functions on I such that

$$f(n^- tg) = \lambda(t) f(g)$$

with $n^- \in I \cap N^-$, $t \in T(\mathbb{Z}_p)$ and $g \in I$. $\mathscr{A}_\lambda(L)$ can be viewed as a subspace of $\mathscr{A}(\Omega_1, L)$. Indeed, let $T(\mathbb{Z}_p)$ act on $\mathscr{A}(\Omega_1, L)$ by the natural translation, then

$$\mathscr{A}_\lambda(L) = \mathscr{A}(\Omega_1, L)[\lambda] := \{\phi \in \mathscr{A}(\Omega_1, L) | t\phi = \lambda(t)\phi\}$$

The $*$-action of Δ^+ on Ω_1 induces a $*$-action of Δ^+ on $\mathscr{A}(\Omega_1, L)$, which commutes with the translation action of $T(\mathbb{Z}_p)$ above. So in particular, we have an $*$-action of Δ^+ on $\mathscr{A}_\lambda(L)$, as for $g \in \Delta^+$ and $\phi \in \mathscr{A}_\lambda(L)$, we have

$$g * \phi([x]) := \phi([x] * g)$$

Define the distribution space

$$\mathscr{D}_\lambda(L) = Hom_{cont}(\mathscr{A}_\lambda(L), L)$$

Considering the dual action of Δ^+ on $\mathscr{D}_\lambda(L)$, we have the next proposition, a detailed statement and proof can be found in [1, lemma 3.2.8].

Proposition 2.2. *$\mathscr{D}_\lambda(L)$ is a compact Frechet space over L. If $\delta \in \Delta^{++}$ then the $*$-action of δ gives a compact operator on $\mathscr{D}_\lambda(L)$*

If $\lambda \in \mathfrak{X}^\iota$, then ι also induces actions on $\mathscr{A}_\lambda$ and on $\mathscr{D}_\lambda$ in the same way as on the spaces $W_{\lambda^{alg}}$ and $\mathscr{W}_\lambda$.

Remark. The theory of compact operators on a orthonormalizable (p-adic) Banach space is originally due to Serre and generalized by Coleman [5]. This notion is generalized to compact Frechet spaces by Urban in [1, Chapter 2], where we can see that most results about compact operators on Banach spaces still hold for compact Frechet spaces.

2.3.2

Let $\mathscr{U}$ be an affinoid subdomain of $\mathfrak{X}$ or $\mathfrak{X}^\iota$ and $n \geq n(\mathscr{U})$. We associate the rigid space $(\Omega_m)_n^{rig}$ to Ω_m as the definition of $T(\mathbb{Z}_p)_n^{rig}$, and define $\mathscr{A}_{\mathscr{U},n}(L)$ as the set of rigid analytic L-valued functions on $\mathscr{U} \times (\Omega_1)_n^{rig}$ satisfying

$$f(\lambda, [tn]) = \lambda(t) f(\lambda, [n])$$

for all $\lambda \in \mathscr{U}(L)$, $t \in T(\mathbb{Z}_p)_n^{rig}(L)$ and $n \in N(\mathbb{Z}_p)^{rig}(L)$.

It is not hard to see that

$$\mathscr{A}_{\mathscr{U},n} = \mathscr{O}(\mathscr{U}) \hat{\otimes} \mathscr{A}_n(N(\mathbb{Z}_p))$$

So $\mathscr{A}_{\mathscr{U},n}$ are $\mathscr{O}(\mathscr{U})$-orthonormalizable Banach spaces and the inclusions $\mathscr{A}_{\mathscr{U},n} \subset \mathscr{A}_{\mathscr{U},n+1}$ are completely continuous.

We can also extend the $*$-action of Δ^+ on Ω_1 to an action on $(\Omega_1)_n^{rig}$ for all $n \geq 0$. By lemma 2.1, we have

$$\mathscr{A}_{\mathscr{U},n} = \{ f \in \mathscr{O}((\Omega_1)_n^{rig}) \otimes \mathscr{O}(\mathscr{U}) | t(f \otimes 1) = f \otimes t, t \in T(\mathbb{Z}_p)_n^{rig} \}.$$

This gives the $*$-action on $\mathscr{A}_{\mathscr{U},n}$

Now set

$$\mathscr{A}_{\mathscr{U}} = \bigcup_{n \geq n(\mathscr{U})} \mathscr{A}_{\mathscr{U},n},$$

and $\mathscr{D}'_{\mathscr{U},n} := Hom_{(O)(\mathscr{U})}(\mathscr{A}_{\mathscr{U},n}, (O)(\mathscr{U}))$ the continuous $\mathscr{O}(\mathscr{U})$-dual of $\mathscr{A}_{\mathscr{U},n}$. Then we have a canonical injective map

$$\mathscr{O}(\mathscr{U})\hat{\otimes}_L \mathscr{D}_n(N(\mathbb{Z}_p), L) \to \mathscr{D}'_{\mathscr{U},n}$$

We then denote $\mathscr{D}_{\mathscr{U},n}$ the image of this map, and define

$$\mathscr{D}_{\mathscr{U}} := \varprojlim \mathscr{D}_{\mathscr{U},n}$$

$\mathscr{A}_{\mathscr{U}}$ and $\mathscr{D}_{\mathscr{U}}$ can be viewed as Δ^+-modules. Moreover, from the results above we have

Proposition 2.3. *(a) $\mathscr{A}_{\mathscr{U}} \otimes_\lambda L \cong \mathscr{A}_\lambda(L)$ and $\mathscr{D}_{\mathscr{U}} \otimes_\lambda L \cong \mathscr{D}_\lambda(L)$ via specialization. (b) If $\delta \in \Delta^{++}$, then the $*$-action of δ gives a compact operator on the $\mathscr{O}(\mathscr{U})$-projective compact Frechet spaces $\mathscr{D}_{\mathscr{U}}$*

All of these results can be found in [1, section 3.4].

Remark. (a) The $*$-action of Δ^+ on $\mathscr{D}$ is compatible with the natural action of I on it. (b) From now on, we fix $K = K^p I$, and view $\mathscr{D}$ as a K-module via the projection $K \to I$. Choosing Γ_i with respect to K, we can view $\mathscr{D}$ as a Γ_i-module and compute the cohomology $H(\Gamma_i \backslash \mathscr{H}_G, \mathscr{D})$.

Chapter 3

ACTIONS ON RESOLUTIONS AND COHOMOLOGY GROUPS

3.1 resolution for arithmetic subgroups

3.1.1

Now let's consider the local symmetric space $S_G(K)$. If M is a $(G(\mathbb{Q}), K)$-module with Z_K acting trivially, then it defines a non-trivial local system M on $S_G(K)$. It is well known that the cohomology of M can be computed by the cohomology of arithmetic subgroups.

$$H^\bullet(S_G(K), M) = \oplus_i H^\bullet(\Gamma_i, M)$$

In our situation, M is one of $W_{\lambda^{alg}}(L)$, $\mathscr{W}_\lambda(L)$, $\mathscr{D}_\lambda(L)$ and $\mathscr{D}_\mathfrak{U}(L)$. We can find a perfect complex $R\Gamma^\bullet(K, M)$ computing the right side of the equality, such that every $R\Gamma^q(K, M)$ is a finite copy of M.

Lemma 3.1. *For $\forall i$, there exist $\mathbb{Z}[\Gamma_i]$-free of finite type resolutions of the trivial Γ_i-module $\mathbb{Z}$ of length d. We choose one as*

$$0 \to C_d(\Gamma_i) \to \cdots \to C_1(\Gamma_i) \to C_0(\Gamma_i) \to \mathbb{Z} \to 0$$

where $C_i(\Gamma_i)$ are free $\mathbb{Z}[\Gamma_i]$-modules of finite rank.

This is a standard lemma obtained by considering the triangulation of the Borel-Serre compactification of $\Gamma_i \backslash \mathscr{H}_G$. A detailed proof could be found in [1, Lemma 4.2.2].

Now set

$$C^\bullet(\Gamma_i, M) := Hom_{\Gamma_i}(C_\bullet(\Gamma_i), M)$$

and

$$R\Gamma^\bullet(K^p I, M) := \oplus_i C^\bullet(\Gamma_i, M)$$

$R\Gamma^j(K^p I, M)$ is isomorphic to finitely many copies of M and

$$h^j(R\Gamma^\bullet(K^p I, M)) = H^j(S_G(K), M).$$

We have functoriality for $R\Gamma^\bullet(K, M)$, (see [1, 4.2]). Suppose $\varphi : K' \to K$ is a group morphism bertween two open compact subgroups of $G(\mathbb{A}_f)$ and $\varphi^\# : M \to M'$ is a morphism between K-module M and K'-module M', satisfying $\varphi^\#(\varphi(k')m) = k'\varphi^\#(m)$ for any $k' \in K'$ and $m \in M$, then the pair $(\varphi, \varphi^\#)$ induces a morphism $\varphi^\bullet : R\Gamma^\bullet(K, M) \to R\Gamma^\bullet(K', M')$, which is defined up to homotopy.

Now write $f = f^p \otimes u_t \in \mathscr{H}_p(K^p)$ such that it is projected to $\mathscr{U}_p$ with image in $\mathbb{Z}_p[T^+/T(\mathbb{Z}_p)]$ is t. For $t \in T^+$, it induces a morphism $R\Gamma(t)$ on $R\Gamma^\bullet(K.M)$ as follow:

$$R\Gamma^\bullet(K, M) \to R\Gamma^\bullet(tKt^{-1}, M) \to R\Gamma^\bullet(K \cap tKt^{-1}, M) \to R\Gamma^\bullet(K, M)$$

where the first map is given by pair $(ad(t^{-1}), m \mapsto t * m)$, the second is given by the restriction map from $K \cap tKt^{-1}$ to tKt^{-1}, and the last one is given by the corestriction as writing

$$K = \sqcup_j k_j.(K \cap tKt^{-1}).$$

We can check that $R\Gamma(t_1) \circ R\Gamma(t_2) = R\Gamma(t_1 t_2)$, so this deduces an action of the Hecke algebras on $R\Gamma^\bullet(K^p I, M)$.

Remark. (*a*) The Hecke action on $R\Gamma^\bullet(K^pI, M)$ defined above (up to homotopy) indeces an action on the cohomology. This is coincident with the usual Hecke action on the cohomology.

(*b*) The morphism on the complex induced by the pair $(\varphi, \varphi^\#)$ preserve the properties of $\varphi^\#$. For our application, if $t \in T^{++}$, then f acts on $R\Gamma^\bullet(K^pI, \mathscr{D}_\lambda(L))$ as a compact operator. This is from proposition 3.2 and the fact that every $R\Gamma^q(K, M)$ is a finite copy of M.

(*c*) If $M = \mathscr{D}_\lambda$, with $\lambda \in \mathscr{U}$, then, by proposition 2.3, such a resolution can be viewed as the specialization of $R\Gamma^\bullet(K^pI, \mathscr{D}_\mathscr{U})$ at λ. Moreover, for $\mathscr{U}' \subset \mathscr{U}$, we have the natural restriction morphism $\mathscr{O}(\mathscr{U}) \to \mathscr{O}(\mathscr{U}')$, then we can view $R\Gamma^\bullet(K^pI, \mathscr{D}_{\mathscr{U}'})$ as the restriction of $R\Gamma^\bullet(K^pI, \mathscr{D}_\mathscr{U})$. All these restriction are compatible with the action of Hecke operators.

3.1.2

If $\lambda \in \mathfrak{X}^\iota$, we have a pair $\iota : K \to K$ and $\iota : M \to M$.

Lemma 3.2. *For $g \in I$, $x \in M$, where $M = W^\vee_{\lambda^{alg}}(L)$, $\mathscr{W}^\vee_\lambda(L)$, $\mathscr{D}_\lambda(L)$ or $\mathscr{D}_\mathfrak{U}(L)$. Then*

$$g * x^\iota = (g^\iota * x)^\iota$$

Therefore, by the functoriality, ι induces an morphism on $R\Gamma^\bullet(K, M)$.

The proof is straightforward from the definition.

Now let's consider the semi-product ${}^\iota\mathscr{H}_p(K^p) := \mathscr{H}_p(K^p) \rtimes <\iota>$. When we identify $\mathscr{H}_p(K^p)$ and $<\iota>$ as subgroups of ${}^\iota\mathscr{H}_p(K^p)$, we denote the products of $f \in \mathscr{H}_p(K^p)$ and ι by $f \times \iota$ and $\iota \times f$.

Lemma 3.3. *We can extend the actions of $\mathscr{H}_p(K^p)$ and ι on $R\Gamma^\bullet(K, M)$ to an action of ${}^\iota\mathscr{H}_p(K^p)$.*

Proof. We only have to check that the actions of $\mathscr{H}_p(K^p)$ and ι on $R\Gamma^\bullet(K, M)$ are compatible with the identity $\iota \times f = f^\iota \times \iota$ which is given by the definition of semi-product ${}^\iota\mathscr{H}_p(K^p) := \mathscr{H}_p(K^p) \rtimes <\iota>$. That is, we have to check that as morphisms on $R\Gamma(K^pI, M)$,

$$\iota \circ R\Gamma(t) \circ \iota^{-1} = R\Gamma(t^\iota)$$

This follows from the definition of the fuctoriality and the fact that ι permutes the set $\{k_j\}$ as defined in 3.1.1. □

We will focus on the operators $\iota \times f$ with $f = f^p \otimes u_t$, $t \in T^{++}$. A Hecke operator f in this form is called admissible, and the subalgebra of $\mathscr{H}_p$ generated by admissible opertors is denoted by $\mathscr{H}_p'$. Notice that the operator $\iota \times f$ is compact for every admissible f.

3.2 twisted action on finite slope cohomology

In this section we work on the distribution spaces $\mathscr{D}_\lambda(L)$, with L a finite extension of $\mathbb{Q}_p$ and $\lambda \in \mathfrak{X}$. We use the notations and definitions of compact operators on p-adic Frechet spaces and finite slope decompositions following [5] and [1]. The lemma below ensures us to consider the finite slope part of the cohomology $H^\bullet(S_G(K^pI), \mathscr{D}_\lambda(L))$ defined by admissible Hecke operators .

Lemma 3.4. *Let A be a $\mathbb{Q}_p$-Banach algebra, M a compact projective Frechet A-module, and f a compact A-linear operator of M. Then the Fredholm determinant $R(f, X)$ is entire over A. Assume that $R(f, X) = Q(X)S(X)$ over A with Q and S relatively prime and Q is a Fredholm polynomial with invertible leading coefficient, then we have a decomposition of M*

$$M = N_f(Q) \oplus F_f(Q)$$

of close submodules satisfing:

(a) $Q^*(f)$ annihilates $N_f(Q)$ and is invertible on $F_f(Q)$

(b) the projector on $N_f(Q)$ is given by $E_Q(f)$ with $E_Q(X) \in XA\{\{X\}\}$ whose coefficients are polynomials of those of Q and S.

Moreover, if A is semi-simple, then $N_f(Q)$ is of finite rank, and the characteristic polynomial of f on $N_f(Q)$ is Q. In particular, for $h \in \mathbb{Q}$, we may choose $Q(x)$ such that $N_f(Q) = M^{\leq h}$, the $\leq h$-slope decomposition of M.

Proof. By the definition of projective compact Frechet space we can write $M = \varprojlim M_n$ with M_n are projective A-banach modules. And for n sufficiently large $R(f, X) = det(1 - Xf|M_n)$. The proposition is known for projective Banach modules due to Serre and Coleman, see [12] and [5]. So we have such decompositions $M_n = N_{n,f}(Q) \oplus F_{n,f}(Q)$ for those n. Then take the projective limit, we have the decomposition $M = N_f(Q) \oplus F_f(Q)$. Actually, by the definition of a compact operator, we can prove that for n sufficiently large, $N_{n,f}(Q)$ are isomorphic, so the last statement follows. □

Now we fix $f = f_p \otimes u_t \in R_{\mathscr{S},p}$ admissible with $t \in T^{++}$. So f defines a compact operator on the cohomology groups $H^\bullet(S_G(K^pI), \mathscr{D}_\lambda(L))$, so for any $h \in \mathbb{Q}_+$, we can consider the slope less than h part of of the cohomology, denoted by $H^\bullet(S_G(K^pI), \mathscr{D}_\lambda(L))^{\leq h}$ and the finite slope part of $H^\bullet(S_G(K^pI), \mathscr{D}_\lambda(L))$ by $H^\bullet_{fs}(S_G(K^pI), \mathscr{D}_\lambda(L))$ which is defined by $\varinjlim H^\bullet(S_G(K^pI), \mathscr{D}_\lambda(L))^{\leq h}$. Since $R_{\mathscr{S},p}$ is in the center of the Hecke algebra, $H^\bullet_{fs}(S_G(K^pI), \mathscr{D}_\lambda(L))$ is defined independent of the choice of u_t. We have

Lemma 3.5. *ι is well defined on $H^\bullet_{fs}(S_G(K^pI), \mathscr{D}_\lambda(L))$, so the action of $R_{\mathscr{S},p}$ on $H^\bullet_{fs}(S_G(K^pI), \mathscr{D}_\lambda(L))$ extends to an action of ${}^\iota R_{\mathscr{S},p}$.*

To prove this lemma we need the lemma below, which is [1, lemma 2.3.2]

Lemma 3.6. *Let M, M' be two L-Banach (or Frechet) spaces, u and u' be two endomorphism of M and M' respectively. Let $M = M_u^{\leq h} \oplus M_1$ and $M' = M'^{\leq h}_{u'} \oplus M'_1$ be their $\leq h$-slope decompositions respectively. Now suppose f is a continuous linear map from M to M' such that $f \circ u = u' \circ f$, then f respects the slope decompositions.*

Proof. Since we have $\iota \times f = f^\iota \times \iota$, by the lemma above we know

$$\iota : H^q_{fs}(S_G(K^pI), \mathscr{D}_\lambda(L))^{\leq h}_{f^\iota} \to H^q_{fs}(S_G(K^pI), \mathscr{D}_\lambda(L))^{\leq h}_{f}$$

Assume that ι has order l, we have

$$\iota : \bigcap_{i=1}^{l} H^q_{fs}(S_G(K^pI), \mathscr{D}_\lambda(L))^{\leq h}_{f^{\iota^i}} \to \bigcap_{i=1}^{l} H^q_{fs}(S_G(K^pI), \mathscr{D}_\lambda(L))^{\leq h}_{f^{\iota^i}}$$

Now take the inductive limit with respect to h, since the finite slope part is not dependent on f, we get the lemma. □

From now on, if it is necessary, we will denote the action of ${}^{\iota}\mathscr{H}_p(K^p)$ on the cohomology groups by $*$.

For later use we need some comparsion results between the cohomolgy of $\mathscr{D}_\lambda(L)$ and of $W^\vee_{\lambda^{alg}}(L)$, this may also been viewed as a classicity result. The idea is traced back to Hida, and our version is a twisted analogue to Urban [1, proposition 4.3.10]. The point is, since we need involve the twisted action of ι, we have to consider $H^q_{fs}(S_G, -)^{\leqslant h}_\iota$ instead. However, the proof is of no difference.

