ON LAGRANGE-HERMITE INTERPOLATION*

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1. Introduction. Let the \( p(n + 1) \) numbers \( y_{i}^{(m)} \), \( 0 \leq i \leq n, \ 0 \leq m \leq p - 1 \), be given. It is well known that there exists a unique polynomial \( P_{n,p}(t) \) of degree \( p(n + 1) - 1 \) such that

\[
P_{n,p}(x_i) = y_{i}^{(m)}, \quad 0 \leq i \leq n, \quad 0 \leq m \leq p - 1.
\]

A classical problem is to find a formula for \( P_{n,p}(t) \) in the form

\[
P_{n,p}(t) = \sum_{i=0}^{n} \sum_{m=0}^{p-1} C_{m,i}^{n,p}(t)y_{i}^{(m)}.
\]

The conditions on the \( C_{m,i}^{n,p}(t) \) are that

\[
D_i^r C_{m,i}^{n,p}(x_r) = \delta_{j,m} \delta_{r,i}, \quad 0 \leq r \leq n, \quad 0 \leq j \leq p - 1,
\]

where \( D_i \equiv \frac{d}{dt} \) and \( \delta_{j,m} \) is a Kronecker symbol. These conditions are used by Householder [5, pp. 193–195] to derive the formulas for \( p = 1, 2 \). The formula for \( p = 3 \) is given by Salzer [9]. The solution for \( n = 0 \) is given by Taylor’s formula.

Many authors have reported on the case where \( p \) depends on \( i \). General prescriptions for a solution in this more general case may be found in Fort [2, pp. 85–88], Greville [3], Hermite [4], Krylov [6, pp. 45–49], Kuntzmann [7, pp. 167–169], and Spitzbart [12]; but these prescriptions do not determine the structure of the interpolating polynomial. By restricting ourselves to the case where \( p \) is independent of \( i \), which is the most important case in practice, we can determine the structure. Salzer [10] discovered some of the properties of \( P_{n,p}(t) \) by semiempirical means.

We shall obtain, by a partial fraction expansion, a solution of surprising simplicity. [See (3.6), (3.7), or (3.8).] The solution depends upon the Bell polynomials which we now discuss.

2. Bell polynomials. Let \( g = g(t) \) and define \( B_n \) by

\[
e^{-\omega g} D_t^n e^{\omega g} = B_n(\omega) = B_n(\omega; g_1, \cdots, g_n), \quad g_i \equiv g^{(i)}.
\]

\( B_n \) is a polynomial in \( \omega \) with coefficients which are polynomials in \( g_i \). Define \( U_{n,k} \) by

\[
B_n(\omega; g_1, \cdots, g_n) = \sum_{k=0}^{n} U_{n,k}(g_1, \cdots, g_{n-k+1})\omega^k.
\]

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Then

\[ U_{n,k} = \frac{1}{k!} D_{x_k} B_n(0). \]

The \( B_n(1; g_1, \ldots, g_n) \) were studied by Bell [1]. (See also Schlömilch [11, p. 4].) An explicit formula for \( B_n \) is

\[ B_n = n! \sum \omega^j \prod_{i=1}^{n} \frac{1}{b_i!} \left( \frac{g_i}{i!} \right)^{b_i}, \]

where \( j = \sum_{i=1}^{n} b_i \) and where the sum is taken over all nonnegative integers \( b_i \) for which \( \sum_{i=1}^{n} i b_i = n \).

Let \( F(t) = f[g(t)] \). Then

\[ F^{(n)} = \sum_{k=0}^{n} f^{(k)} U_{n,k}(g', \ldots, g^{(n-k+1)}), \]

or

\[ F^{(n)} = B_n(f; g', \ldots, g^{(n)}), \quad f^{(k)} = f^k. \]

Generating functions and symbolic recurrence relations for the Bell polynomials may be found in Bell [1] and Riordan [8, pp. 35–38, 45–48]. The first five \( B_n \) are:

\[ \begin{align*}
B_0 &= 1, \\
B_1 &= \omega g_1, \\
B_2 &= \omega^2 g_1^2 + \omega g_2, \\
B_3 &= \omega^3 g_1^3 + 3\omega^2 g_1 g_2 + \omega g_3, \\
B_4 &= \omega^4 g_1^4 + 6\omega^3 g_1^2 g_2 + \omega^2(4g_1 g_3 + 3g_2^2) + \omega g_4.
\end{align*} \]

3. The formula for the interpolatory polynomial. Let \( P(t)/Q(t) \) be a proper rational function and let \( Q(t) \) have a zero of multiplicity \( p \) at \( x_i \). Let

\[ \begin{align*}
\frac{1}{Q(t)} &= \sum_{j=1}^{p} \frac{\alpha_{p,j}}{(t - x_i)^j} + \eta(t), \\
\frac{P(t)}{Q(t)} &= \sum_{j=1}^{p} \frac{\beta_{p,j}}{(t - x_i)^j} + \lambda(t).
\end{align*} \]

Then it is easy to show that

\[ \beta_{p,p-j} = \sum_{k=0}^{j} \alpha_{p,p-k} \frac{P^{(j-k)}(x_i)}{(j - k)!}. \]
This result is the key to the solution of the Lagrange-Hermite interpolation problem. It permits us to write the interpolatory polynomial as a linear combination of the $y_i^{(m)}$.

