Graph Structure and Coloring

Matthieu Plumettaz

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ABSTRACT

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We denote by $G = (V, E)$ a graph with vertex set V and edge set E. A graph G is claw-free if no vertex of G has three pairwise nonadjacent neighbours. Claw-free graphs are a natural generalization of line graphs. This thesis answers several questions about claw-free graphs and line graphs.

In 1988, Chvátal and Sbihi [15] proved a decomposition theorem for claw-free perfect graphs. They showed that claw-free perfect graphs either have a clique-cutset or come from two basic classes of graphs called elementary and peculiar graphs. In 1999, Maffray and Reed [26] successfully described how elementary graphs can be built using line graphs of bipartite graphs and local augmentation. However gluing two claw-free perfect graphs on a clique does not necessarily produce claw-free graphs. The first result of this thesis is a complete structural description of claw-free perfect graphs. We also give a construction for all perfect circular interval graphs. This is joint work with Chudnovsky, and these results first appeared in [8].

Erdős and Lovász conjectured in 1968 that for every graph G and all integers $s, t \geq 2$ such that $s + t - 1 = \chi(G) > \omega(G)$, there exists a partition (S, T) of the vertex set of G such that $\chi(G|S) \geq s$ and $\chi(G|T) \geq t$. This conjecture is known in the graph theory community as the Erdős-Lovász Tihany Conjecture. For general graphs, the only settled cases of the conjecture are when s and t are small. Recently, the conjecture was proved for a few special classes of graphs: graphs with stability number 2 [2], line graphs [24] and quasi-line graphs [2]. The second part of this thesis considers the conjecture for claw-free graphs and presents some progresses on it. This is joint work with Chudnovsky and Fradkin, and it first appeared in [5].

Reed's ω, Δ, χ conjecture proposes that every graph satisfies $\chi \leq \lceil \frac{1}{2}(\Delta + 1 + \omega) \rceil$; it is known to hold for all claw-free graphs. The third part of this thesis considers a local strengthening of this conjecture. We prove the local strengthening for line graphs, then note that previous results immediately tell us that the local strengthening holds for all quasi-line graphs. Our proofs lead to

polytime algorithms for constructing colorings that achieve our bounds: The complexity are $O(n^2)$ for line graphs and $O(n^3m^2)$ for quasi-line graphs. For line graphs, this is faster than the best known algorithm for constructing a coloring that achieves the bound of Reed's original conjecture. This is joint work with Chudnovsky, King and Seymour, and it originally appeared in [7].

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Acknowledgments

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To my parents

Chapter 1

Introduction

1.1 Perfect graphs

A graph G is a mathematical object used to model pairwise relations among a collection of entities. This collection of entities is called the vertex set and is denoted by $V(G)$. The edge set, denoted by $E(G)$, represents the relations between pairs of elements of $V(G)$. Graphs have numerous applications to a wide variety of fields, from finding the shortest path between two cities on a GPS, to managing inventory in a warehouse or detecting particular molecules in biology. The major part of the thesis will be about with structural graph theory. Structural graph theory tries to understand families of graphs. When someone studies a particular problem, it is generally possible to characterize some properties of the underlying family of graphs. One of our main goals is to understand what are the basic graphs in a given family. In particular, we want to describe a family in terms of well-understood graphs and construction steps. This description can then lead to a better understanding of how to approach a problem both from a theoretical and an algorithmic point of view.

For two elements $x, y \in V(G)$, we say that x is *adjacent* to y if $xy \in E(G)$ and x is *non-adjacent* to y if $xy \notin E(G)$. A *clique* in G is a set $X \subseteq V(G)$ such that every two members of X are adjacent. A set $X \subseteq V(G)$ is a *stable set* in G if every two members of X are non-adjacent. A set $S \subseteq V(G)$ is an *anti-matching* if every vertex in S is non-adjacent to at most one vertex of S. A *brace* is a clique of size 2, a *triangle* is a clique of size 3 and a *triad* is a stable set of size 3. Note that all the graphs that we consider in this thesis are finite.

We say that H is a *subgraph* of G with vertex set X, if every pair of vertices in X that are adjacent in H are also adjacent in G. For $X \subseteq V(G)$, we define the subgraph $G|X$ *induced on* X as the subgraph with vertex set X and such that x is adjacent to y in $G|X$ if and only if x is adjacent to y in G. For a graph H, we say that H is an *induced subgraph* of G if there exists $X \subseteq V(G)$ such that $G|X = H$. A k-coloring of G is a map $c : V(G) \rightarrow \{1, \ldots, k\}$ such that for every pair of adjacent vertices $v, w \in V(G)$, $c(v) \neq c(w)$. For simplicity, we may also refer to a k-coloring as a coloring. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest integer such that there exits a $\chi(G)$-coloring of G. The *clique number* of G, denoted by $\omega(G)$, is the size of a maximum clique in G, and the *stability number* of G, denoted by $\alpha(G)$ is the size of the maximum stable set in G. A graph G is said to be *perfect* if for every induced subgraph G' of G, the chromatic number of G' is equal to the clique number of G'. The complement of a graph G is the graph $\overline{G}$ with vertex set $V(G)$ and such that x is adjacent to y in $\overline{G}$ if and only if x is non-adjacent to y in G.

Perfect graphs were introduced in 1960 by Claude Berge and are a central family of graphs because they are the graphs that behave 'perfectly' in terms of coloring. For any graph G, $\omega(G)$ is always a trivial lower bound on the chromatic number. Perfect graphs are the family of graphs that match this bound and are closed under taking induced subgraph. When Claude Berge introduced the family of perfect graphs, he also introduced another family of graphs - that we now call Berge graphs. To give a formal description, we first need a few more definitions. A *path* in G is a subgraph P with n vertices for $n \geq 1$, whose vertex set can be ordered as $\{p_1, \ldots, p_n\}$ such that p_i is adjacent to p_{i+1} for $1 \leq i < n$. A *cycle* in G is a subgraph C with n vertices for some $n \geq 3$, whose vertex set can be ordered as $\{c_1, \ldots, c_n\}$ such that c_i is adjacent to c_{i+1} for $1 \leq i < n$, and c_n is adjacent to c_1. We say that a cycle C is a *hole*, if $n > 3$ and if for all $1 \leq i, j \leq n$ with $i + 2 \leq j$ and $(i, j) \neq (1, n)$, c_i is non-adjacent to c_j. The *length* of C is the number of vertices of C. We say that a graph G is *Berge* if G does not contain any odd holes and $\overline{G}$ does not contain any odd holes. Claude Berge stated two conjectures when introducing perfect graphs. The first one, known as the Weak Perfect Graph Conjecture, states that a graph G is perfect if an only if $\overline{G}$ is perfect. It was proved to be true by Lovász [25]. The second conjecture, known as the Strong Perfect Graph Conjecture, states that a graph is perfect if and only if it is Berge. This conjecture remained open for more than 40 years before Chudnovsky, Robertson, Seymour and Thomas proved it in 2002 [9]. Those two results are stated bellow.

1.1.1 (**Weak Perfect Graph Theorem.** Lovász [25]). *A graph G is perfect if and only if $\overline{G}$ is perfect.*

1.1.2 (**Strong Perfect Graph Theorem.** Chudnovsky, Robertson, Seymour and Thomas [9]). *A graph is perfect if and only it is Berge.*

1.2 Claw-free perfect graphs

The *neighborhood* of a vertex v is the set $N(v)$ of vertices adjacent to v. Vertices of $N(v)$ are called *neighbors* of v. Given a multigraph G, the *line graph of G*, denoted by $L(G)$, is the graph with vertex set $V(L(G)) = E(G)$ in which two vertices are adjacent precisely if their corresponding edges in H share an endpoint. We say that a graph G' is a *line graph* if for some multigraph G, $L(G)$ is isomorphic to G'. A vertex is *simplicial* if its neighborhood is a clique, and a vertex is *bisimplicial* if its neighborhood is the union of two cliques. A graph G is *quasi-line* if every vertex v of G is bisimplicial. A *claw* is the graph with four vertices and three edges where the edges are all incident to a single vertex (see Figure 1.1). A graph G is *claw-free* if it contains no induced claw. It is easy to observe that every line graph is quasi-line and every quasi-line graph is claw-free. In Figure 1.2, we give two examples that show that there are quasi-line graphs that are not line graphs and claw-free graphs that are not quasi-line graphs.

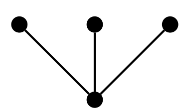

Figure 1.1: This illustration show a claw.

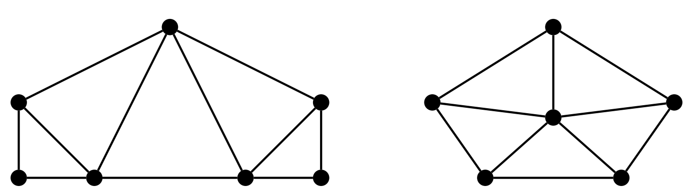

Figure 1.2: The graph on the left is a quasi-line graph but is not a line graph. The graph on the right is a claw-free graph but is not a quasi-line graph.

Several attempts have been made to describe claw-free perfect graphs: first by Chvátal and Sbihi in 1988 [15], and then by Maffray and Reed in 1999 [26]. However, these results only showed how to decompose claw-free perfect graphs, but did not show how to construct them explicitly. Indeed, Chvátal and Sbihi result uses clique-cutsets to decompose claw-free perfect graphs into two basic classes of graphs. But the inverse operation of gluing two graphs on a clique might produce a claw in the resulting graph. In order to obtain a structural theorem, understanding how to decompose graphs is only a first step, but it still only gives an incomplete picture of the family. We wish to be able to build all graphs in a family from smaller graphs in such a way that every graph we construct is in the family.

The results presented in Chapter 3 give a complete description of the structure of claw-free perfect graphs. In fact, by the Strong Perfect Graph Theorem 1.1.2, we study claw-free Berge graphs because in many cases it is easier to prove that a graph is Berge than to prove that the graph is perfect. Actually we will work with slightly more general objects called trigraphs which will be defined in Chapter 2. Chudnovsky and Seymour proved a structural theorem for general claw-free graphs [13] and quasi-line graphs in [14]. Later we will show that every perfect claw-free graph is a quasi-line graph, however not all quasi-line graphs are perfect. Our result refines the characterization of quasi-line graphs from [14] to obtain a precise description of perfect quasi-line graphs.

1.3 Conjectures related to the chromatic number

Finding the exact value of the chromatic number of a graph is a fundamental algorithmic and theoretical problem in graph theory. Attempt to bound the value of $\chi(G)$ for families of graphs have been made since the beginning of graph theory. One of the most famous example is probably the *Four Color Theorem*. In the 18th century, the following question has been raised: Is it was true that the chromatic number of planar graphs is 4? A graph is *planar* if it can be drawn on a plan with no edge crossing each other. It is easy to build a planar graph that needs 4 colors, but it took more than a century until Appel and Haken proved in 1976 the following:

1.3.1 (**Four Color Theorem.** Appel and Haken [1]). *Let G be a planar graph, then $\chi(G) \leq 4$.*

In the last 50 years, many conjectures and many theorems related to the chromatic number have been stated. We present now one of them, a conjecture that Erdős and Lovász made in 1968.

Conjecture 1 (Erdős-Lovász Tihany). *For every graph G with $\chi(G) > \omega(G)$ and for every two integers $s, t \geq 2$ with $s + t = \chi(G) + 1$, there is a partition (S, T) of the vertex set $V(G)$ such that $\chi(G|S) \geq s$ and $\chi(G|T) \geq t$.*

Currently, the only settled cases of the conjecture are $(s, t) \in \{(2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (3, 5)\}$ [3; 28; 33; 34]. Recently, Balogh, Kostochka, Prince and Stiebitz proved Conjecture 1 for quasi-line graphs. In Chapter 4, we consider the Erdős-Lovász Tihany Conjecture for claw-free graphs. We prove a slightly weakened version of Conjecture 1 for this class of graphs. Our proof relies on a structure theorem of claw-free graphs by Chudnovsky and Seymour [13]

The *degree* $d(v)$ of a vertex $v \in V(G)$ is the number of vertices adjacent to v in G. For a graph G, we define the *maximal degree* by $\Delta(G) = \max_{v \in V(G)}\{d(v)\}$. The chromatic number of G is trivially bounded above by $\Delta(G) + 1$ and below by $\omega(G)$. Reed's ω, Δ, χ Conjecture proposes, roughly speaking, that $\chi(G)$ falls in the lower half of this range:

Conjecture 2 (Reed). *For any graph G,*

$$\chi(G) \leq \left\lceil \tfrac{1}{2}(\Delta(G) + 1 + \omega(G)) \right\rceil .$$

One of the first classes of graphs for which this conjecture was proved is the class of line graphs [23]. Already for line graph the conjecture is tight. We show in Figure 1.3 examples of line graphs for which the conjecture holds with equality.

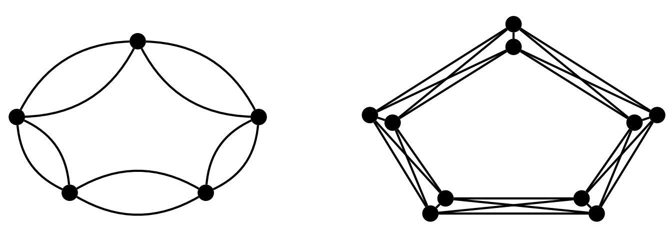

Figure 1.3: Example of a line graph for which Conjecture 2 is tight. The graph on the right is the line graph of the graph on the left.

The proof of Conjecture 2 for line graphs was later extended to quasi-line graphs [21; 22] and claw-free graphs [21]. In his thesis, King proposed a strengthening of Reed's Conjecture, giving a bound in terms of local parameters. For a vertex v, let $\omega(v)$ denote the size of the largest clique containing v.

Conjecture 3 (King [21]). *For any graph G,*

$$\chi(G) \leq \max_{v \in V(G)} \left\lceil \tfrac{1}{2}(d(v) + 1 + \omega(v)) \right\rceil .$$

In Chapter 5 we prove that Conjecture 3 holds for line graphs. Then using methods similar to [22], we extend the result to quasi-line graphs. Furthermore our proofs yield polytime algorithms for constructing a proper coloring achieving the bound of the theorem: $O(n^2)$ time for a line graph on n vertices, and $O(n^3 m^2)$ time for a quasi-line graph on n vertices and m edges.

The thesis is organized as follow. In Chapter 2, we introduce trigraphs and notions associated with them. In Chapter 3, we present and prove our structural theorem for claw-free perfect graphs. In Chapter 4, we explore the Erdős-Lovász Tihany for claw-free graphs. Finally in Chapter 5, we prove Conjecture 3 for quasi-line graphs.

Chapter 2

Trigraphs

Trigraphs are a generalization of graphs that are useful for studying problems about forbidden induced subgraphs. Trigraphs will be extensively used in Chapter 3. The majority of graph notions can be directly extended to trigraphs, and will be described in this chapter. All graphs and trigraphs considered in this thesis are finite.

A *trigraph* G consists of a finite set $V(G)$ of vertices, and a map $\theta_G : V(G)^2 \to \{-1, 0, 1\}$, satisfying:

- for all $v \in V(G)$, $\theta_G(v, v) = 0$.
- for all distinct $u, v \in V(G)$, $\theta_G(u, v) = \theta_G(v, u)$
- for all distinct $u, v, w \in V(G)$, at most one of $\theta_G(u, v), \theta_G(u, w)$ equals 0.

For distinct $u, v \in V(G)$, we say that u, v are *strongly adjacent* if $\theta_G(u, v) = 1$, *strongly antiadjacent* if $\theta_G(u, v) = -1$, and *semiadjacent* if $\theta_G(u, v) = 0$. We say that u, v are *adjacent* if they are either strongly adjacent or semiadjacent, and *antiadjacent* if they are either strongly antiadjacent or semiadjacent. Also, we say that u is *adjacent* to v if u, v are adjacent, and that u is *antiadjacent* to v if u, v are antiadjacent. For a vertex a and a set $B \subseteq V(G) \setminus \{a\}$, we say that a is *complete* (resp. *anticomplete*) to B if a is adjacent (resp. antiadjacent) to every vertex in B. For two disjoint $A, B \subset V(G)$, we say that A is *complete* (resp. *anticomplete*) to B if every vertex in A is complete (resp. anticomplete) to B. Similarly, we say that a is strongly complete to B if a is strongly adjacent to every member of B, and so on.

Let G be a trigraph. A *clique* is a set $X \subseteq V(G)$ such that every two members of X are adjacent and X is a *strong clique* if every two members of X are strongly adjacent. A set $X \subseteq V(G)$ is a *stable set* if every two members of X are antiadjacent and X is a *strong stable set* if every two members of X are strongly antiadjacent. A *triangle* is a clique of size 3, and a *triad* is a stable set of size 3.

For a trigraph G and $X \subseteq V(G)$, we define the trigraph $G|X$ *induced on* X as follows. Its vertex set is X, and its adjacency function is the restriction of θ_G to X^2. We say that G *contains* H, and H is a *subtrigraph* of G if there exists $X \subseteq V(G)$ such that H is isomorphic to $G|X$.

A *claw* is a trigraph H such that $V(H) = \{x, a, b, c\}$, x is complete to $\{a, b, c\}$ and $\{a, b, c\}$ is a triad. A trigraph G is said to be *claw-free* if G does not contains a claw.

A *path* in G is a subtrigraph P with n vertices for $n \geq 1$, whose vertex set can be ordered as $\{p_1, \ldots, p_n\}$ such that p_i is adjacent to p_{i+1} for $1 \leq i < n$ and p_i is antiadjacent to p_j if $|i - j| > 1$. We generally denote P with the following sequence $p_1 - p_2 - \ldots - p_n$ and say that the path P is from p_1 to p_n. For a path $P = p_1 - \ldots - p_n$ and $i, j \in \{1, \ldots, n\}$ with $i < j$, we denote by $p_i - P - p_j$ the *subpath* P' of P defined by $P' = p_i - p_{i+1} - \ldots - p_j$.

A *cycle* (resp. *anticycle*) in G is a subtrigraph C with n vertices for some $n \geq 3$, whose vertex set can be ordered as $\{c_1, \ldots, c_n\}$ such that c_i is adjacent (resp. antiadjacent) to c_{i+1} for $1 \leq i < n$, and c_n is adjacent (resp. antiadjacent) to c_1. We say that a cycle (resp. anticycle) C is a *hole* (resp. *antihole*), if $n > 3$ and if for all $1 \leq i, j \leq n$ with $i + 2 \leq j$ and $(i, j) \neq (1, n)$, c_i is antiadjacent (resp. adjacent) to c_j. We will generally denote C with the following sequence $c_1 - c_2 - \ldots - c_n - c_1$. The *length* of C is the number of vertices of C. Vertices c_i and c_j are said to be *consecutive* if $i + 1 = j$ or $\{i, j\} = \{1, n\}$.

A trigraph G is said to be *Berge* if no subtrigraph of G is a hole, and no subtrigraph of G is an antihole. In Chapter 3, we study perfect graphs, which by the strong perfect graph theorem [9], is equivalent to studying Berge graphs. We will in fact work with the slightly more general Berge trigraphs. Since it is easier in many cases to prove that a trigraph is Berge than to prove that the trigraph is perfect, we will only deal with Berge trigraphs.

A trigraph G is *cobipartite* if there exist nonempty subsets $X, Y \subseteq V(G)$ such that X and Y are strong cliques and $X \cup Y = V(G)$.

For $X, A, B, C \subseteq V(G)$, we say that $\{X|A, B, C\}$ *is a claw* if there exist $x \in X$, $a \in A$,

$b \in B$ and $c \in C$ such that $G|\{x, a, b, c\}$ is a claw and x is complete to $\{a, b, c\}$. Similarly, for $X_1, \ldots, X_n \subseteq V(G)$, we say that $X_1 - X_2 - \ldots - X_n - X_1$ is a *hole* (resp. *antihole*) if there exist $x_i \in X_i$ such that $x_1 - x_2 - \ldots - x_n - x_1$ is a hole (resp. antihole). To simplify notation, we will generally forget the bracket delimiting a singleton, i.e. instead of writing $\{\{x\}|A, \{y\}, B\}$ we will simply denote it by $\{x|A, y, B\}$.

Let A, B be disjoint subsets of $V(G)$. The set A is called a *homogeneous set* if A is a strong clique, and every vertex in $V(G)\backslash A$ is either strongly complete or strongly anticomplete to A. The pair (A, B) is called a *homogeneous pair* in G if A, B are nonempty strong cliques, and for every vertex $v \in V(G)\backslash(A \cup B)$, v is either strongly complete to A or strongly anticomplete to A, and either strongly complete to B or strongly anticomplete to B.

Let V_1, V_2 be a partition of $V(G)$ such that $V_1 \cup V_2 = V(G)$, $V_1 \cap V_2 = \emptyset$, and for $i = 1, 2$ there is a subset $A_i \subseteq V_i$ such that:

- A_i and $V_i \backslash A_i$ are not empty for $i = 1, 2$,
- $A_1 \cup A_2$ is a strong clique,
- $V_1 \backslash A_1$ is strongly anticomplete to V_2, and V_1 is strongly anticomplete to $V_2 \backslash A_2$.

The partition (V_1, V_2) is called a *1-join* and we say that G *admits a 1-join* if such a partition exists.

Let $A_1, A_2, A_3, B_1, B_2, B_3$ be disjoint subsets of $V(G)$. The 6-tuple $(A_1, A_2, A_3|B_1, B_2, B_3)$ is called a *hex-join* if $A_1, A_2, A_3, B_1, B_2, B_3$ are strong cliques and

- A_1 is strongly complete to $B_1 \cup B_2$, and strongly anticomplete to B_3, and
- A_2 is strongly complete to $B_2 \cup B_3$, and strongly anticomplete to B_1, and
- A_3 is strongly complete to $B_1 \cup B_3$, and strongly anticomplete to B_2, and
- $\bigcup_i (A_i \cup B_i) = V(G)$.

Let G be a trigraph. For $v \in V(G)$, we define the *neighborhood* of v, denoted $N(v)$, by $N(v) = \{x : x \text{ is adjacent to } v\}$. The trigraph G is said to be a *quasi-line trigraph* if for every $v \in V(G)$, $N(v)$ is the union of two strong cliques.

A trigraph H is a *thickening* of a trigraph G if for every $v \in V(G)$ there is a nonempty subset $X_v \subseteq V(H)$, all pairwise disjoint and with union $V(H)$, satisfying the following:

- for each $v \in V(G)$, X_v is a strong clique of H,
- if $u, v \in V(G)$ are strongly adjacent in G then X_u is strongly complete to X_v in H,
- if $u, v \in V(G)$ are strongly antiadjacent in G then X_u is strongly anticomplete to X_v in H,
- if $u, v \in V(G)$ are semiadjacent in G then X_u is neither strongly complete nor strongly anticomplete to X_v in H.

Next we present some definitions related to quasi-line graphs that have been introduced in [14]. To develop our structural results in Chapter 3, we need a few more definitions that refine and extend the concepts used in [14] and will be presented at the same time.

A *stripe* is a pair (G, Z) of a trigraph G and a subset Z of $V(G)$ such that $|Z| \leq 2$, Z is a strong stable set, $N(z)$ is a strong clique for all $z \in Z$, no vertex is semiadjacent to a vertex in Z, and no vertex is adjacent to two vertices of Z.

G is said to be a *linear interval trigraph* if its vertex set can be numbered $\{v_1, \ldots, v_n\}$ in such a way that for $1 \leq i < j < k \leq n$, if v_i, v_k are adjacent then v_j is strongly adjacent to both v_i, v_k. Given such a trigraph G and numbering $v_1, \ldots, v_n$ with $n \geq 2$, we call $(G, \{v_1, v_n\})$ a *linear interval stripe* if G is connected, no vertex is semiadjacent to v_1 or to v_n, there is no vertex adjacent to both v_1, v_n, and v_1, v_n are strongly antiadjacent. By analogy with intervals, we will use the following notation, $[v_i, v_j] = \{v_k\}_{i \leq k \leq j}$, $(v_i, v_j) = \{v_k\}_{i<k<j}$, $[v_i, v_j) = \{v_k\}_{i \leq k < j}$ and $(v_i, v_j] = \{v_k\}_{i<k \leq j}$. Moreover we will write $x_i < x_j$ if $i < j$.

Let Σ be a circle in the sphere, and let $F_1, \ldots, F_k \subseteq \Sigma$ be homeomorphic to the interval $[0, 1]$, such that no two of $F_1, \ldots, F_k$ share an end-point. Now let $V \subseteq \Sigma$ be finite, and let G be a trigraph with vertex set V in which, for distinct $u, v \in V$,

- if $u, v \in F_i$ for some i then u, v are adjacent, and if also at least one of u, v belongs to the interior of F_i then u, v are strongly adjacent,
- if there is no i such that $u, v \in F_i$ then u, v are strongly antiadjacent.

Such a trigraph G is called a *circular interval trigraph*. We will denote by F_i^* the interior of F_i.

Let G have four vertices say w, x, y, z, such that w is strongly adjacent to x, y is strongly adjacent to z, x is semiadjacent to y, and all other pairs are strongly antiadjacent. Then the pair $(G, \{w, z\})$ is a *spring* and the pair $(G \backslash w, \{z\})$ is a *truncated spring*.

Let G have three vertices say v, z_1, z_2 such that v is strongly adjacent to z_1 and z_2, and z_1, z_2 are strongly antiadjacent. Then the pair $(G, \{z_1, z_2\})$ is a *spot*, the pair $(G, \{z_1\})$ is a *one-ended spot* and the pair $(G \setminus z_2, \{z_1\})$ is a *truncated spot*.

Let G be a circular interval trigraph, and let $\Sigma, F_1, \ldots, F_k$ be as in the corresponding definition. Let $z \in V(G)$ belong to at most one of $F_1, \ldots, F_k$; and if $z \in F_i$ say, let no vertex be an endpoint of F_i. We call the pair $(G, \{z\})$ a *bubble*.

If H is a thickening of G, where X_v $(v \in V(G))$ are the corresponding subsets, and $Z \subseteq V(G)$ and $|X_v| = 1$ for each $v \in Z$, let Z' be the union of all X_v $(v \in Z)$; we say that (H, Z') is a *thickening* of (G, Z).

The following construction is slightly different from how *linear interval joins* have been defined for general quasi-line graphs [14], but the resulting graphs are exactly the same. We may also assume that if (G, Z) is a stripe then $V(G) \neq Z$. Any trigraph G that can be constructed in the following manner is called a *linear interval join*.

- Let $H = (V, E)$ be a graph, possibly with multiple edges and loops.
- Let $\eta : (E \times V) \cup E \to 2^{V(G)}$.
- For every edge $e = x_1 x_2 \in E$ (where $x_1 = x_2$ if e is a loop)
  - Let (G_e, Y_e) be either
    - a spot or a thickening of a linear interval stripe if e is not a loop, or
    - the thickening of a bubble if e is a loop.

    Moreover let ϕ_e be a bijection between Y_e and the endpoints of e.
  - Let $\eta(e, x_j) = N(\phi_e(x_j))$ for $j = 1, 2$ and $\eta(e, v) = \emptyset$ if v is not an endpoint of e.
  - Let $\eta(e) = V(G_e) \setminus Y_e$.
- Construct G with $V(G) = \bigcup_{e \in E} \eta(e)$, $G|\eta(e) = G_e \setminus Y_e$ for all $e \in E$ and such that $\eta(f, x)$ is strongly complete to $\eta(g, x)$ for all $f, g \in E$ and $x \in V$ (in particular if x is an endpoint of both f and g, then the sets $\eta(f, x)$ and $\eta(g, y)$ are nonempty and strongly complete to each other).

Moreover, we call the graph H used in the construction of a linear interval join G the *skeleton* of G, and we say that e has been *replaced* by (G_e, Y_e).

Let G be a circular interval trigraph. The trigraph G is a *structured circular interval trigraph* if, for some even integer $n \geq 4$, $V(G)$ can be partitioned into pairwise disjoint strong cliques $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ such that (all indices are modulo n):

(S1) $\bigcup_i (X_i \cup Y_i) = V(G)$.

(S2) $X_i \neq \emptyset \; \forall \; i$.

(S3) Y_i is strongly complete to X_i and X_{i+1} and strongly anticomplete to $V(G) \setminus (X_i \cup X_{i+1} \cup Y_i)$.

(S4) If $Y_i \neq \emptyset$ then X_i is strongly complete to X_{i+1}.

(S5) Every vertex in X_i has at least one neighbor in X_{i+1} and at least one neighbor in X_{i-1}.

(S6) X_i is strongly complete to X_{i+1} or X_{i-1} and possibly both, and strongly anticomplete to $V(G) \setminus (X_i \cup X_{i-1} \cup X_{i+1} \cup Y_i \cup Y_{i-1})$.

A bubble (G, Z) is said to be a *structured bubble* if G is a structured circular interval trigraph.

We need to define one important class of Berge circular interval trigraphs. Let G be a trigraph with vertex set the disjoint union of sets $\{a_1, a_2, a_3\}, B_1^1, B_1^2, B_1^3, B_2^1, B_2^2, B_2^3, B_3^1, B_3^2, B_3^3$ such that $|B_i^j| \leq 1$ for $1 \leq i, j \leq 3$ with adjacency as follows (all additions are modulo 3):

- For $i = 1, 2, 3$, $B_i^1 \cup B_i^2 \cup B_i^3$ is a strong clique.
- For $i = 1, 2, 3$, B_i^i is strongly complete to $\bigcup_{k=1}^{3} (B_{i+1}^k \cup B_{i+2}^k)$.
- For $1 \leq i, j \leq 3$ with $i \neq j$, B_i^j is strongly complete to $\bigcup_{k=1}^{3} B_j^k$.
- For $i = 1, 2, 3$, B_i^{i+1} and B_{i+2}^{i+1} are either both empty or both nonempty, and if they are both nonempty then B_i^{i+1} is not strongly complete to B_{i+2}^{i+1}.
- For $i = 1, 2, 3$, a_i is strongly complete to $\bigcup_{k=1}^{3} (B_i^k \cup B_{i+1}^k)$ and a_i is strongly anticomplete to $\bigcup_{k=1}^{3} B_{i+2}^k$.
- a_1 is antiadjacent to a_3, and a_2 is strongly anticomplete to $\{a_1, a_3\}$.

- If a_1 is semiadjacent to a_3 then $B_3^1 \cup B_2^1 = \emptyset$.
- There exist $x_i \in V(G) \cap (B_i^1 \cup B_i^2 \cup B_i^3)$ for $i = 1, 2, 3$, such that $\{x_1, x_2, x_3\}$ is a triangle.

We define $\mathcal{C}$ to be the class of all such trigraphs G. We will prove in 3.2.7 that all trigraphs in $\mathcal{C}$ are Berge and circular inteveral. Moreover we define $\mathcal{C}'$ to be the set of all pairs $(H, \{z\})$ such that there exists $i \in \{1, 2, 3\}$ with $z \in X_{a_i}$, H is a thickening of a trigraph in $\mathcal{C}$ with $B_{i+1}^{i+2} \cup B_i^{i+2} = \emptyset$ and such that $N(z) \cap (X_{a_{i+1}} \cup X_{a_{i+2}}) = \emptyset$ (with X_{a_i} as in the definition of a thickening).

A *signing* of a graph $G = (V, E)$ is a function $s : E \to \{0, 1\}$. The value $v(C)$ of a cycle C is $v(C) = \sum_{e \in C} s(e)$. A graph, possibly with multiple edges and loops, is said to be *evenly signed* by s if for all cycles C in G, C has an even value, and in that case the pair (G, s) is said to be an *evenly signed graph*.

We need to define three classes of graphs that are going to play an important role in the structure of claw-free perfect graphs.

$\mathcal{F}_1$: Let (G, s) be a pair of a graph G (possibly with multiple edges and loops) and a signing s of G such that:

- $V(G) = \{x_1, x_2, x_3\}$,
- there is an edge e_{ij} between x_i and x_j with $s(e_{ij}) = 1$ for all $\{i, j\} \subset \{1, 2, 3\}$ with $i \neq j$,
- if e and f are such that $s(e) = s(f) = 0$, then e is parallel to f.

We define $\mathcal{F}_1$ to be the class of all such pairs (G, s).

$\mathcal{F}_2$: Let (G, s) be a pair of a graph G (possibly with multiple edges and loops) and a signing s of G such that $|V(G)| = 4|$, all pairs of vertices of G are adjacent and $s(e) = 1$ for all $e \in E(G)$. We define $\mathcal{F}_2$ to be the class of all such pairs (G, s).

$\mathcal{F}_3$: Let (G, s) be a pair of a graph G (possibly with multiple edges and loops) and a signing s of G such that:

- $V(G) = \{x_1, x_2, \dots, x_n\}$ with $n \geq 4$,
- there is an edge e_{12} between x_1 and x_2 with $s(e) = 1$,
- $\{x_1, x_2\}$ is complete to $\{x_3, \dots, x_n\}$,

- $\{x_3, \ldots, x_n\}$ is a stable set,
- if $s(e) = 0$, then e is an edge between x_1 and x_2.

We define $\mathcal{F}_3$ to be the class of all such pairs (G, s).

An *even structure* is a pair (G, s) of a graph G and a signing s such that for all blocks A of G, $(A, s|_{E(A)})$ is either a member of $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$ or an evenly signed graph.

Here is a construction; a trigraph G that can be constructed in this manner is called an *evenly structured linear interval join.*

- Let $H = (V, E)$ and the signing s be an even structure.
- Let $\eta : (E \times V) \cup E \to 2^{V(G)}$.
- For every edge $e = x_1x_2 \in E$ (where $x_1 = x_2$ if e is a loop),
  - Let (G_e, Y_e) be:
    - a spot if e is in a cycle, $x_1 \neq x_2$ and $s(e) = 1$,
    - a thickening of a spring if e is in a cycle, $x_1 \neq x_2$, and $s(e) = 0$,
    - a trigraph in $\mathcal{C}'$ if e is a loop,
    - either a spot or a thickening of a linear interval stripe if e is not in a cycle.
  - Let ϕ_e be a bijection between the endpoints of e and Y_e.
  - Let $\eta(e, x_j) = N(\phi_e(x_j))$ for $j = 1, 2$ and $\eta(e, v) = \emptyset$ if v is not an endpoint of e.
  - Let $\eta(e) = V(G_e)\backslash Y_e$.
- Construct G with $V(G) = \bigcup_{e \in E} \eta(e)$, $G|\eta(e) = G_e \backslash Y_e$ for all $e \in E$ and such that $\eta(f, x)$ is complete to $\eta(g, x)$ for all $f, g \in E$ and $x \in V$ (in particular if x is an endpoint of both f and g, then the sets $\eta(f, x)$ and $\eta(g, y)$ are nonempty and strongly complete to each other).

As for the linear interval join, we call the graph H used in the construction of an evenly structured linear interval join G the *skeleton* of G, and we say that e has been *replaced* by (G_e, Y_e) .

