What is the complexity of surface integration?

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Abstract

We study the worst case complexity of computing ε-approximations of surface integrals. This problem has two sources of partial information: the integrand f and the function g defining the surface. The problem is nonlinear in its dependence on g. Here, f is an r times continuously differentiable scalar function of l variables, and g is an s times continuously differentiable injective function of d variables with l components. We must have $d \le l$ and $s \ge 1$ for surface integration to be well-defined. Surface integration is related to the classical integration problem for functions of d variables that are $\min\{r, s-1\}$ times continuously differentiable. This might suggest that the complexity of surface integration should be $\Theta((1/\varepsilon)^{d/\min\{r,s-1\}})$. Indeed, this holds when $d < l$ and $s = 1$, in which case the surface integration problem has infinite complexity. However, if $d \le l$ and $s \ge 2$, we prove that

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the complexity of surface integration is $O((1/\varepsilon)^{d/\min\{r,s\}})$. Furthermore, this bound is sharp whenever $d < l$.

1 Introduction

Surface integration is an important problem of applied mathematics. One example is the calculation of surface area. In addition, the solutions of many problems (such as elliptic boundary value problems, see, e.g., Stakgold 1998) can be expressed as surface integrals. This explains why it is natural to study the complexity of computing an ε-approximation of surface integrals.

A surface integral is defined by two functions. The first is an integrand, which is a scalar function f of l variables. The second is a function g defining the surface. This function g depends on d variables, and must have l components. The surface is well-defined iff g is a C^1-injection. Hence, we must have $d \le l$. These functions f and g may have different smoothness. In this paper, we assume that f and g are r and s times continuously differentiable, respectively. Obviously, we must have $r \ge 0$ and $s \ge 1$.

The numerical computation of surface integrals involves two sources of partial information, arising from evaluating the functions f and g at points in their domains. The surface integration problem is nonlinear in its dependence on the function g. However, for a fixed g, the problem is linear in its dependence on f.

A surface integral may be expressed as a classical integral of a d-variate scalar function, which is $\min\{r, s-1\}$ times continuously differentiable. The classical integration problem is a linear problem, and is one of the most-studied problems in information-based complexity (IBC). In particular, it is known that the ε-complexity of classical integration for d-variate functions that are t times continuously differentiable is proportional to $(1/\varepsilon)^{d/t}$; see Bakhvalov (1959), as well as the references in Novak (1988), Traub et al. (1988), and Traub and Werschulz (1998). Thus, it would appear plausible that the ε-complexity of surface integration should be proportional to $(1/\varepsilon)^{d/\min\{r,s-1\}}$.

Surprisingly enough, this is not usually the case. This result holds only when $d < l$ and $s = 1$. In this case, the complexity of surface integration becomes infinite, and so the problem is unsolvable. This is not particularly surprising. Indeed, the definition of the surface integral involves partial derivatives of g. Hence when $s = 1$, we have the minimal amount of smoothness required for the surface integral to be well-defined, and so the integrand appearing in the transformed problem cannot be approximated arbitrarily closely. This is similar to many other IBC problems, for which minimal smoothness implies infinite complexity, see, e.g., the many instances in Traub and Woźniakowski (1980) and Werschulz (1991).

We now suppose that $d \le l$ and $s \ge 2$. Then we prove that the ε-complexity is at most proportional to $(1/\varepsilon)^{d/\min\{r,s\}}$. Note that we have s in the denominator of the exponent, rather than the $s-1$ that we would expect. This surprising result holds because the surface integral can be expressed as a sum of integrals of smoother integrands, provided that we do integrations by parts. This technique yields upper bounds on the error that involve second derivatives of g; hence these bounds are of the proper magnitude only when $s \ge 2$.

Are these bounds sharp? We give a partial answer in the affirmative. More precisely, when $d < l$, we prove that the ε-complexity is proportional to $(1/\varepsilon)^{d/\min\{r,s\}}$. Hence these bounds are sharp in this case.

What happens when $d = l$? We have only a partial result, for the case $d = l = 1$. We find that the ε-complexity is proportional to $(1/\varepsilon)^r$. Note that this is independent of s, the smoothness of g, and holds even when $s = 1$. Sharpness of the upper bound for the case $d = l \ge 2$ is an open problem.

As mentioned above, one important instance of the surface integration problem is the calculation of surface area. One might hope to get better complexity bounds for the surface area problem, since the integrand is identically one and we only have one source of partial information, namely the surface itself. Unfortunately, this is not the case. The proof of the lower bound for surface integration holds for constant integrands. Hence the complexity of surface area is of the same order as surface integration.

We pose some open problems.

In this paper, we study what are probably the most commonly-used classes of smooth functions. An alternative would be to use classes of functions having bounded mixed derivatives of orders r and s. In this case, the dependence of complexity on d should be less drastic, by analogy with known results for the classical integration problem, see, e.g., Temlyakov (1989), as well as the references in Traub and Werschulz (1998). Modulo logarithmic factors, we expect that the complexity should be proportional to $(1/\varepsilon)^{1/\min\{r,s\}}$.

This paper concentrates on establishing sharp exponents of $1/\varepsilon$. In particular, we have ignored any dependence of Θ-factors on d and l. As long as d is small, this dependence is not crucial. However, for large d relative to r and s, we have the curse of dimension. Hence, surface integration joins the club whose members are the many computational problems suffering from this curse. As with these other problems, this curse can be broken by switching to the randomized setting, since the classical Monte Carlo method can be used. So, the complexity of surface integration in the randomized setting is at most proportional to $(1/\varepsilon)^2$, even for $r = 0$ and $s = 1$. For the classical integration problem, the curse of dimension can be broken, even in the worst case setting, if we consider integrands from weighted classes of functions (Sloan and Woźniakowski, 1998). Our hope is that the same can be done for surface integration.

We briefly outline this paper. Section 2 contains a precise definition of the problem for general classes of integrands and surfaces. In Section 3, we establish relations between the surface and classical integration problems for general classes. Finally, in Section 4, we turn attention to the classes of smooth functions described above. We present an algorithm, establish its error bound, and prove its optimality in the case $d < l$.

2 Problem formulation

Before describing the problem to be solved, we first recall the definition of surface integrals; see Edwards (1973, p. 334 ff.) for further discussion. Let d and l be given positive integers, with $d \le l$. Let $I = [0, 1]$ denote the unit interval. For a C^1 injection $g\colon I^d \to \mathbb{R}^l$, we say that

$$\Sigma_g = \{\, g(x) : x \in I^d \,\}$$

is a d-dimensional *surface* in $\mathbb{R}^l$.

For any such function g, the *gradient* $\nabla g\colon I^d \to \mathbb{R}^{l\times d}$ is defined by

$$[(\nabla g)(x)]_{i,j} = (\partial_j g_i)(x) \qquad \text{for } i \in \{1, \dots, l\},\ j \in \{1, \dots, d\}, \text{ and } x \in I^d,$$

where ∂_j denotes the partial derivative in the jth coordinate direction and g_i is the ith component of g.