Let

$$
\pi(t) = \prod_{i=0}^{n} (t - x_i), \quad Q(t) = \pi^{p}(t),
$$

(3.2)

$$
R_i(t) = \frac{\pi(t)}{t - x_i}, \quad L_i(t) = \frac{R_i(t)}{R_i(x_i)}.
$$

We calculate the contribution to $P_{n,p}(t)$ due to $x_i$ and then sum on $i$. We have

$$
P_{n,p}(t) \equiv \frac{Q(t) P_{n,p}(t)}{Q(t)} = Q(t) \sum_{i=1}^{p} \beta_{p,i}\frac{P^{(i-k)}(x_i)}{(i-k)!} + \rho(t),
$$

$$
\beta_{p,i}\frac{P^{(i-k)}(x_i)}{(i-k)!} = \sum_{k=0}^{i} \alpha_{p,i-k} \frac{P^{(i-k)}(x_i)}{(j-k)!}, \quad \alpha_{p,i-k} = \frac{1}{k!} D_k R_i^{-p}(x_i).
$$

Using

$$
P^{(i-k)}(x_i) = y_i^{(i-k)},
$$

we obtain, after some manipulation,

$$
P_{n,p}(t) = L_i^p(t) \sum_{m=0}^{p-1} y_i^{(m)} \frac{(t - x_i)^m}{m!} \sum_{r=0}^{p-1-m} \frac{(t - x_i)^r}{\nu!} R_i^p(x_i) D_i^\nu R_i^{-p}(x_i) + \rho(t).
$$

(3.3)

Let

$$
S_r \equiv S_r(x_i) = (-1)^r(\nu - 1)! \sum_{r=0}^{n} \frac{1}{(x_i - x_r)^r}.
$$

(3.4)

It follows from (2.1), (3.2), and (3.4) that

$$
R_i^p(x_i) D_i^\nu R_i^{-p}(x_i) = B_r(p; S_1, \ldots, S_r).
$$

(3.5)

Using (3.5) and adding the contributions from all the $x_i$, we obtain as a solution to our problem

$$
P_{n,p}(t) = \sum_{i=0}^{n} L_i^p(t) \sum_{m=0}^{p-1} \frac{(t - x_i)^m}{m!} y_i^{(m)} \sum_{r=0}^{p-1-m} \frac{(t - x_i)^r}{\nu!} \sum_{r=0}^{n} \frac{1}{(x_i - x_r)^r} B_r(p; S_1, \ldots, S_r).
$$

(3.6)

Thus the essence of the $p$th order Lagrange-Hermite formula is contained in the $B_r(p; S_1, \ldots, S_r)$, $0 \leq r \leq p - 1$. Let

$$
G_{p,i,m} = \sum_{r=0}^{p-1-m} \frac{(t - x_i)^r}{\nu!} B_r(p; S_1, \ldots, S_r).
$$
Observe that $G_{p,i,m}$ may be obtained from the polynomial $G_{p,i,0}$ by truncating the highest $m$ terms. Hence for each $p$, $P_{n,p}(t)$ may be easily obtained from $G_p \equiv G_{p,i,0}$. The first five $G$ are:

\[ G_1 = 1, \]
\[ G_2 = 1 + (t - x_i)2S_1, \]
\[ G_3 = 1 + (t - x_i)3S_1 + \frac{1}{2}(t - x_i)^2[3^2S_1^2 + 3S_2], \]
\[ G_4 = 1 + (t - x_i)4S_1 + \frac{1}{2}(t - x_i)^2[4^2S_1^2 + 4S_2] + \frac{1}{6}(t - x_i)^3[4^3S_1^3 + 3\cdot 4^2S_1S_2 + 4S_3], \]
\[ G_5 = 1 + (t - x_i)5S_1 + \frac{1}{2}(t - x_i)^2[5^2S_1^2 + 5S_2] + \frac{1}{6}(t - x_i)^3[5^3S_1^3 + 3\cdot 5^2S_1S_2 + 5S_3] + \frac{1}{24}(t - x_i)^4[5^4S_1^4 + 6\cdot 5^2S_1^2S_2 + 5^2(4S_1S_3 + 3S_2^2) + 5S_4]. \]

Equation (3.6) may be written in a number of other ways. Let

\[ T_v \equiv T_v(x_i) = (v - 1)! \sum_{r=0,r \neq i}^{n} \left( \frac{x_i - t}{x_i - x_r} \right)^v. \]

Then

\[ (3.7) \quad P_{n,p}(t) = \sum_{i=0}^{n} L_i^p(t) \sum_{m=0}^{p-1} \frac{(t - x_i)^m}{m!} y_i^{(m)} \sum_{r=0}^{p-1-m} \frac{1}{\nu!} B_r(p; T_1, \ldots, T_v). \]