Chapter 3

The Structure of Claw-Free Perfect Graphs

The class of claw-free perfect graphs has been extensively studied in the past. The first structural result for this class was obtained by Chvátal and Sbihi [15]. In particular, in 1998, they proved that every claw-free Berge graph can be decomposed via clique-cutsets into two types of graphs: 'elementary' and 'peculiar'. We say that a graph G admits a *clique-cutset* A, B if A and B are subset of $V(G)$ such that $A \cap B$ is a clique, $A \cup B = V(G)$ and there is no edge between $A \backslash B$ and $B \backslash A$. If a graph G admits a clique-cutset A, B, it is a classical technique to decompose G into $G|A$ and $G|B$. The structure of peculiar graphs was determined precisely by their definition, but that was not the case for elementary graphs. In 1999, Maffray and Reed [26] proved that an elementary graph is an augmentation of the line graph of a bipartite multigraph, thereby giving a precise description of all elementary graphs. Their result, together with the result of Chvátal and Sbihi, gave an alternative proof of Berge's Strong Perfect Graph Conjecture for claw-free Berge graphs (the first proof was due to Parthasarathy and Ravindra [30]). However, this still does not describe the class of claw-free perfect graphs completely, as gluing two claw-free Berge graphs together via a clique-cutset may introduce a claw. In this chapter we use trigraphs and a previous result on claw-free graphs by Chudnovsky and Seymour [14] to obtain a full characterization of claw-free perfect graphs. That is, we give an explicit construction describing all claw-free perfect graphs, using a technique that generalizes the construction of line graphs.

This chapter is organized as follows. In Section 3.1, we prove a few introductory results and state our main theorem 3.1.4. The proof of 3.1.4 is broken down in several cases depending on the underlying structure of the graph. Each section analyzes a different case of the main theorem. In Section 3.2, we study circular interval trigraphs that contain special triangles. Section 3.3 examines circular interval trigraphs that contain a hole of length 4 while Section 3.4 covers the case when a circular interval trigraph contains a long hole. In Section 3.5, we analyze linear interval joins. Finally, in Section 3.6, we gather our results and prove 3.1.4.

3.1 Preliminary results

We start by proving two easy facts.

3.1.1. *Every claw-free Berge trigraph is a quasi-line trigraph.*

Proof. Let G be a claw-free Berge trigraph and let $v \in V(G)$. Since G is claw-free, we deduce that $G|N(v)$ does not contain a triad. Since G is Berge, we deduce that $G|N(v)$ does not contain a odd antihole. Thus $G|N(v)$ is cobipartite and it follows that $N(v)$ is the union of two strong cliques. This proves 3.1.1. □

3.1.2. *Let G be a trigraph and H be a thickening of G. If F is a thickening of H then F is a thickening of G.*

Proof. For $v \in V(H)$, let X_v^F be the strong clique in F as in the definition of a thickening. For $v \in V(G)$, let X_v^H be the strong clique in H as in the definition of a thickening. For $v \in V(G)$, let $Y_v \subseteq V(F)$ be defined as $Y_v = \bigcup_{y \in X_v^H} X_y^F$. Clearly, the sets Y_v are all nonempty, pairwise disjoint and their union is $V(F)$. Since X_v^H is a strong clique, we deduce that Y_v is a strong clique for all $v \in V(G)$. If $u, v \in V(G)$ are strongly adjacent (resp. antiadjacent) in G, then X_u^H is strongly complete (resp. anticomplete) to X_v^H in H and thus Y_u is strongly complete (resp. anticomplete) to Y_v in F. If $u, v \in V(G)$ are semiadjacent in G, then X_u^H is neither strongly complete nor strongly anticomplete to X_v^H in H and hence Y_u is neither strongly complete nor strongly anticomplete to Y_v in F. This proves 3.1.2. □

The following theorem is the main characterization of quasi-line graphs [14]. It is the starting point of our structure theorem for claw-free perfect graphs.

3.1.3. *Every connected quasi-line trigraph G is either a linear interval join or a thickening of a circular interval trigraph.*

We can now state our main theorem:

3.1.4. *Every connected Berge claw-free trigraph is either an evenly structured linear interval join or a thickening of a trigraph in $\mathcal{C}$.*

The goal of this chapter is to prove 3.1.4, but first we can prove an easy result about evenly signed graphs. Here is an algorithm that will produce an even signing for a graph:

Algorithm 1

- Let T be a spanning tree of G and root T at some $r \in V(G)$.
- Arbitrarily assign a value from $\{0, 1\}$ to $s(e)$ for all $e \in T$.
- For every $e = xy \in E(G)\backslash T$, let $s(e) = \sum_{f\in P_x} s(f) + \sum_{f\in P_y} s(f) \pmod 2$ where P_i is the path from r to i in T.

3.1.5. *Algorithm 1 produces an evenly signed graph* (G, s).

Proof. Let C be a cycle in G. First, we notice that for an edge e in T, $s(e)$ can be expressed with the same formula used to calculate the signing of an edge outside of T. In fact we have that for all $e \in E(G)$, $s(e) = \sum_{f\in P_x} s(f) + \sum_{f\in P_y} s(f) \pmod 2$. Thus,

$$\sum_{e=xy\in E(C)} s(e) = \sum_{xy\in E(C)} \left(\sum_{e\in P_x} s(e) + \sum_{e\in P_y} s(e) \right) =$$

$$= 2 \cdot \sum_{x\in V(C)} \left(\sum_{e\in P_x} s(e) \right) = 0 \pmod 2$$

which concludes the proof of 3.1.5. □

The result of 3.1.5 shows that if we have a graph G, we can find all signings s such that (G, s) is an evenly signed graph by using Algorithm 1 with all possible assignments for $s(e)$ on the tree T.

3.2 Essential Triangles

In order to prove 3.1.4, we first prove the following:

3.2.1. *Let G be a Berge circular interval trigraph. Then either G is a linear interval trigraph, or a cobipartite trigraph, or a thickening of a member of $\mathcal{C}$, or G is a structured circular interval trigraph.*

Before going further, more definitions are needed. Let G be a circular interval trigraph defined by Σ and $F_1, \ldots, F_k \subseteq \Sigma$. Let $T = \{c_1, c_2, c_3\}$ be a triangle. We say that T is *essential* if there exist $i_1, i_2, i_3 \in \{1, \ldots, k\}$ such that $c_1, c_2 \in F_{i_1}$, $c_2, c_3 \in F_{i_2}$ and $c_3, c_1 \in F_{i_3}$, and such that $F_{i_1} \cup F_{i_2} \cup F_{i_3} = \Sigma$. Let x, y, q be three points of Σ. We denote by $\Sigma^q_{x,y}$ the subset of Σ such that there exists a homeomorphism $\phi : \Sigma^q_{x,y} \to [0,1]$ with $\phi(x) = 0$ and $\phi(y) = 1$ and such that $q \in \Sigma^q_{x,y}$. Moreover let $\overline{\Sigma}^q_{x,y} = (\Sigma \backslash \Sigma^q_{x,y}) \cup \{x, y\}$.

The following two lemmas are basic facts that will be extensively used to prove 3.2.1.

3.2.2. *Let G be a circular interval trigraph defined by Σ and $F_1, \ldots, F_k$. Let $x, y, a, b \in V(G)$ such that $x \in \overline{\Sigma}^y_{a,b}$. If x is antiadjacent to a and b, then y is strongly antiadjacent to x.*

Proof. Assume not. Since x is adjacent to y, we deduce that there exists F_i such that $x, y \in F_i$. It follows that at least one of $a, b \in F_i^*$. By symmetry we may assume that $a \in F_i^*$, but it implies that a is strongly adjacent to x, a contradiction. This proves 3.2.2. □

3.2.3. *Let G be a circular interval trigraph defined by Σ and $F_1, \ldots, F_k$. Let $x, y, z \in V(G)$ such that x is adjacent to y and x is antiadjacent to z. Then there exists F_i such that $\overline{\Sigma}^z_{x,y} \subseteq F_i$.*

Proof. Since x is adjacent to y there is F_i such that $x, y \in F_i$. Since z is antiadjacent to x, we deduce that $z \notin F_i^*$. Thus we conclude that $\overline{\Sigma}^z_{x,y} \subseteq F_i$. This proves 3.2.3. □

3.2.4. *Let G be a circular interval trigraph defined by Σ and $F_1, \ldots, F_k$, and let $C = c_1 - c_2 - \ldots - c_n - c_1$ be a hole. Then the vertices of C are in order on Σ.*

Proof. Assume not. By symmetry, we may assume that c_1, c_2, c_3, c_4 are not in order on Σ, and thus we may assume that $c_4 \in \Sigma^{c_2}_{c_1,c_3}$. Since c_3 is antiadjacent to c_1 and since c_2 is complete to $\{c_1, c_3\}$, we deduce that there exist F_i and F_j, possibly $F_i = F_j$, such that $\overline{\Sigma}^{c_3}_{c_1,c_2} \subseteq F_i$ and $\overline{\Sigma}^{c_1}_{c_2,c_3} \subseteq F_j$. If $c_4 \in \overline{\Sigma}^{c_3}_{c_1,c_2}$, then since $c_4 \in F_i^*$, we deduce that c_4 is strongly complete to $\{c_1, c_2\}$, a contradiction.

If $c_4 \in \overline{\Sigma}^{c_1}_{c_3,c_2}$, then since $c_4 \in F^*_j$, we deduce that c_4 is strongly complete to $\{c_2, c_3\}$, a contradiction. This proves 3.2.4. □

3.2.5. *Let G be a circular interval trigraph defined by Σ and $F_1, \ldots, F_k$. If G is not a linear interval trigraph, then there exists an essential triangle or a hole in G.*

Proof. Let F_{i_1} be such that $F_{i_1} \cap V(G)$ is maximal and let $y \notin F_{i_1}$. Let $x_0, x_1 \in F_{i_1}$ such that $\overline{\Sigma}^{y}_{x_0,x_1} \cap F_{i_1}$ is maximal.

Let x_2 and F_{i_2} be such that $x_2 \in F_{i_2}$, $x_2 \notin F_{i_1}$ and $\overline{\Sigma}^{x_0}_{x_1,x_2}$ is maximal.

Starting with $j = 3$ and while $x_{j-1} \notin F_{i_1}$, let x_j and F_{i_j} be such that $x_j \in F_{i_j}$, $x_j \notin F_{i_k}$, for any $k < j$ and $\overline{\Sigma}^{x_1}_{x_{j-1},x_j}$ is maximal. Since G is not a linear interval trigraph, there exists $k > 1$ such that $x_k \in F_{i_1}$.

Assume first that $k = 3$. Clearly $F_{i_1} \cup F_{i_2} \cup F_{i_3} = \Sigma$, $x_0, x_1 \in F_{i_1}$, $x_1, x_2 \in F_{i_2}$ and $x_0, x_2 \in F_{i_3}$. Hence $T = \{x_0, x_1, x_2\}$ is an essential triangle.

Assume now that $k > 3$. Clearly x_{j-1} is adjacent to x_j for $j = 1, \ldots, k-1$ and x_{k-1} is adjacent to x_0. By the choice of F_{i_1} and x_0, x_1, we deduce that x_{k-1} is strongly antiadjacent to x_1. By the choice of F_{i_j}, x_{j-1} is antiadjacent to $x_{j+1 \bmod k}$ for all $j = 1, \ldots, k-1$. Hence by 3.2.2, C is a hole. This concludes the proof of 3.2.5. □

3.2.6. *Let G be a circular interval trigraph and C a hole. Let $x \in V(G) \setminus V(C)$, then x is strongly adjacent to two consecutive vertices of C.*

Proof. Let G be defined by Σ and $F_1, \ldots, F_k$ and let $C = c_1 - c_2 - \ldots - c_l - c_1$. By 3.2.4, there exists j such that $x \in \overline{\Sigma}^{c_{j+2}}_{c_j,c_{j+1}}$. Since c_j is adjacent to c_{j+1} and antiadjacent to c_{j+2}, we deduce that there exists $i \in \{1, \ldots, k\}$ such that $\overline{\Sigma}^{c_{j+2}}_{c_j,c_{j+1}} \subseteq F_i$. Hence x is strongly adjacent to c_j and c_{j+1}. This proves 3.2.6. □

In the remainder of this section, we focus on circular interval trigraphs that contain an essential triangle. For the rest of the section, addition is modulo 3.

3.2.7. *Every trigraph in $\mathcal{C}$ is a Berge circular interval trigraph.*

Proof. Let G be in $\mathcal{C}$. We let the reader check that G is indeed a circular interval trigraph, it can easily be done using the following order of the vertices on a circle:

$$B^3_1, B^1_1, B^2_1, a_1, B^1_2, B^2_2, B^3_2, a_2, B^2_3, B^3_3, B^1_3, a_3$$

(1) *There is no odd hole in G.*

Assume there is an odd hole $C = c_1 - c_2 - \ldots - c_n - c_1$ in G. Since B_i^i is strongly complete to $V(G)\backslash\{a_{i+1}\}$, it follows that $V(C) \cap B_i^i = \emptyset$ for all i. Since $G|(B_1^2 \cup B_1^3 \cup B_2^1 \cup B_2^3 \cup B_3^1 \cup B_3^2)$ is a cobipartite trigraph, we deduce that $|\{a_1, a_2, a_3\} \cap V(C)| \geq 1$.

Assume first that a_1, a_3 are two consecutive vertices of C. We may assume that $c_1 = a_1$ and $c_2 = a_3$. Since c_n is adjacent to c_1 and antiadjacent to c_2, we deduce that $c_n \in B_2^1 \cup B_2^3$. Symmetrically, $c_3 \in B_3^1 \cup B_3^2$. As a_1 is semiadjacent to a_3, it follows that $B_2^1 \cup B_3^1 = \emptyset$. Hence, c_3 is strongly adjacent to c_n, a contradiction.

Thus, we may assume that $c_1 = a_i$ and $\{c_2, c_n\} \cap \{a_1, a_2, a_3\} = \emptyset$. Since c_2 is antiadjacent to c_n, and c_1 is complete to $\{c_2, c_n\}$, we deduce that $\{c_2, c_n\} = B_i^{i+2} \cup B_{i+1}^{i+2}$. Without loss of generality, let $c_2 \in B_i^{i+2}$ and $c_n \in B_{i+1}^{i+2}$. Since c_{n-1} is antiadjacent to c_2, we deduce that $c_{n-1} = a_{i+1}$. Symmetrically, we deduce that $c_3 = a_{i+2}$. Since a_{i+2} is not consecutive with a_{i+1} in C, we deduce that $n > 5$. But $|\{x \in V(G) : x \text{ antiadjacent to } c_2\}| \leq 2$, a contradiction. This proves (1).

(2) *There is no odd antihole in G.*

Assume there is an odd antihole $C = c_1 - c_2 - \ldots - c_n$ in G. By (1), we may assume that C has length at least 7. Since B_i^i is strongly complete to $V(G)\backslash\{a_{i+1}\}$, it follows that $V(C) \cap B_i^i = \emptyset$ for all i.

Assume first that a_1 is semiadjacent to a_3. Then $B_3^1 \cup B_2^1 = \emptyset$. Since $|V(G)\backslash(B_1^1 \cup B_2^2 \cup B_3^3)| = 7$, we deduce that $V(C) = (\{a_1, a_2, a_3\} \cup B_1^2 \cup B_1^3 \cup B_3^2 \cup B_2^3)$. But a_2 has only two neighbors in $(\{a_1, a_2, a_3\} \cup B_1^2 \cup B_1^3 \cup B_3^2 \cup B_2^3)$, a contradiction. This proves that a_1 is strongly antiadjacent to a_3.

Assume now that $|V(C) \cap \{a_1, a_2, a_3\}| = 1$. We may assume that $a_1 \in V(C)$ and it follows that $V(C) = \{a_1\} \cup \bigcup_{j \neq k} B_j^k$. But $G|(\{a_i\} \bigcup_{j \neq k} B_j^k)$ is not an antihole of length 7, since the vertex of B_1^2 has 5 strong neighbors in $(\{a_i\} \bigcup_{j \neq k} B_j^k)$, a contradiction.

Hence we may assume that $|V(C) \cap \{a_1, a_2, a_3\}| \geq 2$. Since there is no triad in C, we deduce that $|C \cap \{a_1, a_2, a_3\}| = 2$ and by symmetry we may assume that $c_1 = a_1$, $c_2 = a_2$ and $a_3 \notin C$. But since $B_1^2 \cup B_1^3$ is strongly anticomplete to a_2 and $B_3^1 \cup B_3^2$ is strongly anticomplete to a_1, we deduce that $\{c_4, c_5, c_6\} \subseteq B_2^1 \cup B_2^3$, a contradiction. This proves (2).

Now by (1) and (2), we deduce 3.2.7. □

3.2.8. *Let G be a Berge circular interval trigraph such that G is not cobipartite. If G has an essential triangle, then G is a thickening of a trigraph in $\mathcal{C}$.*

Proof. Let $\{x_1, x_2, x_3\}$ be an essential triangle and let F_1, F_2, F_3 be such that $x_1 \in F_1 \cap F_3$, $x_2 \in F_1 \cap F_2$, $x_3 \in F_2 \cap F_3$ and $F_1 \cup F_2 \cup F_3 = \Sigma$.

(1) *x_i is not in a triad for $i = 1, 2, 3$.*

Assume x_1 is in a triad. Then there exist y, z such that $\{x_1, y, z\}$ is a triad. Since $x_1 \in F_1 \cap F_3$, we deduce that $y, z \in F_2^*$ and so y is strongly adjacent to z, a contradiction. This proves (1).

By (1) and as G is not a cobipartite trigraph, there exists a triad $\{a_1^*, a_2^*, a_3^*\}$ and we may assume that $a_i^* \in F_i \backslash (F_{i+1} \cup F_{i+2})$, $i = 1, 2, 3$. Let $\overline{a}_i \in F_i \cap \Sigma^{x_i}_{a_i^*, a_{i+2}^*}$ and $\overline{a}'_i \in F_i \cap \Sigma^{x_{i+1}}_{a_i^*, a_{i+1}^*}$ such that $\overline{a}_i, \overline{a}'_i$ are in triads and $\Sigma^{a_i^*}_{\overline{a}_i, \overline{a}'_i}$ is maximal. Let $\mathcal{A}_i = \Sigma^{a_i^*}_{\overline{a}_i, \overline{a}'_i}$, $\mathcal{B}_i = \Sigma^{x_i}_{a_i^*, a_{i+2}^*} \backslash (\mathcal{A}_i \cup \mathcal{A}_{i+2})$, $A_i = V(G) \cap \mathcal{A}_i$ and $B_i = V(G) \cap \mathcal{B}_i$. By the definition of $\overline{a}_1, \overline{a}_2, \overline{a}_3, \overline{a}'_1, \overline{a}'_2, \overline{a}'_3$, no vertex in $B_1 \cup B_2 \cup B_3$ is in a triad.

(2) *$\{\overline{a}_1, \overline{a}_2, \overline{a}_3\}$ and $\{\overline{a}'_1, \overline{a}'_2, \overline{a}'_3\}$ are triads.*

By the definition, $\overline{a}_1$ is in a triad. Let $\{\overline{a}_1, a_2, a_3\}$ be a triad, then we assume that $a_i \in A_i$, $i = 2, 3$. By 3.2.2, $\overline{a}_1$ is non adjacent to $\overline{a}_3$. Now, using symmetry, we deduce that $\{\overline{a}_1, \overline{a}_2, \overline{a}_3\}$ and $\{\overline{a}'_1, \overline{a}'_2, \overline{a}'_3\}$ are triads. This proves (2).

(3) *For all $x \in A_i$ there exist $y \in A_{i+1}, z \in A_{i+2}$ such that $\{x, y, z\}$ is a triad.*

By symmetry, we may assume that $x \in A_1$. If $|A_1| = 1$, then $x = a_1^*$ and $\{a_1^*, a_2^*, a_3^*\}$ is a triad.

Therefore, we may assume that $\overline{a}_1 \neq \overline{a}'_1$. By (2) and 3.2.2, x is antiadjacent to $\overline{a}'_2$ and $\overline{a}_3$. We may assume that $\{x, \overline{a}'_2, \overline{a}_3\}$ is not a triad, then $\overline{a}'_2$ is strongly adjacent to $\overline{a}_3$. By (2) and 3.2.2, $\overline{a}_2$ is strongly antiadjacent to $\overline{a}'_3$. Since $x - \overline{a}_2 - \overline{a}'_2 - \overline{a}_3 - \overline{a}'_3 - x$ is not a hole of length 5, we deduce that x is not strongly complete to $\{\overline{a}_2, \overline{a}'_3\}$. But now one of $\{x, \overline{a}'_2, \overline{a}'_3\}$, $\{x, \overline{a}_2, \overline{a}_3\}$ is a triad. This proves (3).

(4) *$\{x_1, x_2, x_3\}$ is a triangle such that $x_i \in B_i$ for $i = 1, 2, 3$.*

By (3), $x_i \notin A_1 \cup A_2 \cup A_3$ for $i = 1, 2, 3$. By the definition of B_i, it follows that $x_i \in B_i$ for $i = 1, 2, 3$. Moreover, $\{x_1, x_2, x_3\}$ is an essential triangle. This proves (4).

(5) $(A_1, A_2, A_3|B_1, B_2, B_3)$ *is a hex-join.*

By the definition of $A_1, A_2, A_3, B_1, B_2, B_3$, they are clearly pairwise disjoint and $\bigcup_i (A_i \cup B_i) = V(G)$. Clearly A_i is a strong clique as $\mathcal{A}_i \subset F_i,\ i = 1, 2, 3$.

If there exist $y_i, y'_i \in B_i$ such that y_i is antiadjacent to y'_i, then $\{y_i, y'_i, a^*_{i+1}\}$ is a triad by 3.2.2, a contradiction. Thus B_i is a strong clique for $i = 1, 2, 3$.

By symmetry, it remains to show that B_1 is strongly anticomplete to A_2 and strongly complete to A_1. Since $B_1 \subset \overline{\Sigma}^{a^*_2}_{a^*_1, a^*_3}$, we deduce that B_1 is strongly anticomplete to A_2 by 3.2.2 and (3).

Suppose there is $a_1 \in A_1$ and $b_1 \in B_1$ such that a_1 is antiadjacent to b_1. By (3), let $a_2 \in A_2$ and $a_3 \in A_3$ be such that $\{a_1, a_2, a_3\}$ is a triad. Since a_2 is anticomplete to $\{a_1, a_3\}$, and $b_1 \in \overline{\Sigma}^{a_2}_{a_1, a_3}$, we deduce by 3.2.2 that b_1 is strongly antiadjacent to a_2. Thus $\{a_1, a_2, b_1\}$ is a triad, a contradiction as $b_1 \in B_1$. This concludes the proof of (5).

(6) *There is no triangle* $\{a_1, a_2, a_3\}$ *with* $a_i \in A_i,\ i = 1, 2, 3$

Let $a_i \in A_i,\ i = 1, 2, 3$ be such that a_1 is adjacent to $a_i,\ i = 2, 3$. By (3), let $c_i \in A_i,\ i = 2, 3$ such that $\{a_1, c_2, c_3\}$ is a triad. By 3.2.3, $c_2 \in \overline{\Sigma}^{a_1}_{a_2, a_3}$. By symmetry, $c_3 \in \overline{\Sigma}^{a_1}_{a_2, a_3}$. Since $\{a_2|a_1, c_2, c_3\}$ is not a claw, we deduce that c_3 is strongly antiadjacent to a_2. By (2) and as $a_2 \in \overline{\Sigma}^{\overline{a}'_3}_{\overline{a}'_2, \overline{a}'_1}$, $\overline{a}'_3$ is antiadjacent a_2. Since $a_3 \in \overline{\Sigma}^{a_2}_{c_3, \overline{a}'_3}$ and by (2), a_3 is strongly antiadjacent to a_2. This proves (6).

For the rest of the proof of 3.2.8, let $\{j, k, l\} = \{1, 2, 3\}$.

(7) *There is no induced 3-edge path* $w - x - y - z$ *such that* $w \in A_j,\ x, y \in A_k,\ z \in A_l$.

Assume that $w - x - y - z$ is an induced 3-edge path such that $w \in A_1,\ x, y \in A_2,\ z \in A_3$. Now by (5), $w - x - y - z - x_1 - w$ is a hole of length 5, a contradiction. This proves (7).

(8) *For* $i = 1, 2, 3$*, let* $y_i \in A_i$*. Then* y_k *is strongly antiadjacent to at least one of* y_j, y_l.

Suppose there exist $y_i \in A_i,\ i = 1, 2, 3$ such that y_2 is adjacent to y_1 and y_3. By (6), y_1 is strongly antiadjacent to y_3. By (3), there exist $z_1, z_3 \in A_2$ such that z_1 is antiadjacent to y_1 and z_3 is antiadjacent to y_3. Since $\{y_2|y_1, y_3, z_3\}$ and $\{y_2|y_1, y_3, z_1\}$ are not claws, we deduce that y_1 is strongly adjacent to z_3, and y_3 is strongly adjacent to z_1. But $y_1 - z - 3 - z_1 - y_3$ is a 3-edge path, contrary to (7). This proves (8).

(9) *A_j is strongly anticomplete to at least one of A_k, A_l.*

Assume not. By symmetry, we may assume there are $x \in A_1$, $y, z \in A_2$ and $w \in A_3$ such that x is adjacent to y and z is adjacent to w. By (8), x is strongly antiadjacent to w, y is strongly antiadjacent to w, and z is strongly antiadjacent to x; and in particular $y \neq z$. But now $x-y-z-w$ is am induced 3-edge path, contrary to (7). This proves (9).

(10) *For $i = 1, 2, 3$, let $b_i \in B_i$ such that b_k is adjacent to b_l. Then b_j is strongly adjacent to at least one of b_k, b_l.*

By symmetry, we may assume that $j = 1$, $k = 2$ and $l = 3$. Since $b_1 - a_3^* - b_3 - b_2 - a_1^* - b_1$ is not a hole of length 5, by (5) we deduce that b_1 is strongly adjacent to at least one of b_2, b_3. This proves (10).

(11) *Let $x \in B_j$, then x is strongly complete to one of B_k, B_l.*

Assume there is $y \in B_k$ such that x is antiadjacent to y. Let $z \in B_l$. If y is antiadjacent to z, then x is strongly adjacent to z since $\{x, y, z\}$ is not a triad. By (10), if y is strongly adjacent to z, then x is strongly adjacent to z. Thus x is strongly complete to B_l. This proves (11).

By (9) and symmetry, we may assume that A_2 is strongly anticomplete to $A_1 \cup A_3$.

Let B_i^i be the set of all vertices of B_i that are strongly complete to $B_{i+1} \cup B_{i+2}$. For $j \neq i$, let B_i^j be the set of all vertices of $B_i \backslash B_i^i$ that are strongly complete to B_j. By (11), we deduce that $B_i = \bigcup_{j=1}^{3} B_i^j$.

(12) *If $B_j^k = \emptyset$, then $B_l^k = \emptyset$.*

Assume that B_j^k is empty. It implies that B_l^k is strongly complete to $B_j \cup B_k$, contrary of the definition of B_l^l and B_l^k. This proves (12).

Now, we observe that A_2, B_1^1, B_2^2, B_3^3 are homogeneous sets and (A_1, A_3), (B_1^2, B_3^2), (B_2^3, B_1^3), (B_3^1, B_2^1) are homogeneous pairs. If A_1 is strongly anticomplete to A_3, then by (4) and (12), G is a thickening of a member of $\mathcal{C}$. Thus, we may assume that A_1 is not strongly anticomplete to A_3. Since $A_1 - A_3 - B_3^1 - A_2 - B_2^1 - A_1$ is not a hole of length 5, we deduce that either $B_2^1 = \emptyset$ or $B_3^1 = \emptyset$. By (12), it follows that $B_2^1 \cup B_3^1$ is empty. Using (4) and (12), we deduce that G is a thickening of a member of $\mathcal{C}$. This concludes the proof of 3.2.8. □

3.3 Holes of Length 4

Next we examine circular interval trigraphs that contain a hole of length 4.

3.3.1. *Let G be a Berge circular interval trigraph. If G has a hole of length 4 and no essential triangle, then G is a structured circular interval trigraph.*

Proof. In the following proof, the addition is modulo 4. Let G be defined by Σ and $F_1, \ldots, F_k$. Let $x_1^* - x_2^* - x_3^* - x_4^* - x_1^*$ be a hole of length 4. We may assume that $x_i^*, x_{i+1}^* \in F_i,\ i = 1, 2, 3, 4$.

(1) *x_i^* is strongly antiadjacent to x_{i+2}^*.*

Assume not. By symmetry we may assume that x_1^* is adjacent to x_3^*. Moreover, we may assume that there exists $i \in \{1, \ldots, k\}$ such that $\Sigma_{x_1^*, x_3^*}^{x_2^*} \subseteq F_i$. If $i = 4$, it implies that $\{x_1^*, x_2^*, x_3^*, x_4^*\} \subset F_4$, and thus $x_1^* - x_2^* - x_3^* - x_4^* - x_1^*$ is not a hole, a contradiction. Symmetrically, we may assume that $i \neq 3$. But now $\{x_1^*, x_3^*, x_4^*\}$ is an essential triangle since $F_i \cup F_3 \cup F_4 = \Sigma$, a contradiction. This proves (1).

For $i = 1, 2, 3, 4$, let $\mathcal{X}_i, \mathcal{Y}_i \subset \Sigma$ and $X_i, Y_i \subset V(G)$ be such that:

(H1) each of $\mathcal{X}_i, \mathcal{Y}_i$ is homeomorphic to $[0, 1)$,

(H2) $X_i \subseteq V(G) \cap \mathcal{X}_i$, $Y_i \subseteq V(G) \cap \mathcal{Y}_i$, $i = 1, 2, 3, 4$,

(H3) $\bigcup_i (\mathcal{X}_i \cup \mathcal{Y}_i) = \Sigma$,

(H4) $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{X}_4, \mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4$ are pairwise disjoint,

(H5) $\mathcal{Y}_i \subseteq \overline{\Sigma}_{x_i^*, x_{i+1}^*}^{x_{i+2}^*}$, $i = 1, 2, 3, 4$,

(H6) $x_i^* \in X_i$, $i = 1, 2, 3, 4$,

(H7) $X_1, X_2, X_3, X_4, Y_1, Y_2, Y_3, Y_4$ are disjoints strong cliques satisfying (S2)-(S6),

(H8) $\bigcup_i (X_i \cup Y_i)$ is maximal.

By (1), such a structure exists. We may assume that $V(G) \backslash \bigcup_i (X_i \cup Y_i)$ is not empty. Let $x \in V(G) \backslash \bigcup_i (X_i \cup Y_i)$. For $S \subseteq V(G) \backslash \{x\}$, we denote by S^C the subset of S that is complete to x, and by S^A the subset of S that is anticomplete to x.

For $i = 1,2,3,4$, let $x_i^l, x_i^r \in X_i$ be such that $x_{i-1}^*, x_i^l, x_i^r, x_{i+1}^*$ are in this order on Σ and such that $\overline{\Sigma}_{x_i^l, x_i^r}^{x_{i+1}^*}$ is maximal.

(2) $\{x_i^r, x_{i+1}^l\}$ *is complete to* $X_i \cup X_{i+1}$.

By (S5), there exists $a \in X_i$ such that a is adjacent to x_{i+1}^r. By 3.2.3 and (S6), there exists F_l such that $\{a, x_i^r\} \cup X_{i+1} \subseteq F_l$ and thus x_i^r is complete to X_{i+1}. By symmetry, x_{i+1}^l is complete to X_i. This proves (2) by (H7).

(3) *If* X_i *is not complete to* X_{i+1}, *then* x_i^l *is strongly antiadjacent to* x_{i+1}^r.

Let $a \in X_i$ and $b \in X_{i+1}$ be such that a is strongly antiadjacent to b. By 3.2.2 and (S6), a is strongly antiadjacent to x_{i+1}^r. By 3.2.2 and (S6), x_{i+1}^r is strongly antiadjacent to x_i^l. This proves (3).

(4) $x \notin \overline{\Sigma}_{x_i^l, x_i^r}^{x_{i+1}^l}$ *for all* i.

Assume not. We may assume that $x \in \overline{\Sigma}_{x_1^l, x_1^r}^{x_2^l}$. For $i = 1,2,3,4$, let $Y_i' = Y_i$, for $i = 2,3,4$, let $X_i' = X_i$ and let $X_1' = X_1 \cup \{x\}$. Since $Y_2 \cup Y_3 \cup X_3$ is strongly anticomplete to $\{x_1^r, x_1^l\}$ by (S3) and (S6), we deduce by 3.2.2 that x is strongly anticomplete to $Y_2 \cup Y_3 \cup X_3$. Since x_1^r is adjacent to x_4^r by (2), we deduce by 3.2.3 that x is strongly complete to Y_4 and not strongly anticomplete to X_4. By symmetry, x is strongly complete to Y_1 and not strongly anticomplete to X_2. Since x_1^l is strongly adjacent to x_1^r, we deduce that X_1' is a strong clique. If X_1 is strongly complete to X_2, it follows from 3.2.3 that x is strongly complete to X_2. By symmetry, if X_1 is strongly complete to X_4, then x is strongly complete to X_4. The above arguments show that $X_1', \ldots, X_4', Y_1', \ldots, Y_4'$ are disjoint cliques satisfying (S2)-(S6). Moreover, $\mathcal{X}_i, \mathcal{Y}_i$ $i = 1,2,3,4$ clearly satisfy (H1)-(H5) with X_i', Y_i' $i = 1,2,3,4$, contrary to the maximality of $\bigcup_i (X_i \cup Y_i)$. This proves (4).

By (4) and by symmetry, we may assume that $x \in \overline{\Sigma}_{x_1^r, x_2^l}^{x_3^*}$ and therefore $x \in F_1$. By 3.2.2 and (S3), x is strongly anticomplete to Y_3. Since $x \in F_1$, we deduce that x is strongly complete to Y_1.

(5) X_3^C *is strongly anticomplete to* X_4^C.

Assume not. We may assume there exist $x_3 \in X_3^C$ and $x_4 \in X_4^C$ such that x_3 is adjacent to x_4. By (S6), x_3 is strongly antiadjacent to x_1^* and therefore by 3.2.3 there exists F_i, $i \in \{1, \ldots, k\}$, such that $x, x_3 \in F_i$ and $x_1^* \notin F_i$. By symmetry, there exists F_j, $j \in \{1, \ldots, k\}$ such that $x, x_4 \in F_j$

and $x_2^* \notin F_j$. Moreover, as $x_2^* \in F_i$, we deduce that $F_i \neq F_j$. By (S6), x_i^* is strongly anticomplete to x_{i+2} for $i = 1, 2$. Now, since x_3 is adjacent to x_4, we deduce from 3.2.3 that there exists F_l such that $x_3, x_4 \in F_l$ and $l \in \{1, \ldots, k\} \backslash \{i, j\}$. Since $\overline{\Sigma}_{x,x_3}^{x_4} \subseteq F_i$, $\overline{\Sigma}_{x,x_4}^{x_3} \subseteq F_j$ and $\overline{\Sigma}_{x_3,x_4}^{x} \subseteq F_k$, we deduce that $F_i \cup F_j \cup F_k = \Sigma$. Hence, $\{x, x_3, x_4\}$ is an essential triangle, a contradiction. This proves (5).

(6) *At least one of X_3^C, X_4^C is empty.*

Assume not. Let $a \in X_4^C$. By 3.2.3 and since a is strongly anticomplete to X_2, we deduce that there is F_i, $i \in \{1, \ldots, k\}$, such that $\{a, x_4^r, x\} \in F_i$ and thus $x_4^r \in X_4^C$. Symmetrically, $x_3^l \in X_3^C$. By (5), x_4^r is strongly antiadjacent to x_3^l. By (S6), X_1 is strongly complete to X_4, and X_2 is strongly complete to X_3. By (2) and (5), x is anticomplete to $\{x_3^r, x_4^l\}$. But now by (2) and (S6), $x - x_4^l - x_2^l - x_4^r - x_3^l - x_1^r - x_3^r - x$ is an antihole of length 7, a contradiction. This proves (6).

