Proof and Reasoning in Secondary School Algebra Textbooks

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ABSTRACT

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The purpose of this study was to determine the extent to which the modeling of deductive reasoning and proof-type thinking occurs in a mathematics course in which students are not explicitly preparing to write formal mathematical proofs. Algebra was chosen because it is the course that typically directly precedes a student’s first formal introduction to proof in geometry in the United States.

The lens through which this study aimed to examine the intended curriculum was by identifying and reviewing the modeling of proof and deductive reasoning in the most popular and widely circulated algebra textbooks throughout the United States. Textbooks have a major impact on mathematics classrooms, playing a significant role in determining a teacher’s classroom practices as well as student activities. A rubric was developed to analyze the presence of reasoning and proof in algebra textbooks, and an analysis of the coverage of various topics was performed. The findings indicate that, roughly speaking, students are only exposed to justification of mathematical claims and proof-type thinking in 38% of all sections analyzed. Furthermore, only 6% of coded sections contained an actual proof or justification that offered the same ideas or reasoning as a proof.

It was found that when there was some justification or proof present, the most prevalent means of convincing the reader of the truth of a concept, theorem, or procedure was through the use of specific examples. Textbooks attempting to give a series of examples to justify or convince the reader of the truth of a concept, theorem, or procedure often fell short of offering a mathematical proof because they lacked generality and/or, in some cases, the inductive step.
While many textbooks stated a general rule at some point, most only used deductive reasoning within a specific example if at all. Textbooks rarely expose students to the kinds of reasoning required by mathematical proof in that they rarely expose students to reasoning about mathematics with generality.

This study found a lack of sufficient evidence of instruction or modeling of proof and reasoning in secondary school algebra textbooks. This could indicate that, overall, algebra textbooks may not fulfill the proof and reasoning guidelines set forth by the *NCTM Principles and Standards* and the *Common Core State Standards*. Thus, the enacted curriculum in mathematics classrooms may also fail to address the recommendations of these influential and policy-defining organizations.
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P. D.
Chapter I

INTRODUCTION: NEED, PURPOSE, AND PROCEDURE

*Given (from the research community):*

- Students perform poorly at tasks involving reasoning, mathematical justification, and proof writing internationally.
- Mathematicians largely point to proof writing as what engaging in mathematics is really all about.
- According to policymakers, students’ ability to reason and prove needs to be developed at all points in their mathematics education.
- Modeling is generally accepted as an effective form of instruction.

**Need for Study**

“Proof” means various things to different people; it also has a range of definitions in different professions. Furthermore, proving something to an adult is different from proving it to a 9th grader. It is often said that scientists prove their theories empirically. For a scientist to say they have proven something, he or she must show that it somehow follows from the result of an experiment that is reproducible. In a court of law, one may be proven innocent or guilty when a lawyer convinces a jury of one’s peers that one is innocent or guilty. In mathematics, “proof” means something entirely different.

To prove something to be true in the world of mathematics, one must show something to be true using logic and reasoning for all possible cases, even when there
are an infinite number of such cases. One must show that given one or many accepted facts, the thing to be proven logically follows from those given facts and must be true. A proof in mathematics is a logical argument, not an empirical or evidential one. This idea is central to the field of mathematics. Haimo (1995) describes proof as the “major distinguishing component” of mathematics, further citing that it is the “essence of [the field of mathematics’] uniqueness” (p. 103).

Proof is of the utmost importance to the field of mathematics, and it is central to all mathematical discovery. Proof writing literally defines mathematics as we know it (Ball & Bass, 2003; Haimo, 1995; Harel & Sowder, 2007; Healy & Hoyles, 1998; Schoenfeld, 1994). Aside from some basic axioms that mathematicians hold to be true, all of mathematics has been built through proof writing. Using the axioms and previously proved theorems, each addition to the field has been defined with proof, an argument using only known facts and deductive reasoning (Öner, 2006). As such, proof is also of great importance to students of mathematics and those entering the field. The National Council of Mathematics Teachers’ (2000) *Principles and Standards for School Mathematics* describe students’ ability to reason and engage in proof writing as “essential to understanding mathematics.”

Despite the clear importance of deductive reasoning and proof writing in the field of mathematics, students of mathematics have great difficulty learning to justify their claims using deductive reasoning (Chazan, 1993; Coe & Ruthven, 1994; Moore, 1994) and struggle to generate their own proofs (Reiss & Renkl, 2002). Although there are difficulties with assessing students’ ability to write proofs and reason deductively, the data from large-scale assessments and research in the field show that only a small number of students demonstrate understandings related to proof and deductive reasoning (Harel & Sowder, 2007; Silver & Carpenter, 1989).

Tinto (1988) suggests that students even fail to comprehend both the general idea of what proof is and its role in the field of mathematics. According to her research,
many secondary school students think proof is used only to verify facts that are already known. Such deep misunderstandings serve to highlight the issue further. Studies focusing on students’ understanding related to proof and deductive reasoning outside of the United States have similar findings (Coe & Ruthven, 1994; Fischbein & Kedem, 1982; Harel & Sowder, 2007, Healy & Hoyles, 1998; Porteous, 1990; Recio & Godino, 2001; Williams, 1980).

There is no one reason why students experience difficulty in tasks involving proof and deductive reasoning. It stands to reason that if students are to be able to write proofs and reason mathematically, then instruction to that end is needed. Models of the kind of reasoning and justification required to write proof need to be provided.

Unfortunately, there are numerous studies that suggest there may be a lack of proof and reasoning in mathematics classrooms. Research currently suggests that many pre-university educators typically ignore both the importance and the role of proof and reasoning in the classroom. A study done by Porter (1993), spanning six states and 18 high schools observing 62 mathematics and science teachers, found that among the mathematics teachers, “on average, no instructional time is allocated to students learning to develop proofs, not even in Geometry” (p. 4). While there exist major differences in teachers’ approach to the teaching of such concepts (Tinto, 1988), such findings are still problematic.

Larger-scale studies further confirm the lack of attention to proof in secondary classrooms. The Third International Mathematics and Science Study (TIMSS) looked at videotapes of 30 eighth-grade classes in three countries, Germany, the U.S., and Japan. The report of the study indicated that:

The most striking finding in this review of 90 classes was the rarity of explicit mathematical reasoning in the classes. The almost total absence of explicit mathematical reasoning in Algebra and Before Algebra courses raises serious questions about the ways in which those subjects are taught ... the total absence of any instances of inductive or deductive reasoning in
the analyzed United States classes cries out for curriculum developers to address this aspect of learning mathematics. (Manaster, 1998, p. 803)

These findings are particularly troubling due to the fact that explicit instruction on proof has been found to increase student performance on proof tasks (Healy & Hoyles, 1998). Furthermore, Silver and Kenney (2000) reported that students whose eighth-grade teachers allocated more time to teaching reasoning in the mathematics classroom yielded higher overall scores on items designed to assess reasoning, justification, and proof on the 1996 National Assessment of Educational Progress (NAEP) when compared to their peers. Such studies not only underscore the importance of explicit instruction on proof and reasoning, but also the importance of exposing students to particular modes of thought prior to their initial proof writing experiences. Attention to reasoning should be present at all levels of mathematics education, especially in proof-writing courses and courses directly preceding students’ introduction to formal mathematical proof (National Council of Mathematics Teachers [NCTM], 2000).

Schoenfeld (1994) points out that even when proof is present in the American curriculum, it is often framed as something that can be separated from mathematics or as a topic under the umbrella of mathematics, when, in fact, it is the essence of engaging in mathematical thought.

Despite the recommendations of numerous policy-making and trend-setting organizations, it appears there is a lack of proof, reasoning, and statement justification in mathematics classrooms. Researchers point out that currently there is a lack of research pertaining to the presence of proof reasoning in current curricula (Sylianides & Silver, 2004).

**Purpose of the Study**

The purpose of this study was to determine the extent to which the modeling of deductive reasoning and proof-type thinking occurs in a mathematics course in which
students are not explicitly preparing to write formal mathematical proofs. Note that the term “modeling” was used to indicate the act of a teacher serving as an example of a type of behavior or engaging in an activity that students can emulate or learn from.

In order to prepare students to write proofs and engage in deductive reasoning, instruction should occur throughout their mathematics education. The *NCTM Principles and Standards* and the recently published *Common Core State Standards* both place great importance on proof and deductive reasoning at all levels of education (NCTM, 2000). If districts and teachers are preparing students according to the guidelines set forth by these trendsetter organizations, then there should be evidence of this in current curricula.

One lens through which the curriculum can be reviewed to identify the modeling of proof and deductive reasoning is through examination of textbooks. Research has shown that textbooks have a major impact on mathematics classrooms (Fujita & Jones, 2003; Nathan, Long, and Aliabi, 2002; Schmalz, 1990; Yerushalmy, Gordon, & Chazan, 1993). Textbooks play a significant role in determining a teacher’s classroom practices (Nathan et al., 1990) as well as influencing student activities (Yerushalmy et al., 1993). As a result, textbooks play an important and powerful role in determining much of students’ educational experiences. Indeed, some scholars believe that textbooks influence and often “shape” students’ mathematical knowledge (Fujita & Jones, 2003). Nathan et al. (2002) found that in addition to having a strong influence on a teacher’s handling of a given topic, textbooks may have a profound influence on teachers’ day-to-day planning.

One aim of the study was to address the lack of research pertaining to the presence of proof reasoning in current curricula (Sylianides & Silver, 2004) by looking for evidence of the modeling of deductive reasoning and proof-type thinking in secondary algebra textbooks.
Research Questions

1. What qualifies as justification or proof in non-proof writing introductory algebra textbooks (before students are explicitly instructed in proof writing)?
2. Using the most widely circulated textbooks as a lens, how often are students exposed to (a) justification of mathematical claims and mathematical reasoning or (b) the concept of proof in their introductory algebra course?
3. What kinds of justification or proof (if any) do students experience in first-year algebra textbooks?
4. When students are exposed to justification in algebra, how often do those justifications contain the same kind of reasoning required by mathematical proof?

Procedures of the Study

Multiple strategies were implemented in order to answer the research questions proposed. An extensive review of the relevant literature was conducted to answer the first research question. This was necessary because, in order to answer the latter research questions, proof must first be defined by identifying its various functions and what qualifies as a proof.

The latter research questions were addressed through analysis of the way in which various topics are handled in secondary school classrooms. To answer the second research question, textbooks’ handling of certain algebra topics was analyzed to see if any kind of justification or proof is provided when a new theorem, property, or procedure is first introduced. Textbooks examined included the most popular and widely circulated algebra textbooks in the United States. Textbooks examined included stated-optioned algebra textbooks from the most influential states, including New York, California, Texas, and Florida.
In cases where some sort of explanation, justification, or sense-making activity is present in the text, this study examined the text further to determine what kind of explanation or proof it was. Was it a mathematical proof or merely an empirical argument? Did it contain the same reasoning as a mathematical proof, but lack the formally accepted form? If it was found that textbooks offer empirical evidence to support mathematical claims (such as using a number of examples to convince a student a theorem “works”), then this could reinforce the findings of researchers such as Chazan (1993), Knuth, Slaughter, Choppin, and Sutherland (2002), and Thompson (1991).
Chapter II

LITERATURE REVIEW

History of Proof

According to Eves (1990), people have practiced mathematics in one form or another since before 3,000 B.C. Primitive cultures were acquainted with the notions of number and magnitude. Such concepts arose due to the practical needs of primitive man. Mathematical proof as we know it did not exist at that time.

According to the Greeks, mathematics originated in ancient Egypt (Eves 1990). However Egyptian mathematics did not focus on proof and reasoning as we do today. Rather, it seems it relied on procedures for calculations or algorithms whose results could be observed. This is similar to mathematics as it existed in ancient Mesopotamia. Conjectures were proved empirically similar to the way in which modern-day science might prove and test a theory (Arsac, 2007).

We have no evidence that mathematicians of antiquity ever attempted to prove mathematical concepts until ancient Greece. As Eves (1990) explains:

For the first time in mathematics ... men began to ask fundamental questions such as “Why are the base angles of an isosceles triangle equal?” and “Why does a diameter of a circle bisect the circle?” The empirical process of the ancient orient, quite sufficient for the question how, no longer sufficed to answer these more scientific inquiries of why. Some attempt at demonstrative methods was bound to assert itself, and the deductive feature, which modern scholars regard as a fundamental characteristic of mathematics, came into prominence. Thus mathematics in the modern sense of the word, was born.... (p.72)
The first person credited with attempting to prove his ideas using logical reasoning was Thales of Miletus (Eves, 1990). Although Thales worked on many of the same problems as his predecessors, his accomplishments were distinguished because he supported his findings using an argument employing logical reasoning. Prior to Thales, scholars had supported claims and findings with intuition and experimentation. Ancient Greece is where mathematics was transformed from an empirical discipline to the “demonstrative” science we know today (Arsac, 2007).

Around the 3rd century BCE, Euclid published *The Elements*. Euclid’s *Elements* consists of 465 propositions codifying the Greeks’ knowledge of geometry, number theory, and, some would argue, algebra (Boyer, 1991). In addition to defining geometry, even as we know it today, Euclid’s seminal work *Elements* also set the standard for modern mathematics and “dominated the mathematics of the western world until late 19th century and, in essence, is still intact in our days” (Harel & Sowder, 2007, p. 12). It is widely considered to be one of the most influential textbooks of all time (Boyer, 1991).

In the last 2,500 years since Thales and Euclid, mathematicians have pursued mathematical knowledge through logical arguments. The Greeks mostly proved concepts geometrically, using constructions and direct proofs. The Pythagoreans were among the first to establish that *all* mathematical statements needed to be proven through reason and logic (Eves, 1990). While they did not engage in proofs that mathematicians would consider sufficient today, they were still using logical reasoning to determine the veracity of their conjectures (Krantz, 2007).

It was not until the 19th and early 20th centuries that mathematical proofs began to acquire the form and rigor currently expected from mathematical proofs in all areas of mathematics (Eves, 1990). Today, axioms, theorems, valid assumptions, and definitions are laid out. Using a specific kind of logic and reasoning, they are shown to lead to a given statement being unequivocally true or false, provided the reader does
not disagree with the initially stated axioms, theorems, valid assumptions, and definitions.

The earliest evidence we have of the teaching of proof and deductive reasoning also dates back to ancient Greece (Stylianou, Blanton, & Knuth, 2009). There is evidence of teaching proof and deductive reasoning found in descriptions of the Pythagorean school, and we still have, intact, Plato’s famous work, the Republic, in which he describes in detail such instruction to be part of the curriculum taught to the future philosopher-rulers (Fowler, 1999).

In modern times in the United States, proof has been primarily taught as a part of high school geometry for over a century (Herbst, 2002). The idea that students should first learn to consume and produce proofs in geometry can be traced back to the late 19th century report of the Committee of Ten. A group of prominent educational leaders at the time, the Committee of Ten was formed to look closely at university curriculum and college admissions in relation to the high school curriculum and how high schools were preparing students for college (Herbst, 2002). In their report, they concluded that one of the reasons to teach mathematics to students is to “train the mind’s powers of conceiving, judging, and reasoning ... in formal geometry we have the best possible arena for training in deductive reasoning” (Hill, 1895, as reported by Stylianou et al., 2009). Historically, this makes sense in light of the aforementioned association of proof and geometry since Euclid published his Elements. Indeed, proof was not associated with other areas of mathematics (such as algebra and arithmetic) until the 19th century (Davis & Hersh, 1981, as reported by Stylianou et al., 2009).