Let

\[ H_{p,i,m,k} = \sum_{r=k}^{p-1-m} \frac{U_{r,k}}{\nu!} (T_1, \ldots, T_{r-k+1}). \]

Then

\[ P_{n,p}(t) = \sum_{i=0}^{n} L_i^p(t) \sum_{m=0}^{p-1} \frac{(t - x_i)^m}{m!} y_i^{(m)} \sum_{k=0}^{p-1-m} H_{p,i,m,k} p^k. \]

A formula for $P_{n,p}(t)$ in which the coefficients are polynomials in the $L_i^{(j)}(x_i)$ may be obtained as follows. Let

\[ R_i^{-p}(t) = f[g(t)], \quad f(u) = u^{-p}, \quad g(t) = R_i(t). \]

Then using (2.2), and with $L_i^{(j)} \equiv L_i^{(j)}(x_i)$,

\[ R^p(x_i)D_t^jR^{-p}(x_i) = \sum_{k=0}^{r} (-1)^k k! C(p + k - 1, k) U_{v,k}(L', \ldots, L^{(r-k+1)}). \]

Hence

\[ (3.8) \quad P_{n,p}(t) = \sum_{i=0}^{n} L_i^p(t) \sum_{m=0}^{p-1} \frac{(t - x_i)^m}{m!} y_i^{(m)} E_{p,i,m}, \]

\[ E_{p,i,m} = \sum_{r=0}^{p-1-m} \frac{(t - x_i)^r}{\nu!} \sum_{k=0}^{r} (-1)^k k! C(p + k - 1, k) \]

\[ \cdot U_{v,k}(L', \ldots, L^{(r-k+1)}). \]
Observe that $E_{p,i,m}$ may be obtained from the polynomial $E_{p,i,0}$ by truncating the highest $m$ terms. Hence for each $p$, $P_{n,p}(t)$ may be easily obtained from $E_p = E_{p,i,0}$. The first five $E_p$ are

\begin{align*}
E_1 &= 1, \\
E_2 &= 1 + (t - x_i)[-2L'], \\
E_3 &= 1 + (t - x_i)[-3L'] + \frac{1}{2}(t - x_i)^2[-3L'' + 12(L')^2], \\
E_4 &= 1 + (t - x_i)[-4L'] + \frac{1}{2}(t - x_i)^2[-4L'' + 20(L')^2] \\
&\quad + \frac{1}{6}(t - x_i)^3[-4L''' + 60(L'L') - 120(L')^3], \\
E_5 &= 1 + (t - x_i)[-5L'] + \frac{1}{2}(t - x_i)^2[-5L'' + 30(L')^2] \\
&\quad + \frac{1}{6}(t - x_i)^3[-5L''' + 90(L'L') - 210(L')^3] \\
&\quad + \frac{1}{24}(t - x_i)^4[-5L^{(4)} + 120L'L''' + 90(L'')^2 - 1260(L')^2L'' + 1680(L')^4].
\end{align*}

As far as calculation with these formulas is concerned, observe that

$$L_i^{(j)}(x_i) = \frac{R_i^{(j)}(x_i)}{R_i(x_i)}.$$ 

The $R_i^{(j)}(x_i), j \geq 0$, may be obtained from $\pi(t)$ by repeated synthetic division.

4. Some applications. The interpolation formula may be used to generalize the Cauchy relations,

$$t^j = \sum_{i=0}^{n} x_i^j I_i(t), \quad j = 0, 1, \ldots, n.$$ 

Corresponding to the case $j = 0$, we have the following generalization.

\begin{equation}
1 = \sum_{i=0}^{n} L_i^p(t) \sum_{r=0}^{p-1} \frac{(t - x_i)^r}{r!} B_r(p; S_1, \ldots, S_r).
\end{equation}

Since the leading coefficient of $t$ on the right side of (4.1) vanishes,

\begin{equation}
\sum_{i=0}^{n} \frac{1}{\pi'(x_i)^p} B_{p-1}(p; S_1, \ldots, S_{p-1}) = 0.
\end{equation}

This generalizes

$$\sum_{i=0}^{n} \frac{1}{\pi'(x_i)} = 0.$$ 

We can derive a formula for the confluent divided difference with the same number of repetitions of all arguments, $f[x_0, p; x_1, p; \cdots; x_n, p]$. 

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(This notation is introduced in Traub [13, pp. 241-242].) Since this divided difference is the coefficient of the highest degree term in (3.6), we obtain

\[ f[x_0, p; x_1, p; \cdots; x_n, p] = \sum_{m=0}^{p-1} \frac{B_{p-1-m}(p; S_1, \cdots, S_{p-1-m})}{m!(p - 1 - m)!} \]  

(4.2)

\[ \cdot \sum_{i=0}^{n} \frac{f^{(m)}(x_i)}{\pi^i(x_i)} \]

REFERENCES