By symmetry, we may assume that x is strongly anticomplete to X_4. By (2) and 3.2.3, x is strongly complete to $X_1 \cup X_2$.

(7) *x is adjacent to x_3^l.*

Assume not. By 3.2.2, x is strongly anticomplete to X_3. Since $x - Y_2 - x_3^r - x_4^r - X_1 - x$ and $x - Y_4 - x_4^l - x_3^l - X_2 - x$ are not holes of length 5, we deduce that x is strongly anticomplete to $Y_2 \cup Y_4$. Since $x - X_2 - X_3 - X_4 - X_1 - x$ is not a cycle of length 5, we deduce that X_1 is strongly complete to X_2. For $i = 1, 2, 3, 4$, let $X_i' = X_i$, for $i = 2, 3, 4$, let $Y_i' = Y_i$, and let $Y_1' = Y_1 \cup \{x\}$. The above arguments show that $X_1', \ldots, X_n', Y_1', \ldots, Y_n'$ are disjoint cliques satisfying (S2)-(S6). Moreover, it is easy to find $\mathcal{X}_i', \mathcal{Y}_i'$, $i = 1, 2, 3, 4$, satisfying (H1)-(H5), contrary to the maximality of $\bigcup_i (X_i \cup Y_i)$. This proves (7).

By 3.2.3 and (7), x is strongly complete to Y_2. For $i = 3, 4$, let $X_i' = X_i$, for $i = 1, 2, 3$, let $Y_i' = Y_i$, let $Y_4' = Y_4^A$, let $X_1' = X_1 \cup Y_4^C$ and let $X_2' = X_2 \cup \{x\}$. The above arguments show that $X_1', \ldots, X_n', Y_1', \ldots, Y_n'$ are disjoint cliques satisfying (S2), (S3) and (S5). To get a contradiction, it remains to show that $X_1', \ldots, X_n', Y_1', \ldots, Y_n'$ satisfy (S4) and (S6).

First we check (S4). Since $X_3' = X_3$, $X_4' = X_4$ and $Y_3' = Y_3$, and since $X_1' \backslash X_1 \subset Y_4$ is strongly complete to X_4, it is enough to check the following:

- If $Y_2 \neq \emptyset$ then X_2' is complete to X_3'.

- If $Y_1 \neq \emptyset$ then X'_1 is complete to X'_2.

For the former, we observe that if x is not strongly complete to X_3, then since $x - Y_2 - X_3 - X_4 - X_1 - x$ is not a hole of length 5, we deduce that Y_2 is empty. For the latter, since x is strongly complete to X_1, it is enought to show that if Y_1 is not empty, then Y_4^C is empty. Since X_3^C is not empty, it follows that $Y_1 \subseteq \overline{\Sigma}_{x,x_1^*}^{x_2^*}$. Now if Y_4^C is not empty, then Y_1 is empty by 3.2.3 and (S4).

To check (S6), we need to prove the following:

(i) If X'_1 is not strongly complete to X'_2 then X'_2 is strongly complete to X'_3.

(ii) If X'_2 is not strongly complete to X'_3 then X'_3 is strongly complete to X'_4.

(iii) If X'_3 is not strongly complete to X'_4 then X'_4 is strongly complete to X'_1.

(iv) If X'_4 is not strongly complete to X'_1 then X'_1 is strongly complete to X'_2.

For (i), first assume that x is not strongly complete to X_3. By 3.2.2, we deduce that x is strongly anticomplete to x_3^r. Since $x - x_2^r - x_3^r - X_4 - Y_4 - x$ and $x - x_2^r - x_3^r - X_4 - X_1 - x$ are not cycles of length 5, we deduce that Y_4^C is empty and that X_1 is strongly complete to X_2. Thus $X'_1 = X_1$ and since x is strongly complete to X_1, it follow that X'_1 is strongly complete to X'_2. So we may assume that x is strongly complete to X_3. By 3.2.3 and (S6), it follows that X_2 is strongly complete to X_3 and thus X'_2 is strongly complete to X'_3. This proves (i).

For (ii), if X'_3 is not strongly complete to X'_4, then by (3) it follows that x_3^l is strongly antiadjacent to x_4^r. Moreover by (S4), X_2 is strongly complete to X_3. Since $x - x_3^l - x_3^r - x_4^r - X_1 - x$ is not a cycle of length 5, we deduce, using (2), that x is strongly complete to X_3 and thus X'_3 is strongly complete to X'_2. This proves (ii).

For (iii) and (iv), we may assume that X'_4 is not strongly complete to X'_1. Since X_4 is strongly complete to Y_4, we deduce that X_4 is not strongly complete to X_1. But by (S6), it implies that X_4 is strongly complete to X_3 and thus X'_4 is strongly complete to X'_3, and (iii) follows. Also by (S6), we deduce that X_1 is strongly complete to X_2. Moreover by (S4), it follows that Y_4 is empty. Since x is strongly complete to X_1, we deduce that X'_1 is strongly complete to X'_2, and (iv) follows.

The above arguments show that $X'_1, \ldots, X'_n, Y'_1, \ldots, Y'_n$ are disjoint cliques satisfying (S2)-(S6). Moreover, it is easy to find $\mathcal{X}'_i, \mathcal{Y}'_i,\ i = 1, 2, 3, 4$, satisfying (H1)-(H5), contrary to the maximality of $\bigcup_i (X_i \cup Y_i)$. This concludes the proof of 3.3.1 □

3.4 Long Holes

In this section, we study circular interval trigraphs that contain a hole of length at least 6.

A result equivalent to 3.4.1 has been proved independently by Kennedy and King [20]. The following was proved in joint work with Varun Jalan.

3.4.1. *Let G be a circular interval trigraph defined by Σ and $F_1, \ldots, F_k \subseteq \Sigma$. Let $P = p_0 - p_1 - \ldots - p_{n-1} - p_0$ and $Q = q_0 - q_1 - \ldots - q_{m-1} - q_0$ be holes. If $n + 1 < m$ then there is a hole of length l for all $n < l < m$. In particular, if G is Berge then all holes of G have the same length.*

Proof. We start by proving the first assertion of 3.4.1. We may assume that the vertices of P and Q are ordered clockwise on Σ. Since P and Q are holes, it follows that $n \geq 4$ and $m > 5$. We are going to prove the following claim which directly implies the first assertion of 3.4.1 by induction.

(1) *There exists a hole of length $m - 1$.*

We may assume that Q and P are chosen such that $|V(Q) \cap V(P)|$ is maximal.

(2) *If there are $i \in \{0, \ldots, m-1\}, j \in \{0, \ldots, n-1\}$ such that*

$$q_i, q_{i+1} \in \overline{\Sigma}_{p_j,p_{j+1}}^{p_{j+2}} \backslash \{p_j, p_{j+1}\}$$

with $q_m = q_0$, $p_n = p_1$ and $p_{n-1} = p_0$, then there is a hole of length $m - 1$ in G.

We may assume that $q_1, q_2 \in \overline{\Sigma}_{p_1,p_2}^{p_3} \backslash \{p_1, p_2\}$. Since q_1 is antiadjacent to q_3, we deduce that $q_3 \notin \overline{\Sigma}_{p_1,p_2}^{p_3}$. Since $p_2 \in \overline{\Sigma}_{q_2,q_3}^{q_1}$, we deduce by 3.2.3 that p_2 is strongly anticomplete to $\{q_0, q_5\}$.

If p_2 is adjacent to q_4, it follows that $Q - q_1 - p_2 - q_4 - Q$ is a hole of length $q - 1$. Thus we may assume that p_2 is strongly antiadjacent to q_4. But then $Q' = Q - q_1 - p_2 - q_3 - Q$ is a hole of length m with $|V(Q') \cap V(P)| > |V(Q) \cap V(P)|$, a contradiction. This proves (2).

By (2) and since $m > n + 1$, we may assume that $|V(P) \cap V(Q)| > 1$. Let $V(P) \cap V(Q) = \{x_0, x_1, \ldots, x_{s-1}\}$. We may assume that $x_0, \ldots, x_{s-1}$ are in clockwise order on Σ. For $i \in \{0, \ldots, s-1\}$, let $A_i = \overline{\Sigma}_{x_i, x_{i+1 \bmod s}}^{x_{i+2 \bmod s}}$. Since $m > n+1$, there exists $k \in \{0, \ldots, s-1\}$ such that $|A_k \cap V(P)| < |A_k \cap V(Q)|$. By (2), it follows that $|A_k \cap V(P)| = |A_k \cap V(Q)| - 1$. Let P' be the subpath of P such that $V(P') = V(P) \cap A_k$. Let Q' be the subpath of Q such that $V(Q') \cap A_k = \{x_i, x_{i+1}\}$. Then $x_1 - P' - x_2 - Q' - x_1$ is a hole of length $m - 1$.

This proves (1) and the first assertion of 3.4.1. Since every hole in a Berge trigraph has even length, the second assertion of 3.4.1 follows immediately from the first. This concludes the proof of 3.4.1. □

3.4.2. *Let G be a Berge circular interval trigraph. If G has a hole of length n with $n \geq 6$, then G is a structured circular interval trigraph.*

Proof. Let G be a Berge circular interval trigraph. Let $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$ be pairwise disjoint cliques satisfying $(S2)-(S6)$ and with $|\bigcup_i (X_i \cup Y_i)|$ maximum. Such sets exist since there is a hole of length n in G. Moreover since G is Berge, it follows that n is even. We may assume that $V(G) \setminus \bigcup_i (X_i \cup Y_i)$ is not empty. Let $x \in V(G) \setminus \bigcup_i (X_i \cup Y_i)$.

For $S \subseteq V(G) \setminus \{x\}$, we denote by S^C the subset of S that is complete to x, and by S^A the subset of S that is anticomplete to x.

(1) *If $y \in X_i^C$ and $z \in X_{i+1}^C$ then y is strongly adjacent to z.*

Assume not. We may assume $y \in X_1^C$ and $z \in X_2^C$ but y is antiadjacent to z. By (S4), $Y_1 = \emptyset$. By (S6), X_2 is strongly complete to X_3, and X_n is strongly complete to X_1. Since $\{x | y, z, \cup_{i=4}^{n-1} X_i \cup_{i=3}^{n-1} Y_i\}$ is not a claw, x is strongly anticomplete to $X_4, \dots, X_{n-1}, Y_3, \dots, Y_{n-1}$. Since $x - z - X_3 - \dots - X_{n-1} - y - x$ is not a hole of length $n+1$, we deduce that x is strongly complete to at least one of X_3 or X_n. Without loss of generality, we may assume that x is strongly complete to X_3. Since $x - X_3 - X_4 - \dots - X_n - x$ is not a hole of length $n-1$, x is strongly anticomplete to X_n. Since $\{X_3 | X_4, Y_2, x\}$ and $\{X_3 | X_2, X_4, x\}$ are not claws, we deduce that x is strongly complete to $Y_2 \cup X_2$.

For $i = 3, \dots, n$, let $X'_i = X_i$, for $i = 1, \dots, n-1$, let $Y'_i = Y_i$. Let $X'_2 = X_2 \cup \{x\}$, $X'_1 = X_1 \cup Y_n^C$ and $Y'_n = Y_n^A$. Then $X'_1, \dots, X'_n, Y'_1, \dots, Y'_n$ are disjoint cliques satisfying $(S2)-(S6)$ but with $|\bigcup_i (X_i \cup Y_i)| < |\bigcup_i (X'_i \cup Y'_i)|$, a contradiction. This proves (1).

(2) *If $X_i^C \neq \emptyset$ and $X_{i+2}^C \neq \emptyset$ then $X_{i+1}^A = \emptyset$.*

Assume not. We may assume $y \in X_n^C$ and $z \in X_2^C$ and $w \in X_1^A$. Since $\{x | y, z, \cup_{i=4}^{n-2} X_i\}$ is not a claw by (S6), x is strongly anticomplete to $X_4, \dots, X_{n-2}$. Assume that $C = x - X_3 - \dots - X_{n-1} - x$ is a hole. Then C has length $n-2$, but w is strongly anticomplete to $V(C) \setminus \{x\}$, contrary to 3.2.6. Thus x is strongly anticomplete to at least one of X_3 or X_{n-1}. By symmetry, we may assume that

x is strongly anticomplete to X_3. Since $x - X_2 - X_3 - \ldots - X_{n-1} - x$ is not a hole length $n-1$, x is strongly anticomplete to X_{n-1}. By (S6) and symmetry, we may assume that X_1 is strongly complete to X_2. But now $\{z|X_3, x, w\}$ is a claw, a contradiction. This proves (2).

(3) *If $X_i^C \neq \emptyset$, then $X_{i+2}^C = \emptyset$.*

Assume not. We may assume there exist $y \in X_n^C$ and $z \in X_2^C$. By (2), x is strongly complete to X_1. Since $\{x|y, z, \cup_{i=4}^{n-2} X_i \cup_{j=3}^{n-2} Y_j\}$ is not a claw by (S6), it follows that x is strongly anticomplete to $X_4, \ldots, X_{n-2}$ and $Y_3, \ldots, Y_{n-2}$.

If $X_3^C \neq \emptyset$, then either $\{x|X_1, X_3, X_{n-1}\}$ is a claw or $x - X_3 - X_4 - \ldots - X_n - x$ is a hole of length $n-1$ and therefore odd, hence x is strongly anticomplete to X_3. By symmetry, x is strongly anticomplete to X_{n-1}. Since $\{z|X_3, x, Y_1\}$ and $\{y|X_{n-1}, x, Y_n\}$ are not claws, x is strongly complete to $Y_1 \cup Y_n$.

For $i = 3, \ldots, n-1$, let $X_i' = X_i$ and for $i = 1, 3, 4, \ldots, n-2, n$, let $Y_i' = Y_i$. Let $X_2' = X_2 \cup Y_2^C$, let $X_1' = X_1 \cup \{x\}$, let $Y_2' = Y_2^A$, let $X_n' = X_n \cup Y_{n-1}^C$ and let $Y_{n-1}' = Y_{n-1}^A$.

Clearly $X_1', \ldots, X_n', Y_1', \ldots, Y_n'$ are disjoint cliques such that $|\bigcup_i (X_i \cup Y_i)| < |\bigcup_i (X_i' \cup Y_i')|$. The above arguments show that $X_1', \ldots, X_n', Y_1', \ldots, Y_n'$ satisfy (S2) and (S5). To get a contradiction, we need to show that $X_1', \ldots, X_n', Y_1', \ldots, Y_n'$ satisfy (S3), (S4) and (S6).

Since $\{x|X_n, Y_1, Y_2^C\}$ is not a claw, we deduce that either $Y_1 = \emptyset$ or $Y_2^C = \emptyset$. In both cases, it implies that Y_1' is strongly complete to X_2'. Symmetrically, Y_n' is strongly complete to X_{n-1}'. Hence, (S3) is satisfied.

It remains to prove the following.

(i) If $Y_1 \neq \emptyset$, then X_1' is strongly complete to X_2'

(ii) If $Y_n \neq \emptyset$, then X_n' is strongly complete to X_1'

(iii) X_2' is strongly complete to at least one of X_3', X_1'.

(iv) X_n' is strongly complete to at least one of X_{n-1}', X_2'.

(v) X_1' is strongly complete to at least one of X_n', X_2'.

Assume that $Y_1 \neq \emptyset$. It implies by (S4), that X_1 is strongly complete to X_2. Since $\{x|Y_n, Y_1, Y_2^C\}$ is not a claw, we deduce that $Y_2^C = \emptyset$. Since $x - Y_1 - X_2^A - X_3 - \ldots - X_n - x$ is not a hole of length $n+1$, we deduce that $X_2^A = \emptyset$ and thus X_1' is strongly complete to X_2'. This proves i) and by symmetry ii) holds.

If $Y_2^C \neq \emptyset$, it follows by (S4) that X_2' is strongly complete to X_3' and iii) holds. Thus we may assume that Y_2^C is empty. If X_2^A is empty, and since by (S6), X_2 is strongly complete to at least one of X_1, X_3, it follows that X_2' is strongly complete to at least one of X_1', X_3'. Thus we may assume that $X_2^A \neq \emptyset$. Since $x - Y_1 - X_2^A - X_3 - \ldots - X_n - x$ is not a hole of length $n+1$, we deduce that $Y_1 = \emptyset$.

Assume that there exist $w \in X_2$ and $v \in X_3$ such that w is antiadjacent to v. Suppose first that $w \in X_2^C$. Since $x - w - X_2^A - v - X_4 - \ldots - X_n - x$ is not a cycle of length $n+1$, we deduce that v is strongly anticomplete to X_2^A. By (S5), there exists $a \in X_2^C$ adjacent to v. But $\{a|x, v, X_2^A\}$ is a claw, a contradiction. Thus we may assume that $w \in X_2^A$ and v is strongly complete to X_2^C. But $\{z|x, v, w\}$ is a claw, a contradiction. Hence X_2 is strongly complete to X_3. This proves iii) and by symmetry iv) holds.

We claim that x is strongly complete to at least one of X_2 or X_n. Suppose that $p \in X_n^A$ and $q \in X_2^A$. By (S5) and (S6), there is $r \in X_1$ that is adjacent to both p and q. But $\{r|p, q, x\}$ is a claw, a contradiction. This proves the claim. By symmetry we may assume that x is strongly complete to X_n. By (1), X_n is strongly complete to X_1. If $Y_{n-1}^C = \emptyset$, it follows that X_1' is strongly complete to X_n' and v) holds. Thus we may assume that $Y_{n-1}^C \neq \emptyset$. Since $\{x|X_1, Y_{n-1}^C, Y_2^C\}$ is not a claw, we deduce that $Y_2^C = \emptyset$. Since $x - Y_{n-1}^C - X_{n-1} - \ldots - X_3 - X_2^A - X_1 - x$ is not a hole of length $n+1$, we deduce that X_2^A is empty. By (1), X_1 is strongly complete to X_2 and thus X_1' is strongly complete to X_2'. This proves v). This concludes the proof of (3).

Let $C = x_1 - x_2 - \ldots - x_n - x_1$ be a hole of length n with $x_i \in X_i$. By 3.2.6, x is strongly adjacent to two consecutive vertices of C. Without loss of generality, we may assume that x is strongly complete to $\{x_1, x_2\}$. By (1), x_1 is strongly adjacent to x_2. By (3), x is strongly anticomplete to $X_3 \cup X_4 \cup X_{n-1} \cup X_n$. Since $G|(\{x\} \bigcup_i X_i)$ does not contain an induced a cycle of length $p \neq n$ by 3.4.1, we deduce that x is strongly anticomplete to X_i for $i = 5, \ldots, n-2$. Similarly, x is strongly anticomplete to $Y_3 \cup \ldots \cup Y_{n-1}$ otherwise there is a hole of length $p \neq n$ in G.

Since $x - Y_2 - X_3 - \ldots - X_n - X_1 - x$ and $x - Y_n - X_n - \ldots - X_2 - x$ are not holes of length $n+1$, we deduce that x is strongly anticomplete to $Y_2 \cup Y_n$.

Since $\{X_2^C|X_1^A, x, X_3\}$ and $\{X_1^C|X_2^A, x, X_n\}$ are not claws, it follows that X_1^A is strongly anticomplete to X_2^C and X_1^C is strongly anticomplete to X_2^A. Suppose there is $a \in X_1^A$. By (S5), there is $b \in X_2^A$ adjacent to a. But $G|(\{x_1, x_2, a, b\}$ is a hole of length 4 strongly anticomplete to X_4, contrary to 3.2.6. Thus $X_1^A = X_2^A = \emptyset$ and by (1), X_1 is strongly complete to X_2. Since $\{X_1|x, Y_1, X_n\}$ is not a claw, we deduce that x is strongly complete to Y_1.

For $i = 1, \ldots, n$, let $X_i' = X_i$, for $i = 2, \ldots, n$, let $Y_i' = Y_i$ and let $Y_1' = Y_1 \cup \{x\}$. The above arguments show that $X_1', \ldots, X_n', Y_1', \ldots, Y_n'$ are cliques satisfying $(S2) - (S6)$ but $|\bigcup_i (X_i \cup Y_i)| < |\bigcup_i (X_i' \cup Y_i')|$, a contradiction. This concludes the proof of 3.4.2. □

We now have all the tools to prove 3.2.1.

Proof of 3.2.1. We may assume that G is not a linear interval trigraph and not a cobipartite trigraph. By 3.2.5, there is an essential triangle or a hole in G. Then by 3.2.8, 3.3.1 and 3.4.2, G is either a structured circular interval trigraph or is a thickening of a trigraph in $\mathcal{C}$. This proves 3.2.1. □

3.5 Some Facts about Linear Interval Join

In this section we prove some lemmas about paths in linear interval stripes.

3.5.1. *Let G be a linear interval join with skeleton H such that G is Berge. Let e be an edge of H that is in a cycle. Let $\eta(e) = V(T)\backslash Z$ where (T, Z) is a thickening of a linear interval stripe $(S, \{x_1, x_n\})$. Then the lengths of all paths from x_1 to x_n in $(S, \{x_1, x_n\})$ have the same parity.*

Proof. Assume not. Let $C = c_0 - c_1 - \ldots - c_n - c_0$ be a cycle in H such that $e = c_0c_n$. For $i = 0, \ldots, n-1$, let $c_ic_{i+1} = e_i$, $(G_{e_i}, \{x_i^1, x_i^2\})$ be such that $\eta(e_i) = V(G_{e_i})\backslash\{x_i^1, x_i^2\}$, $\phi_{e_i}(c_i) = x_i^1$ and $\phi_{e_i}(c_{i+1}) = x_i^2$ as in the definition of a linear interval join. We may assume that $\phi_e(c_n) = x_1$ and $\phi_e(c_0) = x_n$. Let $O = x_1 - o_1 - \ldots - o_{l-1} - x_n$ be an odd path from x_1 to x_n in S and $P = x_1 - p_1 - \ldots - p_{l'-1} - x_n$ be an even path from x_1 to x_n in S. For $i = 0, 1, \ldots, n-1$, let Q_i be a path in G_{e_i} from x_i^1 to x_i^2. Let Q_i' be the subpath of Q_i with $V(Q_i') = V(Q_i)\backslash\{x_i^1, x_i^2\}$.

Let $C_1 = X_{o_1} - \ldots - X_{o_{l-1}} - Q_0' - Q_1' - \ldots - Q_{n-1}' - X_{o_1}$ and $C_2 = X_{p_1} - \ldots - X_{p_{l'-1}} - Q_0' - Q_1' - \ldots - Q_{n-1}' - X_{p_1}$. Then one of C_1, C_2 is an odd hole in G, a contradiction. This proves 3.5.1. □

Before the next lemma, we need some additional definitions. Let $(G, \{x_1, x_n\})$ be a linear interval stripe. The *right path* of G is the path $R = r_0 - \ldots - r_p$ (where $r_0 = x_1$ and $r_p = x_n$) defined inductively starting with $i = 1$ such that $r_i = x_{i^*}$ with $i^* = \max\{j|x_j \text{ is adjacent to } r_{i-1}\}$ (i.e. from r_i take a maximal edge on the right to r_{i+1}). Similarly the *left path* of G is the path $L = l_0 - \ldots - l_p$ (where $l_0 = x_1$ and $l_p = x_n$) defined inductively starting with $i = p-1$ such that $l_i = x_{i^*}$ with $i^* = \min\{j|x_j \text{ is adjacent to } l_{i+1}\}$.

3.5.2. *Let $(G, \{x_1, x_n\})$ be a linear interval stripe and R be the right path of G. If $x, y \in V(R)$, then $x - R - y$ is a shortest path between x and y.*

Proof. Let $P = x - p_1 - \ldots - p_{t-1} - y$ be a path between x and y of length t and let $x - r_l - \ldots - r_{s+l-2} - y = x - R - y$. By the definition of R and since G is a linear interval stripe, we deduce that $r_{l+i-1} \geq p_i$ for $i = 1, \ldots, s-1$. Hence it follows that $s \leq t$. This proves 3.5.2. □

3.5.3. *Every linear interval trigraph is Berge.*

Proof. Let G be a linear interval trigraph with $V(G) = \{v_1, \ldots, v_n\}$. The proof is by induction on the number of vertices. Clearly $H = G|\{v_1, \ldots, v_{n-1}\}$ is a linear interval trigraph, so inductively H is Berge. Since G is a linear interval trigraph, it follows that $N(v_n)$ is a strong clique. But if A is an odd hole or an odd antihole in G, then for every $a \in V(A)$, it follows that $N(a) \cap V(A)$ is not a strong clique. Therefore $v_n \notin V(A)$ and consequently G is Berge. This proves 3.5.3. □

3.5.4. *Let $(G, \{x_1, x_n\})$ be a linear interval stripe. Let S and Q be two paths from x_1 to x_n of length s and q such that $s < q$. Then there exists a path of length m from x_1 to x_n in G for all $s < m < q$.*

Proof. Let G' be a circular interval trigraph obtained from G by adding a new vertex x as follows:

- $V(G') = V(G) \cup \{x\}$,
- $G'|V(G) = G$,
- x is strongly anticomplete to $V(G) \backslash \{x_1, x_n\}$,
- x is strongly complete to $\{x_1, x_n\}$.

Let $s < m < q$, $C_1 = x_1 - S - x_n - x - x_1$ and $C_2 = x_1 - Q - x_n - x - x_1$. Clearly, C_1 and C_2 are holes of length $s+2$ and $q+2$ in G'. By 3.4.1, there exists a hole C_m of length $m+2$ in G'. Since it is easily seen from the definition of linear interval trigraph that there is no hole in G, we deduce that $x \in V(C_m)$. Let $C_m = x - c_1 - c_2 - \ldots - c_{m+1} - x$. Since $N(x) = \{x_1, x_n\}$, we may assume that $c_1 = x_1$ and $c_{m+1} = x_n$. But now $x_1 - c_2 - \ldots - c_m - x_n$ is a path of length m from x_1 to x_n in G. This proves 3.5.4. □

We say that a linear interval stripe $(G, \{x_1, x_n\})$ has *length* p if all paths from x_1 to x_n have length p.

3.5.5. *Let $(G, \{x_1, x_n\})$ be a linear interval stripe of length p. Let $L = l_0 - \ldots - l_p$ and $R = r_0 - \ldots - r_p$ be the left and right paths. Then $r_0 < l_1 \leq r_1 < l_2 \leq r_2 < \ldots < l_{p-1} \leq r_{p-1} < l_p$.*

Proof. Since G is a linear interval trigraph and by the definition of right path, it follows that $r_0 < r_1 < r_2 < \ldots < r_p$.

We claim that if $l_i \in (r_{i-1}, r_i]$, then $l_{i-1} \in (r_{i-2}, r_{i-1}]$. Assume that $l_i \in (r_{i-1}, r_i]$. Since r_{i-1} is adjacent to r_i, we deduce that l_i is adjacent to r_{i-1}. By the definition of the left path, $l_{i-1} \leq r_{i-1}$. Since $r_{i-1} < l_i$ and by the definition of the right path, we deduce that r_{i-2} is strongly antiadjacent to l_i. Since G is a linear interval trigraph, we deduce that $l_{i-1} > r_{i-2}$. This proves the claim.

Now, since $l_p \in (r_{p-1}, r_p]$ and using the claim inductively, we deduce that $r_{i-1} < l_i \leq r_i$ for $i = 1, \ldots, p$. This proves 3.5.5. □

3.5.6. *Let $(G, \{x_1, x_n\})$ be a linear interval stripe of length p. Let $L = l_0 - \ldots - l_p$ and $R = r_0 - \ldots - r_p$ be the left and right paths. Then $[r_0, l_i)$ is strongly anticomplete to $[l_{i+1}, l_p]$ and $[r_0, r_i]$ is strongly anticomplete to $(r_{i+1}, l_p]$ for $i = 0, \ldots, p$.*

Proof. Assume not. By symmetry, we may assume that there exist i, $a \in [r_0, l_i)$ and $b \in [l_{i+1}, l_p]$ such that a is adjacent to b. Since $l_{i+1} \in (a, b]$ and since G is a linear interval trigraph, we deduce that l_{i+1} is adjacent to a. But $a < l_i$, contrary to the definition of the left path. This proves 3.5.6. □

3.5.7. *Let $(G, \{x_1, x_n\})$ be a linear interval stripe of length $p \geq 3$. Let $L = l_0 - \ldots - l_p$ and $R = r_0 - \ldots - r_p$ be the left and right paths. If l_i and r_{i+1} are strongly adjacent for some $0 < i < p$, then G admits a 1-join.*

Proof. Let i be such that l_i and r_{i+1} are strongly adjacent. Since G is a linear interval trigraph, we deduce that $[l_i, r_{i+1}]$ is a strong clique. By 3.5.6, it follows that $[r_0, l_i)$ is strongly anticomplete to $(r_{i+1}, r_p]$.

Suppose there exists $x \in [l_i, r_{i+1}]$ that is adjacent to a vertex $a \in [r_0, l_i)$ and $b \in (r_{i+1}, r_p]$. By 3.5.6, it follows that a is strongly anticomplete to $[l_{i+1}, l_p]$ and thus $x \in [l_i, l_{i+1})$. Symmetrically, $x \in (r_i, r_{i+1}]$. Hence by 3.5.5, we deduce that $x \in (r_i, l_{i+1})$. By the definition of the right path and since a is adjacent to x, we deduce that $a \notin [r_0, r_{i-1}]$. Hence $a \in (r_{i-1}, l_i)$. By symmetry, $b \in (r_{i+1}, l_{i+2})$.

We claim that $P = r_0 - R - r_{i-1} - a - x - b - l_{i+2} - L - l_p$ is a path. Since $r_{i-1} < a$ and by the definition of the right path, we deduce that r_{i-2} is strongly antiadjacent to a. Since $b < l_{i+2}$ and by the definition of the left path, we deduce that b is strongly antiadjacent to l_{i+3}. By 3.5.6 and since $a \in (r_{i-1}, l_i)$ and $b \in (r_{i+1}, l_{i+2})$, it follows that a and b are strongly antiadjacent. Moreover since $x \in (r_i, l_{i+1})$ and by the definition of the left and right path, we deduce that x is strongly anticomplete to $\{r_{i-1}, l_{i+2}\}$. This proves the claim.

But P is an path of length $p+1$, a contradiction. Hence for all $x \in [l_i, r_{i+1}]$, x is strongly anticomplete to at least one of $[r_0, l_i)$, $(r_{i+1}, r_p]$.

Let $V_1 = \{x \in [l_i, r_{i+1}] : x \text{ is strongly anticomplete to } (r_{i+1}, r_p]\}$ and $V_2 = [l_i, r_{i+1}] \backslash V_1$. The above arguments shows that $([r_0, l_i) \cup V_1, (r_{i+1}, r_p] \cup V_2)$ is a 1-join. This proves 3.5.7. □

3.5.8. *Let $(G, \{x_1, x_n\})$ be a linear interval stripe of length p with $p > 3$, then G admits a 1-join.*

Proof. Assume not. Let $L = l_0 - \ldots - l_p$ and $R = r_0 - \ldots - r_p$ be the left and right paths. If $r_2 = l_2$, it follows that r_2 is strongly adjacent to at least one of l_1, r_3, contrary to 3.5.7. Thus by 3.5.5, we may assume that $l_2 < r_2$.

By 3.5.7, we may assume that l_1 is antiadjacent to r_2 and l_2 is antiadjacent to r_3. By 3.5.5, it follows that $l_2 \in (r_1, r_2)$. Since G is a linear interval trigraph, we deduce that l_2 is adjacent to r_2. Hence $l_0 - l_1 - l_2 - r_2 - R - r_p$ is a path of length $p+1$, a contradiction. This proves 3.5.8. □

3.5.9. *Let $(G, \{x_1, x_n\})$ be a linear interval stripe of length three, and (H, Z) a thickening of $(G, \{x_1, x_n\})$. Then either H admits a 1-join or (H, Z) is the thickening of a spring.*

Proof. Let $L = l_0 - l_1 - l_2 - l_3$ and $R = r_0 - r_1 - r_2 - r_3$ be the left and right paths of G. If l_1 is strongly adjacent to r_2 then by 3.5.7, G admits a 1-join, and so does H.

Thus, we may assume that l_1 is not strongly adjacent to r_2. Suppose that there exists $a \in (r_1, l_2)$. Since $a > r_1$, we deduce that a is strongly antiadjacent to r_0. Symmetrically, a is strongly antiadjacent to l_3. By 3.5.5, it follows that $a \in (l_1, l_2)$. Since G is a linear interval trigraph, we deduce that a is adjacent to l_1. Symmetrically, a is adjacent to r_2. Hence $r_0 - l_1 - a - r_2 - l_3$ is a path of length 4, contrary to the fact that $(G, \{x_1, x_n\})$ has length 3. Thus $(r_1, l_2) = \emptyset$.

Since r_0 is strongly adjacent to r_1 and as G is a linear interval trigraph, we deduce that $(r_0, r_1]$ is a strong clique, and moreover, that r_0 is strongly complete to $(r_0, r_1]$. By 3.5.6, it follows that r_0 is strongly anticomplete to $[l_2, l_3]$. By symmetry and since $V(G) = \{r_0, l_3\} \cup (r_0, r_1] \cup [l_2, l_3)$, the above arguments show that $((r_0, r_1], [l_2, l_3))$ is a homogeneous pair. Moreover by 3.5.5, $l_1 \in (r_0, r_1]$ and $r_2 \in [l_2, l_3)$. Since l_1 is antiadjacent to r_2, we deduce that $(r_0, r_1]$ is not strongly complete to $[l_2, l_3)$. Since $r_2 \in [l_2, l_3)$ and by the definition of the right path, we deduce that $(r_0, r_1]$ is not strongly anticomplete to $[l_2, l_3)$.

Now setting $X_w = \{l_0\}$, $X_x = (r_0, r_1]$, $X_y = [l_2, l_3)$ and $X_z = \{r_3\}$, we observe that $(G, \{x_1, x_n\})$ is the thickening of a spring, and therefore (H, Z) is the thickening of a spring. This proves 3.5.9. □

3.6 Proof of the Main Theorem

In this section we collect the results we have proved so far, and finish the proof of the main theorem.

3.6.1. *Let $(G, \{x\})$ be a connected cobipartite bubble. Then $(G, \{x\})$ is a thickening of a truncated spot, a thickening of a truncated spring or a thickening of a one-ended spot.*

Proof. Let X and Y be two disjoint strong cliques such that $X \cup Y = V(G)$. We may assume that $\{x\} \subseteq X$. If $\{x\} \cup N(x) = V(G)$, it follows that $N(x)$ is a homogeneous set. Hence $(G, \{x\})$ is the thickening of a truncated spot.