Proof in school mathematics remained, in large part, a high school geometry topic until the “new math” movement of the 1960s attempted to incorporate proof into all areas of mathematics. In this regard, however, the movement was largely unsuccessful (Hanna, 1995).
Stylianou et al. (2009) report that Wu (1996) argued that the lack of proof in mathematics education was a gross misrepresentation of the field of mathematics and that its absence in the curriculum is “a glaring defect in the present-day mathematics education in high school, namely, the fact that outside geometry, there are essentially no proofs. Even as anomalies in education go, this is certainly more anomalous than others in so much as it presents a totally falsified picture of mathematics itself” (p. 228).

What is Proof?

As many scholars have noted, proof is of the utmost importance to the field of mathematics. Proof is central to all mathematical discovery, and it literally defines mathematics as we know it (Ball & Bass, 2003; Haimo, 1995; Harel & Sowder, 2007; Healy & Hoyles, 1998; Schoenfeld, 1994). Engaging in proof-type thinking and being able to generate proofs is what it means to engage in mathematics at higher levels. Haimo (1995) describes proof as the “major distinguishing component” of mathematics; further citing that it is the “essence of [its] uniqueness” (p. 103). In other words, proof is what sets mathematics apart from other fields. Various scholars, from both the world of mathematics and mathematics education, claim that engaging in proof writing and reasoning is central to both engaging in mathematics and learning mathematics (Ball, Hoyles, Jahke, & Movshovitz-Hadar, 2002; Epp, 1998; Herbst & Brach, 2006).

To prove a claim or theorem in mathematics is not to convince one or many people that the claim is true, nor is it to show it to be true for many cases or under certain conditions. To prove something to be true in the world of mathematics is to show something is true using logic and reasoning for all possible cases, even when there are an infinite number of such cases. It must be shown that given one or many accepted facts, the claim to be proven logically follows from those given facts and must be true. A
proof in mathematics is a logical argument, not an empirical one. As Aleksandrov (1969) states:

We could measure the angles at the base of a thousand isosceles triangles with extreme accuracy, but such a procedure would never provide a mathematical proof of the theorem that the base angles of an isosceles triangle are congruent. Mathematics demands that this result be deduced from the fundamental concepts of geometry, which are precisely formulated in the axioms. (p. 3)

Aside from some basic axioms, which mathematicians hold to be true, all of mathematics has been built in this fashion. Each addition to the field has been defined with what many scholars call a rigorous proof, an argument using only known facts and deductive reasoning (Öner, 2006). Such argument is not required to establish truth in other fields.

From the 4th grader asking why four times three is twelve, to the university professor pursuing more advanced and subtle conjectures, proof is essential to the field of mathematics at all levels. The ability to prove a conjecture to be true, with no doubt, is how the field of mathematics as a body of knowledge grows. Mathematics itself is built upon the ability to prove something to be true or false, without doubt or reservation (Kotelawala, 2007). As a result, within the professional mathematics community, the form and structure to which a valid proof must conform have been codified and formalized to a great degree. Mathematical proofs conform to certain norms and follow certain rules. Proofs are so central to the profession that many of the accepted avenues to proving a given statement have been given names. Dozens of proof techniques exist, such as: proof by induction, proof by exhaustion, proof by construction, and proof by contradiction.

Despite such codification and formalization, there are times when mathematicians disagree about what qualifies as a proof (Kotelawala, 2007). In order for a proof to be truly accepted as valid, a mathematician must get approval from various sources, including their mathematics community, school, or organization, and ultimately
from a peer-reviewed journal comprised of the larger community of mathematicians (Hersh, 1997).

Furthermore, mathematicians often expect more from proof than a mere statement that a given conjecture is true or false. Hanna (2000) has compiled a comprehensive list of the various functions of proof and proving in the mathematics community:

- **verification** (concerned with the truth of a statement)
- **explanation** (providing insight into why it is true)
- **systematization** (the organization of various results into a deductive system of axioms, major concepts, and theorems)
- **discovery** (the discovery or invention of new results)
- **communication** (the transmission of mathematical knowledge)
- **construction** of an empirical theory
- **exploration** of the meaning of a definition or the consequences of an assumption
- **incorporation** of a well-known fact into a new framework and thus viewing it from a fresh perspective.

The final proof itself is clearly concerned with verification and discovery; the end result shows the statement under consideration to be either true or false. Formal mathematical proofs, however, do not necessarily provide explanation, communication, or incorporation as stated above. “Some proofs are by their nature more explanatory than others” (Hanna, 2000, p. 5). Depending on the audience, the question of why a statement is true can be of much more value than merely stating that a statement is true.

The distinction between proofs that prove and proofs that explain is of greater significance when considering the audience. Students of mathematics experience proof differently from those already engaged in research. For the student of mathematics, the
proof’s role in answering the question “why” is much more important. Some scholars have chosen to define what qualifies as a proof by looking at the audience it will serve. Balacheff (1988) distinguishes proofs based on their audience. He explains that we call a proof an explanation which is accepted by a community at a given time. We call a mathematical proof, a proof accepted by mathematicians. As a discourse, mathematical proofs have now a days a specific structure and follow well-defined rules that have been formalized by logicians. (p. 2)

Since a specific community must accept a proof in order for it to be considered valid, proof becomes dependent upon the audience for which it is written.

A good example of how even formal proof writing within the professional mathematical community is audience-dependent comes from mathematicians’ use of the words “clearly” and “similarly.” Mathematicians routinely make use of these words in order to omit or skip parts of a proof they consider to be simple or repetitive (Goetting, 1995). By assuming that parts of a proof are clear to the reader and hence unnecessary to include, mathematicians are assuming their audience has certain knowledge. Without access to this omitted knowledge, the proof would be rendered invalid. Without all parts of a proof, it is not a logical, rigorous proof of the veracity of the statement to be proved. However, since the validity of the proof is based on the audience to which it is being presented, certain gaps in the argument are allowed.

Philosophers of mathematics question proof further, pushing the community’s understanding of proof and truth in mathematics. In his work Proofs and Refutations, Imre Lakatos (1976) challenges the very idea that one can (or should) ever consider a proof to be complete or absolutely true, arguing that one can only say that no counterexample has been found yet. He argues that proof-writing should be seen as more of a dialog between mathematicians, in which counterexamples to conjectures help to broaden, refine, and strengthen mathematical ideas. He rejects the idea of a finished proof and argues that this process of informal proof or conjecture and its
subsequent refutation through counterexample and discourse is how mathematical knowledge truly grows.

Proofs play different roles in different communities. Furthermore, what qualifies as proof depends on the audience to which it is being offered. To this end, proof must be discussed within a context. Mathematical proofs are important to both the professional and the educational mathematics communities, and thus a clear distinction between the role proof plays in these two communities needs to be made.

**Proof in Mathematics Education**

Mathematics students must interact with proofs in two clear ways. First, as they begin to learn mathematics, they must engage in asking “why.” The rules that govern mathematics need not be blindly memorized but can be understood given proof, that is, given a compelling argument as to why things are the way they are. Second, students of mathematics must themselves engage in proof writing, and they must receive some guidance and instruction to this end.

Scholars such as Hanna (1990) and Hersh (1993) cite explanation as the main reason for proof in the classroom. Students do not need to blindly believe or follow the teachings of their mathematics teacher. As is the nature of the subject, mathematics should inspire students to question and require proof for the rules and concepts that come up in the course of their studies. Understanding that mathematics is not just a collection of rules to apply and that it is founded on reasoning is an important message to convey to students (Stacy & Vincent, 2009). For example, the rules that govern arithmetic with exponents and logarithms do not exist because someone decided they should be that way. They exist because exponents and logarithms are built from a system and the laws that govern them logically flow from the fact that multiplication is repeated addition and exponentiation is repeated multiplication. If students fail to make
these connections when they engage in mathematics, then they misunderstand what mathematics truly is. They should be encouraged to ask why things are the way they are, and proof or justification for any claims made should be given.

In her paper, *Proofs that Prove and Proofs that Explain*, Hanna (1989) summarizes the distinction between the two kinds of proofs noted in the title:

There is nevertheless a very important difference between these two kinds of proofs. A proof that proves only shows that a theorem is true; a proof that explains also shows why it is true. A proof that proves may rely on mathematical induction or even on syntactic considerations alone. A proof that explains must provide a rationale based upon the mathematical ideas involved: the mathematical properties that cause the asserted theorem or mathematical statement to be true. (p. 47)

Hanna argues that the mission of a mathematics teacher is to make students understand mathematics and that, in order to do so, proofs that explain should have a more prominent role in mathematics curriculum. She argues that in some cases this may mean a move away from the formal proofs found in many textbooks toward alternative ways of demonstrating the validity of mathematical results. Given the different audience, what might suffice as a proof in the mathematics classroom can be different from that which would suffice in a mathematics journal.

It is essential, as mathematics educators, to have students engaging in proof writing regardless of its form. In order to truly experience feeling what it is to be a mathematician, students must write their own proofs. To this end, some teachers provide students with many opportunities to read proofs. A finished proof, however, often hides the process that went into writing it. In reading a proof and following its argument, it appears that the author sat down and proceeded, without interruption, to follow a series of steps that lead to the conclusion, when, in fact, nothing could be farther from the truth. Kotelawala (2007) cites Polya (1954), who describes this best:

Finished mathematics presented in a finished form appears as purely demonstrative, consisting of proofs only. Yet mathematics in the making resembles any other human knowledge in the making. You have to guess a
mathematical theorem before you prove it; you have to guess the idea of the proof before you carry through the details. You have to combine observations and follow analogies; you have to try and try again. The result of a mathematician’s creative work is demonstrative reasoning, a proof; but the proof is discovered by plausible reasoning, by guessing. If the learning of mathematics reflects to any degree the invention of mathematics, it must have a place for some guessing, for plausible inference. (p. vi)

This situation parallels Thomas Kuhn’s (1962) discussion of the history of science and the writing of textbooks. Textbooks give the impression that science has progressed in a fairly linear fashion since the dawn of the discipline itself. They give the student the impression that discovery A led to discovery B, which led to discovery C, which led to the state of science today. This is misleading and inaccurate. The current scientific paradigm is the result of various scientific revolutions, deviations, and debunked theorems (Kuhn, 1962). Similarly, proofs give the impression that a great mathematical mind simply sat down and wrote the proof line by line, when, in fact, proofs are often the result of countless false starts, guesswork, and tireless attempts.

Another serious problem with proof writing as it often exists in the mathematics classroom is the nature of proofs students in high school and college are often asked to write. It is not uncommon for a proof to read: Prove: Statement A; Givens: Statement B, Statement C. This is far from the act of authentic proof writing. In real proof writing, the mathematician must play with the math, experiment with it, and guess a little to make a conjecture in the first place. Only then does the mathematician begin to play with the mechanics of the proof itself. Here, once again, the mathematician must employ all of his or her knowledge to decide what given facts must be used and what approaches might be appropriate. By giving the statement to be proven and the given facts required to do so, mathematics educators take away much of what engaging in mathematics is all about.

The ability to comprehend and write proofs is of even greater importance today in the wake of Reform Mathematics, sometimes called Standards-Based Mathematics. The
National Council of Mathematics Teachers’ publication of *Curriculum and Evaluation Standards for School Mathematics* in 1989 was the initial driving force behind reforming of how mathematics was taught in the United States. Within the document, the NCTM advocated for making changes to the focus of the mathematics classroom, which had typically been centered on mastery of manual arithmetic and basic skills and procedures, to include more attention to conceptual knowledge and problem solving. In addition to mastering basic skills and having procedural fluency, the NCTM called for students, even in lower grades, to spend more time in the classroom, allowing students discovery knowledge on their own, and called for more of an emphasis on proof (Raimi, 2000; Ross, 2000). Prior to these reforms, the study of algebra was almost exclusively the study of equations and symbolic expressions (Star, Herbel-Eisenmann, & Smith, 2000). The result of these reforms led educators to focus more on conceptual knowledge and a more student-centered classroom, in which students discover “new” mathematics for themselves.

For educators who create student-centered classrooms and environments around the ideas of constructivist theory, proof plays a very central role. For this reason, a constructivist approach to mathematics education may be an effective avenue for giving mathematics students a true education in what it means to be a mathematician. The central idea of constructivist theory is that students are given the opportunity to discover the mathematics on their own, using what they already know.

Typically, in a constructivist approach, students are asked to develop their own rules and “discover” new mathematics. They are led to the discovery of mathematical concepts that are new to them. In this situation, the teacher plays the role of a facilitator and problem-poser, and the student engages in authentic problem solving, mathematical discovery, and ultimately authentic proof writing. When mathematics instruction takes place in a constructivist environment, logic, reasoning, and proof play a central role in students’ learning, and their education mimics the creation of
mathematics itself. The students are led, in a scaffolded way, through a process in which they take what they know and use it to prove new ideas and make conjectures of their own.

As Kotelawala (2007) points out, much of the research put forth in mathematics education suggests that the validity of a proof should be determined by considering the audience or community for whom it was prepared (Almeida, 1996; Hanna, 1989; Hersh, 1997; Smith & Henderson, 1959). Asking students to discover, justify, and explain mathematics, or asking them the question, “How do you know this is true?” is essentially asking students to prove a statement or conjecture (Almeida, 1996). Smith and Henderson (1959) argue that a second grader’s explanation of an arithmetic problem is a proof for the community of second graders to which they were presenting. When one considers the example of a second grader offering a “proof” to their peers, one needs to consider the implications for education.

**Student Performance in Writing Proof and Reasoning**

Despite the clear importance of proof both in mathematics education and in the field of mathematics at large, much of the research suggests that worldwide, at all levels, students struggle when it comes to proof writing (Alcock & Weber, 2005; Balacheff, 1988; Chazan, 1993; Harel & Sowder, 2007; Healy & Hoyles, 2000, Knuth et al., 2002; Recio & Godino, 2001; Silver & Carpenter, 1989; Williams, 1980). Learning to justify mathematical claims and use hypothetical deductive reasoning is an immense challenge for students, especially high school students (Chazan, 1993; Coe & Ruthven, 1994; Moore, 1994), who will be the primary focus of this study. Given this difficulty, it is no surprise that students have significant trouble generating proofs in general (Reiss & Renkl, 2002).
It should be noted that this difficulty in generating and understand proof is not unique to high school students and has been observed with undergraduates as well, even mathematics majors (Harel & Sowder, 1998; Weber, 2001). Selden and Selden (2003) presented undergraduates with various arguments claiming to prove the statement, “If $3$ divides $n^2$, then $3$ divides $n$,” and asked them to determine which arguments qualified as valid proof. They found that the students’ initial responses were no better than guessing. In a similar study, Alcock and Weber (2005) presented 13 undergraduate mathematics majors with a flawed proof in real analysis; less than half of the participants rejected the flawed argument as invalid and only two did so for legitimate reasons.

Looking at secondary school students, large-scale nationwide assessments such as the NAEP provide limited information about students’ understandings regarding proof and deductive reasoning. It appears that only a small percentage of students demonstrate understandings related to proof and deductive reasoning (Harel & Sowder, 2007). For example, examining related test items from the fourth NAEP (1985-1986), Silver and Carpenter (1989) were able to conclude that “most 11th grade students demonstrated little understanding of the nature and methods of mathematical argumentation and proof” (p. 11). Their analysis goes on to paint an even bleaker picture, stating that the students’ poor performance on test items assessing proof and deductive reasoning indicate that the experiences students have in school mathematics “fail to acquaint them with the fundamental nature and methods of the discipline” (p. 18).