Define

$$\sigma_g(x) = \sqrt{\det A(x)} \qquad \forall\, x \in I^d,$$

where

$$A(x) = [(\nabla g)(x)]^T [(\nabla g)(x)] \qquad \forall\, x \in I^d,$$

i.e., $A(x) = [(a_{i,j}(x)]_{i,j=1}^d$ is the $d \times d$ matrix having components

$$a_{i,j}(x) = \sum_{k=1}^{l} (\partial_i g_k)(x)(\partial_j g_k)(x)$$

for $i, j \in \{1, \dots, d\}$ and $x \in I^d$. Note that when $d = l$, this simplifies to

$$\sigma_g(x) = |\det[(\nabla g)(x)]| \qquad \forall\, x \in I^d,$$

If $f\colon D_f \subseteq \mathbb{R}^l \to \mathbb{R}$ is a measurable function whose domain D_f is a superset of Σ_g, then

$$\int_{\Sigma_g} f\, d\sigma = \int_{I^d} (f \circ g)\sigma_g \equiv \int_{I^d} f(g(x))\, \sigma_g(x)\, dx \tag{2.1}$$

is the *surface integral of* f *over* Σ_g.

We now describe the problem to be solved. For given $F \times G$ we want to approximate the *surface integral* operator

$$S(f, g) = \int_{\Sigma_g} f \, d\sigma \qquad \forall [f, g] \in F \times G.$$

Here, we assume that $F \times G$ is chosen such that the surface integral operator S is well-defined. Observe that the presence of g means that S is a *nonlinear* functional. However, for a fixed g, the functional S depends linearly on f.

We compute an approximation $U(f, g)$ to $S(f, g)$ by using information

$$N(f, g) = [f(x^{(1)}), \ldots, f(x^{(m)}), g(x^{(m+1)}), \ldots, g(x^{(n)})] \tag{2.2}$$

of cardinality $m + l(n - m)$, where $x^{(1)}, \ldots, x^{(m)} \in I^l$ and $x^{(m+1)}, \ldots, x^{(n)} \in I^d$. We also allow adaption. That is, the number n of evaluations, as well as the sample points $x^{(1)}, \ldots, x^{(n)}$, may depend on the previously-computed function values of f and g; for details, see, e.g., Traub et al. (1988, Chapter 2).

Remark. Note that the permissible information is function values of f and g. One could also allow the evaluation of derivatives, as well. We restrict ourselves to function values alone, as this makes the exposition much simpler. However, it is easy to see that the results of this paper also hold if arbitrary derivative evaluations are allowed. □

Our approximation U is given by

$$U(f, g) = \phi\big(N(f, g)\big) \tag{2.3}$$

for some mapping $\phi \colon N(F \times G) \to \mathbb{R}$. We measure the quality of an approximation U by its *worst case error*

$$e(U) = \sup_{[f,g] \in F \times G} |S(f, g) - U(f, g)| \,.$$

The cost of computing $U(f, g)$ is defined as cost $\big(U(f, g)\big)$, which is the weighted sum of the total number of function values of f and g, as well as the number of arithmetic operations and comparisons needed to obtain $U(f, g)$. More precisely, we assume that each evaluation of f or of g_i (where $i \in \{1, \ldots, l\}$) costs $\mathbf{c}$. The cost of each arithmetic operation is taken as 1. Then cost $\big(U(f, g)\big)$ for U of the form (2.3) is $\mathbf{c}\,\big(m + l(n - m)\big) + \tilde{n}$, where $\tilde{n}$ is the total number of arithmetic operations and comparisons needed to compute $U(f, g)$. Here $\mathbf{c} \ge 1$, and usually it is realistic to assume that $\mathbf{c} \gg 1$; see once more Traub et al. (1988, Chapter 2) or Traub and Werschulz (1998, Chapter 2) for details. Then

$$\mathrm{cost}(U) = \sup_{[f,g]\in F\times G} \mathrm{cost}\big(U(f,g)\big)$$

is the worst case cost of U. The *ε-complexity* of surface integration in the class $F \times G$ is the minimal cost of computing an ε-approximation, i.e.,

$$\mathrm{comp}(\varepsilon) = \inf\{\,\mathrm{cost}(U) \;:\; U \text{ such that } e(U) \le \varepsilon\,\}.$$

The purpose of this paper is to find sharp estimates of the ε-complexity of surface integration for a number of classes $F \times G$. Our estimates will be sharp only in terms of the power of ε^{-1}, with constants depending, in particular, on d and l.

3 Surface and classical integration

We show that the surface integration problem is related to the classical problem of integration. The latter is defined as an approximation of

$$\mathrm{S}^{\mathrm{int}}(f) = \int_{I^d} f(x)\,dx, \qquad f \in H$$

for some class H of scalar functions defined on I^d. Let $\mathrm{comp}^{\mathrm{int}}(\varepsilon; H)$ denote the ε-complexity of integration, which is defined analogously to surface integration. The classical integration problem has been extensively studied and sharp bounds on its ε-complexity are known for many classes H, see Traub and Werschulz (1998) and Woźniakowski (1999) for surveys.

Relations between surface and classical integration will be presented by inequalities between their complexities. That is, for the given class $F \times G$ we will find corresponding classes H such that the ε-complexity of surface integration for $F \times G$ is bounded from below and above by the ε-complexity of integration for the classes H.

We begin with a lower bound. We assume that the identity embedding or projection function $\mathrm{id} : \mathbb{R}^d \to \mathbb{R}^l$, defined as

$$\mathrm{id}(x) = [x_1, x_2, \ldots, x_d, 0, 0, \ldots, 0] \qquad \forall\, x \in \mathbb{R}^d,$$

belongs to G. Then $\sigma_{\mathrm{id}}(x) \equiv 1$ and

$$S(f, \mathrm{id}) = \mathrm{S}^{\mathrm{int}}(f \circ \mathrm{id}) \qquad \forall\, f \in F. \tag{3.1}$$

Define

$$H(F) = \{ f \circ \mathrm{id} : f \in F \}.$$

The class $H(F)$ is the natural projection of functions from F by taking the first d variables, which amounts to restricting the domain of the functions to I^d. Then (3.1) yields

$$\mathrm{comp}(\varepsilon; F, G) \geq \mathrm{comp}^{\mathrm{int}}\big(\varepsilon; H(F)\big). \tag{3.2}$$

This means that surface integration for $F \times G$ is not easier than integration for $H(F)$. Since the assumption id $\in G$ is quite natural and sharp error estimates of $\mathrm{comp}^{\mathrm{int}}\big(\varepsilon; H(F)\big)$ are known for many F, we have lower bounds on the ε-complexity of surface integration in these cases.