Thus we may assume that $\{x\} \cup N(x) \neq V(G)$. Let $Y_1 = Y \cap N(x)$ and $Y_2 = Y \backslash Y_1$. Then x is strongly complete to Y_1 and strongly anticomplete to Y_2. Observe that $(N(x), Y_2)$ is a homogeneous pair. Since G is connected, we deduce that $|N(x)| \geq 1$ and that $N(x)$ is not strongly anticomplete to Y_2. If $N(x)$ is strongly complete to Y_2, we observe that $(G, \{x\})$ is a thickening of a one-ended spot. And otherwise, we observe that $(G, \{x\})$ is a thickening of a truncated spring. This concludes the proof of 3.6.1. □

3.6.2. *Let $(G, \{z\})$ be a stripe such that G is a thickening of a trigraph in $\mathcal{C}$. Then $(G, \{z\})$ is in $\mathcal{C}'$.*

Proof. Let H be a trigraph in $\mathcal{C}$ such that G is a thickening of H. For $i, j = 1, 2, 3$, let $B_i^j \subseteq V(H)$ and $a_i \in V(H)$ be as in the definition of $\mathcal{C}$. For $i = 1, 2, 3$, let $X_{a_i} \subset V(G)$ be as in the definition of a thickening. For $b \in V(G) \backslash (X_{a_1} \cup X_{a_2} \cup X_{a_3})$ and since there exists i such that $X_{a_i} \cup X_{a_{i+1}} \subseteq N(b)$, and X_{a_i} is not strongly complete to $X_{a_{i+1}}$, we deduce that $b \notin \{z\}$. Thus there exists $k \in \{1, 2, 3\}$ such that $z \in X_{a_k}$. Since $\bigcup_{i=1}^{3}(B_k^1 \cup B_{k+1}^i) \subseteq N(z)$ and since there exists no $c \in X_{a_{k+1}} \cup X_{a_{k+2}}$ with c strongly complete to $\bigcup_{i=1}^{3}(B_k^1 \cup B_{k+1}^i)$, we deduce that $N(z) \cap (X_{a_{k+1}} \cup X_{a_{k+2}}) = \emptyset$. Since B_{k+1}^{k+2} is anticomplete to B_k^{k+2} and $B_{k+1}^{k+2} \cup B_k^{k+2} \subseteq N(z)$, we deduce from the definition of $\mathcal{C}$ that $B_{k+1}^{k+2} \cup B_k^{k+2} = \emptyset$. Hence we deduce that $(G, \{z\})$ is in $\mathcal{C}'$. This proves 3.6.2. □

3.6.3. *Let G be a trigraph and let H be a thickening of G. For $v \in V(G)$, let X_v be as in the definition of thickening of a trigraph. Let $C = c_1 - c_2 - \ldots - c_n - c_1$ be an odd hole or an odd antihole of H. Then $|V(C) \cap X_v| \leq 1$ for all $v \in V(G)$.*

Proof. Assume not. We may assume that $|V(C) \cap X_x| \geq 2$ with $x \in V(G)$.

Assume first that C is a hole. By symmetry, we may assume that $c_1, c_2 \in X_x$. Since c_3 is antiadjacent to c_1 and adjacent to c_2, we deduce that there exists $y \in V(G)$ such that x is semiadjacent to y and $c_3 \in X_y$. By symmetry, and since x is semiadjacent to at most one vertex in G, we deduce that $c_n \in X_y$, a contradiction since X_y is a strong clique.

Assume now that C is an antihole. By symmetry, we may assume that there exists $k \in \{3, \ldots, n-1\}$ such that $c_1, c_k \in X_x$. Moreover we may assume by symmetry that k is even.

(1) *For $i \in \{1, \ldots, k/2\}$, if i is odd then $c_i, c_{k-i+1} \in X_x$, and there exists $y \in V(G)$ such that if i is even then $c_i, c_{k-i+1} \in X_y$.*

By induction on i. By assumption, $c_1, c_k \in X_x$. Since c_2 is adjacent to c_k and antiadjacent to c_1, we deduce that there exists $y \in V(G)$ such that x is semiadjacent to y in G and $c_2 \in X_y$. By symmetry, and since x is semiadjacent to at most one vertex in G, we deduce that $c_{k-1} \in X_y$.

Now let $i \in \{3, \ldots, k/2\}$ and assume first that i is odd. By induction, we may assume that $c_{i-1}, c_{k-i+2} \in X_y$. Since c_i is adjacent to c_{k-i+2} and antiadjacent to c_{i-1}, we deduce that $c_i \in X_x$ since y is semiadjacent only to x in G. By symmetry, we deduce that $c_{k-i+1} \in X_x$. Now if i is even, the same argument holds by symmetry. This proves (1).

By (1), there exists $z \in \{x, y\}$ such that $c_{k/2}, c_{k/2+1} \in X_z$, a contradiction. This concludes the proof of 3.6.3. $\square$

3.6.4. *Let G be a trigraph and let H be a thickening of G. Then G is Berge if and only if H is Berge.*

Proof. If $C = c_1 - c_2 - \ldots - c_n - c_1$ is an odd hole (resp. antihole) in G then $C' = X_{c_1} - X_{c_2} - \ldots - X_{c_n} - X_{c_1}$ is an odd hole (resp. antihole) in H.

Now assume that $C = c_1 - c_2 - \ldots - c_n - c_1$ is an odd hole or an odd antihole in H. By 3.6.3, there is $x_i \in V(G)$ such that $c_i \in X_{x_i}$ for $i = 1, \ldots, n$ and such that $x_i \neq x_j$ for all $i \neq j$. But $x_1 - x_2 - \ldots - x_n - x_1$ is an odd hole or an odd antihole in G. This proves 3.6.4. $\square$

3.6.5. *Let G be a structured circular interval trigraph. Then G is Berge.*

Proof. Assume not. For $i = 1, \ldots, n$, let X_i and Y_i be as in the definition of structured circular interval trigraph. Let $C = c_1 - \ldots - c_n - c_1$ be an odd hole or an odd antihole in G. Since $N(y)$ is a strong clique for all $y \in \bigcup_{i=1}^n Y_i$, we deduce that $V(C) \cap \bigcup_{i=1}^n Y_i = \emptyset$. But by 3.6.3 and (S1)-(S6), we get a contradiction. This proves 3.6.5. $\square$

3.6.6. *Let G be a structured circular interval trigraph. Then G is a thickening of an evenly structured linear interval join.*

Proof. Let $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ and n be as in the definition of structured circular interval trigraph. Throughout this proof, the addition is modulo n.

Let $H = (V, E)$ be a graph and s be a signing such that:

- $V \subseteq \{h_1, h_2, \ldots, h_n\} \cup \{l_1^1, \ldots, l_1^{|Y_1|}\} \cup \ldots \cup \{l_n^1, \ldots, l_n^{|Y_n|}\}$,
- if X_i is not strongly complete to X_{i+1}, then $h_{i+1} \notin V$, and there is exactly one edge e_i between h_i and h_{i+2}, and $s(e_i) = 0$,
- if X_i is strongly complete to $X_{i-1} \cup X_{i+1}$, then there are $|X_i|$ edges $e_i^1, \ldots, e_i^{|X_i|}$ between h_i and h_{i+1}, and $s(e_i^k) = 1$ for $k = 1, \ldots, |X_i|$,
- if $h_i \in V$, there is one edge between h_i and l_{i-1}^k with $s(h_i l_{i-1}^k) = 1$ for $k = 1, \ldots, |Y_{i-1}|$.

Then G is an evenly structured linear interval join with skeleton H and such that each stripe associated with an edge e with $s(e) = 1$ is a spot. This proves 3.6.6. □

We can now prove the following.

3.6.7. *Let G be a linear interval join. Then G is Berge if and only if G is an evenly structured linear interval join.*

Proof.

⇐ Let G be an evenly structured linear interval join. We have to show that G is Berge. By 3.5.3, linear interval stripes are Berge. By 3.2.7 and 3.6.4, trigraphs in $\mathcal{C}'$ are Berge. By 3.6.5, structured bubbles are Berge. Clearly spots, truncated spots, one-ended spots and truncated springs are Berge. By 3.6.4 and due to the construction of evenly structured linear interval join, the only holes created are of even length due to the signing. Thus G is Berge.

⇒ Let G be a Berge linear interval join. Let H be a skeleton of G. We may assume that H is chosen among all skeletons of G such that $|V(H)|$ is maximum and subject to that with $|E(H)|$ maximum. Let (G_e, Y_e), $e = x_1x_2$ (with $x_1 = x_2$ if e is a loop) and $\phi_e : V(e) \to Y_e$ be associated with H as in the definition of linear interval join.

(1) *If (G_e, Y_e) is a thickening of a linear interval stripe such that e is in a cycle in H but e is not a loop, then G_e does not admit a 1-join.*

Assume not. Let $Y_e = \{y, z\}$ and $e = x_1x_2$. We may assume that $\phi_e(x_1) = y$ and $\phi_e(x_2) = z$. Let H' be the graph obtained from H by adding a new vertex a' as follows: $V(H') = V(H) \cup \{a'\}$, $H'|V(H) = H \backslash e$ and a' is adjacent to x_1 and x_2, and to no other vertex.

Let (F_e, Z_e) be a linear interval stripe such that (G_e, Y_e) is a thickening of (F_e, Z_e) and such that F_e admits a 1-join. Let $V_1, V_2, A_1, A_2 \subset V(F_e)$ be as in the definition of 1-join. Moreover let W_1, W_2 be the natural partition of $V(G_e)$ such that $G_e|W_k$ is a thickening of $F_e|W_k$ for $k = 1, 2$ and (W_1, W_2) is a 1-join. We may assume that $V(F_e) = \{v_1, \ldots, v_n\}$, $V_1 = \{v_1, \ldots, v_k\}$ and $V_2 = \{v_{k+1}, \ldots, v_n\}$. Let F_e^1 be such that $V(F_e^1) = \{v_1, \ldots, v_k, v'_{k+1}\}$, $F_e^1|V_1 = F_e$ and v'_{k+1} is complete to A_1 and anticomplete to $V_1 \backslash A_1$. Let (G_e^1, Y_e^1) be the thickening of $(F_e^1, \{v_1, v'_{k+1}\})$ such that $G_e^1 \backslash Y_e^1 = G_e|(W_1 \backslash Y_e)$. Let F_i^2 be such that $V(F_e^2) =$

$\{v'_k, v_{k+1}, \ldots, v_n\}$, $F_e^2|V_2 = F_e$ and v'_k is complete to A_2 and anticomplete to $V_2 \backslash A_2$. Let (G_e^2, Y_e^2) be the thickening of $(F_e^2, \{v'_k, v_n\})$ such that $G_e^2 \backslash Y_e^2 = G_e|(W_2 \backslash Y_e)$.

Now G is a linear interval join with skeleton H' using the same stripes as the construction with skeleton H except for stripe (G_e^1, Y_e^1) and (G_e^2, Y_e^2) associated with the edges $a'x_1$ and $a'x_2$, contrary to the maximality of $|V(H)|$. This proves (1).

Let s be a signing of G such that $s(e) = 1$ if (G_e, Y_e) is a spot, and $s(e) = 0$ if (G_e, Y_e) is not a spot.

It remains to prove that:

(P1) *if e is not a loop and is in a cycle and $s(e) = 0$, then (G_e, Y_e) is a thickening of a spring, and*

(P2) *(H, s) is an even structure,*

(P3) *if e is a loop, then (G_e, Y_e) is a trigraph in $\mathcal{C}'$.*

First we prove (P1). Let $e = x_1x_2$ be in a cycle and such that $s(e) = 0$ and e is not a loop. Let (G_e, Y_e) be a thickening of a linear interval stripe such that e has been replaced by (G_e, Y_e) in the construction. Let $Y_e = \{y, z\}$. We may assume that $\phi_e(x_1) = y$ and $\phi_e(x_2) = z$. By 3.5.1 and 3.5.4, if $e \in H$ is in a cycle, then all paths from y to z have the same length. By (1), (G_e, Y_e) does not admit a 1-join, and thus by 3.5.8 and 3.5.9, (G_e, Y_e) is the thickening of a spring. This proves (P1).

Before proving (P2). We need the following claims.

(2) *Let $C = c_1 - c_2 - c_3 - c_1$ be a cycle in H with edge set $E(C) = \{e_1, e_2, e_3\}$. If $s(e_1) = s(e_2) = 0$ and $s(e_3) = 1$, then there is an odd hole in G.*

By (P1), (G_{e_1}, Y_{e_1}) and (G_{e_2}, Y_{e_2}) are springs. It follows that the springs (G_{e_1}, Y_{e_1}) and (G_{e_2}, Y_{e_2}) together with the spot (G_{e_3}, Y_{e_3}) induce a hole of length 5 in G, a contradiction. This proves (2).

(3) *Let $C = c_1 - c_2 - \ldots - c_n - c_1$ be a cycle in H such that $n > 3$ and such that $\sum_{e \in E(C)} s(e)$ is odd; then there is an odd hole in G.*

The proof of (3) is similar to the proof of (2) and is omitted.

(4) *Let* $\{z_1, z_2, z_3\}$ *be a triangle in* H*. For* $i = 1, 2, 3$*, let* e_i *be an edge between* z_i *and* $z_{i+1 \mod 3}$ *such that* $s(e_i) = 1$*. If* $y \in V(H)\backslash\{z_1, z_2, z_3\}$ *is adjacent to at least two vertices in* $\{z_1, z_2, z_3\}$*, then* $s(f) = 1$ *for every edge* f *with one end* y *and the other end in* $\{z_1, z_2, z_3\}$*.*

Assume that there is an edge e_4 with one end y and the other end in $\{z_1, z_2, z_3\}$ with $s(e_4) = 0$. By symmetry, we may assume that z_1 is an end of e_4. By symmetry, we may also assume that there is an edge e_5 between y and z_2. If $s(e_5) = 0$, we deduce by (2) using $y - z_1 - z_2 - y$ that there is an odd hole in G, a contradiction. But if $s(e_5) = 1$, we deduce by (2) using $y - z_1 - z_3 - z_2 - y$ that there is an odd hole in G, a contradiction. This proves (4).

(5) *Let* A *be a block of* H*. Assume that there is a cycle* $C = c_1 - c_2 - c_3 - c_1$ *in* H *such that* $s(e) = 1$ *for all* $e \in E(C)$*. Then all connected components of* $A\backslash V(C)$ *have size* 1*.*

Let B be a connected components of $A\backslash V(C)$ such that $|B| > 1$. Since $B \cup \{c_1, c_2, c_3\}$ is 2-connected, there are at least 2 vertices in B that are not anticomplete to $\{c_1, c_2, c_3\}$. Similarly, there are at least 2 vertices in $\{c_1, c_2, c_3\}$ that are not anticomplete to B. Hence, we can find $b_i, b_j \in B$ such that b_i is adjacent to c_i and b_j is adjacent to c_j with $i \neq j$. By symmetry, we may assume that $i = 1$ and $j = 2$. Since B is connected, we deduce that there is a path P from b_1 to b_2 in B. But $C_1 = c_3 - c_1 - b_1 - P - b_2 - c_2 - c_3$ and $C_2 = c_1 - b_1 - P - b_2 - c_2 - c_1$ are cycles of length greater than 3 and one of them has an odd value, thus by (3) there is an odd hole in G, a contradiction. This proves (5).

Now we prove (P2). We need to prove that every block of H is either a member of $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$ or an evenly signed graph. Let A be such a block and assume that $(A, s|_A)$ is not an evenly signed graph. It follows that there exists a cycle $C = c_1 - c_2 - \ldots - c_n - c_1$ in A of odd value. By (3) and (2), we deduce that C has length 3 and $s(e) = 1$ for all edges $e \in E(C)$.

By (2), if $|V(A)| = 3$ we deduce that A is a member of $\mathcal{F}_1$. Hence we may assume that there is $c_4 \in A$. By (5) and by symmetry, we deduce that c_4 is adjacent to both c_1 and c_2. By (4), we deduce that $s(e) = 1$ for all edges e between $\{c_1, c_2, c_3\}$ and c_4.

Assume first that c_4 is adjacent to c_3. Assume that $|V(A)| > 4$. Since A is connected, there is $y \in A\backslash\{c_1, c_2, c_3, c_4\}$ such that y is not anticomplete to $\{c_1, c_2, c_3, c_4\}$. Let $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Since there is a cycle $C_{ijk} = c_i - c_j - c_k - c_i$ of length 3 with $s(e) = 1$ for all edges $e \in E(C_{ijk})$, we deduce by (5) that y is not adjacent to c_l. Hence y is anticomplete to $\{c_1, c_2, c_3, c_4\}$, a contradiction. It follows that $|V(A)| = 4$. Assume now that there is an edge e in A with $s(e) = 0$. By symmetry, we may assume that e is between c_1 and c_2. Now $c_1 - c_2 - c_3 - c_4 - c_1$, is a cycle of length 4 of odd value. By (3), it follows that G has an odd hole, a contradiction. Hence $s(e) = 1$ for all edges e in A and we deduce that A is a member of $\mathcal{F}_2$.

Assume now that c_4 is not adjacent to c_3. By (5), we deduce that $E(A \setminus \{c_1, c_2, c_3\}) = \emptyset$. Similarly by (5), it follows that $E(A \setminus \{c_1, c_2, c_4\}) = \emptyset$. Since A is 2-connected, it follows that $\{c_1, c_2\}$ is complete to $V(A) \setminus \{c_1, c_2\}$. By (4), we deduce that $s(f) = 1$ for all edges f between $\{c_1, c_2\}$ and $V(A) \setminus \{c_1, c_2\}$. Hence A is a member of $\mathcal{F}_3$. This proves (P2).

Finally we prove (P3). Let e be a loop. Let (G_e, Y_e) be a thickening of a bubble such that e has been replaced by (G_e, Y_e) in the construction. Let $Y_e = \{y\}$. Let (F, W) be a bubble such that (G_e, Y_e) is a thickening of (F, W). By 3.2.1, F is a linear interval trigraph, a cobipartite trigraph, a structured circular interval trigraph or a thickening of a trigraph in $\mathcal{C}$.

Assume first that F is a linear interval trigraph. Let $\{v_1, \ldots, v_n\}$ be the set of vertices of F. Let $k \in \{1, \ldots, n\}$ be such that $\{v_k\} = W$. For $v_i \in V(F)$, let $X_{v_i} \subset V(G_i)$ be as in the definition of a thickening. Let $l < r$ be such that $N(v_k) = \{v_l, \ldots, v_r\}$. Assume that $1 < l$ and $r < n$. Let H' be the graph obtained from H by adding two new vertices a_1, a_2 as follows: $V(H') = V(H) \cup \{a_1, a_2\}$, $H'|V(H) = H \setminus e$, a_1 and a_2 are adjacent to $\phi_e^{-1}(y)$ and to no other vertex. Let F_l be such that $V(F_l) = \{v_0, v_1, \ldots, v_k\}$, $F_l \setminus v_0 = F|\{v_1, \ldots, v_k\}$ and v_0 is adjacent to v_1 and to no other vertex. Let F_r be such that $V(F_r) = \{v_k, \ldots, v_n, v_{n+1}\}$, $F_r \setminus v_{n+1} = F|\{v_k, \ldots, v_n\}$ and v_{n+1} is adjacent to v_n and to no other vertex. Let (G_e^l, Y_e^l) be the thickening of $(F_l, \{v_0, v_k\})$ such that $G_e^l \setminus Y_e^l = G_e | \bigcup_{j=1}^{k-1} X_{v_j}$. Let (G_e^r, Y_e^r) be the thickening of $(F_r, \{v_k, v_{n+1}\})$ such that $G_e^r \setminus Y_e^r = G_e | \bigcup_{j=k+1}^{n} X_{v_j}$. Now G is a linear interval join with skeleton H' using the same stripes as the construction with skeleton H except for (G_e^l, Y_e^l) and (G_e^r, Y_e^r) instead of (G_e, Y_e), contrary to the maximality of $|V(H)|$. Hence by symmetry, we may assume that $l = 1$. Now let H' be the graph obtained from H by adding a new vertex a' as follows: $V(H') = V(H) \cup \{a'\}$, $H'|V(H) = H \setminus e$ and a' is adjacent to $\phi_e^{-1}(y)$ and to no other vertex. Let F' be such that $V(F') = \{v_1, \ldots, v_n, v_{n+1}\}$, $F'|V(F) = F$ and v_{n+1} is adjacent to v_n and to no other vertex. Let (G'_e, Y'_e) be the thickening of $(F', \{v_1, v_{n+1}\})$

such that $G'_e \backslash Y'_e = G_e \backslash Y_e$. Now G is a linear interval join with skeleton H' using the same stripes as the construction with skeleton H except for (G'_e, Y'_e) instead of (G_e, Y_e), contrary to the maximality of $|V(H)|$. Hence F is not a linear interval trigraph.

Assume now that F is a structured circular interval trigraph. Using the same construction as in the proof of 3.6.6, it is easy to see that there exist H' with $|V(H')| > |V(H)|$ and a set of stripes $\mathcal{S}$, such that G is a linear interval join with skeleton H' using the stripes of $\mathcal{S}$, contrary to the maximality of $|V(H)|$. Hence F is not a structured circular interval trigraph.

Assume now that F is a cobipartite trigraph. Clearly any thickening of a cobipartite trigraph is a cobipartite trigraph. By 3.6.1, (G_e, Y_e) is a thickening of a truncated spot, a thickening of a truncated spring or a thickening of a one-ended spot.

Assume that (G_e, Y_e) is a thickening of a one-ended spot. Let $X_v \subset V(G_e)$ be as in the definition of a thickening. Let H' be the graph obtained from H by adding a new vertex a' as follows: $V(H') = V(H) \cup \{a'\}$, $H'|V(H) = H \backslash e$, there is $|X_v|$ edges between a' and $\phi_e^{-1}(y)$, there is a loop l on a' and a' is adjacent to no other vertex than $\phi_e^{-1}(y)$. Let the stripes associated with the edges between a' and $\phi_e^{-1}(y)$ be spots and let the stripe associated with the loop on a' be a thickening of a truncated spot. Now G is a linear interval join with skeleton H' using the same stripes as the construction with skeleton H except for additional edges, contrary to the maximality of $|V(H)|$. Hence (G_i, Y_i) is not a thickening of a one-ended spot.

Assume now that (G_e, Y_e) is a thickening of a truncated spot. Let H' be the graph obtained from H by adding $|V(G_e)| - 1$ new vertices $a_1, \dots, a_{|V(G_e)|-1}$ as follows: $V(H') = V(H) \cup \{a_1, \dots, a_{|V(G_e)|-1}\}$, $H'|V(H) = H \backslash e$, and for $j \in \{1, \dots, |V(G_e)| - 1\}$, a_j is adjacent to $\phi_e^{-1}(y)$ and to no other vertex. Now G is a linear interval join with skeleton H' using the same stripes as the construction with skeleton H and such that the stripes associated with the added edges are spots, contrary to the maximality of $|V(H)|$. Hence (G_e, Y_e) is not a thickening of a truncated spot.

Assume that (G_e, Y_e) is a thickening of a truncated spring. Let H' be the graph obtained from H by adding a new vertex a' as follows: $V(H') = V(H) \cup \{a'\}$, $H'|V(H) = H \backslash e$, and a' is adjacent to $\phi_e^{-1}(y)$ and no other vertex. Now G is a linear interval join with skeleton H'

using the same stripes as the construction with skeleton H and such that the stripe associated with the edge $a'\phi_e^{-1}(y)$ is a spring, contrary to the maximality of $|V(H)|$. Hence (G_e, Y_e) is not a thickening of a truncated spring.

Finally assume that G_e is a thickening of a trigraph in $\mathcal{C}$. By 3.6.2, it follows that (G_e, Y_e) is in $\mathcal{C}'$. This concludes the proof of (P3).

Hence G is an evenly structured linear interval join.

This concludes the proof of 3.6.7. □

A last lemma is needed for the proof of 3.1.4.

3.6.8.

3.6.9. *Let G be a cobipartite trigraph. Then G is a thickening of a linear interval trigraph.*

Proof. Let Y, Z be two disjoint strong cliques such that $Y \cup Z = V(G)$. Clearly (Y, Z) is a homogeneous pair. Let H be the trigraph such that $V(H) = \{y, z\}$ and

- y is strongly adjacent to z if Y is strongly complete to Z,
- y is strongly antiadjacent to z if Y is strongly anticomplete to Z,
- y is semiadjacent to z if Y is neither strongly complete nor strongly anticomplete to Z.

Now setting $Y = X_y$ a nd $Z = X_z$, we observe that G is a thickening of H. Since H is clearly a linear interval trigraph, it follows that G is a thickening of a linear interval trigraph. This proves 3.6.9. □

Proof of 3.1.4. Let G be a Berge claw-free connected trigraph. By 3.1.3, G is either a linear interval join or a thickening of a circular interval trigraph. By 3.2.1, if G is a thickening of a circular interval trigraph, then G is a thickening of a linear interval trigraph, or a cobipartite trigraph, or a thickening of a member of $\mathcal{C}$, or G is a structured circular interval trigraph. But by 3.6.6, if G is a structured circular interval trigraph, then G is an evenly structured linear interval join. By 3.6.9, if G is a cobipartite trigraph, then G is a thickening of a linear interval trigraph. Moreover, any thickening of a linear interval trigraph is clearly an evenly structured linear interval join. Finally by 3.6.7, if G is a linear interval join, then G is an evenly structured linear interval join. This proves 3.1.4. □

Chapter 4

On the Erdős-Lovász Tihany Conjecture

4.1 Introduction

In 1968, Erdős and Lovász made the following conjecture:

Conjecture 1 (Erdős-Lovász Tihany)**.** *For every graph G with $\chi(G) > \omega(G)$ and any two integers $s, t \geq 2$ with $s+t = \chi(G)+1$, there is a partition (S, T) of the vertex set $V(G)$ such that $\chi(G|S) \geq s$ and $\chi(G|T) \geq t$.*

Let G be a graph such that $\chi(G) > \omega(G)$. We say that a brace $\{u, v\}$ is *Tihany* if $\chi(G \backslash \{u, v\}) \geq \chi(G) - 1$. More generally, if K is a clique of size k in G, then we say that K is *Tihany* if $\chi(G \setminus K) \geq \chi(G) - k + 1$.

The following theorem is the main result of this chapter:

4.1.1. *Let G be a claw-free graph with $\chi(G) > \omega(G)$. Then there exists a clique K with $|K| \leq 5$ such that $\chi(G \backslash K) > \chi(G) - |K|$.*

To prove 4.1.1 we use a structure theorem for claw-free graphs due to Chudnovsky and Seymour that appears in [13] and is described in the next section. Section 4.3 contains some lemmas that serve as 'tools' for later proofs. The next six sections are devoted to dealing with the different outcomes of the structure theorem, proving that a minimal counterexample to 4.1.1 does not fall into any of those classes. In Section 4.4 we deal with the icosahedron and long circular interval graphs, in Section 4.5 with non-2-substantial and non-3-substantial graphs, in Section 4.6 with orientable prismatic graphs, in Section 4.7 with non-orientable prismatic graphs, in Section 4.8 with three-cliqued graphs and finally in Section 4.9 with strip structures. In Section 4.10 all of these results are collected to prove 4.1.1.

4.2 Structure Theorem

The goal of this section is to state and describe the structure theorem for claw-free graphs appearing in [13] (or, more precisely, its corollary). We begin with some definitions which are modified from [13].

Let X, Y be two subsets of $V(G)$ with $X \cap Y = \emptyset$. We say that X and Y are *complete* to each other if every vertex of X is adjacent to every vertex of Y, and we say that they are *anticomplete* to each other if no vertex of X is adjacent to a member of Y. Similarly, if $A \subseteq V(G)$ and $v \in V(G) \setminus A$, then v is *complete* to A if v is adjacent to every vertex in A, and *anticomplete* to A if v has no neighbor in A.

Let $F \subseteq V(G)^2$ be a set of unordered pairs of distinct vertices of G such that every vertex appears in at most one pair. Then H is a *thickening* of (G, F) if for every $v \in V(G)$ there is a nonempty subset $X_v \subseteq V(H)$, all pairwise disjoint and with union $V(H)$ satisfying the following:

- for each $v \in V(G)$, X_v is a clique of H
- if $u, v \in V(G)$ are adjacent in G and $\{u, v\} \notin F$, then X_u is complete to X_v in H
- if $u, v \in V(G)$ are nonadjacent in G and $\{u, v\} \notin F$, then X_u is anticomplete to X_v in H
- if $\{u, v\} \in F$ then X_u is neither complete nor anticomplete to X_v in H.

In this definition of graph thickening, elements of F have the role of pair of vertices semiadjacent in the description of thickening for trigraphs. Here are some classes of claw-free graphs that come up in the structure theorem.

- **Graphs from the icosahedron.** The *icosahedron* is the unique planar graph with twelve vertices all of degree five. Let it have vertices $v_0, v_1, \dots, v_{11}$, where for $1 \le i \le 10$, v_i is adjacent to v_{i+1}, v_{i+2} (reading subscripts modulo 10), and v_0 is adjacent to v_1, v_3, v_5, v_7, v_9, and v_{11} is adjacent to $v_2, v_4, v_6, v_8, v_{10}$. Let this graph be G_0. Let G_1 be obtained from G_0 by deleting v_{11} and let G_2 be obtained from G_1 by deleting v_{10}. Furthermore, let $F' = \{\{v_1, v_4\}, \{v_6, v_9\}\}$.

Let $G \in \mathcal{T}_1$ if G is a thickening of $(G_0, \emptyset)$, $(G_1, \emptyset)$, or (G_2, F) for some $F \subseteq F'$.

- **Fuzzy long circular interval graphs.** Let Σ be a circle, and let $F_1, \ldots, F_k \subseteq \Sigma$ be homeomorphic to the interval $[0, 1]$, such that no two of $F_1, \ldots, F_k$ share an endpoint, and no three of them have union Σ. Now let $V \subseteq \Sigma$ be finite, and let H be a graph with vertex set V in which distinct $u, v \in V$ are adjacent precisely if $u, v \in F_i$ for some i.

  Let F' be the set of pairs $\{u, v\}$ such that u, v are distinct endpoints of F_i for some i. Let $F \subseteq F'$ such that every vertex of G appears in at most one member of F. Then G is a *fuzzy long circular interval graph* if G is a thickening of (H, F).

  Let $G \in \mathcal{T}_2$ if G is a fuzzy long circular interval graph.

- **Fuzzy antiprismatic graphs.** A graph K is *antiprismatic* if for every $X \subseteq V(K)$ with $|X| = 4$, the subgraph induced by X is not a claw and there are at least two pairs of vertices in X that are adjacent. Let H be a graph and let F be a set of pairs $\{u, v\}$ such that every vertex of H is in at most one member of F and

  - no triad of H contains u and no triad of H contains v, or
  - there is a triad of H containing both u and v, and no other triad of H contains u or v.

  Thus F is the set of "changeable edges" discussed in [11]. The pair (H, F) is *antiprismatic* if for every $F' \subseteq F$, the graph obtained from H by changing the adjacency of all the vertex pairs in F' is antiprismatic. We say that a graph G is a *fuzzy antiprismatic graph* if G is a thickening of (H, F) for some antiprismatic pair (H, F).

  Let $G \in \mathcal{T}_3$ if G is a fuzzy antiprismatic graph.

Next, we define what it means for a claw-free graph to admit a "strip-structure". For a multigraph H and $F \in E(H)$, we denote by $\overline{F}$ the set of all $h \in V(H)$ incident with F. Let G be a graph. A *strip-structure* (H, η) of G consists of a multigraph H with $E(H) \neq \emptyset$, and a function η mapping each $F \in E(H)$ to a subset $\eta(F)$ of $V(G)$, and mapping each pair (F, h) with $F \in E(H)$ and $h \in \overline{F}$ to a subset $\eta(F, h)$ of $\eta(F)$, satisfying the following conditions.

(SD1) The sets $\eta(F)$ $(F \in E(H))$ are nonempty and pairwise disjoint and have union $V(G)$.

(SD2) For each $h \in V(H)$, the union of the sets $\eta(F, h)$ for all $F \in E(H)$ with $h \in \overline{F}$ is a clique of G.

(SD3) For all distinct $F_1, F_2 \in E(H)$, if $v_1 \in \eta(F_1)$ and $v_2 \in \eta(F_2)$ are adjacent in G, then there exists $h \in \overline{F_1} \cap \overline{F_2}$ such that $v_1 \in \eta(F_1, h)$ and $v_2 \in \eta(F_2, h)$.

There is also a fourth condition, but it is technical and we will not need it in this thesis.

Let (H, η) be a strip-structure of a graph G, and let $F \in E(H)$, where $\overline{F} = \{h_1, \ldots, h_k\}$. Let $v_1, \ldots, v_k$ be new vertices, and let J be the graph obtained from $G|\eta(F)$ by adding $v_1, \ldots, v_k$, where v_i is complete to $\eta(F, h_i)$ and anticomplete to all other vertices of J. Then $(J, \{v_1, \ldots, v_k\})$ is called the *strip of* (H, η) *at* F. A strip-structure (H, η) is *nontrivial* if $|E(H)| \geq 2$.

We now describe some strips that we will need for the structure theorem of claw-free graph.

$\mathcal{Z}_1$: Let H be a graph with vertex set $\{v_1, \ldots, v_n\}$, such that for $1 \leq i < j < k \leq n$, if v_i, v_k are adjacent then v_j is adjacent to both v_i, v_k. Let $n \geq 2$, let v_1, v_n be nonadjacent, and let there be no vertex adjacent to both v_1 and v_n. Let $F' \subseteq V(H)^2$ be the set of pairs $\{v_i, v_j\}$ such that $i < j$, $v_i \neq v_1$ and $v_j \neq v_n$, v_i is nonadjacent to v_{j+1}, and v_j is nonadjacent to v_{i-1}. Furthermore, let $F \subseteq F'$ such that every vertex of H appears in at most one member of F. Then G is a *fuzzy linear interval graph* if for some H and F, G is a thickening of (H, F) with $|X_{v_1}| = |X_{v_n}| = 1$. Let $X_{v_1} = \{u_1\}$, $X_{v_n} = \{u_n\}$, and $Z = \{u_1, u_n\}$. $\mathcal{Z}_1$ is the set of all pairs (G, Z).

$\mathcal{Z}_2$: Let $n \geq 2$. Construct a graph H as follows. Its vertex set is the disjoint union of three sets A, B, C, where $|A| = |B| = n + 1$ and $|C| = n$, say $A = \{a_0, a_1, \ldots, a_n\}, B = \{b_0, b_1, \ldots, b_n\}$, and $C = \{c_1, \ldots, c_n\}$. Adjacency is as follows. A, B, C are cliques. For $0 \leq i, j \leq n$ with $(i, j) \neq (0, 0)$, let a_i, b_j be adjacent if and only if $i = j$, and for $1 \leq i \leq n$ and $0 \leq j \leq n$, let c_i be adjacent to a_j, b_j if and only if $i \neq j \neq 0$. All other pairs not specified so far are nonadjacent. Now let $X \subseteq A \cup B \cup C \setminus \{a_0, b_0\}$ with $|C \setminus X| \geq 2$. Let $H' = H \setminus X$ and let G be a thickening of (H', F) with $|X_{a_0}| = |X_{b_0}| = 1$ and $F \subseteq V(H')^2$ (we will not specify the possibilities for the set F because they are technical and we will not need them in our proof). Let $X_{a_0} = \{a'_0\}$, $X_{b_0} = \{b'_0\}$, and $Z = \{a'_0, b'_0\}$. $\mathcal{Z}_2$ is the set of all pairs (G, Z).

$\mathcal{Z}_3$: Let H be a graph, and let $h_1 - h_2 - h_3 - h_4 - h_5$ be the vertices of a path of H in order, such that h_1, h_5 both have degree one in H, and every edge of H is incident with one of h_2, h_3, h_4. Let H' be obtained from the line graph of H by making the edges h_2h_3 and h_3h_4 of H (vertices of H') nonadjacent. Let $F \subseteq \{\{h_2h_3, h_3h_4\}\}$ and let G be a thickening of (H', F) with $|X_{h_1h_2}| = |X_{h_4h_5}| = 1$. Let $X_{h_1h_2} = \{u\}$, $X_{h_4h_5} = \{v\}$, and $Z = \{u, v\}$. $\mathcal{Z}_3$ is the set of all pairs (G, Z).