For any $t \in T^+$, and $\lambda \in X^*(T)^+$, we denote

$$N(\lambda, t) := \inf_{w \neq id} |t^{w*\lambda-\lambda}|_p$$

and

$$N^\iota(\lambda, t) := \inf_{i=1}^{l} N(\lambda, t^{\iota^i})$$

Proposition 3.7. *Let $\lambda = \lambda^{alg}\epsilon$ be an arithmetic weight of conductor p^{n_λ}, μ a non-critiacal slope with respect to λ^{alg}. Then for any positive integer $m \geqslant n_\lambda$, we have canonically*

$$H^\bullet(S_G(K), \mathscr{D}_\lambda(L))^{\leqslant\mu}_\iota \cong H^\bullet(S_G(K), W^\vee_\lambda(L)(\epsilon))^{\leqslant\mu}_\iota \tag{3.2.0.1}$$

Similarly, for any $h \leqslant v_p(N^\iota(\lambda, t))$ and $f = f^p \otimes u_t$ with $t \in T^{++}$, then

$$H^\bullet(S_G(K), \mathscr{D}_\lambda(L))^{\leqslant h}_\iota \cong H^\bullet(S_G(K), \mathscr{W}^\vee_{\lambda^{alg}}(L)(\epsilon))^{\leqslant h}_\iota \tag{3.2.0.2}$$

3.3 self-dual representations

An irreducible representation σ of $\mathscr{H}_p$ over a p-adic field $L/\mathbb{Q}_p$ is called p-adic finite slope automorphic of weight $\lambda \in \mathfrak{X}$ and of level K^pI if it appears as a subquotient of $H^q_{fs}(S_G(K), \mathscr{D}_\lambda(L))$ for some integer q. Since $R_{\mathscr{S},p}$ is commutative, the restriction of σ to $R_{\mathscr{S},p}$ is a character, which is denoted by θ_σ.

Now for σ an irreducible automorphic representation of $\mathscr{H}_p(K^p)$, define the twisted action of

σ by ι as σ^ι, which sends $f \in \mathscr{H}_p$ to $\sigma^\iota(f) := \sigma(f^\iota)$. σ is called self-dual with respect to ι if $\sigma^\iota \cong \sigma$. It is easy to see that σ is self-dual with respect to ι if and only if σ can be extended to the semi-product $\mathscr{H}_p \rtimes < \iota >$, since $\iota \times f \times \iota^{-1} = f^\iota$ (see, e.g. [9, Appendix]). Concretely speaking, if $\sigma^\iota \cong \sigma$, we can find an operator $A : V_\sigma \to V_\sigma$ of order l such that $A \circ \sigma = \sigma \circ A$. Then we define the action of ι^i on V_σ by A^i. We denote such an action of ${}^\iota\mathscr{H}_p(K^p)$ extending σ by $\tilde{\sigma}$, and its restriction to ${}^\iota R_{\mathscr{S},p}$ by $\tilde{\theta}_\sigma$. It is easy to see that $\tilde{\sigma}$ is defined up to a character of order i of $< \iota >$. In general, we denote an irreducible representation of ${}^\iota\mathscr{H}_p(K^p)$ (resp. of ${}^\iota R_{\mathscr{S},p}$) by Σ (resp. by Θ). We recall some results in [9, Appendix] in the next lemma:

Lemma 3.8. (a) *Let Σ be irreducible, then its restriction to $\mathscr{H}_p(K^p)$ is irreducible if and only if the trace of Σ is not trivial on $\mathscr{H}_p(K^p) \times \iota$.*

(b) *Let σ be an irreducible automorphic representation of $\mathscr{H}_p(K^p)$, then there are exactly l extensions, say $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_l$, of σ to ${}^\iota\mathscr{H}_p(k^p)$, each two of them are differed by a character of order l and non-isomorphic.*

Now for an irreducible σ and $f \in \mathscr{H}_p(K^p)$, define

$$J_\sigma(f) := tr(\sigma(f))$$

and

$$J_{\tilde{\sigma}_i}(f) := tr(\tilde{\sigma}(f \times \iota))$$

Proposition 3.9. *Let $\lambda \in \mathfrak{X}^\iota(\overline{\mathbb{Q}}_p)$. For every irreducible self-dual finite slope representation σ of G with respect to ι, there are l integer $m_i^q(\sigma, \lambda)$ such that for all $f \in \mathscr{H}_p$,*

$$tr(\iota \times f; H^q_{fs}(S_G(K), \mathscr{D}_\lambda(L))) = \sum_{\sigma^\iota \cong \sigma} \sum_i m_i^q(\sigma, \lambda) J_{\tilde{\sigma}_i}(f)$$

Proof. We have as $\mathscr{H}_p(K^p)$ modules, without counting the multiplicity,

$$H^q_{fs}(S_G(K), \mathscr{D}_\lambda(L)) = \bigoplus_\sigma V_\sigma = \bigoplus_A V_\sigma \bigoplus_B V_\sigma \bigoplus_C V_\sigma$$

where $A = \{\sigma | V_\sigma^{*\iota} = V_\sigma\}$, $B = \{\sigma^\iota \cong \sigma | V_\sigma^{*\iota} \cap V_\sigma = \emptyset\}$ and $C = \{\sigma | \sigma^\iota \ncong \sigma\}$.

If $\sigma \in B$ or C, consider $W_\sigma = \oplus V_\sigma^{*\iota}$, then we see that $*\iota$ permutes the components, so $tr(\iota \times f|W_\sigma)$ is trivial. Then for $\sigma \in B$, we define $m_i^q(\sigma, \lambda) = 0$.

If $\sigma \in A$, (then in particular that σ is self-dual). There must be an irreducible representation Σ of ${}^\iota\mathscr{H}_p(K^p)$ such that $\Sigma = \tilde{\sigma}_i$ for some i the action of ι compatible with $*\iota$. Now for each i, define $m_i^q(\sigma, \lambda)$ be the multiplicity of Σ in A.

With the above setting, the proposition follows directly. □

Remark. (a) We see in the proof of the proposition that not every trace of a self-dual cohomological automorphic representation could appear in the trace of $\iota \times f$ on the finite slope cohomlogy, but only those as restriction of irreducible "cohomological" automorphic representations of ${}^\iota\mathscr{H}_p(K^p)$ (i.e. those $\sigma \in A$).

(b) However, if G admits multiplicity one theorem, (for example, luckily, $G = Gl_n$),then we have $B = \emptyset$ (otherwise since σ is self-dual, that $V_\sigma \cong V_{\sigma^\iota} \cong V_\sigma^{*\iota}$ would appear with multiplicitiy at least two).

Chapter 4

THE FULL EIGENVARIETIES

4.1 spectral variety, local pieces

Let $\mathscr{U}$ be an open affinoid subdomain of $\mathfrak{X}$, $K = K^pI$ as above and $f = f^p \otimes u_t \in R_{\mathscr{S},p}$ with $t \in \Delta^{++}$. We denote

$$R\Gamma(K^pI, \mathscr{D}_{\mathscr{U}}) = \oplus_j R\Gamma^j(K^pI, \mathscr{D}_{\mathscr{U}}).$$

By the theory of completely continuous operator on compact Frechet spaces, we define the Fredholm series $R_{\mathscr{U}}(f, \lambda, X) := \prod_j R^j_{\mathscr{U}}(f, \lambda, X)$, where for $? = \emptyset$ or j , $R^?_{\mathscr{U}}(f, \lambda, X) \in \mathscr{O}(\mathscr{U})^\circ\{\{X\}\}$ is the Fredholm determinant of f acting on $R\Gamma^?(K^pI, \mathscr{D}_{\mathscr{U}})$. By proposition 2.3 and remark 3.1.1, for fixed $\lambda \in \mathscr{U}$, $R^?_{\mathscr{U}}(f, \lambda, X)$ is the Fredholm determinant of f acting on $R\Gamma^?(K^pI, \mathscr{D}_\lambda)$.

We define a power series $R_{\mathfrak{X}}(f, \lambda, X) \in \mathscr{O}(\mathfrak{X})\{\{X\}\}$ such that for any open affinoid $\mathscr{U} \subset \mathfrak{X}$, $R_{\mathscr{U}}(f, \lambda, X)$ is the restriction of $R_{\mathfrak{X}}(f, \lambda, X)$ via the canonical map $\mathscr{O}(\mathfrak{X}) \to \mathscr{O}(\mathscr{U})$. This works by remark 3.1.1.

Let $\mathfrak{J}'(f)$ be the Fredholm variety in $\mathfrak{X} \times \mathbb{A}_1^{rig}$ cut out by $R_{\mathfrak{X}}(f, \lambda, X)$. The projector of $\mathfrak{J}'$ onto $\mathfrak{X}$ is flat. Then $(\lambda, \alpha) \in \mathfrak{J}'(f)(\bar{\mathbb{Q}}_p)$ if and only if α^{-1} is an eigenvalue of f acting on $R\Gamma(K^pI, \mathscr{D}_\lambda)$. We are going to describe the subvariety of $\mathfrak{J}'(f)$ such that those α are actually eigenvalues of the cohomology. To do so, we need apply the Lemma 2.1.

Now write $R_{\mathscr{U}}(f, \lambda, X) = Q_f(X)S_f(X)$ with series $Q_f(X)$ and $S_f(X)$ as in Lemma 3.3. This gives a decomposition of $\mathscr{O}(\mathscr{U})$-modules

$$R\Gamma(K^pI, \mathscr{D}_{\mathscr{U}}) = N_f(Q) \oplus F_f(Q).$$

By the proposition above, $N_f(Q)$ is torsion free of finite rank over $\mathscr{O}(\mathscr{U})$. Denote

$$\mathfrak{S}'(Q) = sp(\mathscr{O}(\mathscr{U})[X]/Q^*(X)),$$

which is an affinoid subdomain of $\mathfrak{J}'(f)$. By Buzzard [8], the collection of $\mathfrak{S}'(Q)$ for all $\mathscr{U}$ and Q is an admissible cover of $\mathfrak{J}'(f)$.

For any j, set

$$N_f^j(Q) := N_f(Q) \cap R\Gamma^j(K^pI, \mathscr{D}_{\mathscr{U}}).$$

Since that $v \in N_f(Q)$ if and only if $Q^*(f)(v) = 0$, $N_f^\bullet(Q)$, is a bounded complex of projective finite rank modules over $\mathscr{O}_{\mathscr{U}}$. If write $N_f^\bullet(Q)_\lambda$ as its specialization at λ, then $H^j(N_f^*(Q)) \hookrightarrow H^j(S_G(K), \mathscr{D}_{\mathscr{U}})$ and $H^j(N_f^*(Q)_\lambda) \hookrightarrow H^j(S_G(K), \mathscr{D}_\lambda)$. As the standard notation, we denote $H = \oplus H^j$.

4.1.1 cohomological non-trivial weight space

We call a weight $\lambda \in \mathfrak{X}(\overline{\mathbb{Q}}_p)$ cohomological non-trivial if $H(S_G(K), \mathscr{D}_\lambda)$ is non-trivial. Our results in this subsection reply on the next important lemma.

Lemma 4.1. *Let A be a noetherian regular domain, $M^\bullet$ a bounded complex of projective finite rank A-modules and $\mathfrak{P}$ a maximal ideal of A. Then*

$$H(M^\bullet/\mathfrak{P}) \neq 0 \Leftrightarrow \mathfrak{P} \in supp_A(H(M^\bullet)).$$

Proof. Since M is finitely generated and projective, the "$\Leftarrow$" direction follows from standard commutative algebra, see, for example, Eisenbud [15, theorem19.2]. So we only need to prove the "$\Rightarrow$" direction. Since A is regular, write $\mathfrak{P} = (x_0, \ldots, x_l)$ with $\{x_0, \ldots, x_l\}$ a regular sequence in A. Since M^* is torsion free, $\{x_0, \ldots, x_l\}$ is also M^*-regular. We now prove the "$\Rightarrow$" part by induction on l.

Assume $\mathfrak{P} = (x_0)$, consider the short exact sequence of complex:

$$0 \to M^\bullet \to M^\bullet \to M^\bullet/\mathfrak{P} \to 0$$

where the first map is given by $\times x_0$. Localize the exact sequence at $\mathfrak{P}$ and take the long exact sequence, we have

$$0 \to H^j(M^\bullet)_\mathfrak{P}/\mathfrak{P} \to H^j(M^\bullet/\mathfrak{P}) \to H^{j+1}(M^\bullet)_\mathfrak{P} \to$$

So $H(M^\bullet/\mathfrak{P}) \neq 0$ implies either $H^j(M^\bullet)_\mathfrak{P}$ or $H^{j+1}(M^\bullet)_\mathfrak{P}$ is not trivial.

Now assume the proposition is true if $\mathfrak{P}$ is generated by a regular sequence of l elements, we need to prove that it is also true for the case $\mathfrak{P} = (x_0, \cdots x_l)$. denote the ideal $(x_0, \cdots x_{l-1})$ by $\mathfrak{P}'$, consider the short exact sequence

$$0 \to M^\bullet/\mathfrak{P}' \to M^\bullet/\mathfrak{P}' \to M^\bullet/\mathfrak{P} \to 0$$

where the first map is given by $\times x_l$. Take long exact sequence and localize at $\mathfrak{P}$ we get $H(M^\bullet_\mathfrak{P}/\mathfrak{P}') \neq 0$. Apply the induction assumption to the $A_\mathfrak{P}$-modules $M^\bullet_\mathfrak{P}$, we have $H(M^\bullet)_\mathfrak{P} \neq 0$.

□

Assuming that λ is cohomological non-trivial and considering all open affinoid neighborhoods of λ and polynomial decompositions over them, we know from the lemma above that there exists $\mathscr{U}$ containing λ and Q in a polynomial decomposition such that $H(N^\bullet_f(Q)) \neq 0$.

Fix $\mathscr{U}$ and Q as above. Since $H(N^\bullet_f(Q))$ is a $\mathscr{O}(\mathscr{U})$-module of finite rank, as in [13], we can associate to it a coherent sheaf, denoted by $H(\widetilde{N^\bullet_f(Q)})$, over $\mathscr{U}$. We denote

$$\mathfrak{W}_Q = supp_{\mathscr{O}(\mathscr{U})} H(\widetilde{N^\bullet_f(Q)})$$

$\mathfrak{W}_Q$ is a rigid subspace of $\mathscr{U}$.

Proposition 4.2. *$\lambda \in \mathfrak{W}_Q(\overline{\mathbb{Q}}_p)$ if and only if $H(N^\bullet_f(Q)_\lambda) \neq 0$.*

The proposition is obtained immediately by applying the lemma above to the complex $N_f^\bullet(Q)$ over $\mathscr{O}(\mathscr{U})$. So $\mathfrak{W}_Q$ can be viewed as local pieces of the cohomological non-trivial weight space. We will construct spectral variety locally over $\mathfrak{W}_Q$.

4.1.2 local spectral varieties

We could further consider $N_f^\bullet(Q)$ as a complex of $\mathscr{O}(\mathscr{U})[X]$-modules, with acting of X via f^{-1}. So $H(N_f^\bullet(Q))$ is a finite generated $\mathscr{O}(\mathscr{U})[X]$-module. As in the last subsection, we can associated to it a coherent sheaf, denoted again by $H(\widetilde{N_f^\bullet(Q)})$, over $\mathscr{U} \times \mathbb{A}_1^{rig}$. We set

$$\mathfrak{S}_Q := supp_{\mathscr{O}(\mathscr{U})[X]} H(\widetilde{N_f^\bullet(Q)}) \subset \mathfrak{S}'(Q)$$

Proposition 4.3. *$\mathfrak{S}_Q$ is locally finite over $\mathfrak{W}_Q$. $s = (\lambda, \alpha) \in \mathfrak{S}_Q(\overline{\mathbb{Q}}_p)$ if and only if $\lambda \in \mathfrak{W}_Q(\overline{\mathbb{Q}}_p)$ and α^{-1} is an eigenvalue of f acting on $H(N_f^\bullet(Q)_\lambda)$.*

Proof. We only need to prove the second statement. To simplify the notation, we denote $\mathscr{O}(\mathscr{U}) := A$. For $s = (\lambda, \alpha) \in \mathscr{U} \times \mathbb{A}^{rig}(\overline{\mathbb{Q}}_p)$, set $\mathfrak{P}_\lambda$ be the maximal ideal of A associated to λ, k_λ the residue field, and J_s the maximal ideal of $A[X]$ associated to s. Consider the following diagram:

$$A \to A[X] \to k_\lambda[X] \to l[X]$$

where l is some finite extension of k_λ in $\overline{\mathbb{Q}}_p$ containing α. Then $\mathfrak{P}_\lambda$ is the pull back of J_s. Denote the $\mathfrak{a}$ be the pull back ideal of $(X - \alpha)$ in $k_\lambda[X]$, then J_s is the pull back of $\mathfrak{a}$. Now the proposition follows from a similar discussion as in the proof of lemma 4.1 and the next trivial lemma:

Lemma 4.4. *Let M be a finite k-vector space, with a k-linear operator f. Consider M as a $k[X]$-module with X acting via f^{-1}. Then for any prime ideal $\mathfrak{p}$ of $k[X]$, $M_\mathfrak{p} \neq 0$ if and only if $det(1 - Xf) \in \mathfrak{p}$.*

□

$\mathfrak{S}_Q$ is the local spectral variety we want, in one word, we have the following diagram:

$$\begin{array}{ccc} \mathfrak{S}_Q & \longrightarrow & \mathfrak{S}'(Q) \\ \downarrow & & \downarrow \\ \mathfrak{W}_Q & \longrightarrow & \mathscr{U} \end{array}$$

where $\mathfrak{W}_Q$ parameters cohomological non-trivial weights and $\mathfrak{S}_Q$ parameters eigenvalues.

Remark. Recall that we chose a resolution in lemma 3.1 to compute the cohomology of the arithmetic groups. By proposition 4.3, our construction of spectral variety does not depend on this choice.

4.2 gluing the local pieces

Firstly, for a fixed open subaffinoid subdomain $\mathscr{U} \subset \mathfrak{X}$, we glue all $\mathfrak{S}_Q$ with Q in some prime decomposition $R_{\mathscr{U}}(f, \lambda, X) = Q(X)S(X)$ to get a spectral variety $\mathfrak{S}_{\mathscr{U}}$.