We now obtain an upper bound. Observe that for $h = (f \circ g)\sigma_g$, we may use (2.1) to see that

$$S(f, g) = \mathrm{S}^{\mathrm{int}}(h).$$

If evaluating partial derivatives were a permissible information operation, then we would have been able to compute $h(x)$ by first computing $g_i(x)$ and $\partial_j g_i(x)$ for $1 \leq i \leq l$ and $1 \leq j \leq d$, and then computing the determinant of $A(x)$. Since only function evaluations are permissible, we will replace the partial derivatives $\partial_j g_i(x)$ by sufficiently-fine difference quotients, obtaining an approximation $\tilde{h}$ to h. The cost of computing $\tilde{h}$ is at most $2l(1 + d)\,\mathbf{c} + \gamma d^2 l$, with an absolute constant γ of order 1. Hence, an approximation to the surface integral $S(f, g)$ can be obtained by approximating $\mathrm{S}^{\mathrm{int}}(\tilde{h})$, with each evaluation of $\tilde{h}$ having cost $2l(1+d)\,\mathbf{c} + \gamma d^2 l$. The function $\tilde{h}$ belongs to the class

$$H(F, G) = \{ (f \circ g)\sigma_g : [f, g] \in F \times G \}.$$

Hence, the surface integration problem for the class $F \times G$ is reduced to the integration problem for the class $H(F, G)$. Observe that the ε-complexity of surface and classical integration can be written in the form $\mathbf{c}\,\mathrm{comp}_1(\varepsilon) + \mathrm{comp}_2(\varepsilon)$. Therefore, the replacement of $\mathbf{c}$ by $2l(1 + d)(1 + \gamma d/\mathbf{c})\,\mathbf{c}$ changes the ε-complexity at most by a factor $2l(1 + d)(1 + \gamma d/\mathbf{c})$. Note that this factor is linear in l and at most quadratic in d. This yields

$$\mathrm{comp}(\varepsilon; F, G) \leq 2l(1 + d)\left(1 + \gamma \frac{d}{\mathbf{c}}\right) \mathrm{comp}^{\mathrm{int}}\big(\varepsilon; H(F, G)\big). \tag{3.3}$$

This means that surface integration for $F \times G$ is essentially not harder than integration for $H(F, G)$. As we shall see later, sharp estimates of $\mathrm{comp}^{\mathrm{int}}\big(\varepsilon; H(F, G)\big)$ are known for some F and G, and this allows us to obtain upper bounds on the ε-complexity of surface integration.

When are the bounds (3.2) and (3.3) sharp? That is, we would like to know when the ε-complexity of surface integration for the class $F \times G$ is essentially the same as the ε-complexity of integration for the class $H(F)$ or for the class $H(F, G)$. One could expect that the bound (3.2) should not be sharp, since $g = \mathrm{id}$ seems like a very easy case for surface integration. After all, the entire difficulty of surface integration is in its nonlinear dependence on g, and for large enough classes G we should expect that for some g, surface integrals are harder to approximate than their classical counterparts.

This expectation can be supported by analyzing the upper bound (3.3) for the class $H(F, G)$. Usually, the ε-complexity of integration depends on the smoothness of the integrands. When we switch from functions f and g to the functions h, we may lose some smoothness, since we have differentiated g. However, if f is less smooth than g, then the lost smoothness of g may not be harmful.

Furthermore, there is one case for which the smoothness of g is irrelevant. (Of course, we must always have at least C^1 smoothness of g, so that surface integrals will be well-defined.) This is the case $l = 1$, which necessarily implies that $d = 1$ and $\sigma_g(x) = |g'(x)|$. Since g is C^1 and $g'(x) \neq 0$, we may use the standard change of variables $t = g(x)$ in (2.1) to conclude that

$$S(f, g) = \int_0^1 f(g(x))\, |g'(x)|\, dx = \begin{cases} \displaystyle\int_{g(0)}^{g(1)} f(t)\, dt & \text{if } g(0) < g(1), \\ \displaystyle -\int_{g(0)}^{g(1)} f(t)\, dt & \text{if } g(0) > g(1). \end{cases}$$

Hence, surface integration reduces to classical integration in this case. The smoothness of g is irrelevant. The only dependence on g is through the interval of integration. For many classes $H(F)$, the ε-complexity depends on the length of the interval of integration, and is finite only for finite lengths. For such $H(F)$, we must restrict the class G to functions for which $g(I)$ is uniformly bounded for all $g \in G$.

We illustrate the last point by taking F as the class of r times continuously differentiable functions f defined on the interval $[0, M]$ and for which all derivatives up to order r are bounded by 1. Here M is a given (large) positive number. To guarantee that $f \circ g$ is well-defined we restrict the class G to functions for which $g(I) \subset [0, M]$. It is known that the ε-complexity of this classical integration problem is $\Theta(M^{1+1/r}\, \mathbf{c}\, \varepsilon^{-1/r})$, with the big Θ-factor independent of M and ε (see, e.g., Novak, 1988, p. 37).

This also proves that the ε-complexity of surface integration is of the same order. In this case the bound (3.2) is sharp.

4 Surface integration for smooth functions

In this section, we study surface integration for classes of smooth functions.

We begin by taking $F = F_{l,r,C_1}$ as the class of functions $f: I^l \to \mathbb{R}$ that are r times continuously differentiable and that satisfy

$$\|f\|_{C^r(I^l)} \le C_1 \tag{4.1}$$

for some positive C_1. Here[1]

$$\|f\|_{C^r(I^l)} = \max_{|\alpha|\le r} \|D^\alpha f\|, \tag{4.2}$$

with $\|\cdot\|$ denoting the max norm. We take $G = G_{d,l,s,C_2,c_2}$ as the class of functions $g: I^d \to I^l$ that are s times continuously differentiable and that satisfy

$$\|g\|_{C^s(I^d)} \le C_2 \tag{4.3}$$

for some positive C_2, and for which

$$\min_{x\in I^d} \sigma_g(x) \ \ge\ c_2 \tag{4.4}$$

for some positive c_2. Here,

$$\|g\|_{C^s(I^d)} = \max_{|\alpha|\le s} \max_{1\le i\le l} \|D^\alpha g_i\|.$$

For simplicity, we assume that $c_2 \le 1 \le C_2$, so that $g = \mathrm{id} \in G$. The smoothness parameters r and s are integers satisfying $r \ge 0$ and $s \ge 1$. Observe that the functions from F have the common domain I^l. The composition $f \circ g$ is well-defined, since $g(I^d) \subset I^l$ for all $g \in G$.

Remark. We briefly comment on the conditions that define our classes F and G. Upper bounds on derivatives, such as (4.1) and (4.3), are typical assumptions for problems studied by information-based complexity (see, e.g., Traub and Werschulz, 1998, and the references cited therein). Lower bounds on derivatives, such as (4.4), occur far less often. Since the surface integral is well-defined if we only require

$$\min_{x\in I^d} \sigma_g(x) > 0, \tag{4.5}$$

one might ask why we need the constant c_2. The first reason is that any lower bounds we prove will be stronger. That is, since (4.4) implies (4.5), lower bounds

[1] We use the standard multi-index notation found in (e.g.) Ciarlet (1978, p. 11). In particular, for a multi-index $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_d]$ we have $D^\alpha f = \partial^{|\alpha|} f/(\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_d} x_d)$.

established for the case (4.4) will automatically hold for the case (4.5). The second reason is more important. When $d < l$, there is a square root in the definition of the surface integral. When we compute derivatives of $\sigma_\rho(x)$, the value of $\sigma_\rho(x)$ appears in the denominator. The bound (4.4) guarantees that the derivatives of $\sigma_\rho(x)$ are bounded, with the norms $\|\sigma_g\|_{C^{s-1}(I^d)}$ for $g \in C^s(I^d)$ being uniformly bounded with a bound proportional to C_2/c_2. However, when $d = l$, the square root does not appear in the definition of the surface integral, and so the condition (4.4) could be replaced by (4.5). □

We are ready to find the ε-complexity of surface integration for the class $F \times G$. First, we shall establish an upper bound, which holds for the case that $r \geq 1$ and $s \geq 2$. Later, we shall establish a lower bound for the case $r \geq 0$, $s \geq 1$, and $d < l$.

