An Alternative Proof of Genericity for the Unitary Group in Three Variables

Chongli Wang

Submitted in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy
in the Graduate School of Arts and Sciences

COLUMBIA UNIVERSITY

2016

Abstract

An Alternative Proof of Genericity for the Unitary Group in Three Variables

Chongli Wang

In this thesis, we prove that local genericity implies globally genericity for the quasi-split unitary group U_3 for a quadratic extension of number fields E/F. We follow [Fli1992] and [GJR2001] closely, using the relative trace formula approach. Our main result is the existence of smooth transfer for the relative trace formulae in [GJR2001], which is circumvented there. The basic idea is to compute the Mellin transform of Shalika germ functions and show that they are equal in the unitary case and the general linear case.

Contents

Acknowledgments

First and foremost, I would like to thank Wei Zhang for his support, patience, and generosity. During the past five years I have been extremely fortunate to have him as my advisor and have benefited time and time again from his enthusiasm and insight, especially for giving me so many interesting problems to work on.

I also would like to thank Dorian Goldfeld, Zhengyu Mao, David Soudry, and Eric Urban for conversations, comments, invitations to speak, classes taught, and, of course, for sitting on my thesis defense.

During my time at Columbia, I have benefited greatly from all kinds of conversations with my friends: Ioan Filip, Zhijie Huang, Rahul Krishna, Pak-Hin Lee, Zheng Liu, Vivek Pal, Changjian Su, Hang Xue, Jingyu Zhao, and Zijun Zhou. I have enjoyed so much talking to you about mathematics and life.

Finally, thanks to my parents and wife for your support all the time. I can achieve nothing without you and your encouragement.

Chapter 1

Introduction

In this thesis, we give an alternative proof of the following result for the quasi-split unitary group G in three variables relative to a quadratic extension E/F in [GJR2001]: a cuspidal automorphic representation π of G is globally generic if and only if π_v is generic at every place v. We use the same approach as in [GJR2001] and [Zha2014b], via the comparison of two relative trace formulae. Our new result is the general existence of smooth transfer for the relative trace formulae in [GJR2001]. The authors in [GJR2001] circumvented this existence problem.

In general we consider a reductive group G, defined and quasi-split over a number field F. We denote its adele ring as $\mathbb{A} = \mathbb{A}_F$. Let π be a cuspidal automorphic representation of $G(\mathbb{A})$ with central character ζ, and fix an embedding $\pi \hookrightarrow L^2_0(G(F)\backslash G(\mathbb{A}), \zeta)$. For $\phi \in \pi^\infty$, define the Whittaker functional

$$\mathcal{W}(\phi) := \int_{N(F)\backslash N(\mathbb{A}_F)} \phi(n)\bar{\theta}(n)\, dn,$$

where N is the maximal unipotent subgroup of G, and θ is a generic character on $N(F)\backslash N(\mathbb{A}_F)$. π is called generic with respect to θ if $\mathcal{W}$ is not zero. If $\mathcal{W}$ exists, it is unique up to a scalar by the uniqueness of Whittaker functional. When this is the case, define the Whittaker function of ϕ on $G_{\mathbb{A}}$ by

$$W_\phi(g) = \mathcal{W}(\rho(g)\phi) = \int_{N(F)\backslash N(\mathbb{A}_F)} \phi(ng)\bar{\theta}(n)\, dn.$$

Then the map $\phi \mapsto W_\phi$ is not trivial. In our case when $G = U_3$ we have multiplicity one ([Rog1990, Theorem 13.3.1, page 201]) and therefore the embedding is unique. It is a general conjecture that inside a tempered L-packet of an automorphic representation, there is exactly one component which is generic with respect to any given θ. We prove this result in the quasi-split case when $G = U_3$, and then the main result follows as a corallary to this result.

For a general space X, we denote $\mathcal{C}(X) = \mathfrak{S}(X)$ to be Schwartz functions on X. Throughout this paper, we make an assumption on the unitary group G similar to [Zha2014b] for the weak base change as follows. An automorphic representation $\Pi = \otimes_v \Pi_v$ of $\mathrm{GL}_3(\mathbb{A}_E)$ is called the weak base change of π, and denoted $\mathrm{BC}(\pi)$, if Π_v is the local base change of π_v for all but finitely many places v where π_v is unramified. Throughout the thesis, we use the following result. For every cuspidal automorphic representation π, the weak base change exists and satisfies the following local-global compatibility at all split places v: the v-component of $\mathrm{BC}(\pi)$ is the local base change of π_v, more precisely if v_1, v_2 are two places of E over v, then $\mathrm{BC}(\pi)_{v_1} = \pi_v$ and $\mathrm{BC}(\pi)_{v_2} = \pi_v^*$.

Remark. In our case when G is quasi-split, this result is true by the work [Mok2015] of Mok following analogous work of Arthur on endoscopic classification.

We choose an arbitrary $f \in \mathcal{C}(G(\mathbb{A}))$ and define the global Bessel distribution by

$$J_\pi(f) := \sum_{\phi \in \mathcal{B}(\pi)} \mathcal{W}(\pi(f)\phi)\overline{\mathcal{W}(\phi)},$$

where ϕ goes through an orthonormal basis $\mathcal{B}(\pi)$ of π. Similarly we can define the local Bessel distribution by

$$J_{\pi_v}(f_v) := \sum_{\phi_v \in \mathcal{B}(\pi_v)} \mathcal{W}_v(\pi_v(f_v)\phi_v)\overline{\mathcal{W}_v(\phi_v)},$$

where ϕ_v goes through an orthonormal basis $\mathcal{B}(\pi_v)$ of π_v, and $\mathcal{W}_v$ is the local nontrivial Whittaker functional which is unique up to a scalar. Suppose $f = \otimes_v f_v$. Then the main result is that

$$J_\pi(f) = c(\pi) \prod_v J^\natural_{\pi_v}(f_v)$$

for some non-zero constant $c(\pi)$ independent of f, where $J^\natural_{\pi_v}(f_v)$ is the normalization of $J_{\pi_v}(f_v)$, in the sense that we normalize it to be 1 over unramified places to make sense of the product. Note that the existence of such a constant follows from the uniqueness of Whittaker functional. We would like to prove an explicit formula for $c(\pi)$ in terms of the nonvanishing of certain L-functions and hopefully the new result on the existence of general smooth transfer can help us to tackle this problem.

Now let us give some heuristics as to how we can prove the nonvanishing of J_π. We can first define a corresponding distribution I_Π for the general linear group by considering the base change (which translates to distinguishedness) and the Whittaker functional. Now by summing over different π and Π, we obtain I and J defined as distributions. Both I and J admit geometric decompositions and can be written as a sum over orbital integrals. Thus by using smooth transfer we can somehow identify $I_\Pi(f')$ and $J_\pi(f)$. The central part of this paper is to show the existence of such smooth transfer (matching) between f and f'. But the decomposition of I_Π is classical and well-known, and therefore by choosing each local component carefully we are able to choose $I_\Pi(f') \neq 0$. Hence the nonvanishing for J_π follows. In Section 2, we will give a more detailed explanation of this relative trace formula approach.

We want to point out at this stage that this result shares the same philosophy with many other works connecting period integrals of automorphic forms with special values of L-functions. The pioneering example we consider is Waldspurger's work [Wal1985] on a formula relating toric periods and the central value of L-functions on GL_2. Later Gross and Prasad formulated a conjecture generalizing Waldspurger's work to higher rank orthogonal groups in [GP1992] and [GP1994]. As in our case we would like to calculate $c(\pi)$ explicitly there are similar refined versions of conjectures of Gross and Prasad formulated by Ichino and Ikeda in [II2010]. Later Gan, Gross and Prasad generalized the conjectures to cover unitary groups and symplectic groups in [GGP2012]. A refined conjecture for unitary groups following this was formulated in [Har2011]. Zhang proved both the Gan–Gross–Prasad conjecture and the refined conjecture for the case of $U_{n-1} \times U_n$ under mild conditions in [Zha2014b] and [Zha2014a]. Later Xue proved the GGP conjecture for $U_n \times U_n$ using similar approaches in [Xue2014].

The plan of the paper is as follows. Section 2 gives an outline of the proof following [GJR2001] closely. Section 3 reviews some basics of Asai L-functions and quotes a result connecting the property of being a base change and distinguishedness. There is also a nonvanishing result for I_Π in that section. Section 4 proves the existence of smooth transfer for split places of F. Sections 5 to 8 prove the existence of smooth transfer for non-split places of F by computing the Mellin transforms of Shalika germ functions. Section 9 gives a result on the separation of spectrum and concludes our results.

Chapter 2

Outline of the proof

Now we give an outline of the proof of the result. Let E/F be a quadratic extension of number fields with Galois group $\{1, {}^-\}$, and G be the quasi-split unitary group over F of three variables, i.e.

$$G(F) = \{g \in \mathrm{GL}_3(E) \mid {}^t\bar{g}wg = w\},$$

where $w = w_G = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Let π be a cuspidal automorphic representation of G. Consider Π, its weak base change to $G' = \mathrm{GL}_3$ over E. Regard G' as a group over F and denote H' to be GL_3 over F. Define the Flicker–Rallis period by

$$\beta(\phi) := \int_{Z(\mathbb{A})H'(F)\backslash H'(\mathbb{A})} \phi(h)\, dh$$

(we remark that when n is even, there is a character involved for U_n, but we will not consider this case). We call an automorphic representation Π of G distinguished if $\beta \neq 0$. We have the following distinction property.

Proposition 2.1. *If Π is the weak base change of π, then Π is distinguished.*

This is proved by connecting both arguments with the existence of a pole at $s = 1$ for the partial Asai L-function of Π.

We also have a local period defined as

$$\beta_v(W_v) := \int_{N_2(F_v)\backslash GL_2(F_v)} W_v \begin{pmatrix} \epsilon_2 h & \\ & 1 \end{pmatrix} dh,$$

where $W_v \in \mathcal{W}(\Pi, \psi_E)$ is a Whitakker function, $\epsilon_2 = \mathrm{diag}(\tau^2, \tau)$ for $\tau + \bar{\tau} = 0$. By [Zha2014a, page 17], we can calculate β_v at the unramified places: if we normalize $W_v(1) = 1$, then

$$\beta_v(W_v) = \mathrm{vol}(K) L(1, \Pi_v, \mathrm{As}).$$

So we can define the normalized local period by

$$\beta_v^\natural(W_v) = \frac{\beta(W_v)}{L(1, \Pi_v, \mathrm{As})}.$$

Now we state a product decomposition for the global Flicker–Rallis period.

Proposition 2.2. *We have an explicit decompostion*

$$\beta(\phi) = \frac{n \cdot \mathrm{Res}_{s=1} L(s, \Pi, \mathrm{As})}{\mathrm{vol}(F^\times \backslash \mathbb{A}^1)} \prod_v \beta_v^\natural(W_v),$$

where in our case $n = 3$ and $W = W_\phi = \otimes_v W_v \in \mathcal{W}(\Pi, \psi_E)$. For a bad place v, $\prod_v \beta_v^\natural(W_v)$ is defined as in [Zha2014a, Remark 6].

Proof. This is exactly the result in [Zha2014a, Proposition 3.2]. □

We define the corresponding global Bessel distribution with respect to G' by

$$I_\Pi(f) := \sum_{\phi \in \mathcal{B}(\Pi)} \beta(\Pi(f)\phi) \overline{\mathcal{W}(\phi)},$$

where ϕ goes through an orthonormal basis $\mathcal{B}(\Pi)$ of Π. We similarly define the local Bessel distribution by

$$I_{\Pi_v}(f_v) := \sum_{\phi_v \in \mathcal{B}(\Pi_v)} \beta_v(\Pi_v(f_v)\phi_v) \overline{\mathcal{W}_v(\phi_v)},$$

where ϕ_v goes through an orthonormal basis $\mathcal{B}(\Pi_v)$ of Π. Then we have the following product decomposition of $I_\Pi(f)$ as a result of the product decomposition of $\beta(\phi)$.

Proposition 2.3. *We have an explicit decomposition*

$$I_\Pi(f) = \frac{n \cdot \mathrm{Res}_{s=1} L(s, \Pi, \mathrm{As})}{\mathrm{vol}(F^\times \backslash \mathbb{A}^1)} \prod_v I_{\Pi_v}^\natural(f_v),$$

where $n = 3$ and $I_{\Pi_v}^\natural(f_v)$ is the normalized $I_{\Pi_v}(f_v)$, i.e. 1 at all unramified places, and at ramified places, *we replace β_v by $\beta_v^\natural$ to get our normalized version.*

Now we recall the relative trace formulae in [GJR2001]. First let us consider the RTF in the unitary case. Recall $G = U_3$ is quasi-split with maximal unipotent subgroup N. We notice that G contains a large open double coset $NwAN$ of N, where A is the subgroup of diagonal elements in G, and we call this set the union of all regular orbits of $N \times N$. We say $f = \otimes_v f_v \in \mathcal{C}(G(\mathbb{A}))$ is nice if there exists one place v such that f_v is supported on the union of regular orbits. For such an f and a character ζ on $Z_G(\mathbb{A})/Z_G(F)$ we consider (the ζ-part of) the kernel function

$$K_f(x,y) := \int_{Z_G(\mathbb{A})/Z_G(F)} \sum_{\gamma \in G(F)} f(x^{-1}\gamma y z)\zeta(z)\, dz,$$

and the distribution

$$J(f) := \int_{[N]\times[N]} K_f(n_1,n_2)\theta(n_1)\theta(n_2)\, dn_1\, dn_2,$$

where $[N] = N(F)\backslash N(\mathbb{A}_F)$, and θ is the generic character of $[N]$ given by

$$\theta\begin{pmatrix} 1 & x & t - \frac{x\bar{x}}{2} \\ 0 & 1 & -\bar{x} \\ 0 & 0 & 1 \end{pmatrix} = \psi_E(x).$$

Now for $\gamma = ud_a$, where $u \in U_1(\mathbb{A})$ and $d_a = \mathrm{diag}(a, 1, \bar{a}^{-1})$, we define the global orbital integral as

$$O(\gamma, f) := \int_{N(\mathbb{A}_F)\times N(\mathbb{A}_F)} f(n_1^{-1} w\gamma n_2)\theta(n_1)\theta(n_2)\, dn_1\, dn_2.$$

Similarly we can define the local orbital integral as

$$O(\gamma, f_v) := \int_{N(F_v)\times N(F_v)} f_v(n_1^{-1} w\gamma n_2)\theta_v(n_1)\theta_v(n_2)\, dn_1\, dn_2.$$

Then we have a trivial decomposition $O(\gamma, f) = \prod_v O(\gamma_v, f_v)$ when $\gamma = (\gamma_v)_v$, $f = \otimes_v f_v$.

Proposition 2.4. *For every nice f, $J(f)$ converges absolutely and we have*

$$J(f) = \sum_{a \in E^\times} \int_{U_1(\mathbb{A})} O(ud_a, f)\zeta(u)\, du,$$

which is a finite sum over a.