Furthermore, other studies suggest that students fail to even comprehend what a proof is. Tinto (1988) found that many high school students in her study had such profound misunderstanding about proof and its role in mathematics that they thought proof was only used to verify facts that were already known.
Other research also indicates that many high school students will use examples to attempt to prove a statement, incorrectly assuming that such an empirical argument is sufficient to establish truth in mathematics (Balacheff, 1988; Chazan, 1993). Thompson (1991) conducted a study with a group of advanced college-bound secondary students enrolled in a course meant to emphasize reasoning and proof and found that a large number of participants attempted to “prove” a statement by giving examples. It seems, however, the problem goes back further. Knuth et al. (2002) found that the seeds of such misunderstandings begin as early as middle school. In their study, they found that roughly 70% of participants, in a group of 350 students, attempted to use examples to prove a given statement.

Learning to generate proof and justify claims mathematically is incredibly challenging, not only for American students, but also for students worldwide. In England and Wales, Healy and Hoyles (1998) conducted a large-scale study involving 2,459 14-15 year olds performing in the upper quartile on nationwide assessments of mathematics. The aim of the study was to assess both student performance in proof writing and students’ conceptions about the meaning and purpose of proof. The study found that the majority of students were “unable to distinguish and describe mathematical properties relevant to a proof and use deductive reasoning in their arguments” (p. 6). The average score on any proof-related test item was less than half, with 14% to 62% of students unable to even begin to formulate a response. “More than a quarter of students had little or no idea of the meaning of proof and what it was for” (Healy & Hoyles, 2000, p. 418). Other researchers, such as Coe and Ruthven (1994) and Porteous (1990), have also looked at British students and have findings that are consistent with Healy and Hoyles’s research or reveal a similarly bleak picture of students’ understandings surrounding proof.

Fischbein and Kedem (1982) found that students failed to understand that once a proof of a claim was constructed, no further examples were needed to verify the claim.
In Spain, Recio and Godino (2001) found that less than half of a group of beginning university students participating in their study were able to prove either of the following statements: “The [absolute] difference between the squares of consecutive natural numbers is odd and equal to the sum of the numbers,” and “The bisectors of adjacent supplementary angles are perpendicular.” About a third could prove both. In Canada, Williams (1980) interviewed 11th grade students and concluded that less than 30% demonstrated understanding of the meaning of mathematical proof. While there are countless other studies looking at various nations around the world (Harel & Sowder, 2007), the fact is clear: students’ ability to write and understand proof and engage in deductive reasoning is a problem not just in the United States but internationally.

Some studies suggest that this problem may extend to teacher knowledge as well. Some research has shown that teachers of mathematics experience difficulty determining the validity of a given proof (Selden & Selden, 2003). Martin and Harel (1989) found that pre-service elementary teachers’ ability to determine the validity of a proof was more based in form than content. Participants in the study were found to accept proofs presented in a two-column format as valid and reject proofs presented in paragraph form. In an interview-based study involving 16 in-service high school teachers, Knuth (2002a) found that participants had significant difficulty determining proof validity. Participants not only struggled with the task of proof validation, but many accepted flawed arguments as valid mathematical proofs.

The current state of research suggests that students of mathematics have difficulty in writing proof and reasoning both domestically and internationally. There are many possible reasons why students experience such difficulty in tasks involving proof and deductive reasoning. If mathematics educators want students to be able to effectively write proofs and reason mathematically, then they need to be given instruction to that end by providing models of the kind of reasoning and justification required to write proof.
**Where’s the Proof?**

There are numerous studies that suggest there may be a lack of proof and reasoning in mathematics classrooms. Research currently suggests that many pre-university educators typically ignore both the importance and the role of proof and reasoning in the classroom. In a study done by Porter (1993) spanning six states and 18 high schools observing 62 mathematics and science teachers, it was found that among the mathematics teachers, “on average, no instructional time is allocated to students learning to develop proofs, not even in Geometry” (p. 4). While there exist major differences in teachers’ approach to the teaching of such concepts (Tinto, 1988), such findings are still troubling and problematic.

Larger-scale studies further confirm the lack of attention to proof in secondary classrooms. The Third International Mathematics and Science Study (TIMSS) looked at videotapes of 30 eighth-grade classes in three countries, Germany, the U.S., and Japan. The report of the study indicated that:

> The most striking finding in this review of 90 classes was the rarity of explicit mathematical reasoning in the classes. The almost total absence of explicit mathematical reasoning in Algebra and Before Algebra courses raises serious questions about the ways in which those subjects are taught ... the total absence of any instances of inductive or deductive reasoning in the analyzed United States classes cries out for curriculum developers to address this aspect of learning mathematics. (Manaster, 1998, p. 803)

These findings are particularly troubling due to the fact that explicit instruction on proof has been found to increase student performance on proof tasks (Healy & Hoyles, 1998). Furthermore, Silver and Kenney (2000) reported that students whose eighth-grade teachers allocated more time to teaching reasoning in the mathematics classroom yielded higher overall scores on items designed to assess reasoning, justification, and proof on the 1996 NAEP when compared to their peers. Such studies not only underscore the importance of explicit instruction on proof and reasoning, but also the importance of exposing students to particular modes of thought prior to their initial
proof writing experiences. As NCTM’s (2000) *Principles and Standards for School Mathematics* explain:

Reasoning and proof cannot simply be taught in a single unit on logic, for example, or by "doing proofs" in geometry. Proof is a very difficult area for undergraduate mathematics students. Perhaps students at the postsecondary level find proof so difficult because their only experience in writing proofs has been in a high school geometry course, so they have a limited perspective (Moore 1994). Reasoning and proof should be a consistent part of students' mathematical experience in prekindergarten through grade 12. Reasoning mathematically is a habit of mind, and like all habits, it must be developed through consistent use in many contexts.

Attention to reasoning should be present at all levels of mathematics education, especially in proof writing courses and courses directly preceding students’ introduction to formal mathematical proof.

Schoenfeld (1994) points out that even when proof is present the curriculum, it is often framed as something that can be separated from mathematics or as a topic under the umbrella of mathematics, when, in fact, it is the essence of engaging in mathematical thought. Schoenfeld’s observation makes perfect sense when placed in a historical context. Until the later part of the 20th century, deductive reasoning and proof was not seen as a topic deserving of attention throughout schooling and was almost exclusively studied in a student's first course in geometry. Such courses were typically modeled on Euclid’s *Elements*. Students typically interacted with proof by way of memorizing key proofs involved in developing Euclidean geometry (Herbst, 2002; Sinclair, 2008).

Despite the recommendations of numerous policymaking and trend-setting organizations, it appears there is a lack of proof, reasoning, and statement justification in mathematics classrooms. The National Council of Mathematics Teachers (NCTM) *Principles and Standards* and the recently published *Common Core State Standards* both place great importance on proof and deductive reasoning at all levels of education (NCTM, 2000). The National Council of Mathematics Teachers’ (2000) *Principles and
Standards for School Mathematics describe students’ ability to reason and engage in proof writing as “essential to understanding mathematics.”

Reasoning and Proof is one of NCTM’s Process Standards. It stipulates that instructional programs from prekindergarten through 12th grade should enable all students to:

- Recognize reasoning and proof as fundamental aspects of mathematics
- Make and investigate mathematical conjectures
- Develop and evaluate mathematical arguments and proofs
- Select and use various types of reasoning and methods of proof. (NCTM, 2000)

Students’ ability to reason deductively is also highlighted as an important part of mathematics education in the recently published Common Core State Standards (National Governors Association Center for Best Practices, 2010). In addition to proof and reasoning appearing repeatedly in the content strands, reasoning appears in 4 of the 8 of the Common Core State Standards’ Standards for Mathematical Practice.

Despite such recommendations, researchers point out that currently there is a lack of research pertaining to the presence of proof reasoning in current curricula (Sylianides & Silver, 2004). In order to prepare students to write proofs and engage in deductive reasoning, instruction should occur throughout their mathematics education. If districts and teachers are preparing students according to the guidelines set forth by these policymaking and trend-setting organizations, then there should be evidence of said preparation in current curricula.

Algebra, Deductive Reasoning, and Proof

A typical student in the United States school system formally studies proof for the first time in high school geometry (Hanna, 1998; Senk, 1985). However, in order to meaningfully engage in proof-type thinking upon commencing the study of geometry,
students’ deductive reasoning skills must be developed at all levels of their mathematics education prior to their formal introduction to proof (NCTM, 2000). Preparing students for geometry is not the only motivation for developing reasoning and proof-type thinking. While these skills are important to the study of proof, they also enable students to interact meaningfully with mathematics at all levels of study.

Deductive reasoning and proof-type thinking are central to the study of algebra (Greenes & Findell, 1998). Without the ability to reason using known mathematics, algebra appears as a collection of facts to be memorized, which it is not. Vance (1998) even describes algebra as a “way of thinking.” In order to truly master algebra and engage in algebraic reasoning, a student must be adept at reasoning deductively (Kriegler, n.d.). Therefore, it makes sense that the study of algebra would begin in close proximity to the study of proof in geometry, where deductive reasoning and proof-type thinking are explicitly required.

While the idea that proof should first be introduced in geometry dates back to the late 19th century report of the Committee of Ten (Herbst, 2002), it may also just be tradition, as proof and geometry have always been associated due to Euclid’s Elements. The idea that proof is intrinsic to all areas of mathematics is relatively new. As reported by Stylianou et al. (2009), Davis and Hersh (1981) point out that proof was not associated with other areas of mathematics other than geometry until the 19th century.

Kriegler (n.d.) states that algebraic reasoning has two major components. They include, “(1) the development of mathematical thinking tools and (2) the study of fundamental algebraic ideas” (p. 1). In other words, algebraic reasoning is (1) the ability to reason deductively and make connections (in addition to other mental activities); and (2) application of fundamental ideas of mathematics. Kriegler is essentially describing the same modes of thought required by proof.

As it is important that students understand why a given algorithm works (Morrow & Kenney, 1998; Skemp, 1978), proof-type thinking and the ability to reason deductively
also help students learn algorithms and procedures associated with many algebraic problems. Without developing a student’s ability to reason using known mathematics, they cannot deduce why many of the algorithms taught in algebra are permitted or work. This is true for many of the most common algebra topics, such as solving an equation for an unknown or solving a system of equations by elimination.

Furthermore, deductive reasoning enables students to use algorithms and procedures correctly as well as select an appropriate procedure or algorithm for a given problem, a skill often referred to as procedural flexibility in the research community (Star, 2001, 2002, 2004). Many national and international assessments show that U.S. students lack procedural flexibility, suggesting that they only know algorithms and procedures by rote and are unable to select an appropriate algorithm or procedure when they are given a novel or unfamiliar problem (e.g., Beaton, Mullis, Martin, Gonzales, Kelly, & Smith, 1996; Schmidt, McKnight, Cogan, Jakwerth, & Houang, 1999, both as reported by Star, 2005).

Deductive reasoning and proof are just as central to the study of algebra as they are to geometry. Even without formal instruction in proof, the deductive reasoning and proof-type thinking required to engage in algebraic reasoning necessitates instruction. In order to prepare students to write proofs and engage in deductive reasoning, instruction need occur throughout their mathematics education, as stated in NCTM’s (2000) Principles and Standards for School Mathematics cited earlier.

If students are being prepared to reason mathematically throughout their education, then there should be evidence of said preparation throughout their education, and especially in algebra, immediately prior to their introduction to proof in geometry.
Textbooks and the Teaching and Learning of Proof and Reasoning

For the purposes of this study, curriculum will be defined as Senk and Thompson (2003) define it: Curriculum is the mathematical content of the textbook or instructional materials. This study will focus on the mathematical content (the curriculum) put forth in various algebra textbooks.

Research has shown that textbooks have a major impact on the mathematics classroom. Textbooks play a significant role in determining a teacher’s classroom practices (Nathan et al., 1990) as well as influencing student activities (Yerushalmy et al., 1993). As a result, textbooks determine much of a student’s educational experience. Indeed, some scholars believe that textbooks influence and often “shape” students’ mathematical knowledge (Fujita & Jones, 2003). Nathan (2002) found that in addition to having strong influence on how a teacher handles a given topic, textbooks may also have profound influence on a teacher’s day-to-day planning.

Begle (1973) explains, “The Textbook has a powerful influence on what students learn…. The evidence indicates that most student learning is directed by the text rather than the teacher” (p. 209). On this point, the research is clear: textbooks play a major role in student learning (Begle, 1973; Nathan et al., 2002; Senk & Thompson 2003).

Textbooks and curriculum are primary factors in shaping students’ understanding of deductive reasoning and proof (Dreyfus, 1999; Hoyles, 1997; Stylianides & Silver, 2004). Hoyles’s (1997) research supports this claim empirically. Öner (2006) explains:

Hoyles observed that meanings students appropriate for proof were shaped and modified by the way the curriculum is organized, which is exemplified in mathematics textbooks that are written for that curriculum. In that curriculum, students are not introduced to definitions and generally not required to produce logical deductions in mathematics. The results of her questionnaire indicated that students’ responses matched this particular notion of proof. (p. 8)

Textbooks play a major role in shaping students' understandings as well as how teachers approach topics in their classrooms.
Despite the clear importance of the content in mathematics textbooks and the strong impact they have on instruction, student learning, and student conceptions, there is a lack of research in the area (Stylianides & Silver, 2004). Many scholars have called for the research community to give more attention to the analysis of textbooks (Baker et al., 2010; Reys, Reys, & Chavez, 2004; Senk & Thompson, 2003; Weiss, Knapp, Hollweg, & Burrill, 2001). The National Research Council (2004) even specifies that more research needs to be done on the content of textbooks with regard to the treatment of standards dealing with mathematical reasoning.

**Importance of the Study**

The purpose of this study was to determine the extent to which the modeling of deductive reasoning and proof-type thinking occurs in a mathematics course in which students are not explicitly preparing to write formal mathematical proofs. It aimed to address the lack of research pertaining to the presence of proof reasoning in current curricula (Sylianides & Silver, 2004). In the United States, the first time a student is likely to encounter proof is in their first-year geometry course, and the course prior to that course is typically algebra (Hanna, 1998; Senk, 1985). If districts and teachers are preparing students to reason about mathematics and write proofs according to the guidelines set forth by various policymaking and trend-setting organizations, then there should be evidence of said preparation in current curricula. Since students are explicitly taught proof in geometry, the non-proof writing course where modeling of proof and reasoning should be strongest and most prominent is algebra.

One way to audit the current enacted curriculum is by looking at textbooks. If students are not exposed to proof or reasoning and justification throughout their mathematics education, then this may begin to explain the voluminous amount of research pointing to their later difficulties with proof writing. Furthermore, if it is found
that textbooks offer empirical evidence to support mathematical claims (such as using a number of examples to convince a student that a theorem “works”), then this could reinforce the findings of researchers such as Chazan (1993), Knuth et al. (2002), and Thompson (1991).
Chapter III

METHODS AND PROCEDURES

Overview

This chapter provides a description of the methods that were used to collect and analyze data for this study. It begins with a restatement of the research questions that guided the study. The second section provides a description of the data collection. It describes how the textbooks used in the study were selected, how the topics analyzed in each textbook were selected, and the methods used for analyzing each textbook section. The third section describes the methods used for analyzing the data. Finally, the fourth section describes the limitations of the methods used to analyze the data.