$\mathcal{Z}_4$: Let H be the graph with vertex set $\{a_0, a_1, a_2, b_0, b_1, b_2, b_3, c_1, c_2\}$ and adjacency as follows: $\{a_0, a_1, a_2\}, \{b_0, b_1, b_2, b_3\}, \{a_2, c_1, c_2\}$, and $\{a_1, b_1, c_2\}$ are cliques; b_2, c_1 are adjacent; and all other pairs are nonadjacent. Let $F = \{\{b_2, c_2\}, \{b_3, c_1\}\}$ and let G be a thickening of (H, F) with $|X_{a_0}| = |X_{b_0}| = 1$. Let $X_{a_0} = \{a'_0\}$, $X_{b_0} = \{b'_0\}$, and $Z = \{a'_0, b'_0\}$. $\mathcal{Z}_4$ is the set of all pairs (G, Z).

$\mathcal{Z}_5$: Let H be the graph with vertex set $\{v_1, \ldots, v_{12}\}$, and with adjacency as follows. $v_1 - \cdots - v_6 - v_1$ is an induced cycle in G of length 6. Next, v_7 is adjacent to v_1, v_2; v_8 is adjacent to v_4, v_5; v_9 is adjacent to v_6, v_1, v_2, v_3; v_{10} is adjacent to v_3, v_4, v_5, v_6, v_9; v_{11} is adjacent to $v_3, v_4, v_6, v_1, v_9, v_{10}$; and v_{12} is adjacent to $v_2, v_3, v_5, v_6, v_9, v_{10}$. No other pairs are adjacent. Let H' be a graph isomorphic to $H \setminus X$ for some $X \subseteq \{v_{11}, v_{12}\}$ and let $F \subseteq \{\{v_9, v_{10}\}\}$. Let G be a thickening of (H', F) with $|X_{a_0}| = |X_{b_0}| = 1$. Let $X_{v_7} = \{v'_7\}$, $X_{v_8} = \{v'_8\}$, and $Z = \{v'_7, v'_8\}$. $\mathcal{Z}_5$ is the set of all pairs (G, Z).

We are now ready to state a structure theorem for claw-free graphs that is an easy corollary of the main result of [13].

4.2.1. *Let G be a connected claw-free graph. Then either*

- *G is a member of $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$, or*
- *$V(G)$ is the union of three cliques, or*
- *G admits a nontrivial strip-structure such that for each strip (J, Z), $1 \leq |Z| \leq 2$, and if $|Z| = 2$, then either*
  - *$|V(J)| = 3$ and Z is complete to $V(J) \setminus Z$, or*
  - *(J, Z) is a member of $\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}_3 \cup \mathcal{Z}_4 \cup \mathcal{Z}_5$.*

4.3 Tools

In this section we present a few lemmas that will then be used extensively in the following sections to prove results on the different graphs used in 4.2.1.Let K be a clique in G. We denote by $C(K)$ the set of common neighbors of the members of K, by $A(K)$ the set of their common non-neighbors, and by $M(K)$ the set of vertices that are mixed on the clique K. Formally,

$$\begin{aligned} C(K) =& \{x \in V(G) \setminus K : ux \in E(G) \text{ for all } u \in K\} \\ A(K) =& \{x \in V(G) : ux \notin E(G) \text{ for all } u \in K\} \\ M(K) =& V(G) \setminus (C(K) \cup A(K)). \end{aligned}$$

We say that a clique K is *dense* if $C(K)$ is a clique and we say that it is *good* if $C(K)$ is an anti-matching.

The following result is taken from [34]. Because it is fundamental to many of our results, we include its proof here for completeness.

4.3.1. *Let G be a graph with chromatic number χ and let K be a clique of size k in G. If K is not Tihany, then every color class of a $(\chi - k)$-coloring of $G \setminus K$ contains a vertex complete to K.*

Proof. Suppose not. Since K is not Tihany, it follows that $G \setminus K$ is $(\chi - k)$-colorable. Let C be a color class of a $(\chi - k)$-coloring of $G \setminus K$ with no vertex complete to K. Define a coloring of $K \cup C$ by giving a distinct color to each vertex of K and giving each vertex of C a color of one of its non-neighbors in K. This defines a k-coloring of $G|(K \cup C)$. Note also that $G \setminus (K \cup C)$ is $(\chi - k - 1)$-colorable. However, this implies that G is $(\chi - 1)$-colorable, a contradiction. This proves 4.3.1. □

The next lemma is one of our most important and basic tool.

4.3.2. *Let G be a graph such that $\chi(G) > \omega(G)$. Let K be a clique of G. If K is dense, then it is Tihany.*

Proof. Suppose that K is not Tihany. Let $\mathcal{C}$ be a $(\chi - k)$-coloring of $G \setminus K$. By 4.3.1, every color class of $\mathcal{C}$ contains a vertex complete to K. Hence, every color class contains a member of $C(K)$ and so $|C(K) \cup K| \geq \chi(G) > \omega(G)$, a contradiction. This proves 4.3.2. □

Let (A, B) be disjoint subsets of $V(G)$. The pair (A, B) is called a *homogeneous pair* in G if A, B are cliques, and for every vertex $v \in V(G) \setminus (A \cup B)$, v is either complete to A or anticomplete to A and either complete to B or anticomplete to B. A *W-join* (A, B) is a homogeneous pair in which A is neither complete nor anticomplete to B. We say that a W-join (A, B) is *reduced* if we can partition A into two sets A^1 and A^2 and we can partition B into B^1, B^2 such that A^1 is complete to B^1, A^2 is anticomplete to B, and B^2 is anticomplete to A. Note that since A is neither complete nor anticomplete to B, it follows that both A^1 and B^1 are non-empty and at least one of A^2, B^2 is non-empty. We call a W-join that is not reduced a *non-reduced* W-join.

Let H be a thickening of (G, F) for some valid $F \subseteq V(G)^2$ and let $\{u, v\} \in F$. Then we notice that (X_u, X_v) is a W-join in H. If for every $\{u, v\} \in F$ we have that (X_u, X_v) is a reduced W-join then we say that H is a *reduced thickening* of G.

The following result appears in [4].

4.3.3. *Let G be a claw-free graph and suppose that G admits a non-reduced W-join. Then there exists a subgraph H of G with the following properties:*

1. *H is a claw-free graph, $|V(H)| = |V(G)|$ and $|E(H)| < |E(G)|$.*

2. *$\chi(H) = \chi(G)$.*

The result of 4.3.3 implies the following:

4.3.4. *Assume that G be a claw-free graph with $\chi(G) > \omega(G)$ that is a minimal counterexample to 4.1.1. Assume also that G is a thickening of (H, F) for some claw-free graph H and $F \subseteq V(H)^2$. Then G is a reduced thickening of (H, F).*

For a clique $K \subseteq V(G)$ and $F \subseteq V(G)^2$, we define $S_F(K) = \{x \in V(G) : \exists k \in K \text{ s.t. } \{x, k\} \in F \text{ and } x \in C(K \backslash k)\}$.

4.3.5. *Let G be a reduced thickening of (H, F) for some claw-free graph H and $F \subseteq V(H)^2$. Let K be a clique in H such that for all $x, y \in C(K) \subseteq V(H)$, $\{x, y\} \notin F$. If $C(K) \cup S_F(K)$ is a clique, then there exists a dense clique of size $|K|$ in G.*

Proof. Let K' be a clique of size $|K|$ in G such that $K' \cap X_v \neq \emptyset$ for all $v \in K$. By the definition of a thickening such a clique exists. Moreover since $C(K) \cup S_F(K)$ is a clique, it follows that K' is dense. This proves 4.3.5. □

The following lemma is a direct corollary of 4.3.2 and 4.3.5.

4.3.6. *Let G be a reduced thickening of (H, F) for some claw-free graph H and $F \subseteq V(H)^2$. Let K be a dense clique in H such that for all $x, y \in C(K)$, $\{x, y\} \notin F$. If $C(K) \cup S_F(K)$ is a clique, then there exists a Tihany clique of size $|K|$ in G.*

The following result helps us handle the case when $C(x)$ is an antimatching for some vertex $x \in V(G)$.

4.3.7. *Let G be a graph with $\chi(G) > \omega(G)$. Let $u, x, y \in V(G)$ such that $ux, uy \in E(G)$, $xy \notin E(G)$ and $x \neq y$. Let $E = \{u, x\}$ and $E' = \{u, y\}$. If $C(E) = C(E')$ then E, E' are Tihany.*

Proof. Suppose that E is not Tihany. Let $\mathcal{C}$ be a $(\chi(G) - 2)$-coloring of $G \setminus \{u, x\}$. Let $C \in \mathcal{C}$ be the color class such that $y \in C$. By 4.3.1, there is a vertex $z \in C$ such that z is complete to E, and so $z \in C(E)$. But y is complete to $C(E)$, a contradiction. Hence E is Tihany and by symmetry, so is E'. □

In particular, if we have a vertex x such that $C(x)$ is an antimatching, we can find a Tihany edge either by 4.3.2 or by 4.3.7.

4.3.8. *Let H be a graph, G a thickening of (H, F) for some valid $F \subseteq H(V)^2$ such that $\chi(G) > \omega(G)$. Let K be a clique of H. Assume that for all $\{x, y\} \in F$ such that $x \in K$, y is complete to $C(K) \backslash \{y\}$. Let $u, v \in C(K)$ with $u \neq v$ be such that u is not adjacent to v and $\{u, v\}$ is complete to $C(K) \backslash \{u, v\}$. Moreover assume that if there exists $E \in F$ with $\{u, v\} \cap E \neq \emptyset$, then $E = \{u, v\}$. Then there exists a Tihany clique of size $|K| + 1$ in G.*

Proof. Assume not. Let K' be a clique of size $|K|$ in G such that $K' \cap X_y \neq \emptyset$ for all $y \in K$. If $\{u, v\} \notin F$, let $a \in X_u$, $A = X_u$, $b \in X_v$ and $B = X_v$. If $\{u, v\} \in F$, let X_u^1, X_u^2, X_v^1 and X_v^2 be as in the definition of reduced W-join. By symmetry, we may assume that X_u^2 is not empty. If X_v^2 is empty, let $a \in X_u^2$, $A = X_u^2$, $b \in X_v^1$ and $B = X_v^1$; and if X_v^2 is not empty, let $a \in X_u^2$, $A = X_u$, $b \in X_v^2$ and $B = X_v$.

Now let $T_a = K' \cup \{a\}$ and $T_b = K' \cup \{b\}$. We may assume that $\chi(G \backslash T_a) = \chi(G \backslash T_b) = \chi(G) - |K| - 1$. By 4.3.1, we may assume that every color class of $G \backslash T_a$ contains a common neighbor of T_a. Since no vertex of B is complete to T_a, and since B is a clique complete to $C(T_a) \backslash A$, it follows that $|A| > |B|$. But similarly, $|B| > |A|$, a contradiction. This proves 4.3.8. □

We need an additional definition before proving the next lemma. Let K be a clique; we denote by $\overline{C}(K)$ the closed neighborhood of K, i.e. $\overline{C}(K) := C(K) \cup K$.

4.3.9. *Let G be a graph such that $\chi(G) > \omega(G)$. Let A and B be cliques such that $2 \leq |A|, |B| \leq 3$ (i.e., each one is a brace or a triangle). If $\overline{C}(A) \cap \overline{C}(B) = \emptyset$ and $\overline{C}(A) \cup \overline{C}(B)$ contains no triads then at least one of A, B is Tihany.*

Proof. Assume not and let $k = \chi(G) - |A|$. By 4.3.1, in every k-coloring of $G \backslash A$ every color class must have a vertex in $C(A)$. As there is no triad in $\overline{C}(A) \cup \overline{C}(B)$, it follows that every vertex of $C(A)$ is in a color class with at most one vertex of $\overline{C}(B)$, thus $\overline{C}(A) > \overline{C}(B)$. By symmetry, it follows that $\overline{C}(A) < \overline{C}(B)$, a contradiction. This proves 4.3.9. □

4.3.10. *Let G be a claw-free graph such that $\chi(G) > \omega(G)$. If G admits a clique cutset, then there is a Tihany brace in G.*

Proof. Let K be a clique cutset. Let $A, B \subset V(G) \backslash K$ such that $A \cap B = \emptyset$ and $A \cup B \cup K = V(G)$. Let $\chi_A = \chi(G|(A \cup K))$ and $\chi_B = \chi(G|(B \cup K))$. By symmetry, we may assume that $\chi_A \geq \chi_B$.

(1) $\chi(G) = \chi_A$

Let $\mathcal{S}_A = (A_1, A_2, \ldots, A_{\chi_A})$ and $\mathcal{S}_B = (B_1, B_2, \ldots, B_{\chi_B})$ be optimal colorings of $G|(A \cup K)$ and $G|(B \cup K)$. Let $K = \{k_1, k_2, \ldots, k_{|K|}\}$. Up to renaming the stable sets, we may assume that $A_i \cap B_i = \{k_i\}$ for all $i = 1, 2, \ldots, |K|$. Then $\mathcal{S} = (A_1 \cup B_1, A_2 \cup B_2, \ldots, A_{\chi_B} \cup B_{\chi_B}, A_{\chi_B+1}, \ldots, A_{\chi_A}\}$ is a χ_A-coloring of G. This proves (1).

Now let $x \in B$ and $y \in K$ be such that $xy \in E(G)$. Then $\chi(G \backslash \{x, y\}) \geq \chi(G|(A \cup K \backslash \{y\}) \geq \chi_A - 1 \geq \chi(G) - 1$. Hence $\{x, y\}$ is a Tihany brace. This proves 4.3.10. □

4.4 The Icosahedron and Long Circular Interval Graphs

4.4.1. *Let $G \in \mathcal{T}_1$. If $\chi(G) > \omega(G)$, then there exists a Tihany brace in G.*

Proof. Let $v_0, v_1, \ldots, v_{11}$ be as in the definition of the icosahedron. Let G_0, G_1, G_2, and F be as in the definition of $\mathcal{T}_1$. Then G is a thickening of either $(G_0, \emptyset)$, $(G_1, \emptyset)$, or (G_2, F) for $F \subseteq \{(v_1, v_4), (v_6, v_9)\}$. For $0 \leq i \leq 11$, let X_{v_i} be as in the definition of thickening (where $X_{v_{11}}$ is empty when G is a thickening of $(G_1, \emptyset)$ or (G_2, F), and $X_{v_{10}}$ is empty when G is a thickening of (G_2, F)). Let $x_i \in X_{v_i}$ and $w_i = |X_{v_i}|$.

First suppose that G is a thickening of $(G_1, \emptyset)$ or (G_2, F). Then $C(\{x_4, x_6\}) = X_{v_4} \cup X_{v_5} \cup X_{v_6}$ is a clique. Therefore, $\{x_4, x_6\}$ is a Tihany brace by 4.3.2.

So we may assume that G is a thickening of $(G_0, \emptyset)$. Suppose that no brace of G is Tihany and let $E = \{x_1, x_3\}$. Then $G \backslash E$ is $(\chi - 2)$-colorable. By 4.3.1, every color class contains at least one vertex from $C(E) = (X_{v_1} \cup X_{v_2} \cup X_{v_3} \cup X_{v_0}) \setminus \{x_1, x_3\}$. Since $\alpha(G) = 3$, it follows that every color class has at most two vertices from $\bigcup_{i=4}^{11} X_{v_i}$. Hence we conclude that

$$w_4 + w_5 + w_6 + w_7 + w_8 + w_9 + w_{10} + w_{11} \leq 2 \cdot (w_1 + w_2 + w_3 + w_0 - 2)$$

A similar inequality exists for every brace $\{x_i, x_j\}$. Summing these inequalities over all braces $\{x_i, x_j\}$, it follows that $(\sum_{i=0}^{11} 20w_i) \leq (\sum_{i=0}^{11} 20w_i) - 120$, a contradiction. This proves 4.4.1. □

4.4.2. *Let $G \in \mathcal{T}_2$. If $\chi(G) > \omega(G)$, then there exists a Tihany brace in G.*

Proof. Let $H, F, \Sigma, F_1, \ldots, F_k$ be as in the definition of $\mathcal{T}_2$ such that G is a thickening of (H, F). Let F_i be such that there exists no j with $F_i \subset F_j$. Let $\{x_k, \ldots, x_l\} = V(H) \cap F_i$ and without loss of generality, we may assume that $\{x_k, \ldots, x_l\}$ are in order on Σ. Since $C(\{x_k, x_l\}) = \{x_{k+1}, \ldots, x_{l-1}\}$, it follows that $\{x_k, x_l\}$ is dense. Hence by 4.3.6 there exists a Tihany brace in G. This proves 4.4.2. □

4.5 Non-2-substantial and Non-3-substantial Graphs

In this section we study graphs where a few vertices cover all the triads. An antiprismatic graph G is *k-substantial* if for every $S \subseteq V(G)$ with $|S| < k$ there is a triad T with $S \cap T = \emptyset$. The *matching number* of a graph G, denoted by $\mu(G)$, is the number of edges in a maximum matching in G. Balogh et al. [2] proved the following theorem.

4.5.1. *Let G be a graph such that $\alpha(G) = 2$ and $\chi(G) > \omega(G)$. For any two integers $s, t \geq 2$ such that $s + t = \chi(G) + 1$ there exists a partition (S, T) of $V(G)$ such that $\chi(G|S) \geq s$ and $\chi(G|T) \geq t$.*

The following theorem is a result of Gallai and Edmonds on matchings and it will be used in the study of non-2-substantial and non-3-substantial graphs.

4.5.2 (Gallai-Edmonds Structure Theorem [17], [19]). *Let $G = (V, E)$ be a graph. Let D denote the set of nodes which are not covered by at least one maximum matching of G. Let A be the set of nodes in $V\backslash D$ adjacent to at least one node in D. Let $C = V\backslash(A \cup D)$. Then:*

i) The number of covered nodes by a maximum matching in G equals $|V| + |A| - c(D)$, where $c(D)$ denotes the number of components of the graph spanned by D.

ii) If M is a maximum matching of G, then for every component F of $G|D$, $E(D) \cap M$ covers all but one of the nodes of F, $E(C) \cap M$ is a perfect matching of $G|C$ and M matches all the nodes of A with nodes in distinct components of D.

4.5.3. *Let G be an antiprismatic graph. Let K be a clique and assume that $u, v \in V(G)\backslash\overline{C}(K)$ are non-adjacent. If $\alpha(G|(C(K) \cup \{u, v\})) = 2$ and $\alpha(G|K \cup \{u, v\}) = 3$, then $G|\overline{C}(K)$ is cobipartite.*

Proof. Since there is no triad in $C(K) \cup \{u, v\}$, we deduce that there is no vertex in $C(K)$ anticomplete to $\{u, v\}$. Since G is claw-free and $\alpha(G|K \cup \{u, v\}) = 3$, it follows that there is no vertex in $C(K)$ complete to $\{u, v\}$. Let $C_u, C_v \subseteq C(K)$ be such that $C_u \cup C_v = C(K)$ and for all $x \in C(K)$, x is adjacent to u and non-adjacent to v if $x \in C_u$, and x is adjacent to v and non-adjacent to u if $x \in C_v$. Since $\alpha(G|(C_v \cup \{u\})) = 2$, we deduce that C_v is a clique and by symmetry C_u is a clique. Hence $\overline{C}(K)$ is the union of two cliques. This proves 4.5.3. □

4.5.4. *Let G be a claw-free graph such that $\chi(G) > \omega(G)$. Let K be a clique such that $\alpha(G\backslash K) \leq 2$. Then there exists a Tihany clique of size at most $|K| + 1$ in G.*

Proof. Assume not. Let $n = |V(G)|$, $w \in C(K)$ and $K' = K \cup \{w\}$ (such a vertex w exists by 4.3.1).

(1) $\chi(G) = n - \mu(G^c)$.

Since K' is not Tihany, it follows that $\chi(G\backslash K') = \chi(G) - |K'|$. Since $\alpha(G\backslash K') \leq 2$, we deduce that $\chi(G\backslash K') \geq \frac{n-|K'|}{2}$, and thus $\chi(G) \geq \frac{n+|K'|}{2}$. Hence in every optimal coloring of G the color classes have an average size strictly smaller than 2, and since G is claw-free, we deduce that there is an optimal coloring of G where all color classes have size 1 or 2. It follows that $\chi(G) \geq n - \mu(G^c)$. But clearly $\chi(G) \leq n - \mu(G^c)$, thus $\chi(G) = n - \mu(G^c)$. This proves (1).

(2) *Let T be a clique of size $|K| + 1$ in G, then $\chi(G\backslash T) = n - |T| - \mu(G^c\backslash T)$.*

Since T is not Tihany, it follows that $\chi(G\backslash T) = \chi(G) - |T| \geq \frac{n+|K'|}{2} - |T| = \frac{n-|T|}{2} = \frac{|V(G\backslash T)|}{2}$. Hence in every optimal coloring of $G\backslash T$, the color classes have an average size smaller than 2, and since G is claw-free, we deduce that there is an optimal coloring of $G\backslash T$ where all color classes have size 1 or 2. It follows that $\chi(G\backslash T) \geq |V(G\backslash T| - \mu(G^c\backslash T)$. Hence $\chi(G\backslash T) = n - |T| - \mu(G^c\backslash T)$. This proves (2).

Let A, D, C be as in 4.5.2. Since $\chi(G) \geq \frac{n+|K'|}{2}$ and $\chi(G) = n - \mu(G^c)$, we deduce that $\mu(G^c) \leq \frac{n-|K'|}{2}$. By 4.5.2 i), we deduce that $\mu(G^c) = \frac{n+|A|-c(D)}{2}$. Thus, it follows that $c(D) \geq |K'|$. Let $D_1, D_2, \ldots, D_{c(D)}$ be the anticomponents of $G|D$. Let $d_i \in D_i$ for $i = 1, \ldots, c(D)$.

(3) $|D_i| = 1$ *for all* i.

Assume not, and by symmetry assume $|D_1| > 1$. Since G is claw-free, we deduce $\alpha(G|D_1) = 2$. Thus there exist $x, y \in D_1$ such that x is adjacent to y. Now $T = \{x, y, d_2, \ldots, d_{|K|}\}$ is a clique of size $|K| + 1$ and by 4.5.2 ii), it follows that $\mu(G^c\backslash T) < \mu(G^c)$. By (1) and (2), it follows that $\chi(G\backslash T) + |T| = n - \mu(G^c\backslash T) > n - \mu(G^c) = \chi(G)$, a contradiction. This proves (3).

Let $T = \{d_1, \ldots, d_{|K|+1}\}$. By (3), it follows that $C(T) \cap D$ is a clique. By 4.3.2, we deduce that $C(T) \cap A \neq \emptyset$. Let $x \in C(T) \cap A$. Now $S = \{d_1, \ldots, d_{|K|}, x\}$ is a clique of size $|K| + 1$ and by 4.5.2 ii), it follows that $\mu(G^c\backslash S) < \mu(G^c)$. By (1) and (2), it follows that $\chi(G\backslash S) + |S| = n - \mu(G^c\backslash S) > n - \mu(G^c) = \chi(G)$, a contradiction. This concludes the proof of 4.5.4. □

4.5.5. *Let H be an antiprismatic graph such that there exists $x \in V(H)$ with $\alpha(H\backslash x) = 2$. Let G be a reduced thickening of (H, F) for some valid $F \subseteq V(G)^2$ such that $\chi(G) > \omega(G)$ and $|X_x| > 1$. Then for all $\{u, v\} \in X_x$, $\chi(G\backslash\{u, v\}) \geq \chi(G) - 1$.*

Proof. Let $u, v \in X_x$. We may assume that $\{u, v\}$ is not Tihany. Let $k = \chi(G\backslash\{u, v\})$ and $\mathcal{S} = (S_1, S_2, \ldots, S_k)$ be a k-coloring of $G\backslash\{u, v\}$. By 4.3.1, $S_i \cap C(\{u, v\}) \neq \emptyset$. Let $I_j = \{i : |S_i| = j\}$ and let $O = C(\{u, v\}) \cap \bigcup_{i \in I_1 \cup I_2} S_i$ and $P = C(\{u, v\}) \cap \bigcup_{i \in I_3} S_i$.

Since $\alpha(H\backslash x) = 2$, it follows that $S_i \cap X_x \neq \emptyset$ for all $i \in I_3$. Hence, P is a clique complete to O and thus $\omega(G|O \cup P) = \omega(G|O) + |I_3|$. Since $\chi(G) > \omega(G)$, we deduce that $\omega(G|O) < |I_1 \cup I_2|$. By 4.5.3 and since $O \subseteq \overline{C}(X_x)$, we deduce that $G|O$ is cobipartite. Hence $\chi(G|O) = \omega(G|O) < |I_1 \cup I_2|$. Thus the coloring $\mathcal{S}$ does not induce an optimal coloring of $G|O$. It follows that there exists an augmenting antipath $P = p_1 - p_2 - \ldots - p_{2l}$ in O. Now let $T_i = \{p_{2i-1}, p_{2i}\}$ for $i = 1, \ldots, l$. Let s be such that $p_1 \in S_s$ and e be such that $p_{2l} \in S_e$. They are the color classes where the augmenting antipath starts and ends. If $|S_s| = 2$, let $T_{l+1} = (\{u\} \cup S_s \backslash p_1)$, otherwise let $T_{l+1} = \{u\}$. If $|S_e| = 2$, let $T_{l+2} = (\{v\} \cup S_e \backslash p_{2l})$, otherwise let $T_{l+2} = \{v\}$. Let $J = \{i | S_i \cap V(P) \neq \emptyset\}$. Clearly $|J| = l + 1$. Now $(T_1, T_2, \ldots, T_{l+2})$ is a $(l+2)$-coloring of $\bigcup_{i \in J} S_i \cup \{u, v\}$, which together with the color classes S_i for $i \notin J$ create a $k+1$-coloring of G, a contradiction. This proves 4.5.5. □

The next lemma is a direct corollary of 4.5.4 and 4.5.5.

4.5.6. *Let H be a non-2-substantial antiprismatic graph. Let G be a reduced thickening of (H, F) for some valid $F \subseteq V(G)^2$ such that $\chi(G) > \omega(G)$. Then there exists a Tihany brace in G.*

Now we look at non-3-substantial graphs.

4.5.7. *Let H be a non-3-substantial antiprismatic graph. Assume that $u, v \in H$ satisfy $\alpha(H \backslash \{u, v\}) = 2$. Let G be a reduced thickening of H such that $\chi(G) > \omega(G)$. If u is not adjacent to v, then there exists a Tihany brace or triangle in G.*

Proof. Assume not. Let $N_u = C(u) \backslash C(\{u, v\})$ and $N_v = C(v) \backslash C(\{u, v\})$. Since H is antiprismatic, it follows that N_u and N_v are antimatchings.

By 4.5.6, we deduce that N_u and N_v are not cliques. Let $x_u, y_u \in N_u$ be non-adjacent, and $x_v, y_v \in N_v$ be non-adjacent. Since $\alpha(H \backslash \{u, v\}) = 2$ and H is antiprismatic, we may assume by symmetry that $x_u x_v, y_u y_v$ are edges, and $x_u y_v, y_u x_v$ are non-edges. Since $\alpha(H \backslash \{u, v\}) = 2$ and H is antiprismatic, it follows that every vertex in $C(\{u, v\})$ is either strongly complete to $x_u x_v$ and strongly anticomplete to $y_u y_v$, or strongly complete to $y_u y_v$ and strongly anticomplete to $x_u x_v$. Let (N_x, N_y) be the partition of $C(\{u, v\})$ such that all $x \in N_x$ are complete to $x_u x_v$ and and all $y \in N_y$ are complete to $y_u y_v$.

Assume first that $N_x \neq \emptyset$ and $N_y \neq \emptyset$. Let $n_x \in N_x$ and $n_y \in N_y$ and let $T_u = \{u, y_u, n_y\}$ and $T_v = \{v, x_v, n_x\}$. Clearly T_u and T_v are triangles.

(1) $\alpha(G | (\overline{C}(T_u) \cup \overline{C}(T_v)) = 2$ *and* $\overline{C}(T_u) \cap \overline{C}(T_v) = \emptyset$.

Assume not. Since $\overline{C}(T_u) \subseteq N_y \cup N_u \cup \{u\}$ and $\overline{C}(T_v) \subseteq N_x \cup N_v \cup \{v\}$, we deduce that $\overline{C}(T_u) \cap \overline{C}(T_v) = \emptyset$. Let $T \in \overline{C}(T_u) \cup \overline{C}(T_v)$ be a triad. By symmetry, we may assume that

$u \in T$. Clearly, $T \backslash u \in N_v$. But since H is antiprismatic, we deduce that $T \backslash u \subseteq C(n_x)$, hence $T \backslash u \notin \overline{C}(T_u) \cup \overline{C}(T_v)$, a contradiction. This proves (1).

Now let $S_u, S_v \in G$ be triangles such that $|S_u \cap X_u| = |S_u \cap X_{y_u}| = |S_u \cap X_{n_y}| = 1$ and $|S_v \cap X_v| = |S_v \cap X_{x_v}| = |S_v \cap X_{n_x}| = 1$. By (1) and 4.3.9 and since G is a reduced thickening of H, we deduce that there is a Tihany triangle in G.

Now assume that at least one of N_x, N_y is empty. By symmetry, we may assume that N_x is empty. Since $C(\{u, x_u\})$ is an antimatching, by 4.3.8 there exists a Tihany triangle in G. This concludes the proof of 4.5.7. □

4.5.8. *Let H be a non-3-substantial antiprismatic graph. Let $u, v \in H$ be such that $\alpha(G \backslash \{u, v\}) = 2$. Let G be a reduced thickening of (H, F) for some valid $F \subseteq V(H)^2$ such that $\chi(G) > \omega(G)$. If u is adjacent to v, then there exists a Tihany clique K in G with $|K| \leq 4$.*

Proof. Assume not. By 4.5.4, we may assume that $|X_u \cup X_v| > 2$. By 4.5.6, we may assume that $|X_u| > 0$ or $|X_v| > 0$. If $|X_u| = 1$, then $G \backslash X_u$ is a reduced thickening of a non-2-substantial antiprismatic graph. By 4.5.5, there exists a brace $\{x, y\}$ in X_v such that $\chi(G \backslash (\{x, y\} \cup X_u)) \geq \chi(G \backslash X_u) - 1$. But $\chi(G \backslash X_u) - 1 \geq \chi(G) - 2$, hence $\{x, y\} \cup X_u$ is a Tihany triangle, a contradiction. Thus $|X_u| > 1$, and by symmetry $|X_v| > 1$.

Let $x_1, y_1 \in X_u$ and $x_2, y_2 \in X_v$, thus $C = \{x_1, x_2, y_1, y_2\}$ is a clique of size 4.

Let $k = \chi(G \backslash C)$ and $\mathcal{S} = (S_1, S_2, \ldots, S_k)$ be a k-coloring of $G \backslash C$. By 4.3.1, it follows that $S_i \cap N(C) \neq \emptyset$. For $j = 1, 2, 3$ let $I_j = \{i : |S_i| = j\}$ and let $O = N(C) \cap \bigcup_{i \in I_1 \cup I_2} S_i$ and $P = N(C) \cap \bigcup_{i \in I_3} S_i$.

Since $\alpha(H \backslash \{u, v\}) = 2$, it follows that $S_i \cap (X_u \cup X_v) \neq \emptyset$ for all $i \in I_3$. Hence, $\omega(G|O \cup P) = \omega(G|O) + |I_3|$. Since $\chi(G) > \omega(G)$, we deduce that $\omega(G|O) < |I_1 \cup I_2|$. By 4.5.3, we deduce that $G|O$ is cobipartite. Hence $\chi(G|O) = \omega(G|O) < |I_1| + |I_2|$. Thus the coloring $\mathcal{S}$ does not induce an optimal coloring of $G|O$. It follows that there exists an augmenting antipath $P = p_1 - p_2 - \ldots - p_{2l}$ in O. Now let $T_i = \{p_{2i-1}, p_{2i}\}$ for $i = 1, \ldots, l$. Let s be such that $p_1 \in S_s$ and e be such that $p_{2l} \in S_e$. They are the color classes where the augmenting antipath starts and ends. Since $S_s \backslash p_1$ is not complete to $\{x_1, y_1\}$, we deduce that there exists $\hat{s} \in \{1, 2\}$ such that $x_{\hat{s}}$ is antiadjacent to $S_s \backslash p_1$. Let $T_{l+1} = \{x_{\hat{s}}\} \cup S_s \backslash p_1$ and $T_{l+2} = \{x_1, x_2\} \backslash x_{\hat{s}}$. Since $S_e \backslash p_{2l}$ is not complete to $\{x_2, y_2\}$, we deduce that there exists $\hat{e} \in \{1,2\}$ such that $x_{\hat{e}}$ is antiadjacent to $S_e \backslash p_{2l}$. Let $T_{l+3} = \{x_{\hat{e}}\} \cup S_e \backslash p_{2l}$ and $T_{l+4} = \{y_1, y_2\} \backslash x_{\hat{e}}$.

Let $J = \{i | S_i \cap V(P) \neq \emptyset\}$. Clearly $|J| = l+1$. Now $(T_1, T_2, \ldots, T_{l+2}, T_{l+3}, T_{l+4})$ is a $(l+4)$-coloring of $\bigcup_{i \in J} S_i \cup \{x_1, x_2, y_1, y_2\}$, which together with the color classes S_i, for $i \notin J$, create a $k+3$-coloring of G, a contradiction. This proves 4.5.8. □

The following lemma is a direct corollary of 4.5.7 and 4.5.8.

4.5.9. *Let H be a non-3-substantial antiprismatic graph. Let G be a reduced thickening of H such that $\chi(G) > \omega(G)$. Then there exists a Tihany clique $K \subset V(G)$ with $|K| \leq 4$.*

4.6 Complements of Orientable Prismatic Graphs

In this section we study the complements of orientable prismatic graphs. A graph is *prismatic* if its complement is antiprismatic. We can also define also prismatic graph in a direct way. A graph G is prismatic if for every triangle $T \subseteq V(G)$ and $x \in V(G) \backslash T$, then $|N(x) \cap T| = 1$. Let G be prismatic and let S, T be two disjoint triangles in G. By definition of G there exists a perfect matching between S and T. An *orientation* $\mathcal{O}$ of G is a choice of a cyclic orientation $\mathcal{O}(T)$ for every triangle T of G such that if $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2, t_3\}$ are disjoint triangles with $\mathcal{O}(S) = s_1 \to s_2 \to s_3 \to s_1$ and $\mathcal{O}(T) = t_1 \to t_2 \to t_3 \to t_1$, then $s_i t_i \in E(G)$ $i = 1, 2, 3$. We say that G is *orientable* if it admits an orientation, and G is *non-orientable* otherwise.

The *core* of a graph G is the union of all the triangles in G. If $\{a, b, c\}$ is a triangle in G and both b, c only belong to one triangle in G, then b and c are said to be *weak*. The *strong core* of G is the subset of the core such that no vertex in the strong core is weak. As proved in [11], if H is a thickening of (G, F) for some valid $F \subseteq V(G)^2$ and $\{x, y\} \in F$, then x and y are not in the strong core.