Assume $Q(X) = Q'(X)S'(X)$, such that Q' and S' are relatively prime polynomials. By definition we have

$$\mathfrak{S}'(Q') \hookrightarrow \mathfrak{S}'(Q)$$

is a subvariety. Apply proposition 6.1 to $N_f(Q)$ with respect to $Q(X) = Q'(X)S'(X)$, we have decomposition

$$N_f(Q) = N_f(Q') \oplus N_f(S'),$$

and

$$H^j(N_f^\bullet(Q)) = H^j(N_f^\bullet(Q')) \oplus H^j(N_f^\bullet(S')).$$

These decompositions are compatible with the action of f, so there is a natural embedding

$$\mathfrak{S}_{Q'} \hookrightarrow \mathfrak{S}_Q.$$

Remark. Considering $R_{\mathscr{U}}(f, \lambda, X) = Q'(X)S(X)$, we get another closed subspace $N_f(Q')$ by Lemma

3.3. It is not hard to see that this coincides to our notation above.

To glue those $\mathfrak{S}_Q$, we only have to show:

Proposition 4.5. *The next diagram is a Cartesian product:*

$$\begin{array}{ccc} \mathfrak{S}_{Q'} & \longrightarrow & \mathfrak{S}'(Q') \\ \downarrow & & \downarrow \\ \mathfrak{S}_Q & \longrightarrow & \mathfrak{S}'(Q) \end{array}$$

Proof. We only have to show the restriction of $supp_{\mathscr{O}(\mathscr{U})[X]}H(\widetilde{N_f^\bullet(Q)})$ to $\mathfrak{S}'(Q')$ is $supp_{\mathscr{O}(\mathscr{U})[X]}H(\widetilde{N_f^\bullet(Q')})$. To show this, we only have to show for a maximal ideal $\mathfrak{P}$ in $\mathscr{O}(\mathscr{U})[X]$ containing $Q'^*(X)$, $H(N_f^\bullet(Q)_\mathfrak{P}) \neq 0$ implies $H(N_f^\bullet(Q')_\mathfrak{P}) \neq 0$. This is true since Q' and S' are relatively prime and $H^j(N_f^\bullet(Q)) = H^j(N_f^\bullet(Q')) \oplus H^j(N_f^\bullet(S'))$. □

Now we show that all $\mathfrak{S}_{\mathscr{U},Q}$ can be glued together to get a spectral variety $\mathfrak{S}_\mathfrak{X}$ over $\mathfrak{X}$.

Proposition 4.6. *Let $\mathscr{U}' \hookrightarrow \mathscr{U}$ be open affinoid subdomains of $\mathfrak{X}$, then*

$$\mathfrak{S}_{\mathscr{U},Q} \times \mathscr{U}' = \mathfrak{S}_{\mathscr{U}',Q_{\mathscr{U}'}}$$

where $Q_{\mathscr{U}'}$ is the restriction of Q to $\mathscr{U}'$ via $\iota : \mathscr{O}(\mathscr{U}) \to \mathscr{O}(\mathscr{U}')$.

Proof. As remark 3.1.1,

$$R\Gamma^?(K^pI, \mathscr{D}_\mathscr{U}) \otimes_{\mathscr{O}(\mathscr{U})} \mathscr{O}(\mathscr{U}') = R\Gamma^?(K^pI, \mathscr{D}_{\mathscr{U}'})$$

Since $\mathscr{O}(\mathscr{U}) \to \mathscr{O}(\mathscr{U}')$ is flat, after tensoring $\mathscr{O}(\mathscr{U}')$, we have

$$R\Gamma^?(K^pI, \mathscr{D}_{\mathscr{U}'}) = (N_f^?(Q)_\mathscr{U} \otimes \mathscr{O}(\mathscr{U}')) \oplus (F_f^?(Q)_\mathscr{U} \otimes \mathscr{O}(\mathscr{U}')).$$

On the other hand, we have

$$R\Gamma^?(K^pI, \mathscr{D}_{\mathscr{U}'}) = N_f^?(Q)_{\mathscr{U}'} \oplus F_f^?(Q)_{\mathscr{U}'}.$$

It follows from Lemma 3.3, the projection from $R\Gamma(K^pI, \mathscr{D}_{\mathscr{U}'})$ to $N_f(Q)_{\mathscr{U}'}$ is $E_{Q_{\mathscr{U}'}}(f)$ and to

$N_f(Q)_{\mathscr{U}} \otimes \mathscr{O}(\mathscr{U}')$ is $\iota E_Q(f)$. This two actually are same, so

$$N_f(Q)_{\mathscr{U}'} = N_f(Q)_{\mathscr{U}} \otimes \mathscr{O}(\mathscr{U}').$$

Again since ι is flat, we have

$$H(N_f^\bullet(Q)_{\mathscr{U}}) \otimes_{\mathscr{O}(\mathscr{U})} \mathscr{O}(\mathscr{U}') = H(N_f^\bullet(Q)_{\mathscr{U}'}).$$

This implies the proposition. □

As a summary of the results of this and the last section, we have:

Theorem 4.7. *Let $f = f^p \otimes u_t$, with $t \in \Delta^{++}$, we construct $\mathfrak{S}_{\mathfrak{X}} = \mathfrak{S}_{\mathfrak{X}}(f)$ as above. For $s = (\lambda, \alpha) \in \mathfrak{X} \times \mathbb{A}_1^{rig}(\overline{\mathbb{Q}}_p)$, $s \in \mathfrak{S}_{\mathfrak{X}}(\overline{\mathbb{Q}}_p)$ if and only if α^{-1} is an eigenvalue of f acting on $H(S_G(K), \mathscr{D}_\lambda)$.*

4.3 eigenvarieties

In this section we construct the full eigenvariety over the spectral variety that is constructed in last two sections, the method is standard [1] [8]. Let $K = K^p I$, $\mathscr{S}$ the finite set of primes as in section 2.1. As in [1, 5.2.1], there is a p-adic space $\mathfrak{R} := \mathfrak{R}_{\mathscr{S},p}$ such that for any finite extension L of $\mathbb{Q}_p$

$$\mathfrak{R}_{\mathscr{S},p}(L) = Hom_{ct,alg}(\overline{R}_{\mathscr{S},p}, L),$$

where $\overline{R}_{\mathscr{S},p}$ is the p-adic completion of $R_{\mathscr{S},p}[u_t^{-1}, t \in T^+]$. Our eigenvariety will be in the p-adic space $\mathfrak{Y} = \mathfrak{Y}_{\mathscr{S},p} := \mathfrak{X} \times \mathfrak{R}$, which has a natural projection to the weight space $\mathfrak{X}$.

Using the notation of Urban, call $f \in R_{\mathscr{S},p}$ admissible if $f = f^p \otimes u_t$, with $t \in T^{++}$. For any admissible f, there is a morphism between ringed spaces $R_f : \mathfrak{Y} \to \mathfrak{X} \times \mathbb{A}_1^{rig}$ given by

$$\mathfrak{Y}(\overline{\mathbb{Q}}_p) \to \mathfrak{X} \times \mathbb{A}_1^{rig}(\overline{\mathbb{Q}}_p)$$

via $(\lambda, \theta) \mapsto (\lambda, \theta(f)^{-1})$, and

$$R_f^* : \mathscr{O}(\mathfrak{X})\{\{X\}\} \to \mathscr{O}(\mathfrak{X})\hat{\otimes}\overline{R}_{\mathscr{S},p}$$

via $X \mapsto f^{-1}$.

Now define

$$\mathfrak{E}_{K^p} := \prod_f R_f^{-1}(\mathfrak{S}_{\mathfrak{X}}(f))$$

with f running over all adimissible operators in $R_{\mathscr{S},p}$. For an admissible f, we extend R_f by $\mathfrak{E}_{K^p} \to \prod_f \mathfrak{S}(f) \to \mathfrak{S}(f)$. Via this we can define a G-topology on $\mathfrak{E}_{K^p}$ to make it a rigid space. Concretely speaking, an open set in $\mathfrak{E}_{K^p}$ is said to be admissinle open if it is union or intersection of open subsets of the form $(R_{f_1} \times \cdots \times R_{f_n})^{-1}(\mathfrak{U})$, where f_i are admissible operators and $\mathfrak{U}$ is an admissible open subset of $\mathfrak{S}(f_1) \times \ldots \mathfrak{S}(f_n)$ (also see [1]). It is clear that

$$\mathfrak{E}(\overline{\mathbb{Q}}_p) = \cap R_f^{-1}(\mathfrak{S}(f)(\overline{\mathbb{Q}}_p)).$$

The next proposition shows that $\mathfrak{E}_{\mathscr{S},p}$ can be viewed as the full eigenvariety since its points do paramtrize all overconvergent cohomological eigensystems.

Proposition 4.8. *Let $K = K^pI$ be as above, $y = (\lambda, \theta) \in \mathfrak{Y}(\overline{\mathbb{Q}}_p)$. Then $y \in \mathfrak{E}(\overline{\mathbb{Q}}_p)$ if and only if $H(S_G(K), \mathscr{D}_\lambda)[\theta] \neq 0$. Moreover, if $y \in \mathfrak{E}(\overline{\mathbb{Q}}_p)$, then there exists an admissible f such that*

$$R_f^{-1}(R_f(y)) \cap \mathfrak{E}(\overline{\mathbb{Q}}_p) = \{y\}.$$

Proof. The first statement follows from theorem 4.7 directly. The second is essentially due to Coleman and Mazur (see [1] as well). Let $y = (\lambda_y, \theta_y) \in \mathfrak{E}(\overline{\mathbb{Q}}_p)$, and $f_t := 1_{K^p} \otimes u_t$ with $t \in T^{++}$, $h = v_p(\theta_y(f_t))$. By Ash and Stevens [3, theorem 6.2.1], there is an open affinoid neighborhood $\mathfrak{V} \ni \lambda_y$ such that $\mathscr{D}_{\mathfrak{V}}$ has h-slope decomposition with respect to f_t. Consider the polynomial decomposition $R_{\mathfrak{V}}(f_t, \lambda, X) = Q(X)S(X)$ corresponding to the h-slope decomposition, and the action of $R_{\mathscr{S},p}$ on complex $N^\bullet_{f_t}(Q)$ upto homotopy. We denote by $\mathscr{K}^b_{pf}(Ban_A)$ the triangulated category of perfect bounded complexes of Banach modules over A modulo homotopy. Let h_Q

be the image of $R_{\mathscr{S},p}$ in $End^b_{pf}(N^\bullet_{f_t}(Q)_{\lambda_y})$, which is defined over some finite extension L of $\mathbb{Q}_p$. Consider $f_1, \ldots, f_n$ in $R_{\mathscr{S},p}$ whose images form a system of generators of h_Q. Now we can repeat the standard arguement as in the proof of proposition 5.2.3 in [1]. □

Just like in [1, section 5.3], we have a second construction of the eigenvariety, in which we construct the local pieces of the full eigenvariety directly. Consider $\mathscr{U}$ and Q as in section 4.1 and 4.2, with respect to a fixed admissible f. We denote $h_{\mathscr{U}}$ be the image of $R_{\mathscr{U}} := R_{\mathscr{S},p} \otimes \mathscr{O}(\mathscr{U})$ in $End^b_{pf}(R\Gamma^\bullet(K^pI, \mathscr{D}_{\mathscr{U}}))$ and $h_{\mathscr{U},Q}$ be the image of $R_{\mathscr{U}}$ in $End^b_{pf}(N^\bullet_f(Q))$. Then clearly the restriction maps $h_{\mathscr{U}}$ surjectively to $h_{\mathscr{U},Q}$ with a kernel. Define

$$\mathfrak{E}'_{\mathscr{U},Q} := sp(h_{\mathscr{U},Q}).$$

Now we have a natural inclusion $\mathfrak{E}'_{\mathscr{U},Q}(\overline{\mathbb{Q}}_p) \hookrightarrow \mathscr{U} \times \mathfrak{R}_{\mathscr{S},p}(\overline{\mathbb{Q}}_p)$. It is not hard to see that $\mathfrak{E}'_{\mathscr{U},Q}(\overline{\mathbb{Q}}_p) \supset R_f^{-1}(\mathfrak{S}_{\mathscr{U},Q}(\overline{\mathbb{Q}}_p))$, indeed, by definition we know that (see, [13, §8]),

$$\mathfrak{S}_{\mathscr{U},Q} = sp(\mathscr{O}(\mathscr{U})[X]/ann(H(N^\bullet_f(Q))))$$

so $\mathscr{O}(\mathscr{U})[X]/Q^* \to h_{\mathscr{U},Q}$ factors through $\mathscr{O}(\mathscr{U})[X]/ann(H(N^\bullet_f(Q)))$.

So we want to find a subvariety of $\mathfrak{E}'_{\mathscr{U},Q}$ whose points are points of some local piece of the eigenvariety we have constructed above. Define:

$$\mathfrak{E}_{\mathscr{U},Q} = supp_{h_{\mathscr{U},Q}} H(\widetilde{N^\bullet_f(Q)})$$

Proposition 4.9.

$$\mathfrak{E}_{\mathscr{U},Q}(\overline{\mathbb{Q}}_p) = R_f^{-1}\mathfrak{S}_{\mathscr{U},Q}(\overline{\mathbb{Q}}_p) \subset (\mathfrak{E} \times \mathscr{U})(\overline{\mathbb{Q}}_p)$$

Proof. For one direction, we have $supp_{h_{\mathscr{U},Q}}(H(N^\bullet_f(Q))) \subset supp_{\mathscr{O}(\mathscr{U})[X]}(H(N^\bullet_f(Q)))$ via the map $\mathscr{O}(\mathscr{U})[X] \to h_{\mathscr{U},Q}$ as $X \mapsto f^{-1}$. For the other direction, let $\mathfrak{P}_\lambda$ be the maximal ideal of $h_{\mathscr{U}}$ corresponding to $y = (\theta, \lambda) \in R_f^{-1}\mathfrak{S}_{\mathscr{U},Q}(\overline{\mathbb{Q}}_p)$. Since $y \in \mathfrak{E}(\overline{\mathbb{Q}}_p)$, we know that $\mathfrak{P}_\lambda \in supp_{h_{\mathscr{U}}}(H(N^\bullet_f(Q)) \oplus H(N^\bullet_f(Q)))$. Since moreover $\theta(f)^{-1}$ happens in $h_{\mathscr{U},Q}$ acting on $H(N^\bullet_f(Q))$, that $\mathfrak{P}_\lambda \in supp_{h_{\mathscr{U},Q}} H(N^\bullet_f(Q))$. This completes the proof. □

The next two propositions show that we can patch $\mathfrak{E}_{\mathscr{U},Q}$ together.

Proposition 4.10. *Let $\mathscr{U}' \subset \mathscr{U}$ be two open affinoid subdomains of $\mathfrak{X}$, the the canonical surjective map $h_{\mathscr{U},Q} \otimes_{\mathscr{O}(\mathscr{U})} \mathscr{O}(\mathscr{U}') \to h_{\mathscr{U}',Q}$ is an isomorphism.*

Proof. Since $\mathscr{O}(\mathscr{U}) \to \mathscr{O}(\mathscr{U}')$ is flat and $N_f(Q)$ is finite presented. This follows from a standard commutative algebra result, see for example [15, proposition 2.10]. □

Now we assume a relative prime decomposition $Q(X) = Q'(X)S'(X)$ over $\mathscr{U}$ as in section 4.2. By the discussion there we have the canoncial inclusion $\mathfrak{S}_{Q'} \hookrightarrow \mathfrak{S}_Q$. We further have

Proposition 4.11. *The next diagram is a Cartesian product:*

$$\begin{array}{ccc} \mathfrak{E}_{\mathscr{U},Q'} & \longrightarrow & \mathfrak{E}_{\mathscr{U},Q} \\ \downarrow & & \downarrow \\ \mathfrak{S}_{\mathscr{U},Q'} & \longrightarrow & \mathfrak{S}_{\mathscr{U},Q} \end{array}$$

Proof. Write $A := \mathscr{O}(\mathscr{U})$, $ann_{A[X]}(H(N_f^\bullet(Q))) := \mathfrak{a}_Q$ and $ann_{h_{\mathscr{U}},Q}(H(N_f^\bullet(Q))) := \mathfrak{b}_Q$ we only have to show

$$(h_Q/\mathfrak{b}_Q) \otimes_{A[X]/\mathfrak{a}_Q} A[X]/\mathfrak{a}_{Q'} = h_{Q'}$$

As in the proof of proposition 4.6, we have $N_f^\bullet(Q) = N_f^\bullet(Q') \oplus N_f^\bullet(S')$ and $H(N_f^\bullet(Q)) = H(N_f^\bullet(Q')) \oplus H(N_f^\bullet(S'))$. So we have a natural surjection from $h_Q/\mathfrak{b}_Q \otimes_{A[X]/\mathfrak{a}_Q} A[X]/\mathfrak{a}_{Q'}$ to $h_{Q'}$ deduced from the projection from $N_f(Q)$ to $N_f(Q')$. Notice we defined $N_f(Q)$ as $A[X]$-module with X acting on via f, an element in $h_Q/\mathfrak{b}_Q \otimes_{A[X]/\mathfrak{a}_Q} A[X]/\mathfrak{a}_{Q'}$ maps to 0 must has coefficients in $\mathfrak{a}_{Q'}$, so is 0. This completes the proof. □

Combining propositions 4.9 to 4.11, we can patch $\mathfrak{E}_{\mathscr{U},Q}$ together to get a reduced rigid variety $\mathfrak{D}_{K^pI}$ whose points are the same to $\mathfrak{E}_{K^pI}(\overline{\mathbb{Q}}_p)$. This implies that $\mathfrak{D}_{K^pI}$ is the reduced closed subspace of $\mathfrak{E}_{K^pI}$. We summarize our results by the next theorem.

Theorem 4.12. *$\mathfrak{E}_{K^pI} \subset \mathfrak{Y}$ is a rigid space over $\mathbb{Q}_p$ of dimension $\leq dim\mathfrak{X}$, whose points are those $y = (\lambda_y, \theta_y)$ such that $H(S_G(K), \mathscr{D}_{\lambda_y})[\theta_y] \neq 0$. Moreover, for any $f \in R_{\mathscr{S},p}$, the projection $\mathfrak{E}_{K^pI} \to \mathfrak{S}_{\mathfrak{X},K^pI,f}$ is surjective and locally finite.*

Proof. The first statement is proposition 4.8, the surjectivity and local finiteness follow from the discussion above proposition 4.9. □

We can apply the results above to families of finite slope automorphic representations. It is not new and is well known from Ash-Stevens [3].