Proof. For nice f the kernel function can be written as

$$K_f(x,y) = \int_{[U_1]} \sum_{\gamma\in N(F)\backslash G(F)/N(F)} \sum_{\gamma_1,\gamma_2\in N(F)} f(x^{-1}\gamma_1^{-1}\gamma\gamma_2 y)\zeta(u)\,du,$$

where the sum over γ is only over the regular orbits, therefore the stabilizers of γ_1 and γ_2 are trivial. Now making a change of variables $x \mapsto \gamma_1 x$, $y \mapsto \gamma_2 y$, we have

$$J(f) = \sum_{\gamma\in N(F)\backslash G(F)/N(F)U_1(F)} \int_{U_1(\mathbb{A})} O(u\gamma, f)\zeta(u)\,du,$$

and on the regular orbits

$$N(F)\backslash N(F)wA(F)N(F)/N(F)U_1(F) \simeq E^\times. \qquad \square$$

Now we consider the general linear case. Recall $G' = \mathrm{Res}_{E/F}\mathrm{GL}_3$ as an F-group and $H' = \mathrm{GL}_3$ over F. For $\tilde{f}' \in \mathcal{C}_c^\infty(\mathrm{GL}_3(\mathbb{A}_E))$ and $w(z) = \zeta(z\bar{z}^{-1})$ we consider (the w-part of) the kernel function

$$K_{\tilde{f}'}(x,y) = \int_{Z(\mathbb{A}_E)/Z(E)} \sum_{\gamma\in GL_3(E)} \tilde{f}'(x^{-1}\gamma y z)\omega(z)\,dz,$$

and the distribution

$$I(\tilde{f}') := \int_{H'(F)Z_{H'}(\mathbb{A}_F)\backslash H'(\mathbb{A}_F)} \int_{N(E)\backslash N(\mathbb{A}_E)} K_{\tilde{f}'}(h,n)\theta'(n)\,dn\,dh,$$

where θ' is the character on $N(E)\backslash N(\mathbb{A}_E)$ defined by

$$\theta'\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \psi_E(x-y).$$

We would like to simplify the integral as follows. Define $\mathcal{S}$ to be the symmetric space as a variety over F with F-points

$$\mathcal{S}(F) = \{s \in \mathrm{GL}_3(E) \mid s\bar{s} = 1\}.$$

By Hilbert Satz-90, we have an isomorphism of affine F-varieties

$$\nu : \mathrm{GL}_{3,F}\backslash \mathrm{Res}_{E/F}\mathrm{GL}_3 \to \mathcal{S}$$

$$g \mapsto \bar{g}^{-1}g.$$

This gives rise to a bijection of F-points

$$\mathrm{GL}_3(F)\backslash\mathrm{GL}_3(E) \simeq \mathcal{S}(F).$$

Now we define a new function $f' \in \mathcal{C}(\mathcal{S})$ for each $\tilde{f}'$ by

$$f'(s) := \int_{H'(\mathbb{A})} \tilde{f}'(hx)\, dh,\ \ \nu(x) = s.$$

The map $\tilde{f}' \mapsto f'$ gives a surjection $\mathcal{C}(G') \to \mathcal{C}(\mathcal{S})$. Consider the right N-action on $\mathcal{S}$ by $s^n = \bar{n}^{-1}sn$. Notice that $\mathcal{S}$ has a large open set consisting of the N-orbits of wA_w, where $A_w = \{\alpha = \mathrm{diag}(a, u, \bar{a}^{-1}) \mid u\bar{u} = 1\}$. We also call this set the regular orbits. Let $\eta \in \mathrm{GL}_3(E)$ such that $\nu(\eta) = w$. Then the regular orbits translate into regular orbits of $\mathrm{GL}_3(E)$ as $\mathrm{GL}_3(F)\eta N_3(E)$. Similar to the unitary case, we say $\tilde{f}' = \otimes_v \tilde{f}'_v$ is nice if for any one place v, $\tilde{f}'_v$ is supported on the regular orbits of $G'(E_v)$. This translates to saying that f' is nice if f'_v is supported on the regular orbits of $\mathcal{S}(F_v)$. Now we have the following

Proposition 2.5. *For nice f' we have*

$$I(\tilde{f}') = \int_{N(E)\backslash N(\mathbb{A}_E)} \int_{U_1(F)\backslash U_1(\mathbb{A})} \sum_{\gamma\in\mathcal{S}(F)} f'(\bar{n}^{-1}\gamma n u)\theta'(n)\, dn\, \zeta(u)\, du.$$

Remark. We will also denote the left hand side as $I(f')$ since it depends on f' rather than $\tilde{f}'$.

Proof. This follows by integrating $\tilde{f}'$ over the group $H'(\mathbb{A}_F)$ and substituting the formula for f' as a function on $\mathcal{S}(\mathbb{A}_F)$. □

Now for $\gamma = ud_a$, where $u \in U_1(\mathbb{A})$ and $d_a = \mathrm{diag}(a, 1, \bar{a}^{-1})$, we define the global orbital integral as

$$O(\gamma, f') := \int_{N(\mathbb{A}_E)} f'(\bar{n}^{-1}w\gamma n)\theta'(n)\, dn.$$

Similarly we can define a local orbital integral

$$O(\gamma, f'_v) := \int_{N(E_v)} f'_v(\bar{n}^{-1}w\gamma n)\theta'_v(n)\, dn.$$

Then we have a trivial decomposition $O(\gamma, f') = \prod_v O(\gamma_v, f'_v)$ when $\gamma = (\gamma_v)_v,\ f = \otimes_v f'_v$.

Proposition 2.6.

$$I(f') = \sum_{a \in E^\times} \int_{U_1(\mathbb{A}_F)} O(ud_a, f')\zeta(u)\, du.$$

Proof. This is simply because our test function f' is supported on the regular orbits and the proof is similar to Proposition 2.4. □

Therefore by comparing Proposition 2.4 and Proposition 2.6, the question of $I(f') = J(f)$ reduces to that of having identical orbital integrals over all $\gamma = ud_a$. We call f and f' smooth transfers (or smooth matchings) of each other if they have this property. Furthermore by factorization we can reduce this property to each local component. In particular, we need to have identical orbital integrals over all local orbits of γ_v, i.e. local smooth transfers. Now the question reduces to the fundamental lemma and the existence of smooth transfers. Notice at first that for f or f' that are supported on the regular orbits, the existence of smooth transfer of each one is automatic simply by the matching of regular orbits. The fundamental lemma is proved in [Jac1992] and [Mao1993]. In [Mao1993] the smooth matching of the entire two Hecke algebras is proved. We prove the existence of smooth transfers for arbitrary smooth functions, even though to prove the result we only need the existence for some special (smooth) functions.

In order to establish the result we need to have spectral decompositions of I and J, i.e. the spectral sides of relative trace formulae. In [GJR2001] Jacquet proved the relative trace formulae in both cases. We first recall some notations introduced in [Rog1990]. Let (M, σ) be a pair consisting of a Levi subgoup M of G and a cuspidal automorphic representation σ of M. A cuspidal datum is an equivalence class of pairs $\kappa = \{(M, \sigma)\}$. We can decompose the kernel function $K(x, y) = \sum_\kappa K_\kappa(x, y)$ and define the corresponding J_κ and I_κ. The detailed definition can be found in [GJR2001].

Proposition 2.7. *We have an absolutely convergent spectral decompostion of $J(f)$ as follows:*

$$J(f) = \sum_\kappa J_\kappa(f),$$

where the sum is over all cuspidal data κ. Implicitly all associated automorphic representations in the sum have central character ζ.

Proof. This is the result of [GJR2001] proved in Section 10. □

Proposition 2.8. *We have an absolutely convergent spectral decompostion of $I(f')$ as follows:*

$$I(f') = \sum_{discrete} I_\kappa(f') + \sum_{\mu_1} I_{\mu_1}(f'),$$

where the first sum is over the cuspidal data κ (corresponding to the discrete spectrum), and the second sum is over all idele class characters μ_1 of E (corresponding to the continuous spectrum). Implicitly all associated automorphic representations in the sum are distinguished and have central character ζ.

Proof. This is the main result of [GJR2001], stated in Section 9, page 64. □

By the separation of spectrum we have the following

Proposition 2.9.

$$I_\Pi(f') = \sum_\pi J_\pi(f),$$

where the sum is over all members of the tempered L-packet with weak base change Π.

From this proposition we can deduce our main results.

Chapter 3

Distinguishedness and nonvanishing of local Bessel distributions

Let E/F be a quadratic extension of number fields with Galois group $\{1, ^-\}$, $G = \mathrm{Res}_{E/F}\mathrm{GL}_n$, and $H = \mathrm{GL}_n(F)$. The L-group of G is

$$^LG := (\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})) \rtimes \{1, ^-\}.$$

The Asai L-function is the Langlands L-function attached to the representation $r : {}^LG \to \mathrm{GL}(\mathbb{C}^n \otimes \mathbb{C}^n)$ where

$$r(g_1, g_2)(v \otimes w) = (g_1 v) \otimes (g_2 w),$$
$$r(^-)(v \otimes w) = w \otimes v.$$

Let Π be a cuspidal automorphic representation of G where $n = 3$. Fix S_0 as before so that Π is unramified outside S. Denote by $L^{S_0}(s, \Pi, \mathrm{As})$ the product of the local factors of the Asai L-function over places outside S_0.

Proposition 3.1. *Let Π be a cuspidal automorphic representation of GL_n with n odd. Suppose the central character ω of Π is trivial on $\mathbb{A}_F^\times$. Then the following are equivalent:*

1. *Π is distinguished by H.*
2. *$L(s, \Pi, \mathrm{As})$ has a pole at $s = 1$.*
3. *$\tilde{\Pi} \simeq \Pi^\sigma$.*

Proof. This is exactly [GJR2001, page 12, Proposition 1]. □

Now we consider a non-split place v_0 of F and state a nonvanishing result of local Bessel distributions I_{Π_v}.

Proposition 3.2. *If Π_v is distinguished by $H(F_{v_0})$, then $I_{\Pi_v}(f'_v) \neq 0$ for some f'_v supported on the regular orbits.*

Proof. This is exactly [GJR2001, page 18, Theorem 1]. □

Chapter 4

Smooth transfer at the split places

Let v be a split place of F. Then $\mathcal{S}(F_v)$ can be identified with pairs $(g_1, g_2) \in \mathrm{GL}_3(F_v) \times \mathrm{GL}_3(F_v)$ such that $g_1 g_2 = 1$, hence with $\mathrm{GL}_3(F_v)$ via $(g_1, g_2) \mapsto g_1$. Given $\tilde{f}'_v = (\tilde{f}'_{v_1}, \tilde{f}'_{v_2})$, set

$$f'_v = \int_{GL_3(F_v)} \tilde{f}'_{v_1}(hg_1)\tilde{f}'_{v_2}(h)\, dh$$

as a function on $\mathcal{C}(\mathcal{S}(F_v))$. Therefore f'_v is simply the convolution of $\tilde{f}'_{v_1}$ and $\check{\tilde{f}}'_{v_1}$, where $\check{g}(x) = g(x^{-1})$, and so the local orbital integral becomes

$$I(\gamma, f'_v) = \int_{N(F_v)\times N(F_v)} f'_v(n_2^{-1}\gamma n_1)\theta(n_1 n_2)\, dn_1\, dn_2.$$

On the unitary side, $G(F_v)$ can be identified with pairs $(g_1, g_2) \in \mathrm{GL}_3(F_v) \times \mathrm{GL}_3(F_v)$ such that $g_2 = w^t g_1^{-1} w$, hence with $\mathrm{GL}_3(F_v)$ via $(g_1, g_2) \mapsto g_1$. Given $f_v \in \mathcal{C}(G(F_v))$, the local orbital integral becomes

$$J(\gamma, f_v) = \int_{N(F_v)\times N(F_v)} f_v(n_2^{-1}\gamma n_1)\theta(n_1 n_2)\, dn_1\, dn_2.$$

By comparing the two orbital integrals, we see that the smooth tranfer exists.

Chapter 5

Existence of germ functions

In this section, we are going to prove the existence of Shalika germ functions in the general linear case, or more precisely in the symmetric space case. The idea of Shalika germ functions was introduced by Jacquet and Ye in [JY1996] for the quadratic base change of GL_3, and was explicitly carried out in [JY1999]. The Shalika germ expansion formula originated from the group case by Harish-Chandra in [HC1970], and Kottwitz wrote a beautiful introduction to this formula in [Kot2005].

Let E/F be a quadratic extension of nonarchimedean local fields. Assume their residue fields are of cardinality q_E and q respectively. Let $^-$ be the nontrivial element in $\mathrm{Gal}(E/F)$, $N = N_3(E)$, $\mathcal{S} = \mathcal{S}_3(F) = \{s \in G' = \mathrm{GL}_3(E) \mid s\bar{s} = 1\}$. N acts on the right on $\mathcal{S}$ by $s^n = \bar{n}^{-1}sn$. Then the N-oribts in $\mathcal{S}$ are parametrized by $\{wa \mid w^2 = 1, waw\bar{a} = 1\}$, where w is a Weyl group element and a is a diagonal element. Let $\psi = \psi_F$ be an additive character of F. For X an F-space, denote $\mathcal{C}(X)$ to be the set of compactly supported locally constant functions on X. Let f' be in $\mathcal{C}(S)$.

Define the generic character θ' by

$$\theta' \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \psi_E(x - y),\ \psi_E(u) = \psi_F(u + \bar{u}).$$

Now we are going to give a full description of the N-orbits of $\mathcal{S}$ and define the corresponding orbital integrals if possible. More precisely, we define orbital integrals for relevant orbits.

- $w = w_G = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $A_w = \left\{ \alpha = \begin{pmatrix} a & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & \bar{a}^{-1} \end{pmatrix} \middle| u\bar{u} = 1 \right\}$. Its stabilizer is trivial, so all orbits are relevant. Hence we define the orbital integral

$$I(w\alpha) = \int_N f'(\bar{n}^{-1} w\alpha n)\theta'(n)\, dn. \tag{5.1}$$

It is absolutely convergent and when f' is fixed, it gives a smooth function on α.

- $w = w_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $A'_w = \left\{ \alpha = \begin{pmatrix} a & 0 & 0 \\ 0 & \bar{a}^{-1} & 0 \\ 0 & 0 & u \end{pmatrix} \middle| u\bar{u} = 1 \right\}$. Its stabilizer is

$$\left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| az = u\bar{y} \right\},$$

so none of the orbits are relevant.

- $w = w_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $A'_w = \left\{ \alpha = \begin{pmatrix} u & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & \bar{a}^{-1} \end{pmatrix} \middle| u\bar{u} = 1 \right\}$. Similar to the last case, there are no relevant orbits.

- $w = e$, $A'_w = \left\{ \alpha = \begin{pmatrix} u_1 & 0 & 0 \\ 0 & u_2 & 0 \\ 0 & 0 & u_3 \end{pmatrix}, u_i\bar{u}_i = 1 \right\}$. Its stabilizer is

$$\left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| u_1 x = u_2\bar{x},\ u_2 y = u_3\bar{y},\ u_3\bar{z} = u_1 z \right\},$$

so the relevant orbits are $A_e = \{\alpha \in A'_w \mid u_2 = -u_1,\ u_3 = u_1\}$. For those $\alpha \in A_e$ the stabilizer is

$$N_\alpha = \left\{ \begin{pmatrix} 1 & t_1 & z \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix} \middle| t_1 + \bar{t}_1 = t_2 + \bar{t}_2 = 0,\ z = \bar{z} \right\}.$$

Let

$$U_e = \left\{ \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| x, y \in F \right\}, \; V_e = \left\{ \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| t + \bar{t} = 0 \right\},$$

so that $U_e \times V_e \cong N/N_\alpha$. Define the orbital integral

$$\begin{aligned} I(\alpha, f') &= \int_{U_e} \int_{V_e} f'(u^{-1} v \alpha v u) \theta'(u) \, du \, dv \\ &= \int f' \begin{pmatrix} u_1 & u_1 x & u_1(t - xy/2) \\ 0 & -u_1 & u_1 y \\ 0 & 0 & u_1 \end{pmatrix} \psi(x+y) \, dx \, dy \, dt. \end{aligned}$$

From now on, $w = w_G$. Notice that in the definition (5.1) of $I(w\alpha)$ for $w = w_G$, replacing ψ by ψ_b, where $\psi_b(x) = \psi(bx)$ for some nonzero $b \in F$, gives

$$\begin{aligned} & \int_N f'(\bar{n}^{-1} w \alpha n) \theta'_b(n) \, dn \\ = & \int_N f'(\bar{n}^{-1} w \alpha n) \theta'(d_b n d_b^{-1}) \, dn \\ = & |b|^{-2} \int_N f'(d_b^{-1} \bar{n}^{-1} w \alpha_b n d_b) \theta'(n) \, dn, \end{aligned}$$

where $\alpha_b = \alpha d_b^{-2}$. Then the question reduces to finding transfer for the function $\hat{f}'(x) = f'(d_b^{-1} x d_b)$ with ψ. Therefore we may assume $\psi = \psi_F$ is an additive character of F of conductor $\mathcal{O}_F$. Denote $e_0 = \mathrm{diag}(1, -1, 1)$. A (Shalika) germ function is a smooth function K on $A^e_{w_G} = \{b \in A_{w_G} \mid \det(b) = 1\}$ satisfying: for any $f' \in \mathcal{C}(S)$, there exists $W^{f'}_w \in \mathcal{C}(A_{w_G})$, such that

$$I(w\alpha, f') = W^{f'}_w(\alpha) + K * W^{f'}_e(\alpha), \tag{5.2}$$

where $W^{f'}_e(z) = I(z, f')$ for $z \in A_e$, and

$$K * W^{f'}_e(\alpha) = \sum_{\alpha = bu} K(b) W^{f'}_e(u e_0), \tag{5.3}$$

where $\alpha \in A_w$ and $u\bar{u} = 1$. Notice $W^{f'}_w$ is uniquely determined by K and f'.