A *three-cliqued claw-free graph* (G, A, B, C) consists of a claw-free graph G and three cliques A, B, C of G, pairwise disjoint and with union $V(G)$. The complement of a tree-cliqued graph is a 3-*colored graph*. Let $n \geq 0$, and for $1 \leq i \leq n$, let (G_i, A_i, B_i, C_i) be a three-cliqued graph, where $V(G_1), \ldots, V(G_n)$ are all nonempty and pairwise vertex-disjoint. Let $A = A_1 \cup \cdots \cup A_n$, $B = B_1 \cup \cdots \cup B_n$, and $C = C_1 \cup \cdots \cup C_n$, and let G be the graph with vertex set $V(G_1) \cup \cdots \cup V(G_n)$ and with adjacency as follows:

- for $1 \leq i \leq n$, $G|V(G_i) = G_i$;
- for $1 \leq i < j \leq n$, A_i is complete to $V(G_j) \setminus B_j$; B_i is complete to $V(G_j) \setminus C_j$; and C_i is complete to $V(G_j) \setminus A_j$; and
- for $1 \leq i < j \leq n$, if $u \in A_i$ and $v \in B_j$ are adjacent then u, v are both in no triads; and the same applies if $u \in B_i$ and $v \in C_j$, and if $u \in C_i$ and $v \in A_j$.

In particular, A, B, C are cliques, and so (G, A, B, C) is a three-cliqued graph and (G^c, A, B, C) is a 3-colored graph; we call the sequence (G_i, A_i, B_i, C_i) $(i = 1, \ldots, n)$ a *worn hex-chain* for (G, A, B, C). When $n = 2$ we say that (G, A, B, C) is a *worn hex-join* of (G_1, A_1, B_1, C_1) and (G_2, A_2, B_2, C_2). Similarly, the sequence (G_i^c, A_i, B_i, C_i) $(i = 1, \ldots, n)$ is a *worn hex-chain* for (G^c, A, B, C), and when $n = 2$, (G^c, A, B, C) is a *worn hex-join* of (G_1^c, A_1, B_1, C_1) and (G_2^c, A_2, B_2, C_2). Note also that every triad of G is a triad of one of $G_1, \ldots, G_n$. If each G_i is claw-free then so is G and if each G_i^c is prismatic then so is G^c.

If (G, A, B, C) is a three-cliqued graph, and $\{A', B', C'\} = \{A, B, C\}$, then (G, A', B', C') is also a three-cliqued graph, that we say is a *permutation* of (G, A, B, C).

A list of the definitions needed for the study of orientable prismatic graphs can be found in appendix A.1. The structure of prismatic graphs has been extensively studied in [11] and [12]; the resulting two main theorems are the following.

4.6.1. *Every orientable prismatic graph that is not 3-colorable is either not 3-substantial, or a cycle of triangles graph, or a ring of five graph, or a mantled $L(K_{3,3})$.*

4.6.2. *Every 3-colored prismatic graph admits a worn chain decomposition with all terms in $\mathcal{Q}_0 \cup \mathcal{Q}_1 \cup \mathcal{Q}_2$.*

In the remainder of the section, we use these two results to prove our main theorem for complements of orientable prismatic graphs. We begins with some results that deal with the various outcomes of 4.6.1.

4.6.3. *Let H be a prismatic cycle of triangles and G be a reduced thickening of $(\overline{H}, F)$ for some valid $F \subseteq V(H)^2$ such that $\chi(G) > \omega(G)$. Then there exists a Tihany brace or triangle in G.*

Proof. Let the set X_i be as in the definition of a cycle of triangles. Up to renaming the sets, we may assume $|\hat{X}_{2n}| = |\hat{X}_4| = 1$. Let $u \in \hat{X}_{2i}$ and $v \in \hat{X}_4$; hence uv is an edge. We have

$$C_H(\{u,v\}) = \bigcup_{j=1 \bmod 3, j\geq 4} X_j \cup R_1 \cup L_3.$$

If $|\hat{X}_2| > 1$, then $|R_1| = |L_3| = \emptyset$ and so $C_H(\{u,v\})$ is a clique. Therefore by 4.3.6, there is a Tihany brace in G. If $|\hat{X}_2| = 1$, the only non-edges in $\overline{G}|C_H(\{u,v\})$ are a perfect anti-matching between R_1 and L_3. Hence by 4.3.8, there is a Tihany triangle in G. This proves 4.6.3 □

4.6.4. *Let H be a ring of five graph. Let G be a reduced thickening of $(\overline{H}, F)$ for some valid $F \subseteq V(H)^2$ such that $\chi(G) > \omega(G)$. Then there is a Tihany triangle in G.*

Proof. Let a_2, b_3, a_4 be as in the definition of a ring of five. $C(\{a_2, b_3, a_4\}) = V_2 \cup V_4$ and thus $\{a_2, b_3, a_4\}$ is a dense triangle. By the definitions of H and F, it follows that $\{a_2, b_3, a_4\} \cap E = \emptyset$ for all $E \in F$. Hence by 4.3.6, there exists a Tihany triangle in G. This proves 4.6.4. □

4.6.5. *Let H be a mantled $L(K_{3,3})$ and G be a reduced thickening of $(\overline{H}, F)$ for some valid $F \subseteq V(H)^2$. If $\chi(G) > \omega(G)$, then there exists a Tihany brace in G.*

Proof. Assume not. Let W, a^i_j, V^i, V_i be as in the definition of mantled $L(K_{3,3})$. Let X^i_j be the clique corresponding to a^i_j in the thickening and $\mathcal{W}$ (resp. $\mathcal{V}_i$, $\mathcal{V}^i$) be the set of vertices corresponding to W (resp. V_i, V^i) in the thickening. Let $x^j_i \in X^j_i$, $\mathcal{V} = \cup^3_{i=1} \mathcal{V}_i \cup \mathcal{V}^i$ and $k = \chi(G)$.

Recall that for a clique K, we define $A(K) = \{x \in V(G) : ux \notin E(G) \text{ for all } x \in K\}$ and $M(K) = V(G) \backslash (C(K) \cup A(K))$. For a brace E in G, let $M_W(E) := M(E) \cap \mathcal{W}$, $M_V(E) := M(E) \cap \mathcal{V}$, $A_W(E) := A(E) \cap \mathcal{W}$ and $A_V(E) := A(E) \cap \mathcal{V}$. Let $E = \{x^j_i, x^{j'}_{i'}\}$ and let S be a color class in a $(k-2)$-coloring of $G \backslash E$.

(1) *If $S \cap A_V(E) \neq \emptyset$, then $|S| \leq 2$.*

Assume not. Let $S = \{x, y, z\}$ and without loss of generality we may assume that $E = \{x^1_1, x^1_2\}$ and $x \in A_V(E) = \mathcal{V}^1$. Since x is complete to $\mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3$ and X^j_i, for $i = 1,2,3$ $j = 2,3$, we deduce that $y, z \notin \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3$ and $y, z \notin X^j_i$, for $i = 1,2,3$ $j = 2,3$. Since there is no triad in $\mathcal{V}^1 \cup \mathcal{V}^2 \cup \mathcal{V}^3$, it follows that $|\{y,z\} \cap (\mathcal{V}^1 \cup \mathcal{V}^2 \cup \mathcal{V}^3)| \leq 1$. Since $X^1_1 \cup X^1_2 \cup X^1_3$ is a clique, we deduce that $|\{y,z\} \cap (X^1_1 \cup X^1_2 \cup X^1_3)| \leq 1$. Hence, we may assume by symmetry that $y \in X^1_1 \cup X^1_2 \cup X^1_3$ and $z \in \mathcal{V}^2 \cup \mathcal{V}^3$. But $X^1_1 \cup X^1_2 \cup X^1_3$ is complete to $\mathcal{V}^2 \cup \mathcal{V}^3$, a contradiction. This proves (1).

(2) *If* $S \cap M_V(E) \neq \emptyset$, *then* $|S| \leq 2$.

Assume not. Let $S = \{x, y, z\}$ and without loss of generality we may assume that $E = \{x_1^1, x_2^1\}$ and $x \in \mathcal{V}_1$. Since x is complete to $\mathcal{V}^1 \cup \mathcal{V}^2 \cup \mathcal{V}^3$ and $X_2^j \cup X_3^j$, for $j = 1, 2, 3$, we deduce that $y, z \notin \mathcal{V}^1 \cup \mathcal{V}^2 \cup \mathcal{V}^3$ and $y, z \notin X_2^j \cup X_3^j$, for $j = 1, 2, 3$. Since there is no triad in $\mathcal{V}_2 \cup \mathcal{V}_3$, it follows that $|\{y, z\} \cap (\mathcal{V}_2 \cup \mathcal{V}_3)| \leq 1$. As $X_1^1 \cup X_1^2 \cup X_1^3$ is a clique, we deduce that $|\{y, z\} \cap (X_1^1 \cup X_1^2 \cup X_1^3)| \leq 1$. Hence we may assume by symmetry that $y \in \mathcal{V}_2 \cup \mathcal{V}_3$ and $z \in X_1^1 \cup X_1^2 \cup X_1^3$. But $\mathcal{V}_2 \cup \mathcal{V}_3$ is complete to $X_1^1 \cup X_1^2 \cup X_1^3$, a contradiction. This proves (2).

By 4.3.1, every color class of a $(k-2)$-coloring of $G \backslash E$ must have a vertex in $C(E)$. By (1) and (2), it follows that color classes with vertices in $A_V(E) \cup M_V(E)$ have size 2. Hence we deduce that $|A_V(E) + |M_V(E)| + \frac{1}{2}|A_W(E)| + \frac{1}{2}|M_W(E)| \leq |C(E)| - 2$. Summing this inequality on all braces $E = \{x_i^j, x_{i'}^{j'}\}$ $i, j = 1, 2, 3$, it follows that

$$3\sum_i (|\mathcal{V}_i| + |\mathcal{V}^i|) + 6\sum_i (|\mathcal{V}_i| + |\mathcal{V}^i|) + \frac{4}{2}\sum_{i,j}|X_i^j| + \frac{8}{2}\sum_{i,j}|X_i^j| < 9\sum_i (|\mathcal{V}_i| + |\mathcal{V}^i|) + 6\sum_{i,j}|X_i^j|,$$

which is a contradiction. This proves 4.6.5. □

4.6.6. *Let* $(H, H_1, H_2, H_3)^c$ *be a path of triangle s and* (I, I_1, I_2, I_3) *an antiprismatic three-cliqued graph. Let* G *be a worn hex-join of* (H, H_1, H_2, H_3) *and* (I, I_1, I_2, I_3), *and* G' *be a reduced thickening of* (G, F) *for some valid* $F \in V(G)^2$ *such that* $\chi(G') > \omega(G')$. *Then there exists a Tihany clique* K *in* G', *with* $|K| \leq 4$.

Proof. Assume not. Let the set X_j of H be as in the definition of a path of triangles and we may assume that $H_i = \cup_{j=i \mod 3} X_j$.

Assume first that $|\hat{X}_{2i}| > 1$ for some i. Let $u \in X_{2i-2}$ and $v \in X_{2i+2}$, so uv is an edge in G. Moreover $\{u, v\}$ is in the strong core. Thus

$$C_G(\{u, v\}) = \bigcup_{\substack{j = 2i+2 \mod 3, \\ j \geq 2i+2}} X_j \cup \bigcup_{\substack{j = 2i-2 \mod 3, \\ j \leq 2i-2}} X_j \cup I_k$$

for $k = 2i + 1 \mod 3$. Hence $C_G(\{u, v\})$ is a clique and so by 4.3.6, there is a Tihany brace in G', a contradiction. Hence we may assume that $|\hat{X}_{2i}| = 1$ for all i.

Assume that $n \geq 3$ and let $u \in \hat{X}_2, v \in \hat{X}_6$. Then uv is an edge in G. Moreover $\{u, v\}$ is in the strong core. Thus

$$C_G(\{u,v\}) = \bigcup_{j=0 \bmod 3, j\geq 6} X_j \cup X_2 \cup R_3 \cup L_5 \cup H_3.$$

Hence $C_G(\{u, v\})$ is an antimatching, and by 4.3.8, there exists a Tihany triangle in G', a contradiction. It follows that $n \leq 2$.

Assume now that $n = 2$. Let $u \in \hat{X}_2, v \in L_5$. Then uv is an edge in G and $C_G(\{u, v\}) = X_2 \cup R_3 \cup L_5 \cup H_3$. Thus $G|C(\{u, v\})$ is a perfect anti-matching between R_3 and L_5. Hence by 4.3.8, there is a Tihany triangle in G', a contradiction.

Thus we deduce that $n = 1$. Assume that $|R_1| = |L_3| = 1$. Let $u \in X_2$ and $v \in R_1 \cup L_3$ be a neighbor of v. Without loss of generality, we may assume that $v \in L_3$. Since $C_G(\{u, v\}) \subseteq X_2 \cup L_3 \cup H_3$ is a clique, it follows by 4.3.6 that there is a Tihany brace in G', a contradiction. Hence we deduce that $|R_1| = |L_3| > 1$. Now, let $u \in R_1$ and $v \in L_3$ be adjacent. By 4.5.6, we may assume that G is not a 2-non-substantial graph. If follows that there exists $x \in I_2$ such that x is in a triad. Thus $C_G(\{u, v, x\})$ is an antimatching, and by 4.3.8, there exists a Tihany clique K in G' with $|K| \leq 4$, a contradiction. This proves 4.6.6. □

4.6.7. *Let (G, A, B, C) be an antiprismatic graph that admit a worn chain decomposition (G_i, A_i, B_i, C_i). Suppose that there exists k such that (G_k, A_k, B_k, C_k) is the line graph of $K_{3,3}$. Let G' be a reduced thickening of (G, F) for some valid $F \in V(G)^2$. If $\chi(G') > \omega(G')$, then there is a Tihany brace in G'.*

Proof. Assume not. Let $\{a^i_j\}_{i,j=1,2,3}$ be the vertices of G_k using the standard notation. Let $X^i_j = X_{a^i_j}$ be the clique corresponding to a^i_j in the thickening. Moreover, let $x^j_i \in X^j_i$, $w^j_i = |X^j_i|$.

Since all of the vertices in the thickening of G_k are in triads, G_k is linked to the rest of the graph by a hex-join.

Note that $G \backslash \{x^1_1, x^1_2\}$ is $(\chi(G)-2)$-colorable. By 4.3.1, it follows that every color class containing a vertex in $X^2_1 \cup X^3_1$ must have a vertex in $X^1_2 \cup X^1_3$. Hence we deduce that $w^2_1 + w^3_1 \leq w^1_2 + w^1_3 - 1$ and by symmetry $w^2_2 + w^3_2 \leq w^1_1 + w^1_3 - 1$. Summing these two inequalities, it follows that

$$w^2_1 + w^3_1 + w^2_2 + w^3_2 < w^1_2 + w^1_1 + 2w^1_3.$$

A similar inequality can be obtained for all edges $x_i^j x_{i'}^j$. Summing them all, we deduce that $4\sum_{ij} w_i^j < 2\sum_{ij} w_i^j + 2\sum_{ij} w_i^j$, a contradiction. This proves 4.6.7 □

4.6.8. *Let H be a 3-colored prismatic graph. Let G be a reduced thickening of $(\overline{H}, F)$ for some valid $F \subseteq V(H)^2$ such that $\chi(G) > \omega(G)$. Then there exists a Tihany brace or triangle in G.*

Proof. By 4.6.2, H admits a worn chain decomposition with all terms in $\mathcal{Q}_0 \cup \mathcal{Q}_1 \cup \mathcal{Q}_2$. If one term of the decomposition is in $\mathcal{Q}_2$ then by 4.6.6, it follows that there is a Tihany clique K in G with $|K| \leq 4$ G. If one term of the decomposition is in $\mathcal{Q}_1$, then by 4.6.7, it follows that there is a Tihany brace in G. Hence we may assume that all terms are in $\mathcal{Q}_0$. Therefore there are no triads in G and thus by 4.5.1, it follows that there is a Tihany brace in G. This proves 4.6.8. □

We can now prove the main result of this section.

4.6.9. *Let H be an orientable prismatic graph. Let G be a reduced thickening of $(\overline{H}, F)$ for some valid $F \subseteq V(H)^2$ such that $\chi(G) > \omega(G)$. Then there exists a Tihany clique K in G with $|K| \leq 4$.*

Proof. If H admits a worn chain decomposition with all terms in $\mathcal{Q}_0 \cup \mathcal{Q}_1 \cup \mathcal{Q}_2$, then by 4.6.8, G admits a Tihany brace or triangle. Otherwise, by 4.6.1, H is either not 3-substantial, a cycle of triangles, a ring of five graph, or a mantled $L(K_{3,3})$.

If H is not 3-substantial, then by 4.5.7, there is a clique K in G with $|K| \leq 4$. If H is a cycle of triangles, then by 4.6.3, there is a Tihany brace or triangle in G. If H is a ring of five graph, then by 4.6.4, there is a Tihany triangle in G. Finally, if H is a mantled $L(K_{3,3})$, then by 4.6.5, there is a Tihany brace in G. This proves 4.6.9. □

4.7 Non-orientable Prismatic Graphs

The definitions needed to understand this section can be found in appendix A.2. The following is a result from [12].

4.7.1. *Let G be prismatic. Then G is orientable if and only if no induced subgraph of G is a twister or rotator.*

In the following two lemmas, we study complements of orientable prismatic graphs. We split our analysis based on whether the graph contains a twister or a rotator as an induced subgraph.

4.7.2. *Let H be an non-orientable prismatic graph. Assume that there exists $D \subseteq V(H)$ such that $G|D$ is a rotator. Let G be a reduced thickening of $(\overline{H}, F)$ such that $\chi(G) > \omega(G)$ for some valid $F \subseteq V(H)^2$. Then there exists a Tihany clique K in G with $|K| \leq 5$.*

Proof. Assume not. Let $D = \{v_1, \dots, v_9\}$ be as in the definition of a rotator. For $i = 1, 2, 3$, let A_i be the set of vertices of $V(H)\backslash D$ that are adjacent to v_i. Since H is prismatic and $\{v_1, v_2, v_3\}$ is a triangle, it follows that $A_1 \cup A_2 \cup A_3 = V(H)\backslash D$.

Let $I_1 = \{\{5,6\},\{5,9\},\{6,8\},\{8,9\}\}$, $I_2 = \{\{4,6\},\{4,9\},\{6,7\},\{7,9\}\}$ and $I_3 = \{\{4,5\},\{4,8\},\{5,7\},\{7,8\}\}$. For $i = 1, 2, 3$ and $\{k, l\} \in I_i$, let $A_i^{k,l}$ be the set of vertices of $V(H)\backslash D$ that are complete to $\{v_i, v_k, v_l\}$. Since $\{v_1, v_2, v_3\}$ and $\{v_i, v_{i+3}, v_{i+6}\}$ are triangles for $i = 1, 2, 3$ and H is prismatic, we deduce that $A_i = \bigcup_{\{k,l\} \in I_i} A_i^{k,l}$ for $i = 1, 2, 3$. For $i = 1, 2, 3$ and $\{k, l\} \in I_i$ and since $\{v_1, v_4, v_7\}, \{v_2, v_5, v_8\}, \{v_3, v_6, v_9\}$ are triangles and H is prismatic, it follows that $A_i^{k,l}$ is anticomplete to v_m for all $m \in \{4,5,6,7,8,9\}\backslash\{i, k, l\}$.

Assume that $A_2^{4,9}$ and $A_3^{4,8}$ are not empty. Since H is prismatic, we deduce that $A_2^{4,9}$ is anticomplete to $A_3^{4,8}$ in H. Let $x \in A_2^{4,9}$ and $y \in A_3^{4,8}$. Then $C_{\overline{H}}(\{v_1, v_5, v_6, x, y\})$ is a clique and $\{v_1, v_5, v_6, x, y\}$ is in the strong core. Hence by 4.3.6, there exists a Tihany clique of size 5 in G.

Assume now that $A_2^{4,9}$ is not empty, but $A_3^{4,8}$ is empty. Let $x \in A_2^{4,9}$. Then $C_{\overline{H}}(\{v_1, v_5, v_6, x\})$ is a clique and $\{v_1, v_5, v_6, x\}$ is in the core. Moreover $\{v_1, v_6, x\}$ is in the strong core. Since $\{v_2, v_5, v_8\}$ is a triad and v_2 is in the strong core, it follows that if there exists $E \in F$ with $v_5 \in E$, then $E = \{v_5, v_8\}$. But v_8 is not adjacent to v_6 in $\overline{H}$. Hence by 4.3.6, there exists a Tihany clique K of size 4 in G.

We may now assume that $A_2^{4,9} = A_3^{4,8} = \emptyset$. Since H is prismatic, it follows that $C_{\overline{H}}(\{v_1, v_5, v_6\})$ is an anti-matching. Moreover $\{v_1, v_5, v_6\}$ is in the core and v_1 is in the strong core. For $i = 2, 3$, since $\{v_i, v_{i+3}, v_{i+6}\}$ is a triad and v_i is in the strong core, it follows that if there exists $E \in F$ with $v_{i+3} \in E$, then $E = \{v_{i+3}, v_{i+6}\}$. But v_8 is not adjacent to v_6 and v_9 is not adjacent to v_5. Hence by 4.3.6, there exists a Tihany triangle in G. This concludes the proof of 4.7.2. □

4.7.3. *Let H be a non-orientable prismatic graph. Assume that there exists $W \subseteq V(H)$ such that $H|W$ is a twister. Further, assume that there is no induced rotator in H. If G is a reduced thickening of $(\overline{H}, F)$ such that $\chi(G) > \omega(G)$, then there exists a Tihany clique K in G with $|K| \leq 4$.*

Proof. Assume not. Let $W = \{v_1, v_2, \dots, v_8, u_1, u_2\}$ be as in the definition of a twister. Throughout the proof, all addition is modulo 8. For $i = 1, \dots, 8$, let $A_{i,i+1}$ be the set of vertices in $V \backslash W$ that are adjacent to v_i and v_{i+1} and let $B_{i,i+2}$ be the set of vertices in $V \backslash W$ that are adjacent to v_i and v_{i+2}. Moreover, let $C \subseteq V \backslash W$ be the set of vertices that are anticomplete to W. Since H is prismatic, we deduce that $\bigcup_{i=1}^{8} (A_{i,i+1} \cup B_{i,i+2}) \cup C = V \backslash W$. Moreover $A_{i,i+1}$ is complete to $\{v_i, v_{i+1}, v_{i+3}, v_{i+6}\}$ and anticomplete to $W \backslash \{v_i, v_{i+1}, v_{i+3}, v_{i+6}\}$. Since H is prismatic, it follows also that $B_{i,i+2}$ is complete to $u_{i \mod 2}\}$ and anticomplete to $W \backslash \{v_i, v_{i+2}, u_{i \mod 2}\}$. Moreover, C is anticomplete to $\{v_1, v_2, \dots, v_8\}$.

(1) *There exists* $i \in \{1, \dots, 8\}$*, such that* $A_{i,i+1}$ *and* $A_{i+3,j+4}$ *are either both empty or both non-empty.*

Assume not. By symmetry we may assume that $A_{1,2}$ is not empty and $A_{4,5}$ is empty. Since $A_{1,2}$ is not empty, we deduce that $A_{6,7}$ is empty. Since $A_{4,5}$ and $A_{6,7}$ are empty, it follows that $A_{7,8}$ and $A_{3,4}$ are not empty. Let $x \in A_{7,8}$ and $y \in A_{3,4}$. Then $G|\{v_8, u_1, v_4, x, v_6, v_3, v_7, v_2, y\}$ is a rotator, a contradiction. This proves (1).

(2) *If* $A_{i,i+1}$ *and* $A_{i+3,i+4}$ *are both non-empty for some* $i \in \{1, \dots, 8\}$*, then there exists a Tihany clique of size* 5 *in* G*.*

Assume that $A_{2,3}$ and $A_{5,6}$ are not empty and let $x \in A_{2,3}$ and $y \in A_{5,6}$. The anti-neighborhood of $\{v_1, v_7, u_2, x, y\}$ in H is a stable set. Moreover, $\{v_1, v_7, u_2, x, y\}$ is in the strong core and hence by 4.3.6 there is a Tihany clique of size 5 in G. This proves (2).

(3) *If* $A_{i,i+1}$ *and* $A_{i+3,i+4}$ *are both empty for some* $i \in \{1, \dots, 8\}$*, then there exists a Tihany clique of size* 4 *in* G*.*

Assume that $A_{2,3}$ and $A_{5,6}$ are both empty. Then the anti-neighborhood of $\{v_1, v_7, u_2\}$ in H is $A_{8,2} \cup A_{2,4} \cup A_{4,6} \cup A_{6,8}$ which is a matching. Moreover u_2 is in the strong core and $\{v_1, v_7\}$ is in the core. Possibly $\{v_1, v_5\}$ and $\{v_3, v_7\}$ are in F, but $A_{2,8} \cup A_{2,4} \cup A_{4,6} \cup A_{6,8} \cup \{v_3, v_7\}$ is also an anti-matching. Hence by 4.3.8, there is a Tihany clique of size 4 in G. This proves (3).

Now by (1), there exists i such that $A_{i,i+1}$ and $A_{i+3,i+4}$ are either both empty or both non-empty. If $A_{i,i+1}$ and $A_{i+3,i+4}$ are both non-empty, then by (2) there is a Tihany clique of size 5 in G. If $A_{i,i+1}$ and $A_{i+3,i+4}$ are both empty, then by (3) there is a Tihany clique of size 4 in G. This concludes the proof of 4.7.3. □

4.7.4. *Let H be a non-orientable prismatic graph. Let G be a reduced thickening of $(\overline{H}, F)$ for some valid $F \subseteq V(G)^2$ such that $\chi(G) > \omega(G)$; then there exists a Tihany clique K in G with $K \leq 5$.*

Proof. By 4.7.1, it follows that there is an induced twister or an induced rotator in H. If there is an induced rotator in H, then by 4.7.2, it follows that there is a Tihany clique of size 5 in G. If there is an induced twister and no induced rotator in H, then by 4.7.3, it follows that there is a Tihany clique of size 4 in G. This proves 4.7.4. □

4.8 Three-cliqued Graphs

In this section we prove 4.1.1 for those claw-free graphs G for which $V(G)$ can be partitioned into three cliques. The definition of three-cliqued graphs has been given at the start of Section 4.6. A list of three-cliqued claw-free graphs that are needed for the statement of the structure theorem can be found in appendix A.3. We begin with a structure theorem from [13].

4.8.1. *Every three-cliqued claw-free graph admits a worn hex-chain into terms each of which is a reduced thickening of a permutation of a member of one of $\mathcal{TC}_1, \ldots, \mathcal{TC}_5$.*

Let (G, A, B, C) be a three-cliqued graph and K be a clique of G. We say that K is *strongly Tihany* if for all three-cliqued graphs (H, A', B', C'), K is Tihany in every worn hex-join $(I, A \cup A', B \cup B', C \cup C')$ of (G, A, B, C) and (H, A', B', C') such that $\chi(I) > \omega(I)$.

A clique K is said to be *bi-cliqued* if exactly two of $K \cap A, K \cap B, K \cap C$ are non-empty and every $v \in K$ is in a triad. A clique K is said to be *tri-cliqued* if $K \cap A, K \cap B, K \cap C$ are all non-empty and every $v \in K$ is in a triad.

4.8.2. *Let K be a dense clique in (G, A_1, A_2, A_3). If both K and $\overline{C}(K)$ are bi-cliqued, then K is strongly Tihany.*

Proof. Let (G', A', B', C') be a three-cliqued claw-free graph and let (H, D, E, F) be a worn hex-join of (G, A, B, C) and (G', A', B', C'). Then in H, $C(K) \cap V(G')$ is a clique that is complete to $C(K) \cap V(G)$. Hence, by 4.3.2, K is Tihany in H and hence H is strongly Tihany. □

4.8.3. *Let K be a dense clique of a three-cliqued graph (G, A, B, C). If K is tri-cliqued, then K is strongly Tihany.*

Proof. Let (G', A', B', C') be a three-cliqued claw-free graph and let (H, D, E, F) be a hex-join of (G, A, B, C) and (G', A', B', C'). Then in H, $C_H(K) \cap V(G') = \emptyset$ and thus $C_H(K)$ is a clique in H. Hence, by 4.3.2, K is strongly Tihany. □

4.8.4. *Let (G, A, B, C) be an element of $\mathcal{TC}_1$ and G' be a reduced thickening of (G, F) for some valid $F \subseteq V(G)^2$. Then there is either a strongly Tihany brace or a strongly Tihany triangle in G'.*

Proof. Let H, v_1, v_2, v_3 be as in the definition of $\mathcal{TC}_1$; so $L(H) = G$. Let V_{12} be the set of vertices of H that are adjacent to v_1 and v_2 but not to v_3 and let V_{13}, V_{23} be defined similarly. Let V_{123} be the set of vertices complete to $\{v_1, v_2, v_3\}$.

Suppose that $V_{ij} \neq \emptyset$ for some i, j. Then let $v_{ij} \in V_{ij}$, and let x_i be the vertex in G corresponding to the edge $v_{ij}v_i$ in H and x_j be the vertex in G corresponding to the edge $v_{ij}v_j$ in H. Then $C_G(\{x_i, x_j\}) = \emptyset$, and thus by 4.3.5 and 4.8.2, there exists a strongly Tihany brace in G'.

So we may assume that $V_{ij} = \emptyset$ for all i, j. Then from the definition of $\mathcal{TC}_1$, it follows that V_{123} is not empty. Let $v \in V_{123}$ and let x_1, x_2, x_3 be the vertices in G corresponding to the edges vv_1, vv_2, vv_3 of H, respectively. Then $C_G(\{x_1, x_2, x_3\}) = \emptyset$ and hence by 4.3.5 and 4.8.3, there exists a strongly Tihany triangle in G'. This proves 4.8.4. □

4.8.5. *Let (G, A, B, C) be an element of $\mathcal{TC}_2$ and let (G', A', B', C') be a reduced thickening of (G, F) for some valid $F \subseteq V(G)^2$. Then there is either a strongly Tihany brace or a strongly Tihany triangle in G'.*

Proof. Let $\Sigma, F_1, \ldots, F_k, L_1, L_2, L_3$ be as in the definition of $\mathcal{TC}_2$. Without loss of generality, we may assume that A is not anticomplete to B. It follows from the definition of G that there exists F_i such that $F_i \cap A$ and $F_i \cap B$ are both not empty. Let $\{x_k, \ldots, x_l\} = V(H) \cap F_i$ and without loss of generality, we may assume that $\{x_k, \ldots, x_l\}$ are in order on Σ.

Let F_i be such that there exists no j with $F_i \subset F_j$. Let $\{x_k, \ldots, x_l\} = V(H) \cap F_i$ and without loss of generality, we may assume that $\{x_k, \ldots, x_l\}$ are in order on Σ. Since $C(\{x_k, x_l\}) = \{x_{k+1}, \ldots, x_{l-1}\}$, it follows that $\{x_k, x_l\}$ is dense. If x_k, x_l are the endpoints of F_i, it follows by 4.3.1 and 4.3.5 that there is a Tihany brace in G. Otherwise, by 4.3.6 there exists a Tihany brace in G. This proves 4.4.2. □

4.8.6. *Let (G, A, B, C) be an element of $\mathcal{TC}_3$ and let (G', A', B', C') be a reduced thickening of (G, F) for some valid $F \in V(G)^2$. Then there is either a strongly Tihany brace or a strongly Tihany triangle in G'.*

Proof. Let $H, A = \{a_0, a_1, \ldots, a_n\}, B = \{b_0, b_1, \ldots, b_n\}$, $C = \{c_1, \ldots, c_n\}$, and X be as in the definition of near-antiprismatic graphs. Suppose that for some i, $a_i, b_i \in V(G)$. Then since $|C \setminus X| \geq 2$, it follows that there exists $j \neq i$ such that $c_j \in V(G)$. Now $T = \{a_i, b_i, c_j\}$ is dense and tri-cliqued in G, and so by 4.3.5 and 4.8.3 there is a strongly Tihany triangle in G'.

So we may assume that for all i, if $a_i \in V(G)$, then $b_i \notin V(G)$. Since by definition of $\mathcal{TC}_3$ every vertex is in a triad, it follows that $c_i \in V(G)$ whenever $a_i \in V(G)$. Now suppose that $a_i, a_j \in V(G)$ for some $i \neq j$. Then $(\{a_i, a_j\}, \{c_i, c_j\})$ is a non-reduced homogeneous pair in G. Hence we may assume that for all $i \neq j$ at most one of a_i, a_j is in $V(G)$. Let $a_i \in V(G)$,; then for some $j \neq i$ we have $c_j \in V(G)$. Now $E = \{a_i, c_j\}$ is dense and bi-cliqued. Moreover $\overline{C}(E)$ is bi-cliqued, hence by 4.3.5 and 4.8.2, it follows that E is a strongly Tihany brace in G'. This proves 4.8.6. □

4.8.7. *Let G be an element of $\mathcal{TC}_5$ and G' be a reduced thickening of (G, F) for some valid $F \subseteq V(G)^2$. Then there exists either a brace $E \in V(G')$ that is strongly Tihany or a triangle $T \in V(G')$ that is strongly Tihany in G'.*

Proof. First suppose that $G \in \mathcal{TC}_5^1$. Let $H, \{v_1, \ldots, v_8\}$ be as in the definition of $\mathcal{TC}_5^1$. If $v_4 \in V(G)$ then $\{v_2, v_4\}$ is dense and bi-cliqued. Moreover $\overline{C}(\{v_2, v_4\})$ is bi-cliqued and thus by 4.3.5 and 4.8.2, there is a strongly Tihany brace in G'. If $v_3 \in G$, then $\{v_3, v_5\}$ is dense and bi-cliqued. Moreover $\overline{C}(\{v_3, v_5\})$ is bi-cliqued and so by 4.3.5 and 4.8.2, there is a strongly Tihany brace in G'. So we may assume that $v_4, v_3 \notin V(G)$. But then the triangle $T = \{v_1, v_6, v_7\}$ is dense and tri-cliqued and thus by 4.3.5 and 4.8.3, there exists a strongly Tihany triangle in G'.

We may assume now that $G \in \mathcal{TC}_5^2$. If $v_3 \in G$ then $\{v_2, v_3\}$ is dense, bi-cliqued and $\overline{C}(\{v_2, v_3\})$ is bi-cliqued. Otherwise, $\{v_2, v_4\}$ is dense, bi-cliqued and $\overline{C}(\{v_2, v_4\})$ is bi-cliqued. In both cases, it follows from 4.3.5 and 4.8.2 that there exists a strongly Tihany brace in G'. This proves 4.8.7. □

We are now ready to prove the main result of this section.

4.8.8. *Let G be a three-cliqued claw-free graph such that $\chi(G) > \omega(G)$. Then G contains either a Tihany brace or a Tihany triangle in G.*

Proof. By 4.8.1, there exist (G_i, A_i, B_i, C_i), for $i = 1, \dots, n$, such that the sequence (G_i, A_i, B_i, C_i) $(i = 1, \dots, n)$ is a worn hex-chain for (G, A, B, C) and such that (G_i, A_i, B_i, C_i) is a reduced thickening of a permutation of a member of one of $\mathcal{TC}_1, \dots, \mathcal{TC}_5$. If there exists $i \in \{1, \dots, n\}$ such that (G_i, A_i, B_i, C_i) is a reduced thickening of a permutation of a member of $\mathcal{TC}_1$, $\mathcal{TC}_2$, $\mathcal{TC}_3$, or $\mathcal{TC}_5$, then by 4.8.4, 4.8.5, 4.8.6, or 4.8.7 (respectively), there is a strongly Tihany brace or a strongly Tihany triangle in G_i, and thus there is a Tihany brace or a Tihany triangle in G. Thus it follows that (G_i, A_i, B_i, C_i) is a reduced thickening of a member of $\mathcal{TC}_4$ for all $i = 1, \dots, n$. Hence G is a reduced thickening of a three-cliqued antiprismatic graph. By 4.6.8, there exists a Tihany brace or triangle in G. This proves 4.8.8. □

4.9 Non-trivial Strip Structures

In this section we prove 4.1.1 for graphs G that admit non-trivial strip structures and appear in [13].