To establish this upper bound, we shall propose an algorithm using $O(n)$ function evaluations and having a good error bound. This will require us to construct a multivariate spline space $\mathscr{S}$, which is a d-fold tensor product

$$\mathscr{S} = S_\mu(\Delta) \otimes \cdots \otimes S_\mu(\Delta).$$

Here,

$$\Delta = \{t_0 = 0 < t_1 < \ldots < t_\ell < t_{\ell+1} = 1\},$$

where

$$t_i = \frac{i}{\ell + 1} \qquad (0 \leq i \leq \ell + 1)$$

with $\ell = \lfloor n^{1/d} \rfloor$, is a uniform partition of $I = [0, 1]$, and

$$\mu = \max\{s - 1, 2\}.$$

Moreover,

$$S_\mu(\Delta) = \left\{ w \in C^{\mu-1}(I) : w\Big|_{[t_{i-1}, t_i]} \in P_\mu \text{ for } 1 \leq i \leq \ell + 1 \right\}$$

is the space of splines having global smoothness $C^{\mu-1}$ and piecewise polynomial degree μ. We emphasize that $s \geq 2$ in what follows. If we let $\mathscr{Q} = \Delta^d$, then we see that functions $z \in \mathscr{S}$ are $C^{\mu-1}(I^d)$ functions, whose restriction to each subcube $K \in \mathscr{Q}$ is a polynomial of degree at most μ in each of the variables $x_1, \ldots, x_d$.

We will need a quasi-interpolation operator for $\mathscr{S}$, which will be built from the quasi-interpolation operator for $S_\mu(\Delta)$. This latter operator takes the form

$$(Qw)(\tau) = \sum_{j=1}^{\mu+l} \lambda_j(w) B_j(\tau).$$

Here, $B_1, \dots, B_{\ell+\mu}$ are univariate B-splines of degree μ, (see Schumaker, 1981, Section 4.4). Moreover, $\lambda_1, \dots, \lambda_{\ell+\mu} \in [C(I)]^*$ are dual functionals with respect to the basis $\{B_1, \dots, B_{\ell+\mu}\}$. That is, $\lambda_1, \dots, \lambda_{\ell+\mu}$ are continuous linear functionals on $C(I)$ such that

$$\lambda_i(B_j) = \delta_{i,j} \qquad (1 \le i, j \le \ell + \mu).$$

Furthermore, for any $w \in C(I)$, the values $\lambda_1(w), \dots, \lambda_{\ell+\mu}(w)$ can be computed using $O(\ell)$ evaluations of w. For further details, see Schumaker (1981, Section 6.4).

Clearly, Q is a linear projector onto the space $S_\mu(\Delta)$. From Schumaker (1981, Corollary 6.26), we find that if $w \in W^{s,\infty}(I)$ and $q \in \{0, 1, 2\}$, then

$$\|w - Qw\|_{W^{q,\infty}(I)} \le C\ell^{-(s-q)} \|w^{(s)}\|_{L_\infty(I)},$$

where C is independent of w and ℓ.

As in Schultz (1969, pg. 172), for a continuous function $z \colon I^d \to \mathbb{R}$, we let $Q_i z$ denote the function obtained by applying the operator Q to $z(x)$, viewed as a function of x_i, while holding the other variables $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_d$ fixed. We then let

$$\overline{z} = Q_d \circ \cdots \circ Q_1 z.$$

denote the $\mathscr{S}$-quasi-interpolant of z. Note that we can compute $\overline{z}$ using $O(\ell^d) = O(n)$ scalar function evaluations. Moreover, using the same method of proof as Schultz (1969, Theorem 5.8), one can show that if $w \in W^{s,\infty}(I^d)$ and $q \in \{0, 1, 2\}$, then

$$\|z - \overline{z}\|_{W^{q,\infty}(I^d)} \le C\, n^{-(s-q)/d} \|z\|_{W^{s,\infty}(I^d)}, \tag{4.6}$$

where C is independent of z and n.

Finally, we let $\mathscr{V} = \mathscr{S}^l$, defining the $\mathscr{V}$-quasi-interpolant of a vector-valued function v as

$$\overline{v} = (\overline{v_1}, \dots, \overline{v_l}).$$

We can now describe our algorithm. For $[f, g] \in F \times G$, let $h = f \circ g$. Let $\overline{h}$ be the $\mathscr{S}$-quasi-interpolant of h, and let $\overline{g}$ be the $\mathscr{V}$-quasi-interpolant of g. We define our algorithm as

$$U_n(f, g) = \int_{I^d} \overline{h}(x) \sigma_{\overline{g}}(x)\, dx.$$

Note that U_n uses information of cardinality $O(n)$ about $[f, g] \in F \times G$, which may be written in the form

$$N_n(f, g) = [f(g(x^{(1)})), \dots, f(g(x^{(m)})), g(x^{(1)}), \dots, g(x^{(m)})],$$

where $m \sim n$. This information is adaptive, since the evaluation points for f depend on the calculated values of g. The information cost of U_n, i.e., the cost of calculating $N_n(f, g)$, is $O(\mathbf{c}\, n)$. We defer a discussion of the combinatory cost of U_n, i.e., the cost of the arithmetic operations required to calculate U_n (or at least a sufficiently precise approximation of same) given $N_n(f, g)$, until later.

We are ready to estimate the error of U_n.

Theorem 4.1. *Let $r \geq 1$ and $s \geq 2$. Then*

$$e(U_n) = O(n^{-\min\{r,s\}/d}).$$

The O-factor depends only on the global parameters l, d, r, s, C_1, C_2 and c_2.

Proof. Let $[f, g] \in F \times G$, and $h = f \circ g$. Then $h \in C^{\min\{r,s\}}(I^d)$. Furthermore, $\|h\|_{C^{\min\{r,s\}}(I^d)}$ is uniformly bounded by a constant that only depends on C_1 and C_2.

