Dynamics, Graph Theory, and Barsotti-Tate Groups
Variations on a theme of Mochizuki

by

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Abstract

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In this dissertation, we study étale correspondence of hyperbolic curves with unbounded dynamics. Mochizuki proved that over a field of characteristic 0, such curves are always Shimura curves. We explore variants of this question in positive characteristic, using graph theory, l-adic local systems, and Barsotti-Tate groups.

Given a correspondence with unbounded dynamics, we construct an infinite graph with a large group of "algebraic" automorphisms and roughly measures the "generic dynamics" of the correspondence. We construct a specialization map to a graph representing the actual dynamics. Along the way, we formulate conjectures that étale correspondences with unbounded dynamics behave similarly to Hecke correspondences of Shimura curves. Using graph theory, we show that type (3,3) étale correspondences verify various parts of this philosophy.

Key in the second half of this dissertation is a recent p-adic Langlands correspondence, due to Abe, which answers affirmatively the *petites camarades* conjecture of Deligne in the case of curves. This allows us the build a correspondence between rank 2 l-adic local systems with trivial determinant and Frobenius traces in $\mathbb{Q}$ and certain height 2, dimension 1 Barsotti-Tate groups. We formulate a conjecture on the fields of definitions of certain compatible systems of l-adic representations. Relatedly, we conjecture that the Barsotti-Tate groups over complete curves in positive characteristic may be "algebraized" to abelian schemes.

Contents

Conventions, Notation, and Terminology

i. p is a prime number and $q = p^d$.

ii. $\mathbb{F}$ is fixed algebraic closure of $\mathbb{F}_p$.

iii. Given a perfect field k of characteristic p, $W(k)$ denotes the Witt vectors of k and σ denotes the canonical lift of Frobenius. We set $\mathbb{Z}_q = W(\mathbb{F}_q)$ and $\mathbb{Q}_q = \mathbb{Z}_q \otimes \mathbb{Q}$.

iv. Whenever we discuss p-adic valuations, we choose the normalization $v_p(p) = 1$.

v. A *curve* C over a field k is a geometrically integral scheme of dimension 1 over k.

vi. When we discuss morphisms of curves over k, we suppose they are always non-constant and generically separable.

vii. A smooth curve C over a field k is said to be *hyperbolic* if $\mathrm{Aut}_{\overline{k}}(C_{\overline{k}})$ is finite.

viii. Given a field k, Ω will always be an algebraically closed field of transcendence degree 1 over k.

ix. In general, X, Y, and Z will be a curves over k, with $M = k(Z)$, $L = k(X)$, and $K = k(Y)$ the function fields. We *fix* a k-algebra embedding $PQ : k(Z) \hookrightarrow \Omega$ that identifies Ω as an algebraic closure of $k(Z)$.

x. The graph $\mathcal{G}_{gen}$ is the *generic graph* of a correspondence, as defined in Section 2.1. The graph $\mathcal{G}_{phys}$ is the *physical graph*, as defined in Section 2.3

xi. On a smooth curve, an F-isocrystal is as in Saavedra Rivano's thesis [**Saa72**]; that is, we always assume there is an underlying *crystal*.

Acknowledgments

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Finally, thanks to Aaron Bernstein and Anaïs Maurer for their friendship.

To my family.

CHAPTER 1

Introduction

1.1. Background

In this dissertation we study the dynamics of étale correspondences of smooth hyperbolic curves. In [**Moc98**], Mochizuki proved that if

$$\begin{array}{ccc} & Z & \\ \swarrow & & \searrow \\ X & & Y \end{array}$$

is an étale correspondence of complex hyperbolic curves with unbounded dynamics, then X, Y, and Z are all Shimura curves. Mochizuki uses a highly non-trivial result of Margulis [**Mar91**], which characterizes Shimura curves via the group theory of discrete subgroups of $PSL_2(\mathbb{R})$. This dissertation is an exploration of analogs of Mochizuki's theorem in characteristic p.

The most basic examples of Shimura varieties are the modular curves. Let $\Gamma \subset SL_2(\mathbb{Z})$ be a congruence subgroup. The non-compact complex curve $X_\Gamma = \mathbb{H}/\Gamma$ be may endowed with the structure of a complex algebraic curve (possibly with stacky structure.) In fact, X_Γ parametrizes elliptic curves with some extra data and may therefore be defined as a moduli problem over $\mathbb{Z}[\frac{1}{N}]$ where N is the level of Γ, as in Katz-Mazur [**KM85**]. A slightly less familiar example comes from moduli spaces of *fake elliptic curves*; these Shimura varieties are projective algebraic curves that may again be defined over $\mathbb{Z}[\frac{1}{N}]$ for appropriate N as we will see in Definition 2.1.12 (see Buzzard [$\mathbf{B^+97}$] for more details.) See

In general, Shimura varieties parameterize Hodge structures and they are naturally equipped with the structure of complex analytic spaces. An important theorem of Baily-Borel shows that when there are no orbifold points, Shimura varieties can be naturally given the structure of quasi-projective algebraic varieties over $\mathbb{C}$ via holomorphic automorphic forms [**BB66**]. Borel then proved that this structure of an algebraic variety is unique [**B$^+$72**]. See Deligne [**Del79**] for a general introduction to Shimura varieties.

When a Shimura variety is a moduli space of abelian varieties with manifestly algebraic conditions (e.g. "PEL": fixing the data of a polarization, endomorphisms, and level), it can be seen directly that it descends to $\overline{\mathbb{Q}}$. Results of Borovoi [**Bor82**], Deligne [**Del79**], Milne [**MS82, Mil83**], and Varshavky [**Var02**] imply that in general Shimura varieties may be defined over $\overline{\mathbb{Q}}$. Recent work of Kisin [**Kis10**] shows that Shimura varieties of abelian type have natural integral models, which opens up the possibility of studying their reduction modulo p. Using the moduli interpretation of PEL-type Shimura varieties, it is straightforward to define them directly over finite fields $\mathbb{F}_q$, at least for appropriate q. There is not as-of-yet a direct definition of general non PEL-type Shimura varieties over $\mathbb{F}_q$.

Jie Xia has recently taken the simplest example of a non PEL-type Shimura curve, what he calls Mumford curves, and given "direct definitions" modulo p [**Xia13a, Xia13c, Xia14**]. The most basic example of Mumford curves parameterize abelian 4-folds with certain extra Hodge classes, as in Mumford's original paper [**Mum69**]. Xia proved a variety of theorems of the following form: given an abelian scheme $\mathcal{A} \to X$ over a curve X over $\overline{\mathbb{F}_p}$, there are certain conditions that ensure that the pair $(\mathcal{A}, X)$ is the reduction of a Mumford curve together with its universal abelian scheme over $W(\overline{\mathbb{F}_p})$. We will use Xia's results extensively in the final sections of this dissertation.

Unlike Xia's results, we *do not* assume the existence of an abelian scheme $\mathcal{A}$ over the curves we study. Instead, we take as our starting point Mochizuki's Theorem. We seek to find minimal hypotheses that ensure than an étale correspondence of smooth hyperbolic curves over a field k of characteristic p with unbounded dynamics is "related to" the reduction modulo p of Shimura curves: Question 2.1.13. A warning: the phrase "related to" is absolutely vital, as we will see in Remark 2.1.14 via the example of Igusa level structure. Question 2.1.13 is somewhat vague, due to the intervening phrase "is related to", and we will give more concrete questions and conjectures throughout this dissertation.

1.2. Summary of contents

In the second chapter, we operate under general assumptions. Section 2.1 gives the central definition of a *correspondence without a core*, Definition 2.1.5, and uses this definition to state Mochizuki's Theorem, Theorem 2.1.8. In Section 2.2, starting from a correspondence without a core, we construct a pair $(\mathcal{G}_{gen}, A)$ of an infinite graph together with a large group of "algebraic" automorphisms. The graph $\mathcal{G}_{gen}$ roughly measures the "generic dynamics" of the correspondence. We are especially interested in Question 2.2.28: given an étale correspondence without a core, is $\mathcal{G}_{gen}$ a tree? Using Serre's theory of groups acting on trees [**SB77**], we prove that if $\mathcal{G}_{gen}$ is a tree, then the pair $(\mathcal{G}_{gen}, A^{PQ})$ behaves similarly to the building of the p-adic group $PSL(2, \mathbb{Q}_p)$.

Section 2.3 constructs specialization maps $\mathcal{G}_{gen} \to \mathcal{G}_{phys}$. These roughly specialize the dynamics from the generic point to closed points and we prove they are surjective when the original correspondence is étale. Motivated by work of Kohel and Sutherland on *Isogeny Volcanoes* of elliptic curves [**Koh96, Sut13**], we speculate on the behavior and asymptotics of these specialization maps in Question 2.3.5.

Section 2.4 develops some basic results for symmetric correspondences and Section 2.5 defines a *clump* (bounded orbit) of a correspondence, in analogy to the supersingular locus of a Hecke correspondence over $\mathbb{F}_p$. Results in Sections 2.4 and 2.5 imply that general symmetric, type (3,3) étale correspondences without a core behave similarly to Hecke correspondences of Shimura curves. Lemma 2.4.8 implies that in this case, the graph $\mathcal{G}_{gen}$ is a tree using graph theory developed by Tutte [**Tut66**]. Corollary 2.5.9 implies that in this case, there is at most one clump, using recent work of Perret and Hallouin [**HP14**]. We find it especially appealing that the work of Perret and Hallouin uses spectral graph theory.

In the third chapter, we specialize our assumptions. In order to continue our exploration of Question 2.1.13, we found it profitable to add Assumption 3.2.1. Roughly speaking, Assumption 3.2.1 supposes that the pair $(\mathcal{G}_{gen}, G_P)$ is isomorphic to the action of $PSL(2, \mathbb{Z}_2)$ on the building of $PGL(2, \mathbb{Q}_2)$. Here, G_P is a certain profinite subgroup of A. We develop some 2-adic group theory in Section 3.1 to show that Assumption 3.2.1 implies that the symmetric étale correspondence

$$\begin{array}{ccc} & Z & \\ {}^{f}\swarrow & & \searrow^{g} \\ X & & X \end{array}$$

comes equipped with a projective 2-adic local system $\mathscr{P}$ such that $f^*\mathscr{P} \cong g^*\mathscr{P}$. We know that this situation arises in nature when considering moduli spaces of fake elliptic curves and we comment on the likelihood of Assumption 3.2.1 in Remark 3.2.2.

Section 3.3 is a brief discussion of Deligne's compatible systems conjecture from Weil II [**Del80**]. The p-adic part of this conjecture has recently been resolved for X a curve by Abe [**Abe11, Abe13**] by proving a p-adic Langlands correspondence. We will use the so-called "crystalline companion", an overconvergent F-isocrystal, extensively for the rest of this dissertation. To use Deligne's conjecture, we first need to lift our projective local system to an actual local system. We sketch an involved argument to do this in Section 3.4, using the full force of Deligne's conjecture for $SL(3)$.

The goal of Sections 3.5-3.11 is the following. Given a rank 2 l-adic local system $\mathscr{L}$ on X with trivial determinant, infinite monodromy, and Frobenius traces in $\mathbb{Q}$, we want to construct a unique *generically versally deformed* Barsotti-Tate (or p-divisible) group $\mathscr{G}$ on X that is "compatible" with $\mathscr{L}$. This $\mathscr{G}$ will have height 2, dimension 1, and be generically ordinary. To do this requires a number of steps.

(1) Section 3.5 sets up a general machinery extension-of-scalars and Galois descent for K-linear abelian categories. For other references, see Deligne [**Del14**] or Sosna [**Sos14**].

(2) Section 3.6 constructs a Brauer-class obstruction for descending certain objects in an abelian K-linear category. This is used to prove a criterion for descent which will be useful later.

(3) Section 3.7 reviews background on F-isocrystals over perfect fields k of characteristic p and explicitly describes the base-changed category $F\text{-Isoc}(k)_L$ for L a p-adic local field.

(4) Section 3.8 specializes the discussion of Section 3.7 to finite fields. Many of the results are likely well-known, but we could not find always find a reference.

(5) Section 3.9 clarifies our conventions for F-crystals and isocrystals over varieties. In particular, we use the conventions of Saavedra-Rivano [**Saa72**].

(6) Section 3.10 is where the above steps come together. Given $\mathscr{L}$ as above, the p-adic companion is an F-isocrystal $\mathscr{E}$ on X with coefficients in $\overline{\mathbb{Q}_p}$. Using the descent criterion of Section 3.6 the explicit description furnished in Section 3.8, we prove that if $p^2|q$, then $\mathscr{E}(-\frac{1}{2})$ descends to $\mathbb{Q}_p$. This is the content of Corollary 3.10.13. Using work of Katz [**Kat79**], de Jong [**dJ95**], and the highly nontrivial slope-bounds of V. Lafforgue [**Laf11**], we then prove that $\mathscr{E}(-\frac{1}{2})$ is the Dieudonné isocrystal of a height 2, dimension 1 Barsotti-Tate group $\mathscr{G}$ on X.

(7) Section 3.11 then uses work of Xia to prove that there is a unique such $\mathscr{G}$ that is *generically versally deformed.*

BT groups are generally non-algebraic, but here we can show there are only finitely many with certain criteria via the Langlands correspondence. In Section 3.12, we speculate on whether or not $\mathscr{G}$ is "algebraic" or "comes from an abelian motive". We pose various concrete instantiations of this question having to do with fields of definition of compatible systems. We prove that certain compatible systems have finite monodromy in Corollary 3.12.7. Here, the information about the p-adic companion is vital, and we are optimistic that Abe's recent resolution of the p-adic part of Deligne's conjecture will have many interesting consequences.

Finally, in Section 3.13, we discuss conditions under which the correspondence, together with $\mathscr{G}$, deform to characteristic 0, using another result of Xia [**Xia13b**]. The hypothesis of Xia's theorem is that $\mathscr{G}$ is *everywhere versally deformed on* X. We discuss how to possibly get around this restriction.

CHAPTER 2

Dynamics and Graph Theory

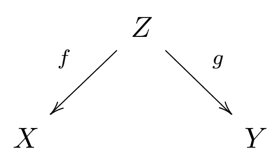
2.1. Correspondences and Cores

Definition 2.1.1. A smooth curve X over a field k is said to be hyperbolic if $\mathrm{Aut}_{\overline{k}}(X_{\overline{k}})$ is finite.

This is equivalent to the usual criterion of $2g-2+r \geq 1$ where g is the geometric genus of the compactification $\overline{X}$ and r is the number of geometric punctures. Over the complex numbers, this is equivalent to X being uniformized by $\mathbb{H}$.

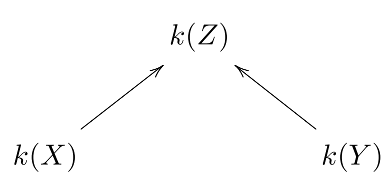
Lemma 2.1.2. *If $X \to Y$ is a non-constant morphism of curves over k where Y is hyperbolic, then X is hyperbolic.*

Definition 2.1.3. A *correspondence of curves* is a diagram



of smooth curves over a field k where f and g are dominant finite separable morphisms. We call such a correspondence of *type* (d,e) if $\deg f = d$ and $\deg g = e$. We call such a correspondence *étale* if both maps are étale. We call a correspondence *minimal* if the associated map $Z \to X \times Y$ is birational onto its image.

To a correspondence we can associate a containment diagram of function fields:

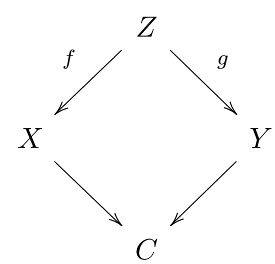


A correspondence is minimal iff there is no proper subfield of $k(Z)$ that contains both $k(X)$ and $k(Y)$.

Remark 2.1.4. Note that we require both f and g to be *finite*; for instance, strict open immersions are not permitted.

Definition 2.1.5. We say a correspondence as above has a *core* if the intersection of the two function fields $k(X) \cap k(Y)$ has transcendence degree 1 over k.

A correspondence has a core if there exists a curve C over k with non-constant maps from X and Y such that the following diagram commutes.



When X and Y are projective, we call the smooth projective curve C associated to the field $k(X) \cap k(Y)$ (considered as a field of transcendence degree 1 over k) the *coarse core* of the correspondence if it exists. When X and Y are not projective, the coarse core is defined to be the complement in C of the images of the points $\overline{X} \backslash X$ and those of $\overline{Y} \backslash Y$, where $\overline{X}$ and $\overline{Y}$ are the compactifications of X and Y respectively.

Remark 2.1.6. Given a correspondence as above, consider the following "many-valued function" $X \dashrightarrow X$ that sends $x \in X$ to the multi-set $f(g^{-1}(g(f^{-1}(x))))$, i.e. start with x, take all pre-images under f, take the image under g, take all pre-images under g and take the image under f. Having a core guarantees that the dynamics of this many-valued function are uniformly bounded.

Problem 2.1.7. If a core exists, is there the notion of a "fine core"? Such a gadget would presumably be a DM stacky curve $\mathcal{W}$ with generically trivial isotropy group such that $\mathcal{W}$ is the 1-colimit of the correspondence in the 1-category of DM stacky curves with generically trivial isotropy group (similar to the curves occurring in the work of Abramovich and Vistoli [**AV02**].)

We have a candidate answer for Problem 2.1.7 in Remark 2.2.3.

General correspondences of curves will not have cores. When we restrict to étale correspondences without cores, there is the following remarkable theorem of Mochizuki [**Moc98**] (due in large part to Margulis [**Mar91**]), which is the starting point of this dissertation.

Theorem 2.1.8. [**Moc98**] *If $X \leftarrow Z \rightarrow Y$ is an étale correspondence of complex hyperbolic curves (i.e. all three curves are uniformised by the upper half plane $\mathbb{H}$) without a core, then X, Y, and Z are all Shimura (arithmetic) curves.*

Remark 2.1.9. Theorem 2.1.8 in particular implies that if $X \leftarrow Z \rightarrow Y$ is an étale correspondence of complex hyperbolic curves, then all of the curves and maps can be defined over $\overline{\mathbb{Q}}$. In particular, there are *no non-trivial deformations* of étale correspondences of hyperbolic curves without a core.

The proof of Theorem 2.1.8 comes down to an explicit description, due to Margulis [**Mar91**], of the arithmetic subgroups Γ of $SL(2,\mathbb{R})$. Given a complex hyperbolic curve C, we have an inclusion

$$\Gamma := \pi_1(C) \rightarrow SL(2,\mathbb{R})$$

We say $\gamma \in SL(2,\mathbb{R})$ commensurates Γ if the discrete group $\gamma\Gamma\gamma^{-1}$ is commensurable with Γ, i.e. their intersection is of finite index in both groups. Define $Comm(\Gamma)$ to be the subgroup commensurating Γ. Margulis has proved that Γ is arithmetic if and only if $[Comm(\Gamma):\Gamma] = \infty$.

Example 2.1.10. The commensurater of $SL(2,\mathbb{Z})$ in $SL(2,\mathbb{R})$ is $SL(2,\mathbb{Q})$. The modular curve $Y(1) = [\mathbb{H}/SL(2,\mathbb{Z})]$ is arithmetic

Exercise 2.1.11. The correspondence of non-projective stacky modular curves $Y(1) \leftarrow Y_0(2) \rightarrow Y(1)$ does not have a core. Here, $Y_0(2)$ is the moduli space of pairs of elliptic curves $(E_1 \overset{2:1}{\rightarrow} E_2)$ with a given degree-2 isogeny between them, and the maps send the isogeny to the source and target elliptic curve respectively. Hint: to prove this over the complex numbers, look at the "orbits" of $\tau \in \mathbb{H}$.

Definition 2.1.12. Let D be an indefinite non-split quaternion algebra over $\mathbb{Q}$ of discriminant d and let $\mathcal{O}_D$ be a fixed maximal order. Let k be a field whose characteristic is prime to d. A fake elliptic curve is a pair (A, i) of an abelian surface A over k and an injective ring homomorphism $i : \mathcal{O}_D \to \mathrm{End}_k(A)$. The abelian surface A is endowed with the unique principal polarization such that the Rosati involution induces the canonical involution on $\mathcal{O}_D$.

Just as one can construct a modular curve parameterizing elliptic curves, there is a Shimura curve X^D parameterizing faking elliptic curves with multiplication by $\mathcal{O}_D$. Over the complex numbers, these are compact hyperbolic curves. Explicitly, if one chooses an isomorphism $D \otimes \mathbb{R} \cong M_{2\times 2}(\mathbb{R})$, take the image of $\Gamma = \mathcal{O}_D^1$ of elements of $\mathcal{O}_D^*$ of norm 1 (for the standard norm on $\mathcal{O}_D$) inside of $SL(2, \mathbb{R})$. This is a discrete subgroup and in fact acts properly discontinuously and cocompactly on $\mathbb{H}$. The quotient $\mathbb{H}/\Gamma$ is the Shimura curve associated to $\mathcal{O}_D$. There is a notion of isogeny of fake elliptic curves which is required to be compatible with the $\mathcal{O}_D$ structure and the associated "fake degree" of an isogeny. See Buzzard [**B**$^+$**97**] or Boutot-Carayol [**BC91**] for more details. These definitions allow us the define Hecke correspondences as in the elliptic modular case. For instance, as long as 2 splits in D, one can define the correspondence

$$\begin{array}{ccc} & X_0^D(2) & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ X^D & & X^D \end{array}$$

where $X_0^D(2)$ parametrizes pairs of fake elliptic curves $(A_1 \to A_2)$ with a given isogeny of fake degree 2 between them and π_1 and π_2 are the projections to the source and target respectively. This is another example of a correspondence of (stacky) hyperbolic curves without a core. To get an example without orbifold points, one can add level structure by picking an open compact subgroup $K \subset \mathbb{A}^f$ of the finite adeles. This correspondence is in fact defined over $\mathbb{Z}[\frac{1}{2d}]$ and so may be reduced mod p for all $p \nmid 2d$.

Motivated by these examples, the orienting question of this thesis is to find a characteristic p analog of Mochizuki's theorem.

Question 2.1.13. *If $X \leftarrow Z \rightarrow Y$ is an étale correspondence of hyperbolic curves without a core over a field k of characteristic p, then is the correspondence is related to the reduction mod p of a Hecke correspondence of Shimura curves?*

Remark 2.1.14. The clause "is related to" in Question 2.1.13 is absolutely vital; in particular, there are examples of étale correspondence of hyperbolic curves without a core that should not deform to characteristic 0. Consider, for instance, the Hecke correspondence

$$Y(1) \xleftarrow{\pi_1} Y_0(2) \xrightarrow{\pi_2} Y(1)$$

of modular curves over $\mathbb{F}_p$, $p \neq 2$. Here, as above, $Y_0(2)$ parametrizes pairs of elliptic curves $E_1 \overset{2:1}{\to} E_2$ with a given degree-2 isogeny between them. By definition, there is a universal elliptic curve $\mathcal{E} \to Y(1)$. Let $\mathcal{G} = \mathcal{E}[p^\infty]$ be the associated p-divisible group over $Y(1)$. Note that $\pi_1^*\mathcal{G} \cong \pi_2^*\mathcal{G}$. Let X be the cover of $Y(1)$ that trivializes the finite flat group scheme $\mathcal{G}[p]_{ét}$ away from the supersingular locus of $Y(1)$. X is branched exactly at the supersingular points. Let Z be the analogous cover of $Y_0(2)$. Then we have the correspondence

$$X \leftarrow Z \rightarrow X$$

which does not have a core (the dynamics of an ordinary point are unbounded) and morally one does not expect this correspondence to lift to characteristic 0. This construction is sometimes referred to as adding *Igusa level structure* in the literature. See Buzzard [**B$^+$97**] for the analogous construction for Shimura curves parameterizing fake elliptic curves.

Remark 2.1.15. Ching-Li Chai has suggested a possible counterexample to Question 2.1.13 using certain leaves on a Hilbert modular variety. However, the counterexample is still related to a Hecke correspondence of Shimura varieties.

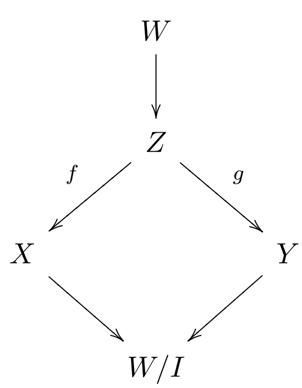
Note 2.1.16. In Corollary 2.2.11 we will reduce Question 2.1.13 to the analogous question with $k = \mathbb{F}$.

2.2. The Generic Graph and a Recursive Tower

We begin with a simple Galois-theoretic observation related to the existence of a core.

Lemma 2.2.1. *Let $X \leftarrow Z \rightarrow Y$ be a correspondence over k where Z is hyperbolic. A core exists if and only if there exists a curve $W \rightarrow Z$ such that the composite maps $W \rightarrow X$ and $W \rightarrow Y$ are both finite Galois.*

Proof. Suppose such a curve W existed. W is hyperbolic because it maps nontrivially to a hyperbolic curve. Then the groups $\mathrm{Aut}(W/X)$ and $\mathrm{Aut}(W/Y)$ are both subgroups of $\mathrm{Aut}_k(W)$, which is a finite group because W is hyperbolic. The group I generated by these Galois groups is therefore finite, and the curve W/I fits into a diagram:



Therefore the core exists. Conversely, if a core of the correspondence exists, call the coarse core C. Let W be a Galois closure of the map $Z \rightarrow C$. Then the composite maps $W \rightarrow X$ and $W \rightarrow Y$ are both Galois. □

Corollary 2.2.2. *Let $X \leftarrow Z \rightarrow Y$ be a correspondence of (possibly non-hyperbolic) curves over $\mathbb{F}$. Then a core exists if and only if there exists a curve $W \rightarrow Z$ such that the composite maps $W \rightarrow X$ and $W \rightarrow Y$ are both Galois.*

Proof. The proof is almost exactly the same as that of Lemma 2.2.1: the key observation is that everything in sight (including every element of $\mathrm{Aut}(W/X)$ and $\mathrm{Aut}(W/Y)$) may be defined over some finite field $\mathbb{F}_q$, therefore the group they generate inside of $\mathrm{Aut}(W)$ consists of automorphisms defined over $\mathbb{F}_q$ and is hence finite. □

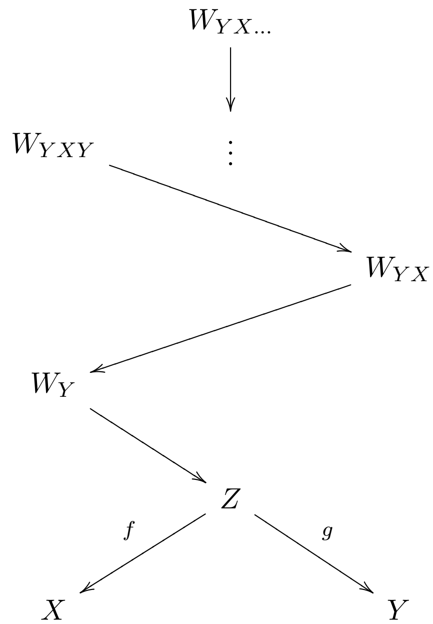
Remark 2.2.3. Lemma 2.2.1 gives a candidate for the fine core, as in Problem 2.1.7, if it exists. Namely, choose such a W and let I be the group generated by the Galois groups $\mathrm{Aut}(W/X)$ and $\mathrm{Aut}(W/Y)$ inside of the finite group $\mathrm{Aut}_k(W)$. A guess for the fine core $\mathcal{C}$ is the stack $[W/I]$. It has generically trivial isotropy group as I acts faithfully on W.