The following research questions were considered in this study:

1. What qualifies as justification or proof in non-proof writing in introductory algebra courses (before students are explicitly instructed in proof writing)?
2. Using the most widely circulated textbooks as a lens, how often are students exposed to (a) justification of mathematical claims and mathematical reasoning or (b) the concept of proof in their introductory algebra course?
3. What kinds of justification or proof (if any) do students experience in first-year algebra textbooks?
4. When students are exposed to justification in algebra, how often do those justifications contain the same kind of reasoning required by mathematical proof?
The first question is addressed in the literature review. To answer the latter research questions, the way in which textbooks handle certain algebra topics will be analyzed. First, it will be determined whether justification or proof is provided when a new theorem, property, or procedure is introduced. Then the type of justification or proof that is provided will be classified.

Data Collection

Selection and Acquisition of Texts

This study examined various sections from a number of introductory algebra textbooks.

The aim in the selection of textbooks was to examine the textbooks students are most commonly exposed to in the United States. To this end, textbooks examined were intended to include some of the most popular and widely circulated algebra textbooks currently in use from some of the largest and most influential states. Textbooks examined included stated-option algebra textbooks and/or texts currently in use from a number of states, including New York, California, Texas, and Florida.

Textbooks were collected in two ways. First, the researcher reached out to various schools in the aforementioned states, requesting they send the name of any textbooks they use to teach algebra, as well as a copy of the book if they had one to spare. Second, the researcher attempted to purchase any uncollected textbooks from the list of stated-option algebra textbooks from each state.

In this way, a collection of textbooks that students are actually exposed to was selected and acquired. It should be noted that the books currently in use in schools are not always the most current version of a textbook, even if the most current version typically appears on the list of stated-option texts. In situations where two versions of a
given text were acquired or selected, the researcher only used the more current text for the study.

Since the goal of the study was to capture what is being done in classrooms, books that are stated-option texts for multiple states were counted once for each state in which it is an approved text. Some publishers release a different version of their textbook for each state. For instance, Glencoe’s *Algebra 1*, which is authored by Holiday et al., has separate editions for Texas, Florida, and New York. These different versions have different titles, and some of the sections are custom-tailored to the curricular guidelines set forth by the respective state. However, with regard to the elementary topics covered in this study, with few exceptions, the actual content of the textbooks does not differ. In almost all cases of textbooks published with a special state-specific edition, combining like terms, solving for an unknown, and operations on exponential expressions with same base (and the other topics covered in this study) are treated identically. Still, these are different textbooks, and this study treated each of them as a different book. This was to allow for the possibility that the books might differ in some sections.

As this study aimed to get a clear picture of the intended curriculum in the United States, this created a problem. Other books, such as *Discovering Algebra*, do not have special state-specific editions (with the exception of a California edition) even though *Discovering Algebra* is used in New York, Florida, and Texas. If it was only counted once, then it would not have as much weight as the Glencoe series of algebra books simply because Glencoe produces state-specific editions, even though *Discovering Algebra* makes up an equal amount of the nation’s intended curriculum.

Thus, books that are approved for use in multiple states have a greater impact because they are used in more classrooms and therefore should have more weight in the study. In order to address this, books that are stated-option texts for multiple states were counted once for each state in which they were an approved text. This was
necessary in order to ensure that the relative impact of each textbook on classrooms in the United States was represented with as much accuracy as possible. Therefore, throughout the study, *Discovering Algebra* was treated as though there are four editions, one for New York, one for Florida, one for Texas, and one for California.

Below is a table containing all textbooks included in this study. Textbooks that have multiple state-specific editions, or which are stated-option texts for multiple states are listed in parentheses in order to condense the list and make it easier to read. (See Appendix A for an expanded list including all publishers and authors.)

Table 1. Textbooks Included in the Study

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<thead>
<tr>
<th>Textbook</th>
<th>(Each state in which the book is used or has a separate edition is noted in parentheses)</th>
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<tbody>
<tr>
<td>Algebra 1 An incremental development (CA, FL, NY)</td>
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<tr>
<td>Algebra 1 Concepts and Skills (FL, NY)</td>
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<tr>
<td>Algebra Concepts and Applications (CA, NY, TX)</td>
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<td>Algebra Connections (TX)</td>
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<tr>
<td>Amsco's Integrated Algebra (NY)</td>
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<td>CME Project Algebra 1 (NY)</td>
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<tr>
<td>Discovering Algebra (CA, FL, NY, TX)</td>
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<tr>
<td>Glencoe Mathematics Algebra 1 (FL, NY, TX)</td>
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<tr>
<td>Holt McDougal Algebra 1 (CA, NY)</td>
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<tr>
<td>Holt, Rinehart and Winston New York Algebra 1</td>
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</tr>
<tr>
<td>McDougal Littell Algebra 1 (FL, NY, TX)</td>
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<td>Prentice Hall Algebra 1 (FL)</td>
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<td>Prentice Hall California Algebra (CA)</td>
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<tr>
<td>Prentice Hall Integrated Algebra (NY)</td>
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<tr>
<td>Springboard Mathematics with Meaning Algebra 1 (FL)</td>
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</table>

**Unit of Analysis**

The unit of analysis for this study was one section of the textbook that covered a single concept, theorem, property, or procedure. The term “topic” was sometimes used to avoid repetition of the phrase “concept, theorem, property, or procedure.” The
phrase “coverage of the topic” or “the textbook” was used to refer to the particular section of the textbook that was allocated to the coverage of the theorem, property, or procedure to be examined. Each textbook was reviewed for the coverage of multiple topics; thus, multiple sections from each book were examined, with each section constituting a single unit of analysis.

Selection of Topics

A preliminary survey of the selected textbooks was performed. All the topics covered were recorded. The topics chosen were those that met the following criteria:

(1) Each topic had to be present in all the curricula across the various states surveyed so as to ensure each topic could be examined in each textbook.

(2) The topics were selected so that some (not all) were in related areas of the curriculum and some were intended to come from different areas of the algebra curriculum. They did not all come from the same unit, although some of the topics were clearly related and might appear in the same unit.

The second point needs some clarification. Combining like terms and the distributive property are clearly related. From a mathematical standpoint, one is, in fact, a consequence of the other. However, both were chosen because some books use one to explain the other. The distributive property is not always used to justify combining like terms. If some books used simply state certain topics in order to prove or justify others, then that was something the study should capture. It should also be noted that these were not all the topics that met the above criteria, merely a selection of those topics that met the criteria. Table 2 provides a brief summary of each topic chosen.
<table>
<thead>
<tr>
<th>Topic</th>
<th>Previous knowledge and/or underlying ideas that could be called upon in order to prove or justify the topic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Combining Like Terms</td>
<td>• Multiplication is repeated addition</td>
</tr>
<tr>
<td></td>
<td>• Distributive property</td>
</tr>
<tr>
<td>Distributive Property</td>
<td>• Multiplication is repeated addition</td>
</tr>
<tr>
<td></td>
<td>• Combining like terms</td>
</tr>
<tr>
<td></td>
<td>• Associative property</td>
</tr>
<tr>
<td>Multiplication of Exponential Expressions</td>
<td>• Exponentiation is repeated multiplication</td>
</tr>
<tr>
<td>Division of Exponential Expressions</td>
<td>• Prime factorization</td>
</tr>
<tr>
<td></td>
<td>• Reducing</td>
</tr>
<tr>
<td></td>
<td>• Cancelling multiple representations of the numeral one.</td>
</tr>
<tr>
<td>Solving for an Unknown in First-Degree Equation</td>
<td>• Adding or subtracting equivalent quantities from both sides of an equal sign will maintain equality</td>
</tr>
<tr>
<td></td>
<td>• Inverse operations undo each other</td>
</tr>
<tr>
<td></td>
<td>• Transitive property of equality.</td>
</tr>
<tr>
<td>Solving Systems of Linear Equations by Elimination</td>
<td>• Adding or subtracting equivalent quantities from both sides of an equal sign will maintain equality</td>
</tr>
<tr>
<td></td>
<td>• Inverse operations undo each other</td>
</tr>
<tr>
<td></td>
<td>• Transitive property of equality.</td>
</tr>
<tr>
<td>Parallel Lines Have the Same Slope.</td>
<td>• Similarity</td>
</tr>
<tr>
<td></td>
<td>• Trigonometry</td>
</tr>
<tr>
<td></td>
<td>• Proof by contradiction</td>
</tr>
<tr>
<td></td>
<td>• Concept of Slope</td>
</tr>
</tbody>
</table>

**Categorization of the Coverage of the Topic**

The main form of data analysis was an assessment of what percentage of topics examined offered justification.

For each textbook, multiple variables were identified. The name of the book, publisher, year it was published, and the page numbers of topics being examined were recorded. For each text, the researcher determined if the textbook’s treatment of each topic offered any kind of explanation, justification, or sense-making activity. The treatment of the topic was categorized and coded as one of four types according to the rubric described in Table 3.
Table 3. Rubric for Classifying the Coverage of the Topic

<table>
<thead>
<tr>
<th>Code</th>
<th>Type</th>
<th>Rubric Used for Coding Each Type</th>
</tr>
</thead>
</table>
| 1    | No justification or reasoning present | • Does not attempt to explain why a given theorem, property, or procedure is true or works.  
• Does not call upon prior knowledge in the context of the new topic.  
• May show examples of how to use a concept, theorem, property, or procedure, but only to show how to use it  
• States the concept, theorem, property, or procedure. |
| 2    | Some form of justification is present | • Attempts in some way to convince the reader of the truth of the concept theorem, property or procedure.  
• May call upon prior knowledge, but only in a cursory way and not within specific examples. For instance, may state a prior theorem in the beginning of the section that could be used in conjunction with reasoning to show why, but does not justify individual steps within an example to show where the theorem or principle is being used.  
• May show examples to demonstrate that a theorem, property, or procedure “works.”  
• May attempt to link the topic to prior knowledge but does not make explicit connections. |
| 3    | Justification models or prompts mathematical reasoning | • Attempts to convince the reader of the truth of the concept, theorem, property or procedure using mathematics and reasoning.  
• May show a specific example and call upon prior knowledge at specific points in the example to show why the theorem, property, or procedure “works” or is true.  
• Shows how the given topic follows from mathematics already known using reasoning.  
• Justifies steps in its argument citing mathematical properties and/or prior knowledge.  
• May attempt to be general, but only by substituting letters for numbers after an argument has been made using a specific example (does not make the argument generally).  
• Does not call upon prior knowledge in its attempts to be general. |
| 4    | Offers proof | • Attempts to convince the reader of the truth of the concept theorem, property or procedure using mathematics and reasoning.  
• Justifies steps in its argument citing mathematical properties and/or prior knowledge.  
• Calls upon prior knowledge to show why the theorem, property, or procedure is true in a general way.  
• Shows a proof, even if it lacks for formal form. |
The first classification represents texts with a lack of justification or reasoning. A typical text that falls into this category may show examples of the property or procedure being used, but it is done more to demonstrate the procedure associated with the topic than to convince the reader of the truth of the theorem, property, or procedure using mathematics and reasoning.

Textbooks that meet the criteria of the second classification offer some kind of justification or modeling of mathematical reasoning. For example, a theorem that could be used to convince the reader of the truth of the theorem, property, or procedure using mathematics and reasoning may be stated, but it is not used to do so. The text that falls into this category may call upon prior knowledge, but only in a cursory way and not within specific examples. It may state or remind the reader of prior knowledge that could be used in conjunction with reasoning to justify the topic, but it does not justify individual steps within an example to show how the theorem or principle is being used. Texts in this category may show examples of the theorem, property, or procedure being used, but only to demonstrate that it “works.”

Texts of the third classification model or prompt mathematical reasoning. The text may show how the given topic follows from mathematics already known using reasoning. In other cases, it may show a specific example that references prior knowledge at specific points in order to show why the theorem, property, or procedure is permitted or true. Although it shows how the given topic follows from mathematics already known using reasoning, it fails to be general. It may attempt to state the theorem, property, or procedure in a general way, but only after justification. Coverage of a given topic classified in the third group indicates that while the text may attempt to model or prompt mathematical reasoning, it does not do so in a general way.

The fourth classification represents texts that offer a sound mathematical proof. For the purposes of this study, a proof shall be defined as any argument or justification that offers the same ideas or reasoning as a proof, even if it lacks the traditional form.
This fourth type of justification is distinguished from the third in that it offers a generalized argument that does not depend on a specific example. In order for a textbook's coverage of a topic to be coded as belonging to the fourth group, it must call upon prior knowledge to show why the theorem, property, or procedure is permitted or true in a general way.

The distinction between the classifications is subtle, but significant. For example, if a text introduces combining like terms by listing examples in order to display that a stated procedure “works,” then it was coded into the second category, since there is some attempt at justifying that the procedure offered is true. If the text offers any attempt to show why those examples work, say by calling upon a student's knowledge that multiplication is repeated addition or that the distributive property can be applied to produce the same results, then it was coded into the third or fourth category, depending on the explicitness and generality of the explanation. In many cases, the distinction between being coded as category 3 and category 4 was an issue of the generality of the argument. If this study finds that students are usually exposed to justification of mathematical claims only within the context of specific examples, then this may start to explain the research of Balacheff (1988) and Chazan (1993), who have shown that it is common for high school students to use examples in an attempt to prove a statement, incorrectly assuming that such an empirical argument is sufficient to establish truth in mathematics.

It should be noted that the above use of the phrases “prior knowledge” or “known mathematics” refers to any topics covered in the textbook prior to the textbook's coverage of the topic being examined.

Some secondary mathematics algebra textbooks introduce topics in the form of a series of scaffolded student-centered activities. When coding texts that assume such an approach, the coders assumed that students were able to complete the tasks as described and included all intended student responses in their analysis of the coverage
of the topic. In other words, if a text is trying to show that a particular theorem is true by leading the reader through a series of scaffolded activities, then the text was coded assuming the reader is capable of responding to the given activities.

Categorization of Student Activities Following the Coverage of the Topic

An examination of the student activities and exercises at the end of each section was performed for two of the topics examined in this study to assess the nature of the problems. The two topics for which the exercises were examined are the division of exponents and the slopes of parallel lines. It was unnecessary to examine student activities and exercises at the end of each section for every topic since research has shown that textbook exercises and student activities have a consistent level of difficulty throughout the text. The researcher examined all questions numbered as multiples of three and coded them according the scheme described below. Textbooks will typically provide answers to the odd-numbered exercises at the end of the book. This could affect the kinds of questions textbooks choose to have odd- or even-numbered. The decision to code student activities that are numbered as multiples of three allowed the researcher to capture both odd and even numbers, without having to code every single item. Only questions pertaining to the section were coded. If more than one topic was covered in a section, then questions relating to a topic other than the one being examined was labeled "Not Applicable" and was ignored in the analysis. If the book had a section, separate from the section containing the coverage of the topic, that is intended to maintain past skills or spiral to previous material, then it was not coded.

Since the formatting of the student activities section can vary greatly from textbook to textbook, two alternate coding schemes were developed. In some cases, the way in which the student activities are numbered can cause the coding of the student activity section to be misrepresented using the above coding schema. To address this
issue, in cases where a text has only 18 or fewer student activities, but has questions or activities containing four or more parts, the researcher simply looked at every third student activity. For instance, if in a text containing 18 or fewer questions, the first activity had 10 sub-activities delineated with letters a-j, then the researcher coded every third activity separately as a different activity or question. In the above case, coding began with an examination of 1c, 1f, 1i, and so on. This is referred to as alternate coding schema 1 in the data.