Let π_0 be a finite slope cuspidal automorphic representation of $G(\mathbb{A})$ occuring in $H^*(S_G(K), \mathbb{W}_{\lambda_0})$ for some K and λ_0. Let σ_0 be a p-stablization of π whose restriction on $R_{\mathscr{S},p}$ is denoted by θ_0, which is a finite slope Hecke eigensystem of weight λ_0. Now fix $h \in \mathbb{Q}^{>0}$, $f = f^p \otimes u_t$ with $t \in T^{++}$, assume that θ_0 is of slope less than h with respect to t. Then by [3, theorem 4.5.1], we can find an open affinoid neighborhood U of λ over which $R\Gamma(K^pI, \mathscr{D}_{\mathscr{U}})$ admits a $\leq h$-slope decomposition. This $\leq h$-slope decomposition then gives a polynomial decomposition $R_{\mathscr{U}}(f, \lambda, X) = Q^h(X)S^h(X)$, and,

Corollary 4.13. *There exists an open affinoid neighborhood $\mathscr{U} \subset \mathfrak{X}$ of λ_0 such that the open affinoid $\mathscr{V} = \mathfrak{E}_{U,Q^h}$ sitting over $\mathscr{U}$ with a homomorphism $\theta_{\mathscr{V}} : R_{\mathscr{S},p} \to \mathscr{O}(\mathscr{V})$ and $y_0 \in \mathscr{V}(\overline{\mathbb{Q}}_p)$ satisfying:*

(a) *the specialization of $\theta_{\mathscr{V}}$ at y_0 is θ_0.*

(b) *for every point $y \in \mathscr{V}$, the specialization of $\theta_{\mathscr{V}}$ at y is an eigensystem with slope less than h.*

Remark. Here we implicitly assumed that a classical eigensystem gives rise to an overconvergent eigensystem. This is detailed studied in [1, section 3.3] via the BGG resolution. Actually, there is a classicity theorem when h is small [1, Proposition 4.3.10], concretely, assume that λ is algebraic and $f = f^p \otimes u_t$ is admissible, then there exists a rational bound $B(\lambda, t) \in \mathbb{Q}_{>0}$ such that for any $h < B(\lambda, t)$, $H^\bullet(S_G(K), \mathscr{D}_\lambda)^{\leq h} \cong H^\bullet(S_G(K), \mathbb{W}_\lambda)^{\leq h}$.

4.4 Urban's conjecture

Now assume π_0, σ_0, θ_0 as above. We can define the Euler-Poincare characteristic of π_0 as in [1] by

$$EP(\sigma_0, \lambda_0, K^p) = \sum_q (-1)^q dim_{\mathbb{C}} Hom(\sigma_0, H^q(S_G(K), \mathbb{W}_{\lambda_0}))$$

If G is a connected reductive group whose archimedean part has a discrete series. In [1], Urban constructed an eigenvariety for G whose $\overline{\mathbb{Q}}_p$-points parameterize those Hecke eigensystems coming from finite slope cuspidal automorphic representations with non-trivial Euler-Poincare characteristics.

The next proposition says that this eigenvarity is actually the union of all irreducible components of $\mathfrak{E}_{K^p}$ which containing classical cuspidal points.

Proposition 4.14. *If G_∞ has a discrete series, then $dim\mathfrak{E}_{K^p} = dim\mathfrak{X}$ and $y_0 = (\theta_0, \lambda_0) \in \mathfrak{E}(K^p)(\overline{\mathbb{Q}}_p)$ is contained in an irreducible compnont of codimension 0 if and only if $EP(\sigma_0, \lambda_0, K^p) \neq 0$*

Proof. If $EP(\sigma_0, \lambda_0, K^p) \neq 0$, then by the results in [1], there is an eigenvariety of dimension $dim\mathfrak{X}$ pass through y_0. Conversely, if y_0 is contained in an irreducible compnont of codimension 0, then we can find a sequence of points $y_n \in \mathfrak{E}_{K^p}(\overline{\mathbb{Q}}_p)$ approaching to y with regular weights λ_n. All these points have nontrivial Euler-poincare characteristics. □

It is interesting to characterize the irreducible components of $\mathfrak{E}_{K^p}$. Here we cite the conjecture formulated by Urban in [1, conjecture 5.7.3] to close our discussion.

Conjecture. *Say $y = (\lambda, \theta) \in \mathfrak{E}_{K^p}(\overline{\mathbb{Q}}_p)$ is contained in only one irreducible component C. Then the image of the projection of C to $\mathfrak{X}$ is of codimension d if and only if there exist two non negative integers p and q such that*

(a) $d = q - p$,

(b) $H^r(S_G(K), \mathscr{D}_\lambda)[\theta]$ is non zero only if $p \leq r \leq q$.

Chapter 5

TWISTED FRANKE'S TRACE FORMULA

In this section, we give a twisted version of Franke's trace formula, which writes the trace for the cohomolgy $H^\bullet(S_G, W_\lambda^\vee(\mathbb{C}))$ in terms of traces of cohomology of Levi subgroups of G. Since we are dealing with the classical module $W_\lambda^\vee(\mathbb{C})$, we assume $\lambda \in \mathfrak{X}^\iota$ is algebraic and regular.

Let's recall some notations first. For any reductive group G with fixed Borel pair (B,T), write $R = R_G$ (resp. $R^+ = R_G^+$) the set of roots (resp. of positive roots), $\mathscr{P}_G$ and $\mathscr{L}_G$ the set of conjugate classes of standard parabolic subgroups and of standard Levi subgroups of G respectively. For $P \in \mathscr{P}_G$, we have the standard Levi decomposion $P = MN$. Write $\check{\mathfrak{a}}_P := X^*(P) \otimes \mathbb{R}$ and $S(\check{\mathfrak{a}}_P)$ its symmetric space, $\check{\mathfrak{a}}_P^G$ the subspace of $\check{\mathfrak{a}}_P$ whose elements are trivial on G.

The Weyl group of G is denoted by $\mathscr{W}_G$, and

$$\mathscr{W}^M := \{w \in \mathscr{W}_G \mid w^{-1}(\alpha) \in R^+,\ \forall \alpha \in R^+ \cap R_M\}$$

Then for any algebraic weight λ and $w \in \mathscr{W}^M$ we have

$$w * \lambda := w(\lambda + \rho_P) - \rho_P$$

a dominant weight and the Kostant decomposition

$$H^q(\mathfrak{n}, W_\lambda^\vee) = \bigoplus_{w\in \mathscr{W}^M \ l(w)=n-q} {W^M_{w*\lambda+2\rho_P}}^\vee$$

where W^M_μ is the irreducible algebraic representation of M of highest weight μ and $n = dim(\mathfrak{n})$.

We also write

$$\mathscr{W}^M_{Eis} := \{w \in \mathscr{W}^M \mid w^{-1}(\beta^\vee) > 0, \forall \beta \in R_P\}$$

When λ is regular, and $w \in \mathscr{W}^M_{Eis}$, the Eisenstein series associated to a class in $H^\bullet(S_M, {W^M_{w*\lambda}}^\vee)$ defines an Eisenstein class in $H^\bullet(S_G, W^\vee_\lambda)$.

Let $\mathfrak{g} := LieG^1_\infty$, χ_λ the character of Z_∞ according to its action on $W^\vee_\lambda(\mathbb{C})$, $Z(\mathfrak{g})$ the center of the universal enveloping algebra of $\mathfrak{g}$ and $\mathscr{I} \subset Z(\mathfrak{g})$ the annihilator of $W_\lambda(\mathbb{C})$. By the standard results of $(\mathfrak{g}, K_\infty)$-modules, (see e.g. [10]) we have

$$\begin{aligned} H^\bullet(S_G, W^\vee_\lambda(\mathbb{C})) &= H^\bullet(\mathfrak{g}, K_\infty; C(G(\mathbb{Q}\backslash G(\mathbb{A}), \chi_\lambda) \otimes W^\vee_\lambda(\mathbb{C})) \\ &= H^\bullet(\mathfrak{g}, K_\infty; A_\lambda \otimes W^\vee_\lambda(\mathbb{C})) \end{aligned}$$

where A_λ is the subspace of V_G, the K-finite smooth functions of uniform modrate growth on $G(\mathbb{Q})Z_\infty\backslash G(\mathbb{A})$, which are annihilated by a power of $\mathscr{I}$.

Now for $P \in \mathscr{P}_G$, write $V_G(P)$ the subspace of V_G consists of those elements which are negligible along all $Q \in \mathscr{P}_G$ such that $Q \neq P$, and $A_{\lambda,P} := A_\lambda \cap V_G(P)$. Then we have, as $(\mathfrak{g}, K_\infty, G(\mathbb{A}_f))$-modules,

$$A_\lambda = \bigoplus_{P\in\mathscr{P}_G} A_{\lambda,P} \tag{5.0.0.1}$$

Furthermore, write $P = MN$, the Levi decompostin and let π be an irreducible cuspidal representation of M with central character χ_π and weight λ, we denote $A_{\lambda,P,\pi}$ be the subspace of $A_{\lambda,P}$

consisting of those $\phi \in A_\lambda$ whose constant term ϕ^P along P belongs to

$$L^2_{cusp}(M(\mathbb{Q})\backslash M(\mathbb{A}), \pi) \otimes S(\mathfrak{a}_P^\vee)$$

Then we have a theorem from Franke and Schwermer (see [16])

Theorem 5.1. *There is a direct decomposition of $(\mathfrak{g}, K_\infty, G(\mathbb{A}_f))$-modules*

$$A_\lambda = \bigoplus_{P\in\mathscr{P}_G} \bigoplus_{\pi} A_{\lambda,P,\pi} \tag{5.0.0.2}$$

Where π runs over all irreducible cuspidal representation of M of weight λ.

Now with the above notations, we have

$$\begin{aligned}
H^\bullet(S_G, W_\lambda^\vee(\mathbb{C})) &= H^\bullet(\mathfrak{g}, K_\infty; C(G(\mathbb{Q}\backslash G(\mathbb{A}), \chi_\lambda) \otimes W_\lambda^\vee(\mathbb{C})) \\
&= H^\bullet(\mathfrak{g}, K_\infty; A_\lambda \otimes W_\lambda^\vee(\mathbb{C})) \\
&= \bigoplus_{P\in\mathscr{P}_G} H^\bullet(\mathfrak{g}, K_\infty; A_{\lambda,P} \otimes W_\lambda^\vee(\mathbb{C})) \\
&= \bigoplus_{P\in\mathscr{P}_G} \bigoplus_{\pi} H^\bullet(\mathfrak{g}, K_\infty; A_{\lambda,P,\pi} \otimes W_\lambda^\vee(\mathbb{C})) \\
&= \bigoplus_{P\in\mathscr{P}_G} \bigoplus_{\pi_{P,\lambda}} H^\bullet(\mathfrak{g}, K_\infty; Ind_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)} Ind_{\mathfrak{p}}^{\mathfrak{g}} V_\pi \otimes \check{\mathfrak{a}}_P^G \otimes W_\lambda^\vee(\mathbb{C})) \\
&= \bigoplus_{P\in\mathscr{P}_G} \bigoplus_{\pi_{P,\lambda}} Ind_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)} H^\bullet(\mathfrak{p}, K_{\mathfrak{m}}; V_\pi \otimes \check{\mathfrak{a}}_P^G \otimes W_\lambda^\vee(\mathbb{C}))
\end{aligned}$$

Where the last second equality follows from [17], and the last equality follows from [10, III, 2.5]. It also follows from [10] that , for every $P = MN$ and $\pi = \pi_{P,\lambda}$, we have a spectral sequence

$$\begin{aligned}
E_2^{p,q}(\pi) &= H^p(\mathfrak{m}, K_M; H^q(\mathfrak{n}, W_\lambda^\vee(\mathbb{C})) \otimes V_\pi \otimes S(\check{\mathfrak{a}}_P^G)) \\
&\Rightarrow H^\bullet(\mathfrak{p}, K_P; W_\lambda^\vee(\mathbb{C}) \otimes V_\pi \otimes S(\check{\mathfrak{a}}_P^G))
\end{aligned}$$

So combining with the Konstant decomposition, we have a spectral sequence

$$\begin{aligned} E_2^{p,q} &= \bigoplus_{P\in\mathscr{P}_G}\bigoplus_{\pi_{P,\lambda}}\bigoplus_{\substack{w\in\mathscr{W}^M_{Ein}\\ l(w)=n_M-q}} Ind_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)} H^p(\mathfrak{m},K_M;W^{M,\vee}_{w*\lambda+2\rho_P}(\mathbb{C})\otimes V_\pi\otimes S(\check{\mathfrak{a}}_P^G)) \\ &\Rightarrow H^\bullet(\mathfrak{g},K_\infty;A_\lambda\otimes W_\lambda^\vee(\mathbb{C}))=H^\bullet(S_G,W_\lambda^\vee(\mathbb{C})) \end{aligned}$$

This spectral sequence respects the Hecke action, and the Franke's trace formula follows by applying the Hecke operators on it. Now we gives a twisted version by involving a study of the action of ι on this spectral sequence as well.

A discussion as Lemma 3.2 shows that ι is well-defined on $H^\bullet(S_G,W_\lambda^\vee(\mathbb{C}))$. Consider the isomorphism between $H^\bullet(S_G,W_\lambda^\vee(\mathbb{C}))$ and $H^\bullet(\mathfrak{g},K_\infty;A_\lambda\otimes W_\lambda^\vee(\mathbb{C}))$, (see [10], [16]) we have ι acts on the later space deduced from the action of ι on A_λ and $W_\lambda(\mathbb{C})$. Where for any $\phi\in A_\lambda$, $\phi^\iota(x)=\phi(x^{\iota^{-1}})$ and for any $\varphi\in W_\lambda(\mathbb{C})$, $\varphi^\iota(x)=\varphi(x^{\iota^{-1}})$.

The first step is to study the action of ι on the right side of (5.0.0.1), before this, we need to mention the next lemma, which can be found in [23]:

Lemma 5.2. *let ι be an automorphism of G of finite order which commutes with the Cartan involution, then there esixts a minimal parabolic P_0, which and whose Levi decomposition are stable under ι.*

So we can denote by $\mathscr{P}_G^\iota$ the set of conjugacy classes of parabolic subgroups which are stable under ι, a remark in [23] indicates that $\mathscr{P}_G^\iota\subsetneq\mathscr{P}_G$ in general.

We denote the image of $A_{\lambda,P}$ by $A_{\lambda,P}^\iota$ and we have

Lemma 5.3.

$$A_{\lambda,P}^\iota=A_{\lambda,P^\iota}$$

Proof. Let $\phi\in A_{\lambda,P}$, by definition, we need to show for any $Q\neq P^\iota\in\mathscr{P}_G$ and any $g\in G(\mathbb{A})$ that $R_g(\phi^\iota)^Q\perp L^2_{cusp}(M_Q(\mathbb{Q})\backslash M_Q(\mathbb{A}))$. However, an explicit computation shows that $(\phi^\iota)^Q=(\phi^{Q^{\iota^{-1}}})^\iota$.

So we have

$$R_g(\phi^\iota)^Q \perp L^2_{cusp}(M_{Q^{\iota^{-1}}}(\mathbb{Q})\backslash M_{Q^{\iota^{-1}}}(\mathbb{A}))^\iota.$$

Consider the map

$$\theta_Q : L^2_{cusp}(M_Q(\mathbb{Q})\backslash M_Q(\mathbb{A})) \to L^2_{cusp}(M_{Q^\iota}(\mathbb{Q})\backslash M_{Q^\iota}(\mathbb{A}))$$

as $\theta_Q(f)(x) := f(x^{\iota^{-1}})$, we see actually

$$L^2_{cusp}(M_Q(\mathbb{Q})\backslash M_Q(\mathbb{A})) = L^2_{cusp}(M_{Q^{\iota^{-1}}}(\mathbb{Q})\backslash M_{Q^{\iota^{-1}}}(\mathbb{A}))^\iota.$$

$\square$

The next step is to study the action of ι on the right side of (5.0.0.2), let $\pi = \pi_{P,\lambda}$ be an irreducible cuspidal representation of M_P. Consider the map θ_P defined as in the proof of the lemma above, then θ deduces an irreducible cuspidal representation of M_{P^ι} from π, we deonote this representation by π^θ. We have

Lemma 5.4.

$$A^\iota_{\lambda,P,\pi} = A_{\lambda,P^\iota,\pi^\theta}$$

Proof. The only point is to check that for any $\phi \in A_{\lambda,P,\pi}$, $(\phi^\iota)^{P^\iota}$ is right translated according to π^θ. This is straightforward by a computation from definitions. $\square$

Now we study the action of ι on $E_2^{p,q}$ of the spectral sequence. Concretely speaking, we have to study the imge of $W^{M,\vee}_{w*\lambda+2\rho_P}(\mathbb{C}) \otimes V_\pi \otimes S(\check{\mathfrak{a}}^G_P)$ under ι. Since we will take trace on the spectral sequence, we only need to consider those P with $P^\iota = P$. It is clear that $\iota : V_\pi \to V_{\pi^\theta}$ and $\iota : S(\check{\mathfrak{a}}^G_P) \to S(\check{\mathfrak{a}}^{G^\iota}_{P^\iota})$, so we only need to study the action of ι on $W^M_{w*\lambda+2\rho_P}(\mathbb{C})$. Furthermore, by writing down explicitly the Konstant decompostion, we know that for every $\varphi \in W^M_{w*\lambda+2\rho_P}(\mathbb{C})$, $\varphi^\iota(x) = \varphi(x^{\iota^{-1}})$, so $(W^M_{w*\lambda+2\rho_P}(\mathbb{C}))^\iota = W^M_{(w*\lambda+2\rho_P)^\iota}(\mathbb{C})$. We collect in the next lemma some basic results we may need:

Lemma 5.5. *Let group G with ι as above. Then ι acts on the Weyl group $\mathscr{W} = N_G(T)/T$ via $[x] \mapsto [x^\iota]$ for any $x \in N_G(T)$.*

(1) *Let* S_α *be a simple reflection in* $\mathscr{W}$ *corresponding to a simple root* α*, then* $(S_\alpha)^\iota = S_{\alpha^\iota}$*. So in particular,* ι *preserves the length.*

(2) *For any* $w \in \mathscr{W}$*,* $(w(\alpha))^\iota = w^\iota(\alpha^\iota)$*.*

(3) *Let* $P \in \mathscr{P}$*, then* $\rho_P^\iota = \rho_P$*.*

Proof. The proof of the lemma is elementary from definitions. Concretely speaking, (1) follows from the fact that S_α is the only nontrivial element in $N_{G_\alpha}(T)/T$ (see, e.g. [18, IV]); (2) follows from a direct computation and the definition above; (3) follows from the definition that $\rho_P(t) := det(Ad(t)|_{\mathfrak{n}_P})^{1/2}$(see, e.g. [10, III]). □

By the previous lemma, we know that $(W^M_{w*\lambda+2\rho_P}(\mathbb{C}))^\iota = W^M_{w^\iota*\lambda+2\rho_P}(\mathbb{C})$. Since λ is regular, we know that ι maps $W^M_{w*\lambda+2\rho_P}(\mathbb{C})$ to itself if and only if $w^\iota = w$. With all above preparation, we have

Theorem 5.6 (Twisted Franke's trace formula)**.** *Assume* λ *is regular and* $f \in C_c^\infty(G(\mathbb{A}_f))$*. With* $P = MN$ *a standard Levi decomposiotion, we can define the constant term of* f *along* M $f_M \in C_c^\infty(M(\mathbb{A}_f))$ *by*

$$f_M(m) := |\delta_P(m)|_{\mathbb{A}} \int_{K_M \times N(\mathbb{A}_f)} f(k^{-1}mnk)dndk.$$

Then we have

$$tr^{st}(\iota \times f|H^\bullet(S_G, W^\vee_\lambda(\mathbb{C}))) = \sum_{P=P^\iota} \sum_{\substack{w \in \mathscr{W}^M_{Ein} \\ w^\iota = w}} (-1)^{l(w)+n_M} tr^{st}(\iota \times f_M|H^\bullet_{cusp}(S_M, W^{M,\vee}_{w*\lambda+2\rho_P}))$$

where the notation st indicates that we are using the standard action of Hecke algebra here.