Lemma 5.1. *Suppose K is a germ function. Then for arbitrary $g_e \in \mathcal{C}(A_e)$, $g_w \in \mathcal{C}(A_{w_G})$, one can find*

$f' \in C(S)$ *such that*

$$W_e^{f'} = g_e, W_w^{f'} = g_w.$$

Proof. Denote $Z_w = \{X \in S \mid X_{31} = X_{32} = X_{21} = 0\}$, $Z_e = \{X \in S \mid X_{31} = 0\}$, and Ω_w (respectively Ω_e) to be the complement in S of Z_w (respectively Z_e). Then

$$Z_w \supset A_e \cdot N \longrightarrow A_e$$

gives a surjective map $\mathcal{C}(Z_w)$ to $\mathcal{C}(A_e)$ sending f_1 to $I(\alpha, f_1)$, and

$$\Omega_e \cong N \times A_w \longrightarrow A_w$$

gives a surjective map $\mathcal{C}(\Omega_e)$ to $\mathcal{C}(A_w)$ sending f_2 to $I(w\alpha, f_2)$.

Now choose $f_1 \in \mathcal{C}(S)$ such that $I(\alpha, f_1) = g_e$. There is $f_2 \in \mathcal{C}(\Omega_e)$ such that

$$I(w\alpha, f_2) = g_w - W_w^{f_1}.$$

The function $f' = f_1 + f_2$ will satisfy the conditions we require. ☐

Proposition 5.2. *Germ functions exist. If K is a germ function, $t \in \mathcal{C}(A_{w_G}^e)$, then $H = K + t$ is also a germ function and all germ functions come from this way.*

Proof. We adopted the similar approach as in [JY1996] Theorem 2.1. For any X an N-invariant F-subspace of S, define

$$\mathcal{C}(X, \theta') = \text{the subspace spanned by } g(x^n) - {\theta'}^{-1}(n)g(x),$$

where $g \in \mathcal{C}(X), n \in N$, i.e. this space is the twisted Jacquet module of the space $\mathcal{C}(X)$. Then $\mathcal{C}(X)_{\theta'} = \mathcal{C}(X)/\mathcal{C}(X, \theta')$ is the covariant. For $f_1, f_2 \in \mathcal{C}(X)$, denote $f_1 \simeq f_2$ if they have the same image in $\mathcal{C}(X)_{\theta'}$. We have a short exact sequence

$$0 \to \mathcal{C}(\Omega_w) \to \mathcal{C}(\mathcal{S}) \to \mathcal{C}(Z_w) \to 0,$$

and hence a short exact sequence

$$0 \to \mathcal{C}(\Omega_w)_{\theta'} \to \mathcal{C}(\mathcal{S})_{\theta'} \to \mathcal{C}(Z_w)_{\theta'} \to 0,$$

because taking covariants is an exact functor by [BH2006, P. 56].

Lemma 5.3. *Given $f' \in \mathcal{C}(S)$, it has image $0 \in \mathcal{C}(Z_w)_{\theta'}$ if and only if*

$$I(\alpha, f') = 0 \text{ for all } \alpha \in A_e.$$

Proof. We only need to consider the "if" part. Denote $A = A'_e$. The N-equivariant map

$$N \times A \longrightarrow Z_w$$

gives a surjective map $\mathcal{C}(N \times A)$ to $\mathcal{C}(Z_w)$ mapping β to f'_β. This induces a map

$$\mathcal{C}(Z_w)^*_{\theta'} \longrightarrow \mathcal{C}(N \times A)^*_{\theta'} \longrightarrow \mathcal{C}(A)^*$$

mapping T to $T_{N\times A}$ then to T_A. Choose β such that f'_β is the restriction of f' to Z_w. Notice that f' has image 0 if and only if $T(f'_\beta) = 0$ for all T in $\mathcal{C}(Z_w)^*_{\theta'}$.

Claim. T_A is supported in A_e.

Suppose γ_1 in $\mathcal{C}(A)$ has support outside A_e. Define a smooth function $n : A \to N$ such that if $\alpha \in \operatorname{supp}(\gamma_1)$, then n_α is in the stabilizer of α in N and $\theta'(n_\alpha) \neq 1$. Also we require $n_\alpha = 1$ outside a big compact open set of A. On small compact open sets this is always possible, so we can patch them together to get a function on A.

Choose $\delta_0 \in \mathcal{C}(N)$ such that

$$\int \delta_0(n)\theta'(n)\, dn = 1,$$

and define

$$\gamma(n, \alpha) = (\delta_0(n) - \delta_0(n_\alpha n))\gamma_1(\alpha).$$

Then $f'_\gamma = 0$ since up to a scalar it is just the integral over stabilizers. Hence,

$$0 = T(f'_\gamma) = T_{N\times A}(\gamma) = T_A(\gamma_A),$$

where

$$\gamma_A(\alpha) = \int_N \gamma(n, \alpha)\theta'(n)\, dn = (1 - {\theta'}^{-1}(n_\alpha))\gamma_1(\alpha).$$

Since the choice n_α only depends on the support of γ_1, γ_A can be any function supported outside A_e on which T_A vanishes. Therefore the claim holds.

Now restrict T_A and β_A to A_e, which we will still denote T_A, and

$$T(f'_\beta) = T_A(\beta_A) = T_A(I(\alpha, f'_\beta)) = 0$$

since all orbital integrals of f' vanish for orbits in A_e. □

Let $G_1 = \{g \in S \mid \det g = -1\}$. Choose $f_0 \in \mathcal{C}(\mathcal{S})$ such that

$$\begin{cases} I(e_0, f_0) = 1, \\ I(je_0, f_0) = 0 \quad \text{for } j^3 = 1, j \neq 1, j \in E. \end{cases}$$

Define $f_1(g) = \sum f_0(h) I(z, f')$, where the sum is over $g = hu$ with $z = ue_0 \in A_e$ and $h \in G_1$. Then

$$I(ue_0, f_1) = \sum_{j^3=1} I(j^{-1}e_0, f_0) I(uje_0, f') = I(ue_0, f').$$

Therefore, $f' \simeq f_1 + f_2$, where $f_2 \in \mathcal{C}(\Omega_w)$. Similar to the proof of the previous lemma, $\mathcal{C}(\Omega_w)_{\theta'} = \mathcal{C}(\Omega_e)_{\theta'}$, we can assume $f_2 \in \mathcal{C}(\Omega_e)$.

Now if $g = \bar{n}^{-1} w\alpha n$ is equal to $\bar{n}^{-1} wbnu$ for $b \in A^e_{w_G}$, $u \in U_1$, then $\alpha = bu$ and thus

$$f_1(\bar{n}^{-1} w\alpha n) = \sum_{\alpha = bu} f_0(\bar{n}^{-1} wbn) I(ue_0, f').$$

Hence,

$$I(w\alpha, f_1) = K * W_e^{f'}(\alpha),$$

where $K(b) = I(wb, f_0)$. Finally, we have

$$\begin{aligned} I(w\alpha, f') &= I(w\alpha, f_1) + I(w\alpha, f_2) \\ &= K * W_e^{f'}(\alpha) + W_w^{f'}(\alpha), \end{aligned}$$

where $W_w^{f'}(\alpha) = I(w\alpha, f_2) \in \mathcal{C}(A_w)$. This completes the proof of existence.

Given $H = K + t$ where $t \in \mathcal{C}(A^e_w)$,

$$I(w\alpha, f') = K * W_e^{f'}(\alpha) + W_w^{f',K}(\alpha) = H * W_e^{f'}(\alpha) + [W_w^{f',K}(\alpha) - t * W_e^{f'}(\alpha)].$$

So for H we can choose $W_w^{f',H}(\alpha) = W_w^{f',K}(\alpha) - t * W_e^{f'}(\alpha) \in \mathcal{C}(A_w)$ to make H a germ function.

Conversely, if H and K are both germ functions, then

$$I(w\alpha, f_0) = K(\alpha) = H(\alpha),$$

when α is outside the support of $W_w^{f',H}$ and $W_w^{f',K}$. $\square$

Chapter 6

Germ functions in the general linear case

In this section and the next section we are going to calculate the Mellin transforms of the corresponding Shalika germ functions. The idea is very similar to the computation in [JY1999], i.e. to write the Shalika germ functions as orbital integrals of carefully chosen characteristic functions, and then to compute their Mellin transforms. This computation is very close to the ones in [JY1999], [Jac1992] and [Mao1993].

We assume in this section and the next section that the residue characteristic of F is not 2. Substituting $\alpha = \begin{pmatrix} a & & \\ & u & \\ & & \bar{a}^{-1} \end{pmatrix} \in A^e_{w_G}$ and $f' = f_0 \in \mathcal{C}(\mathcal{S})$ satisfying

$$\begin{cases} I(e_0, f_0) = 1, \\ I(je_0, f_0) = 0 \text{ for } j^3 = 1, j \neq 1, j \in E, \end{cases}$$

into the germ equation (5.2), and letting $a \to 0$ or $a \to \infty$, we see that

$$K(\alpha) = I(w\alpha, f_0) \text{ for } a \text{ large or small enough.}$$

By Proposition 5.2, germ functions are unique up to compactly supported functions, so we can assume $K(\alpha) = I(w\alpha, f_0)$ for all α. To choose such an f_0, we fix a sufficiently large even integer m and let f_0 be the restriction to S of the characteristic function of

$$\begin{cases} |X_{ij}| \le 1, & i<j; \\ |X_{ij}| \le q_E^{-2m}, & i>j; \\ |X_{31}| \le q_E^{-2m-1-\epsilon}; \\ |X_{11}| = |X_{33}| = 1; \\ |X_{22}+1| \le q_E^{-m}; \end{cases}$$

where $\epsilon = 0$ if E/F is unramified, and $\epsilon = 1$ if E/F is ramified.

Denote $z' = xy - z$, and consider $\alpha \in A_{w_G}$ for the integral

$$\begin{aligned} K(\alpha) = I(w\alpha, f_0) &= \int_N f_0(\bar{n}^{-1} w\alpha n)\theta(n)\,dn \\ &= \int f_0\left(\begin{pmatrix} 1 & -\bar{x} & \bar{z'} \\ & 1 & -\bar{y} \\ & & 1 \end{pmatrix}\begin{pmatrix} & & \bar{a}^{-1} \\ & u & \\ a & & \end{pmatrix}\begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix}\right)\psi_E(x-y)\,dx\,dy\,dz \\ &= \int f_0\left(\begin{pmatrix} a\bar{z'} & -u\bar{x} & \bar{a}^{-1} \\ -a\bar{y} & u & 0 \\ a & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix}\right)\psi_E(x-y)\,dx\,dy\,dz \\ &= \int f_0\begin{pmatrix} a\bar{z'} & -u\bar{x} + ax\bar{z'} & \bar{a}^{-1} - u\bar{x}y + az\bar{z'} \\ -a\bar{y} & u - ax\bar{y} & uy - a\bar{y}z \\ a & ax & az \end{pmatrix}\psi_E(x-y)\,dx\,dy\,dz \\ &= \int f_0\begin{pmatrix} a\bar{z'}\bar{x}\bar{y} & -u\bar{x} + ax\bar{z'}\bar{x}\bar{y} & \bar{a}^{-1} - u\bar{x}y + az\bar{z'}x\bar{x}y\bar{y} \\ -a\bar{y} & u - ax\bar{y} & uy - a\bar{y}zxy \\ a & ax & azxy \end{pmatrix}\psi_E(x-y)|xy|\,dx\,dy\,dz, \end{aligned}$$

where in the last step we replace z by zxy and write $z' = 1 - z$.

Now we fix u, and consider the Mellin transform of I,

$$\hat{I}(\chi, u) = \int \psi_E(x-y)|xy|\chi(a)\,dx\,dy\,dz\,d^\times a.$$

Changing a to $\frac{a}{x\bar{y}}$, we have

$$\hat{I}(\chi, u) = \int \psi_E(x-y)|xy|\chi^{-1}(x\bar{y})\chi(a)\,dx\,dy\,dz\,d^\times a$$

over the domain

$$\begin{cases} z + z' = 1, |az| = |az'| = 1; \\ |u - a\bar{z'}| \le |x|^{-1}, |u - az| \le |y|^{-1}; \\ |a\bar{a}z\bar{z'} - u\bar{a} + 1| \le |ax^{-1}y^{-1}|; \\ |zx| \ge q_E^{2m}, |zy| \ge q_E^{2m}, |zxy| \ge q_E^{2m+1+\epsilon}; \\ |u - a + 1| \le q_E^{-m}. \end{cases}$$

6.1 Computation for ramified characters

Suppose χ is a ramified multiplicative character of conductor $1 + \mathfrak{p}^l$.

Lemma 6.1.

$$\int_{|x|=q_E^t} \psi_E(x)\chi^{-1}(x)\,dx = 0,$$

unless $t = \begin{cases} l, & \text{if } E/F \text{ is unramified;} \\ l+1, & \text{if } E/F \text{ is ramified.} \end{cases}$

Proof. We only prove the ramified case; the argument for the unramified case is similar. Consider the following two cases.

- Case 1: $t < l + 1$. There exists $u \in 1 + \mathfrak{p}^{l-1}$ such that $\chi(u) \neq 1$. Changing variables $x \mapsto ux$, we get

$$\begin{aligned} &\int_{|x|=q^t} \psi_E(x)\chi^{-1}(x)\,dx \\ &= \chi^{-1}(u)\int_{|x|=q^t} \psi_E(ux)\chi^{-1}(x)\,dx \\ &= \chi^{-1}(u)\int_{|x|=q^t} \psi_E(x)\chi^{-1}(x)\,dx, \end{aligned}$$

  because $|ux - x| \le q^{-1}$ implies $\psi_E(ux - x) = 1$. We conclude it is zero.