In other cases, textbooks may offer very few student activities. If a text offered fewer than 10 student activities or exercises following a section, then the coder coded all student activities. This alternative coding scheme is referred to as alternate coding scheme 2 in the data.

The rubric used to look at each student activity/exercise is outlined below. Each question was coded as one of three types: (1) Little to no deductive reasoning required; (2) Some deductive reasoning required; (3) Reasoning and/or proof required. Each of these three categories correlates to the six levels of Bloom’s (1956) taxonomy of cognitive domains, as shown in Table 4.

Table 4. Rubric for Classifying the Student Activities and Exercises Following the Coverage of Each Topic

<table>
<thead>
<tr>
<th>Code</th>
<th>Rubric used for coding each type</th>
<th>Bloom’s Taxonomy (Bloom, 1956)</th>
</tr>
</thead>
</table>
| 1) Little to no deductive reasoning required | • Prompts the student to apply a rule or procedure.  
• Asks questions or presents activities that are mathematically identical to examples done earlier in the section.  
• Prompts the student to restate facts.  
• May require student to use a concept, theorem, property or procedure from a previous section to reduce the activity to an activity that is mathematically identical to examples in the current section.  
• May require the student to recognize when a concept, theorem, property or procedure can or cannot be used. | Knowledge (Draw, Recognize, Count, Reproduce, Memorize, State, Tabulate, Identify, Point, Follow Directions)  
Comprehension (Change, Classify, Convert, Estimate, Interpret, Measure, Show, Express in other terms) |
Table 4 (continued)

<table>
<thead>
<tr>
<th>Code</th>
<th>Rubric used for coding each type</th>
<th>Bloom’s Taxonomy (Bloom, 1956)</th>
</tr>
</thead>
</table>
| 2) Some deductive reasoning required | • Prompts the student to apply a mathematical concept, theorem, property or procedure in a context that is not derivative of examples in the section.  
• Prompts the student to fix a misused concept, theorem, property or procedure (Or decide if it is used correctly).  
• May present the student with word problems or real life contexts that once translated into mathematical statements or modeled mathematically are identical to examples done earlier in the section. | Application (Calculate, Compute, Construct, Demonstrate, Derive, Graph, Manipulate, Operate, Practice, Solve)  
Analysis (Break down, Diagram, Distinguish, Formulate, Group, Order, Separate, Simplify, Sort) |
| 3) Reasoning and/or proof required | • Prompts the student to reason using mathematical concepts presented to verify or refute a mathematical claim or solve a previously unseen problem.  
• Prompts the student to justify or prove new mathematical claims.  
• May prompt the student to make a claim and justify it mathematically.  
• Asks questions or presents activities that lead the student to discover a new concept, theorem, property, or procedure. | Synthesis (Construct, Create, Prove, Deduce, Derive, Develop, Document, Generate, Integrate, Plan, Predict, Prepare, Propose, Specify, Tell)  
Evaluation (Appraise, Choose, Compare, Conclude, Decide, Describe, Evaluate, Justify, Measure, Validate) |

Coding of the Coverage of the Topic

The variables recorded for each text are described in Table 5.

Table 5. Coded Variables

<table>
<thead>
<tr>
<th>Description of Variable</th>
<th>Variable Name</th>
<th>Coding</th>
</tr>
</thead>
<tbody>
<tr>
<td>Text’s Name</td>
<td>TextName</td>
<td>The name of the textbook</td>
</tr>
<tr>
<td>Publisher</td>
<td>Publisher</td>
<td>A text string containing the name of the publisher</td>
</tr>
<tr>
<td>Year of Publication</td>
<td>TextYear</td>
<td>The four numeral year the text was published</td>
</tr>
<tr>
<td>Page which coverage starts</td>
<td>Pagenum</td>
<td>A three numeral page number indicating the page number which the coverage of the topic starts.</td>
</tr>
</tbody>
</table>
| For which state is it a stated-option text | StatedOption | NY - New York  
FL - Florida  
TX - Texas  
CA - California  
NS - Not a stated-option text |
Table 5 (continued)

<table>
<thead>
<tr>
<th>Description of Variable</th>
<th>Variable Name</th>
<th>Coding</th>
</tr>
</thead>
</table>
| Topic Examined          | Topic        | CLT- Combining like terms  
                          |              | DIS- Distributive Property  
                          |              | MOE- Multiplication of Exponential Expressions  
                          |              | DOE - Division of Exponential Expressions  
                          |              | SFU - Solving for an Unknown/Equality Properties  
                          |              | SYS - Solving Systems of Linear Equations by Elimination  
                          |              | PAR- Parallel Lines Have the Same Slope |
| What kind of justification does the text offer when introducing the topic identified | Justification | 1- no justification or reasoning present  
                                                                 |              | 2- some form justification is present  
                                                                 |              | 3- justification models or prompts mathematical reasoning  
                                                                 |              | 4- offers proof (or the same reasoning required for proof but without the proper form) |

The two individuals responsible for coding the textbooks are both current mathematics teachers in New York Public Schools. Both have more than 7 years of teaching experience and both teach or have taught algebra, geometry, and other courses.

The coders initially met and collaboratively coded two textbooks used in the study using the provided classification rubric for coding developed by the researcher (see Table 3 above). Based on feedback from the coders following the initial meeting, the rubric was revised and the coders met again to discuss how to use the revised rubric.

Working completely separately on same three textbooks, consisting of a total of 21 items of analysis, it was found that the coders independently coded the three texts with over 85% agreement, coding 18 of the 21 items of analysis the same. Afterward, meeting and discussing the first round of independent coding, the coders again coded three different textbooks completely separately, this time achieving 100% agreement, coding 21 of the 21 items of analysis the same. The remaining texts were coded by one of the two coders; however, there were instances in which the coders met or discussed the coding of a particular item.
Data Analysis

In order to answer the last three research questions, analysis of the collected data was performed. To address research question 2, a test was performed to see what percentage of texts examined offered each kind of classification. This was done to illuminate how often students are exposed to the idea that mathematical claims, properties, and theorems need justification. In terms of the coded data, this test determined what percentage of the data was coded as each other for categories under the rubric for classification of the topic of coverage (see Table 3).

To address research questions 3 and 4, an analysis of the types of proof offered for each topic was performed only on those textbooks offering some kind of explanation. In terms of the variables, these tests were performed only on data that did not contain a “1- no justification or reasoning present” for the “justification” variable.

To answer research question 3, “What kinds of justification or proof (if any) do students experience in secondary school first year algebra textbooks?” a test was performed only on texts offering some kind of explanation to see what percentage model mathematical reasoning in their coverage of the topic. There was also a discussion of trends observed while coding the data. In addition, a survey of all the topics covered was conducted, and an in-depth discussion of those items coded as 3 or 4 were presented, identifying trends and other findings.

To answer research question 4, “When students are exposed to justification in algebra, how often do those justifications contain the same kind of reasoning required by mathematical proof?” a test was performed to see what percentage offer proof. This can be rephrased using the above stated variables as the percentage of the data containing a “4-offers proof” for the “justification” variable. Furthermore, there was a more in-depth discussion on the kinds of proofs and justification offered by textbooks
coded as belonging to groups 3 and 4. In addition, there was a discussion within each topic regarding what types of proof are being offered.

It is important to note that the textbooks that model mathematical reasoning for a given topic and texts that offer proof for a given topic are not disjoint. When a textbook offers a proof, it is modeling mathematical reasoning.

Finally, there was some discussion of how related topics were treated within each particular textbook examined and the extent to which their particular relationship could be seen in or affected the data. For instance, if one assumes distribution and the distributive property to be true, then it was used to show why one can combine like terms. There was a check to see if it is always the case that textbooks simply state the distributive property without justification, but then provide justification for combining like terms.

There was a discussion involving the level of justification present across the data for solving for an unknown and solving a system of equations by elimination. Both topics rely on the same basic properties of equality. Any attempt to justify or convince the reader of the truth of the properties or procedures related to these two topics would rely on similar prior knowledge. Therefore, a textbook might be consistent in its handling of the two topics with relation to providing justification and modeling mathematical reasoning. The data were analyzed for evidence of this connection.

The rules for multiplying exponential expressions with the same base are clearly related to the rules for dividing exponential expressions with the same base. The level of justification and reasoning modeled in the sections dealing with the rules for multiplying exponential expressions with the same base and the rules for dividing exponential expressions were examined to see if textbooks were consistent with respect to their handling of these two topics. The data were analyzed to see if when a textbook offers justification for one of the two topics, it also offers justification for the other.
Chapter IV

FINDINGS

Addressing the Research Questions

With the exception of the first research question, which was addressed in the review of the relevant literature in Chapter II, each research question is addressed below by discussing the findings of the study.

As previously stated the research questions are:

1. What qualifies as justification or proof in non-proof writing in introductory algebra courses (before students are explicitly instructed in proof writing)?
2. Using the most widely circulated textbooks as a lens, how often are students exposed to (a) justification of mathematical claims and mathematical reasoning or (b) the concept of proof in their introductory algebra course?
3. What kinds of justification or proof (if any) do students experience in first-year algebra textbooks?
4. When students are exposed to justification in algebra, how often do those justifications contain the same kind of reasoning required by mathematical proof?
Research Question 2

*Using the most widely circulated textbooks as a lens, how often are students exposed to (a) justification of mathematical claims and mathematical reasoning or (b) the concept of proof in their introductory algebra course?*

In order to answer the second research question, an analysis of the coverage of the topics in textbooks examined in this study was performed. This information is captured in Figure 1.

Figure 1. Presence of Reasoning and Proof in Algebra Textbooks: Coding Results for All Topics in All Texts

The findings are troubling. Looking at the data broadly reveals that, roughly speaking, students are only exposed to justification of mathematical claims in 38% of all sections covering the selected topics (falling within groups 3 or 4 in Figure 1). Of that 38% of the items examined, only 6% of coded sections contained an actual proof or justification that offers the same ideas or reasoning as a proof (falling within group 4 in Figure 1). In
62% of sections examined, there is little to no explanation or justification for mathematical claims made (falling within groups 1 or 2 in Figure 1).

This broader perspective has been graphically represented in Figure 2 below. Items coded as belonging to group 1 or group 2 were combined as items that do not sufficiently model or prompt mathematical or justification of mathematical claims. Items coded as belonging to group 3 or group 4 were combined as items that do model or exhibit justification of mathematical claims.

Figure 2. Presence of Reasoning and Proof in Algebra Textbooks: Coding Results for All Topics in All Texts with Condensed Categories

While this broad approach to looking at the data does not address many of the subtle nuances among the different kinds of justifications students are exposed to, it makes the overall state of the situation apparent. In 62% of the items examined, students are not exposed to the idea that mathematical claims require justification or
proof. The textbooks do not offer many examples or models of the mathematical thinking required to justify mathematical claims. Conversely, regardless of the level of rigor used, 38% of the items examined model the idea that mathematical claims require justification or proof and model the mathematical reasoning required to do so. At the same time, only 6% of all items examined contain the reasoning required for an actual mathematical proof.

A list of textbooks and the percentage of items coded as belonging to group 3 or group 4 in that textbook is provided. In order to make the list more manageable, textbooks with multiple editions, or textbooks approved for use in multiple states, have been condensed here, even though they were coded separately and treated as different textbooks throughout the findings. It was possible to condense the list in this manner because textbooks with the same publishers and/or authors (with few exceptions) are identical in their handling of the topics included in this study.

Table 6. Percentage of Sections Modeling Mathematical Reasoning and Proof by Textbook

<table>
<thead>
<tr>
<th>Textbook</th>
<th>Percentage of Sections Modeling Mathematical Reasoning &amp; Proof (Those Sections Coded as belonging to Group 3 or 4, n=7)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Algebra 1 An incremental development (CA, FL, NY)</td>
<td>14%</td>
</tr>
<tr>
<td>Algebra 1 Concepts and Skills (FL, NY)</td>
<td>43%</td>
</tr>
<tr>
<td>Algebra Concepts and Applications (CA, NY, TX)</td>
<td>29%</td>
</tr>
<tr>
<td>Algebra Connections (TX)</td>
<td>57%</td>
</tr>
<tr>
<td>Amsco's Integrated Algebra (NY)</td>
<td>100%</td>
</tr>
<tr>
<td>CME Project Algebra 1 (NY)</td>
<td>71%</td>
</tr>
<tr>
<td>Discovering Algebra (CA, FL, NY, TX)</td>
<td>57%</td>
</tr>
<tr>
<td>Glencoe Mathematics Algebra 1 (FL, NY, TX)</td>
<td>57%</td>
</tr>
<tr>
<td>Holt McDougal Algebra 1 (CA, NY)</td>
<td>29%</td>
</tr>
<tr>
<td>Holt, Rinehart and Winston New York Algebra 1</td>
<td>29%</td>
</tr>
<tr>
<td>McDougal Littell Algebra 1 (FL, NY, TX)</td>
<td>0%</td>
</tr>
<tr>
<td>Prentice Hall Algebra 1 (FL)</td>
<td>57%</td>
</tr>
<tr>
<td>Prentice Hall California Algebra (CA)</td>
<td>43%</td>
</tr>
<tr>
<td>Prentice Hall Integrated Algebra (NY)</td>
<td>43%</td>
</tr>
<tr>
<td>Springboard Mathematics with Meaning Algebra 1 (FL)</td>
<td>14%</td>
</tr>
</tbody>
</table>
A 71% in this list means that 71% of the sections analyzed in this book were coded as belonging to group 3 or group 4 in this study (as was the case with CME Project Algebra 1). It is important to note that this list can be deceiving. For instance, a quick glance down the list might lead one to conclude that Amsco's Integrated Algebra, published to teach a course using New York State's integrated algebra curriculum, is exemplary in its attempts to model proof and reasoning. While this textbook does provide justification and reasoning, some of its explanations and proofs would be more difficult for lower-level functioning students to understand. For instance, Amsco's Integrated Algebra provides a formal proof for why two parallel lines must have the same slope, but the proof relies heavily on a trigonometric argument that looks at the angle created by two lines intersecting the x-axis. Such a proof is outside the scope of a more typical course in algebra. This particular proof has been provided later in this chapter and will be discussed at more length below. In order to look beyond the numbers and examine issues such as this in greater depth, a more thorough discussion of each topic of coverage is provided in answering research questions 3 and 4 in sections 4.3 and 4.4, respectively.

**Research Question 3**

*What kinds of justification or proof (if any) do students experience in first-year algebra textbooks?*

The most prevalent form of justification or means of convincing the reader of the truth of theorem, property, or procedure is the use of specific examples. In most cases, textbooks offering explanation most typically provide a number of annotated examples in which the solution (to a problem related to the topic of the section) is shown step-by-step with some explanation adjacent to each step. These explanations typically clarify either what has been done to get to the next step or which property has been applied.
Textbooks attempting to give a series of examples to justify or convince the reader of the truth of a theorem, property, or procedure often fall short of offering a mathematical proof because they lack generality and/or, in some cases, the inductive step. While many textbooks state the general rule at some point, most only use reasoning within a specific example, if at all. The way in which textbooks introduce the shortcut for dividing exponential expressions with the same base clearly illustrates this idea.