Example 2.2.4. Let us see the relevance both of Z being hyperbolic in Lemma 2.2.1 and of the base field being $\mathbb{F}$ in Corollary 2.2.2. Let $Z = \mathbb{P}^1_{\mathbb{F}_p(t)}$ and consider the following finite cyclic subgroups of $PGL(2, \mathbb{F}_p(t))$: H_1 is generated by the unipotent matrix $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$ and H_2 is generated by the transpose matrix $\left(\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right)$. Quotienting Z gives a correspondence $Z/H_1 \leftarrow Z \rightarrow Z/H_2$. Both arrows are Galois, but there is evidently no core because the subgroup of $PGL(2, \mathbb{F}_p(t))$ generated by H_1 and H_2 is infinite. Note that for every specialization of $t \in \mathbb{F}$, the correspondence does in fact have a core by Corollary 2.2.2.

Let $X \leftarrow Z \rightarrow Y$ be a correspondence of curves where Z is hyperbolic (or where $k = \mathbb{F}$). Motivated by Lemma 2.2.1 (resp. Corollary 2.2.2), we perform the following iterative procedure in the absence of a core: take a Galois closure of $Z \rightarrow Y$ and call it W_Y. Because we assumed a core does not exist, the associated map $W_Y \rightarrow X$ cannot be Galois by Lemma 2.2.1. Take a Galois closure of this map and call it W_{YX}. Again, the associated map $W_{YX} \rightarrow Y$ cannot be Galois, so we can take a Galois closure to obtain

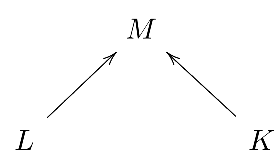
W_{YXY}. Continuing in the fashion, we get an inverse system of curves $W_{YX\dots}$.

(2.2.1)



Note that $W_{YX\dots}$ is Galois over Z. In fact, $W_{YX\dots}$ is Galois over both X and Y because there is a final system of Galois subcovers for each.

We explicate the based function-field perspective on this construction: let



be the associated diagram of function fields, where $M = k(Z)$, $L = k(X)$, and $K = k(Y)$. Recall that $L \cap K$ is a finite extension of k; this is exactly the condition that correspondence does not have a core.

Pick an algebraic closure Ω of M, i.e. let Ω be an algebraically closed field of transcendence degree 1 over k and pick *once and for all* an embedding of k-algebras $PQ : M \hookrightarrow \Omega$. (The notation will be justified later, when PQ will correspond to an edge of a graph.) Let E_K be the Galois closure of M/K in Ω. Then E_K/L is no longer Galois by Lemma 2.2.1. Let E_{KL} be the Galois closure of E_K/L in Ω. Continuing in this fashion, we get an infinite algebraic field extension $E_{KL\dots}$ of E, Galois over both L and K.

Lemma 2.2.5. *$W_{YX\dots}$ is isomorphic to $W_{XY\dots}$. That is, by reversing the roles of X and Y we get mutually final systems of Galois covers.*

PROOF. Equivalently, we must show that $E_{KL\dots} = E_{LK\dots}$ as subfields of Ω. First, note that $E_K \subset E_{LK}$ because E_K is the minimal extension of E in Ω that is Galois over K. Similarly, $E_{KL} \subset E_{LKL}$ because E_{KL} is the minimal extension of E_K in Ω that is Galois over L. Continuing, we see that $E_{KL\dots} \subset E_{LK\dots}$. By symmetry, the reverse inclusion holds as desired. $\square$

Corollary 2.2.6. *The field extension $E_{KL\dots} = E_{LK\dots}$ of M, thought of as a subfield of Ω, is characterized by the property that it is the minimal field extension of M inside of Ω that is Galois over both L and K.*

For brevity, we denote the inverse system $W_{XYX\dots}$ by W_∞. Let E_∞ be the associated function field, considered as a subfield of Ω. Finally, let $F_\infty \subset \Omega$ be the compositum of E_∞ and $\overline{k}$, thought of as subfields of Ω. In what follows, unless otherwise specified we consider $E_\infty \subset F_\infty \subset \Omega$ as inclusions of abstract k-algebras.

Question 2.2.7. *Does the "field of constants" of E_∞ have finite degree over k? That is, does $E \otimes_k \overline{k}$ decompose as an algebra to be the product of finitely many fields? What if the original correspondence is étale?*

If $\mathrm{Gal}(k^{sep}/k)$ is abelian and $\mathrm{Gal}(E_\infty/K)$ has finite abelianization, then Question 2.2.7 has an affirmative answer. In particular, this applies if $k \cong \mathbb{F}_q$ and $\mathrm{Gal}(E_\infty/K)$ is a simple group. We will see in Proposition 2.5.10 that if the correspondence has an *étale clump* , then Question 2.2.7 has an affirmative answer.

Remark 2.2.8. In Diagram 2.2.1, the morphism $W_Y \to Z$ is Galois. By precomposing with $\mathrm{Aut}(W_Y/Z)$, we equip W_Y with $\mathrm{Aut}(W_Y/Z)$-many maps to Z. More generally, all curves $W_{YX\dots X}$ will be naturally equipped with $\mathrm{Aut}(W_{YX\dots X}/Z)$-many maps to Z via precompositions by Galois automorphisms. This will be useful later, e.g. in Remark 2.3.3, when we try to explicitly understand the curves $W_{YX\dots Y}$.

The following Lemma allows us to *specialize* étale correspondences without a core.

Lemma 2.2.9. *Let S be an irreducible scheme of finite type with generic point η. Let X, Y, and Z be proper, smooth, geometrically irreducible curves over S. Suppose Z is "hyperbolic" over S; that is, the genus of a fiber is at least 2. Let $X \leftarrow Z \rightarrow Y$ an finite étale correspondence of schemes commuting with the structure maps to S. If s is a geometric point of S such that*

$$\begin{array}{ccc} & Z_s & \\ \swarrow & & \searrow \\ X_s & & Y_s \end{array}$$

has a core, then $X_\eta \leftarrow Z_\eta \rightarrow Y_\eta$ has a core.

Proof. First of all $X_\eta \leftarrow Z_\eta \rightarrow Y_\eta$ is a correspondence of curves over η. Let us suppose it does not have a core. Then the process of iterated Galois closure, as detailed above, continues endlessly to produce a tower of irreducible curves over η. Any finite étale morphism has a Galois closure. This implies that we can apply the construction of taking iterated Galois closures to the *correspondence of schemes* $X \leftarrow Z \rightarrow Y$ to build a tower $W_{YX\ldots}$ of smooth, proper, irreducible curves over S where all of the maps are finite étale exactly as in Diagram 2.2.1.

The fact that the maps $Z \rightarrow X$, $Z \rightarrow Y$, and $W_{YX\ldots} \rightarrow Z$ are finite étale implies that taking a Galois closure and then restricting to s yields a finite Galois étale cover of Z_s. For example, $(W_Y)_s$ is a (possibly disconnected) finite Galois cover of Y_s that maps surjectively to $(W_s)_{Y_s}$, the Galois closure of the map $Z_s \rightarrow Y_s$.

The curve $(W_{YX\ldots X})_\eta$ is mapped to surjectively by the corresponding curve in the tower over η. Therefore, if we could prove $(W_{YX\ldots X})_\eta$ were disconnected, we would get a contradiction with the original assumption that $X_\eta \leftarrow Z_\eta \rightarrow Y_\eta$ had no core.

As the correspondence specialized to s has a core, Lemma 2.2.1 implies that there exists a curve $W_{YX\ldots Y}$ of our tower such that the fiber $(W_{YX\ldots Y})_s$ is disconnected. We therefore have a smooth proper curve $W_{XY\ldots Y} \rightarrow S$ such that the fiber over s is disconnected. Zariski's connectedness principle implies $(W_{YX\ldots Y})_\eta$ is disconnected (this is where we use properness), contradicting our original assumption that $X_\eta \leftarrow Z_\eta \rightarrow Y_\eta$ had no core. $\square$

Remark 2.2.10. Example 2.2.4 shows that we need to assume that Z is hyperbolic in Lemma 2.2.9.

Corollary 2.2.11. *We may "reduce" the study of Question 2.1.13 to where $k = \mathbb{F}$. That is, given an étale correspondence of hyperbolic curves $X \leftarrow Z \rightarrow Y$ without a core over a field k of characteristic p, we can specialize to an étale correspondence without a core over $\mathbb{F}$.*

Proof. By spreading out, we may ensure that we are in the situation of Lemma 2.2.9. Then the nonexistence of a core implies the same for all of the geometric fibers by Lemma 2.2.9. $\square$

Remark 2.2.12. The argument of Lemma 2.2.9 *does not work* if the correspondence is not assumed to be étale.

Lemma 2.2.9 says that for an étale correspondence of projective hyperbolic curves, the property of "not having a core" specializes. The converse is true in more generality and is rather useful: it implies that one way to answer Question 2.1.13 is to directly lift the correspondence to characteristic 0.

Lemma 2.2.13. *Let $S = Spec(R)$ be the spectrum of a dvr with closed point s and generic point η. Let X, Y, and Z be smooth, projective, geometrically irreducible curves over S and let $X \leftarrow Z \rightarrow Y$ be a diagram of schemes, with morphisms finite and flat, that is a correspondence of curves when restricted to s and to η. If over η the correspondence has a core, then over s the correspondence has a core.*

Proof. Let π be a uniformizer of R. Denote by κ residue field of R and by K the fraction field of R. Pick a non-constant rational function f in the intersection $K(X) \cap K(Y)$; the intersection takes place in $K(Z)$. By multiplying by an appropriate power of π, we can guarantee that f extends to rational functions on the special fiber and in fact that f has nonzero reduction in $0 \neq \overline{f} \in \kappa(X_s) \cap \kappa(Y_s)$. Suppose f is constant modulo π, or equivalently that $f \equiv c (\mathrm{mod} \pi)$ for some $c \in R$. Then $\frac{f-c}{\pi}$ may again be reduced modulo π. If $\frac{f-c}{\pi}$ is non-constant on the special fiber, we are done, so suppose not and repeat the procedure. This procedure terminates because our original choice of $f \in K(X)$ was non-constant and the result will be a non-constant function in $\kappa(X_s) \cap \kappa(Y_s)$. $\square$

Corollary 2.2.14. *Let $X \leftarrow Z \rightarrow Y$ be an étale correspondence of projective hyperbolic curves without a core over $\mathbb{F}$. If this correspondence lifts to a correspondence of curves $\tilde{X} \leftarrow \tilde{Z} \rightarrow \tilde{Y}$ over $W(\mathbb{F})$, then X, Y, and Z are the reductions modulo p of Shimura curves.*

Proof. The lifted correspondence is automatically étale. Lemma 2.2.13 then implies that the general fiber does not have a core. Mochizuki's Theorem 2.1.8 then implies that $\tilde{X}$, $\tilde{Y}$, and $\tilde{Z}$ are all Shimura curves as desired. $\square$

We construct an infinite 2-colored graph $\mathcal{G}_{gen}^{full}$, which we call the full generic graph of the correspondence. The blue vertices of $\mathcal{G}_{gen}^{full}$ are the Ω-valued points of X; more precisely, a blue vertex is given by a horizontal arrow as follows such that the diagram commutes.

$$\begin{array}{ccc} k(X) & \longrightarrow & \Omega \\ & \nwarrow \quad \nearrow & \\ & k & \end{array}$$

Similarly, the red vertices are the Ω-valued points of Y and the edges are the Ω-valued points of Z. A red vertex $p : k(X) \hookrightarrow \Omega$ and blue vertex $q : k(Y) \hookrightarrow \Omega$ are joined by an edge if there exists an embedding $k(Z) \hookrightarrow \Omega$ that restricts to p and to q on the subfields $k(X)$ and $k(Y)$ respectively. Note that $\mathrm{Aut}_k(\Omega)$ naturally acts on the graph $\mathcal{G}_{gen}^{full}$ by post-composition.

Remark 2.2.15. The original correspondence is minimal iff there are no multiple edges in $\mathcal{G}_{gen}^{full}$. (Recall that the morphisms in correspondences were separable by definition.)

Definition 2.2.16. Given any subgraph $H \subset \mathcal{G}_{gen}^{full}$, we define the subfield $E_H \subset \Omega$ by taking the compositum of the subfields $e(k(Z)) \subset \Omega$, $p(k(X)) \subset \Omega$, and $q(k(Y)) \subset \Omega$ corresponding to all of the edges and vertices e, p, and q in H.

There is no reason to believe that $\mathcal{G}_{gen}^{full}$ is connected. We give $\mathcal{G}_{gen}^{full}$ a distinguished blue vertex P, red vertex Q, and edge PQ between them by picking the k-embedding

$$PQ : k(Z) \hookrightarrow \Omega$$

and we set the graph $\mathcal{G}_{gen}$ (the *generic graph*) to be the connected component of $\mathcal{G}_{gen}^{full}$ containing this distinguished edge. All connected components of $\mathcal{G}_{gen}^{full}$ arise in this way and all connected components of $\mathcal{G}_{gen}^{full}$ are isomorphic. We denote by $P(k(X))$ the image of the distinguished blue point P as a subfield of Ω and similarly for $Q(k(Y))$.

Remark 2.2.17. One is tempted to make a converse definition to Definition 2.2.16: given any subfield $E \subset \Omega$ (respectively $E \subset E_\infty$), define $\mathcal{G}_E^{full}$ (respectively $\mathcal{G}_E$) to be the subgraph of $\mathcal{G}_{gen}^{full}$ (respectively $\mathcal{G}_{gen}$) whose points and edges are have image inside of E. This definition is rather poorly behaved; for instance if one starts out with a finite connected subgraph $H \subset \mathcal{G}_{gen}$, takes $E_H \subset E_\infty$, and then looks at the associated graph G_{E_H}, there is no reason to believe that this graph is connected.

Lemma 2.2.18. *Let $H \subset \mathcal{G}_{gen}$ be the full subgraph consisting of all vertices of distance at most n from a fixed vertex v_0; that is, H is the closed ball $H = B(v_0, n)$. Then E_H is Galois over E_{v_0}.*

Proof. First of all, , E_{v_0} is the field corresponding to v_0 as in Definition 2.2.16. We may suppose WLOG that v_0 is a blue vertex, so $E_{v_0} = v_0(k(X))$ as v_0 is by definition a k-embedding $k(X)$ to Ω. In other words, v_0 gives Ω the structure of a $k(X)$-algebra. Now, E_H is the compositum of all of the fields associated to all of the edges and vertices in H in Ω. In particular, if $\mathcal{P} = \{P\}$ is the collection of all paths of length n starting at v_0, then E_H is the compositum of $(E_P)_{P\in\mathcal{P}}$ inside of Ω. Here each E_P and E_H has a natural $k(X)$-algebra structure via v_0.

Consider E_H together with the subfields E_P, $P \in \mathcal{P}$, as abstract $k(X)$-algebras. The compositum of the subfields E_P inside of E_H is exactly E_H. Let ϕ_0 be the original embedding $E_H \hookrightarrow \Omega$. To prove E_H is Galois over $k(X)$, we must show that for every

$$\phi \in \mathrm{Hom}_{k(X)}(E_H, \Omega)$$

the image of ϕ is contained in $\phi_0(E_H)$. Note that ϕ is determined by where all of the E_P are sent. Any ϕ can be obtained from ϕ_0 via an element of $\mathrm{Aut}(\Omega/k(X))$, as Ω is algebraically closed, and so a path P of length n originating at v_0 is sent to another such path P'. In other words, $\phi(E_P) = \phi_0(E_{P'})$ for another path P' of length n originating at v_0. As E_H was the compositum of all such E_P, it follows that the extension $E_H/k(X)$ is Galois as desired. □

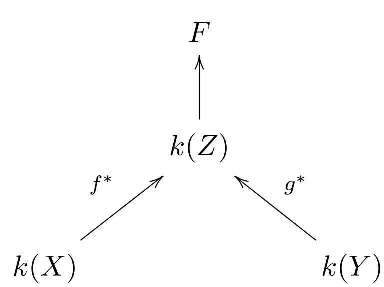
The graph $\mathcal{G}_{gen}$ is a full subgraph of $\mathcal{G}_{gen}^{full}$ so, as in Definition 2.2.16, we can take the associated field $E_{\mathcal{G}_{gen}} \subset \Omega$ given by the compositum of the subfields of Ω associated to the edges. A natural question arises: what is the relation between $E_{\mathcal{G}_{gen}}$ and E_∞? We record a basic observation.

Corollary 2.2.19. *The subfield $E_{\mathcal{G}_{gen}} \subset \Omega$ is Galois over both $P(k(X))$ and $Q(k(Y))$. Therefore $E_\infty \subset E_{\mathcal{G}_{gen}}$*

Proof. The connected graph $\mathcal{G}_{gen}$ is the union of the subgraphs $\cup_n B(P,n)$ of closed balls of radius n around P, so by Lemma 2.2.18 the field $E_{\mathcal{G}_{gen}}$ is Galois over $P(k(X))$. Similarly, $E_{\mathcal{G}_{gen}}$ is Galois over $Q(k(Y))$. Corollary 2.2.6 then implies that $E_\infty \subset E_{\mathcal{G}_{gen}}$ as desired. □

Lemma 2.2.20. *Let $X \leftarrow Z \rightarrow Y$ be a correspondence over k where Z is hyperbolic and embed the function fields into Ω via $PQ : k(Z) \hookrightarrow \Omega$. If there is a subfield $F \subset \Omega$ that is Galois over both $k(X)$ and $k(Y)$, then $E_{\mathcal{G}_{gen}} \subset F$.*

Proof. We have the following diagram of fields



where F is Galois over *both* $k(X)$ and $k(Y)$. The field F is naturally equipped with the structure of a $k(Z)$ algebra. Extend $PQ : k(Z) \hookrightarrow \Omega$ any which way to a $k(Z)$-algebra embedding $\phi : F \to \Omega$. Then the image of any edge adjacent to P in $\mathcal{G}_{gen}$ lives inside of the image $\phi(F)$ because F is Galois over $k(X)$. Similarly, the image of any edge adjacent to Q in $\mathcal{G}_{gen}$ lives inside the image of $\phi(F)$.

Let $v \neq Q$ be a vertex adjacent to P. There exists an automorphism $\alpha \in \mathrm{Gal}(F/P(k(X)))$ that sends $Q(k(Y))$ to E_v because F is Galois over $P(k(X))$. Conjugating by α, we deduce that F is Galois over E_v and hence the image of all edges emanating from v lie in F. By propagating, we get that $E_{\mathcal{G}_{gen}} \subset F$ as desired. □

Corollary 2.2.21. *We have an equality of fields $E_\infty = E_{\mathcal{G}_{gen}}$, considered as subfields of Ω.*

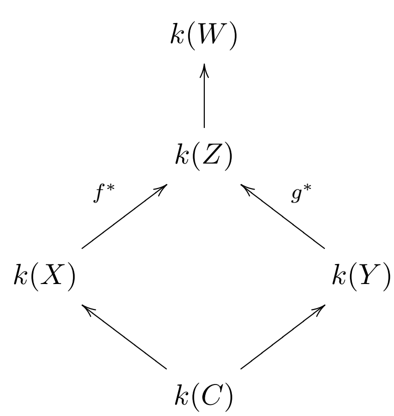
Proof. Combine Lemma 2.2.20 and Corollary 2.2.19. □

The graph $\mathcal{G}_{gen}$ informally reflects the "generic dynamics" of the correspondence. One way of making this precise is the following proposition.

Proposition 2.2.22. *Let $X \leftarrow Z \rightarrow Y$ be a correspondence over k where Z is hyperbolic. This correspondence has no core if and only if $\mathcal{G}_{gen}$ is an infinite graph.*

Proof. If $\mathcal{G}_{gen}$ is finite, then $E_{\mathcal{G}_{gen}}$ is a finite Galois extension of both $k(X)$ and $k(Y)$, so the correspondence has a core by Lemma 2.2.1.

Conversely, if the correspondence had a core, then let C be the coarse core. Let W be as in Lemma 2.2.1, with W Galois over C. We have the following diagram of fields, where we again fix $PQ : k(Z) \hookrightarrow \Omega$ and an extension $\phi : k(W) \hookrightarrow \Omega$.



Let v be blue vertex in $\mathcal{G}_{gen}$ adjacent to Q, given by a k-algebra embedding $k(X) \hookrightarrow \phi(k(W))$ by Lemma 2.2.20. As $\phi(k(W))/Q(k(Y))$ is Galois (because $k(W)/k(Y)$ is Galois and ϕ is an embedding of fields), there exists an automorphism

$$\alpha \in \mathrm{Gal}(\phi(k(W))/Q(k(Y))) \cong \mathrm{Gal}(k(W)/k(Y))$$

that sends $P(k(X))$ to E_v. As $k(C) \subset k(Y)$, this implies that E_v contains the image of $k(C)$. By propagating, we see that for every vertex v of $\mathcal{G}_{gen}$, E_v contains the image of $k(C)$.

Therefore, for every edge $e \in \mathcal{G}_{gen}$, we have that E_e contains the image of $k(C)$. The field extension $k(W)/k(C)$ is finite and separable, so there are only finitely many intermediate fields, and for any given intermediate field F there are only finitely many k-isomorphisms $k(Z) \to F$ because Z is hyperbolic. This implies there are only finitely many edges e in $\mathcal{G}_{gen}$, i.e. that $\mathcal{G}_{gen}$ is finite, as desired. □

Proposition 2.2.23. *For any edge $e \in \mathcal{G}_{gen}$, the field E_e lands in some $E_{KL\ldots K}$. Equivalently, $e(k(Z))$ is contained in a finite extension of $PQ(k(Z))$.*

Proof. To clarify the statement, recall that we defined the tower of fields $E_{KL\ldots K}$ with respect to the embedding $PQ : k(Z) \to \Omega$. The claim is that for any edge e in the connected component of $\mathcal{G}_{gen}^{full}$ containing PQ, the image $e(k(Z))$ lands in an subfield of Ω that is finite over the field $PQ(k(Z))$. This implies the proposition as stated because of the following two facts

- $e(k(Z))$ lands inside of E_∞, which is exhausted by fields of the form $E_{KL\ldots K}$, by Lemma 2.2.20
- $k(Z)$ is finitely generated over k.

Once the rather confusing statement is unraveled, the proposition is straightforward. Any edge e of $\mathcal{G}_{gen}$ is a finite distance, say r, from the edge PQ. We prove the result by induction on r. In the base case of $r = 0$, $e = PQ$ and so $e(k(Z)) = PQ(k(Z))$ is finite over $PQ(k(Z))$. Suppose we know the result for all edges of distance less than r from PQ. The edge e shares a vertex with some edge f of distance $r-1$ from PQ. This implies that the compositum of $e(k(Z))$ and $f(k(Z))$ inside of Ω is a finite extension of both. Let

$F \subset \Omega$ be a finite extension of $PQ(k(Z))$ containing $f(k(Z))$. Then the compositum of F and $e(k(Z))$ is a finite extension of both, as desired. $\square$

Remark 2.2.24. We take a brief digression into the structure of automorphism groups of fields. Let Ω be any field. We endow the group $\mathrm{Aut}(\Omega)$ with the compact-open topology, considering Ω to be a discrete set. Given any finite subset $S \subset \Omega$, the subgroup $\mathrm{Stab}(S) \subset \mathrm{Aut}(\Omega)$ is an open subgroup and as S ranges these form a neighborhood base of the identity in $\mathrm{Aut}(\Omega)$. If $K \subset \Omega$ is a separable Galois extension with K finitely generated over its prime field, the natural map $\mathrm{Gal}(\Omega/K) \subset \mathrm{Aut}(\Omega)$ is an open embedding of topological groups; in other words, the topology just defined is compatible with the usual profinite topology on Galois groups.

Note that this procedure generalizes: if $k \subset \Omega$ is a field extension, we may give the group $\mathrm{Aut}_k(\Omega)$ has the structure of a topological group, where a neighborhood base of the identity is given by $\mathrm{Stab}(S)$ for finite subsets $S \subset \Omega\backslash k$. However, $\mathrm{Aut}_k(\Omega) \subset \mathrm{Aut}(\Omega)$ *is not* an open embedding unless k is finitely generated over its prime field.

Any element $g \in \mathrm{Aut}_k(E_\infty)$ gives a map of graphs $\mathcal{G}_{gen} \to \mathcal{G}_{gen}^{full}$ by post-composition: for instance, an edge $e : k(Z) \to E_\infty \subset \Omega$ is sent to the edge $g \circ e : k(Z) \to E_\infty \subset \Omega$. In fact, the Galois groups $G_P := \mathrm{Gal}(E_\infty/P(k(X)))$ and $G_Q := \mathrm{Gal}(E_\infty/Q(k(Y)))$ actually act on the connected graph; $g \in G_P$ sends an edge $e : k(Z) \to E_\infty \subset \Omega$ to $g \circ e : k(Z) \to E_\infty \subset \Omega$, and $g \circ e$ is an edge of the connected graph $\mathcal{G}_{gen}$ because g fixes P.

Definition 2.2.25. Let $A \subset \mathrm{Aut}_k(E_\infty)$ be the subgroup of $\mathrm{Aut}_k(E_\infty)$ sends $\mathcal{G}_{gen}$ to itself with the induced topology, as in Remark 2.2.24. Let $A^{PQ} \subset A$ be the subgroup of A generated by G_P and G_Q with the induced topology from A.

Remark 2.2.26. The topology on A^{PQ} is uniquely determined from declaring the compact subgroups G_P and G_Q to be open.

By definition, A acts faithfully on $\mathcal{G}_{gen}$: if $g \in A$ acts trivially on $\mathcal{G}_{gen}$, then it acts trivially on the field generated by all of the vertices and the edges of $\mathcal{G}_{gen}$, i.e. it is the trivial automorphism of E_∞. If we give $\mathcal{G}_{gen}$ the discrete topology, A^{PQ} acts continuously on $\mathcal{G}_{gen}$. Let $d = \deg(Z \to X)$ and $e = \deg(Z \to Y)$. Then the degree of a blue vertex is

d and the degree of a red vertex is e. Moreover, G_P acts transitively on the edges coming out of P by Galois theory and similarly G_Q acts transitively on the edges coming out of Q. By conjugating we see that $A^{PQ} \subset \mathrm{Aut}(\mathcal{G}_{gen})$ acts transitively on the edges coming out of any vertex. Therefore the group A^{PQ} acts transitively on the edges of $\mathcal{G}_{gen}$, subject to the constraint that colors of the vertices are preserved. This is recorded in the following corollary.

Corollary 2.2.27. *In the notation above, A^{PQ} and hence also A act transitively on the edges of $\mathcal{G}_{gen}$, subject to the constraint that the colors of the vertices are preserved. We say the pair $(\mathcal{G}_{gen}, A^{PQ})$ is colored-edge-symmetric.*

Question 2.2.28. *If $X \leftarrow Z \rightarrow Y$ is a minimal correspondence with no core, does $\mathcal{G}_{gen}$ have any cycles? What if it is étale?*

The graph $\mathcal{G}_{gen}$ being a tree has consequences for the structure of the group A^{PQ}. To state these, we need a theorem of Serre.