From (4.6), we have the following error estimates:[2]

1. Let $q \in \{0, 1\}$. Then

$$\|h - \overline{h}\|_{W^{q,\infty}(I^d)} \leq C_3\, n^{-(\min\{r,s\}-q)/d} \|h\|_{W^{\min\{r,s\},\infty}(I^d)}. \tag{4.7}$$

2. Let $q \in \{0, 1, 2\}$. Then

$$\|g - \overline{g}\|_{W^{q,\infty}(I^d)} \leq C_4\, n^{-(s-q)/d} \|g\|_{W^{s,\infty}(I^d)}. \tag{4.8}$$

We need to show that

$$\left| S(f, g) - \int_{I^d} \overline{h}(x)\sigma_{\overline{g}}(x)\, dx \right| = O(n^{-\min\{r,s\}/d}). \tag{4.9}$$

Observe that $h\sigma_g - \overline{h}\sigma_{\overline{g}} = (h - \overline{h})\sigma_g + \overline{h}(\sigma_g - \sigma_{\overline{g}})$. Therefore

$$\left| S(f, g) - \int_{I^d} \overline{h}(x)\sigma_{\overline{g}}(x)\, dx \right| \leq |I_1| + |I_2|, \tag{4.10}$$

where

$$I_1 = \int_{I^d} \big(h(x) - \overline{h}(x)\big)\sigma_g(x)\, dx$$

and

$$I_2 = \int_{I^d} \overline{h}(x)\big(\sigma_g(x) - \sigma_{\overline{g}}(x)\big)\, dx.$$

[2] Here, and in what follows, all constants C_i will be positive and independent of f and g (and, thus, of h), and of n.

Now

$$|I_1| \le \|h - \overline{h}\|_{L_\infty(I^d)} \|\sigma_g\|_{L_1(I^d)}.$$

Since $s \ge 2$, the conditions (4.3) and (4.4) defining G imply that there exist C_5 and C_6 such that $\|\sigma_g\|_{L_1(I^d)} \le C_5$ and $\|h\|_{C^{\min\{r,s\}}(I^d)} \le C_6$. Using (4.7), we now see that

$$|I_1| \le C_7\, n^{-\min\{r,s\}/d}. \tag{4.11}$$

We turn to I_2. We claim that

$$|I_2| \le C_8\, n^{-s/d}. \tag{4.12}$$

Indeed, consider the matrices A_g and $A_{\overline{g}}$ defined as in Section 2. They have components given by

$$a_{i,j} = \sum_{k=1}^{l} \partial_i g_k\, \partial_j g_k \qquad (1 \le i,\, j \le d)$$

and

$$\overline{a}_{i,j} = \sum_{k=1}^{l} \partial_i \overline{g}_k\, \partial_j \overline{g}_k \qquad (1 \le i,\, j \le d),$$

respectively. Letting

$$u = \frac{\overline{h}}{\sqrt{\det A_g} + \sqrt{\det A_{\overline{g}}}}\,,$$

we have

$$I_2 = \int_{I^d} u(x)\big(\det A_g(x) - \det A_{\overline{g}}(x)\big)\, dx.$$

Now

$$\det A_g - \det A_{\overline{g}} = \sum_{\mathbf{i}\in\Pi_d} (-1)^{|\mathbf{i}|}(a_{i_1,1} \dots a_{i_d,d} - \overline{a}_{i_1,1} \dots \overline{a}_{i_d,d}),$$

where Π_d is the set of all permutations of $\{1, \dots, d\}$ and $|\mathbf{i}|$ denotes the sign of $\mathbf{i} \in \Pi_d$. Since

$$a_{i_1,1} \dots a_{i_d,d} - \overline{a}_{i_1,1} \dots \overline{a}_{i_d,d} = \sum_{k=1}^{d} \overline{a}_{i_1,1} \dots \overline{a}_{i_{k-1},k-1}(a_{i_k,k} - \overline{a}_{i_k,k}) a_{i_{k+1},k+1} \dots a_{i_d,d},$$

it follows that

$$I_2 = \sum_{\mathbf{i}\in\Pi_d} (-1)^{|\mathbf{i}|} \sum_{k=1}^{d} \int_{I^d} u(x) \overline{a}_{i_1,1}(x) \dots \overline{a}_{i_{k-1},k-1}(x) \times \big(a_{i_k,k}(x) - \overline{a}_{i_k,k}(x)\big) a_{i_{k+1},k+1}(x) \dots a_{i_d,d}(x)\, dx.$$

Writing

$$a_{i,j} - \overline{a}_{i,j} = \sum_{p=1}^{l} \partial_i (g_p - \overline{g}_p)\, \partial_j g_p + \sum_{p=1}^{l} \partial_i \overline{g}_p\, \partial_j (g_p - \overline{g}_p)$$

we find

$$\begin{aligned} I_2 = \sum_{\mathbf{i}\in\Pi_d} (-1)^{|\mathbf{i}|} \sum_{k=1}^{d} \sum_{p=1}^{l} \int_{I^d} u(x) \overline{a}_{i_1,1}(x) \ldots \overline{a}_{i_{k-1},k-1}(x) a_{i_{k+1},k+1}(x) \ldots a_{i_d,d}(x) \\ \times \left[\partial_{i_k} \big(g_p(x) - \overline{g}_p(x)\big)\, \partial_k g_p(x) + \partial_k \big(g_p(x) - \overline{g}_p(x)\big)\, \partial_{i_k} \overline{g}_p(x) \right] dx. \end{aligned}$$

Let

$$b_{\mathbf{i},k,p} = u\, \overline{a}_{i_1,1} \ldots \overline{a}_{i_{k-1},k-1}\, a_{i_{k+1},k+1} \ldots a_{i_d,d}\, \partial_k g_p$$

and

$$\overline{b}_{\mathbf{i},k,p} = u\, \overline{a}_{i_1,1} \ldots \overline{a}_{i_{k-1},k-1}\, a_{i_{k+1},k+1} \ldots a_{i_d,d}\, \partial_{i_k} \overline{g}_p.$$

We find that

$$\begin{aligned} I_2 = \sum_{\mathbf{i}\in\Pi_d} (-1)^{|\mathbf{i}|} \sum_{k=1}^{d} \sum_{p=1}^{l} \bigg(\int_{I^d} b_{\mathbf{i},k,p}(x) \partial_{i_k} \big(g_p(x) - \overline{g}_p(x)\big)\, dx \\ + \int_{I^d} \overline{b}_{\mathbf{i},k,p}(x) \partial_k \big(g_p(x) - \overline{g}_p(x)\big)\, dx \bigg). \end{aligned}$$

For $j \in \{1, \ldots, d\}$, let I_j^{d-1} denote the $(d-1)$-dimensional unit cube in the variables $x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_d$, and let

$$dx_j^{d-1} = dx_1 \ldots dx_{j-1} dx_{j+1} \ldots dx_d.$$

Note that $g_p, \overline{g}_p \in W^{2,\infty}(I^d)$, from which we see that $b_{\mathbf{i},k,p}, \overline{b}_{\mathbf{i},k,p} \in W^{1,\infty}(I^d)$. Hence, we can integrate by parts to obtain

$$\begin{aligned} I_2 = \sum_{\mathbf{i}\in\Pi_d} (-1)^{|\mathbf{i}|} \sum_{k=1}^{d} \sum_{p=1}^{l} \bigg(& \int_{I_{i_k}^{d-1}} \left[b_{\mathbf{i},k,p}(x) \big(g_p(x) - \overline{g}_p(x)\big) \right]_{x_{i_k}=0}^{x_{i_k}=1} dx_{i_k}^{d-1} \\ & - \int_{I^d} (\partial_{i_k} b_{\mathbf{i},k,p})(x) \big(g_p(x) - \overline{g}_p(x)\big)\, dx \\ & + \int_{I_k^{d-1}} \left[\overline{b}_{\mathbf{i},k,p}(x) \big(g_p(x) - \overline{g}_p(x)\big) \right]_{x_k=0}^{x_k=1} dx_k^{d-1} \\ & - \int_{I^d} (\partial_k \overline{b}_{\mathbf{i},k,p})(x) \big(g_p(x) - \overline{g}_p(x)\big)\, dx \bigg). \end{aligned} \tag{4.13}$$