Figure 3. An Excerpt from Prentice-Hall’s Florida Algebra 1 (2011)

Focus Question How are the properties of dividing powers and multiplying powers similar?

You can use repeated multiplication to simplify quotients of powers with the same base. Expand the numerator and the denominator. Then divide out the common factors.

\[
\frac{4^5}{4^3} = \frac{4 \cdot 4 \cdot 4 \cdot 4 \cdot 4}{4 \cdot 4 \cdot 4} = 4^2
\]

This example illustrates the following property of exponents.

Property Dividing Powers With the Same Base

Words To divide powers with the same base, subtract the exponents.

Algebra \( \frac{a^m}{a^n} = a^{m-n} \), where \( a \neq 0 \) and \( m \) and \( n \) are integers

Examples \( \frac{2^6}{2^2} = 2^{6-2} = 2^4 \quad \frac{x^4}{x^2} = x^{4-2} = x^2 \quad \frac{1}{x^3} \)

(Charles et al., 2011)
From a mathematical standpoint, showing why a specific example is true by suggesting a pattern or shortcut does not prove anything outside of the context of that specific case. There are countless occasions in mathematics in which a theorem or pattern holds true for many cases but breaks down at some later point. A series of examples is not enough to construct proof.

Providing specific examples without generality is the most common way in which textbooks examined in this study deviated from or fell short of what a mathematician would consider mathematical justification or proof. This does not mean that these explanations do not expose students to some justification or mathematical reasoning. Although it lacks generality, the above example taken from Pearson’s CME Project Algebra 1 successfully convinces the reader why the rule works for that specific case using prior knowledge and mathematical reasoning. It is hard to deny that this particular example is clearly making a good case for why the rule will often be true. It provides evidence for why one can subtract powers when dividing two exponential expressions that share the same base. This example shows why this works for a particular base, and
the explanation suggests why it would work for any base. While this is not enough to completely satisfy a mathematician, it certainly models mathematical reasoning and relies upon prior knowledge of mathematics.

There is no short answer to research question 3. In general, the most prevalent way justification or proof is presented in first-year algebra texts is through a number of examples (sometimes annotated) that show or suggest that a particular theorem, property, or procedure is true or works. However, to address research question 3 further and in more detail requires a discussion of each particular topic included in this study. This will be done in the next section, while answering research question 4.

**Research Question 4**

*When students are exposed to justification in algebra, how often do those justifications contain the same kind of reasoning required by mathematical proof?*

In order to address question 4 and further elaborate on question 3, this section will examine the 38% of the items of analysis coded as belonging to group 3 (models or prompts mathematical reasoning) or in group 4 (offers proof). In order to discuss the different kinds of justification or proof offered and the extent to which those justifications contain the same kind of reasoning required by mathematical proof, a textbook must offer a certain amount of justification or proof in the first place. Those items coded as belonging to group 2 offer insufficient, incomplete, or otherwise parenthetical justification or reasoning related to proof. Therefore, these items will only be referenced when there is a relevant comparison to items coded as belonging to group 3 or 4. The figure below presents each topic analyzed in the study and the percentage of each one coded as belonging to group 3 or 4.
Following a discussion of each topic of coverage separately, there will be a concluding section discussing overall trends and observations.

**Combining Like Terms**

The most common justifications or explanations for the procedure used to combine like terms rely on the fact that multiplication is repeated addition or, more often, the distributive property. About half of the items analyzed in this study attempted one of the above-mentioned justifications or explanations. Given that this study has found that only 38% of topics analyzed contained sufficient justification or reasoning, textbooks provided justification and reasoning for combining like terms more than most other topics examined in this study.
In textbooks in which the distributive property is covered first, the distributive property is almost always used to justify why like terms can be combined in the typical way. Specific examples are used to illustrate or suggest why we can simply add the coefficients.

A typical example comes from Amsco’s *Integrated Algebra 1* (2007):

*To add like terms, we use the distributive property of multiplication over addition.*

\[
9x + 2x = (9 + 2)x = 11x \\
-16d + 3d = (-16 + 3)d = 13d
\]

(Gantert, 2007)

After showing examples like the one above, most textbooks state the procedure suggested by the example and then immediately begin to use the idea that you can add the coefficients of like terms without further mention of the distributive property.

Few textbooks explicitly ask students to reason through or explain how the distributive property allows for combining like terms outside of the initial examples. That is not to say it does not happen. CME Project’s *Algebra 1* (2009) states that one can use the distributive property to combine like terms and then asks students to “use the distributive property to explain why $3x + 5x = 8x.$” However, such explicit attention to assessing the presentation of mathematical reasoning is rare, even among textbooks that model it. It should be noted that CME Project’s *Algebra 1* will be referenced multiple times in this section, as it is exemplary in this regard.

In textbooks that make extensive use of models such as algebra tiles, combining like terms is often glossed over or assumed. Using algebra tiles involves understanding that $2x + 3x = 5x$ is equivalent to adding 2x tiles to 3x tiles to get 5x tiles. Algebra tiles reduce combining like terms to simple addition of a manipulative, obscuring the fact that students are applying a new mathematical concept that requires justification.
Distribution and the Distributive Property

Most textbooks simply demonstrate the distributive property using a specific example and do not attempt to discuss why it is true. Only 29% of textbooks in this study attempted to offer any justification for why the distributive property is true. When textbooks attempt to give justification, the most common approach is to show that it is true for a specific case of numbers, often in some context where the answer to a question can be calculated in two different ways. This is a typical example of how many textbooks show this:

Rob’s DVD Store sells new and old videos. New DVDs cost $12 and used DVDs cost $5. If 6 people go to the store and each buy 1 new DVD and 1 old DVD, how much money will Rob’s DVD Store make?

<table>
<thead>
<tr>
<th>Method 1</th>
<th>Method 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>6 ($12 + $5)</td>
<td>6 ($12) + 6($5)</td>
</tr>
<tr>
<td>6(17)</td>
<td>72 + 30</td>
</tr>
<tr>
<td>$102</td>
<td>$102</td>
</tr>
</tbody>
</table>

Either method gives you $102. This is an example of the Distributive Property.

(This is similar to examples found in the Glencoe Mathematics Algebra texts.)

After showing a situation that illustrates or suggests the distributive property, most texts just jump right into using it, often simplifying an expression and citing the step in which the property is used. While one could argue that the context illustrates that the distributive property holds true in a real life example, none of the textbooks draw upon prior knowledge to show that it is true. With the exception of those textbooks using an area model, no textbooks examined in this study use the fact that multiplication is repeated addition or the associative property to show it is true in a strictly mathematical sense. One could argue that the use of algebra tiles achieves this, but no textbooks make the argument algebraically in absence of algebra tiles. If any prior mathematical knowledge is used to show why the distributive property is true, then it is usually how to calculate area.
Many of the more thorough and general justifications, and often those coded as belonging to group 3, incorporated some kind of visual representation to have the reader observe and believe the distributive property. The most common approach is to use the area of two adjacent rectangles that share a side to suggest, illustrate, or convince the reader of the distributive property. Often a textbook section coded as belonging to 3 or 4 would achieve a degree of generality by letting one or more of the sides of the rectangles be unknown and simply represented using a variable. As mentioned above, other texts would use a more abstract visual model such as algebra tiles to represent and illustrate the property.

Overall, textbooks struggle to offer justification for the distributive property. Those offering justification primarily use a story or an area model to demonstrate that the property holds true for a specific case. A greater degree of generality is typically achieved when the area model is employed.

**Multiplication of Exponential Expression**

Most textbooks in this study provided some form of justification for the rule associated with multiplication of exponential expressions. These attempts at justifying or generating a rule for multiplying exponential expressions with the same base typically involve a product with variables or numbers as the base and numbers as the exponents. The most common explanations show the expansion of an exponential expression as repeated multiplication. After expanding both exponential expressions, most textbooks condense the two expansions into one exponential expression. It typically looks something like this:

\[ x^3 \cdot x^4 = (x \cdot x \cdot x) \cdot (x \cdot x \cdot x \cdot x) = x^7 \]

In the better explanations (which appear below), the textbook will typically show the expanded product and make explicit connections between the number of factors in each initial exponential expression and the final power of the exponential expression.
They make explicit connections to the shortcut in which one can add the powers of the two original exponential expressions if they have the same base in order to find the power of their product.

Although sections coded as belonging to both groups 2 and 3 state the rule in a general way, those in group 3 differ in that they draw the reader’s attention to how many factors each exponential expression contains, and they make explicit connections between this and the rule. Sections coded as belonging to group 2 typically stop at showing that the rule works.

Other textbooks seemingly leave the discovery of the rule to the reader by providing a student activity in which the students expand both parts of a product of two exponential expressions with the same base in order to find their product. A good example of this comes from Prentice Hall’s *Integrated Algebra* (2008).

Figure 6. An Excerpt from Prentice Hall’s *Integrated Algebra* (2008)
In most cases where this is done, the textbooks undermine their own attempts at allowing the students to discover the mathematics on their own by stating the rule for them immediately following the activity, as with the above example. However, assuming the textbook is being used as a guide, it is possible that a teacher would give such an activity before using the textbook with the students.

Only one textbook offered mathematical proof or a justification with generality for the rules governing the multiplication of exponential expressions with the same base. CME Project’s Algebra 1 handled the topic in a manner similar to many other books coded as belonging to group 3, but argued with a degree of generality that is not found elsewhere in the study.

Figure 7. An Excerpt from Pearson’s CME Project’s Algebra 1 (2009)

While there is no AOAG for exponentiation, there are some basic rules for exponents. In Lesson 6.1, you explored a collection of proposed basic rules. Group I explored one of these rules.

\[ a^b \cdot a^c = a^{b+c} \]

Why is this rule true? Try it with numbers. For example, \((3^2)(3^5)\).

\[
\begin{align*}
(3^2)(3^5) &= (3 \cdot 3) \cdot (3 \cdot 3 \cdot 3 \cdot 3 \cdot 3) \\
&= (3 \cdot 3 \cdot 3 \cdot 3 \cdot 3) \\
&= 3^7
\end{align*}
\]

When you multiply \(3^2 \times 3^5\), there are a total of 7 factors of 3. You use the same process when you find the product \((3^b)(3^c)\).

\[
\begin{align*}
(3^b)(3^c) &= (3 \cdot 3 \cdot \ldots \cdot 3) \cdot (3 \cdot 3 \cdot \ldots \cdot 3) \\
&= (3 \cdot 3 \cdot \ldots \cdot 3) \\
&= 3^{b+c}
\end{align*}
\]

This argument works if the base is 2, –1, \(\frac{1}{2}\), or even a variable, such as \(a\). Now that you have an argument, or proof, you can write the theorem. This simple statement, the Law of Exponents, is very important to the discussion of exponents.

**Theorem 6.1 The Law of Exponents**

For any number \(a\) and positive integers \(b\) and \(c\), \(a^b \cdot a^c = a^{b+c}\).

(Cuoco et al, 2009)
Please note that AOAG stands for the “any-order any-grouping properties of addition and multiplication.” This particular textbook describes a combination of both the commutative and associative properties of multiplication and addition using this acronym. This proof is also one of the few examples encountered in which the textbook explicitly refers to its argument as a proof within the text.

Overall, textbooks attempting to offer justification or proof of the “rules” for multiplying exponential expressions with the same base do so with specific examples, showing the “rule” works for specific (often annotated cases) and suggesting or stating that it would work for any non-zero base.

**Division of Exponential Expression**

Generally speaking, the way in which textbooks handle division of exponential expression with the same base was quite similar to the way they handled multiplication of exponents with the same base. The most common justification of the rules used to divide exponential expressions with the same base involves a quotient with variables or numbers as the base and numbers as the exponents. These explanations show the expansion of an exponential expression as repeated multiplication in both the numerator and the denominator and cancellation of the common factors, as pictured in Figures 3 and 4 above. In more thorough explanations, the textbook makes explicit connections to the shortcut in which one can divide two exponential expressions with the same base by subtracting their exponents.

As with multiplication of exponents with the same base, sections coded as belonging to both group 2 and group 3 state the rule in a general way, but in sections coded as belonging to group 3, the textbook draws the reader’s attention to how many factors are being cancelled and why they can be cancelled, and makes explicit connections between this and the rule. Sections coded as belonging to group 2 typically stop at showing that the rule works.
While 43% of all textbooks analyzed in this study attempted to offer some kind of justification or reasoning for the rules governing the division of exponential expressions with the same base, none of the textbooks analyzed offer a mathematical proof for the rules and procedures governing division of exponential expressions with the same base in a general way. In order to be coded as belonging to group 4, a text would have to make all such connections explicit, but in a general way, rather than within the context of specific problem. As noted above, most sections coded as belonging to group 3 provided a sufficient explanation within a specific example, but failed to argue generally.

**Parallel Lines Have the Same Slope**

Of all the claims analyzed in this study, the statement that parallel lines have the same slope was the least justified, with only 7% offering justification. Of the 28 textbooks examined, 20 of them simply stated that parallel lines have the same slope, six provided little to no justification, and two provided mathematical proof. Since this section is dealing only with those texts offering mathematical reasoning and justification of mathematical claims, it will look more closely at the two texts that offer mathematical proof for the claim that parallel lines have the same slope.

*Amsco’s Integrated Algebra* (2007) proves that parallel lines have the same slope through the use of trigonometry with a formal mathematical proof.
This proof uses prior knowledge garnered in New York State’s Integrated Algebra curriculum to show in complete generality that any two lines that have the same slope are parallel. The prior knowledge required to follow the proof includes trigonometric ratios, slope, and knowledge about the angle relationships created when a transversal crosses two parallel lines. It is possible that this handling of the topic is unique to this textbook because New York State includes trigonometric ratios in its Integrated Algebra curriculum. Such a proof would not be appropriate in a strict algebra course because students would not necessarily be familiar with all the prior knowledge required for it.

The second proof comes from Pearson’s CME Project’s Algebra 1. In this case, the textbook begins by stating, “If two distinct lines have the same slope, then they are parallel” (p. 368) and proceeds to offer a proof, which is labeled as such in the textbook.
This is incredibly rare. What follows is a formal proof by contradiction that relies on a student’s knowledge of slope, substitution, and the slope-intercept form of a line. After proving the claim, the textbook directs the reader to explain why two of the key lines from the proof are true: “If the lines do not intersect, then they are parallel,” and, “If \( b = d \), then the two lines \( y = ax + b \) and \( y = ax + d \) must be the same.” Very few of the textbooks that do offer a proof check for understanding or anticipate student confusion in this manner.

It should be noted that although both texts provide a formal proof, it seems as though the first proof discussed would be more difficult for lower-level functioning students to understand. While proof by contradiction can be difficult to comprehend, the prior knowledge required to understand the second proof is more related to the topic of coverage in the section. Trigonometry is rarely used in conjunction with a study of parallel lines in an algebra course. However, slope, substitution, and slope-intercept form are all relevant and common to the study of lines. Furthermore, by addressing possible points of confusion, the latter textbook puts the emphasis on explaining why the claim is true.