Let (J, Z) be a strip. We say that (J, Z) is a *line graph strip* if $|V(J)| = 3$, $|Z| = 2$ and Z is complete to $V(J) \setminus Z$.

The following two lemmas appear in [4].

4.9.1. *Suppose that G admits a nontrivial strip-structure such that $|Z| = 1$ for some strip (J, Z) of (H, η). Then either G is a clique or G admits a clique cutset.*

4.9.2. *Let G be a graph that admits a nontrivial strip-structure (H, η) such that for every $F \in E(H)$, the strip of (H, η) at F is a line graph strip. Then G is a line graph.*

We now use these lemmas to prove the main result of this section.

4.9.3. *Let G be a claw-free graph with $\chi(G) > \omega(G)$ that is a minimal counterexample to 4.1.1. Then G does not admit a nontrivial strip-structure (H, η) such that for each strip (J, Z) of (H, η), $1 \le |Z| \le 2$, and if $|Z| = 2$ then either $|V(J)| = 3$ and Z is complete to $V(J) \setminus Z$, or (J, Z) is a member of $\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}_3 \cup \mathcal{Z}_4 \cup \mathcal{Z}_5$.*

Proof. Suppose that G admits a nontrivial strip-structure (H, η) such that for each strip (J, Z) of (H, η), $1 \le |Z| \le 2$. Further suppose that $|Z| = 1$ for some strip (J, Z). Then by 4.9.1 either G is a clique or G admits a clique cutset; in the former case G does not satisfy $\chi(G) > \omega(G)$, and in the latter case 4.9.3 follows from 4.3.10. Hence we may assume that $|Z| = 2$ for all strips (J, Z).

If all the strips of (H, η) are line graph strips, then by 4.9.2, G is a line graph and the result follows from [2]. So we may assume that some strip (J_1, Z_1) is not a line graph strip. Let $Z_1 = \{a_1, b_1\}$. Let $A_1 = N_{J_1}(a_1)$, $B_1 = N_{J_1}(b_1)$, $A_2 = N_G(A_1) \setminus V(J_1)$, and $B_2 = N_G(B_1) \setminus V(J_1)$. Let $C_1 = V(J_1) \setminus (A_1 \cup B_1)$ and $C_2 = V(G) \setminus (V(J_1) \cup A_2 \cup B_2)$. Then $V(G) = A_1 \cup B_1 \cup C_1 \cup A_2 \cup B_2 \cup C_2$.

(1) *If $C_2 = \emptyset$ and $A_2 = B_2$, then there is a Tihany clique K in G with $|K| \leq 5$.*

Note that $V(G) = A_1 \cup B_1 \cup C_1 \cup A_2$. Since $|Z_1| = 2$ and (J_1, Z_1) is not a line graph strip, it follows that (J_1, Z_1) is a member of $\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}_3 \cup \mathcal{Z}_4 \cup \mathcal{Z}_5$. We consider the cases separately:

1. If (J_1, Z_1) is a member of $\mathcal{Z}_1$, then J_1 is a fuzzy linear interval graph and so G is a fuzzy long circular interval graph and 4.9.3 follows from [2].

2. If (J_1, Z_1) is a member of $\mathcal{Z}_2, \mathcal{Z}_3$, or $\mathcal{Z}_4$. In all of these cases, A_1, B_1, and C_1 are all cliques and so $V(G)$ is the union of three cliques, namely $A_1 \cup A_2, B_1$, and C_1. Hence, by 4.8.8, there exists a Tihany clique K with $|K| \leq 5$.

3. If (J_1, Z_1) is a member of $\mathcal{Z}_5$. Let $v_1, \ldots, v_{12}, X, H, H', F$ be as in the definition of $\mathcal{Z}_5$ and for $1 \leq i \leq 12$ let X_{v_i} be as in the definition of a thickening. Then A_2 is complete to $X_{v_1} \cup X_{v_2} \cup X_{v_4} \cup X_{v_5}$. Let H'' be the graph obtained from H' by adding a new vertex a_2, adjacent to v_1, v_2, v_4 and v_5. Then H'' is an antiprismatic graph. Moreover, no triad of H'' contains v_9 or v_{10}. Thus the pair (H', F) is antiprismatic, and G is a thickening of (H', F), so 4.9.3 follows from 4.6.9 and 4.7.4.

This proves (1).

By (1), we may assume $C_2 \neq \emptyset$ or $A_2 \neq B_2$. Suppose that $A_2 = B_2$. Then since $C_2 \neq \emptyset$ it follows that A_2 is a clique cutset of G and the result follows from 4.3.10. Hence, we may assume that $A_2 \neq B_2$ and without loss of generality we may assume that $A_2 \setminus B_2 \neq \emptyset$. Let $v \in A_2 \setminus B_2$ and let $w \in A_1 \setminus B_1$. Then $E = \{v, w\}$ is dense and 4.9.3 follows from 4.3.2. □

4.10 Proof of the Main Theorem

We can now prove the main theorem.

Proof of 4.1.1. Let G be a claw-free graph with $\chi(G) > \omega(G)$, and suppose that there does not exist a clique K in G with $|K| \leq 5$ such that $\chi(G \backslash K) > \chi(G) - |K|$. By 4.9.3 and 4.2.1, it follows that either G is a member of $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$ or $V(G)$ is the union of three cliques. By 4.4.1, it follows that G is not a member of $\mathcal{T}_1$. By 4.4.2, it follows that G is not a member of $\mathcal{T}_2$. By 4.6.9 and 4.7.4, we deduce that G is not a member $\mathcal{T}_3$. Hence, it follows that $V(G)$ is the union of three cliques. But by 4.8.8, it follows that there is a Tihany brace or triangle in G, a contradiction. This proves 4.1.1. □

Chapter 5

A Local Strengthening of Reed's Conjecture

5.1 Introduction

The chromatic number is a notion of utmost importance in graph theory. Finding its exact value for a graph is a central problem both from a theoretical and algorithmic point of view. For general graphs, there is a trivial lower and upper bound on the chromatic number that we present now. We include the proof for completeness.

5.1.1. *Let G be a graph. Then $\omega(G) \leq \chi(G) \leq \Delta(G) + 1$.*

Proof. Let K be a clique of size $\omega(G)$. No two vertices of K can have the same color, hence we need at least $\omega(G)$ colors for the vertices of K. It follows that $\chi(G) \geq \omega(G)$.

For the upper bound we will use induction on the number of vertices in G. Clearly if G has one vertex, then $\chi(G) = 1 \leq \Delta(G) + 1$. Now let G be such that $|V(G)| = n$ and assume that for all graph H with $|V(H)| < n$ then $\chi(H) \leq \Delta(H) + 1$. Let $x \in V(G)$. Now $G \backslash x$ has $n - 1$ vertices and so $\chi(G \backslash x) \leq \Delta(G \backslash x) + 1 \leq \Delta(G) + 1$. But $N(x)$ uses at most $\Delta(G)$ colors since $|N(x)| = d(x) \leq \Delta(G)$. Therefore there is at least one color free to extend the coloring of $G \backslash x$ to a coloring of G using at most $\Delta(G) + 1$ colors. This proves 5.1.1. □

In 1998, Reed made the following conjecture.

Conjecture 2 (Reed)**.** *For any graph G,*

$$\chi(G) \leq \left\lceil \tfrac{1}{2}(\Delta(G) + 1 + \omega(G)) \right\rceil.$$

Conjecture 2 has been proved first for line graphs [23] and was then extended to quasi-line graphs [21; 22] and later claw-free graphs [21]. Later, King proposed a local strengthening of Reed's Conjecture.

Conjecture 3 (King)**.** *For any graph G,*

$$\chi(G) \leq \max_{v \in V(G)} \left\lceil \tfrac{1}{2}(d(v) + 1 + \omega(v)) \right\rceil.$$

There are several pieces of evidence that lend credence to Conjecture 3. First is the fact that the result holds for claw-free graphs with stability number at most three [21]. However, for the remaining classes of claw-free graphs, which are constructed as a generalization of line graphs [10], the conjecture has remained open.

The second piece of evidence for Conjecture 3 is that the fractional relaxation holds. The *fractional chromatic number* $\chi_f(G)$ is the optimal value of the following linear program, which is the linear relaxation of the standard integer program formulation of the graph coloring problem.

$$\begin{array}{rcll} \chi_f(G) = & \min & \sum_{S} x_S & \\ & \text{subject to} & \sum_{S \ni v} x_S \geq 1 & \forall\, v \in V(G) \\ & & x_S \in [0,1] & \forall \text{ stable set } S \end{array}$$

It was noted by McDiarmid as an extension of a theorem of Reed [27] that the following holds.

5.1.2 (McDiarmid)**.** *For any graph G,*

$$\chi_f(G) \leq \max_{v \in V(G)} \left(\tfrac{1}{2}(d(v) + 1 + \omega(v)) \right).$$

The main result of this chapter is:

5.1.3. *For any quasi-line graph G,*

$$\chi(G) \leq \max_{v \in V(G)} \left\lceil \tfrac{1}{2}(d(v) + 1 + \omega(v)) \right\rceil.$$

This chapter is organized as follow. In Section 5.2, we prove Conjecture 3 for line graphs. In Section 5.3, we introduce quasi-line graphs and some important concepts. In Section 5.4, we study how quasi-line graphs can be decomposed into smaller pieces that are well understood, and finally in Section 5.5 we put the different pieces together to prove 5.1.3 and discuss some algorithmic notions.

5.2 Line Graphs

In order to prove Conjecture 3 for line graphs, we prove an equivalent statement in the setting of edge colorings of multigraphs. Given distinct adjacent vertices u and v in a multigraph G, we let $\mu_G(uv)$ denote the number of edges between u and v. We let $t_G(uv)$ denote the maximum, over all vertices $w \notin \{u, v\}$, of the number of edges with both endpoints in $\{u, v, w\}$. That is,

$$t_G(uv) := \max_{w \in N(u) \cap N(v)} \left(\mu_G(uv) + \mu_G(uw) + \mu_G(vw)\right).$$

We omit the subscripts when the multigraph in question is clear.

Observe that given an edge e in G with endpoints u and v, the degree of uv in $L(G)$ is $d(u) + d(v) - \mu(uv) - 1$. And since any clique in $L(G)$ containing e comes from the edges incident to u, the edges incident to v, or the edges in a triangle containing u and v, we can see that $\omega(v)$ in $L(G)$ is equal to $\max\{d(u), d(v), t(uv)\}$. Therefore we prove the following theorem, which, aside from the algorithmic claim, is equivalent to proving Conjecture 3 for line graphs:

5.2.1. *Let G be a multigraph on m edges, and let*

$$\gamma_l'(G) := \max_{uv \in E(G)} \Big\lceil \max \Big\{ d(u) + \tfrac{1}{2}(d(v) - \mu(vu)),\ d(v) + \tfrac{1}{2}(d(u) - \mu(uv)), \\ \tfrac{1}{2}(d(u) + d(v) - \mu_G(uv) + t(uv)) \Big\} \Big\rceil. \quad (5.1)$$

Then $\chi'(G) \leq \gamma_l'(G)$, and we can find a $\gamma_l'(G)$-edge-coloring of G in $O(m^2)$ time.

The most intuitive approach to achieving this bound on the chromatic index involves assuming that G is a minimum counterexample, then characterizing $\gamma_l'(G)$-edge-colorings of $G - e$ for an edge e. We want an algorithmic result, so we will have to be a bit more careful to ensure that we can modify partial $\gamma_l'(G)$-edge-colorings efficiently until we find one that we can extend to a complete $\gamma_l'(G)$-edge-coloring of G.

Our $O(m^2)$-time algorithm requires time-efficient data structures, i.e. a combination of lists and matrices. Our algorithm will build the multigraph one edge at a time, maintaining a proper k-edge-coloring at each step (in this case, $k = \gamma'_l(G)$). We may assume we are given vertices 1 to n and a multiset of m edges, which we first sort at a cost of $O(m \log m)$ time, then add to the graph in lexicographic order. We may also assume that G contains no isolated vertices.

As we build the multigraph we maintain an $n \times n$ adjacency matrix, each cell of which contains a list of edges between the two vertices in question. We also maintain a sorted list of neighbors for each vertex, and a sorted list of edges incident to each vertex. Further, we maintain a $k \times n$ color-vertex incidence matrix, and for each vertex v two lists: a list of the colors appearing on an edge incident to v, and a list of the colors not incident to v. All these structures shall be connected with appropriate links. For example, if color c does not appear at vertex v, the corresponding cell of the color-vertex incidence matrix will be linked to the corresponding node in the list of colors absent at v. If c does appear at v, the cell will be linked to the corresponding node in the list of colors incident to v, as well as to the edge incident to v with color c.

To initialize the coloring data structures we first need to determine $\gamma'_l(G)$. After building the multigraph in time $O(nm) \subseteq O(m^2)$, for each edge uv we can determine $d(u)$, $d(v)$, $\mu(uv)$, and $t(uv)$ in $O(n)$ time. So we can determine $\gamma'_l(G)$ in $O(nm)$ time and initialize the structures in $O(nm + \gamma'_l(G)n) \subseteq O(m^2)$ time.

These structures allow us to update efficiently: When we add an edge to the multigraph, the fact that the edges are presorted allows us to update all lists and matrices in constant time. When changing the color of an edge, the interlinkedness of the matrices and lists allows us to update in constant time.

We begin by defining, for a vertex v, a *fan hinged at* v. Let e be an edge incident to v, and let $v_1, \ldots, v_\ell$ be a set of distinct neighbors of v with e between v and v_1. Let $c : E \setminus \{e\} \to \{1, \ldots, k\}$ be a proper edge coloring of $G \setminus \{e\}$ for some fixed k. Then $F = (e; c; v; v_1, \ldots, v_\ell)$ is a *fan* if for every j such that $2 \leq j \leq \ell$, there exists some i less than j such that some edge between v and v_j is assigned a color that does not appear on any edge incident to v_i (i.e. a color *missing* at v_i). We say that F is *hinged at* v. If there is no $u \notin \{v, v_1, \ldots, v_\ell\}$ such that $F' = (e; c; v; v_1, \ldots, v_\ell, u)$ is a fan, we say that F is a *maximal fan*. The *size* of a fan refers to the number of neighbors of the hinge vertex contained in the fan (in this case, ℓ). These fans generalize Vizing's fans, originally used in the proof of Vizing's theorem [37]. Given a partial k-edge-coloring of G and a vertex w, we say that a color is *incident to* w if the color appears on an edge incident to w. We use $\mathcal{C}(w)$ to denote the set of colors incident to w, and we use $\bar{\mathcal{C}}(w)$ to denote $[k] \setminus \mathcal{C}(w)$.

Fans allow us to modify partial k-edge-colorings of a graph (specifically those with exactly one uncolored edge). We will show that if $k \geq \gamma_l'(G)$, then either every maximal fan has size 2 or we can easily find a k-edge-coloring of G. For more general results related to fans, see [35]. We first prove that we can construct a k-edge-coloring of G from a partial k-edge-coloring of $G - e$ whenever we have a fan for which certain sets are not disjoint.

5.2.2. *For some edge e in a multigraph G and positive integer k, let c be a k-edge-coloring of $G-e$. If there is a fan $F = (e; c; v; v_1, \ldots, v_\ell)$ such that for some j, $\bar{\mathcal{C}}(v) \cap \bar{\mathcal{C}}(v_j) \neq \emptyset$, then we can find a k-edge-coloring of G in $O(k+m)$ time.*

Proof. Let j be the minimum index for which $\bar{\mathcal{C}}(v) \cap \bar{\mathcal{C}}(v_j)$ is nonempty. If $j = 1$, then the result is trivial, since we can extend c to a proper k-edge-coloring of G. Otherwise $j \geq 2$ and we can find j in $O(m)$ time. We define e_1 to be e. We then construct a function $f : \{2, \ldots, \ell\} \to \{1, \ldots, \ell - 1\}$ such that for each i, (1) $f(i) < i$ and (2) there is an edge e_i between v and v_i such that $c(e_i)$ is missing at $v_{f(i)}$. We can find this function in $O(k+m)$ time by building a list of the earliest v_i at which each color is missing, and computing f for increasing values of i starting at 2. While doing so we also find the set of edges $\{e_i\}_{i=2}^{\ell}$.

We construct a k-edge-coloring c_j of $G - e_j$ from c by shifting the color $c(e_j)$ from e_j to $e_{f(j)}$, shifting the color $c(e_{f(j)})$ from $e_{f(j)}$ to $e_{f(f(j))}$, and so on, until we shift a color to e. We now have a k-edge-coloring c_j of $G - e_j$ such that some color is missing at both v and v_j. We can therefore extend c_j to a proper k-edge-coloring of G in $O(k+m)$ time. □

5.2.3. *For some edge e in a multigraph G and positive integer k, let c be a k-edge-coloring of $G-e$. If there is a fan $F = (e; c; v; v_1, \ldots, v_\ell)$ such that for some i and j satisfying $1 \leq i < j \leq \ell$, $\bar{\mathcal{C}}(v_i) \cap \bar{\mathcal{C}}(v_j) \neq \emptyset$, then we can find v_i and v_j in $O(k+m)$ time, and we can find a k-edge-coloring of G in $O(k+m)$ time.*

Proof. We can easily find i and j in $O(k+m)$ time if they exist. Let α be a color in $\bar{\mathcal{C}}(v)$ and let β be a color in $\bar{\mathcal{C}}(v_i) \cap \bar{\mathcal{C}}(v_j)$. Note that by 5.2.2, we can assume $\alpha \in \mathcal{C}(v_i) \cap \mathcal{C}(v_j)$ and $\beta \in \mathcal{C}(v)$.

Let $G_{\alpha,\beta}$ be the subgraph of G containing those edges colored α or β. Every component of $G_{\alpha,\beta}$ containing v, v_i, or v_j is a path on ≥ 2 vertices. Thus either v_i or v_j is in a component of $G_{\alpha,\beta}$ not containing v. Exchanging the colors α and β on this component leaves us with a k-edge-coloring of $G - e$ in which either $\bar{\mathcal{C}}(v) \cap \bar{\mathcal{C}}(v_i) \neq \emptyset$ or $\bar{\mathcal{C}}(v) \cap \bar{\mathcal{C}}(v_j) \neq \emptyset$. This allows us to apply 5.2.2 to find a k-edge-coloring of G. We can easily do this work in $O(m)$ time. □

The previous two lemmas suggest that we can extend a coloring more easily when we have a large fan, so we now consider how we can extend a fan that is not maximal. Given a fan $F = (e; c; v; v_1, \ldots, v_\ell)$, we use $d(F)$ to denote $d(v) + \sum_{i=1}^{\ell} d(v_i)$.

5.2.4. *For some edge e in a multigraph G and integer $k \geq \Delta(G)$, let c be a k-edge-coloring of $G - e$ and let F be a fan. Then we can extend F to a maximal fan $F' = (e; c; v; v_1, v_2, \ldots, v_\ell)$ in $O(k + d(F'))$ time.*

Proof. We proceed by setting $F' = F$ and extending F' until it is maximal. To this end we maintain two color sets. The first, $\mathcal{C}$, consists of those colors appearing incident to v but not between v and another vertex of F'. The second, $\bar{\mathcal{C}}_{F'}$, consists of those colors that are in $\mathcal{C}$ and are missing at some fan vertex. Clearly F' is maximal if and only if $\bar{\mathcal{C}}_{F'} = \emptyset$. We can perform this initialization in $O(k + d(F))$ time by counting the number of times each color in $\mathcal{C}$ appears incident to a vertex of the fan.

Now suppose we have $F' = (e; c; v; v_1, v_2, \ldots, v_\ell)$, along with sets $\mathcal{C}$ and $\bar{\mathcal{C}}_{F'}$, which we may assume is not empty. Take an edge incident to v with a color in $\bar{\mathcal{C}}_F$; call its other endpoint $v_{\ell+1}$. We now update $\mathcal{C}$ by removing all colors appearing between v and $v_{\ell+1}$. We update $\bar{\mathcal{C}}_{F'}$ by removing all colors appearing between v and $v_{\ell+1}$, and adding all colors in $\mathcal{C} \cap \bar{\mathcal{C}}(v_{\ell+1})$. Set $F' = (e; c; v; v_1, v_2, \ldots, v_{\ell+1})$. We can perform this update in $d(v_{\ell+1})$ time; the lemma follows. □

We can now prove that if $k \geq \gamma_l'(G)$ and we have a maximal fan of size 1 or at least 3, we can find a k-edge-coloring of G in $O(k + m)$ time.

5.2.5. *For some edge e in a multigraph G and positive integer $k \geq \gamma_l'(G)$, let c be a k-edge-coloring of $G - e$ and let $F = (e; c; v; v_1)$ be a fan. If F is a maximal fan we can find a k-edge-coloring of G in $O(k + m)$ time.*

Proof. If $\bar{\mathcal{C}}(v) \cap \bar{\mathcal{C}}(v_1)$ is nonempty, then we can easily extend the coloring of $G - e$ to a k-edge-coloring of G. So assume $\bar{\mathcal{C}}(v) \cap \bar{\mathcal{C}}(v_1)$ is empty. Since $k \geq \gamma'_l(G) \geq d(v_1)$, $\bar{\mathcal{C}}(v_1)$ is nonempty. Therefore there is a color in $\bar{\mathcal{C}}(v_1)$ appearing on an edge incident to v whose other endpoint, call it v_2, is not v_1. Thus $(e; c; v; v_1, v_2)$ is a fan, contradicting the maximality of F. □

5.2.6. *For some edge e in a multigraph G and positive integer $k \geq \gamma'_l(G)$, let c be a k-edge-coloring of $G - e$ and let $F = (e; c; v; v_1, v_2, \ldots, v_\ell)$ be a maximal fan with $\ell \geq 3$. Then we can find a k-edge-coloring of G in $O(k + m)$ time.*

Proof. Let v_0 denote v for ease of notation. If the sets $\bar{\mathcal{C}}(v_0), \bar{\mathcal{C}}(v_1), \ldots, \bar{\mathcal{C}}(v_\ell)$ are not all pairwise disjoint, then using 5.2.2 or 5.2.3 we can find a k-edge-coloring of G in $O(m)$ time. We can easily determine whether or not these sets are pairwise disjoint in $O(k + m)$ time. Now assume they are all pairwise disjoint; we will exhibit a contradiction, which is enough to prove the lemma.

The number of missing colors at v_i, i.e. $|\bar{\mathcal{C}}(v_i)|$, is $k - d(v_i)$ if $2 \leq i \leq \ell$, and $k - d(v_i) + 1$ if $i \in \{0, 1\}$. Since F is maximal, any edge with one endpoint v_0 and the other endpoint outside $\{v_0, \ldots, v_\ell\}$ must have a color not appearing in $\cup_{i=0}^{\ell} \bar{\mathcal{C}}(v_i)$. Therefore

$$\left(\sum_{i=0}^{\ell} k - d(v_i) \right) + 2 + \left(d(v_0) - \sum_{i=1}^{\ell} \mu(v_0 v_i) \right) \quad \leq \quad k. \tag{5.2}$$

Thus

$$\ell k + 2 - \sum_{i=1}^{\ell} \mu(v_0 v_i) \quad \leq \quad \sum_{i=1}^{\ell} d(v_i). \tag{5.3}$$

But since $k \geq \gamma'_l(G)$, (5.1) tells us that for all $i \in [\ell]$,

$$d(v_i) + \tfrac{1}{2}(d(v_0) - \mu(v_0 v_i)) \quad \leq \quad k \tag{5.4}$$

Thus substituting for k tells us

$$\sum_{i=1}^{\ell} \frac{d(v_0) + 2d(v_i) - \mu(v_0 v_i)}{2} + 2 - \sum_{i=1}^{\ell} \mu(v_0 v_i) \quad \leq \quad \sum_{i=1}^{\ell} d(v_i).$$

So

$$\begin{aligned} 2 + \tfrac{1}{2}\ell d(v_0) - \tfrac{3}{2}\sum_{i=1}^{\ell} \mu(v_0 v_i) \quad &\leq \quad 0 \\ 2 + \tfrac{1}{2}\ell d(v_0) \quad &\leq \quad \tfrac{3}{2}\sum_{i=1}^{\ell} \mu(v_0 v_i) \\ \tfrac{\ell}{2} d(v_0) \quad &< \quad \tfrac{3}{2} d(v_0). \end{aligned}$$

This is a contradiction, since $\ell \geq 3$. □

We are now ready to prove the main lemma of this section.

5.2.7. *For some edge e_0 in a multigraph G and positive integer $k \geq \gamma_l'(G)$, let c_0 be a k-edge-coloring of $G - e$. Then we can find a k-edge-coloring of G in $O(k + m)$ time.*

As we will show, this lemma easily implies 5.2.1. We approach this lemma by constructing a sequence of overlapping fans of size two until we can apply a previous lemma. If we cannot do this, then our sequence results in a cycle in G and a set of partial k-edge-colorings of G with a very specific structure that leads us to a contradiction.

Proof of 5.2.7. We postpone algorithmic considerations until the end of the proof.

Let v_0 and v_1 be the endpoints of e_0, and let $F_0 = (e_0; c_0; v_1; v_0, u_1, \ldots, u_\ell)$ be a maximal fan. If $|\{u_1, \ldots, u_\ell\}| \neq 1$, then we can apply 5.2.5 or 5.2.6. More generally, if at any time we find a fan of size three or more we can finish by applying 5.2.6. So assume $\{u_1, \ldots, u_\ell\}$ is a single vertex; call it v_2.

Let $\bar{\mathcal{C}}_0$ denote the set of colors missing at v_0 in the partial coloring c_0, and take some color $\alpha_0 \in \bar{\mathcal{C}}_0$. Note that if α_0 does not appear on an edge between v_1 and v_2, then α_0 appears between v_1 and a vertex $u \notin \{v_0, v_1, v_2\}$, so there is a fan $(e_0; c_0; v_1; v_0, v_2, u)$ of size 3 and apply 5.2.6 to complete the coloring. So we can assume that α_0 does appear on an edge between v_1 and v_2.

Let e_1 denote the edge between v_1 and v_2 given color α_0 in c_0. We construct a new coloring c_1 of $G - e_1$ from c_0 by uncoloring e_1 and assigning e_0 color α_0. Let $\bar{\mathcal{C}}_1$ denote the set of colors missing at v_1 in the coloring c_1. Now let $F_1 = (e_1; c_1; v_2; v_1, v_3)$ be a maximal fan. As with F_0, we can assume that F_1 exists and is indeed maximal. The vertex v_3 may or may not be the same as v_0.

Let $\alpha_1 \in \bar{\mathcal{C}}_1$ be a color in $\bar{\mathcal{C}}_1$. Just as α_0 appears between v_1 and v_2 in c_0, we can see that α_1 appears between v_2 and v_3. Now let e_2 be the edge between v_2 and v_3 having color α_1 in c_1. We construct a coloring c_2 of $G - e_2$ from c_1 by uncoloring e_2 and assigning e_1 color α_1.

We continue to construct a sequence of fans $F_i = (e_i, c_i; v_{i+1}; v_i, v_{i+2})$ for $i = 0, 1, 2, \ldots$ in this way, maintaining the property that $\alpha_{i+2} = \alpha_i$. This is possible because when we construct c_{i+1} from c_i, we make α_i available at v_{i+2}, so the set $\bar{\mathcal{C}}_{i+2}$ (the set of colors missing at v_{i+2} in the coloring c_{i+2}) always contains α_i. We continue constructing our sequence of fans until we reach some j for which $v_j \in \{v_i\}_{i=0}^{j-1}$, which will inevitably happen if we never find a fan of size 3 or greater. We claim that $v_j = v_0$ and j is odd. To see this, consider the original edge-coloring of $G - e_0$ and note that for $1 \leq i \leq j-1$, α_0 appears on an edge between v_i and v_{i+1} precisely if i is odd, and α_1 appears on an edge between v_i and v_{i+1} precisely if i is even. Thus since the edges of color α_0 form a matching, and so do the edges of color α_1, we indeed have $v_j = v_0$ and j odd. Furthermore $F_0 = F_j$. Let C denote the cycle $v_0, v_1, \ldots, v_{j-1}$. In each coloring, α_0 and α_1 both appear $(j-1)/2$ times on C, in a near-perfect matching. Let H be the sub-multigraph of G consisting of those edges between v_i and v_{i+1} for $0 \leq j \leq j-1$ (with indices modulo j). Let A be the set of colors missing on at least one vertex of C, and let H_A be the sub-multigraph of H consisting of e_0 and those edges receiving a color in A in c_0 (and therefore in any c_i).

Suppose $j = 3$. If some color is missing on two vertices of C in c_0, c_1, or c_2, we can easily find a k-edge-coloring of G since any two vertices of C are the endpoints of e_0, e_1, or e_2. We know that every color in $\bar{\mathcal{C}}_0$ appears between v_1 and v_2, and every color in $\bar{\mathcal{C}}_1$ appears between v_2 and $v_3 = v_0$. Therefore $|E(H_A)| = |A| + 1$. Therefore

$$\begin{aligned}
2\gamma_l'(G) &\geq d_G(v_0) + d_G(v_1) + t_G(v_0v_1) - \mu_G(v_0v_1) \\
&= d_{H_A}(v_0) + d_{H_A}(v_1) + 2(k - |A|) + t_G(v_0v_1) - \mu_G(v_0v_1) \\
&\geq d_{H_A}(v_0) + d_{H_A}(v_1) + 2(k - |A|) + t_{H_A}(v_0v_1) - \mu_{H_A}(v_0v_1) \\
&\geq 2|E(H_A)| + 2(k - |A|) \\
&> 2|A| + 2(k - |A|) = 2k
\end{aligned}$$

This is a contradiction since $k \geq \gamma_l'(G)$. We can therefore assume that $j \geq 5$.

Let β be a color in $A \setminus \{\alpha_0, \alpha_1\}$. If β is missing at two consecutive vertices v_i and v_{i+1}, then we can easily extend c_i to a k-edge-coloring of G. Bearing in mind that each F_i is a maximal fan, we claim that if β is not missing at two consecutive vertices, then either we can easily k-edge-color G, or the number of edges colored β in H_A is at least twice the number of vertices at which β is missing in any c_i.

To prove this claim, first assume without loss of generality that $\beta \in \bar{\mathcal{C}}_0$. Since β is not missing at v_1, β appears on an edge between v_1 and v_2 for the same reason that α_0 does. Likewise, since β is not missing at v_{j-1}, β appears on an edge between v_{j-1} and v_{j-2}. Finally, suppose β appears between v_1 and v_2, and is missing at v_3 in c_0. Then let e_β be the edge between v_1 and v_2 with color

β in c_0. We construct a coloring c_0' from c_0 by giving e_2 color β and giving e_β color α_1 (i.e. we swap the colors of e_β and e_2). Thus c_0' is a k-edge-coloring of $G - e_0$ in which β is missing at both v_0 and v_1. We can therefore extend $G - e_0$ to a k-edge-coloring of G. Thus if β is missing at v_3 or v_{j-3} we can easily k-edge-color G. We therefore have at least two edges of H_A colored β for every vertex of C at which β is missing, and we do not double-count edges. This proves the claim, and the analogous claim for any color in A also holds.

Now we have

$$\sum_{i=0}^{j-1} \mu_{H_A}(v_i v_{i+1}) = |E(H_A)| \ > \ 2\sum_{i=0}^{j-1} (k - d_G(v_i))\,. \tag{5.5}$$

Therefore taking indices modulo j, we have

$$\sum_{i=0}^{j-1} \left(d_G(v_i) + \tfrac{1}{2}\mu_{H_A}(v_{i+1}v_{i+2})\right) \ > \ jk. \tag{5.6}$$

Therefore there exists some index i for which

$$d_G(v_i) + \tfrac{1}{2}\mu_{H_A}(v_{i+1}v_{i+2}) \quad > \quad k. \tag{5.7}$$

Therefore

$$k \geq d_G(v_i) + \tfrac{1}{2}\mu_G(v_{i+1}v_{i+2}) \quad > \quad k. \tag{5.8}$$

This is a contradiction, so we can indeed find a k-edge-coloring of G. It remains to prove that we can do so in $O(k+m)$ time.

Given the coloring c_i, we can construct the fan $F_i = (e_i, c_i; v_{i+1}; v_i, v_{i+2})$ and determine whether or not it is maximal in $O(k+d(F_i))$ time. If it is not maximal, we can complete the k-edge-coloring of G in $O(m)$ time; this will happen at most once throughout the entire process. Therefore we will either complete the coloring or construct our cycle of fans $F_0, \ldots, F_{j-1}$ in $O(\sum_{i=0}^{j-1}(k + d(F_i)))$ time. This is not the desired bound, so suppose there is an index i for which $k > d(F_i)$. In this case we certainly have two intersecting sets of available colors in F_i, so we can apply 5.2.2 or 5.2.3 when we arrive at F_i, and find the k-edge-coloring of G in $O(k+m)$ time. If no such i exists, then $jk = O(\sum_{i=0}^{j-1}(d(F_i))) = O(m)$, and we indeed complete the construction of all fans in $O(k+m)$ time.

Since each F_i is a maximal fan, in c_0 there must be some color $\beta \notin \{\alpha_0, \alpha_1\}$ missing at two consecutive vertices v_i and v_{i+1}, otherwise we reach a contradiction. To find β and i, we first check for any i for which $|\bar{\mathcal{C}}_i| > d(v_{i+1})$, which we can easily do in $O(m)$ time – such an i guarantees a $\beta \in \bar{\mathcal{C}}_i \cap \bar{\mathcal{C}}_{i+1}$, which we can find in $O(k)$ time. If such a trivial i does not exist, we search for a satisfying i by comparing $\bar{\mathcal{C}}_i$ for each i from 0 to j. We can do this in $O(|\bar{\mathcal{C}}_i| + |\bar{\mathcal{C}}_{i+1}|)$ time for each i, and since each i satisfies $|\bar{\mathcal{C}}_i| \leq d(v_{i+1})$, this takes $O(m)$ time in total. Therefore the entire operation takes $O(k+m)$ time. □

We now complete the proof of 5.2.1.

Proof of 5.2.1. Let $k = \gamma_l'(G)$. As noted in Section 5.2, we can compute k in $O(m^2)$ time. Taking the (lexicographically presorted) edges $e_1, \ldots, e_m$ of G, for $i = 0, \ldots, m$ let G_i denote the subgraph of G on edges $\{e_j \mid j \leq i\}$. Since G_0 is empty it is vacuously k-edge-colored. Given a k-edge-coloring of G_i, we can find a k-edge-coloring of G_{i+1} in $O(k+m)$ time by applying 5.2.7. Since $k = \gamma_l'(G) = O(m)$, each augmentation step takes $O(m)$ time, for a total running time of $O(m^2)$. The theorem follows. □

This gives us the following result for line graphs, since for any multigraph G we have $|V(L(G))| = |E(G)|$:

5.2.8. *Given a line graph G on n vertices, we can find a proper coloring of G using $\gamma_l(G)$ colors in $O(n^2)$ time.*

Proof. To $\gamma_l(G)$-color G we first find a multigraph H such that $G = L(H)$, then we apply 5.2.1. As discussed in [21] §4.2.3, we can construct H from G in $O(|E(G)|)$ time using one of a number of known algorithms. □

This is faster than the algorithm of King, Reed, and Vetta [23] for $\gamma(G)$-coloring line graphs, which is given an improved complexity bound of $O(n^{5/2})$ in [21], §4.2.3.