Proof. This could be obtained by simply applying $\iota \times f$ on the spectral sequence $E_2^{p,q}$. All the previous lemmas ensure that only these terms mentioned in the formula above contribute to the trace. □

For the rest of this section, we prove a cuspidal decomposition formula. Assume $\lambda = \lambda^{alg}\epsilon$ is an arithmetic regular weight in $\mathfrak{X}^\iota$, define

$$I_G^{cl}(\iota \times f, \lambda) := tr^*(\iota \times f|H^\bullet(S_G, W^\vee_{\lambda^{alg}}(L)))$$

where $*$ means that here the Hecke action is induced by the $*$-action.

Lemma 5.7. *Assume $f = f^p \otimes u_t \in R_{S,p}$, then*

$$I_G^{cl}(\iota \times f, \lambda) = \lambda(\xi(t)) tr^{st}(\iota \times f | H^\bullet(S_G, W^\vee_{\lambda^{alg}}(\mathbb{C}))).$$

This is obtained by comparing the two actions on the complex computing the cohomology, see [1].

Now we can define the classical cuspidal trace. Assume λ, f as above, $K = K^p I$ chosen as chapter 2, define:

$$I_{G,0}^{cl}(\iota \times f, \lambda) := meas(K^p)\lambda(\xi(t)) tr^{st}(\iota \times f | H^\bullet_{cusp}(S_G(K), W^\vee_\lambda(\mathbb{C})(\epsilon))) \tag{5.0.0.3}$$

Notice here the trace is well defined since ι is well defined on the cuspidal cohomology.

We want to express $I_G^{cl}(\iota \times f, \lambda)$ as a combination of cuspidal traces of Levi subgroups of G, this is the so called cuspidal decomposition formula. For any $w \in \mathscr{W}^M$, define a p-regularized linear map from $\mathscr{H}_p(G)$ to $\mathscr{H}_p(M)$, which sends every f to $f^{reg}_{M,w}$ (see [1, 4.1.9]). Concretely, for $f = f^p \otimes u_t \in \mathscr{H}'_p(G)$, it sends f to

$$f^{reg}_{M,w} := \epsilon_{\xi,w}(t) f^p_M \otimes u_{wtw^{-1}} \tag{5.0.0.4}$$

where the normalizer $\epsilon_{\xi,w}(t) := \xi(t)^{w^{-1}(\rho_P)+\rho_P} |t^{w^{-1}(\rho_P)+\rho_P}|_p$. This is well-defined by the definition of $\mathscr{W}^M$. For general f, the definition is given by linear extension.

Theorem 5.8. *Let f be as above, λ is arithmetic and regular, then:*

$$I_G^{cl}(\iota \times f, \lambda) = \sum_{P=P^\iota} \sum_{\substack{v \in \mathscr{W}^M_{Ein} \\ v^\iota = v}} \sum_{w \in \mathscr{W}^M} (-1)^{l(v)+n_M} \xi(t)^{\lambda - w^{-1} v * \lambda} I^{cl}_{M,0}(f^{reg}_{M,w}, v * \lambda + 2\rho_P)$$

Proof. The proof is essentially same to [1, Lemma4.6.2],we just sketch the proof for the case $\epsilon = 1$ and $f = 1_{K^p} \otimes u_t$ here. By the twisted trace formula and lemma 5.7 above, we know that

$$I_{G,0}^{cl}(\iota \times f, \lambda) = \sum_{P=P^\iota} \sum_{\substack{v \in \mathscr{W}^M_{Ein} \\ v^\iota = v}} (-1)^{l(v)+n_M} \xi(t)^\lambda tr^{st}(\iota \times f_M | H^\bullet_{cusp}(S_M, W^{M,\vee}_{v*\lambda+2\rho_P})) \tag{5.0.0.5}$$

so we have to compute $tr^{st}(\iota \times f_M | H^\bullet_{cusp}(S_M, W^{M,\vee}_{v*\lambda+2\rho_P}))$. To simplify notations, we denote $\mu := v * \lambda + 2\rho_P$ and $\sigma_\mu := H^\bullet_{cusp}(S_M, W^{M,\vee}_\mu(\mathbb{C}))^{K^p}$, then we have, since ι is well defined on σ_μ,

$$tr^{st}(\iota \times f_M|\sigma_\mu) = tr^{st}(\iota \times u_t|Ind^{G(\mathbb{Q}_p)}_{P(\mathbb{Q}_p)}\sigma_\mu).$$

Now recall the decomposition

$$G(\mathbb{Q}_p) = \bigsqcup_{w\in\mathscr{W}^M} P(\mathbb{Q}_p)wI \tag{5.0.0.6}$$

by this we have

$$(Ind^{G(\mathbb{Q}_p)}_{P(\mathbb{Q}_p)}\sigma_\mu)^I \cong \bigoplus_{w\in\mathscr{W}^M} w \bullet \sigma_\mu^{I_M} \tag{5.0.0.7}$$

where $w \bullet \sigma_\mu^{I_M}$ means translating the elements in $\sigma_\mu^{I_M}$ by w according to the decomposition (5.0.0.6). Then a computation combining with (5.0.0.3) shows:

$$\begin{aligned} & tr^{st}(\iota \times u_t|(Ind^{G(\mathbb{Q}_p)}_{P(\mathbb{Q}_p)}\sigma_\mu)^I) \\ = \quad & meas(K^p)^{-1} \sum_{w\in\mathscr{W}^M} [N_w(\mathbb{Z}_p) : tN_wt^{-1}]\xi(t)^{w^{-1}(\mu)} I^{cl}_{M,0}(\iota \times 1_{K^p} \otimes u_{wtw^{-1},M}, \mu) \end{aligned}$$

Notice that $[N_w(\mathbb{Z}_p) : tN_wt^{-1}] = |t^{w^{-1}(\rho_P)+\rho_P}|_p^{-1}$ and we have

$$[N_w(\mathbb{Z}_p) : tN_wt^{-1}]\xi(t)^{w^{-1}(\mu)} = \xi(t)^{w^{-1}v*\lambda}\epsilon_{\xi,w}(t)^{-1}.$$

Notice that

$$(1_{K^p})_M = meas(K^p)1_{K^p},$$

and by the definition (5.0.0.4), we have

$$tr^{st}(\iota \times f_M|H^\bullet_{cusp}(S_M, W^{M,\vee}_{v*\lambda+2\rho_P})) = \sum_w \xi(t)^{w^{-1}v*\lambda} I^{cl}_{M,0}(f^{reg}_{M,w}, v * \lambda + 2\rho_P)$$

Combing this with (5.0.0.3) we conclude the theorem. □

Chapter 6

TWISTED FINITE SLOPE CHARACTER DISTRIBUTIONS

6.1 twisted finite slope character distributions

In [1], Eric Urban developed the theory of finite slope character distributions and used it to construct eigenvarieties. In this section, we apply his idea but a twisted version of his definitions. We also define and study families of twisted FSCDs.

Definition 6.1. *Let ι be an algebraic automorphism of G of finite order l, $L/\mathbb{Q}_p$ a finite extension. An L-valued twisted finite slope character distribution (twisted FSCD) with respect to ι is a $\mathbb{Q}_p$-linear map $J : \mathscr{H}'_p \to L$, such that there exists a collection of irreducible self-dual finite slope automorphic representations $\{\sigma\}$ and a set of l integers $\bar{m}(\sigma) := \{m_i(\sigma) | i = 1, \ldots, l\}$ to each σ satisfying*

(1) for all $t \in T^{++}$, $h \in \mathbb{Q}$ and K^p there finitely many σ such that $\bar{m}(\sigma) \neq 0$, $\sigma^{K^p} \neq 0$ and the slope of $\sigma \leqslant h$.

(2) for any $f \in \mathscr{H}'_p$,

$$J(f) = \sum_{\sigma} \sum_{i} m_i(\sigma) J_{\tilde{\sigma}_i}(f)$$

If for any $f \in \mathscr{H}'_p$ those m_i appearing in the formula above are non-negative, then we say the *twisted FSCD is effective. If a twisted FSCD J is effective, then for any K^p we define*

$$V_J(K^p) = \bigoplus_{\sigma} \bigoplus_{i} V_{\tilde{\sigma}_i}^{m_i(\sigma)}$$

which is a representation space of ${}^{\iota}\mathscr{H}_p(K^p)$ and so that for any f

$$J(f) = tr(\iota \times f | V_J(K^p)).$$

If we define the multiplicity of σ in the FSCD by

$$m_J(\sigma) := m(\sigma) := \sum_i m_i(\sigma)$$

, then as modules of $\mathscr{H}_p(K^p)$,

$$V_J(K^p) = \bigoplus_{\sigma} V_{\sigma}^{m(\sigma)}$$

Given a twisted FSCD J, we could associate to it a Fredholm determinant for any $f \in \mathscr{H}_p(K^p)$ as

$$P_J(X, f) := \prod_{\sigma} \prod_{i} \det(1 - X\tilde{\sigma}_i(\iota \times f))^{m_i(\sigma)}$$

As [1, Lemma 4.1.13], if for all $f \in \mathscr{H}_p'$, $P_J(X, f)$ is an entire power series, then J is effective.

We now define a twisted FSCD, parameterizing the twisted traces of self-dual finite slope cuspidal automorphic representations (as hinted in proposition 3.6). An analytic family of this twisted FSCD over $\mathfrak{X}^{\iota}$ will lead to the eigenvarieties we need.

Let L be a finite extension of $\mathbb{Q}_p$, $\lambda \in \mathfrak{X}^{\iota}(L)$. For $f \in \mathscr{H}_p(K^p)$, define

$$I_G^{\dagger}(\iota \times f, \lambda) := tr(\iota \times f; H_{fs}^{\bullet}(S_G, \mathscr{D}_{\lambda}(L))),$$

then

$$I_G^{\dagger}(\iota \times f, \lambda) = meas(K^p) \times tr(\iota \times f; H_{fs}^{\bullet}(S_G(K^pI), \mathscr{D}_{\lambda}(L)))$$

$$= meas(K^p) \times tr(\iota \times f; R\Gamma^{\bullet}_{fs}(S_G(K^pI), \mathscr{D}_\lambda(L)))$$

Write $\mathscr{P}_G^\iota$ the subset of $\mathscr{P}_G$ whose element P satisfying $P^\iota = P$, and similarly $\mathscr{L}_M^\iota$ and $\mathscr{W}_{Eis}^{M,\iota}$. For $M \in \mathscr{L}_G^\iota$ and $w \in \mathscr{W}_{Eis}^{M,\iota}$, we define twisted finite slope distributions $I_{G,0}^\dagger(\iota \times f, \lambda)$ and $I_{G,M,w}^\dagger(\iota \times f, \lambda)$ by induction to $rk(G)$.

If $rk(G) = 0$, define

$$I_{G,0}^\dagger(\iota \times f, \lambda) = I_{G,G}^\dagger(\iota \times f, \lambda) := I_G^\dagger(\iota \times f, \lambda)$$

Now given a positive integer r and assume the distributions have been defined for cases that $rk(G)$ is less than r, then for proper $M \in \mathscr{L}_G^\iota$ and $f = f^p \otimes u_t$, define

$$I_{G,M,w}^\dagger(\iota \times f, \lambda) := I_{M,0}^\dagger(\iota \times f_{M,w}^{reg}, w * \lambda + 2\rho_P)$$

$$I_{G,M}^\dagger(\iota \times f, \lambda) := \sum_{w \in \mathscr{W}_{Eis}^{M,\iota}} (-1)^{l(w)+dim\mathfrak{n}_M} I_{G,M,w}^\dagger(\iota \times f, \lambda)$$

And then, define:

$$I_{G,0}^\dagger(\iota \times f, \lambda) := I_G^\dagger(\iota \times f, \lambda) - \sum_{proper\ \ M \in \mathscr{L}_G^\iota} I_{G,M}^\dagger(\iota \times f, \lambda)$$

Proposition 6.2. *The $I_{G,?}^\dagger(\iota \times f, \lambda)$ defined above are twisted FSCDs, where $? = M, 0$. Moreover, in any case, if $m_i \neq 0$ then σ_i is self-daul. We denote the Fredholm determinants associated to them by*

$$P_{G,?}^\dagger(f, \lambda, X)$$

Proof. Recall the p-regularization respects the parabolic induction. If σ_M is an irreducible finite slope representation of $\mathscr{H}_{M,p}$, we denote

$$I_{M,w}^G := Ind_M^G(\sigma_f^p) \otimes \theta_{\sigma,w}$$

where $\theta_{\sigma,w}$ is the character of $\mathscr{U}_p$ defined by

$$u_t \mapsto \theta_\sigma(u_{wtw^{-1}}).$$

Since $M \in \mathscr{L}_G^\iota$, if σ_M is an irreducible self-dual representation of M and $\tilde{\sigma}_{M,i}$ is one of its irreducible extensions to $\mathscr{H}_{M,p}$, then $\tilde{\sigma}_{M,i}(\iota)$ also gives an action of ι on $I_{M,w}^G$. Then with the normalizer θ, we have

$$J_{\tilde{\sigma}_{M,i}}(f_{M,w}^{reg}) = tr(\iota \times f | I_{M,w}^G).$$

So by the induction in the definitions above, we only need to show the proposition for $I_G^\dagger(\iota \times f, \lambda)$. This follows from proposition 3.9. □

Now compare $I_{G,0}^\dagger(\iota \times f, \lambda)$ with the classical twisted traces studied in chapter 5.

Lemma 6.3. *Let λ be an algebraic dominant weight and $f = f^p \otimes u_t \in \mathscr{H}_p'(K^p)$, then*

$$I_G^+(\iota \times f, \lambda) \equiv I_G^{cl}(\iota \times f, \lambda) \quad mod \quad N^\iota(\lambda, t) Meas(K^p)$$

where $N^\iota(\lambda, t)$ is defined as in proposition 3.7.

Proof. Notice both sides of the congruence are $mear(K^p) \times \mathbb{Z}_p$-valued by definition. Let $h = v_p(N^\iota(\lambda, t))$, we have

$$I_G^+(\iota \times f, \lambda) \equiv tr(\iota \times f | H_{fs}^\bullet(S_G(K), \mathscr{D}_\lambda(L)))^{\leqslant h} \quad mod \quad N^\iota(\lambda, t) Meas(K^p)$$

$$I_G^{cl}(\iota \times f, \lambda) \equiv tr(\iota \times f | H^\bullet(S_G(K^p I_m), W_\lambda^\vee(L)))^{\leqslant h} \quad mod \quad N^\iota(\lambda, t) Meas(K^p)$$

Then the lemma follows from Proposition 3.7. □

Proposition 6.4. *Let $f = f^p \otimes u_t \in \mathscr{H}_p(K^p)$ and λ regular arithmetic. Then*

$$I_{G,0}^+(\iota \times f, \lambda) \equiv I_{G,0}^{cl}(\iota \times f, \lambda) \quad mod \quad N^\iota(\lambda, t) Meas(K^p) \tag{6.1.0.1}$$

Proof. The proposition can be proved by induction on the rank of G. The case $rk(G)=0$ is just the lemma 6.3 above. Then the induction process follows from the defintion of $I^{+}_{G,0}(\iota\times f,\lambda)$ and Theorem 5.8, noting that if $v\neq w$, then $N^{\iota}(\lambda,t)$ divides $\xi(t)^{\lambda-w^{-1}v*\lambda}$. □

6.2 analytic families of twisted FSCD

In this section we vary $I^{\dagger}_{G,?}(\iota\times f,\lambda)$ according to the weight λ over the analytic weight space $\mathfrak{X}^{\iota}$. According to section 2.2.2 and section 2.3.2, we have

Proposition 6.5. *(a) $\mathscr{A}_{\mathfrak{U}}\otimes_{\lambda}L\cong\mathscr{A}_{\lambda}(L)$ and $\mathscr{D}_{\mathfrak{U}}\otimes_{\lambda}L\cong\mathscr{D}_{\lambda}(L)$ via specialization, and these specializations respect the action of ι.*
(b) If $\delta\in\Delta^{++}$, then the $*$-action of δ gives a compact operator on the $\mathscr{O}(\mathfrak{U})$-projective compact Frechet spaces $\mathscr{D}_{\mathfrak{U}}$. Moreover, the $*$-actions are compatible with the action of ι in the sense of Lemma 3.2.
(c) For $M=\mathscr{D}_{\mathfrak{U}}$, all the results in section 3.1 respect the specializations.

Proof. Checking the results are all straightforward. □

Now let

$$\Lambda_{\mathfrak{X}^{\iota}}:=\varprojlim_{\mathfrak{U}\subset\mathfrak{X}^{\iota}}\mathscr{O}^{0}(\mathfrak{U})\subset\mathscr{O}(\mathfrak{X}^{\iota}),$$

we have

Proposition 6.6. *Let $f\in\mathscr{H}'_{p}(K^{p})$, and write $\Lambda_{\mathfrak{X}^{\iota},\mathbb{Q}_p}:=\Lambda_{\mathfrak{X}^{\iota}}\otimes\mathbb{Q}_p$, then as functions of $\lambda\in\mathfrak{X}^{\iota}$, $I^{\dagger}_{G}(\iota\times f,\lambda)$, $I^{\dagger}_{G,M,w}(\iota\times f,\lambda)$ and $I^{\dagger}_{G,0}(\iota\times f,\lambda)$ are all in $\Lambda_{\mathfrak{X}^{\iota},\mathbb{Q}_p}$. In particular, they are analytic on $\mathfrak{X}^{\iota}$.*

Proof. From the definitions, we only need to prove it for $I^{\dagger}_{G}(\iota\times f,\lambda)$ and the other cases will follow from an induction on $rk(G)$. By proposition 6.5 above, locally $I^{\dagger}_{G}(\iota\times f,-)=meas(K^{p})tr(\iota\times f|R\Gamma^{\bullet}_{fs}(K^{p}I,\mathscr{D}_{\mathfrak{U},n}))$ for some proper $n\geqslant n(\mathfrak{U})$. Actually $meas(K^{p})tr(\iota\times f|R\Gamma^{\bullet}_{fs}(K^{p}I,\mathscr{D}_{\mathfrak{U},n}))\in\mathscr{O}^{0}(\mathfrak{U})$ since f preserves the $\mathscr{O}^{0}(\mathfrak{U})$-lattiace $R\Gamma^{\bullet}(K^{p}I,\mathscr{D}^{0}_{\mathfrak{U},n})$ and ι is algebraic. So we have $I^{\dagger}_{G}(\iota\times f,\lambda)\in\varprojlim_{\mathfrak{U}\subset\mathfrak{X}^{\iota}}\mathscr{O}^{0}(\mathfrak{U})\in\Lambda_{\mathfrak{X}^{\iota}}$ □

6.3 effectivity

This section is aimed to prove the following proposition, which asserts that the twisted FSCD $I^{\dagger}_{G,0}(\iota \times f, \lambda)$ we defined above is essentially effective in case it is not vanishing.