Theorem 2.2.29. *(Serre) Let G be a group acting on a graph X, and let e be an edge of X connecting vertices p and q. Suppose that e is a fundamental domain for the action. Let G_p, G_q, and G_e be the stabilizers in G of p, q, and e respectively. Then the following are equivalent.*

(1) *X is a tree*
(2) *The homomorphism $G_p *_{G_e} G_q \rightarrow G$ induced by the inclusions $G_p \rightarrow G$ and $G_q \rightarrow G$ is an isomorphism*

PROOF. This is a direct translation of Théorm 6 on Page 48 of [**SB77**]. $\square$

Proposition 2.2.30. *Suppose $\mathcal{G}_{gen}$ is a tree. Then the natural map $G_P *_{G_{PQ}} G_Q \rightarrow A^{PQ}$ is an isomorphism of topological groups.*

PROOF. There is no element $a \in A^{PQ}$ that flips any edge e of $\mathcal{G}_{gen}$ because A^{PQ} preserves the coloring. By Corollary 2.2.27, the segment PQ is a fundamental domain for the action of A^{PQ} on $\mathcal{G}_{gen}$. Therefore, by Serre's Theorem, the fact that $\mathcal{G}_{gen}$ is a tree implies the induced map $G_P *_{G_{PQ}} G_Q \to A^{PQ}$ is an isomorphism of abstract groups. The group $G_P *_{G_{PQ}} G_Q$ has a natural topology generated by the topologies of G_P and G_Q, and endowed with this topology the above map is an isomorphism of topological groups. $\square$

When $\mathcal{G}_{gen}$ is a tree, we may describe the pair $(\mathcal{G}_{gen}, A^{PQ})$ in a different way. Given any compact open subgroup $G \subset A^{PQ}$ and any vertex $v \in \mathcal{G}_{gen}$, the orbit $G.v$ is compact and discrete (as we gave $\mathcal{G}_{gen}$ the discrete topology) and is hence finite. Therefore G acts on a finite subtree T of $\mathcal{G}_{gen}$ and hence the action factors through a finite quotient H of G. Therefore we have a finite group H acting on a finite tree T.

Lemma 2.2.31. *A finite group H acting on a finite tree T always as a fixed point (though not necessarily a fixed vertex.)*

Proof. This is well known and Aaron Bernstein gave the following elegant proof.

Let the height $h(v)$ of a vertex v be the maximal distance from v to any leaf. Any automorphism of T preserves heights. If there is a unique vertex v of minimal height, we are done, so suppose there is another vertex w of minimal height. Then v and w must be connected by an edge: if the unique path between them contained an intermediate vertex u, then some thought shows that $h(u) < h(v)$. As T is a tree, there can be *at most two* vertices of minimal height. If there are two, then their midpoint is a fixed point for *any* automorphism of T. $\square$

Therefore there must be a fixed point $p \in T$; here T is thought of as a topological space. If p were not a vertex T, H would in fact fix the two neighboring vertices of the edge p is on because H respects the coloring of the graph. Therefore G fixes at least one vertex v. On the other hand, given any vertex v, the subgroup G_v fixing v is a compact open subgroup. Therefore, the vertices of $\mathcal{G}_{gen}$ may be thought of as maximal compact subgroups G of A^{PQ}.

Corollary 2.2.32. *If $\mathcal{G}_{gen}$ is a tree, any maximal compact open subgroup G of A^{PQ} is conjugate to either G_P or G_Q.*

Proof. The discussion above shows that every maximal compact open subgroup G of A^{PQ} is G_v for some vertex v of $\mathcal{G}_{gen}$. The group G_v is conjugate in A^{PQ} to G_P or G_Q by

Corollary 2.2.27. Finally, G_P is not conjugate to G_Q in A^{PQ} because the action of A^{PQ} on $\mathcal{G}_{gen}$ preserves the coloring. $\square$

We may similarly describe the adjacency relation in $\mathcal{G}_{gen}$: a blue vertex G_1 and a red vertex G_2 are joined by an edge if the intersection $G_1 \cap G_2$ (inside of A^{PQ}) has index d inside of G_1 and index e inside of G_2.

Remark 2.2.33. If $\mathcal{G}_{gen}$ is a tree, then the action of A^{PQ} on $\mathcal{G}_{gen}$ is the conjugation action on the maximal compact subgroups.

2.3. Specialization of Graphs and Special Orbits

Given a correspondence $X \xleftarrow{f} Z \xrightarrow{g} Y$ over a field k, we have defined an undirected 2-colored graph $\mathcal{G}_{gen}^{full}$, the *full generic graph,* using an algebraically closed overfield Ω. We can also define the $\mathcal{G}_{phys}^{full}$, the *full physical graph*, which will be an undirected 2-colored graph, using $\overline{k}$. The goal of this section is to speculate on the behavior of "specialization maps" $s_{\tilde{z}} : \mathcal{G}_{gen} \to \mathcal{G}_{phys,z}$ when the curves are proper; informally, if we think of $\mathcal{G}_{gen}$ as the "graph of generic dynamics", this map specializes to the graph associated to the dynamics of a physical point $z \in Z(\overline{k})$.

Definition 2.3.1. Given a correspondence $X \xleftarrow{f} Z \xrightarrow{g} Y$ of curves over k, the full physical graph $\mathcal{G}_{phys}^{full}$ is the following 2-colored graph. The edges are the points $z \in Z(\overline{k})$, the blue vertices are the points $X(\overline{k})$ and the red vertices are the points $Y(\overline{k})$. Adjacent to $z : \mathrm{Spec}(\overline{k}) \to Z$ is the blue vertex $f \circ z \in X(\overline{k})$ and the red vertex $g \circ z \in Y(\overline{k})$. Given a choice of $z \in Z(\overline{k})$, we denote by the $\mathcal{G}_{phys,z}$ the connected component of $\mathcal{G}_{phys}^{full}$ that contains z.

Recall the construction of $\mathcal{G}_{gen}$: pick an edge $PQ \in Z(\Omega)$ of $\mathcal{G}_{gen}^{full}$ and define $\mathcal{G}_{gen}$ to be the connected component of $\mathcal{G}_{gen}^{full}$ that contains PQ, suppressing the implicit PQ in the notation. The field $E_\infty \subset \Omega$ is the compositum of all of the points and edges of of $\mathcal{G}_{gen}$, thought of as subfields of Ω, by Corollary 2.2.21. Therefore, an edge e of $\mathcal{G}_{gen}$ yields an element of the set $Z(E_\infty)$. Similarly, a blue vertex v_b of $\mathcal{G}_{gen}$ yields an element of $X(E_\infty)$ and a red vertex v_r yields an element of $Y(E_\infty)$.

We first spell out exactly what is fixed in the construction of a specialization map. First of all, we assume the curves X, Y, and Z are proper over k. Pick $z \in Z(\overline{k})$. Then pick a point $\tilde{z} \in W_\infty(\overline{k})$, a geometric point of the scheme W_∞ i.e. a compatible system of geometric points on the tower defining W_∞, lying over z. Taking the image of $\tilde{z}$ gives closed point of the scheme W_∞, and the ring $\mathcal{O}_{W_\infty,\tilde{z}}$ is a valuation ring because it is the filtered colimit of valuation rings. Moreover, the fraction field of $\mathcal{O}_{W_\infty,\tilde{z}}$ is E_∞. The choice of $\tilde{z} : \mathrm{Spec}(\overline{k}) \to W_\infty$ yields a morphism $\pi : \mathcal{O}_{W_\infty,\tilde{z}} \to \overline{k}$. We now construct a specialization map

$$s_{\tilde{z}} : \mathcal{G}_{gen} \to \mathcal{G}_{phys}^{full}$$

Let e be an edge of $\mathcal{G}_{gen}$. As discussed above, e yields an element of $Z(E_\infty)$. We want to describe $s_z(e_\infty)$, the image of e, in $\mathcal{G}_{phys,z}$. We have the following diagram; the dotted arrow exists uniquely because the structure map $Z \to \mathrm{Spec}(k)$ is proper.

$$\begin{array}{ccc} \mathrm{Spec}(E_\infty) & \xrightarrow{\;e\;} & Z \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(\mathcal{O}_{W_\infty,\tilde{z}}) & \longrightarrow & \mathrm{Spec}(k) \end{array}$$

Composing π with the dotted arrow, we get a point $\overline{e} \in Z(\overline{k})$. We set $s_{\tilde{z}}(e) = \overline{e}$. The exact same construction works with (red and blue) vertices, and the result is manifestly a map of graphs. Moreover, as $\mathcal{G}_{gen}$ is connected, so is the image. Finally, the edge $PQ \in Z(E_\infty)$ is sent to z. Therefore, we have constructed a map

$$s_{\tilde{z}} : \mathcal{G}_{gen} \to \mathcal{G}_{phys,z}$$

Lemma 2.3.2. *Let $X \xleftarrow{f} Z \xrightarrow{g} Y$ be an étale correspondence of hyperbolic curves without a core over a field k. Then all of the specialization maps are surjective.*

$$s_{\tilde{z}} : \mathcal{G}_{gen} \to \mathcal{G}_{phys,z}$$

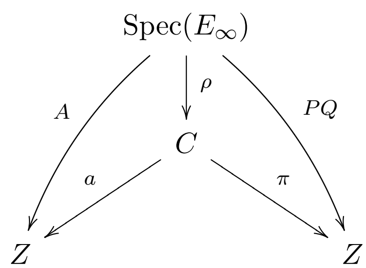
Proof. Because the correspondence is étale, each blue vertex of $\mathcal{G}_{phys}$ is adjacent to $d = \deg(f)$ edges and each red vertex is adjacent to $e = \deg(g)$ edges. It is therefore equivalent to show that no two adjacent edges of $\mathcal{G}_{gen}$ are sent to the same edge in $\mathcal{G}_{phys,z}$.

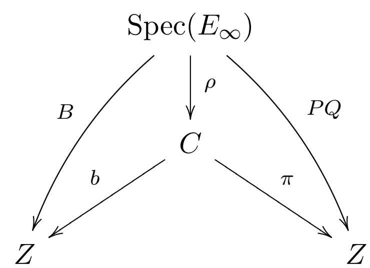
Let A and B be two edges sharing the blue vertex p. We want to show that A and B are not sent to the same edge in $\mathcal{G}_{phys,z}$.

Recall that A and B yield elements of $Z(E_\infty)$ such that $f \circ A = f \circ B = p \in X(E_\infty)$. Proposition 2.2.23 implies that, after possibly enlarging k, there exists an irreducible curve C over k together with maps $\rho : \mathrm{Spec}(E_\infty) \to C$, $\pi : C \to Z$, and $a, b : C \to Z$ such that

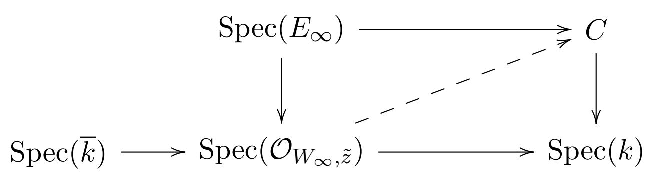
- $\pi \circ \rho = PQ$ considered as elements of $Z(E_\infty)$
- A and B factor through C via a and b.

In the language of Proposition 2.2.23, C is the curve associated to the field $E_{KL\ldots K}$ of transcendence degree 1 over k. More explicitly, we have the following factorizations:

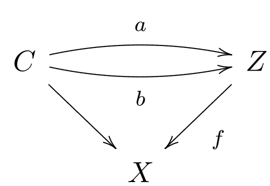




Moreover, the maps π, a, and b are all finite étale. Let us follow the specialization construction. Again, the dotted arrow exists because $C \to \mathrm{Spec}(k)$ is proper.



This diagram gives us a point $x \in C(\overline{k})$ by composition with the dotted arrow. If A and B are identified under the specialization map, $a(x) = b(x) \in Z(\overline{k})$. Now, $f \circ a = f \circ b$ because A and B shared the vertex p, so we have the following diagram.



But C is irreducible and the maps a, b, and f are finite étale, so the assumption that $a(x) = b(x)$ implies that $a = b$ and hence $A = B$, as desired.

$\square$

Remark 2.3.3. The graph $\mathcal{G}_{phys}$ helps describe the tower W_∞. In this remark, we suppose all morphisms are unramified at all points specified. For instance, let $\xi_Y \in W_Y(\overline{k})$ map to $z \in Z(\overline{k})$ which maps to $y \in Y(k)$. Then, as in Remark 2.2.8, there are naturally $\mathrm{Aut}(W_Y/Z)$ many maps from W_Y to Z and we can look at the images of ξ_Y under these maps. In this way, ξ_Y yields the graph of all edges coming out of y in $\mathcal{G}_{phys,z}$. More generally, a point $\xi_{YX\ldots Y} \in W_{YX\ldots Y}(\overline{k})$ which maps to $y \in Y(k)$ under the natural map yields the subgraph of $\mathcal{G}_{phys,z}$ with center y and radius n, where n is the number of letters in the string "$YX\ldots Y$".

We will use this Remark in Proposition 2.5.10 to show that if a *étale clump* exists, then Question 2.2.7 has an affirmative answer.

Consider the Hecke correspondence of open modular curves over $\mathbb{F}_p$

$$\begin{array}{ccc} & Y_0(l) & \\ \swarrow & & \searrow \\ Y(1) & & Y(1) \end{array}$$

The graph $\mathcal{G}_{gen}$ is a tree. For $z \in Y_0(l)(\mathbb{F})$ an ordinary point, both $\mathcal{H}_{phys,z}$ and $\mathcal{G}_{phys,z}$ have at most one cycle: this follows from the work in David Kohel's thesis [**Koh96**], summarized by Andrew Sutherland [**Sut13**]. They call this structure an *Isogeny Volcano.* The cycle comes from the following fact: given an imaginary quadratic field $K/\mathbb{Q}$, there exists an elliptic curve $E/\mathbb{F}$ with multiplication by the maximal order $\mathcal{O}_K$. On the other hand, there are only finitely many supersingular points, and in fact if $\mathcal{G}_{phys,z}$ contains one supersingular point it contains all of them.

Definition 2.3.4. Given an étale correspondence of hyperbolic curves

$$\begin{array}{ccc} & Z & \\ \swarrow & & \searrow \\ X & & Y \end{array}$$

over k without a core, we say a point $z \in Z(\overline{k})$ is *special* if there exists (equivalently for all) $\tilde{z} \in W_\infty(\overline{k})$ over z such that the map $s_{\tilde{z}} : \mathcal{G}_{gen} \to \mathcal{G}_{phys,z}$ is not an isomorphism. We say $z \in Z(\overline{k})$ is *generic* if the it is not special.

Question 2.3.5. *Let $X \leftarrow Z \rightarrow Y$ be an étale correspondence of hyperbolic curves without a core over $\mathbb{F}_q$.*

(1) *Is there always $z \in Z(\mathbb{F})$ that is generic?*
(2) *Is there always a special point that is not part of a clump?*
(3) *Suppose $\mathcal{G}_{gen}$ is free. For every point $z \in Z(\mathbb{F})$, is $\pi_1(\mathcal{G}_{phys,z})$ finitely generated? If $\mathcal{G}_{phys,z}$ is infinite, does $\mathcal{G}_{phys,z}$ have at most one cycle?*
(4) *What is $\lim_{n\to\infty} \frac{|z \in Z(\mathbb{F}_{q^n}) \text{ with } z \text{ generic}|}{|Z(\mathbb{F}_{q^n})|}$?*

Remark 2.3.6. It is very likely that one can show the set of special points is countable. Hence the first part of Question 2.3.5 has an affirmative answer if we replace $\mathbb{F}$ with an uncountable field, for instance $\mathbb{C}$. Because "being a special point with a cycle of less than n" is an algebraic condition, all special points should be defined over the algebraic closure of a prime field.

2.4. Symmetric Correspondences

Definition 2.4.1. A symmetric correspondence is a self-correspondence $X \stackrel{f}{\leftarrow} Z \stackrel{g}{\rightarrow} X$ such that there is an involution $w \in \mathrm{Aut}(Z)$ with $f \circ w = g$, i.e. w swaps f and g. We denote by w^* the induced involution on $k(Z)$.

Note that if the correspondence is minimal, w is unique if it exists. Therefore being symmetric is a property and not a structure of a minimal correspondence.

Lemma 2.4.2. *Let $X \stackrel{f}{\leftarrow} Z \stackrel{g}{\rightarrow} X$ be a symmetric correspondence without a core where Z is hyperbolic. Any $w \in Aut(Z)$ that swaps f and g lifts to an automorphism $\tilde{w}$ of W_∞. We denote by $\tilde{w}^*$ the associated automorphism of $k(W_\infty)$.*

Proof. We proceed exactly as in the discussion at the beginning of Section 2.2: let W_f (resp. W_g) denote a Galois closure of arrow f (resp. g). The automorphism w of Z

swaps f and g and hence we can choose an isomorphism $w_1 : W_g \to W_f$ living over w on Z:

$$\begin{array}{ccc} W_g & \xrightarrow{w_1} & W_f \\ \downarrow & & \downarrow \\ W & \xrightarrow{w} & W \end{array}$$

Similarly, we can chose an isomorphism $w_2 : W_{gf} \to W_{fg}$ living over w on Z, again because w swaps the roles of f and g. Continuing in this fashion, we get an isomorphism of towers

$$\tilde{w} : W_{gf...} \to W_{fg...}$$

By Lemma 2.2.5, $W_{fg...}$ is isomorphic to W_{gf} as a pro-curve over W and we may think of $\tilde{w}$ as an automorphism of W_∞ living over $w \in \mathrm{Aut}(Z)$. □

Remark 2.4.3. Another way of phrasing Lemma 2.4.2 is as follows. If $X \stackrel{f}{\leftarrow} Z \stackrel{g}{\rightarrow} X$ is a symmetric correspondence without a core with Z hyperbolic, then for any choice of symmetry w, the following map is Galois.

$$W_\infty \to Z/ < w >$$

From this perspective, it is clear that the lift $\tilde{w}$ is not unique.

Definition 2.4.4. Let $X \stackrel{f}{\leftarrow} Z \stackrel{g}{\rightarrow} X$ be a symmetric correspondence without a core where Z is hyperbolic. Pick a symmetry w and a lift $\tilde{w}$ to W_∞, which exists by Lemma 2.4.2. Let $\tilde{w}^*$ be the associated automorphism of E_∞. Define $A^{\tilde{w}} \subset \mathrm{Aut}_k(E_\infty)$ be the subgroup generated by G_P, G_Q, and $\tilde{w}^*$. Equivalently, $A^{\tilde{w}}$ is the subgroup of A generated by A^{PQ} (as in Definition 2.2.25) and $\tilde{w}^*$. We give the subgroup $A^{\tilde{w}} \subset A$ the induced topology from A.

Corollary 2.4.5. *Let $X \stackrel{f}{\leftarrow} Z \stackrel{g}{\rightarrow} X$ be a symmetric correspondence without a core where Z is hyperbolic with symmetry w and let $\tilde{w}$ be a lift of the symmetry to W_∞. Then $A^{\tilde{w}}$ and hence A acts transitively on the oriented edges of $\mathcal{G}_{gen}$.*

Proof. Corollary 2.2.27 says that A^{PQ} acts transitively on $\mathcal{G}_{gen}$ subject to the constraint that the colors of the vertices are preserved. Recall that we constructed $\mathcal{G}_{gen}$

by picking an Ω-valued point of Z, $PQ : k(Z) \hookrightarrow \Omega$, and taking the connected component of $\mathcal{G}_{gen}^{full}$ that contains PQ. Once we have chosen PQ we may consider it as a map $PQ : k(Z) \hookrightarrow E_\infty \subset \Omega$ because the choice of PQ determines a subfield $E_\infty \subset \Omega$. The automorphism $\tilde{w}^* \in \mathrm{Aut}(E_\infty)$ swaps the points P and Q. By conjugating we get that $A^{\tilde{w}}$ acts transitively on the edges of $\mathcal{G}_{gen}$, in the usual sense of remembering the endpoints. □

In the following we will occasionally restrict to the case of a symmetric type (3,3) correspondence, where we can prove several results: Lemma 2.4.8, Proposition 2.4.11, and Corollary 2.5.9.

Definition 2.4.6. A Tutte object is a pair (G, A) where G is a 3-regular connected graph and A is a group of automorphisms of G. (G, A) is said to be (sharply) s-transitive if A acts (sharply) transitively on all s-arcs. (G, A) is said to be ∞-transitive if it is s-transitive for all $s \geq 1$.

In this language, the pair $(\mathcal{G}_{gen}, A)$ is a Tutte object that is 1-transitive.

Theorem 2.4.7. *(Tutte) A Tutte object (G, A) which is s-transitive and not $s+1$-transitive is sharply s-transitive.*

Proof. The proof is exactly the same as in 7.72 in Tutte's book Connectivity in Graphs [**Tut66**]. Alternatively, see Djokovi and Miller [**DM80**], Theorem 1, for exactly this statement. □

Lemma 2.4.8. *Given a type (3,3) symmetric correspondence $X \leftarrow Z \rightarrow X$ without core where Z is hyperbolic, then the pair $(\mathcal{G}_{gen}, A^{\tilde{w}})$ is ∞-transitive and $\mathcal{G}_{gen}$ is a tree.*

Proof. Suppose $\mathcal{G}_{gen}$ had a cycle. The graph $\mathcal{G}_{gen}$ is infinite by Proposition 2.2.22. Then the pair $(\mathcal{G}_{gen}, A^{\tilde{w}})$ is 1-transitive, so there exists an n with $(\mathcal{G}_{gen}, A^{\tilde{w}})$ is n-transitive but not $n+1$-transitive. Therefore, to prove $\mathcal{G}_{gen}$ is a tree it suffices to prove that the pair $(\mathcal{G}_{gen}, A^{\tilde{w}})$ is ∞-transitive.

Suppose $(\mathcal{G}_{gen}, A^{\tilde{w}})$ was not ∞-transitive. Then there exists a positive integer n such that $(\mathcal{G}_{gen}, A^{\tilde{w}})$ is n-transitive but not $n+1$-transitive because the graph is infinite, connected and 1-transitive. Theorem 2.4.7 implies that the Tutte object $(\mathcal{G}_{gen}, A^{\tilde{w}})$ is then sharply n-transitive, i.e. there exists a unique automorphism in $A^{\tilde{w}}$ sending any n-arc to any other n-arc. Therefore any automorphism in $A^{\tilde{w}}$ that fixes any given n-arc must be the identity automorphism. To any n-arc R I can associate the field E_R which is the field generated by the images of the points and edges inside of E_∞ as in Definition 2.2.16. Pick the n-arc R through P so that E_R is a finite extension of $P(k(X))$. Note that E_∞ is Galois over E_R. The group $\mathrm{Gal}(E_\infty/E_R)$ acts faithfully on $\mathcal{G}_{gen}$ and fixes R. As $(\mathcal{G}_{gen}, A^{\tilde{w}})$ is sharply n-transitive, the group $\mathrm{Gal}(E_\infty/E_R)$ acts trivially on $\mathcal{G}_{gen}$. Therefore $F = E_\infty$ is a finite extension $k(Z)$, Galois over both $k(X)$ and $k(Y)$, which contradicts Lemma 2.2.1 as we assumed our correspondence did not have a core. □

Lemma 2.4.8 poses the following refinement to Question 2.2.28 on whether or not $\mathcal{G}_{gen}$ is a tree.

Question 2.4.9. *Let $X \leftarrow Z \rightarrow X$ be a minimal, symmetric, étale correspondence with no core and Z hyperbolic. Is the pair $(\mathcal{G}_{gen}, A)$ ∞-transitive?*

We now give the definition of a recursive tower. This tower will be used in Section 2.5.

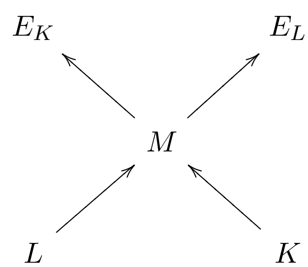
Definition 2.4.10. Given a correspondence $X \overset{f}{\leftarrow} Z \overset{g}{\rightarrow} Y$, we define the recursive tower of (possibly singular) curves S_n as follows.

- Let $Z \times_Y Z = \Delta \cup T_1$. Note that we have a correspondence $Z \overset{p_1}{\leftarrow} T_1 \overset{p_2}{\rightarrow} Z$.
- Let $Z \times_X Z = \Delta \cup T_2$. Note that we have a correspondence $Z \overset{q_1}{\leftarrow} T_2 \overset{q_2}{\rightarrow} Z$.
- Let $S_1 = Z$ and $S_2 = T_1$.
- Let $S_3 = T_1 \times_{p_2,Z,q_1} T_2$.
- Let $S_4 = T_1 \times_{p_2,Z,q_1} T_2 \times_{q_2,Z,p_1} T_1$ and continue in this fashion.

Note that the Ω-points of the reducible curve $Z \times_Y Z$ correspond to pairs of edges in $\mathcal{G}_{gen}^{full}$ that share a common red vertex. Therefore, the Ω-points of $S_2 = T_1$ are adjacent pairs of edges in $\mathcal{G}_{gen}^{full}$ that share a common red vertex, i.e. paths of length two in $\mathcal{G}_{gen}^{full}$ starting from a blue vertex that do not backtrack. Continuing, the Ω-points of S_n will be in bijective correspondence with paths in $\mathcal{G}_{gen}^{full}$ of length n that never backtrack and start with a blue vertex. Informally, we call S_n the curve of "paths of length n that never backtrack".

Proposition 2.4.11. *Given a type (3,3) symmetric correspondence $X \leftarrow Z \rightarrow X$ without core where Z is hyperbolic, $X \overset{f}{\leftarrow} Z \overset{g}{\rightarrow} X$, every curve S_n is irreducible.*

Proof. Because neither map in the correspondence is Galois and the degree is 3, T_1 may be identified with a Galois closure of the map $Z \overset{g}{\rightarrow} X$ and is hence irreducible. Similarly, T_2 is irreducible. Let be E_K the function field of T_1 and E_L the function field of T_2. Correspondingly, we have the following diagram of fields.



Then the "function algebra" of S_n is the k-algebra $E_K \otimes_M E_L \otimes_M E_K \cdots \otimes E_L$ and the goal is to prove this algebra is a field. Note that this algebra has degree $3 \times 2^{n-1}$over L: both $[E_K : M]$ and $[E_L : M]$ are 2. Pick a path R of length n in $\mathcal{G}_{gen}$ originating at P. There are $3 \times 2^{n-1}$ such paths, and A acts transitively on them by Lemma 2.4.8. In fact G_P acts transitively on them because G_P is precisely the subgroup of A fixing P. The stabilizer of R in G_P therefore has index $3 \times 2^{n-1}$ inside of G_P, and we call the field associated to this stabilizer F_R. In fact, F_R is exactly the compositum of the images of all of the edges and vertices of R in E_∞ (if we were strictly following the notation of Definition 2.2.16, we would call this field E_R; we switch to "F_R" to avoid overloading the letter "E".)

There is a natural surjective map of L-algebras $E_K \otimes_M E_L \otimes_M E_K \cdots \otimes E_L \rightarrow F_R$ because F_R is defined to the the compositum of all of the fields which occur in the path R. This an isomorphism because both sides have the same dimension as L-vector spaces. □

Corollary 2.4.12. *Let $X \overset{f}{\leftarrow} Z \overset{g}{\rightarrow} X$ be a minimal symmetric correspondence without a core with Z hyperbolic. Suppose the pair $(\mathcal{G}_{gen}, A)$ is ∞-transitive. Suppose further that the curve S_2 of the correspondence is irreducible. Then all of the curves S_n are irreducible.*

Proof. The proof is identical to that of Proposition 2.4.11. □

2.5. The Directed Physical Graph and Clumps

Definition 2.5.1. Let $X \overset{f}{\leftarrow} Z \overset{g}{\rightarrow} Y$ be a correspondence over a field k. A *clump* S is a finite set of $\overline{k}$ points $S \subset Z(\overline{k})$ such that $f^{-1}(f(S)) = g^{-1}(g(S)) = S$. A clump S is *étale* if f and g are étale at all points of S.