5.3 Quasi-line Graphs

We now leave the setting of edge colorings of multigraphs and consider vertex colorings of simple graphs. As mentioned in the introduction, we can extend Conjecture 3 from line graphs to quasi-line graphs using the same approach that King and Reed used to extend Conjecture 2 from line graphs to quasi-line graphs in [22]. We do not require the full power of Chudnovsky and Seymour's structure theorem for quasi-line graphs [14]. Instead, we use a simpler decomposition theorem from [10]. Our proof of 5.1.3 yields a polytime $\gamma_l(G)$-coloring algorithm; we sketch a bound on its complexity at the end of the section.

We wish to describe the structure of quasi-line graphs. If a quasi-line graph does not contain a certain type of homogeneous pair of cliques, then it is either a circular interval graph or built as a generalization of a line graph – where in a line graph we would replace each edge with a vertex, we now replace each edge with a linear interval graph. We now describe this structure more formally, which is equivalent to the quasi-line trigraph decomposition that we used in Chapter 3. We restate here the decomposition in term of graphs and reintroduce some definition for completeness. It is important to notice that in this chapter, we take the view of gluing strips together into 'composition of linear interval strips', where in Chapter 3 and Chapter 4 we took the opposite approach of decomposing 'linear interval joins' and 'strip structures' into strips.

A *linear interval graph* is a graph $G = (V, E)$ with a *linear interval representation*, which is a point on the real line for each vertex and a set of intervals, such that vertices u and v are adjacent in G precisely if there is an interval containing both corresponding points on the real line. If X and Y are specified cliques in G consisting of the $|X|$ leftmost and $|Y|$ rightmost vertices (with respect to the real line) of G respectively, we say that X and Y are *end-cliques* of G. These cliques may be empty.

Accordingly, a *circular interval graph* is a graph with a *circular interval representation*, i.e. $|V|$ points on the unit circle and a set of intervals (arcs) on the unit circle such that two vertices of G are adjacent precisely if some arc contains both corresponding points. Circular interval graphs are the first of two fundamental types of quasi-line graph. Deng, Hell, and Huang proved that we can identify and find a representation of a circular or linear interval graph in $O(m)$ time [16].

We now describe the second fundamental type of quasi-line graph.

A *linear interval strip* (S, X, Y) is a linear interval graph S with specified end-cliques X and Y. We compose a set of strips as follows. We begin with an underlying directed multigraph H, possibly with loops, and for every every edge e of H we take a linear interval strip (S_e, X_e, Y_e). For $v \in V(H)$ we define the *hub clique* C_v as

$$C_v = \left(\bigcup\{X_e \mid e \text{ is an edge out of } v\}\right) \cup \left(\bigcup\{Y_e \mid e \text{ is an edge into } v\}\right).$$

We construct G from the disjoint union of $\{S_e \mid e \in E(H)\}$ by making each C_v a clique; G is then a *composition of linear interval strips*. Let G_h denote the subgraph of G induced on the union of all hub cliques. That is,

$$G_h = G[\cup_{v\in V(H)} C_v] = G[\cup_{e\in E(H)}(X_e \cup Y_e)].$$

Compositions of linear interval strips generalize line graphs: note that if each S_e satisfies $|S_e| = |X_e| = |Y_e| = 1$ then $G = G_h = L(H)$.

A pair of disjoint nonempty cliques (A, B) in a graph is a *homogeneous pair of cliques* if $|A| + |B| \geq 3$, every vertex outside $A \cup B$ is adjacent to either all or none of A, and every vertex outside $A \cup B$ is adjacent to either all or none of B. Furthermore (A, B) is *nonlinear* if G contains an induced C_4 in $A \cup B$ (this condition is equivalent to insisting that the subgraph of G induced by $A \cup B$ is a linear interval graph).

Chudnovsky and Seymour's structure theorem for quasi-line graphs [10] tells us that any quasi-line graph not containing a clique- cutset is made from the building blocks we just described.

5.3.1. *Any quasi-line graph containing no clique-cutset and no nonlinear homogeneous pair of cliques is either a circular interval graph or a composition of linear interval strips.*

To prove 5.1.3, we first explain how to deal with circular interval graphs and nonlinear homogeneous pairs of cliques, then move on to considering how to decompose a composition of linear interval strips.

We can easily prove Conjecture 3 for circular interval graphs by combining previously known results. Niessen and Kind proved that every circular interval graph G satisfies $\chi(G) = \lceil \chi_f(G) \rceil$ [29], so 5.1.2 immediately implies that Conjecture 3 holds for circular interval graphs. Furthermore Shih and Hsu [32] proved that we can optimally color circular interval graphs in $O(n^{3/2})$ time, which gives us the following result:

5.3.2. *Given a circular interval graph G on n vertices, we can $\gamma_l(G)$-color G in $O(n^{3/2})$ time.*

There are many lemmas of varying generality that tell us we can easily deal with nonlinear homogeneous pairs of cliques; we use the version used by King and Reed [22] in their proof of Conjecture 2 for quasi-line graphs:

5.3.3. *Let G be a quasi-line graph on n vertices containing a nonlinear homogeneous pair of cliques (A, B). In $O(n^{5/2})$ time we can find a proper subgraph G' of G such that G' is quasi-line, $\chi(G') = \chi(G)$, and given a k-coloring of G' we can find a k-coloring of G in $O(n^{5/2})$ time.*

It follows immediately that no minimum counterexample to 5.1.3 contains a nonlinear homogeneous pair of cliques.

5.4 Decomposing Quasi-line Graphs

Decomposing graphs on clique-cutsets for the purpose of finding vertex colorings is straightforward and well understood.

For any monotone bound on the chromatic number for a hereditary class of graphs, no minimum counterexample can contain a clique-cutset, since we can simply "paste together" two partial colorings on a clique-cutset. Tarjan [36] gave an $O(nm)$-time algorithm for constructing a clique-cutset decomposition tree of any graph, and noted that given k-colorings of the leaves of this decomposition tree, we can construct a k-coloring of the original graph in $O(n^2)$ time. Therefore if we can $\gamma_l(G)$-color any quasi-line graph containing no clique-cutset in $O(f(n,m))$ time for some function f, we can $\gamma_l(G)$-color any quasi-line graph in $O(f(n,m)+nm)$ time.

If the multigraph H contains a loop or a vertex of degree 1, then as long as G is not a clique, it will contain a clique-cutset.

A canonical interval 2-join is a composition by which a linear interval graph is attached to another graph. Canonical interval 2-joins arise from compositions of strips, and can be viewed as a local decomposition rather than one that requires knowledge of a graph's global structure as a composition of strips.

Given four cliques X_1, Y_1, X_2, and Y_2, we say that $((V_1,X_1,Y_1),(V_2,X_2,Y_2))$ is an *interval 2-join* if it satisfies the following:

- $V(G)$ can be partitioned into nonempty V_1 and V_2 with $X_1 \cup Y_1 \subseteq V_1$ and $X_2 \cup Y_2 \subseteq V_2$ such that for $v_1 \in V_1$ and $v_2 \in V_2$, v_1v_2 is an edge precisely if $\{v_1,v_2\}$ is in $X_1 \cup X_2$ or $Y_1 \cup Y_2$.

- $G|V_2$ is a linear interval graph with end-cliques X_2 and Y_2.

If we also have X_2 and Y_2 disjoint, then we say $((V_1,X_1,Y_1),(V_2,X_2,Y_2))$ is a *canonical interval 2-join*. The following decomposition theorem is a straightforward consequence of the structure theorem for quasi-line graphs:

5.4.1. *Let G be a quasi-line graph containing no nonlinear homogeneous pair of cliques. Then one of the following holds.*

- *G is a line graph*
- *G is a circular interval graph*
- *G contains a clique-cutset*
- *G admits a canonical interval 2-join.*

Therefore to prove 5.1.3 it only remains to prove that a minimum counterexample cannot contain a canonical interval 2-join. Before doing so we must give some notation and definitions.

We actually need to bound a refinement of $\gamma_l(G)$. Given a canonical interval 2-join $((V_1, X_1, Y_1), (V_2, X_2, Y_2))$ in G with an appropriate partitioning V_1 and V_2, let G_1 denote $G|V_1$, let G_2 denote $G|V_2$ and let H_2 denote $G|(V_2 \cup X_1 \cup Y_1)$. For $v \in H_2$ we define $\omega'(v)$ as the size of the largest clique in H_2 containing v and not intersecting both $X_1 \setminus Y_1$ and $Y_1 \setminus X_1$, and we define $\gamma_l^j(H_2)$ as $\max_{v \in H_2} \lceil d_G(v) + 1 + \omega'(v) \rceil$ (here the superscript j denotes *join*). Observe that $\gamma_l^j(H_2) \leq \gamma_l(G)$. If $v \in X_1 \cup Y_1$, then $\omega'(v)$ is $|X_1| + |X_2|$, $|Y_1| + |Y_2|$, or $|X_1 \cap Y_1| + \omega(G|(X_2 \cup Y_2))$.

The following lemma is due to King and Reed and first appeared in [21]; we include the proof for the sake of completeness.

5.4.2. *Let G be a graph on n vertices and suppose G admits a canonical interval 2-join $((V_1, X_1, Y_1), (V_2, X_2, Y_2))$. Then given a proper l-coloring of G_1 for any $l \geq \gamma_l^j(H_2)$, we can find a proper l-coloring of G in $O(nm)$ time.*

Since $\gamma_l^j(H_2) \leq \gamma_l(G)$, this lemma implies that no minimum counterexample to 5.1.3 contains a canonical interval 2-join.

It is easy to see that a minimum counterexample cannot contain a simplicial vertex (i.e. a vertex whose neighborhood is a clique). Therefore in a canonical interval 2-join $((V_1, X_1, Y_1), (V_2, X_2, Y_2))$ in a minimum counterexample, all four cliques X_1, Y_1, X_2, and Y_2 must be nonempty.

Proof. We proceed by induction on l, observing that the case $l = 1$ is trivial. We begin by modifying the coloring so that the number k of colors used in both X_1 and Y_1 in the l-coloring of G_1 is maximal. That is, if a vertex $v \in X_1$ gets a color that is not seen in Y_1, then every color appearing in Y_1 appears in $N(v)$. This can be done in $O(n^2)$ time. If l exceeds $\gamma_l^j(H_2)$ we can just remove a color class in G_1 and apply induction on what remains. Thus we can assume that $l = \gamma_l^j(H_2)$ and so if we apply induction we must remove a stable set whose removal lowers both l and $\gamma_l^j(H_2)$.

We use case analysis; when considering a case we may assume no previous case applies. In some cases we extend the coloring of G_1 to an l-coloring of G in one step. In other cases we remove a color class in G_1 together with vertices in G_2 such that everything we remove is a stable set, and when we remove it we reduce $\gamma_l^j(v)$ for every $v \in H_2$; after doing this we apply induction on l. Notice that if $X_1 \cap Y_1 \neq \emptyset$ and there are edges between X_2 and Y_2 we may have a large clique in H_2 which contains some but not all of X_1 and some but not all of Y_1; this is not necessarily obvious but we deal with it in every applicable case.

Case 1. $Y_1 \subseteq X_1$.

H_2 is a circular interval graph and X_1 is a clique-cutset. We can $\gamma_l(H_2)$-color H_2 in $O(n^{3/2})$ time using 5.3.2. By permuting the color classes we can ensure that this coloring agrees with the coloring of G_1. In this case $\gamma_l(H_2) \leq \gamma_l^j(H_2) \leq l$ so we are done. By symmetry, this covers the case in which $X_1 \subseteq Y_1$.

Case 2. $k = 0$ and $l > |X_1| + |Y_1|$.

Here X_1 and Y_1 are disjoint. Take a stable set S greedily from left to right in G_2. By this we mean that we start with $S = \{v_1\}$, the leftmost vertex of X_2, and we move along the vertices of G_2 in linear order, adding a vertex to S whenever doing so will leave S a stable set. So S hits X_2. If it hits Y_2, remove S along with a color class in G_1 not intersecting $X_1 \cup Y_1$; these vertices together make a stable set. If $v \in G_2$ it is easy to see that $\gamma_l^j(v)$ will drop: every remaining vertex in G_2 either loses two neighbors or is in Y_2, in which case S intersects every maximal clique containing v. If $v \in X_1 \cup Y_1$, then since X_1 and Y_1 are disjoint, $\omega'(v)$ is either $|X_1| + |X_2|$ or $|Y_1| + |Y_2|$; in either case $\omega'(v)$, and therefore $\gamma_l^j(v)$, drops when S and the color class are removed. Therefore $\gamma_l^j(H_2)$ drops, and we can proceed by induction.

If S does not hit Y_2 we remove S along with a color class from G_1 that hits Y_1 (and therefore not X_1). Since $S \cap Y_2 = \emptyset$ the vertices together make a stable set. Using the same argument as before we can see that removing these vertices drops both l and $\gamma_l^j(H_2)$, so we can proceed by induction.

Case 3. $k = 0$ and $l = |X_1| + |Y_1|$.

Again, X_1 and Y_1 are disjoint. By maximality of k, every vertex in $X_1 \cup Y_1$ has at least

$l-1$ neighbors in G_1. Since $l = |X_1| + |Y_1|$ we know that $\omega'(X_1) \leq |X_1| + |Y_1| - |X_2|$ and $\omega'(Y_1) \leq |X_1| + |Y_1| - |Y_2|$. Thus $|Y_1| \geq 2|X_2|$ and similarly $|X_1| \geq 2|Y_2|$. Assume without loss of generality that $|Y_2| \leq |X_2|$.

We first attempt to l-color $H_2 - Y_1$, which we denote by H_3, such that every color in Y_2 appears in X_1 – this is clearly sufficient to prove the lemma since we can permute the color classes and paste this coloring onto the coloring of G_1 to get a proper l-coloring of G. If $\omega(H_3) \leq l - |Y_2|$, then this is easy: we can $\omega(H_3)$-color the vertices of H_3, then use $|Y_2|$ new colors to recolor Y_2 and $|Y_2|$ vertices of X_1. This is possible since Y_2 and X_1 have no edges between them.

Define b as $l - \omega(H_3)$; we can assume that $b < |Y_2|$. We want an $\omega(H_3)$-coloring of H_3 such that at most b colors appear in Y_2 but not X_1. There is some clique $C = \{v_i, \dots, v_{i+\omega(H_3)-1}\}$ in H_3; this clique does not intersect X_1 because $|X_1 \cup X_2| \leq l - \frac{1}{2}|Y_1| \leq l - |Y_2| < l - b$. Denote by v_j the leftmost neighbor of v_i. Since $\gamma_l^j(v_i) \leq l$, it is clear that v_i has at most $2b$ neighbors outside C, and since $b < |Y_2| \leq \frac{1}{2}|X_1|$ we can be assured that $v_i \notin X_2$. Since $\omega(H_3) > |Y_2|$, $v_i \notin Y_2$.

We now color H_3 from left to right, modulo $\omega(H_3)$. If at most b colors appear in Y_2 but not X_1 then we are done, otherwise we will "roll back" the coloring, starting at v_i. That is, for every $p \geq i$, we modify the coloring of H_3 by giving v_p the color after the one that it currently has, modulo $\omega(H_3)$. Since v_i has at most $2b$ neighbors behind it, we can roll back the coloring at least $\omega(H_3) - 2b - 1$ times for a total of $\omega(H_3) - 2b$ proper colorings of H_3.

Since $v_i \notin Y_2$ the colors on Y_2 will appear in order modulo $\omega(H_3)$. Thus there are $\omega(H_3)$ possible sets of colors appearing on Y_2, and in $2b+1$ of them there are at most b colors appearing in Y_2 but not X_1. It follows that as we roll back the coloring of H_3 we will find an acceptable coloring.

Henceforth we will assume that $|X_1| \geq |Y_1|$.

Case 4. $0 < k < |X_1|$.

Take a stable set S in $G_2 - X_2$ greedily from left to right. If S hits Y_2, we remove S from G, along with a color class from G_1 intersecting X_1 but not Y_1. Otherwise, we remove S along with a color class from G_1 intersecting both X_1 and Y_1. In either case it is a simple matter to confirm that $\gamma_l^j(v)$ drops for every $v \in H_2$ as we did in Case 2. We proceed by induction.

Case 5. $k = |Y_1| = |X_1| = 1$.

In this case $|X_1| = k = 1$. If G_2 is not connected, then X_1 and Y_1 are both clique-cutsets and we can proceed as in Case 1. If G_2 is connected and contains an l-clique, then there is some $v \in V_2$ of degree at least l in the l-clique. Thus $\gamma_l^j(H_2) > l$, contradicting our assumption that $l \geq \gamma_l^j(H_2)$. So $\omega(G_2) < l$. We can $\omega(G_2)$-color G_2 in linear time using only colors not appearing in $X_1 \cup Y_1$, thus extending the l-coloring of G_1 to a proper l-coloring of G.

Case 6. $k = |Y_1| = |X_1| > 1$.

Suppose that k is not minimal. That is, suppose there is a vertex $v \in X_1 \cup Y_1$ whose closed neighborhood does not contain all l colors in the coloring of G_1. Then we can change the color of v and apply Case 4. So assume k is minimal.

Therefore every vertex in X_1 has degree at least $l + |X_2| - 1$. Since $X_1 \cup X_2$ is a clique, $\gamma_l^j(H_2) \geq l \geq \frac{1}{2}(l + |X_2| + |X_1| + |X_2|)$, so $2|X_2| \leq l - k$. Similarly, $2|Y_2| \leq l - k$, so $|X_2| + |Y_2| \leq l - k$. Since there are $l - k$ colors not appearing in $X_1 \cup Y_1$, we can $\omega(G_2)$-color G_2, then permute the color classes so that no color appears in both $X_1 \cup Y_1$ and $X_2 \cup Y_2$. Thus we can extend the l-coloring of G_1 to an l-coloring of G.

These cases cover every possibility, so we need only prove that the coloring can be found in $O(nm)$ time. If k has been maximized and we apply induction, k will stay maximized: every vertex in $X_1 \cup Y_1$ will have every remaining color in its closed neighborhood except possibly if we recolor a vertex in Case 6. In this case the overlap in what remains is $k - 1$, which is the most possible since we remove a vertex from X_1 or Y_1, each of which has size k. Hence we only need to maximize k once. We can determine which case applies in $O(m)$ time, and it is not hard to confirm that whenever we extend the coloring in one step our work can be done in $O(nm)$ time. When we apply induction, i.e. in Cases 2, 4, and possibly 6, all our work can be done in $O(m)$ time. Since $l < n$ it follows that the entire l-coloring can be completed in $O(nm)$ time. □

5.5 Putting the pieces together and Algorithmic Considerations

We are now ready to prove 5.1.3.

Proof of 5.1.3. Let G be a minimum counterexample. By 5.3.3, it follows that G contains no nonlinear homogeneous pair of cliques. By 5.2.1, we deduce that G is not a line graph and 5.3.2 implies that G is not a circular interval graph. By 5.4.2, it follows that G does not admit a canonical interval 2-join. Therefore by 5.4.1, G cannot exist. □

It is fairly clear that our proof of 5.1.3 gives us a polytime coloring algorithm. Here we sketch a bound of $O(n^3m^2)$ on its running time.

We proceed by induction on n. We reduce to the case containing no nonlinear homogeneous pair of cliques by applying 5.3.3 $O(m)$ times in order to find a quasi-line subgraph G' of G such that $\chi(G) = \chi(G')$, and given a k-coloring of G', we can find a k-coloring of G in $O(n^2m^2)$ time. We must now color G'. Following Section 5.4, we need only consider graphs containing no clique-cutsets since $n^3m^2 \geq nm$.

If G' is a circular interval graph we can determine this and $\gamma_l(G)$-color it in $O(n^{3/2})$ time. If G' is a line graph we can determine this in $O(m)$ time using an algorithm of Roussopoulos [31], then $\gamma_l(G)$-color it in $O(n^2)$ time. Otherwise, G' must admit a canonical interval 2-join. In this case Lemma 6.18 in [21], due to King and Reed, tells us that we can find such a decomposition in $O(n^2m)$ time.

This canonical interval 2-join $((V_1, X_1, Y_1), (V_2, X_2, Y_2))$ leaves us to color the induced subgraph G_1 of G', which has at most $n-1$ vertices and is quasi-line. Given a $\gamma_l(G)$-coloring of G_1 we can $\gamma_l(G)$-color G' in $O(nm)$ time, then reconstruct the $\gamma_l(G)$-coloring of G in $O(n^2m^2)$ time. The induction step takes $O(n^2m^2)$ time and reduces the number of vertices, so the total running time of the algorithm is $O(n^3m^2)$.

Remark: This bound does not use recent, more sophisticated results on decomposing quasi-line graphs, such as those found in [6] and [18]. We suspect that by applying these results carefully, one should be able to reduce the running time of the entire $\gamma_l(G)$-coloring algorithm to $O(m^2)$.

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Appendix A

Appendix

A.1 Orientable prismatic graphs

- $\mathcal{Q}_0$ is the class of all 3-coloured graphs (G, A, B, C) such that G has no triangle.
- $\mathcal{Q}_1$ is the class of all 3-coloured graphs (G, A, B, C) where G is isomorphic to the line graph of $K_{3,3}$.
- $\mathcal{Q}_2$ is the class of all canonically-coloured path of triangles graphs.
- **Path of triangles.** A graph G is a *path of triangles* graph if for some integer $n \geq 1$ there are pairwise disjoint stable subsets $X_1, \ldots, X_{2n+1}$ of $V(G)$ with union $V(G)$, satisfying the following conditions (P1)-(P7).

(P1) For $1 \leq i \leq n$, there is a nonempty subset $\hat{X}_{2i} \subseteq X_{2i}$; $|\hat{X}_2| = |\hat{X}_{2n}| = 1$, and for $0 < i < n$, at least one of $\hat{X}_{2i}, \hat{X}_{2i+2}$ has cardinality 1.

(P2) For $1 \leq i < j \leq 2n+1$

(1) if $j - i = 2$ modulo 3 and there exist $u \in X_i$ and $v \in X_j$, nonadjacent, then either i, j are odd and $j = i + 2$, or i, j are even and $u \notin \hat{X}_i$ and $v \notin \hat{X}_j$;

(2) if $j - i \neq 2$ modulo 3 then either $j = i + 1$ or X_i is anticomplete to X_j.

(P3) For $1 \leq i \leq n+1$, X_{2i-1} is the union of three pairwise disjoint sets L_{2i-1}, M_{2i-1}, R_{2i-1}, where $L_1 = M_1 = M_{2n+1} = R_{2n+1} = \emptyset$.

(P4) If $R_1 = \emptyset$ then $n \geq 2$ and $|\hat{X}_4| > 1$, and if $L_{2n+1} = \emptyset$ then $n \geq 2$ and $|\hat{X}_{2n-2}| > 1$.

(P5) For $1 \leq i \leq n$, X_{2i} is anticomplete to $L_{2i-1} \cup R_{2i+1}$; $X_{2i} \backslash \hat{X}_{2i}$ is anticomplete to $M_{2i-1} \cup M_{2i+1}$; and every vertex in $X_{2i} \backslash \hat{X}_{2i}$ is adjacent to exactly one end of every edge between R_{2i-1} and L_{2i+1}.

(P6) For $1 \leq i \leq n$, if $|\hat{X}_{2i}| = 1$, then

(1) R_{2i-1}, L_{2i+1} are matched, and every edge between $M_{2i-1} \cup R_{2i-1}$ and $L_{2i+1} \cup M_{2i+1}$ is between R_{2i-1} and L_{2i+1};

(2) the vertex in $\hat{X}_{2i}$ is complete to $R_{2i-1} \cup M_{2i-1} \cup L_{2i+1} \cup M_{2i+1}$;

(3) L_{2i-1} is complete to X_{2i+1} and X_{2i-1} is complete to R_{2i+1}

(4) if $i > 1$ then $M_{2i-1}, \hat{X}_{2i-2}$ are matched, and if $i < n$ then $M_{2i+1}, \hat{X}_{2i+2}$ are matched.

(P7) For $1 < i < n$, if $|\hat{X}_{2i}| > 1$ then

(1) $R_{2i-1} = L_{2i+1} = \emptyset$;

(2) if $u \in X_{2i-1}$ and $v \in X_{2i+1}$, then u, v are nonadjacent if and only if they have the same neighbour in $\hat{X}_{2i}$.

Let $A_k = \bigcup(X_i : 1 \leq i \leq 2n+1 \text{ and } i = k \mod 3)$ $(k = 0, 1, 2)$. Then (G, A_1, A_2, A_3) is a *canonically-coloured path of triangles graphs.*

- **Cycle of triangles.** A graph G is a *cycle of triangles* graph if for some integer $n \geq 5$ with $n = 2$ modulo 3, there are pairwise disjoint stable subsets $X_1, \ldots, X_{2n}$ of $V(G)$ with union $V(G)$, satisfying the following conditions (C1)-(C6) (reading subscripts modulo $2n$):

(C1) For $1 \leq i \leq n$, there is a nonempty subset $\hat{X}_{2i} \subseteq X_{2i}$, and at least one of $\hat{X}_{2i}, \hat{X}_{2i+2}$ has cardinality 1.

(C2) For $i \in \{1, \ldots, 2n\}$ and all k with $2 \leq k \leq 2n-2$, let $j \in \{1, \ldots, 2n\}$ with $j = i + k$ modulo 2n:

(1) if $k = 2$ modulo 3 and there exist $u \in X_i$ and $v \in X_j$, nonadjacent, then either i, j are odd and $k \in \{2, 2n-2\}$, or i, j are even and $u \notin \hat{X}_i$ and $v \notin \hat{X}_j$;

(2) if $k \neq 2$ modulo 3 then X_i is anticomplete to X_j.

(Note that $k = 2$ modulo 3 if and only if $2n - k = 2$ modulo 3, so these statements are symmetric between i and j.)

(C3) For $1 \leq i \leq n+1$, X_{2i-1} is the union of three pairwise disjoint sets L_{2i-1}, M_{2i-1}, R_{2i-1}.

(C4) For $1 \leq i \leq n$, X_{2i} is anticomplete to $L_{2i-1} \cup R_{2i+1}$; $X_{2i} \backslash \hat{X}_{2i}$ is anticomplete to $M_{2i-1} \cup M_{2i+1}$; and every vertex in $X_{2i} \backslash \hat{X}_{2i}$ is adjacent to exactly one end of every edge between R_{2i-1} and L_{2i+1}.

(C5) For $1 \leq i \leq n$, if $|\hat{X}_{2i}| = 1$, then

(1) R_{2i-1}, L_{2i+1} are matched, and every edge between $M_{2i-1} \cup R_{2i-1}$ and $L_{2i+1} \cup M_{2i+1}$ is between R_{2i-1} and L_{2i+1};

(2) the vertex in $\hat{X}_{2i}$ is complete to $R_{2i-1} \cup M_{2i-1} \cup L_{2i+1} \cup M_{2i+1}$;

(3) L_{2i-1} is complete to X_{2i+1} and X_{2i-1} is complete to R_{2i+1}

(4) $M_{2i-1}, \hat{X}_{2i-2}$ are matched and $M_{2i+1}, \hat{X}_{2i+2}$ are matched.

(C6) For $1 \leq i \leq n$, if $|\hat{X}_{2i}| > 1$ then

(1) $R_{2i-1} = L_{2i+1} = \emptyset$;

(2) if $u \in X_{2i-1}$ and $v \in X_{2i+1}$, then u, v are nonadjacent if and only if they have the same neighbour in $\hat{X}_{2i}$.

- **Ring of five.** Let G be a graph with $V(G)$ the union of the disjoint sets $W = \{a_1, \ldots, a_5, b_1, \ldots, b_5\}$ and $V_0, V_1, \ldots, V_5$. Let adjacency be as follows (reading subscripts modulo 5). For $1 \leq i \leq 5$, $\{a_i, a_{i+1}; b_{i+3}\}$ is a triangle, and a_i is adjacent to b_i; V_0 is complete to $\{b_1, \ldots, b_5\}$ and anticomplete to $\{a_1, \ldots, a_5\}$; $V_0, V_1, \ldots, V_5$ are all stable; for $i = 1, \ldots, 5$, V_i is complete to $\{a_{i-1}, b_i, a_{i+1}\}$ and anticomplete to the remainder of W; V_0 is anticomplete to $V_1 \cup \cdots \cup V_5$; for $1 \leq i \leq 5$ V_i is anticomplete to V_{i+2}; and the adjacency between V_i, V_{i+1} is arbitrary. We call such a graph a *ring of five*.

- **Mantled $L(K_{3,3})$.** Let G be a graph with $V(G)$ the union of seven sets

$$W = \{a_i^j : 1 \leq i, j \leq 3\}, V^1, V^2, V^3, V_1, V_2, V_3,$$

with adjacency as follows. For $1 \leq i, j, i', j' \leq 3$, a_i^j and $a_{i'}^{j'}$ are adjacent if and only if $i' \neq i$ and $j' \neq j$. For $i = 1, 2, 3$, V^i, V_i are stable; V^i is complete to $\{a_i^1, a_i^2, a_i^3\}$, and anticomplete to the remainder of W; and V_i is complete to $\{a_1^i, a_2^i, a_3^i\}$ and anticomplete to the remainder of

W. Moreover, $V^1 \cup V^2 \cup V^3$ is anticomplete to $V_1 \cup V_2 \cup V_3$, and there is no triangle included in $V^1 \cup V^2 \cup V^3$ or in $V_1 \cup V_2 \cup V_3$. We call such a graph G a *mantled* $L(K_{3,3})$.

A.2 Non-orientable prismatic graphs

- **A rotator**. Let G have nine vertices $v_1, v_2, \ldots, v_9$, where $\{v_1, v_2, v_3\}$ is a triangle, $\{v_4, v_5, v_6\}$ is complete to $\{v_7, v_8, v_9\}$, and for $i = 1, 2, 3$, v_i is adjacent to v_{i+3}, v_{i+6}, and there are no other edges. We call G a *rotator*.

- **A twister**. Let G have ten vertices $u_1, u_2, v_1, \ldots, v_8$, where u_1, u_2 are adjacent, for $i = 1, \ldots, 8$ v_i is adjacent to $v_{i-1}, v_{i+1}, v_{i+4}$ (reading subscripts modulo 8), and for $i = 1, 2$, u_i is adjacent to $v_i, v_{i+2}, v_{i+4}, v_{i+6}$, and there are no other edges. We call G a *twister* and u_1, u_2 is the *axis* of the twister.

A.3 Three-cliqued graphs

- **A type of line trigraph**. Let v_1, v_2, v_3 be distinct nonadjacent vertices of a graph H, such that every edge of H is incident with one of v_1, v_2, v_3. Let v_1, v_2, v_3 all have degree at least three, and let all other vertices of H have degree at least one. Moreover, for all distinct $i, j \in \{1, 2, 3\}$, let there be at most one vertex different from v_1, v_2, v_3 that is adjacent to v_i and not to v_j in H. Let A, B, C be the sets of edges of H incident with v_1, v_2, v_3 respectively, and let G be a line trigraph of H. Then (G, A, B, C) is a three-cliqued claw-free trigraph; let $\mathcal{TC}_1$ be the class of all such three-cliqued trigraphs such that every vertex is in a triad.

- **Long circular interval trigraphs**. Let G be a long circular interval trigraph, and let Σ be a circle with $V(G) \subseteq \Sigma$, and $F_1, \ldots, F_k \subseteq \Sigma$, as in the definition of long circular interval trigraph. By a *line* we mean either a subset $X \subseteq V(G)$ with $|X| \leq 1$, or a subset of some F_i homeomorphic to the closed unit interval, with both end-points in $V(G)$. Let L_1, L_2, L_3 be pairwise disjoint lines with $V(G) \subseteq L_1 \cup L_2 \cup L_3$; then $(G, V(G) \cap L_1, V(G) \cap L_2, V(G) \cap L_3)$ is a three-cliqued claw-free trigraph. We denote by $\mathcal{TC}_2$ the class of such three-cliqued trigraphs with the additional property that every vertex is in a triad.

- **Near-antiprismatic trigraphs**. Let H be a near-antiprismatic trigraph, and let A, B, C, X be as in the deffnition of near-antiprismatic trigraph. Let $A' = A \backslash X$ and define B', C' similarly; then (H, A', B', C') is a three-cliqued claw-free trigraph. We denote by $\mathcal{TC}_3$ the class of all three-cliqued trigraphs with the additional property that every vertex is in a triad.

- **Antiprismatic trigraphs**. Let G be an antiprismatic trigraph and let A, B, C be a partition of $V(G)$ into three strong cliques; then (G, A, B, C) is a three-cliqued claw-free trigraph. We denote the class of all such three-cliqued trigraphs by $\mathcal{TC}_4$. (In [11] Chudnovsky and Seymour described explicitly all three-cliqued antiprismatic graphs, and their "changeable" edges; and this therefore provides a description of the three-cliqued antiprismatic trigraphs.) Note that in this case there may be vertices that are in no triads.

- **Sporadic exceptions.**

  - Let H be the trigraph with vertex set $\{v_1, \ldots, v_8\}$ and adjacency as follows: v_i, v_j are strongly adjacent for $1 \leq i < j \leq 6$ with $j - i \leq 2$; the pairs v_1v_5 and v_2v_6 are strongly antiadjacent; v_1, v_6, v_7 are pairwise strongly adjacent, and v_7 is strongly antiadjacent to v_2, v_3, v_4, v_5; v_7, v_8 are strongly adjacent, and v_8 is strongly antiadjacent to $v_1, \ldots, v_6$; the pairs v_1v_4 and v_3v_6 are semiadjacent, and v_2 is antiadjacent to v_5. Let $A = \{v_1, v_2, v_3\}, B = \{v_4, v_5, v_6\}$ and $C = \{v_7, v_8\}$. Let $X \subseteq \{v_3, v_4\}$; then $(H \backslash X, A \backslash X, B \backslash X, C)$ is a three-cliqued claw-free trigraph, and all its vertices are in triads.

  - Let H be the trigraph with vertex set $\{v_1, \ldots, v_9\}$, and adjacency as follows: the sets $A = \{v_1, v_2\}$, $B = \{v_3, v_4, v_5, v_6, v_9\}$ and $C = \{v_7, v_8\}$ are strong cliques; v_9 is strongly adjacent to v_1, v_8 and strongly antiadjacent to v_2, v_7; v_1 is strongly antiadjacent to v_4, v_5, v_6, v_7, semiadjacent to v_3 and strongly adjacent to v_8; v_2 is strongly antiadjacent to v_5, v_6, v_7, v_8 and strongly adjacent to v_3; v_3, v_4 are strongly antiadjacent to v_7, v_8; v_5 is strongly antiadjacent to v_8; v_6 is semiadjacent to v_8 and strongly adjacent to v_7; and the adjacency between the pairs v_2v_4 and v_5v_7 is arbitrary. Let $X \subseteq \{v_3, v_4, v_5, v_6\}$, such that

    * v_2 is not strongly anticomplete to $\{v_3, v_4\} \backslash X$
    * v_7 is not strongly anticomplete to $\{v_5, v_6\} \backslash X$

  - * if $v_4, v_5 \notin X$ then v_2 is adjacent to v_4 and v_5 is adjacent to v_7.

Then $(H \setminus X, A, B \setminus X, C)$ is a three-cliqued claw-free trigraph.

We denote by $\mathcal{TC}_5$ the class of such three-cliqued trigraphs (given by one of these two constructions) with the additional property that every vertex is in a triad.