Proposition 6.7. *There exists a multipler $e_G \neq 0$, such that $e_G I^{cl}_{G,0}(\iota \times f, \lambda)$ and $e_G I^{\dagger}_{G,0}(\iota \times f, \lambda)$ are effective.*

By this proposition, we denote the representation space associated to $e_G I^{\dagger}_{G,0}(\iota \times f, \lambda)$ by $V^{\dagger,\lambda}_{G,0}$.

Since regular dominant weights are dense in the weight space, we only need to prove for λ regular dominant. Let's compute firstly $I^{cl}_{G,0}(\iota \times f, \lambda)$. We have

$$\begin{aligned} & meas(K^p)^{-1} I^{cl}_{G,0}(\iota \times f, \lambda) \\ = \; & \lambda(\xi(t)) tr^{st}(\iota \times f | H^{\bullet}_{cusp}(S_G(K^p I), \mathbb{W}^{\vee}_{\lambda}(\mathbb{C}))) \\ = \; & \sum_{\pi} \lambda(\xi(t)) m_{cusp}(\pi) tr^{st}(\iota \times f | H^{\bullet}(\mathfrak{g}, K_{\infty}; V_{\pi} \otimes \mathbb{W}^{\vee}_{\lambda}(\mathbb{C}))) \\ = \; & \sum_{\pi} m_{cusp}(\pi) tr^{st}(\iota | H^{\bullet}(\mathfrak{g}, K_{\infty}; V_{\pi_{\infty}} \otimes \mathbb{W}^{\vee}_{\lambda}(\mathbb{C}))) tr(\iota \times f | V_{\pi_f}) \end{aligned}$$

Where the summation running over all self-dual cuspidal representations π, such that $V^{\iota}_{\pi} = V_{\pi}$. $m_{cusp}(\pi)$ indicating the muliplicity of π in the cuspidal cohomology. The equations follow from a standard computation of $(\mathfrak{g}, K)$-cohomology, ([10]) together with Proposition 3.9 and proposition 6.4.

So we only need to compute

$$tr^{st}(\iota | H^{\bullet}(\mathfrak{g}, K_{\infty}; V_{\pi_{\infty}} \otimes \mathbb{W}^{\vee}_{\lambda}(\mathbb{C}))) = \sum_{i} (-1)^i tr^{st}(\iota | H^{i}(\mathfrak{g}, K_{\infty}; V_{\pi_{\infty}} \otimes \mathbb{W}^{\vee}_{\lambda}(\mathbb{C})))$$

This is called the Lefschetz number, which is denoted here by $L(\iota, \pi, \lambda)$, and is useful in proving nonvanishing results of cohomology. There are many works involving the computation of Lefschetz numbers ([9], [21], [22], [23]), and all the computation could be traced back to the original paper by Vogan and Zukerman [20]. We summarize the following Theorem from [20]:

Theorem 6.8 (Vogan-Zukerman)**.** *Let π be a cuspidal representation such that $H^{\bullet}(\mathfrak{g}, K_{\infty}; V_{\pi_{\infty}} \otimes$*

$\mathbb{W}^\vee_\lambda(\mathbb{C})) \neq 0$*, then there exists a* θ*-stable parabolic subalgebra* $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ *such that*

$$dim\mathbb{W}^\vee_\lambda(\mathbb{C})/\mathfrak{u}\mathbb{W}^\vee_\lambda(\mathbb{C}) = 1$$

and V_{π_∞} *is of the form* $A_\mathfrak{q}(w_0\lambda)$*. Moreover, we have*

$$H^i(\mathfrak{g}, K_\infty; V_{\pi_\infty} \otimes \mathbb{W}^\vee_\lambda(\mathbb{C})) \cong Hom_{\mathfrak{l}\cap\mathfrak{k}}(\wedge^{i-R_\mathfrak{q}}(\mathfrak{l}\cap\mathfrak{k}), \mathbb{C})$$

Here we also recall some notaions from [20]. θ is the usual Cartan involution of G, so we have a decomposition

$$\mathfrak{g} = \mathfrak{p} + \mathfrak{k}.$$

For a θ stable parabolic $\mathfrak{q}$ and an admissible weight λ, $A_\mathfrak{q}(\lambda)$ is an irreducible $\mathfrak{g}$-module whose restriction on $\mathfrak{k}$ contains a represntation $\mu(\mathfrak{q}, \lambda)$, and $\mu(\mathfrak{q}, \lambda)$ is the representation of K_∞ of highest weight $\lambda + 2\rho(\mathfrak{p} \cap \mathfrak{u})$. We also denote $R_\mathfrak{q} = dim\mathfrak{p} \cap \mathfrak{u}$. Finally w_0 is the Weyl element of the largest length.

Corollary 6.9. *Assume the conditions as the Theorem above and that* λ *is regualar, then* $\mathfrak{u}$ *is maximal unipotent and so* $\mathfrak{q}$ *is Borel.*

Proof. This is a simple observation from the previous theorem. $dim\mathbb{W}^\vee_\lambda(\mathbb{C})/\mathfrak{u}\mathbb{W}^\vee_\lambda(\mathbb{C}) = 1$ imples that $\mathbb{W}^\vee_\lambda(\mathbb{C})$ can be realized in $Ind^\mathfrak{g}_\mathfrak{q}(\chi)$ with χ a character of $\mathfrak{l}$ whose restriction to $\mathfrak{t}$ is $-w_0\lambda$. However, since λ, so is $-w_0\lambda$, is regular, $-w_0\lambda$ cannot be extended (see e.g. [19]). So $\mathfrak{u}$ is maximal. □

The corollary implies that for any regular weight λ there are only finitely many cuspidal π such that $H^\bullet(\mathfrak{g}, K_\infty; V_{\pi_\infty} \otimes \mathbb{W}^\vee_\lambda(\mathbb{C})) \neq 0$, and their infinity part $\pi_\infty = A_\mathfrak{b}(w_0\lambda)$. Wrire $R = R_\mathfrak{b}$, then for each such π

$$\begin{aligned} tr^{st}(\iota|H^\bullet(\mathfrak{g}, K_\infty; V_{\pi_\infty} \otimes \mathbb{W}^\vee_\lambda(\mathbb{C}))) &= \sum_i (-1)^i tr(\iota|Hom_{\mathfrak{l}\cap\mathfrak{k}}(\wedge^{i-R_\mathfrak{q}}(\mathfrak{l}\cap\mathfrak{p}), \mathbb{C})) \\ &= (-1)^R \sum_i (-1)^i tr(\iota|Hom_{\mathfrak{l}\cap\mathfrak{k}}(\wedge^{i}(\mathfrak{l}\cap\mathfrak{p}), \mathbb{C})) \end{aligned}$$

So $I^{cl}_{G,0}(\iota\times f,\lambda)$ is not vanishing if and only if ι has non-trivial alternating trace on $Hom_{\mathfrak{l}\cap\mathfrak{k}}(\mathfrak{l}\cap\mathfrak{k},\mathbb{C})$, which is actually independent on π. In case it is not vanishing, if we denote

$$e_G = meas(K^p)^{-1}((-1)^R\sum_i(-1)^i tr(\iota|Hom_{\mathfrak{l}\cap\mathfrak{k}}(\wedge^i(\mathfrak{l}\cap\mathfrak{p}),\mathbb{C})))^{-1}$$

we have $e_G I^{cl}_{G,0}(\iota\times f,\lambda)$ is effective.

Now denote $P^{cl}_{G.0}(\iota\times f,\lambda,X)$ and $P^{\dagger}_{G.0}(\iota\times f,\lambda,X)$ be the Fredholm series associated to $e_G I^{cl}_{G,0}(\iota\times f,\lambda)$ and $e_G I^{\dagger}_{G,0}(\iota\times f,\lambda)$ respectively, we also denote

$$P^{\dagger-cl}_{G,0}(\iota\times f,\lambda,X)=\frac{P^{\dagger}_{G.0}(\iota\times f,\lambda,X)}{P^{cl}_{G.0}(\iota\times f,\lambda,X)}$$

the next lemma is a direct consequence of Proposition 6.4.

Lemma 6.10. *If λ is regular, then $P^{\dagger-cl}_{G,0}(\iota\times f,\lambda,X)$ is a meromorphic function on $\mathbb{A}^1_{rig}(\mathbb{C}_p)$, and all its zeros and poles are lying in*

$$\{x\in\mathbb{C}_p||x|_p\geq N^{\iota}(\lambda,t)\}$$

Now we choose a closed affinoid subdomain $\mathfrak{U}\subset\mathfrak{X}^{\iota}$ which contains one hence dense algebraic weights. Shrink $\mathfrak{U}$ if necessary so that we can write $P^{\dagger}_{G.0}(\iota\times f,\lambda,X)$ as a quotient of relatively prime Fredholm series , say $T(\iota\times f,\lambda,X)/B(\iota\times f,\lambda,X)$, with both $T(\iota\times f,\lambda,X)$ and $B(\iota\times f,\lambda,X)$ are in $\mathscr{O}(\mathfrak{U}\times\mathbb{A}^1_{rig})$. Now if $B(\iota\times f,\lambda,X)\neq 1$, we get a Fredholm subvariety cut out by $T(\iota\times f,\lambda,X)$ and $B(\iota\times f,\lambda,X)$, say, $\mathfrak{W}=Z(B)-Z(T)$. Since the projection Pr from $Z(B)$ to $\mathfrak{U}$ is flat, that $Pr(\mathfrak{W})$ also contains a dense subset of algebraic weights.

Now let $w=(\lambda,x)\in\mathfrak{W}(\overline{\mathbb{Q}}_p)$ with λ algebraic, we can choose $w'=(\lambda',x')$ p-adically close to w such that λ' is algebraic regular dominant and $|x'|_p<N^{\iota}(\lambda',t)$. So by Lemma 6.10 above, we see that x' is a pole of $P^{cl}_{G.0}(\iota\times f,\lambda',X)$. However, since $e_G I^{cl}_{G,0}(\iota\times f,\lambda')$ is effective, we know that $P^{cl}_{G.0}(\iota\times f,\lambda',X)$ is actually entire. This implies that $P^{\dagger}_{G.0}(\iota\times f,\lambda,X)$ is entire and so $e_G I^{\dagger}_{G,0}(\iota\times f,\lambda)$ is also effective.

Chapter 7

TWISTED EIGENVARIETIES

We construct the twisted eigenvarieties in section, we cannot simply apply Urban's theory but have to make some adaption. Throughout this section, we assume that $e_G \neq 0$.

7.1 twisted spectral varieties

We consider the effective twisted FSCD $e_G I^{\dagger}_{G.0}(\iota \times f, \lambda)$, and the Fredholm power series $P^{\dagger}_{G.0}(\iota \times f, \lambda, X)$ associate to it.

Proposition 7.1 (twisted spectral varieties)**.** *For any $f = f^p \otimes u_t \in \mathscr{H}'_p(K^p)$ with $t \in T^{++}$, we can define a twisted spectral variety $\mathfrak{S}^{\iota}(f) \subset \mathfrak{X}^{\iota} \times \mathbb{A}^1_{rig}$, such that $(\lambda, \alpha) \in \mathfrak{S}^{\iota}(f)(\overline{\mathbb{Q}}_p)$ if and only if α^{-1} is an eigenvalue of $\iota \times f$ on $V^{\dagger,\lambda}_{G,0}(K^p)$. Actually, $\mathfrak{S}^{\iota}(f)$ can be seen as the Fredholm hypersurface cut out by $P^{\dagger}_{G.0}(\iota \times f, \lambda, X)$.*

Remark. This proposition is essentially same to [1, Proposition5.1.6], the difference here is that we do not get a spectral variety parameterizing Hecke eigenvalues but eigenvalues of $\iota \times f$. The way to get such a spectral variety as a rigid space from a Fredholm series is due to Coleman-Mazur [6] and Buzzard [7].

7.2 full eigenvariety

We summmarize the results in chapter 4 here. Given K^p as before, let $\widetilde{R}_{\mathscr{S},p}$ be the p-adic completion of $R_{\mathscr{S},p}[u_t^{-1}, t \in T^+]$, and then we have an analytic space $\mathfrak{R}_{\mathscr{S},p}$, such that for any $L/\mathbb{Q}_p$,

$$\mathfrak{R}_{\mathscr{S},p}(L) = Hom_{ct\ alg}(\widetilde{R}_{\mathscr{S},p}, L).$$

Then $\mathfrak{R}_{\mathscr{S},p}(L)$ contains those characters of finite slope. The canonical p-adic topology on $\mathfrak{R}_{\mathscr{S},p}$ is indeuced by the metric $|\theta - \theta'| = \sup_{f \in R_{\mathscr{S},p}} |\theta(f) - \theta'(f)|_p$.

Now denote $\mathfrak{Y} = \mathfrak{X} \times \mathfrak{R}_{\mathscr{S},p}$, the full eigenvariety is a rigid space $\mathfrak{E}_{K^p} \subset \mathfrak{Y}$ with a projection onto $\mathfrak{X}$ such that $(\lambda, \theta) \in \mathfrak{E}_{K^p}(L)$ if and only if $H^\bullet(S_G(K^p)I, \mathscr{D}_\lambda(L))[\theta] \neq 0$. In particulr, for any $f \in R_{\mathscr{S},p}$, $R_f(\theta) := \theta(f)^{-1}$ is an eigenvalue of f acting on $H^\bullet(S_G(K^pI), \mathscr{D}_\lambda(L))$. So any f gives a projection from $\mathfrak{E}_{K^p}$ to a subspace $\mathfrak{S}(f)$ of $\mathfrak{X} \times \mathbb{A}^1_{rig}$ parameterizing Hecke eigenvalues of f. $\mathfrak{S}(f)$ is known as a spectral variety. We also know that $\mathfrak{E}_{K^p}$ is seperated and $dim\mathfrak{E}_{K^p} = dim\mathfrak{X}$.

7.3 multiplicities

We define and study some multiplicities here.

If θ is a character of $R_{\mathscr{S},p}$, we denote by $m^{\iota,\dagger}(\theta, \lambda, K^p)$ the multiplicity of θ in $V_{G,0}^{\dagger,\lambda}(K^p)$. Let σ be a finite slope automorphic representation, then we denote by $m_?^{\iota,\dagger}(\sigma, \lambda, K^p)$ (resp. $m_?^{\iota,cl}(\sigma, \lambda, K^p)$) be its multiplicity in the distribution $meas(K^p)^{-1} I_?^\dagger(\iota \times f, \lambda)$ (resp. $I_?^{cl}(\iota \times f, \lambda)$). By Proposition 3.9, if θ or σ is not self-dual, then the multiplicity is 0. We can also define the twisted Euler-Poincare characteristic with an arithmetic weight $\lambda = \lambda^{alg}\epsilon$. Let σ be a selfdual finite slope automorphic representation, then its twisted Euler-Poincare characteristic with respect to ι is defined as

$$m^\iota_{EP}(\sigma, \lambda) \tag{7.3.0.1}$$

$$:= \sum_q (-1)^q tr(\iota | Hom_{\mathscr{H}_p}(\sigma, \varinjlim_{K^p} H^q(S_G(K^pI), \mathbb{W}^\vee_{\lambda^{alg}}(\mathbb{C})(\epsilon)))) \tag{7.3.0.2}$$

$$= \sum_i \sum_q (-1)^q tr(\iota | Hom_{\iota\mathscr{H}_p}(\tilde{\sigma}_i, \varinjlim_{K^p} H^q(S_G(K^pI), \mathbb{W}^\vee_{\lambda^{alg}}(\mathbb{C})(\epsilon)))) \tag{7.3.0.3}$$

Lemma 7.2. *If σ is a selfdual finite slope automorphic cuspidal representation, which is non-critical with respect to λ^{alg}, then*

$$m^{\iota}_{EP}(\sigma,\lambda) = m_0^{\iota,cl}(\sigma,\lambda,K^p) = m_0^{\iota,\dagger}(\sigma,\lambda,K^p)$$

Proof. The first identity follows from a direct computation. As in last section, we have

$$\begin{aligned} & meas(K^p)^{-1} I^{\dagger}_{G,0}(\iota\times f,\lambda) \\ = & \sum_{\pi} m_{cusp}(\pi) L(\iota,\pi,\lambda) tr(\iota\times f|\pi_f) \\ = & \sum_{\pi} m_{cusp}(\pi) L(\iota,\pi,\lambda) \sum_{\sigma=\sigma^{\iota}} tr(\iota|Hom_{\mathscr{H}_p(K^p)}(\sigma,\pi_f)) J_{\sigma^{\iota}}(f) \\ = & \sum_{\sigma=\sigma^{\iota}} \sum_{\pi} m_{cusp}(\pi) L(\iota,\pi,\lambda) tr(\iota|Hom_{\mathscr{H}_p(K^p)}(\sigma,\pi_f)) J_{\sigma^{\iota}}(f) \end{aligned}$$

So

$$m^{cl}_{0,\iota}(\sigma,\lambda,K^p) = \sum_{\pi} m_{cusp}(\pi) L(\iota,\pi,\lambda) tr(\iota|Hom_{\mathscr{H}_p(K^p)}(\sigma,\pi_f))$$

Noticing that the Lefschetz number $L(\iota,\pi,\lambda)$ is nontrivial only if π_∞ is tempered, and in this case $m_{cusp}(\pi) = m_{L^2}(\pi)$, then a similar computation shows that $m^{\iota}_{EP}(\sigma,\lambda)$ also equals $\sum_{\pi} m_{cusp}(\pi) L(\iota,\pi,\lambda) tr(\iota|Hom_{\mathscr{H}_p}$

The second equality follows from a similar dicussion as [1, Corollary 4.6.5] and (7.3.0.3). ☐

7.4 twisted eigenvarieties

We construct the twisted eigenvariety in the section.

Similar to 7.2. given K^p, define $\tilde{R}^{\iota}_{\mathscr{S},p}$ the p-adic completion of $R^{\iota}_{\mathscr{S},p}[u_t^{-1}, t\in T^+]$ and an analytic space $\mathfrak{R}^{\iota}_{\mathscr{S},p}$ such that for any $L/\mathbb{Q}_p$,

$$\mathfrak{R}^{\iota}_{\mathscr{S},p}(L) = \{\tilde{\theta}\in Hom_{ct\ gp}(\tilde{R}^{\iota}_{\mathscr{S},p}, L) | \tilde{\theta} \text{ is linear w.r.t. } R_{\mathscr{S},p}\}.$$

Since $R^{\iota}_{\mathscr{S},p} = R_{\mathscr{S},p}\rtimes\langle\iota\rangle$, and ι is of finite order l, we view $\tilde{R}^{\iota}_{\mathscr{S},p}$ as an l pieces union of $\tilde{R}_{\mathscr{S},p}$. Then give $\mathfrak{R}^{\iota}_{\mathscr{S},p}(L)$ the topology induced from the sup-norm topology on $\mathfrak{R}_{\mathscr{S},p}(L)$ as in 7.2.