If $X \overset{f}{\leftarrow} Z \overset{g}{\rightarrow} Y$ has a core, then as in Remark 2.1.6 every $z \in Z(\overline{k})$ is contained in a clump. In the language of Remark 2.1.6, a clump is a *bounded orbit*.

Note that in the case of a Hecke correspondence of projective modular curves over a finite field $\mathbb{F}_q$, there are exactly two clumps: the supersingular locus and the cusps. The former is a clump because there are only finitely many isomorphism classes of supersingular elliptic curves and any elliptic curve isogenous to a supersingular elliptic curve is again supersingular. Hecke correspondences between modular curves are ramified at the cusps. Related to Question 2.1.13, we can pose the following question.

Question 2.5.2. *Given an étale correspondence $X \overset{f}{\leftarrow} Z \overset{g}{\rightarrow} Y$ of projective curves without a core over a finite field $\mathbb{F}_q$, is there exactly one clump?*

Hallouin and Perret examine a related question [**HP14**], and they prove boundedness of certain types of clumps. To state their result precisely, we need some notation.

Definition 2.5.3. Let $X \overset{f}{\leftarrow} Z \overset{g}{\rightarrow} X$ be a self-correspondence over a field k. The physical directed graph $\mathcal{H}_{phys} = \mathcal{H}_{phys}(\overline{k})$ is the graph whose vertices are the points $X(\overline{k})$ and for which there is an oriented edge from $P \in X(\overline{k})$ to $Q \in X(\overline{k})$ if there is a $z \in Z(\overline{k})$ with $f(z) = P$ and $g(z) = Q$. If k'/k is an algebraic field extension, we set $\mathcal{H}_{phys}(k')$ to be the directed subgraph of $\mathcal{H}_{phys}$ with vertices given by the set $X(k')$ and edges by the set $Z(k')$.

If $X \overset{f}{\leftarrow} Z \overset{g}{\rightarrow} X$ is a type (d, d) correspondence of projective curves, then $\mathcal{H}_{phys}$ is d-regular if and only if the correspondence is étale. Here d-regular means the in-degree and the out-degree of every vertex is d.

Definition 2.5.4. Let $X \overset{f}{\leftarrow} Z \overset{g}{\rightarrow} X$ be a self-correspondence over a field k. The singular tower C_n is defined on the level of points as follows.

$$C_n = \{(z_1, z_2, \ldots, z_n) \in Z^n | g(z_1) = f(z_2), g(z_2) = f(z_3), \ldots, g(z_{n-1}) = f(z_n)\}$$

Scheme-theoretically, $C_n = Z \times_{g,X,f} Z \times \cdots \times_{g,X,f} Z$ where the product is over n copies of Z.

Note the similarity to the definition of our first tower S_n. A key difference is that S_n forbade backtracking while the tower C_n does not.

Remark 2.5.5. The geometric points of C_n are in bijection with paths of length n in $\mathcal{H}_{phys}$.

Theorem 2.5.6. *(Perret and Hallouin* [**HP14**]*) Let* $X \xleftarrow{f} Z \xrightarrow{g} X$ *be a type* (d, d) *correspondence over* $\mathbb{F}_q$ *with* X *smooth, projective, and absolutely irreducible and* Z *absolutely irreducible. Suppose* C_n *are all irreducible. Then there is at most one finite d-regular strongly connected component in the physical directed graph* $\mathcal{H}_{phys}$ *of* $X \xleftarrow{f} Z \xrightarrow{g} X$*.*

Perret and Hallouin prove this by spectral graph theory on the graphs $\mathcal{H}_{phys}(\mathbb{F}_{q^r})$; in particular they use the Perron-Frobenius theorem.

Remark 2.5.7. We cannot directly apply Theorem 2.5.6 to the situation of a symmetric type (d, d) correspondence without a core: the symmetry w implies that $C_2 = Z \times_{g,X,f} Z$ has as an irreducible component Γ_w, the graph of the map $Z \xrightarrow{w} Z$, because points of the form $(z, w(z)) \in Z \times Z$ satisfy $g(z) = f(w(z))$.

As above, consider a symmetric correspondence $X \xleftarrow{f} Z \xrightarrow{g} X$ with symmetry w and apply the recursive tower construction S_n. Let A be the curve such that $A \cup \Gamma_w = Z \times_{g,X,f} Z$. We have a correspondence

$$Z \xleftarrow{\pi_1} A \xrightarrow{\pi_2} Z$$

Apply the recursive tower construction C_n to this second correspondence. For instance, $C_2 = A \times_{\pi_2, Z, \pi_1} A$. We recommend the reader to prove Claim 2.5.8 themselves; it is easier to prove than it is to read the proof.

Claim 2.5.8. In the situation above, $C_n = S_{n+1}$

PROOF. We prove the following stronger claim by induction: there is an isomorphism $C_n \cong S_{n+1}$ that commutes with the natural "projection onto the last factor of Z" maps coming from the construction of the two recursive towers. As a base case, $C_1 = A$. $S_2 = T_1$ is the curve such that $Z \times_{g,X,g} Z = \Delta \cup S_2$ by definition. Applying w to the second component of S_2 gives $Z \times_{g,X,f} Z = \Gamma_w \cup (1 \times w)_* S_2$. This exhibits an isomorphism $C_1 \cong S_2$ commuting with projection onto the last factor.

Now suppose we have proven that $C_{n-1} \cong S_n$in a way commuting with the "project onto the last Z-factor" maps. Then $C_n = C_{n-1} \times_{\pi_{n-1},Z,\pi_1} A$ and either $S_n \cong S_{n-1} \times_{q_2,Z,p_1} T_1$ or $S_n \cong S_{n-1} \times_{p_2,Z,q_1} T_2$ depending on whether n is odd or even respectively. Here the morphisms p_1 and q_2 from $S_n \to Z$ are taken on the "last component" as is obvious from the construction of the curves in Definition 2.4.10. Without loss of generality, let us suppose n is odd. Then applying w to the first factor of Z in $T_1 \subset Z \times_{f,X,f} Z$ produces an isomorphism $T_1 \cong A$ such that p_1 corresponds to π_1 and p_2 corresponds to π_2. Therefore, by induction hypothesis $C_n \cong S_{n+1}$ in a way compatible with "projection onto the last factor of Z", as desired. □

The upshot of Claim 2.5.8 is that the recursive tower S_{n+1} of a symmetric correspondence $X \overset{f}{\leftarrow} Z \overset{g}{\rightarrow} X$ of type (d,d) with symmetry w is the same as the recursive tower C_n of the correspondence of type (d',d') $Z \leftarrow A \rightarrow Z$ where $A \cup \Gamma_w = Z \times_{g,X,f} Z$. It is clear that physical directed graph of $X \overset{f}{\leftarrow} Z \overset{g}{\rightarrow} X$ has a finite strongly connected d-regular subgraph if and only if the physical directed graph of $Z \leftarrow A \rightarrow Z$ has a finite strongly connected d'-regular subgraph. We have the following corollary.

Corollary 2.5.9. *Let $X \overset{f}{\leftarrow} Z \overset{g}{\rightarrow} X$ be a symmetric type $(3,3)$ correspondence without a core over a finite field $\mathbb{F}_q$. Then $\mathcal{H}_{phys}$ has at most one finite strongly connected trivalent subgraph. In particular, if the correspondence is étale, $\mathcal{H}_{phys}$ is trivalent and hence there is at most one clump.*

There is a relationship between the existence of étale clumps and Question 2.2.7.

Proposition 2.5.10. *Let $X \leftarrow Z \rightarrow Y$ be a correspondence of curves without a core with Z hyperbolic. If an étale clump exists, then the degree of the maximal "field of constants" of E_∞ is finite over k. In other words, Question 2.2.7 has an affirmative answer.*

PROOF. If a étale clump exists, then all of the points of the clump are defined over a finite extension of fields k'/k. There are therefore k'-valued points of all of the curves $W_{YX\ldots Y}$, as in Remark 2.3.3. This implies that all of the $W_{YX\ldots Y}$ and hence W_∞ and E_∞ have field of constants contained in k'. The field of constants of E_∞ is then finite over k as desired. □

CHAPTER 3

Local Systems and Barsotti-Tate Groups

3.1. Some 2-adic Group Theory

In order to further explore Question 2.1.13, we will place an additional hypothesis: Assumption 3.2.1. In this section, we will develop some elementary 2-adic group theory that will later be used in conjunction with Assumption 3.2.1. Throughout this section we will freely reference concepts from Analytic Pro-P Groups by Dixon, Du Sautoy, Mann, and Segal [**DDSMS03**]. For the definition of a p-adic analytic group, see 8.14 on Page 185 of [**DDSMS03**]. We will be mostly concerned with the groups $SL(2,\mathbb{Z}_2)$ and its quotient $PSL(2,\mathbb{Z}_2) = SL(2,\mathbb{Z}_2)/\{\pm 1\}$. We will also consider the non-compact group $PSL(2,\mathbb{Q}_2) = SL(2,\mathbb{Q}_2)/\{\pm 1\}$. The compact group $SL(2,\mathbb{Z}_2)$ is topologically generated by the standard matrices $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Moreover, these satisfy the relations $S^4 = (ST)^3 = 1$ inside of $SL(2,\mathbb{Z}_2)$.

Definition 3.1.1. The subgroup $\Gamma_0(2^n) \subset PSL(2,\mathbb{Z}_2)$ is the subgroup of matrices that reduce to an upper triangular matrix modulo 2^n. Similarly, the subgroup $\Gamma^0(2^n)$ is the subgroup of matrices that reduce to a lower triangular matrix modulo 2^n. The subgroup $\Gamma(2^n) \subset PSL(2,\mathbb{Z}_2)$ is the kernel of the reduction map to $PSL(2,\mathbb{Z}/2^n\mathbb{Z}) = SL(2,\mathbb{Z}/2^n\mathbb{Z})/\{\pm 1\}$. (Note: our notation is different that of [**DDSMS03**]: they use Γ_n when we use $\Gamma(2^n)$.)

Proposition 3.1.2. *The open non-normal index 3 subgroups $H \subset PSL(2,\mathbb{Z}_2)$ are all conjugate to $\Gamma_0(2)$.*

Proof. To give such an H amounts to giving continuous surjective homomorphism $PSL(2,\mathbb{Z}_2) \to S_3$ by looking at the action on the cosets. Now, $PSL(2,\mathbb{Z}_2)$ is topologically generated by the standard matrices S and T; therefore any such homomorphism is determined by where S and T are sent. The relations imply that S must go to a transposition and ST must go to a 3-cycle. Therefore S and T are sent to distinct transpositions inside of S_3. Any ordered pair of distinct transpositions in S_3 are conjugate, so there is only one conjugacy class of continuous surjective homomorphisms $PSL(2,\mathbb{Z}_2) \to S_3$. The open non-normal index 3 subgroups are the inverse images of the three order-2 subgroups of S_3. These subgroups are all conjugate in S_3 so the open non-normal index 3 subgroups of $PSL(2,\mathbb{Z}_2)$ are also all conjugate. □

Definition 3.1.3. (Definition 3.1, page 48 of [**DDSMS03**]) A pro-p group G is powerful if p is odd and $G/\overline{G^p}$ is abelian or if $p = 2$ and $G/\overline{G^4}$ is abelian. Here $\overline{G^p}$ denotes the topological closure of the group generated by p-powers of elements of G.

Proposition 3.1.4. *The subgroup $\Gamma(4) \subset PSL(2,\mathbb{Z}_2)$ is a powerful pro-2 group.*

Proof. It is straightforward to check that in fact $\Gamma(2)$ is a pro-2 group by using the base for the neighborhoods of 1 given by $\Gamma(2^i)$. To show $\Gamma(4)$ is powerful, we will check two things: that every element in $\Gamma(2^i)$ is the square of an element in $\Gamma(2^{i-1})$ for $i \geq 3$ and then that $[\Gamma(4),\Gamma(4)] \subset \Gamma(16)$. Note that $-1 \notin \Gamma(4)$, so I may consider $\Gamma(4)$ as a subgroup of $SL(2,\mathbb{Z}_2)$.

Given an element $I + A \in \Gamma(2^i)$, where A vanishes modulo 2^i, the formal square root is given by the binomial theorem

$$\sqrt{I+A} = I + \frac{1}{2}A + \frac{1}{2!}\left(\frac{1}{2} \times \frac{-1}{2}\right)A^2 + \frac{1}{3!}\left(\frac{1}{2} \times \frac{-1}{2} \times \frac{-3}{2}\right)A^3 + \dots$$

As we assumed $i \geq 3$, the sum makes sense and converges. We need only check the determinant of the right hand side is 1 rather than -1. Now, all of the higher terms are divisible by 4, so we may rewrite the right hand side as $I + 4B$ for some matrix B. If B has eigenvalues λ_1 and λ_2, then the determinant of $I+4B$ is $(1+4\lambda_1)(1+4\lambda_2) \equiv 1(4)$ and is hence not -1. Therefore, every element in $\Gamma(2^i)$ is the square of an element in $\Gamma(2^{i-1})$ as long as $i \geq 3$.

Now suppose $I + 4A$ and $I + 4B$ are elements of $\Gamma(4)$. The commutator is of the form

$$(I + 4A + 4B + 16AB)(I + 4A + 4B + 16BA)^{-1}$$

By expanding out in a geometric series, we see that $[\Gamma(4), \Gamma(4)] \subset \Gamma(16)$. By the above argument, $\Gamma(16) \subset \Gamma(4)^4$. Therefore, $[\Gamma(4), \Gamma(4)] \subset \Gamma(4)^4$ and $\Gamma(4)$ is therefore powerful. □

Definition 3.1.5. A topological group G is uniformly powerful if it is powerful finitely generated pro-p group that is torsion-free. For brevity, we will call G uniform.

Remark 3.1.6. This is not the definition given in [**DDSMS03**] (Definition 4.1 on page 61.) However, by Theorem 4.5 on page 62, it is equivalent.

Corollary 3.1.7. *The subgroup* $\Gamma(4) \subset PSL(2, \mathbb{Z}_2)$ *is uniform.*

Proof. See Theorem 5.2 on page 88 of [**DDSMS03**]: this suffices because $\Gamma(4)$ in my sense is a subgroup of the group that they compute (for $GL(2, \mathbb{Z}_2)$). □

The utility of the concept of being a "uniform pro-p group" is the following theorem, which is immediate from Theorem 9.10 and Theorem 9.11 on pages 226 and 229 of [**DDSMS03**] respectively.

Theorem 3.1.8. *Let G be a uniform pro-p group and H be a p-adic analytic group. Then any continuous (equivalently analytic) homomorphism $\Phi : G \to H$ is uniquely determined by the induced homomorphism on Lie algebras $\phi : \mathfrak{g} \to \mathfrak{h}$.*

Theorem 3.1.8 implies that any continuous homomorphism $\Phi : \Gamma(4) \to PSL(2, \mathbb{Q}_2)$ is completely determined by the induced map on the Lie algebras. The induced homomorphism on Lie algebras is a map $\phi : \mathfrak{sl}(2, \mathbb{Q}_2) \to \mathfrak{sl}(2, \mathbb{Q}_2)$. If our original homomorphism were an open embedding, the induced map on Lie algebras is an isomorphism and may therefore be thought of as an automorphism of $\mathfrak{sl}(2, \mathbb{Q}_2)$.

Lemma 3.1.9. *Given a field k of characteristic not 2, $Aut(\mathfrak{sl}(2, k)) \cong PGL(2, k)$.*

Now, $PGL(2, \mathbb{Q}_2)$ acts on the 2-adic group $SL(2, \mathbb{Q}_2)$ faithfully by conjugation. This action descends to a faithful action of $PGL(2, \mathbb{Q}_2)$ on $PSL(2, \mathbb{Q}_2)$ because the center of a group is a characteristic subgroup. In particular, we see that any automorphism of $\mathfrak{sl}(2, \mathbb{Q}_2)$ in fact comes from an automorphism of $PSL(2, \mathbb{Q}_2)$. We therefore have the following corollary.

Corollary 3.1.10. *All abstract open embeddings* $\Gamma(4) \to PSL(2, \mathbb{Q}_2)$ *that are group homomorphisms are isomorphic, via composing with an automorphism of* α *of* $PSL(2, \mathbb{Q}_2)$*, to the standard embedding* $\Gamma(4) \subset PSL(2, \mathbb{Q}_2)$*.*

$$\begin{array}{ccc} \Gamma(4) & \subset & PSL(2, \mathbb{Q}_2) \\ & \overset{\Phi}{\searrow} & \uparrow \alpha \\ & & PSL(2, \mathbb{Q}_2) \end{array}$$

Proof. On the level of Lie algebras, $\phi : \mathfrak{sl}(2, \mathbb{Q}_2) \to \mathfrak{sl}(2, \mathbb{Q}_2)$ may be composed with $\alpha = \phi^{-1}$ to get the identity map. As discussed, $\alpha \in PGL(2, \mathbb{Q}_2)$ and therefore comes from an automorphism of the global group $PSL(2, \mathbb{Q}_2)$. Finally, the top inclusion is the standard inclusion by Theorem 3.1.8. □

Lemma 3.1.11. *Any nontrivial unipotent element* $u \in PGL(2, k)$*, where* k *is a field of characteristic not 2, has a unique square root in* $PGL(2, k)$*.*

Proof. First suppose $k = \overline{k}$. The action of u on $\mathbb{P}^1(k)$ fixes exactly one point, which we may suppose to be ∞. Then the transformation u acts as $z \mapsto z + \lambda$ for some $\lambda \in k^*$ because it is unipotent. Every element of $PGL(2, k)$ has a fixed point on $\mathbb{P}^1(k)$, so any square root v must also fix only ∞. Then v will be of the form $z \mapsto z + \mu$ or $z \mapsto -z + \mu$ for some $\mu \in k^*$. It is easy to see the latter cannot happen: v^2 would be the identity and we assumed $u \neq Id$. Therefore v is of the form $z \mapsto z + \frac{\lambda}{2}$ as desired.

Now, for general k; supposing u fixes ∞ amounts to conjugating u to upper triangular form: $AuA^{-1} = T$, where A and T have entries in $\overline{k}$. The fact that $A^{-1}TA$ has coefficients in k implies that $A^{-1}\sqrt{T}A$ has coefficients in k, for the unique square root of T in $PGL(2, \overline{k})$, by explicit computation. Therefore, $A^{-1}\sqrt{T}A$ is the unique square root of u in $PGL(2, k)$. □

Proposition 3.1.12. *Any open embedding* $i : \Gamma_0(2) \to PSL(2, \mathbb{Q}_2)$ *that is a group homomorphism and restricts to the standard embedding on* $\Gamma(4)$ *is the standard embedding. The same statement is true with* $\Gamma_0(2)$ *replaced by* $PSL(2, \mathbb{Z}_2)$*.*

Proof. The group $\Gamma_0(2)$ is topologically generated by the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

for instance, see Serre [**SB77**]. Similarly, $PSL(2, \mathbb{Z}_2)$ is topologically generated by the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

On the other hand, elements of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

with $x \neq 0$ have exactly one square root in $PSL(2, \mathbb{Q}_2)$ by Lemma 3.1.11 or direct computation. Therefore, knowing that i sends $\left(\begin{smallmatrix} 1 & 4 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 4 & 1 \end{smallmatrix}\right)$ to themselves implies that i sends the above two generators to themselves as desired. □

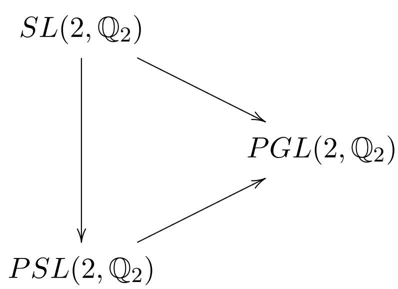
Corollary 3.1.13. *The group* $Aut(PSL(2, \mathbb{Z}_2))$*, itself a 2-adic analytic Lie group, is naturally a subgroup of* $PGL(2, \mathbb{Q}_2)$*. The same is true of the group* $Aut(\Gamma_0(2))$*.*

Proof. Any automorphism Φ of $PSL(2, \mathbb{Z}_2)$ gives an automorphism ϕ of the Lie algebra $\mathfrak{sl}(2, \mathbb{Q}_2)$, that is an element of $PGL(2, \mathbb{Q}_2)$ by Lemma 3.1.9. By Theorem 3.1.8, ϕ determines $\Phi|_{\Gamma(4)} : \Gamma(4) \hookrightarrow PSL(2, \mathbb{Z}_2)$. Now, any element $g \in PSL(2, \mathbb{Z}_2)$ whose square is unipotent is in fact unipotent by an explicit computation. As $\phi \in PGL(2, \mathbb{Q}_2)$, Φ sends the matrices $\left(\begin{smallmatrix} 1 & 4 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 4 & 1 \end{smallmatrix}\right)$ to unipotent elements. Just as in Proposition 3.1.12, Lemma 3.1.11 now shows that Φ is uniquely determined by $\Phi|_{\Gamma(4)}$. We therefore have an embedding $\text{Aut}(PSL(2, \mathbb{Z}_2)) \subset PGL(2, \mathbb{Q}_2)$. The exact same argument works for $\Gamma_0(2)$. □

3.2. A Projective 2-adic Local System

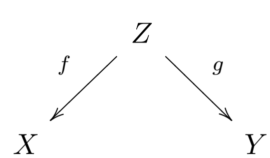
In what follows, we will make use of a construction called the "building" of a semisimple algebraic group over a p-adic field; it is an analogue of the symmetric space G/K for a split real Lie group G. In the case of $PGL(2, \mathbb{Q}_p)$ the building is in fact a rooted tree, and one recovers the action of $PGL(2, \mathbb{Q}_p)$ on $\mathbb{P}^1_{\mathbb{Q}_p}$ by looking at the induced action on the "ends" of the tree. We outline the construction of the tree $\mathcal{T}$ for $PGL(2, \mathbb{Q}_2)$ here for completeness.

The group $GL(2, \mathbb{Q}_2)$ acts on the vector space $V \cong \mathbb{Q}_2^{\oplus 2}$. The vertices of the $\mathcal{T}$ are homothety classes of lattices $\mathbb{Z}_2^{\oplus 2} \hookrightarrow V$; two vertices v_1 and v_2 are connected by an edge if there exists lattices L_1 and L_2 in the equivalence classes v_1 and v_2 respectively such that $L_1 \subset L_2$ and $[L_1 : L_2] = 2$. Therefore $\mathcal{T}$ is a trivalent tree. Give $\mathcal{T}$ the discrete topology. The group $GL(2, \mathbb{Q}_2)$ acts continuously on $\mathcal{T}$, and our original choice of identification of V with $\mathbb{Q}_2^{\oplus 2}$ gives $\mathcal{T}$ a canonical vertex, namely the lattice $\mathbb{Z}_2^{\oplus 2} \subset \mathbb{Q}_2^{\oplus 2}$. This vertex is fixed by the maximal compact subgroup $GL(2, \mathbb{Z}_2) \subset GL(2, \mathbb{Q}_2)$. Note that the action is trivial on the center $\mathbb{Q}_2^*$, hence we get an action of $PGL(2, \mathbb{Q}_2)$ on $\mathcal{T}$. We also get similar actions $SL(2, \mathbb{Q}_2)$ and $PSL(2, \mathbb{Q}_2)$ on $\mathcal{T}$ from the following diagram.



Let G be any one of these 2-adic analytic groups. The stabilizer G_v of a vertex $v \in \mathcal{T}$ is a maximal compact subgroup of G and all maximal compact subgroups of G arise this way. In particular, we may think of the vertices of $\mathcal{T}$ as maximal compact subgroups of G. Similarly, the adjacency relation of vertices in $\mathcal{T}$ is as follows: v and u are connected by an edge if $G_v \cap G_u$ is an index 3 subgroup in both G_v and G_u. The action of G on $\mathcal{T}$ in this description is the conjugation action on maximal compact subgroups.

Let us briefly recall the setup of Section 2.2 to fix notation. Suppose



is a correspondence of curves over k without a core where Z is hyperbolic. Let Ω be an algebraically closed field of transcendence degree 1 over k. Fix an embedding $PQ : k(Z) \hookrightarrow \Omega$ to build the graph $\mathcal{G}_{gen}$. This may equivalently be thought of as a geometric (generic) point $\overline{z} : \mathrm{Spec}(\Omega) \to Z$. Composing with f, we get a geometric point $\overline{x}_f : \mathrm{Spec}(\Omega) \to X$ and similarly composing with g we get a geometric point $\overline{x}_g : \mathrm{Spec}(\Omega) \to Z$. Let $k(X)_f \subset k(Z)$ be the image of $f^* : k(X) \to k(Z)$ and similarly let $k(X)_g \subset k(Z)$ be the image of g^*. We have the associated diagram of fields

$$\begin{array}{ccc} & \Omega & \\ & \uparrow & \\ & E_\infty & \\ & \uparrow & \\ & k(Z) & \\ \nearrow & & \nwarrow \\ k(X)_f & & k(Y)_g \end{array}$$

Here $E_\infty$is as in Corollary 2.2.21. Let G_P, G_Q, and G_{PQ} denote the groups $\mathrm{Gal}(E_\infty/k(X)_f)$, $\mathrm{Gal}(E_\infty/k(Y)_g)$, and $\mathrm{Gal}(E_\infty/k(Z))$ respectively and let $i_P : G_{PQ} \hookrightarrow G_P$ and $i_Q : G_{PQ} \hookrightarrow G_Q$ be the canonical open embeddings given by Galois theory.

Assumption 3.2.1. *Let*

$$\begin{array}{ccc} & Z & \\ {}^{f}\swarrow & & \searrow^{g} \\ X & & X \end{array}$$

be an étale symmetric type (3,3) correspondence of hyperbolic curves without a core. Lemma 2.4.8 implies that $\mathcal{G}_{gen}$ is a tree. Let $\mathcal{T}$ be the building of $PGL(2, \mathbb{Q}_2)$.

We assume that we can pick isomorphisms $G_P \cong PSL(2, \mathbb{Z}_2)$ and $\mathcal{G}_{gen} \cong \mathcal{T}$ such that the induced action of G_P on $\mathcal{G}_{gen}$ is carried to the action of $PSL(2, \mathbb{Z}_2)$ on $\mathcal{T}$ under these isomorphisms.

Remark 3.2.2. We comment on the likelihood of Assumption 3.2.1. In the case of type a symmetric type (3,3) correspondence without a core, Lemma 2.4.8 implies that the pair $(\mathcal{G}_{gen}, A^{\tilde{w}})$ is ∞-transitive and $\mathcal{G}_{gen}$ is a tree. Moreover, Proposition 2.2.30 implies then that

$$G_P *_{G_{PQ}} G_Q \cong A^{PQ} \subset A^{\tilde{w}}$$

exactly as $PSL(2,\mathbb{Z}_2) *_{\Gamma_0(2)} PSL(2,\mathbb{Z}_2) \cong PSL(2,\mathbb{Q}_2) \subset PGL(2,\mathbb{Q}_2)$. We were interested in what mileage we could get out of Assumption 3.2.1. In particular, we wanted to find some minimal hypotheses to answer Question 2.1.13, the orienting question of this dissertation.

Assumption 3.2.1 and Proposition 3.1.2 therefore imply that we can pick an isomorphism of $i_P : G_{PQ} \hookrightarrow G_P$ with $\Gamma_0(2) \subset PSL(2,\mathbb{Z}_2)$ in a way compatible with the action on $\mathcal{G}_{gen}$ and $\mathcal{T}$. Let $\phi : G_P \to PSL(2,\mathbb{Z}_2)$ be such an isomorphism. Now, we assumed our original correspondence was symmetric. Lift (non-canonically) the symmetry w^* to a symmetry $\tilde{w}^*$ of E_∞. The automorphism $\tilde{w}^*$ is not involutive, but its square fixes $k(Z)$.