**Solving for an Unknown**

Only 21% of the textbooks analyzed in this study attempt to justify or explain the procedures taught to solve for an unknown. The vast majority of texts state the addition property of equality, the subtraction property of equality, the multiplication property of equality, and the division property of equality. In many texts, the properties are simply stated and then used in one or two specific examples either to justify a procedure, such as adding or subtracting the same quantity from both sides of the equation to isolate the unknown, or to simply show that the properties hold true. Of the texts that attempt to explain why these properties are true, most do so with one brief example in the form of a word problem or using a model such as algebra tiles or a physical scale. One
example of this can be found in Glencoe Mathematics' Texas Algebra 1 (Holliday et al., 2007):

Suppose a boys’ soccer team has 15 members and the girls’ soccer team has 15 members. If each team adds 3 new players, the number of members on the boys’ and girls’ teams would still be equal. This example illustrates the Addition Property of Equality.

If any attention is given to explaining why the properties are true, it is only done with the addition property of equality, with no mention of the three other properties. In other words, while the other properties are sometimes stated, they are rarely necessary to reason through the example.

Other textbooks, especially those using a scale as a model to understand equality, focus more on the idea of inverse operations and how they “undo” each other. These texts tend not to state the equality properties explicitly. Rather, they allude to them by stating that you need to keep an equation balanced like you would a physical scale by doing the same thing to both sides.

In the textbooks that provide sufficient justifications or explanations of the procedures used to solve for an unknown, examples are annotated to show exactly where and when one of the properties of equality is invoked. For example, instead of simply showing that one can solve the equation $x - 5 = 17$ by adding 5 to both sides of the equal sign, textbooks coded as belonging to group 3 show or explain that the reason one can solve the equation by adding 5 to both sides is because of the addition property of equality.

It should also be noted that, in almost every textbook examined, the section in which students learn to solve an equation for an unknown is the section in which they are first introduced to the aforementioned properties of equality, if those properties are stated at all. Furthermore, while in some cases textbooks dedicate half a page to a page to these properties, little or no attention is given to these properties within the student activity section.
Overall, the 21% of sections related to solving for an unknown that provide or model justification do so vis-à-vis the equality properties and the use of annotated examples.

**Solving Systems of Equations by Elimination**

Of the 43% of textbooks that offer explanation of the procedure used to solve systems of equations by elimination, many use an approach similar to how the same textbook approached solving for an unknown. For instance, all of the Holt McDougal *Algebra 1* textbooks use a physical balance as a model when teaching how to solve for an unknown, then return to the balance (and the idea of equality) to justify combining two equations to solve a system of equations by elimination.

The percentage of textbooks offering justification and reasoning in the section dealing with solving systems of equations by elimination is similar to the overall findings of the study. The way in which textbooks offering justification and reasoning provide justification and reasoning is fairly consistent. Almost all of these textbooks do so by revisiting the addition and subtraction properties of equality. In textbooks offering the same kinds of mathematical reasoning required by a proof, the properties are extended in a general way to include multiple terms or quantities. One example comes from Prentice-Hall’s California *Algebra 1* (2009):

*The Addition and Subtraction Properties of Equality can be extended to state,*

*If \( a = b \) and \( c = d \), then \( a + c = b + d \).  If \( a = b \) and \( c = d \), then \( a - c = b - d \)*

(Bellman et al., 2008)

Another example of this comes from Pearson’s CME Project’s *Algebra 1* (2009): “If \( X = A \) and \( Y = B \), then \( X + Y = A + B \), where \( A, B, X, \) and \( Y \) can be any mathematical expressions” (Cuoco et al., 2009). This distinction shows, in a general way, the subtle differences
between the way in which the addition and subtraction properties of equality are used when solving for an unknown, as opposed to the way in which they are used when solving a system of equations by elimination (which in Algebra 1 is almost exclusively two equations and two unknowns).

After stating the addition and subtraction properties of equality, a typical textbook solves a number of systems, explicitly stating what is being done at each step and why it is permitted, citing when the addition or subtraction property of equality is used. Prentice Hall’s California *Algebra 1* provides as an example of this.

Figure 9. An Excerpt from Prentice Hall's California *Algebra 1* (2009)

![Adding Equations](image)

Less thorough explanations often state what is being done in a particular step, but fail to explain why it is being done. Others skip some of the more salient aspects of the
aforementioned justifications, such as explicitly stating that the sum of the two terms that are to be eliminated will always be zero.

A Closer Look at Related Topics

Some of the topics analyzed in this study are related. If one assumes the distributive property to be true, then it can be used to show why one can combine like terms. Solving for an unknown and solving a system of equations by elimination both rely on the same basic properties of equality. The rules for multiplying exponential expressions with the same base are clearly related to the rules for dividing exponential expressions with the same base. The following three sections will look more closely at these relationships in the context of the data. The discussion that follows will continue to look at the data from a broad perspective. Items coded as belonging to group 1 or group 2 will be combined as items that do not sufficiently model or exhibit justification of mathematical claims and mathematical reasoning. Items coded as belonging to group 3 or group 4 will be combined together as items that do model or exhibit justification of mathematical claims.

Distribution and Combining Like Terms

One concern or point of interest for the study is the fact that if one assumes distribution, then it can be used to show why one can combine like terms. If it was found that all books simply state the distributive property and then use it to show why like terms can be combined, then the data would be skewed because a particular textbook would always get coded as belonging to group 1 or 2 for distribution and then always get coded as belonging to group 3 or 4 for combining like terms. After analyzing the data, it is clear that this is not the case. Thirteen of the textbooks examined in this study provided a similar level of justification for both combining like terms and
distribution, meaning that the text either provided justification for both topics or provided justification for neither topic. Fifteen of the textbooks examined in this study only provided justification for one of the two topics. Therefore, this does not present an issue or problem with the data.

**Solving for an Unknown and Solving a System of Equations by Elimination**

The procedures for solving for an unknown and solving a system of equations by elimination both rely on the same basic properties of equality. Any attempt to justify or convince the reader of the truth of the properties or procedures related to these two topics would rely on similar prior knowledge. Therefore, a textbook might be consistent in the extent to which it provides justification and modeling mathematical reasoning for these two topics.

In terms of the level of reasoning modeled, analysis of the data shows that it was not the case that each textbook provided a similar level of justification and reasoning for both topics. That is to say, not all textbooks had internal consistency in their handling of these two topics. Roughly half of the textbooks analyzed (44%) provided a similar level of justification and reasoning for both solving for an unknown and solving a system of equations by elimination. This means that each of these textbooks was consistent across the two topics, either attempting to justify or convince the reader of the truth of the properties or procedures used in relation to both topics or not doing so for both of the topics. On the other hand, roughly half of the textbooks analyzed (56%) treated each topic differently, attempting to justify or convince the reader of the truth of the properties or procedures used in relation to one of the topics, but then not doing so for the other. The fact that the procedures and properties associated with solving for an unknown and solving a system of equations by elimination both rely on the same basic ideas about equality does not appear to affect the way in which textbooks treat the two topics.
Multiplication of Exponential Expressions and Division of Exponential Expressions

The level of justification and reasoning modeled in the sections dealing with the rules for multiplying exponential expressions with the same base and the rules for dividing exponential expressions with the same base is consistent across textbooks. While the percentage of textbooks that modeled mathematical reasoning and proof was comparable to the study overall, 71% of the individual textbooks treated both topics similarly in that the textbook provided justification or proof for both topics or it did not provide justification or proof for both topics.

This is in contrast to the inconsistency within most textbooks analyzed for the level of reasoning present when solving for an unknown and solving a system of equations by elimination. One possible explanation is the proximity of the two topics within a given textbook. That is to say, in all textbooks analyzed, solving for an unknown and solving a system of equations by elimination are covered in different units of the textbook, whereas multiplication of exponential expressions with the same base and division of exponential expressions with the same base are always covered in the same unit in the textbook.

Student Activities Following the Coverage of the Topic

An examination of the student activities and exercises at the end of each section was performed for two of the topics examined in this study to assess the nature of the problems. The two topics in which student activities were examined were the division of exponents and the slopes of parallel lines. Looking only at these two topics and using the Rubric for Classifying the Student Activities and Exercises (see Table 4 in Chapter III), it was found that the level of reasoning and proof required by the student activities following the coverage of the topic was minimal.
Of all the student exercises examined, 83% of student activities required little to no deductive reasoning, 16% required some deductive reasoning, and approximately 1% required students to engage in deductive reasoning and/or proof writing.

These findings make it clear that, generally speaking, students are exposed to justification and reasoning much more often in the section containing the coverage of the topic than in the student activities section following the coverage of the topic. Overall, the student activities following the coverage of the topic require little reasoning. The vast majority of questions (roughly 83% of all items sampled) are simply student activities that were virtually identical to examples found earlier in the section except with different numbers. Most student activities could be completed by simply copying from examples in the section preceding the student activities.

**Discussion**

Overall, the findings are bleak. In 62% of the items examined, students were not exposed to the idea that mathematical claims require justification or proof. When students were exposed to some level of justification, the reasoning employed did not always model that which is required by the mathematical community.

This is problematic from multiple perspectives. While it is clearly an issue related to students’ understanding of, and ability to engage in, proof and proof-type thinking, it is also problematic for a student’s understanding of algebra. In order for students to choose and apply an appropriate procedure or theorem within the discipline of algebra, a student must understand why said procedure or theorem is true or works (Morrow & Kenney, 1998; Skemp, 1978). Such deductive reasoning is required to achieve the necessary level of procedural flexibility to be successful when faced with a novel or challenging problem (Star, 2001, 2002, 2004). If students are only presented with justification and reasons for why a given theorem, property, or procedure is true 38% of
the time, then they are likely to experience difficulty with achieving the required level of procedural flexibility.

In most cases, first-year algebra textbooks offer little to no justification or proof for mathematical claims. Often, in lieu of a real proof or mathematical explanation for why a given theorem, property, or procedure is true or works, students are exposed to a list of examples with solutions. As such, textbooks offer empirical evidence to support mathematical claims. A proof in mathematics is a logical argument, not an empirical or evidential one. Empirical evidence is not proof. By repeatedly making empirical arguments to justify mathematical claims, educators following the curriculum set forth by the majority of textbooks are essentially “teaching” students that an empirical argument is sufficient to establish truth in mathematics. This reinforces the findings of researchers such as Chazan (1993), Knuth et al. (2002), and Thompson (1991), who have conducted various studies identifying students’ incorrect belief that an empirical argument is sufficient to establish truth in mathematics. It is possible that such widespread use of empirical arguments to establish truth in the intended curriculum is the root of this problem.

Only 38% of the sections analyzed in this study offered sufficient justification and reasoning for mathematical claims. Although these explanations might also employ the use of specific examples, they use them to a different end. In many cases in which justification or reasoning is present, textbooks present an annotated solution to a specific example, explaining both what has been done to solve the problem and why it works or why it is allowed, drawing upon prior mathematical knowledge.

Although the distinction between the two ways in which textbooks use examples may seem subtle, it is the emphasis that is important. Textbooks found to provide justification and reasoning in a particular section make meaningful use of specific examples in order to discuss the underlying mathematics and/or the reason the theorem, property, or procedure is true or works. Conversely, other textbooks offer
examples as a kind of empirical evidence to suggest that a mathematical statement is true.

The *NCTM Principles and Standards* and the recently published *Common Core State Standards* both place great importance on proof and deductive reasoning at all levels of education from kindergarten through university (NCTM, 2000). This study has found a lack of evidence to support claims that the intended curriculum, as seen through the lens of the most popular and widely circulated textbooks, fulfills the guidelines set forth by these organizations. Moreover, if the intended curriculum does not sufficiently focus on the importance of proof and deductive reasoning in algebra, the course directly prior the course in which such topics will be receive explicit instruction, then it raises questions about the attention it receives in preceding mathematics courses and the extent to which our intended curriculum is addressing these important issues central to the study of mathematics.
Chapter V

SUMMARY, CONCLUSIONS, AND RECOMMENDATIONS

Summary

The purpose of this study was to determine the extent to which the modeling of deductive reasoning and proof-type thinking occurs in a mathematics course in which students are not explicitly preparing to write formal mathematical proofs.

The lens through which this study aimed to examine the intended curriculum was by identifying and reviewing the modeling of proof and deductive reasoning in the most popular and widely circulated textbooks throughout the United States. Research has shown that textbooks have a major impact on mathematics classrooms. A plentitude of research shows that textbooks have a major impact on teachers’ practices, student activities, and teacher day-to-day planning (Fujita & Jones, 2003; Nathan et al., 2002; Yerushalmy et al., 1993). In order to address the lack of research pertaining to the presence of proof-like reasoning in current curricula (Sylianides & Silver, 2004), this study looked for evidence of the modeling of deductive reasoning and proof-type thinking in secondary algebra textbooks. To this end, the following research questions were pursued:

1. What qualifies as justification or proof in non-proof writing introductory algebra textbooks (before students are explicitly instructed in proof writing)?
2. Using the most widely circulated textbooks as a lens, how often are students exposed to (a) justification of mathematical claims and mathematical reasoning or (b) the concept of proof in their introductory algebra course?

3. What kinds of justification or proof (if any) do students experience in first-year algebra textbooks?

4. When students are exposed to justification in algebra, how often do those justifications contain the same kind of reasoning required by mathematical proof?

The first research question was addressed through a careful review of the relevant literature. To address the remaining research questions, fundamental algebra topics from the most popular and widely circulated algebra textbooks in the United States were analyzed to see if any kind of justification or proof was provided when a new concept, theorem, property, or procedure was first introduced. In cases where some sort of explanation, justification, or sense-making activity was present in the text, the section was examined further to classify what kind of explanation or proof was present according to a rubric developed by the researcher.

Conclusions

Conclusions were based on both an extensive literature review and data garnered from an in-depth analysis of selected topics in the most popular and widely circulated textbooks currently used in the United States.

Research Question 1

What qualifies as justification or proof in non-proof writing introductory algebra textbooks (before students are explicitly instructed in proof writing)?
A proof in mathematics is a logical argument, not an empirical one (Aleksandrov, 1969). Within the professional mathematical community, the form and structure to which a valid proof must conform have been codified and formalized to a great degree. Mathematical proofs conform to certain norms and follow certain rules. Despite such codification and formalization mathematicians disagree, at times, about what qualifies as a proof (Kotelawala, 2007). In order for a proof to be truly accepted as valid, a mathematician must get approval from various sources, including their mathematics community, school or organization, and ultimately a peer-reviewed journal comprised of a larger community of mathematicians (Hersh, 1997).

Students of mathematics experience proof differently from those already a part of the professional mathematics community. For the student of mathematics, the proof’s role in answering the question “why” is much more important (Hanna, 2000). Some scholars have chosen to define what qualifies as a proof by looking at the audience it will serve. Balacheff (1988) distinguishes proofs based on their audience. He explains that

we call a proof an explanation which is accepted by a community at a given time. We call a mathematical proof, a proof accepted by mathematicians. As a discourse, mathematical proofs have now a days a specific structure and follow well-defined rules that have been formalized by logicians. (p. 2)

Since a specific community must accept a proof in order to be considered valid, proof becomes dependent upon the audience for which it is written.

A good example of how even formal proof writing within the professional mathematics community is audience-dependent comes from mathematicians’ use of the words “clearly” and “similarly.” Mathematicians routinely use these words in order to omit or skip parts of a proof they consider to be simple or repetitive (Goetting, 1995). By assuming that parts of a proof are clear to the reader and hence unnecessary to include, mathematicians are assuming their audience has certain knowledge, knowledge that, if they did not possess it, would render the proof invalid. Without all parts of a proof, it is
not a logical, rigorous proof of the veracity of the statement to be proved. However, since the validity of the proof is based on the audience to which it is being presented, certain gaps in the argument are allowed.