Consider the morphism

$$\mathfrak{i} : \mathfrak{R}^{\iota}_{\mathfrak{S},p}(L) \longrightarrow \mathfrak{R}_{\mathfrak{S},p}(L)$$

given by restricting a character $\tilde{\theta}$ of $\tilde{R}^{\iota}_{\mathscr{S},p}$ to $\tilde{R}_{\mathscr{S},p}$, say, $\theta = \tilde{\theta}|_{\tilde{R}_{\mathscr{S},p}}$, as the definition of the topology, this morphism is finite and continuous.

Now define $\mathfrak{Y}^{\iota} := \mathfrak{X}^{\iota} \times \mathfrak{R}^{\iota}_{\mathscr{S},p}$. And for any $f \in R_{\mathscr{S},p}$, define an morphism on ringed spaces

$$R^{\iota}_f : \mathfrak{Y}^{\iota} \longrightarrow \mathfrak{X}^{\iota} \times \mathbb{A}^1_{rig}$$

as $(\lambda, \tilde{\theta}) \mapsto (\lambda, \tilde{\theta}(\iota \times f)^{-1})$ on L-points, and

$$R^{\iota *}_f : \mathscr{O}(\mathfrak{X}^{\iota})(X) \longrightarrow \mathscr{O}(\mathfrak{X}^{\iota})\hat{\otimes}\tilde{R}^{\iota}_{\mathscr{S},p}$$

as $\sum a_n X^n \mapsto \sum a_n(\iota \times f)^{-n}$ on the function rings.

Now define

$$\tilde{\mathfrak{E}}^{\iota} = \prod_f (R^{\iota}_f)^{-1}(\mathscr{S}^{\iota}(f))$$

and

$$\mathfrak{E}^{\iota} = \mathfrak{i}(\tilde{\mathfrak{E}}^{\iota})$$

With the above notations, we have:

Theorem 7.3 (twisted eigenvarities)**.** *Given K^p as above, there is a subvariety $\mathfrak{E}^{\iota}_{K^p}$ of $\mathfrak{E}_{K^p}$, satisfying:*

(*a*) *For any $(\lambda, \theta) \in \mathfrak{E}_{K^p}(\overline{\mathbb{Q}}_p)$, (λ, θ) is in $\mathfrak{E}^{\iota}_{K^p}(\overline{\mathbb{Q}}_p)$ if and only if θ has a non-trivial twisted Euler-Poincare characteristic and is self-dual with respect to ι.*

(*b*) *Every irreducible component of $\mathfrak{E}^{\iota}_{K^p}(\overline{\mathbb{Q}}_p)$ projects surjectively onto a Zariski subset of $\mathfrak{X}^{\iota}$.*

(*c*) *$\mathfrak{E}^{\iota}_{K^p}$ is equidimensional with the same dimension to $\mathfrak{X}^{\iota}$.*

Proof. (*a*) Write $V = V^{\dagger,\lambda}_{G,0}(K^p) = \oplus V_i^{m^{\iota}(\sigma_i,\lambda)}$ with V_i are self-dual irreducible finite slope representation σ_i of $\mathscr{H}_p(K^p)$, so in particular, $R_{S,p}$ acts on V_i by a character and V_i admits an action by ι. Moreover, write $V_i = \oplus V_i^{\zeta}$ be the spectral decomposition under ι as vector spaces, $R^{\iota}_{S,p}$ acts on each V_i^{ζ}. If $m^{\iota,\dagger}(\theta, \lambda, \iota) \neq 0$, then there is some V_i such that θ appears as $R_{S,p}$ acting on V_i, in particular, θ appears in some V_i^{ζ}. Define $\tilde{\theta}$ be the extension of θ to $R^{\iota}_{S,p}$ by setting $\tilde{\theta}(\iota) = \zeta$. Then for any $f \in R_{S,p}$, $\tilde{\theta}(\iota \times f) = \zeta\theta(f)$ is clearly an eigenvalue.

For the other direction, we need an adaption of the proof of [1, Proposition 5.2.3]. Let $(\lambda, \tilde{\theta})$ be in $\tilde{\mathfrak{E}}^{\iota}(L)$. Fix $t \in T^{++}$, and set $h = v_p(\tilde{\theta}(u_t))$ and set

$$W = V^{\leq h}$$

then we have $\mathscr{H}_p(K^p)$ actin on W since $R_{\mathscr{S},p}$ is in the center of $\mathscr{H}_p(K^p)$. Moreover, since every σ appearing in V is selfdual, that $\mathscr{H}_p(K^p)^{\iota}$ acts on W. Consider $R_{S,p} \to End_L(W)$ with image A finitely generated by $\{f_1, \cdots f_r\}$, and Ω be the set consisting of $\tilde{\theta}(u_t), \tilde{\theta}(f_i)$, and all eigenvalues of u_t, $\iota \times u_t$, f_i, $\iota \times f_i$ on W. Similar, we define R such that for any $\alpha, \alpha' \in \Omega$, $v_p(\alpha - \alpha') \leq v_p(R)$. and operators $h_1 = f_1$, $h_{i+1} = f_{i+1}(1 + Rh_i)$, $f = u_t(1 + Rh_r)$.

By those definitions, we can find $w_f \in V$ and some σappeaing in V, (we denote $\theta = \sigma|_{R_{S,p}}$), such that

$$\tilde{\theta}(\iota \times f)w_f = \sigma(\iota \times f)w_f$$

In particular, w_f is an eigenvector of $\sigma(\iota \times u_t)$ and $\sigma(\iota)$, we just denote their eigenvalue by a_f and b_f respectively. Since ι is of finite order, b_f is a unit.

Now by our definition again, we have $v_p(b_f)+v_p(\theta(u_t))+v_p(\theta(1+Rh_r)) = v_p(\tilde{\theta}(\iota))+v_p(\tilde{\theta}(u_t))+v_p(\tilde{\theta}(1+Rh_r))$. Since b_f and $\tilde{\theta}(\iota)$ are units, and our definition on R, we have that $v_p(\theta(u_t)) = v_p(\tilde{\theta}(u_t))$. This implies that σ acutally appears in W.

On the other hand, we have $(a_p - \tilde{\theta}(\iota \times u_t))\theta(1 + Rh_r) = \tilde{\theta}(\iota \times u_t)(\tilde{\theta}(1 + Rh_r) - \theta(1 + Rh_r))$. This implies $v_p(a_p - \tilde{\theta}(\iota \times u_t)) > v_p(R)$. So we have $a_p = \tilde{\theta}(\iota \times u_t)$ and $\theta(h_r) = \tilde{\theta}(h_r)$. Repeating the process, we have $\tilde{\theta}(f_i) = \theta(f_i)$ for all f_i, so $\tilde{\theta}|_{R_{\mathscr{S},p}} = \theta$.

(b) For any $f \in R_{,p}$, we have

$$\begin{array}{ccccc}
\tilde{\mathfrak{E}}^{\iota}_{K^p}(\overline{\mathbb{Q}}_p) & \xrightarrow{\mathrm{i}} & \mathfrak{E}^{\iota}_{K^p}(\overline{\mathbb{Q}}_p) & \hookrightarrow & \mathfrak{E}_{K^p}(\overline{\mathbb{Q}}_p) \\
\downarrow R^{\iota}_f & & \downarrow R_f & \swarrow R_f & \\
\mathfrak{S}^{\iota}_f(\overline{\mathbb{Q}}_p) & & \mathfrak{X}^{\iota} \cap \mathfrak{S}_f(\overline{\mathbb{Q}}_p) & & \\
\downarrow & \swarrow & & & \\
\mathfrak{X}^{\iota} & & & &
\end{array}$$

With a discussion as in the proof of (a) and [1, Proposition 5.2.3], we have a similar result to Proposition 4.8 for $\tilde{\mathfrak{E}}^{\iota}_{K^p}(\overline{\mathbb{Q}}_p)$, then the proof is the same to [1, Corollary 5.3.8].

(c) This follows from Theorem 4.11 and (b) above.

□

7.5 a bigger twisted eigenvariety

As we have seen in Theorem 7.3 above, the twisted eigenvariety $\mathfrak{E}^{\iota}(\overline{\mathbb{Q}}_p)$ is just a p-adic space which parameterizes those self-dual Hecke eigensystems with non-trivial twisted Euler-Poincare characteristics. In this section we constructed a twisted eigenvariety $\mathfrak{G}^{\iota}$ by adapting the method we used in Chapter 4. This is a rigid space whose $\overline{\mathbb{Q}}_p$-points parameterizes more self-dual Hecke eigensystems occurring in $H^{\bullet}(S_G(K^pI), \mathscr{D}_{\lambda})$ with $\lambda \in \mathfrak{X}^{\iota}$. We will see actually $\mathfrak{E}^{\iota}(\overline{\mathbb{Q}}_p)(\overline{\mathbb{Q}}_p) \subset \mathfrak{G}^{\iota}(\overline{\mathbb{Q}}_p)$ of codimension 0, and we can actually construct $\mathfrak{E}^{\iota}$ inside $\mathfrak{G}^{\iota}$. So it is a rigid variety. The proofs of the most propositions are just the same to chapter 4.

7.5.1 Spectral varieties

Let $\mathscr{U}$ be an open affinoid subdomain of $\mathfrak{X}^{\iota}$, as in section 4.1, we want to study the action of $R_{\mathscr{S},p}$ and $*\iota$ on $R\Gamma(K^pI, \mathscr{D}_{\mathscr{U}})$ as defined in section 3.1. We have by Proposition 2.3 and Remark 3.1.1 that, for any $f \in R_{\mathscr{S},p}$ admissible, there is a power series $P_{\mathfrak{X}^{\iota}}(f, \lambda, X) \in \mathscr{O}(\mathfrak{X}^{\iota})\{\{X\}\}$ whose restriction on $\mathscr{U} \subset \mathfrak{X}^{\iota}$ is written as $P_{\mathscr{U}}(f, \lambda, X)$, such that for any $\lambda \in \mathscr{U}$, its specialization at λ is the Fredholm determinant of f acting on $R\Gamma(K^pI, \mathscr{D}_{\lambda})$.

Proposition 7.4 (Urban)**.** *Let $j : N \hookrightarrow M$ a continuous injection of L-Banach spaces. Let u_N and u_M be respectively compact endomorphisms of N and M such that $j \circ u_N = u_M \circ j$. Then*

$M/j(N)$ has slope decomposition with respect to $u_{M/N} = u_M \ (\ mod\ j(N)\ )$, and

$$\det(1 - Xu_M) = \det(1 - Xu_N)\det(1 - Xu_{M/N})$$

This is [1, Proposition 2.3.9]. If we consider $M = N = R\Gamma(K, \mathscr{D}_{\mathscr{U}})$ and $j = *\iota$, then we have

$$P_{\mathscr{U}}(f, \lambda, X) = P_{\mathscr{U}}(f^{\iota}, \lambda, X) \tag{7.5.1.1}$$

Moreover, if we have the $\mathscr{O}(\mathscr{U})$-module decomposition

$$R\Gamma(K, \mathscr{D}_{\mathscr{U}}) = N_f(Q) \oplus F_f(Q)$$

corresponding to a polynomial decomposition $P_{\mathscr{U}}(f, \lambda, X) = Q(X)S(X)$. Then consider $N = N_f(Q)$, $M = R\Gamma(K, \mathscr{D}_{\mathscr{U}})$, $j = *\iota$, $u_N = f$ and $u_M = f^{\iota}$, we have $N_{f^{\iota}}(Q) = *\iota(N_f(Q))$.

So we define for given admissible f and $Q(X)$

$$N_{f,\iota}(Q) = \bigcap_{i=1}^{l} N_{f^{\iota^i}}(Q) \tag{7.5.1.2}$$

Remark. By our definition, it is obviously that $*\iota$ is well-defined on $N_{f,\iota}(Q)$. So we can consider the action of ${}^{\iota}R_{\mathscr{S},p}$ on $N_{f,\iota}(Q)$. Surly $N_{f,\iota}$ could be 0, but it is not always the case. For example, if σ is an irreducible representation of $\mathscr{H}_p$ lying in the set A as in the proof of Proposition 3.9. Then there is some $Q(X)$ of degree large enough such that $V_{\sigma} \subset N_{f^{\iota}}(Q)$ for each i.

Now just like Proposition 4.2. we define

$$\mathfrak{W}^{\iota}_Q = supp_{\mathscr{O}(\mathscr{U})} H(\widetilde{N^{\bullet}_{f,\iota}(Q)}) \tag{7.5.1.3}$$

Then $\lambda \in \mathfrak{W}^{\iota}_Q(f)(\overline{\mathbb{Q}}_p)$ if and only if $H(N^{\bullet}_{f,\iota}(Q)) \neq 0$. Same to section 4.2, we may glue those $\mathfrak{W}^{\iota}_Q$ to get a cohomological non-trivial weight space $\mathfrak{W}^{\iota}$, then, we have

Proposition 7.5.

$$\mathfrak{W}^{\iota} = \mathfrak{X}^{\iota} \cap \mathfrak{W} \tag{7.5.1.4}$$

Proof. $\lambda \in \mathfrak{W}$, if and only if $H_{fs}(S_G(K), \mathscr{D}_\lambda) \neq 0$. However,

$$H_{fs}(S_G(K), \mathscr{D}_\lambda) = \varinjlim_h H(S_G(K), \mathscr{D}_\lambda)_f^{\leq h} = \varinjlim_h \bigcap_i H(S_G(K), \mathscr{D}_\lambda)_{f^{\iota^i}}^{\leq h}$$

This implies that $\lambda \in \mathfrak{W}^\iota_{Q_h}$, with Q_h the polynomial for some slope decomposition with slope h large enough. □

Now we are going to construct the local spectral variety in this case. Let $f \in R_{\mathscr{S},p}$ admissible, define the set $[f]^\iota := \{f^{\iota^i} \times \iota^j | 1 \leq i, j \leq l\}$. For any $g \in [f]^\iota$, just as in section 4.1.2, define

$$\mathfrak{S}^\iota_{Q,g} := supp_{\mathscr{U}(U)[g]} \widetilde{H^\bullet_{f,\iota}(Q)}$$

Then same to Proposition 4.3, we have

Proposition 7.6. *$\mathfrak{S}^\iota_{Q,g}$ is locally finite over $\mathfrak{W}^\iota_Q$. $s = (\lambda, \alpha) \in \mathfrak{S}^\iota_{Q,g}(\overline{\mathbb{Q}}_p)$ if and only if $\lambda \in \mathfrak{W}^\iota_Q(\overline{\mathbb{Q}}_p)$ and α^{-1} is an eigenvalue of f acting on $H^\bullet_{f,\iota}(Q)$.*

Now same to section 4.2, given $[f]^\iota$ and $g \in [f]^\iota$, we can glue the local spectral varieties $\mathfrak{S}^\iota_{Q,g}$ with respect to ploynomials Q and open affinoid domains $\mathscr{U}$, as a summary, we have:

Theorem 7.7. *We can construct the spectral variety $\mathfrak{S}^\iota_g = \mathfrak{S}^\iota_{\mathfrak{X}^\iota,g}$ as a rigid subspace of $\mathfrak{X}^\iota \times \mathbb{A}_1^{rig}$, such that, $s = (\lambda, \alpha) \in \mathfrak{S}^\iota_g(\overline{\mathbb{Q}}_p)$ if and only if $\lambda \in \mathfrak{W}^\iota(\overline{\mathbb{Q}}_p)$ and α^{-1} is an eigenvalue of g acting on $H^\bullet_{fs}(S_G(K), \mathscr{D}_\lambda)$.*

7.5.2 the bigger eigenvariety

In this subsection we build the bigger eigenvariety over the spectral varieties constructed in the last subsection as in section 4.3. Denote $R_{\mathscr{S},p}[\iota]$ be the algebra generated by ι over $R_{\mathscr{S},p}$ under the relation $\iota \times f \times \iota^{-1} = f^\iota$. Now we can define a p-adic space $\mathfrak{B} = \mathfrak{B}_{\mathscr{S},p}$ such that for any $L/\mathbb{Q}_p$,

$$\mathfrak{B}(L) = \hom_{alg\ ct}(\overline{R}_{\mathscr{S},p}[\iota], L)$$

where $\overline{R}_{\mathscr{S},p}$ is defined as in section 4.3. Define $\mathfrak{Z} = \mathfrak{Z}_{\mathscr{S},p} := \mathfrak{X}^\iota \times \mathfrak{B}$, for any admissible f and $g \in [f]^\iota \subset R_{\mathscr{S},p}[\iota]$, we define the morphism of ringed space $R_g : \mathfrak{Z} \to \mathfrak{X}^\iota \times \mathbb{A}_1^{rig}$ in exactly the same way as in 4.3, then define

$$\tilde{\mathfrak{D}}^\iota := \prod_{[f]^\iota} \prod_{g \in [f]^\iota} R_g^{-1} \mathfrak{S}_g^\iota$$

it topology is also defined as in 4.3.

Then we have a parallel result to Proposition 4.8:

Proposition 7.8. *Assume $\tilde{y} = (\lambda, \tilde{\theta})$ is in $\mathfrak{Z}(\overline{\mathbb{Q}}_p)$, then $\tilde{y} \in \tilde{\mathfrak{D}}^\iota(\overline{\mathbb{Q}}_p)$ if and only if $H^\bullet(S_G(K), \mathscr{D}_\lambda)[\tilde{\theta}] \neq 0$ as ${}^\iota R_{\mathscr{S},p}$-modules.*

Now consider $\mathscr{U} \subset \mathfrak{X}^\iota$ and $P_{\mathscr{U}}(f, X) = Q(X)S(X)$ as above, define $h^\iota_{\mathscr{U}}$ be the image of $R^\iota_{\mathscr{U}} := \mathscr{O}(\mathscr{U}) \otimes R_{\mathscr{S},p}[\iota]$ in $End^b_{pf}(R\Gamma^\bullet(K^pI, \mathscr{D}_{\mathscr{U}}))$ and $h^\iota_{\mathscr{U},Q}$ be the image of $R^\iota_{\mathscr{U}}$ in $End^b_{pf}(N_{f,\iota}(Q))$. Define

$$\tilde{\mathfrak{G}}^{\iota\prime}_{\mathscr{U}} := sp(h^\iota_{\mathscr{U}})$$

and

$$\tilde{\mathfrak{G}}^\iota_{\mathscr{U},Q} := supp_{h^\iota_{\mathscr{U},Q}} H(\widetilde{N^\bullet_{f,\iota}(Q)})$$

Then we have

Proposition 7.9.

$$\tilde{\mathfrak{G}}^\iota_{\mathscr{U},Q}(\overline{\mathbb{Q}}_p) = \prod_{g \in [f]^\iota} R_g^{-1} \mathfrak{S}^\iota_{Q,g}(\overline{\mathbb{Q}}_p)$$

The proof of the proposition is same to Proposition 4.9, moreover, just like the discussion from Proposition 4.9 to Proposition 4.11, we can patch $\tilde{\mathfrak{G}}^\iota_{\mathscr{U},Q}$ with respect to $\mathscr{U}$ and Q and take product over $[f]^\iota$'s to get a rigid variety $\tilde{\mathfrak{G}}^\iota$, whose points are same to $\tilde{\mathfrak{D}}^\iota(\overline{\mathbb{Q}}_p)$.