The two groups G_P and G_Q are conjugate inside of $A^{\tilde{w}} \subset A$ by means of $\tilde{w}^*$; that is, there is an isomorphism $Ad_{\tilde{w}^*} : G_Q \to G_P$. This isomorphism does not commute with i_P and i_Q. We have an isomorphism $\phi \circ Ad_{\tilde{w}^*} : G_Q \to PSL(2,\mathbb{Z}_2)$. By transporting the map $i_Q : G_{PQ} \hookrightarrow G_Q$ via $\phi|_{G_{PQ}}$ on the left and $\phi \circ Ad_{\tilde{w}^*}$ on the right we get a (possibly non-standard) open embedding $i' : \Gamma_0(2) \hookrightarrow PSL(2,\mathbb{Z}_2)$.

$$\begin{array}{ccc} G_{PQ} & \xrightarrow{i_Q} & G_Q \\ \phi \downarrow & & \downarrow \phi\circ Ad_{\tilde{w}^*} \\ \Gamma_0(2) & \underset{i'}{\dashrightarrow} & PSL(2,\mathbb{Z}_2) \end{array} \tag{3.2.1}$$

In Equation 3.2.1 we have constructed a homomorphism $i' : \Gamma_0(2) \to PSL(2,\mathbb{Z}_2)$ which is an abstract open embedding of index 3. We will prove the open embedding i' is isomorphic to the standard open embedding $\Gamma_0(2) \subset PSL(2,\mathbb{Z}_2)$.

Corollary 3.2.3. *The embedding $G_{PQ} \hookrightarrow G_Q$ is isomorphic to the standard embedding $\Gamma_0(2) \hookrightarrow PSL(2,\mathbb{Z}_2)$.*

Proof. We know that $G_{PQ} \cong \Gamma_0(2)$ and $G_Q \cong PSL(2,\mathbb{Z}_2)$. Composing with the natural inclusion, we get a

$$i' : \Gamma_0(2) \to PSL(2,\mathbb{Z}_2) \to PSL(2,\mathbb{Q}_2)$$

The discussion after Corollary 3.1.10 implies that we may compose with an automorphism α of $PSL(2,\mathbb{Q}_2)$ to get a map $i : \Gamma_0(2) \to PSL(2,\mathbb{Z}_2) \to PSL(2,\mathbb{Q}_2)$ that restricts to the standard embedding on $\Gamma(4)$. Proposition 3.1.12 then implies that i is the standard embedding on both $\Gamma_0(2)$ and $PSL(2, \mathbb{Z}_2)$. Therefore the original map $G_{PQ} \hookrightarrow G_Q$ is isomorphic to the standard embedding. □

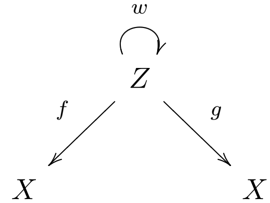
Corollary 3.2.4. *Given Assumption 3.2.1, the group A^{PQ} (as in Definition 2.2.25) is isomorphic to $PSL(2, \mathbb{Q}_2)$.*

Proof. Proposition 2.4.11 implies that $\mathcal{G}_{gen}$ is a tree. By Proposition 2.2.30 we see that the natural map $G_P *_{G_{PQ}} G_Q \to A^{PQ}$ is an isomorphism. On the other hand, $PSL(2, \mathbb{Z}_2) *_{\Gamma_0(2)} PSL(2, \mathbb{Z}_2) \cong PSL(2, \mathbb{Q}_2)$ (for instance by using Serre's theorem for the action of the group $PSL(2, \mathbb{Q}_2)$ on $\mathcal{T}$ as in [**SB77**].) □

Lemma 3.2.5. *Given Assumption 3.2.1, the action of A^{PQ} on $\mathcal{G}_{gen}$ is isomorphic to the standard action of $PSL(2, \mathbb{Q}_2)$ on $\mathcal{T}$. This isomorphism can be chosen to be compatible such that the restricted action of G_P on $\mathcal{G}_{gen}$ is carried to the standard action of the subgroup $PSL(2, \mathbb{Z}_2) \subset PSL(2, \mathbb{Q}_2)$ on $\mathcal{T}$.*

Proof. As we know $\mathcal{G}_{gen}$ is a tree, Remark 2.2.33 and the surrounding discussion provide a description of $\mathcal{G}_{gen}$ and the action of A^{PQ}. The vertices of $\mathcal{G}_{gen}$ are in correspondence with the maximal compact subgroups of A^{PQ} and two vertices G_1 and G_2 are connected by an edge if $G_1 \cap G_2$ has index 3 in both G_1 and G_2. The induced action is the conjugation action of A^{PQ}. Now, the "standard" action of $PSL(2, \mathbb{Q}_2)$ on $\mathcal{T}$ may be described exactly the same way as explicated at the beginning of the section. By Corollary 3.2.4 we are done. □

To continue, we must clarify the role of basepoints in the fundamental groups. Suppose we are given a symmetric étale correspondence with symmetry w



There are canonical surjections $\pi_1(X, \overline{x}_f) \to G_P$ and $\pi_1(X, \overline{x}_g) \to G_Q$. Recall that w^* denotes the induced involution of $k(Z)$ associated to the map $w : Z \to Z$. Lemma 2.4.2

shows that w^* lifts to an element $\tilde{w}^* \in \mathrm{Aut}_k(E_\infty)$. Because Ω is an algebraic closure of E_∞, $\tilde{w}^*$ lifts to an element $\overline{w}^* \in \mathrm{Aut}_k(\Omega)$. Neither lift is in any way unique or canonical.

Remark 3.2.6. It is important that we distinguish between w and w^*; if we were sloppy about the distinction, the formulas involving w might be off by an inverse.

Lemma 3.2.7. *Let Z be a normal connected variety over a field k, let $\alpha \in Aut_k(Z)$, and let $\overline{z} : Spec(\Omega) \to Z$ be a geometric generic point. Given $\overline{\alpha^*} \in Aut_k(\Omega)$ lifting α^*, we get an isomorphism of $\pi_1(Z, \overline{z})$ and $\pi_1(Z, \alpha \circ \overline{z})$.*

Proof. Note that $\overline{\alpha}^* \notin \mathrm{Gal}(\Omega/k(Z))$ but $\overline{\alpha}^*$ fixes $k(Z)$ as a set. Then $Ad_{\overline{\alpha}^*}$ gives an outer automorphism of $\mathrm{Aut}(\Omega/k(Z))$, thought of as a subgroup of $\mathrm{Aut}(\Omega)$. We then have the following diagram.

$$\begin{array}{ccc} \mathrm{Aut}(\Omega/k(Z)) & \longrightarrow & \pi_1(Z, \alpha \circ \overline{z}) \\ {\scriptstyle Ad_{\overline{\alpha}^*}} \downarrow \wr & & \vdots\, \wr \\ \mathrm{Aut}(\Omega/k(Z)) & \longrightarrow & \pi_1(Z, \overline{z}) \end{array}$$

We justify why the dotted arrow exists. The choice $\overline{z} : \mathrm{Spec}(\Omega) \to Z$ is equivalent to the choice of a k-algebra homomorphism $\overline{z}^* : k(Z) \to \Omega$. Therefore, the notion of the "maximal unramified extension" $k(Z)^{unr} \subset \Omega$ makes sense. Note that $\overline{\alpha}^*$ sends the maximal unramified extension $k(Z)^{unr} \subset \Omega$ to itself, i.e. $\overline{\alpha}^*$ restricts to an automorphism $\alpha^{unr,*}$ of $k(Z)^{unr}$. Moreover, $\pi_1(Z, \overline{z})$ is canonically isomorphic to $\mathrm{Gal}(k(Z)^{unr}/k(Z))$, i.e. the subgroup of $\mathrm{Aut}(k(Z)^{unr})$ that fixes the image of $\overline{z}^*$. Similarly, $\pi_1(Z, \alpha \circ \overline{z})$ is canonically isomorphic to the subgroup of $\mathrm{Aut}(k(Z)^{unr})$ that fixes the image of $\overline{z}^* \circ \alpha^* = \overline{\alpha}^* \circ \overline{z}^*$, i.e. the image of $\overline{z}^*$. Therefore the dotted arrow exists and may be thought of as $Ad_{\alpha^{unr,*}}$. We leave it to the reader to check that $Ad_{\alpha^{unr,*}}$ is independent, up to inner automorphism, of the choice of $\alpha^{unr,*}$ lifting α^*. $\square$

Definition 3.2.8. Given Assumption 3.2.1, let $\phi : G_P \to PSL(2, \mathbb{Z}_2)$ be an isomorphism as in Lemma 3.2.5. There is a canonical surjection $\pi_1(X, \overline{x}_f) \twoheadrightarrow G_P$. Define ρ to be the composition of these two homomorphisms, $\rho : \pi_1(X, \overline{x}_f) \to PSL(2, \mathbb{Z}_2)$ and define $\varrho : \pi_1(X, \overline{x}_f) \to PSL(2, \mathbb{Q}_2)$ to be ρ composed with the natural inclusion $PSL(2, \mathbb{Z}_2) \subset PSL(2, \mathbb{Q}_2)$.

Remark 3.2.9. We may informally think of ϱ as a projective rank 2 local system on X with values in $\mathbb{Q}_2$.

Lemma 3.2.10. *Given Assumption 3.2.1, $f^*\varrho \cong g^*\varrho$ as projective $\mathbb{Q}_2$ local systems. That is, there exists $\alpha \in PGL(2,\mathbb{Q}_2)$ such that the following diagram commutes.*

$$\begin{array}{ccc} & & PSL(2,\mathbb{Q}_2) \\ & \overset{f^*\varrho}{\nearrow} & \\ \pi_1(Z,\overline{z}) & & \downarrow Ad_\alpha \\ & \underset{g^*\varrho}{\searrow} & \\ & & PSL(2,\mathbb{Q}_2) \end{array}$$

Proof. This lemma is straightforward once is completely explicit about basepoints. First of all, on the level of fundamental groups we have a natural open embedding

$$f_* : \pi_1(Z,\overline{z}) \to \pi_1(X,\overline{x}_f)$$

given by Galois theory as $\overline{x}_f = f\circ\overline{z}$. On the other hand, we have a natural open embedding

$$g_* : \pi_1(Z, w\circ\overline{z}) \to \pi_1(X,\overline{x}_f)$$

again as $\overline{x}_f = g\circ(w\circ\overline{z})$. The operations of f^* and g^* on a homomorphism $\rho : \pi_1(X,\overline{x}_f) \to G$ are the restrictions along f_* and g_* respectively. To make sense of the formula $f^*\varrho \cong g^*\varrho$ we must therefore identify $\pi_1(Z,\overline{z})$ and $\pi_1(z, w\circ\overline{z})$. We do this as in Lemma 3.2.7, via a lift $\overline{w}^*$. We then have the following diagram.

$$\begin{array}{ccccc} \pi_1(Z, w\circ\overline{z}) & \longrightarrow & \mathrm{Gal}(E_\infty/k(Z)) & \xrightarrow[\sim]{\phi} & \Gamma_0(2) \\ \downarrow & & Ad_{\tilde{w}^*}\downarrow\wr & & \vdots\wr \\ \pi_1(Z,\overline{z}) & \longrightarrow & \mathrm{Gal}(E_\infty/k(Z)) & \xrightarrow[\phi]{\sim} & \Gamma_0(2) \end{array}$$

By Corollary 3.1.13, the dotted arrow is given by Ad_α for some element $\alpha \in PGL(2,\mathbb{Q}_2)$. Therefore $f^*\varrho \cong g^*\varrho$ via conjugation by an element $\alpha \in PGL(2,\mathbb{Q}_2)$ as desired. □

Note 3.2.11. We recall what it means for two representations to be isomorphic. Let π be a group and let ρ_1 and ρ_2 be representations to $GL_n(F)$ for F a field. We say that $\rho_1 \cong \rho_2$ if the two homomorphisms to $GL_n(F)$ are conjugate. Note that this is stronger than demanding that ρ_1 and ρ_2 are abstractly isomorphic; we demand that they are isomorphic by an *inner automorphism of* $GL_n(F)$. For instance, suppose π is a finite group and $\rho : \pi \to GL_n(\mathbb{C})$ is a representation with $n > 2$. Then the "inverse transpose" map is an automorphism of $GL_n(\mathbb{C})$ that conjugates the character and in general does not yield an isomorphic representation. Similarly, when considering projective representations, we say ρ_1 and ρ_2 are isomorphic if the abstract homomorphisms are related by an *inner automorphism* of the underlying $PGL_n(F)$ or $PSL_n(F)$, namely conjugation by an element of $PGL_n(F)$. This notion is admittedly ad hoc for the group PSL_n, which is not an algebraic group. Fortunately, here $PSL_2(\mathbb{Q}_2) \subset PGL_2(\mathbb{Q}_2)$ so this is compatible with the group-theoretic notion.

3.3. An Interlude on Compatible Systems of l-adic representations

Definition 3.3.1. Let X be a normal connected variety over $\mathbb{F}_q$ with geometric point $\overline{x}$. For each closed point $c \in |X|$, let $[Fr_c] \subset \pi_1(X, \overline{x})$ denote the Frobenius conjugacy class. Let l and l' be prime numbers (possibly the same) that are not p. Two continuous representations $\rho : \pi_1(X, \overline{x}) \to GL(n, \overline{\mathbb{Q}_l})$ and $\rho' : \pi_1(X, \overline{x}) \to GL(n, \overline{\mathbb{Q}_{l'}})$ are said to be *compatible* if

- There exists choices of embeddings $i : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}_l}$ and $i' : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}_{l'}}$ such that for each closed point $c \in |X|$ the characteristic polynomials of $\rho(Fr_c)$ and $\rho'(Fr_c)$ are both in $\overline{\mathbb{Q}}[t]$ and in fact are the same.

Deligne's conjecture 1.2.10 of Weil II [**Del80**] is the following.

Conjecture 3.3.2. *(Deligne) Let X be a normal connected variety over a finite field k of cardinality p^f with a geometric point $\overline{x} \to X$. Let $l \neq p$ be a prime. Let $\mathscr{L}_l$ be an irreducible l-adic local system with trivial determinant on X. The choice of $\overline{x}$ allows us to think of this as a representation $\varrho_l : \pi_1(X, \overline{x}) \to SL(n, \overline{\mathbb{Q}_l})$. Then*

(1) *ϱ_l is pure of weight 0.*

(2) *There exists a number field E such that for all closed points $x \in |X|$, the polynomial $det(1 - \varrho_l(F_x)t)$, has all of its coefficients in E. Here F_x is a Frobenius element of x. In particular, the eigenvalues of $\varrho_l(F_x)$ are all algebraic numbers.*

(3) *For each place $\lambda \nmid p$, the inverse roots α of $det(1 - \varrho_l(F_x)t)$ are λ-adic units in $\overline{E_\lambda}$.*

(4) *For each $\lambda | p$, the λ-adic valuations of the inverse roots α satisfy*

$$|\frac{v(\alpha)}{v(Nx)}| \leq \frac{n}{2}$$

(5) *After possibly replacing E by a finite extension, for each $\lambda \nmid p$ there exists a λ-adic local system $\rho_\lambda : \pi_1(X, \overline{x}) \to SL(n, E_\lambda)$ that is compatible with ρ_l.*

(6) *For $\lambda | p$, there exists a crystalline companion to ρ_l.*

Definition 3.3.3. Let X be a normal connected variety over $\mathbb{F}_q$ with geometric point $\overline{x}$ and let E be a number field. An E-compatible system is a compatible system of representations $\rho_\lambda : \pi_1(X, \overline{x}) \to GL(n, E_\lambda)$ for each $\lambda \nmid p$.

Deligne was inspired by the work of Drinfeld on the Langlands Correspondence for $GL(2)$. In particular, if one could prove that ϱ_l was motivic, one would immediately know that ϱ_l gives rise to a compatible family of l'-adic representations: that is, there exist representations $\varrho_{l'} : \pi_1(X, \overline{x}) \to SL(n, \overline{\mathbb{Q}_{l'}})$ compatible with ϱ_l for each l' prime to p. Note that this is slightly weaker than Part 5 of the conjecture, which guarantees that there exists a number field E such that ρ_l fits into an E-compatible system. Nonetheless, this is still quite striking: an l-adic local system is, in this formulation, simply a continuous homomorphism from $\pi_1(X)$, and the topologies on $\mathbb{Q}_l$ and $\mathbb{Q}_{l'}$ are completely different.

By work of Deligne, Drinfeld, and Lafforgue, if X is a curve all such local systems are motivic. Moreover, in this case Chin has proved Part 5 of the conjecture. See, for instance, Theorem 4.1 of Chin [**Chi04**].

Crew [**Cre92**] formulated Part 6 of Deligne's conjecture as follows: for every $\lambda | p$, there is an *overconvergent F-isocrystal on X with coefficients in E_λ* that is compatible in the appropriate sense. As Crew notes, this notion seems to be an appropriate analog for a p-adic local system on a variety in characteristic p. Abe has recently constructed a sufficiently robust theory of p-adic cohomology to prove a p-adic Langlands correspondence and hence answer affirmatively part 6 of Deligne's conjecture when X is a curve [**Abe11, Abe13**]. In fact, he has proven that any absolutely irreducible overconvergent F-isocrystal on a curve X over a finite field gives rise to a cuspidal automorphic representation and hence by Drinfeld and Lafforgue's work to a compatible system of l-adic local systems. In certain circumstances, we may use this p-adic companion to construct a Barsotti-Tate group, as in Theorem 3.11.6. To do this, we need to first discuss a general "extension of scalars" and "descent" mechanism for abelian categories. This will be the topic of Section 3.5.

Remark 3.3.4. Part (4) of the conjecture is not tight for $n = 2$. The case to have in mind is the universal elliptic curve over $Y(1)$ over $\mathbb{F}_{p^2}$. By taking relative l-adic cohomology, one gets a weight 1 rank-2 $\mathbb{Z}_l$ local system on $Y(1)$. Twist by $\overline{\mathbb{Q}_l}(1/2)$ to get a local system which is pure of weight 0. For an ordinary $\mathbb{F}_q$ point, the ratio in question is either $-1/2$ or $1/2$ and for a supersingular point it is 0. This motivates the bound for a weight 0 rank 2 local system, as proven by L. Lafforgue (Theorem VII.6.(iv) of [**Laf02**]).

Theorem 3.3.5. *(L. Lafforgue) Let C be a smooth geometrically connected curve over $\mathbb{F}_q$ and let $\mathscr{L}$ be an absolutely irreducible rank 2 l-adic local system on C with trivial determinant. Then for all closed points $x \in |C|$, the eigenvalues α of F_x satisfy*

$$|\frac{v(\alpha)}{v(Nx)}| \leq \frac{1}{2}$$

3.4. Lifting projective local systems

Throughout this section, we continue to operate under Assumption 3.2.1 with the further assumption that $k \cong \mathbb{F}_q$. Let ρ be the integral 2-adic representation $\pi_1(X) \to PSL(2, \mathbb{Z}_2)$ and let ϱ be the rational representation as in Definition 3.2.8. Lemma 3.2.10 shows that $f^*\varrho \cong g^*\varrho$ as "projective local systems" on Z. In order to apply the Langlands correspondence we would like to lift ϱ to an $SL(2, \mathbb{Q}_2)$-valued representation.

In general, there is an obstruction to doing lifting projective representations valued in $H^2_{cont}(\pi_1(X), \mathbb{G}_m)$. As X is a projective, hyperbolic curve over $\mathbb{F}_q$, X is a $K(G, 1)$ for the étale topology and hence

$$H^2_{cont}(\pi_1(X), \mathbb{G}_m) \cong H^2_{\acute{e}t}(X, \mathbb{G}_m)$$

Our example involves the group $PSL(2, \mathbb{Z}_2)$, so the obstruction in fact lives in $H^2_{ét}(X, \mu_2)$. On the other hand, a class $\xi \in H^2_{ét}(X, \mu_2)$ gives rise to μ_2-gerbe $\mathcal{X}_\xi$, and so one option is to consider the correspondence of stacky curves

$$\begin{array}{ccc} & \mathcal{Z}_\xi & \\ \swarrow & & \searrow \\ \mathcal{X}_\xi & & \mathcal{X}_\xi \end{array}$$

By definition, ρ will lift to an actual representation on all of these stacks. Unfortunately, the Langlands correspondence is as of yet unknown stacky curves, and so this approach will not be fruitful for us.

Another option is to construct a *covering of the correspondence* on which the projective local systems lift. We sketch this approach.

(1) Consider the adjoint representation $\pi_1(X) \to SL(3, \mathbb{Q}_2)$. Because original $PSL(2, \mathbb{Z}_2)$-representation is surjective, the adjoint will be absolutely irreducible.

(2) Deligne's Conjecture 3.3.2 implies there exists a number field E such that for every $\lambda \nmid p$ there is a compatible $\pi_1 \to SL(3, E_\lambda)$.

(3) There exists a place λ such that the residual representation, "mod λ", on Z is absolutely irreducible. This implies there is only one lattice up to homothety.

(4) Moreover, one can pick such a λ such that the image of residual monodromy is a group of even order. (Abiding by the slogan "Most finite groups have even order.")

(5) Trivialize the "mod λ" part of the representation to get a cover of the correspondence. This works because there is a unique lattice, so there is no ambiguity in what we mean by the "mod λ" representation.

$$\begin{array}{ccc} & Z' & \\ {}^{f'}\swarrow & & \searrow^{g'} \\ X' & & X' \end{array}$$

This is exactly analogous to constructing "mixed level structure", as in Katz-Mazur [**KM85**]. As the monodromy group has even order, after a finite extension of the ground field, this kills the Brauer class. In particular, X' has a rank 2 $\mathbb{Q}_2$-valued local system $\mathscr{L}_2$ with trivial determinant such that $f'^*\mathscr{L}_2$ and $g'^*\mathscr{L}_2$ differ by a (possibly trivial) quadratic character.

(6) Now there are two options: we may either kill the quadratic character by taking a double cover of Z' (to get a symmetric type (6,6) étale correspondence) or we can try to choose λ more judiciously to ensure that the quadratic character above is trivial.

Conclusion 3.4.1. Given Assumption 3.2.1, we can lift our correspondence to an étale correspondence

$$\begin{array}{ccc} & Z' & \\ {}^{f'}\swarrow & & \searrow^{g'} \\ X' & & X' \end{array}$$

together with a rank 2 $\mathbb{Q}_2$-local system $\mathscr{L}_2$ on X' such that

- $f^*\mathscr{L}_2 \cong g^*\mathscr{L}_2$.
- $\mathscr{L}_2$ has big image and is hence absolutely irreducible.
- $\mathscr{L}_2$ has trivial determinant.

3.5. Extension of Scalars and Galois descent for abelian categories

Definition 3.5.1. Let $\mathcal{C}$ be a K-linear additive category, where K is a field. Let L/K be a finite field extension. We define the base-changed category $\mathcal{C}_L$ as follows:

- Objects of $\mathcal{C}_L$ are pairs (C, f), where C is an object of $\mathcal{C}$ and $f : L \to End_{\mathcal{C}}C$ is a homomorphism of K-algebras. We call such an f an L-structure on C.
- Morphisms of $\mathcal{C}_L$ are morphisms of $\mathcal{C}$ that are compatible with the L-structure.

It is easy to check (and done in Sosna [**Sos14**]; see also Section 4 of Deligne [**Del**] or Section 5 of the "official" version of that paper [**Del14**]) that $\mathcal{C}_L$ is an L-linear additive category and that if $\mathcal{C}$ was abelian, so is $\mathcal{C}_L$. It is somewhat more involved to prove that if $\mathcal{C}$ is a K-linear rigid abelian $\otimes$ category, $\mathcal{C}_L$ is an L-linear rigid abelian $\otimes$ category with natural $\otimes$ and dual (see Section 4.3 of [**Del**] for elegant definitions of these.) Moreover, if $\mathcal{C}$ is a K-linear Tannakian category (not necessarily assumed to be neutral), then $\mathcal{C}_L$ is an L-linear Tannakian category. This last fact is a nontrivial theorem; see, for instance, Theorem 4.4 of Deligne [**Del**] for an "elementary" proof.

Proposition 3.5.2. *If $\mathcal{C}$ and $\mathcal{D}$ are K-linear categories, then $(\mathcal{C} \oplus \mathcal{D})_L$ is equivalent to $\mathcal{C}_L \oplus \mathcal{D}_L$.*

PROOF. Let $C \oplus D$ be an object of $\mathcal{C} \oplus \mathcal{D}$. Then $End_{\mathcal{C}\oplus\mathcal{D}}(C \oplus D) = End_{\mathcal{C}}C \times End_{\mathcal{D}}D$ as K-algebras. Therefore, the data of a K-algebra map $j : L \to End_{\mathcal{C}\oplus\mathcal{D}}(C \oplus D)$ is the same as the data of a pair of K-algebra maps $j_C : L \to End_{\mathcal{C}}C$ and $j_D : L \to End_{\mathcal{D}}D$. The natural functor $(C \oplus D, j) \mapsto (C, j_c) \oplus (D, j_d)$ is an equivalence of categories. □

Remark 3.5.3. Deligne [**Del**] defines $\mathcal{C}_L$ in a slightly different way. Given an object X of $\mathcal{C}$ and a finite dimensional K-vector space V, define $V \otimes X$ to represent the functor $Y \mapsto V \otimes_K Hom_K(Y, X)$. Given a finite extension L/K, an object with L-structure is defined to be an object X of $\mathcal{C}$ together with a morphism $L \otimes X \to X$ in $\mathcal{C}$. Then the base-changed category $\mathcal{C}_L$ is defined to be the category of objects equipped with an L-structure and with morphisms respecting the L-structure.

Note that there is an induction functor: $\mathrm{Ind}_K^L : \mathcal{C} \to \mathcal{C}_L$. Let $\{\alpha\}$ be a basis for L/K.

$$\mathrm{Ind}_K^L M = (\bigoplus_{\alpha} M, f)$$

where f is induced by the action of L on the $\{\alpha\}$. In Deligne's formulation, $\mathrm{Ind}_K^L M$ is just $L \otimes M$ and the L-structure is the multiplication map $L \otimes_K L \otimes M \to L \otimes M$. The Induction functor has a natural right adjoint: Res_K^L which simply forgets about the L-structure. If M is an object of $\mathcal{C}$, we sometimes denote by M_L the object $\mathrm{Ind}_K^L M$ for shorthand. If C is a K-linear abelian category, induction and restriction are exact functors.

Proposition 3.5.4. *Let A and B be objects of a K-linear category $\mathcal{C}$ and let L be a finite extension of K. $Hom_{\mathcal{C}_L}(A_L, B_L) \cong L \otimes_K Hom_{\mathcal{C}}(A, B)$.*

PROOF. By the adjunction, $\mathrm{Hom}_{\mathcal{C}_L}(A_L, B_L) \cong \mathrm{Hom}_{\mathcal{C}}(A, \mathrm{Res}_{L/K}(B_L))$. On the other hand, $\mathrm{Res}_{L/K}(B_L)$ is naturally isomorphic to the object $L \otimes B$. By the defining property of $L \otimes B$, $\mathrm{Hom}_{\mathcal{C}}(A, \mathrm{Res}_{L/K}(B_L)) \cong L \otimes_K \mathrm{Hom}_{\mathcal{C}}(A, B)$ as desired. □

Remark 3.5.5. The Induction functor allows us to describe more general extensions-of-scalars. Namely, let E be an *algebraic* field extension of K. We define $\mathcal{C}_E$ to be the 2-colimit of the categories $\mathcal{C}_L$ where L ranges over the subfields of E finite over K. In particular, we can define a category $\mathcal{C}_{\overline{K}}$ and an induction functor

$$Ind_K^{\overline{K}} : \mathcal{C} \to \mathcal{C}_{\overline{K}}$$

Definition 3.5.6. Let $\mathcal{C}$ be a K-linear abelian category. We say that an object M is absolutely irreducible if $Ind_K^{\overline{K}}(M)$ is an irreducible object of $\mathcal{C}_{\overline{K}}$.