Thus, much of the research put forth in mathematics education suggests that the validity of a proof should be determined by considering the audience or community for whom it was prepared (Almeida, 1996; Hanna, 1989; Hersh, 1997; Kotelawala, 2007; Smith & Henderson, 1959). Asking students to discover, justify, and explain mathematics, or asking them the question, “How do you know this is true?” is essentially asking students to prove a statement or conjecture (Almeida, 1996). Smith and Henderson (1959) even argue that a second grader’s explanation of an arithmetic problem is a proof for the community of second graders to which they were presenting.

An explanation provided by an algebra textbook, therefore, qualifies as a proof if, and only if, it is a logical argument, not an empirical one, based in mathematics that convinces the reader of the truth of the claim.

**Research Question 2**

Using the most widely circulated textbooks as a lens, how often are students exposed to (a) justification of mathematical claims and mathematical reasoning or (b) the concept of proof in their introductory algebra course?

In order to answer the second research question, an analysis of the coverage of various topics in textbooks examined in this study was performed. This information is captured in Figure 1, which appears earlier. The data show that, roughly speaking, students are only exposed to justification of mathematical claims in 38% of all sections covering the selected topics (falling within category 3 or 4 in Figure 1). Of that 38% of the items examined, only 6% of coded sections contained an actual proof or justification that offered the same ideas or reasoning as a proof (falling within category 4 in
Figure 1). In 62% of sections examined, there was little to no explanation or justification for mathematical claims made.

An analysis of the activities and exercises at the end of each section performed for two of the analyzed topics (the division of exponents and the slopes of parallel lines) also revealed that little to no deductive reasoning was required for the vast majority of student activities and exercises. Of all the student exercises examined, 83% of student activities required little to no deductive reasoning, 16% required some deductive reasoning, and approximately 1% required students to engage in reasoning and/or proof.

**Research Question 3**

What kinds of justification or proof (if any) do students experience in first-year algebra textbooks?

When there is some justification or proof present, the most prevalent form of justification or means of convincing the reader of the truth of a concept, theorem, property, or procedure is the use of specific examples. In most cases, textbooks offering explanations typically provide a number of annotated examples in which the solution (to a problem related to the topic of the section) is shown step-by-step with some explanation adjacent to each step. These explanations typically clarify either what has been done to get to the next step or which property has been applied.

Textbooks attempting to give a series of examples to justify or convince the reader of the truth of a theorem, property, or procedure often fell short of offering a mathematical proof because they lacked generality and/or, in some cases, the inductive step. While many textbooks stated the general rule at some point, most only used reasoning within a specific example, if at all.
Research Question 4

When students are exposed to justification in algebra, how often do those justifications contain the same kind of reasoning required by mathematical proof?

Students are rarely being exposed to the kinds of reasoning required by mathematical proof in that they are rarely exposed to reasoning about mathematics with generality. Students are most commonly exposed to the kinds of deductive reasoning required by proof within the context of a specific problem. In other words, as opposed to using known mathematics to show why a particular theorem, property, or procedure is true with generality, it is very common that students will be exposed to reasoning about mathematics within a specific case. For example, as opposed to a textbook modeling a mathematical explanation of why $x^y \cdot x^z = x^{y+z}$ for $x \neq 0$, students are much more likely to be exposed to a mathematical explanation of why $2^3 \cdot 2^4 = 2^7$ as way of suggesting why $x^y \cdot x^z = x^{y+z}$ for $x \neq 0$. While both arguments use deductive reasoning and prior knowledge, the latter is essentially offering empirical evidence for a mathematical claim, and empirical evidence is not proof.

Recommendations

Documents from major policy influencing organizations such as the NCTM Principles and Standards and the recently published Common Core State Standards both place great importance on proof and deductive reasoning at all levels of education from kindergarten through university (NCTM, 2000). By examining the intended curriculum through the lens of the most popular and widely circulated textbooks, this study found a lack of attention to deductive reasoning and proof in introductory algebra.

The main focus of this study was on textbooks’ coverage of certain topics in algebra. The rubric developed and used by the researchers to code the textbooks’ coverage of the topic appears to be an effective measure that could be used again. It is
possible that modifications would need to be made if a different area of mathematics (such as geometry) were examined. The rubric created by the researchers to look at student activities following the coverage of the topic needs revision. Although each of these three categories correlates to the six levels of Bloom’s taxonomy of cognitive domains, the general lack of deductive reasoning required by student activities following the coverage of the topic would indicate that a rubric with finer differentiation among the lower cognitive domains may have been more illuminating.

The coders experienced some difficulty in dealing with books that attempted to explain a concept, theorem, property, or procedure through a series of scaffolded student activities within the coverage of the topic. For the purposes of this study, the coders assumed that students would always be able to reason through such scaffolded activities successfully and draw the textbook’s intended conclusions (since this study looked at the intended curriculum). This assumption may have been generous to the textbook authors. It is entirely possible that students would not have been able to follow the textbooks’ activities. Therefore, the textbook would have been coded differently. While the final findings of the study do not show sufficient attention to proof and reasoning in textbooks, the actual findings may have been bleaker if such an assumption on the part of the researcher had not been made.

This study found that through the use of specific examples, in lieu of a real proof or mathematical explanation, students are routinely exposed to empirical evidence to support mathematical claims. By repeatedly making empirical arguments to justify mathematical claims, educators following the curriculum set forth by the majority of textbooks are essentially “teaching” students that an empirical argument is sufficient to establish truth in mathematics. This reinforces the findings of researchers such as Chazan (1993), Knuth et al. (2002), and Thompson (1991), who have conducted various studies identifying students’ incorrect belief that an empirical argument qualifies as a proof in mathematics. It is possible that such widespread use of empirical arguments to
establish truth in the intended curriculum is the root of this problem. While it would be difficult to do research to establish a causal relationship, more research is needed to explicitly show a connection or correlation here. Teachers in the field should also be cautious of how they use examples in the classroom, how often they use prior mathematical knowledge to justify a new theorem, property, or procedure, and the methods they use to show or “prove” claims to their students.

Only 38% of the sections analyzed in this study offered sufficient justification and reasoning for mathematical claims. Although these explanations might also employ the use of specific examples, they use them to a different end. In many cases in which justification or reasoning is present, textbooks present an annotated solution to a specific example, explaining both what has been done to solve the problem and why it works or why it is allowed, drawing upon prior mathematical knowledge.

Although the distinction between the two ways in which textbooks use examples may seem subtle, it is the emphasis that is important. Textbooks found to provide justification and reasoning in a particular section make meaningful use of specific examples in order to discuss the underlying mathematics and/or the reason the theorem, property, or procedure is true or works. Conversely, other textbooks offer examples as a kind of empirical evidence to suggest that a mathematical statement is true.

Immediately prior to publication, Thompson, Senk, and Johnson (2012) published a study exploring proof and reasoning in Algebra I, Algebra II, and Pre-Calculus textbooks. As opposed to looking for the modeling of proof and reasoning in introductory algebra textbooks (prior to formal instruction in proof), they looked in Algebra I, Algebra II, and Pre-Calculus textbooks for opportunities to learn reasoning and proof-type thinking in sections dealing with properties of exponents, logarithms, and polynomials. They found that, when looking at the coverage of the topic across all textbooks in all three courses, approximately 50% of properties, procedures, concepts,
and theorems related to the three aforementioned topics were somehow justified. When looking only at the seven Algebra I books included in Thompson et al.’s study, it was reported that 34.2% of all topics were justified with a general argument, as compared with this study’s finding that roughly 38% of all mathematical claims were sufficiently justified. These numbers are remarkably similar. It should be noted, however, that while Thompson et al.’s study illuminates issues that are clearly related to this study, it is not entirely meaningful to compare the overall findings of this study with Thompson et al.’s as they did not distinguish between the different ways in which textbooks make use of specific examples as this study did.

One strength of Thompson et al.’s (2012) study was that, while it only looked at three topics, it looked at those three topics across three different courses. A limitation of this study was that it only looked at introductory algebra textbooks to identify the presence of modeling and instruction of deductive reason and proof-type thinking. It is suggested that this study be replicated vertically throughout a curriculum or a textbook series to see the full scope of students’ exposure to the modeling of deductive reasoning and proof-type thinking.

If students of mathematics are going to learn to justify or prove their mathematical statements using logic and deductive reasoning, then they need to be exposed to such arguments and/or justifications. Educators need to model both the types of thinking involved in proof and the fact that mathematical statements can be justified using prior knowledge. When possible, educators should justify major mathematical claims.

It is still important for textbooks to provide models of how to enact various procedures. However, just as students benefit from seeing solutions to example problems, so would their ability to reason mathematically benefit from exposure to logical arguments using known mathematics. It is important for textbook authors to be thoughtful about how and why they are using examples.
An implication of this study might lead to NCTM taking a more active role in not only identifying areas of importance in mathematics instruction, as they have typically done, but also in promoting them. It is suggested that NCTM offer a "stamp of approval," or some kind of quality grade to those texts that sufficiently address important topics such as deductive reasoning and proof in mathematics instruction. This is a common practice in other organizations that endorse products that support their initiatives. Furthermore, teacher preparation programs should prepare teachers to evaluate texts and instructional materials against *NCTM Principles and Standards* and the *Common Core State Standards* so that teachers can make more informed and educated decisions about the materials they use in their classrooms.

It is not suggested that textbooks need to offer models of proof and reasoning in every section of every chapter. To do so is not always possible or prudent. However, it is important that textbooks repeatedly and consistently expose students to deductive reasoning and proof-type thinking or, at the very least, the idea that new mathematical claims need to be justified mathematically. If a textbook omits a justification or proof for a mathematical claim, then it should either present it as an axiom or explain that, while a mathematical justification for the claim is outside the scope of the course, it is possible.

This study did not explicitly look at the educational value of each proof, but rather just its presence in the curriculum. Hanna (2000) points out that while all proof is concerned with verification and discovery, formal mathematical proofs do not necessarily provide explanation, communication, or incorporation. “Some proofs are by their nature more explanatory than others.” Some of the textbooks analyzed in this study provided justification for claims in a manner that was not likely to be of much use to a typical algebra student. As textbook authors and curriculum writers begin to respond to the recommendations of the *NCTM Principles and Standards* and the *Common Core State Standards* and the curriculum includes more instruction on proof
and reasoning, further analysis will be required to evaluate the extent to which a given justification is appropriate for a student at a given level.

A plenitude of research suggests that students of all levels, both foreign (Fischbein & Kedem, 1982; Recio & Godino, 2001; Williams, 1980) and domestic (Chazan, 1993; Coe & Ruthven, 1994; Moore, 1994; Reiss & Renkl, 2002; Silver & Carpenter, 1989; Tinto, 1988), lack competency and facility with both understanding and generating proof. This study examined the intended curriculum through the lens of textbooks to seek evidence of the exposure of students to proof and deductive reasoning throughout their mathematics education as dictated by the NCTM Principles and Standards and the Common Core State Standards. Specifically, this study sought to find evidence of such instruction in the last course prior to formal proof instruction. This study’s failure to find sufficient evidence of instruction or modeling of proof and reasoning in secondary school algebra textbooks suggests that, overall, textbook companies are not preparing their textbooks according to the guidelines set forth by the NCTM Principles and Standards and the Common Core State Standards. This suggests that the enacted curriculum also fails to address the recommendations of these trendsetting organizations.

More research is needed in actual classrooms. It is possible that, while textbooks fail to provide sufficient instruction and modeling of proof-type thinking and deductive reasoning, it is present in the enacted curriculum. It is possible that teachers add such instruction on their own. To this end, further research into the enacted curriculum is needed.

If textbook materials are not incorporating reasoning and proof, it is possible that teachers are deviating from or adding to the curricula provided by these textbooks. On the other hand, it is also possible that school districts and schools nationwide are failing to address the recommendations set forth in the NCTM Principles and Standards and the Common Core State Standards. A survey of schools nationwide could be done to see
if and how schools are preparing teachers to address these recommendations. It is possible that such instruction is not in their focus or that they have not provided current teachers with professional development to this end. Further observation-based research is necessary to confirm or audit whether reported teacher training and professional development have been provided, and further, to ascertain the nature of instruction of proof and reasoning in algebra classrooms. Simple modifications to instruction, such as always discussing why a new mathematical concept or property is true using known mathematics, could help to address the recommendations.

This study has found a lack of evidence that the intended algebra curriculum, as seen through the most popular and widely circulated textbooks, provides sufficient modeling and instruction of deductive reasoning and proof-type thinking. While research suggests that textbooks are a valid indication of what happens in mathematics classrooms, actual observation of the enacted curriculum across the nation is needed. Such research is time-consuming, expensive, and absolutely necessary.
REFERENCES


Strutchens, M. E., & Blume, G. W. (1997). What do students know about geometry? In P. M. Kenney & E. A. Silver (Eds.), Results from the sixth mathematics assessment (pp. 165-193).


## Appendix A

### EXPANDED LIST OF TEXTBOOKS INCLUDED IN THIS STUDY

<table>
<thead>
<tr>
<th>Title</th>
<th>Publisher</th>
<th>Author(s) / Lead Developer</th>
<th>Year</th>
<th>Used in</th>
</tr>
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<tr>
<td>Algebra 1</td>
<td>Holt McDougal</td>
<td>Burger, Chard, Kennedy, Leinwand, Renfro, Roby, Waits</td>
<td>2011</td>
<td>CA, NY</td>
</tr>
<tr>
<td>Algebra 1</td>
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<td>Larson, Boswell, Kanold, Stiff</td>
<td>2008</td>
<td>CA, FL, TX</td>
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<td>Algebra 1 An incremental development</td>
<td>Saxon</td>
<td>Saxon</td>
<td>2003</td>
<td>CA, NY, FL</td>
</tr>
<tr>
<td>Algebra 1 Concepts and Skills</td>
<td>McDougal Littell</td>
<td>Larson, Boswell, Kanold, Stiff</td>
<td>2004</td>
<td>FL, NY</td>
</tr>
<tr>
<td>Algebra Concepts and Applications</td>
<td>Glencoe McGraw-Hill</td>
<td>Cummings, Malloy, McClain</td>
<td>2007</td>
<td>CA, TX</td>
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<td>Dietiker, Baldinger</td>
<td>2008</td>
<td>TX</td>
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<td>Amsco's Integrated Algebra</td>
<td>Amsco</td>
<td>Gantert</td>
<td>2007</td>
<td>NY</td>
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<tr>
<td>CME Project Algebra 1</td>
<td>Person</td>
<td>Cuoco</td>
<td>2009</td>
<td>NY</td>
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<tr>
<td>Discovering Algebra</td>
<td>Key Curriculum Press</td>
<td>Murdock, Kamischke, Kamischke</td>
<td>2007</td>
<td>FL, NY, TX</td>
</tr>
<tr>
<td>Discovering Algebra California Edition</td>
<td>Key Curriculum Press</td>
<td>Murdock, Kamischke, Kamischke</td>
<td>2007</td>
<td>CA</td>
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<td>Glenco Mathematics Texas Algebra 1</td>
<td>Glencoe</td>
<td>Hoiiday, Marks, Cuevas, Casy, Day, Carter, Hayek</td>
<td>2007</td>
<td>TX</td>
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<td>Prentice Hall Florida Algebra</td>
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<td>Bellman, Bragg, Charles</td>
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<td>Barnett, Nelson</td>
<td>2010</td>
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