- Case 2: $t > l + 1$. There exists $u \in \mathfrak{p}^{-2}$ such that $\psi_E(u) \neq 1$. Changing variables $x \mapsto x + u$, we get

$$\int_{|x|=q^t} \psi_E(x)\chi^{-1}(x)\,dx$$

$$= \psi_E(u) \int_{|x|=q^t} \psi_E(x)\chi^{-1}(x+u)\,dx$$
$$= \psi_E(u) \int_{|x|=q^t} \psi_E(x)\chi^{-1}(x)\,dx,$$

because $|\frac{u+x}{x} - 1| \leq q^{-l}$ implies $\chi(\frac{u+x}{x}) = 1$. We conclude it is zero. □

Proposition 6.2. *If E/F is an unramified extension, then we have*

$$\hat{I}(\chi,u) = \chi(-u)(1-q_E^{-1})^{-1}\left(\int_{|x|=q_E^l} \psi_E(x)\chi(x^{-1})\,dx\right)^2$$
$$\times \begin{cases} 0, & \text{if } l \leq m \text{ and } |1+u| > q_E^{-m}; \\ \int \chi^{-1}(\frac{1}{2}+\beta\sqrt{\tau})\,d\beta, & \text{otherwise}; \end{cases}$$

where the integration domain for β is:

- $|\beta| \geq q_E^{2m-l}$ *when* $l \leq m$ *and* $|1+u| \leq q_E^{-m}$*;*
- $|\beta| \geq q_E^{2m-l}$ *and* $|u(-\frac{1}{2}+\beta\sqrt{\tau}) + \frac{1}{2} + \beta\sqrt{\tau}| \leq q_E^{-m}|\frac{1}{2}+\beta\sqrt{\tau}|$ *when* $m < l < 2m$*;*
- $|u(-\frac{1}{2}+\beta\sqrt{\tau}) + \frac{1}{2} + \beta\sqrt{\tau}| \leq q_E^{-m}|\frac{1}{2}+\beta\sqrt{\tau}|$ *when* $l \geq 2m$*.*

Proof. By Lemma 6.1 one can assume $|x| = |y| = q_E^l$. Then

$$\hat{I}(\chi,u) = \chi(-1)q_E^{2l} \int_{|x|=q_E^l} \psi_E(x)\chi(x^{-1})\,dx \int_{|y|=q_E^l} \psi_E(y)\chi(y^{-1})\,dy \times A$$

where

$$A = \int \chi(a)\,d^\times a\,dz,$$

over

$$\begin{cases} z + z' = 1, |az| = 1, |z| = |1-z|; \\ |u - az| \leq q_E^{-l}, |u - a\bar{z'}| \leq q_E^{-l}; \\ |z| \geq q_E^{2m-l}; \\ |a\bar{a}z\bar{z'} - u\bar{a} + 1| \leq |a|q_E^{-2l}; \\ |u - a + 1| \leq q_E^{-m}. \end{cases}$$

Note

$$|u - a\bar{z'}| \leq q_E^{-l}$$

is equivalent to

$$|z+\bar{z}-1| \le q_E^{-l}|z|.$$

- Case 1: $l \le m$. The conditions become

$$\begin{cases} |z| = |1-z|, |z+\bar{z}-1| \le q_E^{-l}|z|; \\ |u-az| \le q_E^{-l}; \\ |z| \ge q_E^{2m-l}; \\ |\bar{a}\bar{z}(az-u) + (1-a\bar{a}z\bar{z})\bar{z}| \le q_E^{-2l}; \\ |u-a+1| \le q_E^{-m}. \end{cases}$$

Note that $|a| = \frac{1}{|z|} \le q_E^{l-2m} \le q_E^{-m}$, so $|1+u| \le q_E^{-m}$. Making $a \to au/z$, the conditions become

$$\begin{cases} |z| = |1-z|, |z+\bar{z}-1| \le q_E^{-l}|z|, |z| \ge q_E^{2m-l}; \\ |1-a| \le q_E^{-l}, |(a-1)\bar{a} + \bar{z}(1-a\bar{a})| \le q_E^{-2l}. \end{cases}$$

Suppose $z = \alpha + \beta\sqrt{\tau}$, $a = 1 + \varpi^l(x + y\sqrt{\tau})$, where τ is a unit and ϖ is a uniformizer in F[1]. Then

$$A = \chi(u)(1-q_E^{-1})^{-1}q_E^{-l}\int \chi^{-1}(\frac{1}{2} + \beta\sqrt{\tau})\, d\alpha\, d\beta\, dx\, dy,$$

over

$$\begin{cases} |\beta| \ge q_E^{2m-l}, |\alpha - \frac{1}{2}| \le q_E^{-l}|\beta|; \\ |-x + \alpha(2x + \varpi^l(x^2 - y^2\tau)| \le q_E^{-l}; \\ |y + \beta(2x + \varpi^l(x^2 - y^2\tau)| \le q_E^{-l}; \\ |x| \le 1, |y| \le 1. \end{cases}$$

[1]Later in the ramified case, τ will denote a uniformizer of F, and $\varpi^2 = \tau$.

Changing x to $2x+\varpi^l(x^2-y^2\tau)$, this becomes

$$\begin{cases} |\beta| \geq q_E^{2m-l}, |\alpha-\frac{1}{2}| \leq q_E^{-l}|\beta|; \\ |(\alpha-1/2)x| \leq q_E^{-l}; \\ |y+\beta x| \leq q_E^{-l}; \\ |x| \leq 1, |y| \leq 1. \end{cases}$$

Changing x to $x\beta^{-1}$, we have

$$A = \chi(u)(1-q_E^{-1})^{-1} \int |\beta|^{-1/2}\chi^{-1}(\frac{1}{2}+\beta\sqrt{\tau})\, d\alpha\, d\beta\, dx\, dy,$$

over

$$\begin{cases} |\beta| \geq q_E^{2m-l}, |\alpha-\frac{1}{2}| \leq q_E^{-l}|\beta|; \\ |x| \leq |\beta|, |\alpha-1/2||x| \leq q_E^{-l}|\beta|, |y+x| \leq q_E^{-l}, |y| \leq 1. \end{cases}$$

Replacing x by $x-y$ and simplifying, we get

$$\begin{cases} |\beta| \geq q_E^{2m-l}, |\alpha-\frac{1}{2}| \leq q_E^{-l}|\beta|; \\ |x| \leq q_E^{-l}, |y| \leq 1. \end{cases}$$

In conclusion,

$$A = \chi(u)(1-q_E^{-1})^{-1}q_E^{-2l} \int \chi^{-1}(\frac{1}{2}+\beta\sqrt{\tau})\, d\beta.$$

Hence

$$\hat{I}(\chi,u) = \chi(-u)(1-q_E^{-1})^{-1}\left(\int_{|x|=q_E^l} \psi_E(x)\chi(x^{-1})\, dx\right)^2 \times \int_{|\beta|\geq q_E^{2m-l}} \chi^{-1}(\frac{1}{2}+\beta\sqrt{\tau})d\beta$$

if $|1+u| \leq q^{-m}$, and zero otherwise.

- Case 2: $m < l < 2m$. The conditions become

$$\begin{cases} |z| = |1-z|, |z+\bar{z}-1| \le q_E^{-l}|z|; \\ |u-az| \le q_E^{-l}; \\ |z| \ge q_E^{2m-l}; \\ |\bar{a}\bar{z}(az-u) + (1-a\bar{a}z\bar{z})\bar{z}| \le q_E^{-2l}; \\ |az-a+1| \le q_E^{-m}. \end{cases}$$

Changing $a \to \frac{au}{z}$, we get

$$\begin{cases} |z| = |1-z|, |z+\bar{z}-1| \le q_E^{-l}|z|; \\ |1-a| \le q_E^{-l}; \\ |z| \ge q_E^{2m-l}; \\ |\bar{a}(a-1) + (1-a\bar{a})\bar{z}| \le q_E^{-2l}; \\ |au(1-\frac{1}{z})+1| \le q_E^{-m}. \end{cases}$$

Suppose $z = \alpha + \beta\sqrt{\tau}$, $a = 1 + \varpi^l(x+y\sqrt{\tau})$. Notice $|1-\frac{1}{z}+\frac{\bar{z}}{z}| \le q_E^{-l}$ implies $1-\frac{1}{z}$ is a unit, so in the last condition a can be replaced by 1, i.e.

$$|u(z-1)+z| \le q_E^{-m}|z|.$$

Notice $|z| \ge q_E^{2m-l} > 1$. Those conditions becomes

$$\begin{cases} |\beta| \ge q_E^{2m-l}, |\alpha - \frac{1}{2}| \le |\beta| q_E^{-l} \\ |-x+\alpha(2x+\varpi^l(x^2-y^2\tau)| \le q_E^{-l}; \\ |y+\beta(2x+\varpi^l(x^2-y^2\tau)| \le q_E^{-l}; \\ |x| \le 1, |y| \le 1; \\ |u(-\frac{1}{2}+\beta\sqrt{\tau}) + \frac{1}{2} + \beta\sqrt{\tau}| \le q_E^{-m}|\frac{1}{2}+\beta\sqrt{\tau}|. \end{cases}$$

Notice the first four conditons are similar to Case 1 and the final condition is purely in terms of β.

Similarly we have

$$\hat{I}(\chi,u)=\chi(-u)(1-q_E^{-1})^{-1}\left(\int_{|x|=q_E^l}\psi_E(x)\chi(x^{-1})\,dx\right)^2\times\int\chi^{-1}(\frac{1}{2}+\beta\sqrt{\tau})\,d\beta,$$

where the domain of integration over β is

$$\begin{cases}|\beta|\geq q_E^{2m-l};\\ |u(-\frac{1}{2}+\beta\sqrt{\tau})+\frac{1}{2}+\beta\sqrt{\tau}|\leq q^{-m}|\frac{1}{2}+\beta\sqrt{\tau}|.\end{cases}$$

- Case 3: $l\geq 2m$. This is similar to Case 2, except now $|z|\geq q_E^{2m-l}$ becomes vacuous. Hence first two conditons give either

$$|\beta|>1,|\alpha-\frac{1}{2}|\leq|\beta|q_E^{-l}$$

or

$$|\beta|\leq 1,|\alpha-\frac{1}{2}|\leq q_E^{-l}.$$

For the first case, the computation is similar to Case 2 and gives the contribution

$$\chi(-u)(1-q_E^{-1})^{-1}\left(\int_{|x|=q_E^l}\psi_E(x)\chi(x^{-1})\,dx\right)^2\times\int\chi^{-1}(\frac{1}{2}+\beta\sqrt{\tau})\,d\beta,$$

where the domain of integration over β is

$$\begin{cases}|\beta|>1;\\ |u(-\frac{1}{2}+\beta\sqrt{\tau})+\frac{1}{2}+\beta\sqrt{\tau}|\leq q_E^{-m}|\frac{1}{2}+\beta\sqrt{\tau}|.\end{cases}$$

For the second case, after applying a similar change of coordinates, we arrive at

$$\begin{cases}|\beta|\leq 1,|\alpha-\frac{1}{2}|\leq q_E^{-l};\\ |(\alpha-1/2)x|\leq q_E^{-l};\\ |y+\beta x|\leq q_E^{-l};\\ |x|\leq 1,|y|\leq 1;\\ |u(-\frac{1}{2}+\beta\sqrt{\tau})+\frac{1}{2}+\beta\sqrt{\tau}|\leq q_E^{-m}|\frac{1}{2}+\beta\sqrt{\tau}|.\end{cases}$$

Changing y to $y-\beta x$ and simplifying, this becomes

$$\begin{cases} |\beta| \leq 1, |\alpha - \frac{1}{2}| \leq q_E^{-l}, |x| \leq 1, |y| \leq q_E^{-l}; \\ |u(-\frac{1}{2}+\beta\sqrt{\tau}) + \frac{1}{2} + \beta\sqrt{\tau}| \leq q_E^{-m}|\frac{1}{2}+\beta\sqrt{\tau}|. \end{cases}$$

This gives the contribution

$$\chi(-u)(1-q_E^{-1})^{-1}\left(\int_{|x|=q_E^l} \psi_E(x)\chi(x^{-1})\,dx\right)^2 \times \int \chi^{-1}(\frac{1}{2}+\beta\sqrt{\tau})\,d\beta.$$

where the domain of integration over β is

$$\begin{cases} |\beta| \leq 1; \\ |u(-\frac{1}{2}+\beta\sqrt{\tau}) + \frac{1}{2} + \beta\sqrt{\tau}| \leq q_E^{-m}|\frac{1}{2}+\beta\sqrt{\tau}|. \end{cases}$$

Combining everything, we see

$$\hat{I}(\chi, u) = \chi(-u)(1-q_E^{-1})^{-1}\left(\int_{|x|=q_E^l} \psi_E(x)\chi(x^{-1})\,dx\right)^2 \times \int \chi^{-1}(\frac{1}{2}+\beta\sqrt{\tau})\,d\beta,$$

where the domain of integration over β is

$$|u(-\frac{1}{2}+\beta\sqrt{\tau}) + \frac{1}{2} + \beta\sqrt{\tau}| \leq q_E^{-m}|\frac{1}{2}+\beta\sqrt{\tau}|. \qquad \square$$

Proposition 6.3. *If E/F is a ramified extension, and m is an even integer, then we have*

$$\hat{I}(\chi, u) = \chi(-u)(1-q^{-1})^{-1}\left(\int_{|x|=q^{l+1}} \psi_E(x)\chi(x^{-1})\,dx\right)^2 \times A,$$

where A is given as follows.

1. *If l is even, then*

$$A = \begin{cases} 0, & \text{if } l \leq m-1 \text{ and } |1+u| > q^{-m}; \\ \int \chi^{-1}(\frac{1}{2}+\beta\sqrt{\tau})\,d\beta, & \text{otherwise;} \end{cases}$$

where the integration domain for β is:

- $|\beta| \geq q^{2m-l}$ *when* $l \leq m-1$ *and* $|1+u| \leq q^{-m}$*;*
- $|\beta| \geq q^{2m-l}$ *and* $|u(-\frac{1}{2}+\beta\sqrt{\tau})+\frac{1}{2}+\beta\sqrt{\tau}| \leq q^{-m}|\frac{1}{2}+\beta\sqrt{\tau}|$ *when* $m-1 < l < 2m-1$*;*
- $|\beta| > 1$ *and* $|u(-\frac{1}{2}+\beta\sqrt{\tau})+\frac{1}{2}+\beta\sqrt{\tau}| \leq q^{-m}|\frac{1}{2}+\beta\sqrt{\tau}|$ *when* $l \geq 2m-1$.

2. *If* l *is odd, then*

$$A = \begin{cases} 0, & \text{if } l < 2m-1; \\ \int \chi^{-1}(\frac{1}{2}+\beta\sqrt{\tau}),\, d\beta & \text{otherwise;} \end{cases}$$

where the integration domain for β *is:*

$$|\beta| \leq 1, |u(-\frac{1}{2}+\beta\sqrt{\tau})+\frac{1}{2}+\beta\sqrt{\tau}| \leq q^{-m}|\frac{1}{2}+\beta\sqrt{\tau}|.$$

Proof. By Lemma 6.1 one can assume $|x| = |y| = q^{l+1}$.

Suppose l is odd.

- Case 1: $l \leq m-1$. We use the same notations as in Case 1 in Proposition 6.2, except l is replaced by $l+1$. Then we arrive at

$$A = q^{l+1}\int |\beta|^{-1/2}\chi^{-1}(\frac{1}{2}+\beta\sqrt{\tau})\, d\alpha\, d\beta\, dx\, dy,$$

over

$$\begin{cases} |\beta| \geq q^{2m-l+1}, |\alpha - \frac{1}{2}| \leq q^{-l-3}|\beta|; \\ |x| \leq q^{-l-1}; |y| \leq 1; \\ |1+u| \leq q^{-m}. \end{cases}$$

Therefore

$$A = q^{-1}\int_{|\beta|\geq q^{2m-l+1}} \chi^{-1}(\frac{1}{2}+\beta\sqrt{\tau})\, d\beta.$$

Lemma 6.4. *For* $t \geq 1$,

$$\int_{|\beta|=q^t} \chi^{-1}(\frac{1}{2}+\beta\sqrt{\tau})\, d\beta = 0.$$

Proof. For $l \geq 3$, choose $v' = 1+\varpi^{l-1}(u_0+\varpi v_0)$ such that $\chi(v') \neq 1$, $|u_0|, |v_0| \leq 1$, so that

$$v' = (1+\varpi^{l-1}u_0)+\varpi^l v_0.$$

Therefore, $\chi(v) \neq 1$, where $v = 1 + \varpi^{l-1}u_0$. Notice

$$\frac{\frac{1}{2} + \beta v\varpi}{(\frac{1}{2} + \beta\varpi)v} - 1 = -\frac{\frac{1}{2}\varpi^{l-1}u_0}{(\frac{1}{2} + \beta\varpi)v} \in \mathfrak{p}^l.$$

Now changing β to βv in the integral,

$$\int_{|\beta|=q^t} \chi^{-1}(\frac{1}{2} + \beta\sqrt{\tau})\, d\beta = \chi^{-1}(v) \int_{|\beta|=q^t} \chi^{-1}(\frac{1}{2} + \beta\sqrt{\tau})\, d\beta,$$

which gives that the integral equals 0.

For $l = 1$, choose $v' = u_0 + \varpi v_0$ such that $\chi(v') \neq 1$, $|u_0|, |v_0| \leq 1$, so that $v = u_0$ satisfies $\chi(v) \neq 1$. Now everything is the same as above. ☐

Applying this lemma, we get that in this case the integral is 0.

- Case 2: $m - 1 < l < 2m - 1$. Since m is even, l is odd, so $l \geq m + 1$. Following the same computation as in Case 2 in Proposition 6.2, we get

$$A = q^{-1} \int \chi^{-1}(\frac{1}{2} + \beta\sqrt{\tau})\, d\beta,$$

where β is over

$$\begin{cases} |\beta| \geq q^{2m-l+1}; \\ |u(-\frac{1}{2} + \beta\sqrt{\tau}) + \frac{1}{2} + \beta\sqrt{\tau}| \leq q^{-m}|\frac{1}{2} + \beta\sqrt{\tau}|. \end{cases}$$

Lemma 6.5. *For $t \geq 1$,*

$$\int \chi^{-1}(\frac{1}{2} + \beta\sqrt{\tau})\, d\beta = 0,$$

where β is over

$$\begin{cases} |\beta| = q^t; \\ |u(-\frac{1}{2} + \beta\sqrt{\tau}) + \frac{1}{2} + \beta\sqrt{\tau}| \leq q^{-m}|\frac{1}{2} + \beta\sqrt{\tau}|. \end{cases}$$

Proof. Choose v as in Lemma 6.4. Everything goes through except we have to check that under $\beta \mapsto \beta' = \beta v$, the second condition is unchanged. Notice the difference is given by

$$(u(-\frac{1}{2} + \beta'\sqrt{\tau}) + \frac{1}{2} + \beta'\sqrt{\tau}) - (u(-\frac{1}{2} + \beta\sqrt{\tau}) + \frac{1}{2} + \beta\sqrt{\tau}) = \beta\varpi^l u_0 u + \beta\varpi^l u_0,$$

whose absolute value is smaller than or equal to $q^{-m-1}|\beta|(l \geq m+1)$, which is the right hand side in the second condition, so it is unchanged and the integral is 0. □

Applying this lemma, we get that in this case the integral is 0.