If L/K is a Galois extension, note that we have a natural (strict) action of $G :=$ $\mathrm{Gal}(L/K)$ on the category $\mathcal{C}_L$ by twisting the L-structure. That is, if $g \in G$, ${}^g(C, f) :=$ $(C, f \circ g^{-1})$. The group G "does nothing" to maps: a map $\phi : (C, f) \to (C', f')$ in $\mathcal{C}_L$ is just a map $\phi_K : C \to C'$ in $\mathcal{C}$ that commutes with the L-actions, and $g \in G$ acts by fixing the underlying ϕ_K and just twisting the L-structure: ${}^g\phi : (C, f \circ g^{-1}) \to (C', f' \circ g^{-1})$.

A computation shows that if $\lambda \in L$ is considered as a scalar endomorphism of M, then the endomorphism ${}^g\lambda : {}^gM \to {}^gM$ is the scalar $g(\lambda)$. If $\mathcal{C}$ is a K-linear rigid abelian $\otimes$ category, then this action is compatible in every way imaginable with the inherited rigid abelian $\otimes$ structure on $\mathcal{C}_L$. For instance, ${}^g(M^*) \cong ({}^gM)^*$ and ${}^g(M \otimes N) \cong ({}^gM) \otimes ({}^gN)$ canonically. Similarly, if $F : \mathcal{C} \to \mathcal{D}$ is a K-linear functor between K-linear categories, we have a canonical extension functor $F_L : \mathcal{C}_L \to \mathcal{D}_L$ and ${}^gF_L(M) \cong F_L({}^gM)$.

Definition 3.5.7. Suppose $\mathcal{C}$ is a abelian K-linear category and L/K is a finite Galois extension with group G. Let $\mathcal{C}_L$ be the base-changed category. We define the category of descent data, $(\mathcal{C}_L)^G$, as follows. The objects of $(\mathcal{C}_L)^G$ are pairs $(M, \{c_g\})$ where M is an object of $\mathcal{C}_L$ and the $c_g : M \to {}^gM$ are a collection of isomorphisms for each $g \in G$ that satisfies the cocycle condition. The morphisms of $(\mathcal{C}_L)^G$ are maps $(f_g : {}^gM \to {}^gN)$ that are equivariant with respect to the c_g.

Lemma 3.5.8. *If $\mathcal{C}$ is an abelian K-linear category and L/K is a finite Galois extension, then Galois descent holds. That is, $\mathcal{C}$ is equivalent to the category $(\mathcal{C}_L)^G$.*

Proof. This is Lemma 2.7 in [**Sos14**]. □

We say objects and morphisms in the essential image of Ind_K^L "descend". We may use the technology of descent to compare the notions of semi-simplicity (intrinsic to any abelian category) with absolute semi-simplicity.

Proposition 3.5.9. *Let $\mathcal{C}$ be a K-linear abelian category with K a characteristic 0 field. Suppose M is an object of $\mathcal{C}$ that is absolutely semi-simple, i.e. $M_{\overline{K}}$ is a semi-simple object of $\mathcal{C}_{\overline{K}}$. Then M is semi-simple.*

Proof. Let $i : N \hookrightarrow M$ be a sub-object of M in $\mathcal{C}$. By assumption, there is an $f : M_{\overline{K}} \to N_{\overline{K}}$ splitting i. This f is defined over some finite Galois extension L/K with group G. Averaging f over G (using that K is characteristic 0), we get a section of i_L that is invariant under G. By Lemma 3.5.8, this section is defined over K, as desired. □

Proposition 3.5.10. *Let $\mathcal{C}$ be a K-linear abelian category with K a perfect field. Suppose M is a semi-simple object of $\mathcal{C}$ and that $End_{\mathcal{C}}(M)$ is a finite dimensional K-algebra. Then $M_{\overline{K}}$ is a semi-simple object of $\mathcal{C}_{\overline{K}}$.*

Proof. We may suppose M is irreducible. If $M_{\overline{K}}$ were irreducible, we would be done, so suppose it has is not irreducible. The hypothesis on the endomorphism algebra ensures that $M_{\overline{K}}$ has a finite Jordan-Holder filtration, so in particular $M_{\overline{K}}$ has proper simple sub-object N, defined over some finite Galois extension L/K (this is where we use K being perfect.) Then all of the Galois conjugates are all are also simple sub-objects of $M_{\overline{K}}$. The sum $S_{\overline{K}}$ of all simple sub-objects of $M_{\overline{K}}$ therefore descends to an object S of $\mathcal{C}$ by Lemma 3.5.8. As we assumed M was irreducible, S is M. On the other hand, $S_{\overline{K}}$ is the maximal sub-object of $M_{\overline{K}}$ that is semi-simple. Therefore $M_{\overline{K}}$ is semi-simple, as desired. For a proof with slightly different hypotheses on $\mathcal{C}$, see Lemme 4.2 of Deligne [**Del**]. □

Corollary 3.5.11. *Let $\mathcal{C}$ be a K-linear abelian category with K a characteristic 0 field. Let M be an object of $\mathcal{C}$ with $End_{\mathcal{C}}M$ a finite dimensional K-algebra. Then M is semi-simple if and only if it is absolutely semi-simple.*

We record for later a generalization of Schur's Lemma.

Lemma 3.5.12. *Let $\mathcal{C}$ be a K-linear abelian category such that the Hom groups of $\mathcal{C}$ are finite dimensional K-vector spaces. Let M be an absolutely irreducible object of $\mathcal{C}$. Then $End_{\mathcal{C}}(M) \cong K$.*

Proof. As M is absolutely irreducible, $\mathrm{End}_{\mathcal{C}_{\overline{K}}} M_{\overline{K}}$ is a division algebra over $\overline{K}$. The hypothesis ensures that this division algebra is finite dimensional over $\overline{K}$ and hence is exactly $\overline{K}$. Now, by Proposition 3.5.4, $\mathrm{End}_{\mathcal{C}}(M) \otimes_K \overline{K} \cong \overline{K}$ as algebras. By descent, $\mathrm{End}_{\mathcal{C}}(M) \cong K$ as desired. $\square$

3.6. A 2-cocycle obstruction for descent

Suppose $\mathcal{C}$ is an abelian K-linear category and L/K a finite Galois extension with group G. Let $\mathcal{C}_L$ be the base-changed category. Suppose an object $M \in \mathrm{Ob}(\mathcal{C}_L)$ is isomorphic to all of its twists by $g \in G$ and that the natural map $L \tilde{\to} \mathrm{End}_{\mathcal{C}_L} M$ is an isomorphism. (This latter restriction will be relaxed later in the important Remark 3.6.7.) We will define a cohomology class $\xi_M \in H^2(G, L^*)$ which measures whether or not M descends to $\mathcal{C}$; in particular, $\xi_M = 0$ if and only if M descends to K. This construction is well-known in other contexts, see for instance [**Gui10, Wei56**].

Definition 3.6.1. For each $g \in G$ pick an isomorphism $c_g : M \to {}^gM$. The function $\xi_{M,c} : G \times G \to L^*$ depending on the choices $\{c_g\}$ is defined as follows:

$$\xi_{M,c}(g,h) = c_{gh}^{-1} \circ {}^g c_h \circ c_g \in Aut_{\mathcal{C}_L}(M) \cong L^*$$

Proposition 3.6.2. *The function $\xi_{M,c}$ is a 2-cocycle.*

Proof. We need to check

$${}^{g_1}\xi(g_2, g_3)\xi(g_1, g_2g_3) = \xi(g_1g_2, g_3)\xi(g_1, g_2)$$

We may think of the right hand side as a scalar function $M \to M$, which allows us to write it as

$$\begin{aligned}
\xi(g_1g_2,g_3)\xi(g_1,g_2) =& c_{g_1g_2g_3}^{-1} \circ {}^{g_1g_2}c_{g_3} \circ c_{g_1g_2} \circ c_{g_1g_2}^{-1} \circ {}^{g_1}c_{g_2} \circ c_{g_1} \\
=& c_{g_1g_2g_3}^{-1} \circ {}^{g_1}({}^{g_2}c_{g_3} \circ c_{g_2}) \circ c_{g_1} \\
=& c_{g_1g_2g_3}^{-1} \circ {}^{g_1}c_{g_2g_3} \circ {}^{g_1}(c_{g_2g_3}^{-1} \circ {}^{g_2}c_{g_3} \circ c_{g_2}) \circ c_{g_1} \\
=& c_{g_1g_2g_3}^{-1} \circ {}^{g_1}c_{g_2g_3} \circ {}^{g_1}\xi(g_2,g_3) \circ c_{g_1} \\
=& c_{g_1g_2g_3}^{-1} \circ {}^{g_1}c_{g_2g_3} \circ c_{g_1} \circ {}^{g_1}\xi(g_2,g_3) \\
=& \xi(g_1,g_2g_3)\, {}^{g_1}\xi(g_2,g_3)
\end{aligned}$$

In the penultimate line, we may commute the c_{g_1} and ${}^{g_1}\xi(g_{2,}g_3)$ because the latter is in L. □

Remark 3.6.3. If $\xi_{M,c} = 1$, then the collection $\{c_g\}$ form a descent datum for M which is effective by Galois descent for abelian categories, Lemma 3.5.8.

Proposition 3.6.4. *Given an $M \in Ob\mathcal{C}_L$ as above, if $\xi_{M,c}$ is a coboundary, then M is in the essential image of Ind_K^L, i.e. M descends.*

PROOF. If $\xi_{M,c}$ is a coboundary, there exists a function $\alpha : G \to L^*$ such that

$$\xi_{M,c}(g,h) = \frac{{}^g\alpha(h)\alpha(g)}{\alpha(gh)}$$

Now, set $c'_g = \frac{c_g}{\alpha(g)} : M \to {}^gM$ and note that the c'_g are a descent datum for M because the associated $\xi_{M,c'} = 1$. □

Proposition 3.6.5. *Given $M \in \mathcal{C}_L$ as above and two choices $\{c_g\}$ and $\{c'_g\}$ of isomorphisms, $\xi_{M,c}$ and $\xi_{M,c'}$ differ by a coboundary and thus give the same class in $H^2(G, L^*)$. We may therefore unambiguously write ξ_M for the cohomology class associated to M.*

PROOF. Note that $(c'_g)^{-1} \circ c_g : M \to M$ is in L^*. This ratio will be a function $\alpha : G \to L^*$ exhibiting the ratio $\frac{\xi_{M,c}}{\xi_{M,c'}}$ as a coboundary. □

Corollary 3.6.6. *Let $\mathcal{C}$ be an abelian K-linear category, let L/K be a finite Galois extension with group G, and let $\mathcal{C}_L$ be the base-changed category. Let M be an object of $\mathcal{C}_L$ such that*

(1) *$M \cong {}^gM$ for all $g \in G$*

(2) *The natural map $L \to End_{\mathcal{C}_L}(M)$ is an isomorphism.*

Then the cocycle ξ_M (as in Definition 3.6.1) is 0 in $H^2(K, \mathbb{G}_m)$ if and only if M descends.

Proof. Combine Propositions 3.6.4 and 3.6.5. □

Remark 3.6.7. We did not have to assume that $L \to End_{\mathcal{C}_L}(M)$ was an isomorphism for a cocycle to exist. A necessary assumption is that there exists a collection $\{c_g\}$ of isomorphisms such that $\xi_{M,c}(g,h) \in L^*$ for all $g, h \in G$. The key is that H^2 exists as long as the coefficients are abelian. Note, however, that in this level of generality there is no guarantee that cohomology class is unique: it depends very much on the choice of the isomorphisms $\{c_g\}$. Therefore, this technique *will not* be adequate to prove that objects do not descend; we can only prove than an object does descend by finding a collection $\{c_g\}$ whose associated ξ_c is a coboundary.

Note 3.6.8. Note that if $\mathcal{C}_{\mathbb{R}}$ is the category of real representations of a compact group and $\mathcal{C}_{\mathbb{C}}$ is the complexification, namely the category of complex representations of a compact group, then this 2-cocycle has a more classical name: "Frobenius-Schur Indicator". It tests whether an irreducible complex representation of a compact group with a real character can be defined over $\mathbb{R}$. If not, the representation is called quaternionic.

Now, suppose $\mathcal{C}$ is a K-linear rigid abelian $\otimes$ category. If $M \in \mathrm{Ob}(\mathcal{C}_L)$ such that ${}^gM \cong M$ for all $g \in G$ and $L \to End_{\mathcal{C}_L}M$ is an isomorphism, then the same is true for M^*. Moreover, choosing $\{c_g\}$ for M gives the natural choice of $\{(c_g^*)^{-1}\}$ for M^* so $\xi_M^{-1} = \xi_{M^*}$.

In general, if M and N are as above with choices of isomorphisms $\{c_g : M \to {}^gM\}$ and $\{d_g : N \to {}^gN\}$ with associated cohomology classes ξ_M and ξ_N respectively, then we can cook up a cohomology class to possibly detect whether $M \otimes N$ descends, $\xi_M\xi_N$, using the isomorphism $c_g \otimes d_g : M \otimes N \to {}^gM \otimes {}^gN \cong {}^g(M \otimes N)$. This is interesting because in general $M \otimes N$ might have endomorphism algebra larger than L; in particular, we weren't guaranteed the existence of a cohomology class $\xi_{M\otimes N}$, as discussed in Remark 3.6.7.

Lemma 3.6.9. *Let $\mathcal{C}$ be a K-linear rigid abelian $\otimes$ category, L/K a finite Galois extension with group G, and $\mathcal{C}_L$ the base-changed category. Let $M \in Ob\mathcal{C}_L$ have endomorphism algebra L and suppose ${}^gM \cong M$ for all $g \in G$. Then $\underline{End}(M) \cong M \otimes M^*$ descends to $\mathcal{C}_K$.*

PROOF. As noted above, if ξ is the cocycle associated to M, then ξ^{-1} is the cocycle associated to M^*. Then $1 = \xi\xi^{-1}$ is a cocycle associated to $M\otimes M^*$, whence it descends. $\square$

Note 3.6.10. A related classical fact: let V be a finite dimensional complex representation V of a compact group G. Then the representation $\underline{\mathrm{End}}(V) \cong V \otimes V^*$ is defined over $\mathbb{R}$. One proof of this uses the fact that $\underline{\mathrm{End}}(V)$ has an invariant symmetric form: the trace.

We now give two criteria for descent. Though the second is strictly more general than the first, the hypotheses are more complicated and we found it helpful to separate the two.

Lemma 3.6.11. *Let $F : \mathcal{C} \to \mathcal{D}$ be a K-linear functor between abelian K-linear categories. Let L/K be a finite Galois extension with group G and let $F_L : \mathcal{C}_L \to \mathcal{D}_L$ be the base-changed functor. Let $M \in Ob(\mathcal{C}_L)$ be an object such that ${}^gM$ is isomorphic to M for $g \in G$ and $End_{\mathcal{C}_L}M \cong L$. Suppose $End_{\mathcal{D}_L}F_L(M) \cong L$. Then M descends to $\mathcal{C}$ if and only if $F_L(M)$ descends to $\mathcal{D}$.*

PROOF. I claim that if we choose isomorphisms $\{d_g : F_L(M) \to {}^gF_L(M) \cong F_L({}^gM)\}$ we can choose isomorphisms $\{c_g : M \to {}^gM\}$ such that $d_g = F_L(c_g)$. This follows because $End_{\mathcal{C}_L}M \tilde{\to} End_{\mathcal{D}_L}F_L(M)$ implies that

$$Hom_{\mathcal{C}_L}(M, {}^gM) \to Hom_{\mathcal{D}_L}(F_L(M), {}^gF_L(M))$$

is an isomorphism. Therefore the cocycles $\xi_{F(M)}$ and ξ_M are the same. $\square$

Lemma 3.6.12. *Let $F : \mathcal{C} \to \mathcal{D}$ be a K-linear functor between K-linear abelian categories and let L/K be a finite Galois extension with group G. Let $F_L : \mathcal{C}_L \to \mathcal{D}_L$ be the base-changed functor. Suppose $M \in Ob(\mathcal{C}_L)$ with $L \cong End_{\mathcal{C}_L}M$ and ${}^gM \cong M$ for all $g \in G$. Further suppose $F_L(M) \cong N_1 \oplus N_2$ satisfying the following two conditions*

- $L \cong End_{\mathcal{D}_L} N_1$
- *There is no nonzero morphism* $N_1 \to {}^gN_2$ *in* $\mathcal{D}_L$ *for any* $g \in G$.

Then M *(and* N_2*) descend if and only if* N_1 *descends.*

Proof. The composition

$$\begin{aligned} Hom_{\mathcal{C}_L}(M, {}^gM) \to & Hom_{\mathcal{D}_L}(F_L(M), {}^gF_L(M)) \\ & \cong Hom_{\mathcal{D}_L}(N_1 \oplus N_2, {}^gN_1 \oplus {}^gN_2) \to Hom_{\mathcal{D}_L}(N_1, {}^gN_1) \end{aligned}$$

is a homomorphism of L-vector spaces. In fact, the map is nonzero because an isomorphism $c_g : M \to {}^gM$ is sent to the isomorphism $F_L(c_g)$. By the second assumption, this projects to an isomorphism $N_1 \to {}^gN_1$. By the first assumption on N_1, the total composition is an isomorphism. Therefore a collection $\{n_g : N_1 \to {}^gN_1\}$ is canonically the same as a collection $\{m_g : M \to {}^gM\}$ and thus we have $\xi_M = \xi_{N_1}$. □

We now examine the relation between the rank of M and the order of its induced Brauer class when $\mathcal{C}$ is assumed to be a K-linear Tannakian category. Recall that Tannakian categories are not necessarily neutral, i.e. they do not always admit a fiber functor to Vect_K. For the remainder of this section, K is supposed to be a field of characteristic 0. Recall that Tannakian categories have a natural notion of rank (see, for instance, Deligne [**Del07**].) If P is an object of rank 1, there is a natural diagram

$$End(P) \cong P \otimes P^* \rightleftarrows \mathrm{K}$$

where the top arrow is evaluation (i.e. the trace) and the bottom arrow comes from the K-vector-space structure of $End(P)$. As $P \otimes P^*$ has rank 1, these two maps identify $End(P)$ isomorphically with K.

Corollary 3.6.13. *Let* K *be a field of characteristic 0, let* $\mathcal{C}$ *be a* K*-linear Tannakian category and let* P *be a rank-1 object. Then* $K \cong End(P)$.

Proposition 3.6.14. *Let* K *be a field of characteristic 0 and let* $\mathcal{C}$ *be a* K*-linear Tannakian category which is further supposed to be neutral, i.e. comes equipped with fiber functor*

$F : \mathcal{C} \to Vect_K$. Let L/K be a finite Galois extension and let $\mathcal{C}_L$ be the base-changed category. Let $P \in Ob\mathcal{C}_L$ be a rank-1 object. If ${}^gP \cong P$ for all $g \in G$ then P descends.

Proof. Denote by F_L the base-changed fiber functor. By definition, P being rank 1 means that $F_L(P)$ is a rank 1 L-vector space, so $L \cong End\big(F_L(P)\big) \cong End(P)$. All vector spaces descend, so by Lemma 3.6.11, P descends as well. □

Lemma 3.6.15. *Let K be a field of characteristic 0, let $\mathcal{C}$ be a K-linear Tannakian category, and let L/K be a finite Galois extension. Suppose $M \in Ob\mathcal{C}_L$ has rank r, ${}^gM \cong M$ for all $g \in G$, and $L \cong End_{\mathcal{C}_L}M$. If $\bigwedge^r M$ descends (which is automatically satisfied e.g. if $\mathcal{C}$ is neutral by Proposition 3.6.14), then ξ_M is r-torsion in $H^2(G, L^*)$. In particular, there exists a degree r extension of K over which M is defined.*

Proof. Pick $\{c_g : M \to {}^gM\}$ giving the cohomology class ξ_M. The isomorphisms

$$\{c_g^{\otimes r} : M^{\otimes r} \to ({}^gM)^{\otimes r} \cong {}^g(M^{\otimes r})\}$$

preserve the space of anti-symmetric tensors and restrict to give isomorphisms

$$c_g^{\otimes r} : \bigwedge^r M \to {}^g\bigwedge^r M$$

The cohomology class associated to $c_g^{\otimes r}$ is ξ^r, and this cohomology class is unique because $L \cong End \bigwedge^r M$ as $\bigwedge^r M$ is a rank 1 object. As we assumed $\bigwedge^r M$ descends, we deduce that $\xi^r = 0 \in H^2(G, L^*)$. □

3.7. F-isocrystals on perfect fields

Throughout this section, let k be a perfect field. We write $W(k)$ for the Witt vectors of k and $K(k)$ for the field of fractions. We also denote σ the lift of the absolute Frobenius automorphism of k. Our references in this section will be Ehud de Shalit's notes [**dS**] and Grothendieck's course [**GdMDdm74**].

Definition 3.7.1. An F-crystal on k is a pair (L, F) with L a finite free $W(k)$-module and $F : L \to L$ a σ-linear injective map. A morphism of F-crystals $\Phi : (L, F) \to (L', F')$ is a map $L \to L'$ making the square commute. We denote the category of F-crystals by F-Crys(k).

This category is a $\mathbb{Z}_p$-linear additive category with an internal $\otimes$: $(L, F) \otimes (L', F') := (L \otimes L', F \otimes F')$. It notably does not have dual objects. The category of F-isocrystals will remedy this.

Definition 3.7.2. An F-isocrystal on k is a pair (V, F) with V a finite dimensional vector space over $K(k)$ and $F : V \to V$ a σ-linear bijective map. A morphism of F-isocrystals $\Phi : (V, F) \to (V', F')$ is a map $V \to V'$ that makes the square commute. We denote the category of F-isocrystals on k by F-Isoc(k).

F-Isoc(k) has internal homs, duals and tensor products given by the natural formulas. For instance, $(V, F) \otimes (V', F') := (V \otimes V', F \otimes F')$. Similarly, the dual $(V, F)^*$ is defined as (V^*, F^*) where V^* is the dual of V as a $K(k)$ vector space and F^* is defined by the formula $F^*(f)(v) = f(F(v))$. The rank of an F-isocrystal (V, F) is $\dim_{K(k)} V$. The category of F-isocrystals on k is a $\mathbb{Q}_p$-linear non-neutral Tannakian category with fiber functor to $Vect_{K(k)}$.

Given an F-isocrystal $E = (V, F)$ on k and a perfect field extension $i : k \hookrightarrow k'$, we define the pull back $E_{k'} = i^*E$ to be $(V \otimes_{K(k)} K(k'), F')$ where F' acts by F on V and σ on $K(k')$. The Dieudonne-Manin classification theorem classifies F-isocrystals over algebraically closed fields by their slopes".

Definition 3.7.3. Let $\lambda = \frac{s}{r}$ be a rational number with $r > 0$. We define the F-isocrystal E^λ over $\mathbb{F}_p$ as $(\mathbb{Q}_p[F]/(F^r - p^s), *F)$. For k a perfect field we define the $E^\lambda_k := i_k^* E^\lambda$ for the canonical inclusion $i_k : \mathbb{F}_p \hookrightarrow k$. The rank of E^λ is r.

Proposition 3.7.4. *Suppose $\lambda \neq \mu$. Then $Hom_{F\text{-}Isoc(k)}(E^\lambda_k, E^\mu_k) = 0$.*

Proof. This is Proposition 3.3 of [**dS**] □

Theorem 3.7.5. *(Manin-Dieudonne) Let k be an algebraically closed field of characteristic p. Then the category F-Isoc(k) is a semisimple abelian category with simple objects the E^λ_k. In particular, we have a direct sum decomposition of the abelian category*

$$F\text{-}Isoc(k) \cong \bigoplus_{\lambda \in \mathbb{Q}} F\text{-}Isoc(k)^\lambda$$

where F-Isoc$(k)^\lambda$ is the full abelian subcategory of F-Isoc(k) generated by $E^\lambda_{\bar{k}}$. We call objects in F-Isoc$(k)^\lambda$ "isoclinic of slope λ".

Definition 3.7.6. Let k be a perfect field and let E be an F-isocrystal over k. Over $\bar{k}$, $E_{\bar{k}} \cong \oplus(E^\lambda_{\bar{k}})$. The slopes of E are the elements of the multiset $\{\lambda\}$ that occur in this decomposition.

Corollary 3.7.7. *Let k be a perfect field of characteristic p and let E be an F-isocrystal on k. Let $\lambda = \frac{s}{r}$ be a slope of E written in lowest terms. Then λ occurs a multiple of r times.*

An F-isocrystal E is called isoclinic of slope λ if only one distinct λ occurs, possibly with multiplicity. The category F-Isoc$(k)^\lambda$ is defined to be the full abelian subcategory of F-isocrystals that are isoclinic of slope λ. If $E^{\lambda_1}$and E^{λ_2} are simple F-isocrystals on k with slopes λ_1 and λ_2 respectively, then $E^{\lambda_1} \otimes E^{\lambda_2}$ is isoclinic of slope $\lambda_1 + \lambda_2$ and $(E^{\lambda_1})^*$ is simple of slope $-\lambda_1$.

Remark 3.7.8. In Proposition 5.3 of [**dS**], there is another equivalent definition of isoclinic of slope λ. An object (V, F) of F-Isoc(k) is said to be isoclinic with slope $\lambda = \frac{s}{r}$ if the sum of all $\mathcal{O}_{K(k)}$-submodules M of V with $F^r(M) = p^s M$ is V.

Proposition 3.7.9. *Let k be a perfect field. There is a direct sum decomposition of abelian categories:*

$$F\text{-}Isoc(k) \cong \bigoplus_\lambda F\text{-}Isoc(k)^\lambda$$

In summary, given an F-isocrystal M on k, there exists a canonical direct sum decomposition $M \cong \oplus M^\lambda$ into isoclinic factors. For further details, see Proposition 5.3 of [**dS**].

We now discuss extension of scalars of F-isocrystals and the inherited slope decomposition. As noted above, F-Isoc(k) is an abelian $\mathbb{Q}_p$-linear tensor category. Given any p-adic local field $L/\mathbb{Q}_p$ we define F-Isoc$(k)_L$ to be the base-changed category. We refer to objects of this category as "F-isocrystals on k with coefficients in L". By the decomposition of abelian categories F-Isoc$(k) \cong \oplus F$-Isoc$(k)^\lambda$, Proposition 3.5.2 implies

$$F\text{-Isoc}(k)_L \cong \bigoplus \left(F\text{-Isoc}(k)^\lambda\right)_L$$

We may therefore unambiguously refer to the "abelian category of λ-isoclinic F-isocrystals with coefficients in L", which we denote by F-Isoc$(k)^\lambda_L$. This direct sum decomposition also allows us to speak about the slope decomposition (and the slopes) of F-isocrystals with coefficients in L. The category F-Isoc$(k)_L$ has a natural notion of rank. If $E^{\lambda_1}$and E^{λ_2} are simple F-isocrystals on k (with coefficients in a field L) with slopes λ_1 and λ_2 respectively, then $E^{\lambda_1} \otimes E^{\lambda_2}$ is isoclinic of slope $\lambda_1 + \lambda_2$ and $(E^{\lambda_1})^*$ is simple of slope $-\lambda_1$.