We now in the position to define the morphism

$$\mathfrak{i} : \tilde{\mathfrak{D}}^\iota \to \mathfrak{E}$$

Given $\mathscr{U}$, Q as above, we define locally

$$\mathfrak{i} : \tilde{\mathfrak{G}}^{\iota}_{\mathscr{U},Q} := supp_{h^{\iota}_{\mathscr{U},Q}} H(\widetilde{N^{\bullet}_{f,\iota}(Q)}) \to \mathfrak{E}_{\mathscr{U},Q} := supp_{h_{\mathscr{U},Q}} H(\widetilde{N^{\bullet}_{f}(Q)})$$

deduced by the inclusions $h_{\mathscr{U},Q} \hookrightarrow h^{\iota}_{\mathscr{U},Q}$ and $H(N^{\bullet}_{f,\iota}(Q)) \hookrightarrow H(N^{\bullet}_{f}(Q))$. So on the $\overline{\mathbb{Q}}_p$-points, $\mathfrak{i}$ sends $(\lambda, \tilde{\theta})$ to $(\lambda, \theta = \tilde{\theta}|_{R_{\mathscr{S},p}})$. Finally we define:

$$\mathfrak{G}^{\iota} := \mathfrak{i}(\tilde{\mathfrak{G}}^{\iota}) \tag{7.5.2.1}$$

Theorem 7.10. *Assume $y = (\lambda, \theta)$ is in $\mathfrak{E}(\overline{\mathbb{Q}}_p)$, then $y \in \mathfrak{G}^{\iota}(\overline{\mathbb{Q}}_p)$ implies that θ is self-dual with respect to ι. For any $f \in R_{\mathscr{S},p}$, the $R_f : \mathfrak{G}^{\iota} \to \mathfrak{S}^{\iota}_{\mathfrak{W}^{\iota},f}$ is locally finite and surjective. So in particular,* $\dim \mathfrak{G}^{\iota} \leq \dim \mathfrak{X}^{\iota}$.

This is directly follows from the definition.

Remark. (*a*) Generally, $\mathfrak{G}^{\iota}$ is NOT the full eigenvariety parameterizing all self-dual Hecke eigensystems. Actually, it parameterizes those θ such that, as a one-dimensional subspace of $H(S_G(K), \mathscr{D}_{\lambda})$, the $*\iota$ maps V_{θ} to itself.

(*b*) However, if θ has non-trivial twisted Euler-Poincare characteristic, then by a discussion as in the proof of Theorem 7.3 (*a*), θ can be realized in some irreducible self-dual representation σ and with $*\iota$ acted by some eigenvalue. So in particular, $\mathfrak{E}^{\iota}(\overline{\mathbb{Q}}_p)(\overline{\mathbb{Q}}_p) \subset \mathfrak{G}^{\iota}(\overline{\mathbb{Q}}_p)$ of codimension 0.

(*c*) We can actually redo our construction in section 7.4 in $\tilde{\mathfrak{D}}^{\iota}$ instead of in $\mathfrak{Y}^{\iota}$. If so, we do have a rigid space whose points are the points of $\mathfrak{E}^{\iota}$.

Chapter 8

THE CASE OF Gl_n

In this section we study the case when $G = Gl_n$ over $\mathbb{Q}$, and ι is the Cartan invoultion defined by transpose inverse. The results may be related to the conjecture of Asn-Pollack-Stevens [2].

As the general study in last chapters, by considering the twisted twisted FSCDs $I^{\dagger}_{G,0}(\iota \times f, \lambda)$, we can construct an eigenvarity $\mathfrak{E}^{\iota}$ siting over $\mathfrak{X}^{\iota}$ if we can prove it is non-trivial. This is the next proposition:

Proposition 8.1. *Assumptions as above, $\lambda \in \mathfrak{X}^{\iota}$ is regular. Let σ be a fnite slope cuspidal automorphic representation, which is self-dual with respect to ι. Further assume that σ is non-critical with respect to λ. Then $m^{\iota}_{EP}(\sigma, \lambda) \neq 0$*

Proof. We use the notations in Theorem 6.8. By the computation in Lemma 7.2, we know

$$m^{\iota}_{EP}(\sigma, \lambda) = \sum_{\pi} m_{cusp}(\pi) L(\iota, \pi, \lambda) tr(\iota | Hom_{\mathscr{H}_p(K^p)}(\sigma, \pi_f))$$

Since we are dealing with Gl_n, the cohomological packet at infinity for λ has only one element, which is of the form $A_{\mathfrak{b}}(\lambda)$ as in Theorem 6.8 (see also [24]). So

$$m^{\iota}_{EP}(\sigma, \lambda) = L(\iota, A_{\mathfrak{b}}(\lambda), \lambda) \sum_{\pi_{\infty} = A_{\mathfrak{b}}(\lambda)} m_{cusp}(\pi) tr(\iota | Hom_{\mathscr{H}_p(K^p)}(\sigma, \pi_f))$$

and we only have to show that

$$0 \neq L(\iota, A_{\mathfrak{b}}(\lambda), \lambda) = (-1)^{R_{\mathfrak{b}}} \sum_i (-1)^i tr(\iota | Hom_{\mathfrak{l}\cap\mathfrak{k}}(\wedge^i(\mathfrak{l}\cap\mathfrak{p}), \mathbb{C}))$$

where ι acting on $Hom_{\mathfrak{l}\cap\mathfrak{k}}(\wedge^i(\mathfrak{l}\cap\mathfrak{p}), \mathbb{C})$ deduced from its action on $\wedge^i(\mathfrak{l}\cap\mathfrak{p})$. However, recall that our ι is given by the Cartan involution, so it acts on $\wedge^i(\mathfrak{l}\cap\mathfrak{p})$ by scalar $(-1)^i$. This implies $L(\iota, A_{\mathfrak{b}}(\lambda), \lambda) = (-1)^{R_{\mathfrak{b}}} \sum dim \wedge^i (\mathfrak{l}\cap\mathfrak{p}) = (-1)^R 2^n \neq 0$. □

Remark. The proposition 8.1 above implies that the twisted eigenvariety $\mathfrak{E}^{\iota}$ we constructed paremeterizes all self-dual finite slope cuspidal automorphic representations in this case.

Remark. The proposition is not always true for an arbitary ι. For instance, let $G = Gl_n$ with n is even and J the anti-diagonal matrix defined by

$$J = (\delta_{i,n+1-j})_{1\leq i,j\leq n}$$

Then J defines an involution ι' on G such that for any $g \in G$

$$g^{\iota'} = J^t g^{-1} J^{-1}$$

Now let (B, T) be the Borel pair given by upper-trangular matrice, it is stable under ι'. Now every algebraic weight λ of g is determined by n integers $(r_1, \cdots, r_n)$ such that

$$\lambda(diag(t_1, \cdots, t_n)) = t_1^{r_1} \cdots t_n^{r_n},$$

So $\lambda \in \mathfrak{X}^{\iota'}$ if and only if $r_i + r_{n+1-i} = 0$ for any i, and we know further that $dim\mathfrak{X}^{\iota} = [\frac{n}{2}]$.

Similarly, the computation of the Lefschetz number is reduced to the computation of $\sum_i (-1)^i tr(\iota' | Hom_{\mathfrak{l}\cap\mathfrak{k}}(\wedge^i(\mathfrak{l}\cap$
Since now ι' is the composition of $ad(J)$ with the Cartan involution,

$$\begin{aligned} & \sum_i (-1)^i tr(\iota' | Hom_{\mathfrak{l}\cap\mathfrak{k}}(\wedge^i(\mathfrak{l}\cap\mathfrak{p}), \mathbb{C})) \\ = & \sum_i tr(ad(J) | Hom_{\mathfrak{l}\cap\mathfrak{k}}(\wedge^i(\mathfrak{l}\cap\mathfrak{p}), \mathbb{C})) \\ = & \sum_{j=0}^{n/2} (-1)^j C_{n/2}^j = 0 \end{aligned}$$

Before going further we fix the Borel pair (B,T) as in the remark above. Surely it is not stable under ι, however, by Lemma 5.2, a Borel pair we can choose to be stable is actually conjugate to it. So below when we say ι acting on B we actually mean it acts via a conjugate. In particular, ι acts on T as $diag(t_1,\cdots,t_n)\mapsto diag(t_n^{-1},\cdots,t_1^{-1})$.

We also care about the situation called essentially selfdual, in our context, an automorphic representation π is essentially selfdual with respect to ι if there exists an algebraic character χ such that $\chi\circ\det\otimes\pi^\iota\cong\pi$. It is similarly defined for a finite slope automorphic representation σ and a weight λ.

Now a weight $\lambda\in\mathfrak{X}$ is defined by n characters $\chi_1,\cdots,\chi_n$ as

$$\lambda: diag(t_1,\cdots,t_n)\mapsto\chi_1(t_1)\cdots\chi_n(t_n)$$

it follows that we can cut out a subspace of essentially selfdual weights in $\mathfrak{X}$ by the relations $\chi_i\chi_{n+1-i}=\chi_j\chi_{n+1-j}$ for any $0\le i,j\le n$. We donote this space $\mathfrak{X}^e$, which if of dimension $[\frac{n}{2}]+1$. Moreover, its subspace $\mathfrak{X}^\iota_\chi$ of weights essentially selfdual with respect to χ is cut out by the relation $\chi_i\chi_{n+1-i}=\chi_j\chi_{n+1-j}=\chi$, of dimension $[\frac{n}{2}]$.

Considering conjecture 1.1, it is natural to ask if we can construct an eigenvariety which parameterizes all essentially selfdual finite slope cuspidal representations over $\mathfrak{X}^e$. It maybe also important to remark here that every cuspidal automorphic representation is actually sitting over $\mathfrak{X}^e$, by a theorem of Borel-Wallach [10].

It turns out that we can construct $\mathfrak{E}^e$, the essentially selfdual eigenvariety by applying our method to the group $\tilde{G}=Gl_n\times Gl_1$, with an finite order automorphism $\mu:(g,x)\mapsto(g^\iota,\det(g)x)$. Concretely speaking, in this case the weight space $\tilde{\mathfrak{X}}=\mathfrak{X}\times\mathfrak{B}^1$, and the μ-invariant subspace $\tilde{\mathfrak{X}}^\mu$ is given by $\{\tilde{\lambda}=(\lambda,\chi)|\lambda\in\mathfrak{X}^\iota_\chi,\chi\in\mathfrak{B}^1\}$, whose first projection bijectively to $\mathfrak{X}^e$.

Now choose any compact $\tilde{K}=k\times K_1$ of $\tilde{G}$ with K_1 a compact subgroup of Gl_1, we have

$$S_{\tilde{G}}(\tilde{K})=S_G(K)\times(\mathbb{Q}^\times\backslash\mathbb{A}_{\mathbb{Q}}^\times/\hat{\mathbb{Z}}\mathbb{R}^\times)=S_G(K)$$

and we have the Hecke algebra

$$\mathscr{H}_p(\tilde{K}^p) = \mathscr{H}_p(K^p) \times (C_c^\infty(\mathbb{A}^{\times p}/\hat{\mathbb{Z}}^p)) \otimes \mathbb{Z}_p$$

where the last $\mathbb{Z}_p$ appears as constant functions. Now for any $f \in R_{\mathscr{S},p}(\tilde{K}^p)$ we write $f = (f_G, f_1)$ with $f_G = f_G^p \otimes u_t$ $t \in T^{++}$ and $f_1 \in \mathbb{Z}_p^\times$. So we can define the twisted FSCDs $I_?^{cl}(\mu \times f, \tilde{\lambda})$ and $I_?^\dagger(\mu \times f, \tilde{\lambda})$ as before, where $? = \tilde{G}, (\tilde{G}, 0), (\tilde{G}, \tilde{M}, 0)$ when they are making sense.

We can also compute the twisted Euler-Poincare characteristic in this case, acutally for a given weight $\tilde{\lambda} = (\lambda, \chi)$, an irreducible finite slope representation $\tilde{\sigma}$ makes $m_{EP}^\mu(\tilde{\sigma}, \tilde{\lambda}) \neq 0$ only if $\tilde{\sigma} = \sigma \otimes \chi$, and so

$$\begin{aligned}
m_{EP}^\mu(\tilde{\sigma}, \tilde{\lambda}) &= \sum_{\tilde{\pi}} m_{cusp}(\tilde{\pi}) L(\mu, \tilde{\pi}_\infty, \tilde{\lambda}) tr(\mu | Hom_{\mathscr{H}_p}(\tilde{\sigma}, \tilde{\pi}_f)) \\
&= \sum_{\pi} m_{cusp}(\pi) L(\mu, \tilde{\pi}_\infty, \tilde{\lambda}) tr(\iota | Hom_{\mathscr{H}_p}(\sigma, \pi_f)) \\
&= (-1)^R \sum_{\pi} m_{cusp}(\pi) tr(\iota | Hom_{\mathscr{H}_p}(\sigma, \pi_f)) \\
&= m_{EP}^\iota(\sigma, \lambda) \neq 0
\end{aligned}$$

Then a totally same disscusion as Proposition 8.1 imples that we have an eigenvariety parameterizing all irreducible finite slope automorphic representations $\tilde{\sigma}$ of $\tilde{G}$ which are selfdual with respect to μ. Now noticing that every finite slope automorphic representation $\tilde{\sigma}$ of $\tilde{G}$ is given by a finite slope automorphic representation σ of Gl_n and a character χ of Gl_1, and $\tilde{\sigma}$ is selfdual wit respect to μ if and only if σ is essentially selfdual with respect to χ, we have

Theorem 8.2. *For $G = Gl_n$ and ι the Cartan involution, we have an eigenvariety $\mathfrak{E}^e \subset \mathfrak{E}$ such that*

- *(a) there are two projections $p_1 : \mathfrak{E}^e \to \mathfrak{X}^e$ and $p_2 : \mathfrak{E}^e \to \mathfrak{B}^1$, such that $y = (\lambda_y, \theta_y) \in \mathfrak{E}^e(\bar{\mathbb{Q}}_p)$ if and only if θ is an eigencharacter of G of weight $\lambda_y = p_1(y)$ and θ is essentially selfdual with respect to $\chi_y = p_2(y)$*

- *(b) $\mathfrak{E}^e$ is equidimensional of dimension $[\frac{n}{2}] + 1$*

(c) For any $\chi \in \mathfrak{B}^1$, *set* $\mathfrak{E}^e_\chi = p_2^{-1}(\chi)$, *then* $\mathfrak{E}^e_\chi$ *is the eigenvariety of* G *parameterizing irreducible finite slope eigencharacters which are essentially selfdual with respect to* χ. *In particular,* $\mathfrak{E}^e_0 = \mathfrak{E}^\iota$.

Remark. If we analyze the twisted FSCDs $I^?_?(\mu \times f, \tilde{\lambda})$ carefully, we can related them to $I^?_?(\iota \times f_G, \lambda)$, for instance, for $? = \emptyset$ or 0

$$\begin{aligned} I^{cl}_{\tilde{G},?}(\mu \times f, \tilde{\lambda}) &= \tilde{\lambda}(\xi(t, f_1))tr(\mu \times f|H^\bullet_?(S_{\tilde{G}}, \mathbb{W}^\vee_{\tilde{\lambda}}(\mathbb{C}))) \\ &= \lambda(\xi(t))\chi(f_1)tr(\mu \times f|H^\bullet_?(S_{\tilde{G}}, \mathbb{W}^\vee_{\tilde{\lambda}}(\mathbb{C})) \\ &= \lambda(\xi(t))\chi(f_1)\chi(det(f_G))tr(\lambda \times f_G|H^\bullet_?(S_G, (\mathbb{W}_\lambda(\mathbb{C}) \otimes_\chi \mathbb{C})^\vee) \end{aligned}$$

So in particular, if we set χ in $\tilde{\lambda}$ to be trivial, then

$$I^{cl}_{\tilde{G},?}(\mu \times f, \tilde{\lambda}) = I^{cl}_{G,?}(\lambda \times f_G, \lambda)$$

and if for given χ we apply Urban's theoy directly to

$$I^\chi_{G,0}(\iota \times f, \lambda) := \lambda(\xi(t))\chi(f_1)\chi(det(f_G))tr(\lambda \times f_G|H^\bullet_?(S_G, (\mathbb{W}_\lambda(\mathbb{C}) \otimes_\chi \mathbb{C})^\vee)$$

we get $\mathfrak{E}^e_\chi$ directly. For an irreducible finite slope representation σ of Gl_n which is essentially selfdual with respect to χ, and $f \in R_{\mathscr{S},p}$ we define

$$J^\chi_{\iota\times\sigma}(f) := \chi(\det(f))tr(\iota \times f|V_\sigma \otimes_\chi \mathbb{C})$$

Now let θ be a classical cuspidal finite slope Hecke eigen character of Gl_n with regular weight λ and of weight K, following [2], we define θ to be p-adically arithmetic rigid, if it is not contained in any irreducible component of $\mathfrak{E}_{K^p}$ containing a Zariski dense subset of finite slope Hecke eigen characters of arithmetic weights (such a component is called an arithmetic component). So conjecturely ([2]) θ is not essentially selfdual then it is p-adically arithmetic rigid. It is easy to see our Theorem above gives the next corollary, which is the inverse of the conjecture:

Corollary 8.3. *If θ_0 as above is further essentially selfdual, then it is not p-adically arithmetic rigid, with an arithmetic component in $\mathfrak{E}^e$.*

Proof. By the theorem above we have that $(\lambda_0, \theta_0) \in \mathfrak{E}^e(\bar{\mathbb{Q}}_p)$. Now consider the subset Σ of $\mathfrak{E}^e(\bar{\mathbb{Q}}_p)$ containing those (λ, θ) such that λ is arithmetic and θ is non-critical with respect to λ. Σ is Zariski dense in $\mathfrak{E}^e(\bar{\mathbb{Q}}_p)$ since its projection to $\mathfrak{X}^e$ contains an arithmetic point λ. Moreover, by Lemma 7.2, those points in Σ are classical and correspond to cuspidal Hecke eigen characters. □

The next theorem shows that the smooth hyperthesis on arithmetic points of an eigenvarity may give some hint on APG's conjecture.

Theorem 8.4. *Assume that every arithmetic point on eigenvariety is smooth. If θ_0 is not p-adically arithmetic rigid, and its arithmetic component contains an arithmetic, essentially selfdual Hecke eigen character, then this arithmetic component contains a Zariski dense subset of essentially selfdual Hecke eigen characters.*

Proof. By [2], (λ_0, θ_0) is contained in an arithmetic component of dimension $[\frac{n}{2}]+1$ over $\mathfrak{X}^e$. By our assumption, this component intersects with $\mathfrak{E}^e$ at some smooth point and $\mathfrak{E}^e$ is also of dimension $[\frac{n}{2}]+1$. So the arithmetic component contains an irreducible component of $\mathfrak{E}^e$. Then, the theorem follows form the corollary above. □

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