- Case 3: $l \geq 2m-1$. Similar to Case 3 in Proposition 6.2 when E/F is unramified, we have two parts:

  1. $|\beta| > 1$, $|\alpha - \frac{1}{2}| \leq |\beta| q^{-l-3}$;
  2. $|\beta| \leq 1$, $|\alpha - \frac{1}{2}| \leq q^{-l-1}$.

  Similar to Case 2 which we just computed, the integral over the first part vanishes. The second part gives

$$A = \int \chi^{-1}(\frac{1}{2} + \beta\sqrt{\tau})\, d\beta,$$

  where β is over

$$|\beta| \leq 1, |u(-\frac{1}{2} + \beta\sqrt{\tau}) + \frac{1}{2} + \beta\sqrt{\tau}| \leq q^{-m}|\frac{1}{2} + \beta\sqrt{\tau}|.$$

Now suppose l is even.

- Case 1: $l \leq m-1$. We use the same notations as in Case 1 in Proposition 6.2, except l is replaced by $l+1$, and $\bar{a} = 1 - \varpi^{l+1}(x - y\sqrt{\tau})$. Then

$$A = q^{l+1} \int \chi^{-1}(\frac{1}{2} + \beta\sqrt{\tau})\, d\alpha\, d\beta\, dx\, dy,$$

  over

$$\begin{cases} |\beta| \geq q^{2m-l}, |\alpha - \frac{1}{2}| \leq q^{-l-2}|\beta|; \\ |x + \varpi^2\beta(2y - \varpi^l(x^2 - y^2\tau))| \leq q^{-l-2}; \\ |y - \alpha(2y - \varpi^l(x^2 - y^2\tau))| \leq q^{-l}; \\ |x| \leq 1, |y| \leq 1. \end{cases}$$

  Changing y to $2y - \varpi^l(x^2 - y^2\tau)$, the conditions become

$$\begin{cases} |\beta| \geq q^{2m-l}, |\alpha - \frac{1}{2}| \leq q^{-l-2}|\beta|; \\ |x + \varpi^2\beta y| \leq q^{-l-2}; \\ |(\alpha - 1/2)y| \leq q^{-l}; \\ |x| \leq 1, |y| \leq 1. \end{cases}$$

Now changeing y to $\frac{y}{\varpi^2\beta}$, we get

$$A = q^{l+2} \int |\beta|^{-1/2} \chi^{-1}(\frac{1}{2} + \beta\sqrt{\tau})\, d\alpha\, d\beta\, dx\, dy,$$

over

$$\begin{cases} |\beta| \geq q^{2m-l}, |\alpha - \frac{1}{2}| \leq q^{-l-2}|\beta|; \\ |y| \leq q^{-2}|\beta|, |(\alpha - 1/2)y| \leq q^{-l-2}|\beta|, |x| \leq 1, |x+y| \leq q^{-l-2}. \end{cases}$$

Changing $x \to x + y$ and simplifying, we arrive at

$$A = \int_{|\beta| \geq q^{2m-l}} \chi^{-1}(\frac{1}{2} + \beta\sqrt{\tau})\, d\beta.$$

- Case 2: $m - 1 < l < 2m - 1$. Similar to Case 2 in Proposition 6.2, we have

$$A = \int \chi^{-1}(\frac{1}{2} + \beta\sqrt{\tau})\, d\beta,$$

where β is over

$$\begin{cases} |\beta| \geq q_E^{2m-l}; \\ |u(-\frac{1}{2} + \beta\sqrt{\tau}) + \frac{1}{2} + \beta\sqrt{\tau}| \leq q^{-m}|\frac{1}{2} + \beta\sqrt{\tau}|. \end{cases}$$

- Case 3: $l \geq 2m - 1$. Similar to Case 3 in Proposition 6.2 when E/F is unramified, we have two parts:

1. $|\beta| > 1$, $|\alpha - \frac{1}{2}| \leq |\beta| q^{-l-2}$;
2. $|\beta| \leq 1$, $|\alpha - \frac{1}{2}| \leq q^{-l-2}$.

As in Cases 1 and 2, the first part contributes

$$\int \chi^{-1}(\frac{1}{2} + \beta\sqrt{\tau})$$

to A, where β is over

$$\begin{cases} |\beta| > 1; \\ |u(-\frac{1}{2} + \beta\sqrt{\tau}) + \frac{1}{2} + \beta\sqrt{\tau}| \leq q^{-m}|\frac{1}{2} + \beta\sqrt{\tau}|;. \end{cases}$$

For the second part, we work similarly as in Case 1 and denote its contribution to A as A_2. Then

$$A_2 = q^{l+1} \int \chi^{-1}(\frac{1}{2} + \beta\sqrt{\tau})\, d\alpha\, d\beta\, dx\, dy,$$

over

$$\begin{cases} |\beta| \le 1, |\alpha - \frac{1}{2}| \le q^{-l-2}; \\ |x + \varpi^2 \beta y| \le q^{-l-2}; \\ |(\alpha - 1/2)y| \le q^{-l}; \\ |x| \le 1, |y| \le 1; \\ |u(-\frac{1}{2} + \beta\sqrt{\tau}) + \frac{1}{2} + \beta\sqrt{\tau}| \le q^{-m}|\frac{1}{2} + \beta\sqrt{\tau}|. \end{cases}$$

Making the change of variables $x \to x + \varpi^2 \beta y$ and simplifying, we get

$$A_2 = q^{-1} \int_{|\beta| \le 1} \chi^{-1}(\frac{1}{2} + \beta\sqrt{\tau})\, d\beta.$$

Lemma 6.6.

$$\int \chi^{-1}(\frac{1}{2} + \beta\sqrt{\tau})\, d\beta = 0$$

over

$$\begin{cases} |\beta| \le 1; \\ |u(-\frac{1}{2} + \beta\sqrt{\tau}) + \frac{1}{2} + \beta\sqrt{\tau}| \le q^{-m}|\frac{1}{2} + \beta\sqrt{\tau}|. \end{cases}$$

Proof. Choose $v' = 1 + \varpi^{l-1}(u_0 + \varpi v_0)$, $|u_0|, |v_0| \le 1$, such that $\chi(v') \ne 1$. Then

$$(\frac{1}{2} + \beta\varpi)v' \equiv \frac{1}{2} + (\beta + \frac{1}{2}\varpi^{l-2}u_0)\varpi \mod \mathfrak{p}^l.$$

If we make $\beta \mapsto \beta' = \beta + \frac{1}{2}\varpi^{l-2}u_0 (l \ge 2)$, the difference is

$$(u(-\frac{1}{2} + \beta'\sqrt{\tau}) + \frac{1}{2} + \beta'\sqrt{\tau}) - (u(-\frac{1}{2} + \beta\sqrt{\tau}) + \frac{1}{2} + \beta\sqrt{\tau}) = \frac{1}{2}\varpi^{l-1}uu_0 + \frac{1}{2}\varpi^{l-1}u_0 \in \mathfrak{p}^{l-1}.$$

Since $l - 1 \ge 2m - 2 \ge m$, its absolute value is less than or equal to q^{-m}, which is the right hand side of the second condition. Therefore,

$$\int \chi^{-1}(\frac{1}{2} + \beta\sqrt{\tau})\, d\beta = \int \chi^{-1}(\frac{1}{2} + \beta'\sqrt{\tau})\, d\beta = \chi^{-1}(v') \int \chi^{-1}(\frac{1}{2} + \beta\sqrt{\tau})\, d\beta,$$

which implies the integral is 0. □

By this lemma, we conclude $A_2 = 0$ and hence the proposition. □

6.2 Computation for unramified characters

Now, assume $\chi(z) = |z|_E^s$ is an unramified multiplicative character. Denote $X = q_E^{-s}$. Recall the conditions above are

$$\begin{cases} z + z' = 1, |az| = |az'| = 1; \\ |u - a\bar{z'}| \le |x|^{-1}, |u - az| \le |y|^{-1}; \\ |a\bar{a}z\bar{z'} - u\bar{a} + 1| \le |ax^{-1}y^{-1}|; \\ |zx| \ge q_E^{2m}, |zy| \ge q_E^{2m}, |zxy| \ge q_E^{2m+1+\epsilon}; \\ |u - a + 1| \le q_E^{-m}. \end{cases}$$

Proposition 6.7. *If E/F is an unramified extension, then*

$$\hat{I}(\chi, u) = \begin{cases} \hat{I}_t + (q-1)\frac{q_E^m X^{2m+1}}{1-qX}, & \text{if } |1+u| \le q_E^{-m}; \\ 0, & \text{otherwise}; \end{cases}$$

where

$$\hat{I}_t = (1 - q_E^{-1})^{-1} \left(\int_{|x|=q_E} \chi^{-1}(x)\psi_E(x)\, dx \right)^2 \times \int_{|\beta| \ge q_E^{2m-1}} \chi^{-1}(\frac{1}{2} + \beta\sqrt{\tau})\, d\beta.$$

Proof. We restrict to the case $|x| \le q_E$, $|y| \le q_E$.

Notice $|z| \ge q_E^{2m-1}$ implies $|a| \le q_E^{1-2m} \le q_E^{-m}$. Hence the final condition is equivalent to

$$|1 + u| \le q_E^{-m}.$$

Divide the domain of integration into the following four regions:

1. $|x| = |y| = q_E$;
2. $|x| \le 1$, $|y| = q_E$;
3. $|x| = q_E$, $|y| \le 1$;
4. $|x| \le 1$, $|y| \le 1$.

The first region is similar to Case 1 in Proposition 6.2 (where $l = 1$) and gives the contribution

$$\hat{I}_t = (1 - q_E^{-1})^{-1} \left(\int_{|x|=q_E} \chi^{-1}(x)\psi_E(x)\, dx \right)^2 \times \int_{|\beta| \ge q_E^{2m-1}} \chi^{-1}(\frac{1}{2} + \beta\sqrt{\tau})\, d\beta.$$

The contribution of the second region is

$$\hat{I}_{m1} = \int \psi_E(x-y)|xy|\chi^{-1}(x\bar{y})\chi(a)\,dx\,dy\,dz\,d^\times a,$$

which is the sum over r of the contributions $\hat{I}_r$ over

$$\begin{cases} |x| = q_E^{-r}, |y| = q_E, r \geq 0; \\ |u - az| \leq q_E^{-1}; \\ |z| \geq q_E^{2m+r}; \\ |\bar{a}\bar{z}(az-u) + (1-a\bar{a}z\bar{z})\bar{z}| \leq q_E^{r-1}; \end{cases}$$

Then

$$\hat{I}_r = -(1-q_E^{-1})q_E^{1-2r}X^{1-r}\int \chi(a)\,d^\times a\,dz.$$

Changing a to au/z,

$$\hat{I}_r = -(1-q_E^{-1})q_E^{1-2r}X^{1-r}\int \chi(a)\chi^{-1}(z)\,d^\times a\,dz,$$

over

$$\begin{cases} |1-a| \leq q_E^{-1}; \\ |z| \geq q_E^{2m+r}; \\ |\bar{a}(a-1) + (1-a\bar{a})\bar{z}| \leq q_E^{r-1}. \end{cases}$$

The third condition simplifies to

$$|(1-a\bar{a})\bar{z}| \leq q_E^{r-1}.$$

Suppose $|z| = q_E^{2m+r+s}$, $s \geq 0$.

Lemma 6.8. *For $b \geq 1$, the integral*

$$\int d^\times a$$

over

$$|1-a| \leq q_E^{-1}, |1-a\bar{a}| \leq q_E^{-b}$$

is equal to $(1-q_E^{-1})^{-1}q^{-1-b}$.

Proof. This is [Jac1992, Lemma 1]. ∎

Applying this lemma with $b = 2m + s + 1$, we get

$$\begin{aligned}\hat{I}_r &= -(1 - q_E^{-1}) q_E^{1-2r} X^{1-r} \sum_{s \geq 0} (1 - q_E^{-1})^{-1} q^{-2-s-2m} X^{2m+r+s} (q_E^{2m+r+s} - q_E^{2m+r+s-1}) \\ &= -(1 - q_E^{-1}) q_E^m X^{1+2m} \frac{1}{1 - qX} q_E^{-r},\end{aligned}$$

and

$$\hat{I}_{m1} = \sum_{r \geq 0} \hat{I}_r = -q_E^m \frac{X^{1+2m}}{1 - qX}.$$

By symmetry, the contribution $\hat{I}_{m2}$ from the third region is equal to $\hat{I}_{m1}$.

Finally, we compute the contribution $\hat{I}_b$ from the fourth region:

$$\hat{I}_b = \int \psi_E(x - y) |xy| \chi^{-1}(x\bar{y}) \chi(a) \, dx \, dy \, dz \, d^\times a,$$

over

$$\begin{cases} |x| \leq 1, |y| \leq 1, |az| = 1; \\ |z| \geq q_E^{2m+1} |x|^{-1} |y|^{-1}; \\ |\bar{a}\bar{z}(az - u) + (1 - a\bar{a}z\bar{z})\bar{z}| \leq |x|^{-1} |y|^{-1}. \end{cases}$$

Changing a to au/z, these become

$$\begin{cases} |x| \leq 1, |y| \leq 1, |a| = 1; \\ |z| \geq q_E^{2m+1} |xy|^{-1}; \\ |1 - a\bar{a}| \leq |xyz|^{-1}; \end{cases}$$

and the integral becomes

$$\hat{I}_b = \int |xy| \chi^{-1}(xyz) \, dx \, dy \, dz \, d^\times a.$$

Lemma 6.9. *For $b \geq 1$, the integral*

$$\int d^\times a$$

over

$$|a| = 1, |1 - a\bar{a}| \leq q_E^{-b}$$

is $(1 - q^{-1})^{-1} q^{-b}$.

Proof. This is [Jac1992, Lemma 2]. ∎

Now we set

$$|x| = q_E^{-r}, |y| = q_E^{-s}, |z| = q_E^{2m+1+s+r+t}, r, s, t \geq 0,$$

and apply this lemma with $b = 2m + t + 1$ to get

$$\begin{aligned}
\hat{I}_b &= \sum_{r,s,t} (1-q^{-1})^{-1} q^{-2m-t-1} q_E^{2m+t+1} (1-q_E^{-1})^3 q_E^{-r-s} X^{2m+t+1} \\
&= q_E^m X^{2m+1} \sum_t (1-q^{-1})^{-1} q^{1+t} X^t (1-q_E^{-1})^3 \sum_{s,t} q_E^{-s-t} \\
&= (1+q) \frac{q_E^m X^{2m+1}}{1-qX}.
\end{aligned}$$

Combining everything we have

$$\begin{aligned}
\hat{I} &= \hat{I}_t + \hat{I}_{m1} + \hat{I}_{m2} + \hat{I}_b \\
&= \hat{I}_t - 2q_E^m \frac{X^{1+2m}}{1-qX} + (1+q) \frac{q_E^m X^{2m+1}}{1-qX} \\
&= \hat{I}_t + (q-1) \frac{q_E^m X^{2m+1}}{1-qX},
\end{aligned}$$

where

$$\hat{I}_t = (1-q_E^{-1})^{-1} \left(\int_{|x|=q_E} \chi^{-1}(x) \psi_E(x)\, dx \right)^2 \times \int_{|\beta| \geq q_E^{2m-1}} \chi^{-1}(\frac{1}{2} + \beta\sqrt{\tau})\, d\beta.$$

This finishes the proof of the proposition. ∎

Proposition 6.10. *If E/F is a ramified extension, then*

$$\hat{I}(\chi, u) = \begin{cases} \dfrac{X^{2m+3} q^{m+3}}{1-qX^2}, & \text{if } |1+u| \leq q^{-m}; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We restrict to the case $|x| \leq q^2$, $|y| \leq q^2$.