Lest this discussion seem abstract, there is the following concrete description of F-Isoc$(k)_L$.

Proposition 3.7.10. *The category F-Isoc$(k)_L$ is equivalent to the category of pairs (V, F) where V is a finite free module over $K(k) \otimes_{\mathbb{Q}_p} L$ and $F : V \to V$ is a $\sigma \otimes 1$-linear bijective map. The rank of (V, F) is the rank of V as a free $K(k) \otimes L$-module.*

Remark 3.7.11. Note that $K(k) \otimes L$ is *not necessarily* a field; rather, it is a direct product of fields and $\sigma \otimes Id$ permutes the factors. It is a field iff L and $K(k)$ are linearly disjoint over $\mathbb{Q}_p$. This occurs, for instance, if L is totally ramified over $\mathbb{Q}_p$ or if the maximal finite subfield of k is $\mathbb{F}_p$.

Proof. By definition, an object $((V', F'), f)$ of F-Isoc$(k)_L$ consists of (V', F') an F-isocrystal on k and a $\mathbb{Q}_p$-algebra homomorphism $f : L \to End_{F\text{-Isoc}(k)}(V', F')$. Recall that V' is a finite dimensional $K(k)$ vector space and F' is a σ-linear bijective map. This gives V' the structure of a finite module (not a priori free) over $K(k) \otimes_{\mathbb{Q}_p} L$. The bijection F' commutes with the action of L, hence F' is $\sigma \otimes 1$-linear. We need only prove that V' is a free $K(k) \otimes_{\mathbb{Q}_p} L$-module.

Let L° be the maximal unramified subfield of L and let M be the maximal subfield of L° contained in $K(k)$. This notion is unambiguous: M is the unramified extension of $\mathbb{Q}_p$ with residue field the intersection of the maximal finite subfield of k and the residue field of the local field L. Note that L and $K(k)$ are linearly disjoint over M, so $K(k) \otimes_M L$ is a field. Let r be the degree of the extension $M/\mathbb{Q}_p$. Then $K(k) \otimes_{\mathbb{Q}_p} L$ is the direct product $\prod_{i=1}^r (K(k) \otimes_M L)_i$ and $\sigma \otimes 1$ permutes the factors transitively. Because of this direct product decomposition, V' can be written as $\prod_{i=1}^{r} V_i'$ with each V_i' a $K(k) \otimes_M L$-vector space. As F' is $\sigma \otimes 1$ linear and bijective, F' transitively permutes the factors V_i' and hence the dimension of each V_i' as a $K(k) \otimes_M L$ vector space is the same. This implies that V is a free $K(k) \otimes_{\mathbb{Q}_p} L$-module. □

Definition 3.7.12. Fix once and for all a compatible family $(p^{\frac{1}{n}}) \in \overline{\mathbb{Q}_p}$ of roots of p. Let $\overline{\mathbb{Q}_p}(-\frac{a}{b})$ be the following rank 1 object in $F\text{-Isoc}(\mathbb{F}_p)_{\overline{\mathbb{Q}_p}}$ (using the description furnished by Proposition 3.7.10)

$$(\overline{\mathbb{Q}_p} < v >, *p^{\frac{a}{b}})$$

Abusing notation, given any perfect field k of characteristic p, we similarly denote the pullback to k by $\overline{\mathbb{Q}_p}(-\frac{a}{b})$. In terms of Proposition 3.7.10, it is given by the rank 1 module $K(k) \otimes \overline{\mathbb{Q}_p} < v >$ with $F(v) = p^{\frac{a}{b}} v$ and extended $\sigma \otimes 1$-linearly.

3.8. F-isocrystals on finite fields

We describe in a more concrete way the slopes of an object M in $F\text{-Isoc}(\mathbb{F}_q)_L$ where L is a p-adic local field, using the explicit description given by Proposition 3.7.10. Our conventions are: $q = p^d$, $\mathbb{Z}_q := W(\mathbb{F}_q)$ and $\mathbb{Q}_q := \mathbb{Z}_q \otimes \mathbb{Q}$. As always, σ denotes the lift of the absolute Frobenius, here viewed as a ring automorphism of $\mathbb{Z}_q$ or $\mathbb{Q}_q$. Whenever we use the phrase "p-adic valuation", we always mean the valuation v_p normalized such that $v_p(p) = 1$.

Proposition 3.8.1. *Let (V, F) be an F-isocrystal on $\mathbb{F}_q$ with coefficients in L a p-adic local field. Then F^d is a $\mathbb{Q}_q \otimes L$-linear endomorphism of V. Let $P_F(t) \in \mathbb{Q}_q \otimes L[t]$ be the characteristic polynomial of F^d acting on V. Then $P_F(t) \in L[t]$.*

Proof. The ring automorphism $\sigma \otimes 1$ has order d so F^d is a linear endomorphism on the free $\mathbb{Q}_q \otimes L$-module V. The characteristic polynomial $P_F(t)$ a priori has coefficients in $\mathbb{Q}_q \otimes L$, so we must show that the coefficients of $P_F(t)$ are invariant under $\sigma \otimes 1$. Recall that the coefficients of the characteristic polynomial of an operator A are, up to sign, the traces of the exterior powers of A. As there is a notion of $\bigwedge$ for F-isocrystals it is enough to show that trace(F^d) is invariant under $\sigma \otimes 1$.

To do this, pick a $\mathbb{Q}_q \otimes L$ basis $\{v_i\}$ of V. Let $S := (s_{ij})$ be the "matrix" of F in this basis, i.e. $F(v_i) = \sum_j s_{ij} v_j$. Then an easy computation shows that the matrix of F^d in this basis is given by

$$({}^{\sigma^{d-1}\otimes 1}S)({}^{\sigma^{d-2}\otimes 1}S)\cdots({}^{\sigma\otimes 1}S)(S)$$

Then ${}^{\sigma\otimes 1}\mathrm{trace}(F^d) = \mathrm{trace}({}^{\sigma\otimes 1}(F^d)) = \mathrm{trace}\big((S)({}^{\sigma^{d-1}\otimes 1}S)\cdots({}^{\sigma^2\otimes 1}S)({}^{\sigma\otimes 1}S)\big) = \mathrm{trace}(F^d)$ because $\mathrm{trace}(AB) = \mathrm{trace}(BA)$. □

Proposition 3.8.2. *Let (V, F) be an F-isocrystal on $\mathbb{F}_q$ (with coefficients in $\mathbb{Q}_p$). Then F^d is a $\mathbb{Q}_q$-linear endomorphism of V hence has a monic characteristic polynomial $P_F(t) \in \mathbb{Q}_p[t]$ by the previous proposition. The slopes of (V, F) are $\frac{1}{d}$ times the p-adic valuations of the roots of $P_F(t)$.*

Proof. This is in Katz's paper [**Kat79**], see the remark after Lemma 1.3.4 where he cites Manin. Alternatively, see Grothendieck's letter to Barsotti from 1970 at the end of [**GdMDdm74**]. (Note that Grothendieck differs in choice of normalization: he choses $v(q) = 1$, which allows him to avoid the factor of $\frac{1}{d}$ above.) Alternatively, see Corollaire VI.3.4.2.2 of [**Saa72**] where it is deduced from the elegant equivalence of (Tannakian) categories: $F\text{-Isoc}(\mathbb{F}_q)_{\mathbb{Q}_q} \cong F\text{-Isoc}(\mathbb{F}_p)_{\mathbb{Q}_q}$. □

Proposition 3.8.3. *Let (V, F) be an F-isocrystal on $\mathbb{F}_q$ with coefficients in L. Then F^d is a $\mathbb{Q}_q \otimes L$-linear endomorphism of V with characteristic polynomial $P_F(t) \in L[t]$. The slopes of (V, F) are $\frac{1}{d}$ times the p-adic valuations of the roots of $P_F(t)$.*

Proof. Consider the diagram

$$\bigoplus_{\lambda} F\text{-Isoc}(k)^{\lambda} \cong F - \mathrm{Isoc}(k) \rightleftarrows F\text{-Isoc}(k)_L \cong \bigoplus_{\lambda} F - \mathrm{Isoc}(k)^{\lambda}_L$$

where the top functor is induction and the bottom functor is restriction. By definition these functors respect the slope decomposition; for instance, they send isoclinic objects to isoclinic objects. Let n be the degree of $L/\mathbb{Q}_p$. Given $(V, F) \in \mathrm{Ob}F\text{-Isoc}(\mathbb{F}_q)_L$, let $(V', F') = Res^L_{\mathbb{Q}_p}(V, F)$ be the restriction, an F-isocrystal on $\mathbb{F}_q$ with coefficients in $\mathbb{Q}_p$. On the one hand, the slopes of (V', F') are the slopes of (V, F) repeated n times. On the other hand, $P_{F'}(t)$ is the norm of $P_F(t)$ with regards to the map $\mathbb{Q}_p[t] \hookrightarrow L[t]$, so the multiset of p-adic valuations of the roots $P_{F'}(t)$ is the multiset of p-adic valuations of the roots of $P_F(t)$ repeated n times. By Proposition 3.8.2, we know that the p-adic valuations of the roots of $P_{F'}(t)$ are the slopes of (V', F'), so by the above discussion the same is true for (V, F). $\square$

Example 3.8.4. The slope of $\overline{\mathbb{Q}_p}(-\frac{a}{b})$, an object of F-Isoc$(\mathbb{F}_p)_{\overline{\mathbb{Q}_p}}$, is $\frac{a}{b}$.

We now classify rank 1 objects $(V, F) \in \mathrm{Ob}\ F\text{-Isoc}(\mathbb{F}_q)_L$, again using the explicit description in Proposition 3.7.10. As discussed above, the eigenvalue of F^d is in L. The slogan of Proposition 3.8.5 is that this eigenvalue determines (V, F) up to isomorphism. This discussion will allow us to classify semi-simple F-isocrystals on $\mathbb{F}_q$.

Let v be a free generator of V over $\mathbb{Q}_q \otimes_{\mathbb{Q}_p} L$. Then $F(v) = av$ with $a \in (\mathbb{Q}_q \otimes_{\mathbb{Q}_p} L)^*$ and $F^d(v) = (\mathrm{Nm}_{\mathbb{Q}_q \otimes L/L} a)v$ where the norm is taken with respect to the cyclic Galois morphism $L \to \mathbb{Q}_q \otimes L$. Therefore, any $\lambda \in L$ that is in the image of the norm map $\mathrm{Nm} : (\mathbb{Q}_q \otimes L)^* \to L^*$ can be realized as the eigenvalue of F^d. We know that $E^{\lambda} \otimes E^{-\lambda}$ is rank 1 and has unique eigenvalue 1, so to prove that F^d determines a rank 1 F-isocrystal, we need only prove that there is a unique rank 1 F-isocrystal with F^d the identity map.

Proposition 3.8.5. *Let (V, F) be a rank 1 F-isocrystal on $\mathbb{F}_q$ with coefficients in L. Suppose F^d is the identity map. Then (V, F) is isomorphic to the trivial F-isocrystal, i.e. I can pick a generator $v \in V$ such that $F(v) = v$.*

Proof. Suppose $F(v) = \lambda v$ where $\lambda \in (\mathbb{Q}_q \otimes L)^*$. Then $F^d(v) = \mathrm{Nm}_{\mathbb{Q}_q \otimes L/L}(\lambda)v$, where the Norm map is defined with respect to the Galois morphism of algebras $L \hookrightarrow Q \otimes L$. This Galois group is cyclic, generated by $\sigma \otimes 1$. Now, $F(av) = ({}^{\sigma\otimes 1}a)\lambda v$ and we want to find an $a \in (\mathbb{Q}_q \otimes L)^*$ such that $F(av) = av$, i.e. $\frac{a}{{}^{\sigma\otimes 1}a} = \lambda$ where λ has norm 1. This is guaranteed by Hilbert's Theorem 90 for the cyclic morphism of algebras $L \hookrightarrow Q \otimes L$. $\square$

Corollary 3.8.6. *Let $M = (V, F)$ be a rank-1 F-isocrystal on $\mathbb{F}_q$ with coefficients in L. If the eigenvalue of F^d lives in finite index subfield $K \subset L$ and is a norm in K with regards to the algebra homomorphism $K \to \mathbb{Q}_q \otimes K$, then M descends to K.*

Proof. Let the eigenvalue of F^d be λ and let $a \in \mathbb{Q}_q \otimes K$ have norm λ. Consider the rank 1 object E of F-Isoc$(\mathbb{F}_q)_K$ given on a basis element e by $F(e) = a^{-1}e$. Then

$E_L \otimes M$ has F^d being the identity map. Proposition 3.8.5 implies that $E_L \otimes M$ is the trivial F-Isocrystal, i.e. that $M \cong (E_L)^* \cong (E^*)_L$. Therefore M descends as desired. $\square$

Example 3.8.7. Consider the object $\overline{\mathbb{Q}_p}(-\frac{1}{2})$ of $F\text{-Isoc}(\mathbb{F}_{p^2})_{\overline{\mathbb{Q}_p}}$. The eigenvalue of F^2 is $p \in \mathbb{Q}_p$, but p is not in the image of the norm map and hence the object does not descend to an object of $F\text{-Isoc}(\mathbb{F}_{p^2})$. One can also see this by virtue of the fact that no non-integral fraction can occur as the slope of a rank 1 object of $F\text{-Isoc}(k)$. Note that the object does in fact descend to $F\text{-Isoc}(\mathbb{F}_p)_{\mathbb{Q}_p(\sqrt{p})}$.

Remark 3.8.8. Consider $\overline{\mathbb{Q}_p}(-\frac{a}{b})$ as an object of $F\text{-Isoc}(\mathbb{F}_q)_{\overline{\mathbb{Q}_p}}$ where $\frac{a}{b}$ is in lowest terms. It is isomorphic to its Galois twists by the group $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ if and only if $p^b \mid q$. Indeed, the isomorphism class of this F-isocrystal is determined by the eigenvalue of F^d, where $p^d = q$, which is $p^{\frac{ad}{b}}$. This is in $\mathbb{Q}_p$ if and only if $b \mid d$.

Corollary 3.8.6 poses a natural question. Let K be a p-adic local field and let $q = p^f$. Which elements of K are in the image of the norm map?

$$\text{Norm} : (\mathbb{Q}_q \otimes K)^* \to K^*$$

If $K = \mathbb{Q}_p$, then $K^* \cong p^{\mathbb{Z}} \times \mathbb{Z}_p^*$, and the image of the norm map is exactly $p^{d\mathbb{Z}} \times \mathbb{Z}_p^*$. More generally, if $K/\mathbb{Q}_p$ is a totally ramified extension with uniformizer ϖ, then $K^* \cong \varpi^{\mathbb{Z}} \times \mathcal{O}_K^*$, $\mathbb{Q}_q \otimes K$ is a field that is unramified over K, and the image of the norm map is exactly $\varpi^{d\mathbb{Z}} \times \mathcal{O}_K^*$. On the other extreme, if $K \cong \mathbb{Q}_q$, then the following proposition shows that the norm map is surjective.

Proposition 3.8.9. *Let L/K be a cyclic extension of fields, with Gal(L/K) generated by g and of order n. Consider the induced Galois morphism of algebras: $L \to L \otimes_K L$. The image of the norm map for this extension is surjective.*

PROOF. $L \otimes_K L \cong \prod_{x \in G} L$, where the L-algebra structure is given by the first (identity) factor. Then g acts by cyclically shifting the factors and the norm of an element $(\dots, l_x, \dots)$ is just $\prod_{x \in G} l_x$. Therefore the norm map is surjective. $\square$

The above proposition (along with local class field theory) suggests that in general the image of the norm map is rather complicated to classify. However, in our case we can say the following.

Lemma 3.8.10. *Let K be a p-adic local field and let $q = p^d$. Then $\mathcal{O}_K^*$ is in the image of the norm map $(\mathbb{Q}_q \otimes K)^* \to K^*$.*

PROOF. $\mathbb{Q}_q \otimes K \cong \prod K'$ where K' is an unramified extension of K and the norm of an element $(\alpha) \in \prod K'$ is the product of the individual norms of the components with respect to the unramified extension K'/K. On the other hand, the image of the norm map $K'^* \to K^*$ certainly contains $\mathcal{O}_K^*$ as K'/K is unramified. □

Corollary 3.8.11. *Let (V, F) be a rank-1 F-isocrystal on $\mathbb{F}_q$ with coefficients in L. If the eigenvalue of F^d lives in finite index subfield $K \subset L$ and (V, F) has slope 0, then (V, F) descends to K.*

Remark 3.8.12. This makes sense because the slope 0 a.k.a unit root part of F-Isoc(k) is a neutral Tannakian subcategory (equivalent to p-adic representation of G_k.) One may therefore apply the "neutral rank-1 descent" Proposition 3.6.14.

We now discuss simple objects in F-Isoc$(\mathbb{F}_q)_{\overline{\mathbb{Q}_p}}$.

Proposition 3.8.13. *Let $(V, F) \in Ob(F\text{-}Isoc(\mathbb{F}_q))_{\overline{\mathbb{Q}_p}}$ be a simple object. Then (V, F) has rank 1.*

PROOF. We prove the contrapositive: if (V, F) has rank greater than 1, it is not simple. V is a free $\mathbb{Q}_q \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}_p}$ module and $F : V \to V$ is a $\sigma \otimes 1$-linear bijection. The ring $\mathbb{Q}_q \otimes \overline{\mathbb{Q}_p}$ is isomorphic to $\prod(\overline{\mathbb{Q}_p})_i$ where i runs over the $\mathbb{Q}_p$-algebra homomorphisms $i : \mathbb{Q}_q \to \overline{\mathbb{Q}_q}$ and $\sigma \otimes 1$ acts as the cyclic shift. Therefore $V \cong \prod V_i$ as well and F, being $\sigma \otimes 1$-linear, cyclically shifts the factors. F^d is a $\overline{\mathbb{Q}_p}$-linear automorphism of V_1 and hence has an eigenvector v_1 with eigenvalue λ. Let $v_i = F^i v$. Note that $v_i \in V_i$ because F cyclically shifts the factors V_i. The sub-F-isocrystal generated by v_1 is generated over $\overline{\mathbb{Q}_p}$ by $v_1, v_2, \ldots, v_{d-1}$ and hence has rank 1 as an F-isocrystal. Therefore (V, F) is not simple. □

Remark 3.8.14. Proposition 3.8.13 is *not necessarily true* if one replaces $\mathbb{F}_q$ with an arbitrary perfect field of characteristic p. For instance, let k be the perfection of the field $\mathbb{F}_p(t)$, i.e. $k \cong \mathbb{F}_p(t^{\frac{1}{p^\infty}})$. Then the rank 2 F-isocrystal given on a basis by the matrix

$$\begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}$$

is absolutely irreducible.

Proposition 3.8.5 implies that an absolutely semi-simple F-isocrystal (V, F) over $\mathbb{F}_q$ with coefficients in L is determined up to isomorphism by $P_F(t)$. Now, Corollary 3.5.11 implies that an object (V, F) is absolutely semi-simple if and only if it is semi-simple. Therefore a semi-simple F-isocrystal on $\mathbb{F}_q$ is entirely determined by the characteristic polynomial of Frobenius.

Note that the semi-simplicity hypothesis is necessary; for instance an F-isocrystal on $\mathbb{F}_p$ is a $\mathbb{Q}_p$ vector space V and a linear endomorphism F, and one can choose two non-conjugate endomorphisms that have the same characteristic polynomial). Alternatively phrased, $P_F(t)$ determines the isomorphism class of the semi-simplification $(V, F)^{ss}$.

Proposition 3.8.15. *Let L/K be a finite Galois extension with group G with K a p-adic local field. Let (V, F) be an F-isocrystal on $\mathbb{F}_q$with coefficients in L and denote $({}^gV, {}^gF) := {}^g(V, F)$. Then*

$$P_{{}^gF}(t) = {}^gP_F(t)$$

Proof. First of all, V is a finite free $\mathbb{Q}_q \otimes L$ module. Let $g \in G$ and consider the object ${}^g(V, F)$. The underlying sets V and ${}^gV$ may be naturally identified, and if $v \in V$, we write ${}^gv$ for the corresponding element of ${}^gV$; here $l.({}^gv) = {}^g(g^{-1}(l).v)$. Pick a free basis $\{v_i\}$ of V and let the "matrix" of F in this basis be S. Then the "matrix" of ${}^gF$ in the basis $\{{}^gv_i\}$ is ${}^gS$. Moreover, the actions of G and σ commute, so $({}^gF)^d = {}^g(F^d)$. Therefore, $P_{{}^gF}(t) = {}^gP_F(t)$ as desired. □

Corollary 3.8.16. *Let L/L_0 be a Galois extension of p-adic local fields with group G. Let $E = (V, F)$ be a semi-simple object of F-Isoc$(\mathbb{F}_q)_L$. Then $P_F(t) \in L_0[t]$ implies that ${}^gE \cong E$ for all $g \in G$.*

Proof. Immediate from Proposition 3.8.15 and the fact that a semi-simple F-isocrystal is determined up to isomorphism by its characteristic polynomials. □

3.9. Crystals and Isocrystals on Varieties

Let X be a smooth variety in characteristic p. Berthelot has defined the absolute crystalline site on X. Let $Crys(X)$ be the category of crystals in finite locally free modules. By functoriality of the crystalline topos, the absolute frobenius $Frob : X \to X$ gives a functor $Frob^* : Crys(X) \to Crys(X)$.

Definition 3.9.1. An F-crystal on X is a pair (M, F) where M is a crystal in finite locally free modules over the crystalline site and $F : Frob^* M \to M$ is an injective map of crystals.

The category F-Crys(X) is a $\mathbb{Z}_p$-linear category with an internal $\otimes$ but without internal homs or duals in general. There is a object $\mathbb{Z}_p(-1)$ which given by the pair $(\mathcal{O}_{cris}, p)$. We denote by $\mathbb{Z}_p(-n)$ the nth tensor power of $\mathbb{Z}_p(-1)$.

Definition 3.9.2. A Dieudonné crystal is a pair (M, F, V) where (M, F) is an F-crystal in finite locally free modules and $V : M \to Frob^* M$ is a map of crystals such that $V \circ F = p$ and $F \circ V = p$.

We define the category F-Crys$^+(X)$ to be F-Crys(X) with the morphism modules tensored with $\mathbb{Q}_p$, i.e. we formally adjoin $\frac{1}{p}$ to all endomorphism rings. In F-Crys$^+(X)$ the image of the object $\mathbb{Z}_p(-1)$ is denoted $\mathbb{Q}_p(-1)$. The category F-Isoc(X) is given by formally inverting $\mathbb{Q}_p(-1)$; the dual to $\mathbb{Q}_p(-1)$ is called the Tate object $\mathbb{Q}_p(1)$. See section 3 in Saavedro Rivano's thesis [**Saa72**] for further details. If $\mathcal{E}$ is an object of F-Isoc(X), we denote by $\mathcal{E}(n)$ the object $\mathcal{E} \otimes \mathbb{Q}_p(n)$. For any object $\mathcal{E}$ of F-Isoc(X) there is an n such that $\mathcal{E}(-n)$ is effective, i.e. is in the essential image of the functor

$$F\text{-Crys}^+(X) \to F\text{-Isoc}(X)$$

If X is a proper variety, an F-isocrystal is supposed to be the $\mathbb{Q}_p$ analogue of a lisse sheaf. F-Isoc(X) is a $\mathbb{Q}_p$-linear Tannakian category that is non-neutral in general. The category of F-isocrystals with coefficients in L, where L is a p-adic local field, is by definition the base-changed category F-Isoc$(X)_L$. There is a notion of the rank of an F-isocrystal that satisfies that expected constraints given by $\otimes$ and $\oplus$.

We note that if we wanted to define "convergent" or "overconvergent" F-isocrystals, we would have to work substantially harder; in particular, there isn't always a globally defined "crystal". Therefore, in our definitions we stick to unadorned F-isocrystals.

Remark 3.9.3. The category F-Isoc(X) has finite-dimensional Hom spaces and every object has finite length.

3.10. F-isocrystals and l-adic local systems on Curves

Let C be a smooth connected complete curve over $\mathbb{F}_q$ and let $\bar{c}$ be a geometric point. Let $\mathscr{L}$ be an irreducible l-adic local system on C with trivial determinant, which one may think of as a continuous representation $\varrho_l : \pi_1(C, \bar{c}) \to SL(n, \overline{\mathbb{Q}_l})$. A Baire Category argument shows that in fact this representation is defined with coefficients in a finite extension of $\mathbb{Q}_l$. We recall Deligne's Conjecture 3.3.2, which tells us that we can find a number field E such that ρ_l fits into an E-compatible system: for every place $\lambda \nmid p$, there is a compatible local system $\mathscr{L}_\lambda$ with coefficients in E_λ and for every place $\lambda | p$, there is a compatible F-isocrystal $\mathscr{E}_\lambda$ on C with coefficients in E_λ.

Example 3.10.1. *The number field E can in general be larger than the field extension of $\mathbb{Q}$ generated by the coefficients of the characteristic polynomials of all Frobenius elements F_x.* For instance, let D be a non-split quaternion algebra over $\mathbb{Q}$ that is split at ∞ and let l and p be finite primes where D splits. The Shimura curve X^D exists as a smooth complete curve (in the sense of stacks) over $\mathbb{F}_p$ and it admits a universal abelian surface $f : \mathcal{A} \to X^D$ with multiplication by $\mathcal{O}_D$. Then $R^1 f_* \mathbb{Q}_l$ is a rank 4 $\mathbb{Q}_l$ local system with an action of $D \otimes \mathbb{Q}_l$. As we assumed l was a split prime, $D \otimes \mathbb{Q}_l \cong M_{2\times 2}(\mathbb{Q}_l)$ and so we can use Morita equivalence, i.e. look at the image of

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_{2\times 2}(\mathbb{Q}_l)$$

to get a rank 2 local system $\mathscr{L}_l$ with coefficients in $\mathbb{Q}_l$; moreover, $\mathscr{L}_l$ has all Frobenius traces in $\mathbb{Q}$ and is pure of weight 1. The local system $\mathscr{L}_l$ has open image and is hence absolutely irreducible

On the other hand, let l' be a prime where D ramifies (these exist as we assumed D was not split). Let $D'_l = D \otimes \mathbb{Q}_l$ be the unique non-split quaternion algebra over $\mathbb{Q}_l$. By picking a geometric point $\bar{c}$ of X^D, we may think of $R^1 f_* \mathbb{Q}_{l'}$ as rank 4 $\mathbb{Q}_{l'}$-representation V of $\pi_1 = \pi_1(X^D, \bar{c})$ with a commuting action of $D_{l'}$. The action of $D_{l'}$ here is the regular representation, thinking of $D_{l'}$ as a rank 4 $\mathbb{Q}_{l'}$-vector space. In particular, the commutant of π_1 in $\mathrm{Mat}_{4\times 4}(\mathbb{Q}_{l'})$ contains $D_{l'}$.

Suppose $\mathscr{L}_{l'}$ was a rank 2 $\mathbb{Q}_{l'}$-local system compatible with $\mathscr{L}_l$, necessarily absolutely irreducible because $\mathscr{L}_l$ has big image. Then $\mathscr{L}_{l'} \oplus \mathscr{L}_{l'} \cong R^1 f_* \mathbb{Q}_{l'}$. Put another way, $V \cong W \oplus W$, where W is a rank 2 $\mathbb{Q}_{l'}$-representation of π_1 that is absolutely irreducible. Then the commutant of π_1 in $\mathrm{Mat}_{4\times 4}(\mathbb{Q}_{l'})$ would be $\mathrm{Mat}_{2\times 2}(\mathbb{Q}_{l'})$, which cannot possibly contain $D_{l'}$. Therefore $\mathscr{L}_{l'}$ does not exist. In particular, even though *all Frobenius traces are in* $\mathbb{Q}$*, not all of the* l*-adic companions can be defined over* $\mathbb{Q}_l$. In other words, they do not form a $\mathbb{Q}$-compatible system. However, we see from this examples that only finitely many of the l-adic companions cannot be defined over $\mathbb{Q}_l$.