Let

$$\kappa_{f,g} = \max_{1\le k,p\le l} \max_{\mathbf{i}} \max \Big\{ \|b_{\mathbf{i},k,p}\|_{L_1(I^d)}, \|\overline{b}_{\mathbf{i},k,p}\|_{L_1(I^d)}, \\ \|\partial_{i_k} b_{\mathbf{i},k,p}\|_{L_1(I^d)}, \|\partial_k \overline{b}_{\mathbf{i},k,p}\|_{L_1(I^d)} \Big\}.$$

Using (4.13), we see that there exists a positive constant C_9, such that

$$|I_2| \le C_9\, \kappa_{f,g} \max_{1\le p\le l} \|g_p - \overline{g}_p\|_{L_\infty(I^d)}. \tag{4.14}$$

Since $s \ge 2$, we may use (4.8) to see that there exists a positive constant C_{10}, such that

$$\kappa_{f,g} \le C_{10}.$$

Using this bound and (4.8) in (4.14), we obtain (4.12), as claimed. Finally, using (4.11) and (4.12), we have (4.9), which establishes the theorem. □

So, the algorithm U_n, whose information cost is $O(\mathbf{c}\, n)$, provides an approximation whose error is $O(n^{-\min\{r,s\}/d})$. Let us now discuss the implementation of U_n.

Clearly, $U_n(f,g)$ is an integral, whose integrand is a function that has a special form; its restriction to any subcube K is the product of a polynomial $\overline{h}$ and the square root $\sigma_{\overline{g}}$ of a polynomial. If it were not for the presence of this square root, we would be able to evaluate $U_n(f,g)$ exactly by using a quadrature formula having sufficiently large degree of accuracy. Note that when $d = l$, the factor $\sigma_{\overline{g}}$ is the square root of the square of the Jacobian determinant of $\overline{g}$, which is merely $\pm\sigma_{\overline{g}}$. Hence $U_n(f,g)$ can be evaluated exactly with total cost $O(\mathbf{c}\, n)$ when $d = l$.

What happens when $d < l$? We consider two cases.

Suppose first that $r \le s - 1$. Then we can resort to classical techniques. Recall (see, e.g., Novak, 1988, p. 36) that if $v \in C^r(I^d)$, then we can calculate an approximation $I_n^*(v)$ at cost $O(\mathbf{c}\, n)$, for which

$$\int_{I^d} v(x)\, dx - I_n^*(v) = O\big(\|v\|_{C^r(I^d)} n^{-r/d}\big).$$

So simply take

$$U_n^*(f,g) = I_n^*(h\sigma_g).$$

Then we can calculate $U_n^*(f,g)$ at cost proportional to n. However, since $h\sigma_g \in C^r(I^d)$ and $\min\{r,s\} = r$, it now follows that

$$S(f,g) - U_n^*(f,g) = O(n^{-r/d}) = O(n^{-\min\{r,s\}/d}).$$

Observe that $U_n^*(f,g)$ can also be treated as an approximation of $U_n(f,g)$, since

$$U_n(f,g) - U_n^*(f,g) = O(n^{-r/d}) = O(n^{-\min\{r,s\}/d}).$$

Now let us consider the case $r > s-1$. On each subcube $K \in \mathcal{Q}_n$, the quasi-interpolant $\overline{g}$ is a polynomial of degree $\mu = \max\{s-1, 3\}$, and so $\sigma_{\overline{g}}$ is the square root of a polynomial having degree 2μ. We handle the square root as follows:

For any index j, let $\eta = \det A_{\overline{g}}(x^{(j)})$ and $\xi = \det A_{\overline{g}}(x)$, where $A_{\overline{g}} = [\nabla\overline{g}\,]^T[\nabla\overline{g}\,]$. By our assumptions on the class G, we have $0 < c_2 \le \xi, \eta \le C_2$. We can expand

$$\sqrt{\xi} = \sqrt{\eta} + \sum_{t=1}^{\mu} \beta_t(\eta)(\xi-\eta)^t + \Theta\big((\xi-\eta)^{\mu+1}\big),$$

where the $\beta_j(\cdot)$ are well-known functions and the Θ-constant depends only on c_2 and C_2.

We now define our algorithm for the case $r > s-1$ as

$$U_n^*(f,g) = \sum_{K\in\mathcal{Q}_n} U_{n,K}^*(f,g),$$

where

$$U_{n,K}^*(f,g) = \int_K \overline{h}(x)\Bigg(\sqrt{\det A_{\overline{g}}(x^{(K)})} + \sum_{t=1}^{\mu} \beta_t\big(\det A_{\overline{g}}(x^{(K)})\big)\cdot\big(\det A_{\overline{g}}(x) - \det A_{\overline{g}}(x^{(K)})\big)^t\Bigg)\,dx$$

for each subcube $K \in \mathcal{Q}_n$. Here, $x^{(K)}$ is any evaluation point in K; for example, it might be chosen to be as close as possible to the center of K.

We then have

Theorem 4.2. *Let $r \ge 1$ and $s \ge 2$. Then $U_n^*(f,g)$ can be calculated in cost $O(\mathbf{c}\,n)$, and*

$$e(U_n^*) = O(n^{-\min\{r,s\}/d}).$$

The O-factors depend only on the global parameters l, d, r, s, C_1, C_2 and c_2.

Proof. Note that $\det A_{\overline{g}}(x)$ is a polynomial in x. Each term of the outer sum can be calculated exactly with cost independent of n, since we are integrating polynomials. So, $\mathrm{cost}(U_n^*) = O(\mathbf{c}\,n)$.

To calculate the error, note that

$$|e(U_n) - e(U_n^*)| \le C_{11}\Bigg(\sum_{K \in \mathcal{Q}_n} \int_K |\overline{h}(x)|\,|\det A_{\overline{g}}(x^{(K)}) - \det A_{\overline{g}}(x)|^{\mu+1}\,dx \Bigg).$$

Now $\det A_{\overline{g}}$ has a uniformly bounded first derivative, so that there is a positive constant C_{12} such that

$$|\det A_{\overline{g}}(x^{(K)}) - \det A_{\overline{g}}(x)|^{\mu+1} \le C_{12}\|x - x^{(K)}\|^{\mu+1}_{\ell_\infty(\mathbb{R}^d)} = O(n^{-s/d}),$$

since $\mu + 1 \ge s$. Moreover, $\overline{h}$ is uniformly bounded. Since r and s are integers for which $r > s - 1$, it follows that $r \ge s$, so that $s = \min\{r, s\}$. Hence

$$e(U_n) - e(U_n^*) = O(n^{-\min\{r,s\}/d}).$$

Using Theorem 4.1, we see that $e(U_n^*) = \Theta(n^{-\min\{r,s\}/d})$, as required. $\square$

We now prove a lower bound that holds for the case $d < l$. Let

$$e(n; F, G) = \inf\{\, e(U) : U \text{ using information (2.2)} \,\}$$

denote the minimal error for the surface integration problem, over all algorithms using information of the form (2.2), with fixed n and varying m. We have

Theorem 4.3. *Let $d < l$.*

1. *If $s = 1$, then*
$$e(n; F, G) = \Omega(1).$$
2. *If $s \ge 2$, then*
$$e(n; F, G) = \Omega(n^{-\min\{r,s\}/d}).$$

The Ω-factors depend only on the global parameters l, d, r, s, C_1, C_2 and c_2.