Notice $|z| \geq q^{2m-2}$ implies $|a| \leq q^{2-2m} \leq q^{-m}$. Hence the final condition is equivalent to

$$|1+u| \leq q^{-m}.$$

By symmetry between x and y, it suffices to consider the following six regions:

1. $|x| = |y| = q^2$;

2. $|x| = q^2$, $|y| = q$;

3. $|x| = q^2$, $|y| \leq 1$;

4. $|x| = q$, $|y| = q$;

5. $|x| = q$, $|y| \leq 1$;

6. $|x| \leq 1$, $|y| \leq 1$.

Denote the contributions of these regions to $\hat{I}(\chi, u)$ as $\hat{I}_{22}$, $\hat{I}_{21}$, $\hat{I}_{20}$, $\hat{I}_{11}$, $\hat{I}_{10}$, $\hat{I}_{00}$ respectively.

For $\hat{I}_{22}$, the calculation is similar to Case 1 in Proposition 6.2 (with $l = 1$), and it gives

$$\begin{aligned}
\hat{I}_{22} &= (1-q^{-1})^{-1}q^{-1}\left(\int_{|x|=q^2}\psi_E(x)\chi(x^{-1})\,dx\right)^2 \times \int_{|\beta|\geq q^{2m}}\chi^{-1}(\frac{1}{2}+\beta\sqrt{\tau})\,d\beta \\
&= (1-q^{-1})^{-1}qX^4 \times \sum_{s\geq 0} q^{m+s}(1-q^{-1})X^{2m+2s-1} \\
&= \frac{q^{m+1}X^{2m+3}}{1-qX^2}.
\end{aligned}$$

For $\hat{I}_{11}$, the calculation is similar to Case 1 in Proposition 6.2 (with $l = 0$), and it gives

$$\begin{aligned}
\hat{I}_{11} &= (1-q^{-1})^{-1}\left(\int_{|x|=q}\psi_E(x)\chi(x^{-1})\,dx\right)^2 \times \int_{|\beta|\geq q^{2m+2}}\chi^{-1}(\frac{1}{2}+\beta\sqrt{\tau})\,d\beta \\
&= (1-q^{-1})q^2X^2 \times \sum_{s\geq 0} q^{m+s+1}(1-q^{-1})X^{2m+2s+1} \\
&= (1-q^{-1})\frac{(q^{m+3}-q^{m+2})X^{2m+3}}{1-qX^2}.
\end{aligned}$$

For $\hat{I}_{21}$, we make a similar change of variables and end up with

$$\begin{aligned}
\hat{I}_{21} &= \int_{|x|=q^2}\psi_E(x)|x|\chi^{-1}(x)\,dx\int_{|y|=q}\psi_E(y)|y|\chi^{-1}(y)\,dy \times A \\
&= -q^5(1-q^{-1})X^3 \times A,
\end{aligned}$$

where

$$A = \int \chi^{-1}(z)\,d^\times a\,dz$$

over

$$\begin{cases} |z| \geq q^{2m-1}; \\ |1-a| \leq q^{-2}, |z+\bar{z}-1| \leq q^{-1}|z|; \\ |\bar{a}(a-1)+(1-a\bar{a})\bar{z}| \leq q^{-3}. \end{cases}$$

Setting $z = \alpha + \beta\varpi$, $a = 1 + \varpi^2(x + y\varpi)$, $|x|$, $|y| \leq 1$, the conditions become

$$\begin{cases} |\beta| \geq q^{2m}, |\alpha - \frac{1}{2}| \leq q^{-2}|\beta|; \\ |x| \leq 1, |y| \leq 1; \\ |x - \alpha(2x + \varpi^2(x^2 - y^2\varpi^2))| \leq q^{-2}; \\ |\beta(2x + \varpi^2(x^2 - y^2\varpi^2))| \leq 1; \end{cases}$$

and the integral becomes

$$\begin{aligned} A &= \int \chi^{-1}(z)\, d^\times a\, dz \\ &= (1-q^{-1})^{-1}q^{-2} \int \chi^{-1}(z)\, dx\, dy\, d\alpha\, d\beta. \end{aligned}$$

Making the change of variables $x \mapsto 2x + \varpi^2(x^2 - y^2\varpi^2)$,

$$\begin{cases} |\beta| \geq q^{2m}, |\alpha - \frac{1}{2}| \leq q^{-2}|\beta|; \\ |(\alpha - \frac{1}{2})x| \leq q^{-2}, |\beta x| \leq 1, |x| \leq 1, |y| \leq 1. \end{cases}$$

Making $x \mapsto x/\beta$, we have

$$A = (1-q^{-1})^{-1}q^{-2} \int |\beta|^{-1/2}\chi^{-1}(z)\, dx\, dy\, d\alpha\, d\beta$$

over

$$\begin{cases} |\beta| \geq q^{2m}, |\alpha - \frac{1}{2}| \leq q^{-2}|\beta|; \\ |x| \leq 1, |y| \leq 1. \end{cases}$$

Therefore, we get

$$A = (1-q^{-1})^{-1}q^{-3} \sum_{s \geq 0} X^{2m+2s-1}q^{m+s}(1-q^{-1}) = q^{m-3}X^{2m-1}\frac{1}{1-qX^2},$$

and hence

$$\hat{I}_{21} = -(1-q^{-1})\frac{q^{m+2}X^{2m+2}}{1-qX^2}.$$

For $\hat{I}_{20}$, we consider the contribution $\hat{I}_{20}^r$ of $|x| = q^{-r}$, $|y| = q^2$, $r \geq 0$. As above, we have

$$\begin{aligned}\hat{I}_{20}^r &= \int_{|x|=q^{-r}} \psi_E(x)|x|\chi^{-1}(x)\,dx \int_{|y|=q^2} \psi_E(y)|y|\chi^{-1}(y)\,dy \times A \\ &= -q^{3-2r}(1-q^{-1})X^{2-r} \times A,\end{aligned}$$

where

$$A = \int \chi^{-1}(z)\,d^\times a\,dz$$

over

$$\begin{cases} |z| \geq q^{2m+r}; \\ |1-a| \leq q^{-2}; \\ |\bar{a}(a-1) + (1-a\bar{a})\bar{z}| \leq q^{r-2}. \end{cases}$$

The last condition simplifies to

$$|(1-a\bar{a})\bar{z}| \leq q^{r-2}.$$

Lemma 6.11. *Let $b \geq 2$ be an even integer. Then the integral*

$$\int d^\times a$$

over

$$|1-a| \leq q^{-2}, |1-a\bar{a}| \leq q^{-b}$$

is

$$(1-q^{-1})^{-1}q^{-\frac{b}{2}-1}.$$

Proof. This is very similar to Lemma 6.8 and Lemma 6.9. We provide the proof here for completeness. If we write $a = 1 + \varpi^2(x + y\varpi)$, $|x|$, $|y| \leq 1$, then the conditions become

$$|x| \leq 1, |y| \leq 1, |\varpi^2(2x + \varpi^2(x^2 - y^2\varpi))| \leq q^{-b}.$$

Now under $x \mapsto 2x + \varpi^2(x^2 - y^2\varpi)$, the integral becomes

$$q^{-2}(1-q^{-1})^{-1}\int_{|\varpi^2 x|\le q^{-b}} dx\,dy = (1-q^{-1})^{-1}q^{-\frac{b}{2}-1}. \qquad \square$$

Now assume $|z| = q^{2m+r+2s}$ or $q^{2m+r+2s+1}$, $s \geq 0$. According to the lemma,

$$\begin{aligned} A =& \sum_s X^{2m+r+2s}q^{2m+r+2s}(1-q^{-1}) \times (1-q^{-1})^{-1}q^{-1}q^{-m-s-1} \\ &+ \sum_s X^{2m+r+2s+1}q^{2m+r+2s+1}(1-q^{-1}) \times (1-q^{-1})^{-1}q^{-1}q^{-m-s-2} \\ =& X^{2m+r}q^{m+r-2}\sum_s (1+X)X^{2s}q^s \\ =& \frac{X^{2m+r}q^{m+r-2}(1+X)}{1-qX^2}. \end{aligned}$$

Therefore, we have

$$\hat{I}_{20} = \sum_r \hat{I}_{20}^r = -\frac{X^{2m+2}(1+X)}{1-qX^2}q^{m+1}(1-q^{-1})\sum_r q^{-r} = -\frac{X^{2m+2}(1+X)q^{m+1}}{1-qX^2}.$$

For $\hat{I}_{10}$, we consider the contribution of $\hat{I}_{10}^r$ of $|x| = q^{-r}$, $|y| = q$, $r \geq 0$. As above we have

$$\hat{I}_{10}^r = (1-q^{-1})^2 q^{2-2r}X^{1-r} \times A,$$

where

$$A = \int \chi^{-1}(z)\,d^\times a\,dz$$

over

$$\begin{cases} |z| \geq q^{2m+r+1}; \\ |1-a| \leq q^{-1}; \\ |(1-a\bar{a})\bar{z}| \leq q^{r-1}. \end{cases}$$

Lemma 6.12. *Let $b \geq 2$ be an even integer. Then the integral*

$$\int d^\times a$$

over

$$|1-a| \leq q^{-1}, |1-a\bar{a}| \leq q^{-b}$$

is

$$(1-q^{-1})^{-1}q^{-\frac{b}{2}}.$$

Proof. This is similar to Lemma 6.11. ∎

Now assume $|z| = q^{2m+r+1+2s}$ or $q^{2m+r+2+2s}$, $s \geq 0$. Then

$$\begin{aligned}
A =& \sum_s X^{2m+r+1+2s} q^{2m+r+1+2s}(1-q^{-1}) \times (1-q^{-1})^{-1} q^{-m-s-1} \\
&+ \sum_s X^{2m+r+2+2s} q^{2m+r+2+2s}(1-q^{-1}) \times (1-q^{-1})^{-1} q^{-m-s-2} \\
=& X^{2m+1+r} q^{m+r} \sum_s (1+X) X^{2s} q^s \\
=& \frac{X^{2m+1+r} q^{m+r}(1+X)}{1-qX^2}.
\end{aligned}$$

Hence,

$$\hat{I}_{10} = \sum_r \hat{I}_{10}^r = \frac{(1-q^{-1})^2}{1-qX^2} X^{2m+2}(1+X) q^{m+2} \sum_r q^{-r} = \frac{(1-q^{-1})X^{2m+2}(1+X)q^{m+2}}{1-qX^2}.$$

For $\hat{I}_{00}$, we consider the contribution $\hat{I}_{00}^{r,s}$ of $|x| = q^{-r}$, $|y| = q^{-s}$, $r, s \geq 0$. As above we have

$$\hat{I}_{00}^{r,s} = (1-q^{-1})^2 q^{-2r-2s} X^{-r-s} \times A,$$

where

$$A = \int \chi^{-1}(z)\, d^\times a\, dz$$

over

$$\begin{cases} |z| \geq q^{2m+2+r+s}; \\ |a| = 1; \\ |(1-a\bar{a})\bar{z}| \leq q^{r+s}. \end{cases}$$

Lemma 6.13. *Let $b \geq 2$ be an even integer. Then the integral*

$$\int d^\times a$$

over

$$|a| = 1, |1 - a\bar{a}| \leq q^{-b}$$

is

$$2(1 - q^{-1})^{-1} q^{-\frac{b}{2}}.$$

Proof. $a\bar{a} \equiv 1 \mod \mathfrak{p}$ implies $a \equiv 1$ or $-1 \mod \mathfrak{p}$. Hence the integral is twice the result of Lemma 6.12. □

Now assume $|z| = q^{2m+2+r+s+2t}$ or $q^{2m+2+r+s+1+2t}$, $t \geq 0$. Then

$$\begin{aligned}
A &= 2\sum_t X^{2m+2+r+s+2t} q^{2m+2+r+s+2t}(1-q^{-1}) \times (1-q^{-1})^{-1} q^{-m-t-1} \\
&\quad + 2\sum_t X^{2m+3+r+s+2t} q^{2m+3+r+s+2t}(1-q^{-1}) \times (1-q^{-1})^{-1} q^{-m-t-2} \\
&= 2X^{2m+2+r+s} q^{m+r+s+1} \sum_t X^{2t} q^t (1+X) \\
&= 2X^{2m+2+r+s} q^{m+r+s+1} \frac{1+X}{1-qX^2}.
\end{aligned}$$

$$\begin{aligned}
\hat{I}_{00} &= \sum_{r,s} \hat{I}_{00}^{r,s} \\
&= 2X^{2m+2} q^{m+1} \frac{1+X}{1-qX^2} (1-q^{-1})^2 \sum_{r,s} q^{-r-s} \\
&= \frac{2X^{2m+2} q^{m+1}(1+X)}{1-qX^2}.
\end{aligned}$$

In conclusion, by symmetry between x and y, we have

$$\begin{aligned}
\hat{I}(\chi, u) &= \hat{I}_{22} + \hat{I}_{11} + \hat{I}_{00} + 2(\hat{I}_{20} + \hat{I}_{10} + \hat{I}_{21}) \\
&= \frac{X^{2m+2} q^{m+1}}{1-qX^2} (X + (q-1)^2 X + 2X + 2 + 2[-X - 1 + (q-1)(X+1) - (q-1)]) \\
&= \frac{X^{2m+3} q^{m+3}}{1-qX^2}.
\end{aligned}$$

This finishes the proof of the proposition. □

Chapter 7

Germ functions in the unitary case

For the unitary case, denote G to be the standard unitary group of rank 3, i.e.

$$G = \{g \in \mathrm{GL}_3(E) \mid {}^t\bar{g}w_G g = w_G\}.$$

The unipotent subgroup N consists of matrices of the form

$$n = \begin{pmatrix} 1 & x & t - \frac{x\bar{x}}{2} \\ 0 & 1 & -\bar{x} \\ 0 & 0 & 1 \end{pmatrix}, \ t + \bar{t} = 0.$$

Define

$$\theta(n) = \psi_E(x).$$

Consider the action of $N \times N$ on G by

$$x^{(n_1,n_2)} = n_1^{-1} x n_2.$$

By the (relative) Bruhat decomposition, the $N \times N$-orbits have representatives given by $w\alpha$, where w is an element of the (relative) Weyl group and α is a diagonal matrix. Now we give a complete description of the $N \times N$-orbits.

- $w = w_G = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $A_w = \left\{ \alpha = \begin{pmatrix} a & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & \bar{a}^{-1} \end{pmatrix} \middle| u\bar{u} = 1 \right\}$. For a given α, its stabilizer is trivial, so all orbits are relevant. We define the orbital integral

$$J(w\alpha, f) = \int_{N\times N} f(n_1^{-1} w\alpha n_2)\theta(n_1 n_2)\, dn_1\, dn_2.$$

- $w = e$, $A'_e = \left\{ \alpha = \begin{pmatrix} a & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & \bar{a}^{-1} \end{pmatrix} \middle| u\bar{u} = 1 \right\}$. For a given α, its stabilizer is

$$\left\{ (n_1, n_2) \middle| n_i = \begin{pmatrix} 1 & x_i & t_i - \frac{x_i \bar{x}_i}{2} \\ 0 & 1 & -\bar{x}_i \\ 0 & 0 & 1 \end{pmatrix}, ax_2 = ux_1, t_1 = a\bar{a}t_2, t_i + \bar{t_i} = 0, i = 1, 2 \right\},$$

so the relevant orbits are $A_e = \{\alpha = \mathrm{diag}(u_1, -u_1, u_1) \mid u_1\bar{u_1} = 1\}$. For $\alpha \in A_e$, the stabilizer is

$$(N \times N)_\alpha = \left\{ (n_1, n_2) \middle| u_i = \begin{pmatrix} 1 & x_i & t_i - \frac{x_i \bar{x}_i}{2} \\ 0 & 1 & -\bar{x}_i \\ 0 & 0 & 1 \end{pmatrix}, x_1 + x_2 = 0, t_1 = t_2 \right\}.$$

Let

$$n_1 = \begin{pmatrix} 1 & x & t - \frac{x\bar{x}}{2} \\ 0 & 1 & -\bar{x} \\ 0 & 0 & 1 \end{pmatrix},\ n_2 = \begin{pmatrix} 1 & x & -t - \frac{x\bar{x}}{2} \\ 0 & 1 & -\bar{x} \\ 0 & 0 & 1 \end{pmatrix},\ t + \bar{t} = 0,$$

and define the orbital integral

$$\begin{aligned} J(\alpha, f) &= \int f(n_1^{-1}\alpha n_2)\theta(n_1 n_2)\, dx\, dt \\ &= \int f \begin{pmatrix} u_1 & u_1 x & -u_1(t + \frac{x\bar{x}}{2}) \\ 0 & -u_1 & u_1\bar{x} \\ 0 & 0 & u_1 \end{pmatrix} \psi_E(x)\, dx\, dt. \end{aligned}$$

We similarly have a Shalika germ function L on $A^e_{w_G}$, i.e. a function satisfying: for any $f \in \mathcal{C}(G)$, there exists $W^f_w \in \mathcal{C}(A_{w_G})$, such that

$$J(w\alpha, f) = W^f_w(\alpha) + L * W^f_e(\alpha), \tag{7.1}$$

where $W^f_e(z) = J(z, f)$ for $z \in A_e$, and $L * W^f_e(\alpha)$ is defined in the same way as (5.3).

Let f_0' be the restriction to G of the characteristic function of the following set K_m:

$$\begin{cases} |X_{11}| = |X_{33}| = 1; \\ |X_{ij}| \leq 1, i < j; \\ |X_{ij}| \leq q_E^{-2m}, i > j; \\ |X_{31}| \leq q_E^{-2m-1-\epsilon}; \\ |X_{22}+1| \leq q_E^{-m}; \end{cases}$$

where $\epsilon = 0$ when E/F is unramified and $\epsilon = 1$ when E/F is ramified. By the germ equation (7.1), we have

$$L(\alpha) = J(w\alpha, f_0').$$

We compute that

$$\begin{aligned} J(w\alpha, f_0') &= \int_{N' \times N'} f_0'(n_1^{-1} w\alpha n_2)\theta(n_1 n_2)\, dn_1\, dn_2 \\ &= \int \theta(n_1^{-1} n_2)\, dn_1\, dn_2, \end{aligned}$$

where the integral is over

$$n_1 w\alpha n_2 \in K_m.$$

We make the change of variables

$$n_1 w\alpha = kn'$$

where $k \in K_m$ and $n' \in N'$, so that

$$J(w\alpha, f_0') = \int \theta^{-1}(n_1 n')\, dn_1,$$

where $n_1 w\alpha = kn'$ as above.

Suppose

$$n_1 = \begin{pmatrix} 1 & x & t - \frac{x\bar{x}}{2} \\ 0 & 1 & -\bar{x} \\ 0 & 0 & 1 \end{pmatrix},\ t + \bar{t} = 0.$$

Then the condition

$$n_1 w\alpha = \begin{pmatrix} (t-\frac{x\bar{x}}{2})a & ux & \bar{a}^{-1} \\ -\bar{x}a & u & 0 \\ a & 0 & 0 \end{pmatrix} \in K_m N'$$

is equivalent to

$$\begin{cases} |(t-\frac{x\bar{x}}{2})a| = 1; \\ |ax| \le q_E^{-2m}, |a| \le q_E^{-2m-1-\epsilon}; \\ |1+u+\frac{ux\bar{x}}{t-\frac{x\bar{x}}{2}}| \le q_E^{-m}. \end{cases}$$

Under these conditions,

$$n_1' w\alpha = \begin{pmatrix} (t-\frac{x\bar{x}}{2})a & 0 & 0 \\ -a\bar{x} & u+\frac{ux\bar{x}}{t-\frac{x\bar{x}}{2}} & 0 \\ a & -\frac{ux}{t-\frac{x\bar{x}}{2}} & \frac{1}{\bar{a}(-t-\frac{x\bar{x}}{2})} \end{pmatrix} \begin{pmatrix} 1 & b & c \\ 0 & 1 & -\bar{b} \\ 0 & 0 & 1 \end{pmatrix} \in K_m N',$$

where

$$b = \frac{ux}{a(t-\frac{x\bar{x}}{2})},$$
$$c = \frac{1}{a\bar{a}(t-\frac{x\bar{x}}{2})},$$

so that

$$c + \bar{c} = -b\bar{b}.$$

We conclude that

$$J(w\alpha, f_0') = \int \psi_E\left(-x - \frac{ux}{a(t-\frac{x\bar{x}}{2})}\right) dx\, dt.$$

Substituting $t \mapsto tx\bar{x}$, this becomes

$$J(w\alpha, f_0') = \int \psi_E\left(-x - \frac{u}{a\bar{x}(t-\frac{1}{2})}\right) |x|\, dx\, dt,$$

over

$$\begin{cases} |(t-\frac{1}{2})ax\bar{x}| = 1; \\ |ax| \le q_E^{-2m}, |a| \le q_E^{-2m-1-\epsilon}; \\ \left|1+u+\frac{u}{t-\frac{1}{2}}\right| \le q_E^{-m}. \end{cases}$$

Now fix u, take a multiplicative character χ, and consider the Mellin transform of J

$$\hat{J}(\chi, u) = \int J(w\alpha)\chi(a)\, d^{\times}a.$$

Making the change of variables $x \mapsto -x$, $a \mapsto \frac{u}{a\bar{x}(t-\frac{1}{2})}$, $t \mapsto -t$, we have

$$\hat{J}(\chi, u) = \chi(-u)(1-q_E^{-1})^{-1} \int \psi_E(x)\chi^{-1}(\bar{x})\, dx \int \psi_E(a)\chi^{-1}(a)\, da \int_{t+\bar{t}=0} \chi^{-1}(t+\frac{1}{2})\, dt$$

over

$$\begin{cases} |a| = |x|; \\ |a(t+\frac{1}{2})| \geq q_E^{2m}; \\ |ax(t+\frac{1}{2})| \geq q_E^{2m+1+\epsilon}; \\ \left|1+u-\frac{u}{t+\frac{1}{2}}\right| \leq q_E^{-m}. \end{cases}$$

7.1 Computation for ramified characters

Suppose χ is ramified of conductor $\mathfrak{p}^l$,

Proposition 7.1. *If E/F is an unramified extension, then we have*

$$\begin{aligned} \hat{J}(\chi, u) = & \chi(-u)(1-q_E^{-1})^{-1} \left(\int_{|x|=q_E^l} \psi_E(x)\chi(x^{-1})\, dx \right)^2 \\ & \times \begin{cases} 0, & \text{if } l \leq m \text{ and } |1+u| > q_E^{-m}; \\ \int \chi^{-1}(\frac{1}{2}+\beta\sqrt{\tau})\, d\beta, & \text{otherwise}; \end{cases} \end{aligned}$$

where the integration domain for β is:

- $|\beta| \geq q_E^{2m-l}$ *when* $l \leq m$ *and* $|1+u| \leq q_E^{-m}$*;*
- $|\beta| \geq q_E^{2m-l}$ *and* $|u(-\frac{1}{2}+\beta\sqrt{\tau}) + \frac{1}{2}+\beta\sqrt{\tau}| \leq q_E^{-m}|\frac{1}{2}+\beta\sqrt{\tau}|$ *when* $m < l < 2m$*;*
- $|u(-\frac{1}{2}+\beta\sqrt{\tau}) + \frac{1}{2}+\beta\sqrt{\tau}| \leq q_E^{-m}|\frac{1}{2}+\beta\sqrt{\tau}|$ *when* $l \geq 2m$.

Proof. We can restrict to the case $|a| = |x| = q_E^l$. Let $t = \beta\sqrt{\tau}$.

- Case 1: $l \leq m$. We observe that $|t+\frac{1}{2}| \geq q_E^{2m-l} \geq q_E^m$, hence $|1+u| \leq q_E^{-m}$ and the integral for β is taken over $|\beta| \geq q_E^{2m-l}$. We obtain

$$\hat{J}(\chi, u) = \chi(-u)(1 - q_E^{-1})^{-1} \left(\int_{|x|=q_E^l} \psi_E(x)\chi(x^{-1})\, dx \right)^2 \times \int_{|\beta| \geq q_E^{2m-l}} \chi^{-1}(\frac{1}{2} + \beta\sqrt{\tau})\, d\beta.$$

- Case 2: $m < l < 2m$. We observe that $|t + \frac{1}{2}| \geq q_E^{2m-l} > 1$, hence the integral for β is taken over

$$\begin{cases} |\beta| \geq q_E^{2m-l}; \\ |u(-\frac{1}{2} + \beta\sqrt{\tau}) + \frac{1}{2} + \beta\sqrt{\tau}| \leq q_E^{-m} |\frac{1}{2} + \beta\sqrt{\tau}|. \end{cases}$$

We obtain

$$\hat{J}(\chi, u) = \chi(-u)(1 - q_E^{-1})^{-1} \left(\int_{|x|=q_E^l} \psi_E(x)\chi(x^{-1})\, dx \right)^2 \times \int \chi^{-1}(\frac{1}{2} + \beta\sqrt{\tau})\, d\beta$$

over

$$\begin{cases} |\beta| \geq q_E^{2m-l}; \\ |u(-\frac{1}{2} + \beta\sqrt{\tau}) + \frac{1}{2} + \beta\sqrt{\tau}| \leq q_E^{-m} |\frac{1}{2} + \beta\sqrt{\tau}|. \end{cases}$$

- Case 3: $l \geq 2m$. The only remaining condition over β is

$$|u(-\frac{1}{2} + \beta\sqrt{\tau}) + \frac{1}{2} + \beta\sqrt{\tau}| \leq q_E^{-m} |\frac{1}{2} + \beta\sqrt{\tau}|.$$

We obtain

$$\hat{J}(\chi, u) = \chi(-u)(1 - q_E^{-1})^{-1} \left(\int_{|x|=q_E^l} \psi_E(x)\chi(x^{-1})\, dx \right)^2 \times \int \chi^{-1}(\frac{1}{2} + \beta\sqrt{\tau})\, d\beta$$

over

$$|u(-\frac{1}{2} + \beta\sqrt{\tau}) + \frac{1}{2} + \beta\sqrt{\tau}| \leq q_E^{-m} |\frac{1}{2} + \beta\sqrt{\tau}|.$$

□

Proposition 7.2. *If E/F is a ramified extension, and m is an even integer, then we have*

$$\hat{J}(\chi, u) = \chi(-u)(1 - q^{-1})^{-1} \left(\int_{|x|=q^{l+1}} \psi_E(x)\chi(x^{-1})\, dx \right)^2 \times A,$$

where A is given as follows.

1. If l is even, then

$$A = \begin{cases} 0, & \text{if } l \leq m-1 \text{ and } |1+u| > q^{-m}; \\ \int \chi^{-1}(\frac{1}{2} + \beta\sqrt{\tau})\, d\beta, & \text{otherwise}; \end{cases}$$

where the integration domain for β is:

- $|\beta| \geq q^{2m-l}$ *when* $l \leq m-1$ *and* $|1+u| \leq q^{-m}$*;*
- $|\beta| \geq q^{2m-l}$ *and* $|u(-\frac{1}{2} + \beta\sqrt{\tau}) + \frac{1}{2} + \beta\sqrt{\tau}| \leq q^{-m}|\frac{1}{2} + \beta\sqrt{\tau}|$ *when* $m-1 < l < 2m-1$*;*
- $|\beta| > 1$ *and* $|u(-\frac{1}{2} + \beta\sqrt{\tau}) + \frac{1}{2} + \beta\sqrt{\tau}| \leq q^{-m}|\frac{1}{2} + \beta\sqrt{\tau}|$ *when* $l \geq 2m-1$*.*

2. If l is odd, then

$$A = \begin{cases} 0, & \text{if } l < 2m-1; \\ \int \chi^{-1}(\frac{1}{2} + \beta\sqrt{\tau})\, d\beta, & \text{otherwise}; \end{cases}$$

where the integration domain for β is

$$|\beta| \leq 1, |u(-\frac{1}{2} + \beta\sqrt{\tau}) + \frac{1}{2} + \beta\sqrt{\tau}| \leq q^{-m}|\frac{1}{2} + \beta\sqrt{\tau}|.$$

Proof. By Lemma 6.1 one can assume $|x| = |y| = q^{l+1}$.

Suppose l is odd.

- Case 1: $l \leq m-1$. Similar to Case 1 in Proposition 7.1 (with l replaced by $l+1$), we arrive at

  $$\hat{J}(\chi, u) = \chi(-u)(1-q^{-1})^{-1} \left(\int_{|x|=q^{l+1}} \psi_E(x)\chi(x^{-1})\, dx \right)^2 \times \int_{|\beta| \geq q^{2m-l+1}} \chi^{-1}(\frac{1}{2} + \beta\sqrt{\tau})\, d\beta,$$

  but by Lemma 6.4,

  $$\int_{|\beta| \geq q_E^{2m-l+1}} \chi^{-1}(\frac{1}{2} + \beta\sqrt{\tau})\, d\beta = 0.$$

  Hence the integral vanishes.

- Case 2: $m-1 < l < 2m-1$. Similar to Case 2 in Proposition 7.1 (with l replaced by $l+1$), we arrive at

  $$\hat{J}(\chi, u) = \chi(-u)(1-q^{-1})^{-1} \left(\int_{|x|=q^{l+1}} \psi_E(x)\chi(x^{-1})\, dx \right)^2 \times \int \chi^{-1}(\frac{1}{2} + \beta\sqrt{\tau})\, d\beta,$$

where β is over

$$\begin{cases} |\beta| \geq q^{2m-l+1}; \\ |u(-\frac{1}{2}+\beta\sqrt{\tau})+\frac{1}{2}+\beta\sqrt{\tau}| \leq q^{-m}|\frac{1}{2}+\beta\sqrt{\tau}|; \end{cases}$$

but by Lemma 6.5,

$$\int \chi^{-1}(\frac{1}{2}+\beta\sqrt{\tau})\,d\beta = 0.$$

So the integral vanishes.

- Case 3: $l \geq 2m-1$. Similar to Case 3 in Proposition 7.1 (with l is replaced by $l+1$), we arrive at

$$\hat{J}(\chi, u) = \chi(-u)(1-q^{-1})^{-1}\left(\int_{|x|=q^{l+1}} \psi_E(x)\chi(x^{-1})\,dx\right)^2 \times \int \chi^{-1}(\frac{1}{2}+\beta\sqrt{\tau})\,d\beta,$$

where β is over

$$|u(-\frac{1}{2}+\beta\sqrt{\tau})+\frac{1}{2}+\beta\sqrt{\tau}| \leq q^{-m}|\frac{1}{2}+\beta\sqrt{\tau}|;$$

but by Lemma 6.5,

$$\int \chi^{-1}(\frac{1}{2}+\beta\sqrt{\tau})\,d\beta = 0,$$

over

$$\begin{cases} |\beta| > 1; \\ |u(-\frac{1}{2}+\beta\sqrt{\tau})+\frac{1}{2}+\beta\sqrt{\tau}| \leq q^{-m}|\frac{1}{2}+\beta\sqrt{\tau}|. \end{cases}$$

So the integral is over

$$|\beta| \leq 1, |u(-\frac{1}{2}+\beta\sqrt{\tau})+\frac{1}{2}+\beta\sqrt{\tau}| \leq q^{-m}|\frac{1}{2}+\beta\sqrt{\tau}|.$$

Now suppose l is even.