Proof. Using the notation of Section 3, we have $H(F) = F_{d,r,C_1}$. Using (3.2) and the known complexity result (Novak, 1988, p. 37) on classical integration for the class F_{d,r,C_1}, we get a lower bound of $e(n; F, G) = \Omega(n^{-r/d})$. Hence it remains to show that

$$e(n; F, G) = \begin{cases} \Omega(1) & \text{if } s = 1, \\ \Omega(n^{-s/d}) & \text{if } s \ge 2. \end{cases} \tag{4.15}$$

We now take $f \equiv C_1$, which belongs to F, and

$$g(x) = [\tfrac{1}{4}ax_1^2, x_2, \ldots, x_d, x_1, 0, \ldots, 0],$$

where

$$a = \begin{cases} 0 & \text{if } s = 1 \text{ or } s = 2, \\ 1 & \text{if } s \geq 3. \end{cases}$$

We stress that the $(d+1)$st component of g is x_1, and that g is well-defined since $l \geq d+1$. The function g is infinitely differentiable and $\|g\|_{C^s(I^d)} = 1 \leq C_2$. It is easy to check that

$$\sigma_g(x) = \sqrt{1 + \tfrac{1}{4}a^2 x_1^2} \geq 1 \geq c_2.$$

Therefore $g \in G$.

To find a lower bound on $e(n; F, G)$ we use the known estimate (see, e.g. Traub et al., 1988, p. 45)

$$e(n; F, G) \geq \inf \left\{ \tfrac{1}{2} d(N, f) : N \text{ of the form (2.2)} \right\},$$

where

$$d(N, f) = \sup\{ |S(f, g) - S(f, \overline{g})| : \overline{g} \in G \text{ and } N(f, \overline{g}) = N(f, g) \}.$$

Now choose information N of the form (2.2), so that f is evaluated at the points $x^{(1)}, \ldots, x^{(m)}$ and g is evaluated at the points $x^{(m+1)}, \ldots, x^{(n)}$, where these evaluation points may have been chosen adaptively. We need to specify a function $\overline{g} \in G$ satisfying $N(f, \overline{g}) = N(f, g)$.

As in Novak (1988), for a positive number b define

$$w(x) = \begin{cases} b \prod_{j=1}^{d} \left(x_j (1 - x_j) \right)^{s+1} & \text{for } x \in I^d, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $w \in C^s(\mathbb{R}^d)$ for any positive b. We choose b such that $w \in F_{d,s,1}$. Without loss of generality, we suppose that $n - m = \frac{1}{2} p^d$ for some integer p. Divide I^d into $2(n-m) = p^d$ cubes $A_1, \ldots, A_{2(n-m)}$ with mesh-size p^{-1}. Let $y^{(i)}$ be the element of A_i with the smallest components. Define

$$w_i(x) = p^{-s} w\Big(p\big(x - y^{(i)}\big) \Big). \tag{4.16}$$

Then the support of w_i is the cube A_i and it is easy to check that $w_i \in F_{d,s,1}$. Now choose

$$z = \sum_{i \in J} w_i, \tag{4.17}$$

where J is the set of indices i of all cubes A_i containing no g-evaluation points $x^{(m+1)}, \ldots, x^{(n)}$. Since we have $n - m$ such evaluation points and $2(n-m)$ cubes, there are at least $n-m$ indices in J. Since w_i's have disjoint supports, $z \in F_{d,s,1}$. Furthermore z has zero information; that is, $z(x^{(j)}) = 0$ for $j \in \{m+1, \ldots, n\}$.

We are ready to define $\overline{g}$ as

$$\overline{g}(x) = [\tfrac{1}{4}ax_1^2 + \tfrac{1}{2}z(x), x_2, \ldots, x_d, x_1, 0, \ldots, 0].$$

The function $\overline{g}$ belongs to $C^s(I^d)$ and

$$\|\overline{g}\|_{C^s(I^d)} = \max\{\tfrac{1}{2}(a+1), 1\} = 1 \le C_2.$$

It is not hard to check that

$$\sigma_{\overline{g}}(x) = \sqrt{\det\left(I + \mathbf{a}(x)\mathbf{a}(x)^T\right)},$$

where I is the $d \times d$ identity matrix and $\mathbf{a}(x)$ is the column vector of partial derivatives of the first component $\overline{g}_1$ of the function $\overline{g}$. The determinant of $I + \mathbf{a}(x)\mathbf{a}(x)^T$ is simply $1 + \sum_{j=1}^d (\partial_j \overline{g}_1)^2(x)$. Hence,

$$\sigma_{\overline{g}}(x) = \sqrt{1 + \tfrac{1}{4}a^2x_1^2 + \tfrac{1}{2}ax_1(\partial_1 z)(x) + \tfrac{1}{4}\sum_{j=1}^{d}(\partial_j z)^2(x)} \ge 1 \ge c_2.$$

This proves that $\overline{g} \in G$. Obviously, $N(f, \overline{g}) = N(f, g)$. Therefore

$$d(N, f) \ge |S(f, \overline{g}) - S(f, g)|, \tag{4.18}$$

with

$$\begin{aligned} S(f, g) &= C_1 \int_{I^d} \sqrt{1 + \tfrac{1}{4}a^2x_1^2}\, dx\,, \\ S(f, \overline{g}) &= C_1 \int_{I^d} \sqrt{1 + \tfrac{1}{4}a^2x_1^2 + \tfrac{1}{2}ax_1(\partial_1 z)(x) + \tfrac{1}{4}\sum_{j=1}^{d}(\partial_j z)^2(x)}\, dx\,. \end{aligned} \tag{4.19}$$

We first consider the case $s \le 2$, so that $a = 0$. We then have

$$\begin{aligned} S(f, \overline{g}) - S(f, g) &= C_1 \int_{I^d} \left(\sqrt{1 + \tfrac{1}{4}\sum_{j=1}^{d}(\partial_j z)^2(x)} - 1 \right) dx \\ &= \tfrac{1}{4}C_1 \int_{I^d} \frac{\sum_{j=1}^d (\partial_j z)^2(x)}{\sqrt{1 + \frac{1}{4}\sum_{j=1}^d (\partial_j z)^2(x)} + 1}\, dx \\ &\ge \frac{C_1}{2(\sqrt{4+d}+2)} \int_{I^d} \sum_{j=1}^{d} (\partial_j z)^2(x)\, dx. \end{aligned} \tag{4.20}$$