- Case 1: $l \leq m-1$. Similarly, we require $|1+u| \leq q^{-m}$, under which we get

$$\hat{J}(\chi, u) = \chi(-u)(1-q^{-1})^{-1}\left(\int_{|x|=q^{l+1}} \psi_E(x)\chi(x^{-1})\,dx\right)^2 \times \int_{|\beta|\geq q_E^{2m-l}} \chi^{-1}(\frac{1}{2}+\beta\sqrt{\tau})\,d\beta.$$

- Case 2: $m-1 < l < 2m-1$. Similarly we get

$$\hat{J}(\chi, u) = \chi(-u)(1-q^{-1})^{-1}\left(\int_{|x|=q^{l+1}} \psi_E(x)\chi(x^{-1})\,dx\right)^2 \times \int \chi^{-1}(\frac{1}{2}+\beta\sqrt{\tau})\,d\beta,$$

where β is over

$$\begin{cases} |\beta| \geq q_E^{2m-l}; \\ |u(-\frac{1}{2}+\beta\sqrt{\tau})+\frac{1}{2}+\beta\sqrt{\tau}| \leq q^{-m}|\frac{1}{2}+\beta\sqrt{\tau}|. \end{cases}$$

- Case 3: $l \geq 2m-1$. Similarly we get

$$\hat{J}(\chi, u) = \chi(-u)(1-q^{-1})^{-1}\left(\int_{|x|=q^{l+1}} \psi_E(x)\chi(x^{-1})\,dx\right)^2 \times \int \chi^{-1}(\frac{1}{2}+\beta\sqrt{\tau})\,d\beta,$$

where β is over

$$|u(-\frac{1}{2}+\beta\sqrt{\tau})+\frac{1}{2}+\beta\sqrt{\tau}| \leq q^{-m}|\frac{1}{2}+\beta\sqrt{\tau}|.$$

By Lemma 6.6,

$$\int \chi^{-1}(\frac{1}{2}+\beta\sqrt{\tau})\,d\beta = 0$$

over

$$\begin{cases} |\beta| \leq 1; \\ |u(-\frac{1}{2}+\beta\sqrt{\tau})+\frac{1}{2}+\beta\sqrt{\tau}| \leq q^{-m}|\frac{1}{2}+\beta\sqrt{\tau}|. \end{cases}$$

Hence the integral is over

$$\begin{cases} |\beta| > 1; \\ |u(-\frac{1}{2}+\beta\sqrt{\tau})+\frac{1}{2}+\beta\sqrt{\tau}| \leq q^{-m}|\frac{1}{2}+\beta\sqrt{\tau}|. \end{cases}$$

This finishes the proof of the proposition. □

7.2 Computation for unramified characters

Now let $\chi = |z|_E^s$ be an unramified character and $X = q_E^{-s}$.

Proposition 7.3. *If E/F is an unramified extension, then*

$$\hat{J}(\chi, u) = \begin{cases} \hat{J}_t + (q-1)\frac{q_E^m X^{2m+1}}{qX-1}, & \text{if } |1+u| \leq q_E^{-m}; \\ 0, & \text{otherwise}; \end{cases}$$

where

$$\hat{J}_t = (1-q_E^{-1})^{-1}\left(\int_{|x|=q_E}\psi_E(x)\chi^{-1}(x)\,dx\right)^2 \times \int_{|\beta|\geq q_E^{2m-1}}\chi^{-1}(\beta\sqrt{\tau}+\frac{1}{2})\,d\beta.$$

Proof. We can restrict to the domain $|a|=|x|\leq q_E$. Note that in this case $|t+\frac{1}{2}|\geq q_E^{2m-1}$, and therefore the fourth condition becomes $|1+u|\leq q_E^{-m}$.

Denote by $\hat{J}_t$ the contribution from $|a|=|x|=q_E$. Similar to Case 1 in Proposition 7.1,

$$\hat{J}_t = (1-q_E^{-1})^{-1}\left(\int_{|x|=q_E}\psi_E(x)\chi^{-1}(x)\,dx\right)^2 \times \int_{|\beta|\geq q_E^{2m-1}}\chi^{-1}(\beta\sqrt{\tau}+\frac{1}{2})\,d\beta.$$

The contribution $\hat{J}_b$ coming from $|a|=|x|=q_E^{-r}$, where $r\geq 0$, is

$$\begin{aligned}
&(1-q_E^{-1})X^{-2r}q_E^{-2r}\int_{|t+1/2|\geq q_E^{2m+2r+1}}\chi^{-1}(t+1/2)\,dt\\
&=(1-q_E^{-1})X^{-2r}q_E^{-2r}\sum_{s\geq 0}X^{2m+2r+1+s}q^{2m+2r+1+s}(1-1/q)\\
&=X^{2m+1}q_E^m(q-1)\frac{1-q_E^{-1}}{qX-1}\cdot q_E^{-r}.
\end{aligned}$$

Summing over r, we deduce

$$\hat{J}_b = (q-1)\frac{q_E^m X^{2m+1}}{qX-1}.$$

In conclusion,

$$\hat{J}(\chi,u) = \hat{J}_t + (q-1)\frac{q_E^m X^{2m+1}}{qX-1},$$

where

$$\hat{J}_t = (1-q_E^{-1})^{-1}\left(\int_{|x|=q_E}\psi(x)\chi^{-1}(x)\,dx\right)^2 \times \int_{|\beta|\geq q_E^{2m-1}}\chi^{-1}(\beta\sqrt{\tau}+\frac{1}{2})\,d\beta. \qquad \square$$

Proposition 7.4. *If E/F is a ramified extension, then*

$$\hat{J}(\chi,u) = \begin{cases}\frac{q^{m+3}X^{2m+3}}{1-qX^2}, & \textit{if } |1+u|\leq q^{-m};\\ 0, & \textit{otherwise.}\end{cases}$$

Proof. We can restrict to the domain $|a|=|x|\leq q^2$. Note that in this case $|t+\frac{1}{2}|\geq q^{2m-2}$, therefore fourth condition becomes $|1+u|\leq q^{-m}$.

Denote by $\hat{J}_t$ the contribution from $|a| = |x| = q^2$. Similar to Case 1 in Proposition 7.1,

$$\begin{aligned}
\hat{J}_t &= (1-q^{-1})^{-1}\left(\int_{|x|=q^2} \psi_E(x)\chi^{-1}(x)\,dx\right)^2 \times \int_{|\beta|\geq q^{2m}} \chi^{-1}(\beta\sqrt{\tau}+1/2)\,d\beta \\
&= (1-q^{-1})^{-1}q^2X^4 \times \sum_s X^{2m+2s-1}q^{m+s}(1-q^{-1}) \\
&= \frac{q^{m+2}X^{2m+3}}{1-qX^2}.
\end{aligned}$$

Denote by $\hat{J}_b$ the contribution from $|a| = |x| \leq q$, and by $\hat{J}_b^r$ the contribution from $|a| = |x| \leq q^{-r}$, where $r \geq -1$. Similarly we have

$$\begin{aligned}
\hat{J}_b^r &= (1-q^{-1})^{-1}\left(\int_{|x|=q^{-r}} \psi_E(x)\chi^{-1}(x)\,dx\right)^2 \times \int_{|\beta|\geq q^{2m+4+2r}} \chi^{-1}(\beta\sqrt{\tau}+1/2)\,d\beta \\
&= (1-q^{-1})q^{-2r}X^{-2r} \times \sum_{s\geq 0} X^{2m+3+2r+2s}q^{m+2+r+s}(1-q^{-1}) \\
&= (1-q^{-1})^2q^{m+2-r}X^{2m+3}\sum_s X^{2s}q^s \\
&= \frac{(1-q^{-1})^2q^{m+2-r}X^{2m+3}}{1-qX^2}.
\end{aligned}$$

Summing over r, we get

$$\hat{J}_b = \sum_{r\geq -1}\hat{J}_b^r = (1-q^{-1})\frac{q^{m+3}X^{2m+3}}{1-qX^2}.$$

In conclusion,

$$\hat{J}(\chi,u) = \hat{J}_t + \hat{J}_b = \frac{q^{m+3}X^{2m+3}}{1-qX^2}. \qquad \square$$

Chapter 8

Comparison of germ functions

Comparing all the previous cases, in particular, comparing Proposition 6.2 with Proposition 7.1, Proposition 6.3 with Proposition 7.2, Proposition 6.7 with Proposition 7.3 and Proposition 6.10 with Proposition 7.4 we conclude with the following.

Proposition 8.1. *For every fixed u,*

$$\hat{I}(\chi, u) = \hat{J}(\chi, u).$$

Thus,

$$I(w\alpha, f_0) = J(w\alpha, f_0').$$

In particular, if we take $u = \frac{\bar{a}}{a}$ where $\alpha \in A^e_{w_G}$, we have the

Corollary 8.2. *For all $\alpha \in A^e_{w_G}$, we have*

$$K(\alpha) = L(\alpha).$$

Given f' in $\mathcal{C}(\mathcal{S})$, we say that f in $\mathcal{C}(G)$ is the smooth transfer of f' (and vice versa) if they satisfy

$$I(w\alpha, f') = J(w\alpha, f)$$

for all α in A_w.

Proposition 8.3. *For any f in $\mathcal{C}(\mathcal{S})$, its smooth transfer f' in $\mathcal{C}(G')$ exists, and vice versa.*

Proof. By the unitary analogue of Lemma 5.1, given f we can find f' such that

$$W_e^f = W_e^{f'}, W_w^f = W_w^{f'}.$$

Now by Corollary 8.2 we know that the germ functions K and L are the same. Combining this with the identical germ equations (5.2) and (7.1), we see that f' is the transfer of f. $\square$

Chapter 9

Conclusion

Lemma 9.1. *Suppose we have two matching functions $f = \otimes_v f_v \in \mathcal{C}(G(\mathbb{A}_F))$ and $f' = \otimes_v f'_v \in \mathcal{C}(G'(\mathbb{A}_F))$. If Π is the weak base change of one stable L-packet of G, then we have*

$$I_\Pi(f') = \sum_\pi J_\pi(f),$$

where the sum is over all π in the same L-packet.

Proof. This is the result in [Fli1992, Proposition 28]. □

We refer to [Rog1990, page 201] for the definition of stable L-packets and the result [Rog1990, Theorem 1.3.3] that the bage change of a stable tempered L-packet of a cuspidal representation is a single cuspidal automorphic representation.

Theorem 9.2. *Every stable tempered L-packet of a cuspidal representation of $G = U_3$ contains a generic representation.*

Proof. Consider the weak base change Π to $G' = \mathrm{GL}_3(\mathbb{A}_E)$. By Proposition 3.1 we see that Π is distinguished, and hence $I_\Pi(f') \neq 0$ for some $f' = \otimes_v f'_v \in \mathcal{C}(\mathcal{S}(\mathbb{A}_F))$. By Proposition 3.2 and the factorization of $I_\Pi(f')$ into local components, we can modify f'_v at the archimedean places and even places v such that f'_v is supported on the regular orbits and $I_{\Pi_v}(f'_v)$ is not zero. Therefore we maintain the nonvanishing of $I_\Pi(f')$. Take $f = \otimes_v f_v \in \mathcal{C}(G(\mathbb{A}_F))$ to be the transfer of f'. By Lemma 9.1, we see that the sum $\sum_\pi J_\pi(f)$ is not zero, and thus $J_\pi(f)$ is not zero for some π in the L-packet, which means π is generic. □

Using a similar approach by considering the noncuspidal discrete part of the RTF in Proposition 2.8, we can prove that the same result holds for the endoscopic L-packets of a cuspidal representation. Now we are in a position to prove a local statement.

Theorem 9.3. *Consider a quadratic extension E'/F' of local fields. Then every tempered local L-packet of $G(F)$ contains exactly one generic component.*

Proof. By [GRS1997] the only unknown case is when the local L-packet reduces to a single supercuspidal representation. Say $E' = E_v$ and $F' = F_{v_0}$ for a quadratic extension of number fields E/F. Consider any irreducible cuspidal representation π of $G(\mathbb{A}_F)$ with π_{v_0} equal to this supercuspidal representation. By the previous theorem we know there is a generic cuspidal π' inside the L-packet containing π, so in particular $\pi_{v_0} = \pi'_{v_0}$ is generic. ☐

Finally we prove that local genericity implies globally genericity.

Theorem 9.4. *Suppose that π is a cuspidal automorphic representation of $G(\mathbb{A}_F)$, such that π_{v_0} is generic for every place v_0. Then π is globally generic.*

Proof. Let π' be a generic cuspidal automorphic representation inside the L-packet of π. By the local result of Theorem 9.3 we know that $\pi_{v_0} = \pi'_{v_0}$ for all v_0. Thus $\pi = \pi'$. ☐

Bibliography

[BH2006] C. J. Bushnell and G. Henniart, *The local Langlands conjecture for* GL(2), Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 335, Springer-Verlag, Berlin, 2006. MR2234120 ↑17

[Fli1992] Y. Z. Flicker, *Distinguished representations and a Fourier summation formula*, Bull. Soc. Math. France **120** (1992), no. 4, 413–465. MR1194271 ↑2, 58

[GGP2012] W. T. Gan, B. H. Gross, and D. Prasad, *Symplectic local root numbers, central critical L values, and restriction problems in the representation theory of classical groups*, Astérisque **346** (2012), 1–109. Sur les conjectures de Gross et Prasad. I. MR3202556 ↑3

[GJR2001] S. Gelbart, H. Jacquet, and J. Rogawski, *Generic representations for the unitary group in three variables*, Israel J. Math. **126** (2001), 173–237. MR1882037 ↑2, 1, 3, 6, 9, 10, 12

[GP1992] B. H. Gross and D. Prasad, *On the decomposition of a representation of* SO_n *when restricted to* SO_{n-1}, Canad. J. Math. **44** (1992), no. 5, 974–1002. MR1186476 ↑3

[GP1994] ———, *On irreducible representations of* $\mathrm{SO}_{2n+1}\times\mathrm{SO}_{2m}$, Canad. J. Math. **46** (1994), no. 5, 930–950. MR1295124 ↑3

[GRS1997] S. Gelbart, J. Rogawski, and D. Soudry, *Endoscopy, theta-liftings, and period integrals for the unitary group in three variables*, Ann. of Math. (2) **145** (1997), no. 3, 419–476. MR1454699 ↑59

[Har2011] R. N. Harris, *A Refined Gross-Prasad Conjecture for Unitary Groups*, ProQuest LLC, Ann Arbor, MI, 2011. Thesis (Ph.D.)–University of California, San Diego. MR2890098 ↑3

[HC1970] Harish-Chandra, *Harmonic analysis on reductive p-adic groups*, Lecture Notes in Mathematics, Vol. 162, Springer-Verlag, Berlin-New York, 1970. Notes by G. van Dijk. MR0414797 ↑14

[II2010] A. Ichino and T. Ikeda, *On the periods of automorphic forms on special orthogonal groups and the Gross-Prasad conjecture*, Geom. Funct. Anal. **19** (2010), no. 5, 1378–1425. MR2585578 ↑3

[Jac1992] H. Jacquet, *Relative Kloosterman integrals for* GL(3). *II*, Canad. J. Math. **44** (1992), no. 6, 1220–1240. MR1192415 ↑9, 21, 36, 38

[JY1996] H. Jacquet and Y. Ye, *Distinguished representations and quadratic base change for* GL(3), Trans. Amer. Math. Soc. **348** (1996), no. 3, 913–939. MR1340178 ↑14, 17

[JY1999] ———, *Germs of Kloosterman integrals for* GL(3), Trans. Amer. Math. Soc. **351** (1999), no. 3, 1227–1255. MR1443878 ↑14, 21

[Kot2005] R. E. Kottwitz, *Harmonic analysis on reductive p-adic groups and Lie algebras*, Harmonic analysis, the trace formula, and Shimura varieties, 2005, pp. 393–522. MR2192014 ↑14

[Mao1993] Z. Mao, *Relative Kloosterman integrals for* GL(3). *III*, Canad. J. Math. **45** (1993), no. 6, 1211–1230. MR1247543 ↑9, 21

[Mok2015] C. P. Mok, *Endoscopic classification of representations of quasi-split unitary groups*, Mem. Amer. Math. Soc. **235** (2015), no. 1108, vi+248. MR3338302 ↑2

[Rog1990] J. D. Rogawski, *Automorphic representations of unitary groups in three variables*, Annals of Mathematics Studies, vol. 123, Princeton University Press, Princeton, NJ, 1990. MR1081540 ↑1, 9, 58

[Wal1985] J.-L. Waldspurger, *Sur les valeurs de certaines fonctions L automorphes en leur centre de symétrie*, Compositio Math. **54** (1985), no. 2, 173–242. MR783511 ↑3

[Xue2014] H. Xue, *The Gan-Gross-Prasad conjecture for* $\mathrm{U}(n) \times \mathrm{U}(n)$, Adv. Math. **262** (2014), 1130–1191. MR3228451 ↑3

[Zha2014a] W. Zhang, *Automorphic period and the central value of Rankin-Selberg L-function*, J. Amer. Math. Soc. **27** (2014), no. 2, 541–612. MR3164988 ↑3, 5

[Zha2014b] ______, *Fourier transform and the global Gan-Gross-Prasad conjecture for unitary groups*, Ann. of Math. (2) **180** (2014), no. 3, 971–1049. MR3245011 ↑1, 2, 3