Fix an index j. Since the supports A_i of the functions w_i are disjoint, the standard rule for changing variables in multiple integrals imply that

$$\begin{aligned}\int_{I^d} (\partial_j z)^2(x)\,dx &= \sum_{i\in J} \int_{A_i} (\partial_j w_i)^2(x)\,dx \\ &= \frac{1}{p^{2s}} \sum_{i\in J} \int_{A_i} p^2 (\partial_j w)^2 \Big(p\big(x - y^{(i)}\big)\Big)\,dx \\ &= \frac{1}{p^{2(s-1)+d}} \sum_{i\in J} \int_{I^d} (\partial_j w)^2(x)\,dx \\ &= \frac{|J|}{p^{2(s-1)+d}} \int_{I^d} (\partial_j w)^2(x)\,dx.\end{aligned}$$

Summing the previous result over the indices j and recalling that $|J| \ge n - m = \frac{1}{2}p^d$, we find

$$\int_{I^d} \sum_{j=1}^{d} (\partial_j z)^2(x)\,dx \ge \frac{1}{2p^{2(s-1)}} \int_{I^d} |(\nabla w)(x)|^2\,dx.$$

Inserting this result into (4.20) and using (4.18), we see that

$$d(N, f) \ge S(f, \overline{g}) - S(f, g) = \Omega(p^{-2(s-1)}).$$

Now if $s = 1$, we see that

$$d(N, f) = \Omega(1),$$

whereas when $s = 2$, we may use $n \ge n - m = \frac{1}{2}p^d$ to see that

$$d(N, f) = \Omega(p^{-2}) = \Omega(n^{-s/d}).$$

Since N is arbitrary information of cardinality at most n, this establishes (4.15) for $s \le 2$.

We now consider the case $s \ge 3$, for which we have $a = 1$. Let

$$\zeta(x) = 1 + \tfrac{1}{4}x_1^2 \qquad \text{and} \qquad \eta(x) = \tfrac{1}{2}x_1\,(\partial_1 z)(x) + \tfrac{1}{4}\sum_{j=1}^{d} (\partial_j z)^2(x).$$

Using (4.19), we see that

$$d(N, f) \ge S(f, g) - S(f, \overline{g}) = \int_{I^d} \left(\sqrt{\zeta(x)} - \sqrt{\zeta(x) + \eta(x)}\right)\,dx.$$

Since $\sqrt{1+x} \le 1 + \frac{1}{2}x$ for all $x \ge -1$, we get

$$d(N, f) \ge -\int_{I^d} \frac{\eta(x)}{2\sqrt{\zeta(x)}} = \tfrac{1}{4}(I_1 - I_2), \tag{4.21}$$

where

$$I_1 = -\int_{I^d} \frac{x_1}{\sqrt{1+\frac{1}{4}x_1^2}} (\partial_1 z)(x)\, dx$$

and

$$I_2 = \int_{I^d} \frac{\sum_{j=1}^d (\partial_j z)^2(x)}{2\sqrt{1+\frac{1}{4}x_1^2}}\, dx.$$

Since z vanishes on the boundary of I^d, we calculate I_1 using an integration by parts, finding that

$$\begin{aligned} I_1 &= \int_{I^d} \partial_1 \left(\frac{x_1}{\sqrt{1+\frac{1}{4}x_1^2}} \right) z(x)\, dx = \int_{I^d} \frac{z(x)}{(1+\frac{1}{4}x_1^2)^{3/2}}\, dx \\ &\ge \left(\tfrac{4}{5}\right)^{3/2} \int_{I^d} z(x)\, dx = \left(\tfrac{4}{5}\right)^{3/2} \sum_{i\in J} \int_{A_i} w_i(x)\, dx\,. \end{aligned}$$

But for any index i, we have

$$\int_{A_i} w_i(x)\, dx = \frac{1}{p^s} \int_{A_i} w\Big(p\big(x - y^{(i)}\big)\Big)\, dx = \frac{1}{p^{s+d}} \int_{I^d} w(x)\, dx.$$

Since $|J| \ge \frac{1}{2}p^d$, we thus find that

$$I_1 \ge \left(\tfrac{4}{5}\right)^{3/2} \frac{1}{2p^s} \int_{I^d} w(x)\, dx. \tag{4.22}$$

We now look at I_2. We find that

$$|I_2| \le \tfrac{1}{2} \sum_{j=1}^{d} \int_{I^d} (\partial_j z)^2(x)\, dx.$$

From (4.16) and (4.17), it follows that $\|\partial_j z\| = O(p^{-(s-1)})$. Thus

$$|I_2| = O(p^{-2(s-1)}).$$

Since $s \ge 3$ implies $2(s-1) > s$, we conclude that

$$I_1 - I_2 = \Omega(p^{-s}).$$

Using $n \geq n - m = \frac{1}{2} p^d$ we have

$$d(N, f) \geq I_1 - I_2 = \Omega(n^{-s/d}).$$

Since N is arbitrary information of cardinality at most n, this establishes (4.15) for $s \geq 3$, and completes the proof. □

Combining Theorems 4.2 and 4.3, and using the results at the end of Section 3, we have

Theorem 4.4. *The following results hold for the surface integration problem with $F = F_{l,r,C_1}$ and $G = G_{d,l,s,C_2,c_2}$:*

1. *Let $l = 1$, so that $d = 1$ necessarily. Then*

$$e(n; F, G) = \Theta(n^{-r})$$

and

$$\mathrm{comp}(\varepsilon; F, G) = \Theta(\varepsilon^{-1/r}).$$

The Θ-factors depend only on r, C_1 and C_2.

2. *Let $l \geq 2$.*

   (a) *Suppose that $d < l$. If $r = 0$ or $s = 1$, then there exists $\varepsilon_0 > 0$ such that*

$$e(n; F, G) \geq \varepsilon_0 \qquad \forall\, n \geq 0,$$

and so

$$\mathrm{comp}(\varepsilon) = \infty \qquad \forall\, \varepsilon < \varepsilon_0.$$

However, if $r \geq 1$ and $s \geq 2$, then

$$e(n; F, G) = \Theta(n^{-\min\{r,s\}/d}),$$

and

$$\mathrm{comp}(\varepsilon; F, G) = \Theta(\varepsilon^{-d/\min\{r,s\}}).$$

The Θ-factors depend only on d, l, r, s, C_1, C_2, and c_2.

   (b) *Suppose that $d = l$. If $r \geq 1$ and $s \geq 2$, then*

$$e(n; F, G) = O(n^{-\min\{r,s\}/d}),$$

and

$$\mathrm{comp}(\varepsilon; F, G) = O(\varepsilon^{-d/\min\{r,s\}}).$$

The O-factors depend only on d, r, s, C_1, C_2, and c_2.

Note that in the case $d = l \geq 2$, we have only an upper bound on the complexity of surface integration. It is open whether this bound is sharp.

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