Motivated by Example 3.10.1, we have the following conjecture. We believe this conjecture should follow from standard motivic conjectures, only one knows that compatible systems are motivic.

Conjecture 3.10.2. *Let* X *be a smooth, geometrically connected normal variety over* $\mathbb{F}_q$ *and* L *be a number field. Let* $\{\mathscr{L}_\lambda\}_{\lambda \nmid p}$ *be a* L*-compatible system of absolutely irreducible rank* r *local systems with trivial determinant and all characteristic polynomials of Frobenius in a subfield* $K \subset L$*. Then for all but finitely many* λ*,* $\mathscr{L}_\lambda$ *is defined over* K_λ*.*

We believe this conjecture should follow from standard motivic conjectures, once one knows that absolutely irreducible L-compatible systems are motivic in a suitable sense.

Theorem 3.10.3. *(Brauer-Nesbitt) Let Γ be a group and E a characteristic 0 field. Suppose ρ_1 and ρ_2 are two semi-simple finite dimensional representations of Γ with coefficients in E. Then $\rho_1 \cong \rho_2$ if and only if the trace functions are the same.*

Lemma 3.10.4. *Let C be a normal variety over $\mathbb{F}_q$. Let $\mathscr{L}$ be an absolutely irreducible rank n l-adic local system, defined over L, an l-adic local field. Suppose all Frobenius traces are in L_0, an l-adic subfield of L. Further suppose there is a point $x \in C$ such that there is an eigenvalue α of F_x that occurs with multiplicity 1 and $\alpha \in L_0$. Then $\mathscr{L}$ can be defined over L_0.*

Proof. We may suppose the field extension L/L_0 is Galois with group G. The absolute irreducibility hypothesis ensures, by Schur's Lemma (e.g. Lemma 3.5.12), that $L \cong \mathrm{End}(\mathscr{L}_l)$. Let χ_l be the trace of the representation of $\pi_1(C)$ associated to $\mathscr{L}$ (suppressing the implicit basepoint.) By continuity and Chebotarev's Density theorem, χ_l is a function with values in L_0. The Brauer-Nesbitt Theorem 3.10.3 ensures that ${}^g\mathscr{L} \cong \mathscr{L}$ for all $g \in G$.

Let $i : x \to C$ denote the inclusion and consider the restriction functor i^* from the category of L-valued local systems on C to the category L-valued local systems on x. Now, write $i^*\mathscr{L} \cong N_1 \oplus N_2$ where N_1 is the rank one local system with Frobenius eigenvalue α; the direct sum decomposition holds because α occurs with multiplicity 1 in the spectrum of F_x. There is no map $N_1 \to {}^gN_2$ for any $g \in G$, again because α occurs with multiplicity 1, and $L \cong \mathrm{End}(N_1)$. Finally, N_1 descends to L_0 as $\alpha \in L_0$. We may therefore use Lemma 3.6.12 to deduce that $\mathscr{L}$ (and as a corollary N_2) can be descended to L_0. $\square$

Remark 3.10.5. After writing this, we discovered virtually identical group-theoretic analog in the literature: Proposition 7 of Chin [**Chi03**].

Exercise 3.10.6. Let $\mathscr{L}_l$ be an absolutely irreducible rank 2 l-adic local system on C with trivial determinant and infinite monodromy. Suppose all of the Frobenius traces of $\mathscr{L}_l$ are in $\mathbb{Q}$. Show that there does not exist a single closed point $x \in |C|$ such that the characteristic polynomial of F_x is separable and splits over $\mathbb{Q}$. In other words, we cannot possibly use Lemma 3.10.4 in conjunction with a single closed point x to prove that the compatible system $\{\mathscr{L}_l\}$ descends to a $\mathbb{Q}$-compatible system.

We now discuss the p-adic picture. We first record an analog of the Brauer-Nesbitt Theorem 3.10.3 for overconvergent F-isocrystals.

Lemma 3.10.7. *(Abe) Absolutely irreducible overconvergent F-isocrystals with trivial determinant are determined by characteristic polynomials of Frobenius elements.*

Remark 3.10.8. The category F-Isoc(X) is a $\mathbb{Q}_p$-linear Tannakian category. An "absolutely irreducible F-isocrystal" refers to an object of $F\text{-Isoc}(X)_{\overline{\mathbb{Q}_p}}$ of the base-changed category that is irreducible. We do not discuss the overconvergence condition.

Proof. The unique proposition in Section 3 Abe [**Abe11**] shows that certain types of overconvergent F-isocrystals are determined by characteristic polynomials of Frobenius elements. It is shown in [**Abe13**] that all absolutely irreducible overconvergent F-isocrystals with trivial determinant satisfy the relevant hypotheses, which is an analog of ι-mixedness for F-isocrystals. □

Lemma 3.10.9. *Let C a smooth connected curve over $\mathbb{F}_q$, and $\mathscr{E}$ an irreducible rank n overconvergent F-isocrystal on C with coefficients in $\overline{\mathbb{Q}_p}$ such that $\mathscr{E}$ has trivial determinant. By work of Lafforgue and Abe on Conjecture 3.3.2, there is a number field E such that the following holds: for every place $\lambda \nmid p$ there is an E_λ-valued local system $\mathscr{L}_\lambda$ compatible with $\mathscr{E}$ and for every place $\lambda | p$ there is an F-isocrystal $\mathscr{E}_\lambda$ with coefficients in E_λ compatible with our original $\mathscr{E}$. (One of these $\mathscr{E}_\lambda$ will be isomorphic to the original $\mathscr{E}$ after tensoring with $\overline{\mathbb{Q}_p}$.)*

Suppose that for all $\lambda | p$ and for all closed points $x \in |C|$, $i_x^* \mathscr{E}_\lambda$ is an isoclinic F-isocrystal on x. Then "the representation has finite image" in the sense that for every $\lambda \nmid p$, the associated λ-adic representation has finite image. Equivalently, the "motive" can be trivialized by a finite étale cover $C' \to C$

Remark 3.10.10. The theorem also implies that for each $\lambda | p$ the pullback to C' of the F-isocrystal $\mathscr{E}_\lambda$ is also trivial.

Proof. Part 3 of Conjecture 3.3.2 implies that the eigenvalues of F_x are λ-adic units for all $\lambda \nmid p$. On the other hand, for each $\lambda | p$ and for every closed point $x \in |C|$, $i_x^* \mathscr{E}_\lambda$ being isoclinic and having trivial determinant implies the slopes of $i_x^* \mathscr{E}_\lambda$ are 0 and hence that the eigenvalues of F_x are λ-adic units. As eigenvalues of F_x are algebraic numbers, this implies that they are all roots of unity. Moreover, each of these roots of unity lives in a degree n extension of E and there are only finitely many roots of unity that live in such extensions: there are only finitely many roots of unity with fixed bounded degree over $\mathbb{Q}$. Therefore there are only finitely many eigenvalues of F_x as x ranges through the closed points of C.

Now, pick $\lambda \nmid p$ and consider the associated representation $\varrho_\lambda : \pi_1(C, \bar{c}) \to SL(n, E_\lambda)$. By the above discussion, there exists some integer k such that for every closed point $x \in |C|$, the generalized eigenvalues $\varrho(F_x)$ are kˆth roots of unity. But Frobenius elements are dense and ϱ_λ is a continuous homomorphism, so that the same is true for the entire image of ϱ_λ. The image of ϱ_λ therefore only has finitely many traces.

Burnside proved that if $G \subset GL(n, \mathbb{C})$ has finitely many traces and the associated representation is irreducible, then G is finite: see, for instance the proof of 19.A.9 on page 231 of [**Row08**] which directly implies this statement. Thus the entire image of ϱ_λ is finite, as desired. □

Lemma 3.10.11. *Let $L/\mathbb{Q}_p$ be a finite extension, C a smooth connected curve over $\mathbb{F}_q$, and $\mathscr{E}$ an absolutely irreducible F-isocrystal on C of rank n with coefficients in L. Suppose that there exists a closed point $i : x \to C$ such that $i^*\mathscr{E}$ has 0 as a slope with multiplicity 1. Suppose further that for all closed points $c \in |C|$, the characteristic polynomials of Frobenius has coefficients in L_0 for some p-adic subfield $L_0 \subset L$. Then $\mathscr{E}$ can be descended to an F-isocrystal with coefficients in L_0.*

Proof. We proceed exactly as in Lemma 3.10.4. By enlarging L we may suppose that L/L_0 is Galois; let $G = \mathrm{Gal}(L/L_0)$. First, Abe's Lemma 3.10.7 implies that a semi-simple overconvergent F-isocrystal on a smooth curve is determined by the characteristic polynomials of the Frobenius elements at all closed points. In particular, it implies that for any $g \in G$, ${}^g\mathscr{E} \cong \mathscr{E}$ because the characteristic polynomials at all closed points agree by Proposition 3.8.15 and the fact that $\mathscr{E}$ is irreducible (and hence semi-simple.) Now let us use Lemma 3.6.12 for the restriction functor

$$F\text{-Isoc}(C) \to F\text{-Isoc}(x)$$

Because 0 occurs as a slope with multiplicity 1 in $i^*\mathscr{E}$ we can write $i^*\mathscr{E} \cong N_1 \oplus N_2$ by the isoclinic decomposition. Here N_1 has rank 1 and unique slope 0 while no slope of N_2 is 0. There are no maps between N_1 and any Galois twist of N_2 because twisting an F-isocrystal does not change the slope. The endomorphism algebra of any rank-1 object in $F\text{-Isoc}(\mathbb{F}_q)_L$ is L by Corollary 3.6.13. Now, $\text{End}(\mathscr{E}) \cong L$ by Lemma 3.5.12, a.k.a. Schur's Lemma, because $\mathscr{E}$ is absolutely irreducible.

Finally, we must argue that N_1 descends to L_0. Let $x = \text{Spec}(\mathbb{F}_{p^d})$. The fact that the slope 0 occurs exactly once in $i^*\mathscr{E}$ implies that the eigenvalue α of F^d on N_1 is an element of L_0. Moreover, $\alpha \in \mathcal{O}_{L_0}^*$ because N_1 is slope 0. By Corollary 3.8.11, N_1 descends to L_0. The hypotheses of Lemma 3.6.12 are all satisfied and we may conclude that $\mathscr{E}$ (and N_2) descends to L_0. □

Proposition 3.10.12. *Let C be a smooth connected complete curve over $\mathbb{F}_q$ and let $\mathscr{L}$ be an absolutely irreducible rank 2 l-adic local system with trivial determinant, infinite image, and all Frobenius traces in a number field E. There exists*

1. *A positive rational number $\frac{s}{r}$*
2. *A finite field extension $\mathbb{F}_{q'}/\mathbb{F}_q$ with C' denoting the base change of C to $\mathbb{F}_{q'}$ (q' will be the least power of q divisible by p^r)*
3. *A place $\lambda|p$ of E*
4. *An object $\mathscr{E}$ of F-Isoc$(C')_{E_\lambda}$*

such that $\mathscr{E} \otimes \overline{\mathbb{Q}_p}(-\frac{s}{r})$ is compatible with $\mathscr{L}$.

The point of Proposition 3.10.12 is that the field of definition of $\mathscr{E}$ is E_λ, a completion of the field of traces, in (4).

Proof. As we assumed $\mathscr{L}$ had infinite image, Lemma 3.10.9 and Deligne's Conjecture 3.3.2 imply that there is an extension L/E, WLOG Galois, and a place $\lambda|p$ of L such that there exists an object $\mathscr{M} \in F\text{-Isoc}(C)_{L_\lambda}$ that is compatible with $\mathscr{L}$ and such that the general point is not isoclinic. We abuse notation and denote the restriction of λ to E by

λ again. The object $\mathscr{M}$ is isomorphic to its twists by $\mathrm{Gal}(L_\lambda/E_\lambda)$ by Abe's Lemma 3.10.7 because the characteristic polynomials of all closed points are in E. Pick a closed point $i : x \to C$ such that $i^*\mathscr{M}$ has slopes $(-\frac{s}{r}, \frac{s}{r})$. Let q' be the smallest power of q that is divisible by p^r and let C' denote the base change of C to $\mathbb{F}_{q'}$.

Consider the twist $\mathscr{E} := \mathscr{M} \otimes \overline{\mathbb{Q}_p}(-\frac{s}{r})$, thought of as an F-isocrystal on C' with coefficients in $\overline{\mathbb{Q}_p}$. We have enlarged q to q' where $p^r|q'$ and Remark 3.8.8 says that $\overline{\mathbb{Q}_p}(-\frac{s}{r})$, considered as object of $F-\mathrm{Isoc}(\mathbb{F}_{q'})_{\overline{\mathbb{Q}_p}}$, is isomorphic to all of its twists by $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$. Therefore, the characteristic polynomials of the Frobenius elements F_x on $\mathscr{E}$ are elements of $E_\lambda[t]$. Abe's Lemma 3.10.7 then implies that $\mathscr{E}$ is isomorphic to all of its Galois twists by $\mathrm{Gal}(\overline{\mathbb{Q}_p}/E_\lambda)$. At the point "$x$", the slopes are now $(0, \frac{2s}{r})$. Apply Lemma 3.10.11 to descend $\mathscr{E}$ to the field of traces E_λ, as desired. □

We now specialize to the case where $\mathscr{L}$ is a rank 2 l-adic local system with trivial determinant, infinite image, and having all Frobenius traces in a number field E where p splits completely. Proposition 3.10.12 implies that, up to extension of the ground field $\mathbb{F}_q$, we can find an F-isocrystal $\mathscr{E}$ with *coefficients in* $\mathbb{Q}_p$ that is compatible with $\mathscr{L}$ up to a twist. Moreover, by construction, there is a point x such that $\mathscr{E}_x$ is not isoclinic and the slopes are $(0, \frac{2s}{r})$. On the one hand the slope of the determinant of $\mathscr{E}_x$ is necessarily an integer because the coefficients of $\mathscr{E}$ are $\mathbb{Q}_p$. Therefore $\frac{2s}{r}$ is a positive integer. On the other hand, by Lafforgue's slope bounds (Theorem 3.3.5), $\frac{2s}{r} \leq 1$, so $\frac{s}{r} = \frac{1}{2}$ and $\det(\mathscr{E}) \cong \mathbb{Q}_p(-1)$. We record this analysis in the following important corollary.

Corollary 3.10.13. *Let C be a smooth connected complete curve over $\mathbb{F}_q$ and let $\mathscr{L}$ be an absolutely irreducible rank 2 l-adic local system with trivial determinant, infinite image, and all Frobenius traces in $\mathbb{Q}$. Suppose $p^2 \mid q$. Then there exists a unique absolutely irreducible F-isocrystal $\mathscr{E}$ with coefficients in $\mathbb{Q}_p$ that is compatible with $\mathscr{L} \otimes \mathbb{Q}_l(-\frac{1}{2})$. By construction, $\mathscr{E}$ is generically ordinary i.e. there exists a closed point $x \in |C|$ such that $\mathscr{E}_x$ has slopes $(0, 1)$.*

Proof. Only the uniqueness needs to be proved. As $p^2 \mid q$, the character $\overline{\mathbb{Q}_l}(\frac{1}{2})$ in fact descends to a character $\mathbb{Q}_l(\frac{1}{2})$ over $\mathbb{F}_q$ (because q is a quadratic residue mod l) and the coefficients of the characteristic polynomials of the Frobenius elements on $\mathscr{L} \otimes \mathbb{Q}_l(-\frac{1}{2})$

are all in $\mathbb{Q}$. By Abe's Lemma 3.10.7, any absolutely irreducible F-isocrystal on C is uniquely determined by these characteristic polynomials as there is only one embedding $\mathbb{Q} \to \overline{\mathbb{Q}_p}$. □

Lemma 3.10.14. *Let C a smooth connected complete curve over $\mathbb{F}_q$, and $\mathscr{E}$ an absolutely irreducible rank-2 F-isocrystal on C with coefficients in $\mathbb{Q}_p$. Suppose $\mathscr{E}$ has determinant $\mathbb{Q}_p(-1)$. If for all closed points $x \in |C|$ the restriction $\mathscr{E}_x$ is not isoclinic, then $\mathscr{E}$ is not irreducible.*

Proof. As we are assuming $\mathscr{E}$ is rank 2 and has coefficients in $\mathbb{Q}_p$, if either of the slopes of $\mathscr{E}_x$ were non-integral, the slopes would have to be $(\frac{1}{2}, \frac{1}{2})$. If the slopes were integral, they just have to be integers (a, b) that sum to 1. On other other hand, the Slope Bounds of 3.3.5 shows that in this case, the slopes must be $(0, 1)$. As we assumed $\mathscr{E}_x$ is never isoclinic, this implies that every $\mathscr{E}_x$ has slopes $(0, 1)$ for all closed points $x \in |C|$. The slope filtration (see Katz [**Kat79**]) is therefore non-trivial: the point is that if at every $x \in |C|$, the slope 0 occurs, then there is in fact a rank 1 sub-object whose slopes at all closed points x are 0. □

Remark 3.10.15. There is a subtle point to Lemma 3.10.14. Let X be be the modular curve $Y(1)$ minus the supersingular points, over $\mathbb{F}_p$. The universal elliptic curve yields an (overconvergent) F-isocrystal $\mathscr{E}$ on X where all the points are ordinary. By Lemma 3.10.14, $\mathscr{E}$ is not irreducible as an F-isocrystal. However, it is irreducible as an overconvergent F-isocrystal. The point is that the slope filtration does not give a sub-object which is overconvergent. In particular, the slope filtration is not motivic. However, for proper curves this issue does not arise: the slope filtration will automatically give a convergent (and hence overconvergent) F-isocrystal.

Combining Lemma 3.10.14 with the $\mathbb{Q}_p$ companion $\mathscr{E}$ constructed in Corollary 3.10.13, we see that $\mathscr{E}$ is generically ordinary and has (finitely many) supersingular points. In particular, the slopes of $\mathscr{E}_x$ for $x \in |C|$ are either $(0, 1)$ or $(\frac{1}{2}, \frac{1}{2})$.

Let C be smooth curve over $\mathbb{F}_q$ and let $\mathscr{E}$ be an F-isocrystal on C. We are interested in when $\mathscr{E}$ "comes from an F-crystal". That is, there is a functor $F\text{-Cris}(C) \to F\text{-Isoc}(C)$ as in Section 3.9 and we would like to characterize the essential image of this functor.

Lemma 3.10.16. *Let C be a smooth geometrically connected curve over $\mathbb{F}_q$ and let $\mathscr{E}$ be an F-isocrystal on C. Then $\mathscr{E}$ comes from an F-crystal if and only if all of the slopes at all closed points $c \in \mid C \mid$ are positive. Furthermore, $\mathscr{E}$ comes from a Dieudonné crystal if and only if all of the slopes at all closed points $c \in \mid C \mid$ are between 0 and 1.*

Proof. This is the trick in Katz [**Kat79**]. Strictly speaking, Katz proves that if all of the slopes of an F-crystal $\mathscr{M}$ are bigger than some positive number λ, then $\mathscr{M}$ is isogenous to an F-crystal divisible by p^λ. From our conventions in Section 3.9, $\mathscr{E} \otimes \mathbb{Q}_p(-n)$ has an underlying F-crystal $\mathscr{M}$ for all $n \gg 0$. Katz's trick shows then that $\mathscr{M}$ is isogenous to an F-crystal $\mathscr{M}'$ that is divisible by p^n. Dividing F by p^n on $\mathscr{M}'$ produces the required F-crystal.

Finally, given any F-crystal $\mathscr{M}$ with slopes no greater than 1, define $V = F^{-1} \circ p$. This makes $\mathscr{M}$ into a Dieudonné crystal. □

Definition 3.10.17. Let S be a normal scheme over k, a field of characteristic p. We define the category $BT(S)$ to be the category of Barsotti-Tate ("p-divisible") groups on S.

For an introduction to p-divisible groups and their contravariant Dieudonné theory, see Grothendieck [**GdMDdm74**] or Berthelot-Breen-Messing [**BBM82**]. In particular, there exist Dieudonné functors: given a BT group $\mathscr{G}/S$ one can construct a Dieudonné crystal $\mathbb{D}(\mathscr{G})$ over S. The main theorem of [**dJ95**] is the following.

Theorem 3.10.18. *(de Jong) Let S be a smooth scheme over k, a field of characteristic p. Then the category $BT(S)$ is equivalent to the category of Dieudonné crystals on S via $\mathbb{D}$.*

Definition 3.10.19. Let k be a field of characteristic p and let G be a BT group over k. We say G is *ordinary* if $G[p]$ has no local-local part. We say G is *supersingular* if all of the slopes of $\mathbb{D}(G)$ are $\frac{1}{2}$.

If G is a height 2 dimension 1 BT group over k, then G is ordinary iff the slopes of $\mathbb{D}(G)$ are $(0,1)$, $(0,0)$, or $(1,1)$. On the other hand, G is supersingular if and only if $G[p]$ is local-local.

Corollary 3.10.20. *Let C be a smooth connected complete curve over $\mathbb{F}_q$ and let $\mathscr{L}$ be an absolutely irreducible rank two l-adic local system with trivial determinant, infinite image, and all Frobenius traces in $\mathbb{Q}$. There is a BT group $\mathscr{G}$ on C with the following properties*

- *$\mathscr{G}$ has height 2 and dimension 1.*
- *$\mathscr{G}$ is generically ordinary with supersingular points: There is a non-empty dense open subset $U \subset C$ such that for every closed point $u \in U$, $\mathscr{G}_u$ has slopes $(0,1)$ and for every closed point $t \in C \setminus U$, $\mathscr{G}_t$ has slopes $(\frac{1}{2}, \frac{1}{2})$.*
- *The Dieudonné crystal $\mathbb{D}(\mathscr{G})$ is compatible with $\mathscr{L}(-\frac{1}{2})$.*

$\mathscr{G}$ has the following weak uniqueness property: the F-isocrystal $\mathbb{D}(\mathscr{G})$ is unique.

Proof. The uniqueness comes from the following facts: an absolutely irreducible F-isocrystal with trivial determinant is uniquely specified by Frobenius eigenvalues (Theorem 3.10.7), $\mathbb{D}(\mathscr{G})$ is absolutely irreducible because it has both ordinary and supersingular points, and there is a unique embedding $\mathbb{Q} \hookrightarrow \overline{\mathbb{Q}_p}$. □

3.11. Kodaira-Spencer

This section discusses the deformation theory of BT groups in order to obtain uniqueness statements in Corollary 3.10.20. The main references are Illusie [**Ill85**] and Xia [**Xia13b**].

Let S be a smooth scheme over a perfect field k. Given a BT group $\mathscr{G}$ over S and a closed point $s \in S$ there is a map of formal schemes $d_s : S^{/s} \to Def(\mathscr{G}_s)$ to the universal deformation space of $\mathscr{G}_s$. If $\mathscr{G}$ is height 2 and dimension 1, then the dimension of the universal deformation space is 1, as in the familiar elliptic modular case. There is also a related Kodaira-Spencer map: $KS_s : T_{S,s} \to V_s$ where V_s is a vector space related to (the co-Lie complex of) $\mathscr{G}_s$. Illusie (A.2.3.6 of [**Ill85**]) proves that, in the case of $\mathscr{G}$ being height 2 and dimension 1, KS_s is an isomorphism if and only if d_s is formally étale. In this case, we say $\mathscr{G}$ is versally deformed at s.

Lemma 3.11.1. *(Xia's Frobenius Untwisting Lemma) Let S be a smooth curve over a perfect field k of characteristic p. Let $\mathscr{G}$ be a height 2 dimension 1 BT group over. Then $\mathscr{G}/S$ has trivial KS map at the generic point if and only if there exists a BT group $\mathscr{H}$ on S such that $\mathscr{H}^{(p)} \cong \mathscr{G}$.*

Proof. This is Theorem 4.13 in [**Xia13b**]. □

Definition 3.11.2. Let $\mathscr{G}$ be a BT group on an irreducible scheme S, smooth over a perfect field k. We say $\mathscr{G}$ is *generically versally deformed* if there is a non-empty open $U \subset S$ such that for every closed point $u \in U$, $\mathscr{G}$ is versally deformed at u. We say $\mathscr{G}$ is *everywhere versally deformed* if for every closed point $s \in S$, $\mathscr{G}$ is versally deformed at s.

Example 3.11.3. Let us recall the Igusa level structures described in Remark 2.1.14. Let $Y(1) = \mathcal{M}_{1,1}$. There is a universal elliptic curve $\mathcal{E} \to Y(1)$. Let $\mathscr{G} = \mathcal{E}[p^\infty]$ be the associated p-divisible group over $Y(1)$. Here, $\mathscr{G}$ is height 2, dimension 1, and *everywhere versally deformed* on $Y(1)$. Let X be the cover of $Y(1)$ that trivializes the finite flat group scheme $\mathscr{G}[p]^{ét}$ away from the supersingular locus of $Y(1)$. X is branched exactly at the supersingular points. Pulling back $\mathscr{G}$ to X yields a BT group that is generically versally deformed but *not everywhere versally deformed* on X.

Xia's Lemma 3.11.1 allows us to set up the following useful equivalence for characterizing when a height 2 dimension 1 BT group $\mathscr{G}$ is generically versally deformed.

Lemma 3.11.4. *Let S be a smooth curve over a perfect field k of characteristic p. Let $\mathscr{G}$ be a height 2, dimension 1 BT group over S. Let η be the generic point of S. Suppose $\mathscr{G}_\eta$ is ordinary. Then the following are equivalent*

(1) *The KS map is 0.*
(2) *There exists a BT group $\mathscr{H}$ on S such that $\mathscr{H}^{(p)} \cong \mathscr{G}$.*
(3) *There exists a finite flat subgroup scheme $N \subset \mathscr{G}$ over S such that N has order p and is generically étale.*
(4) *The connected-étale exact sequence $\mathscr{G}_\eta[p]^\circ \to \mathscr{G}_\eta[p] \to \mathscr{G}_\eta[p]^{ét}$ over the generic point splits.*

Proof. The equivalence of (1) and (2) is Xia's Lemma 3.11.1. Now, let us assume (2). Then there is a Verschiebug map $V_{\mathscr{H}} : \mathscr{G} \to \mathscr{H}$ whose kernel is generically étale and has order p because we assumed $\mathscr{G}$ (and hence $\mathscr{H}$) were ordinary. Therefore (3) is satisfied. Conversely, given (3), N is p-torsion (Deligne proved that all finite flat commutative group schemes are annihilated by their order. For a proof, see section 1 of [**TO70**].) Therefore we have a factorization:

$$\begin{array}{ccc} \mathscr{G} & \xrightarrow{\times p} & \mathscr{G}/N \\ & \searrow \quad \nearrow & \\ & \mathscr{G} & \end{array}$$

Set $\mathscr{H} = \mathscr{G}/N$. As we assumed N was generically étale and $\mathscr{G}$ was generically ordinary, the map $\mathscr{G} \to \mathscr{G}/N$ may be identified with Verschiebung: $\mathscr{H}^{(p)} \to \mathscr{H}$. In particular, $\mathscr{H}^{(p)} \cong \mathscr{G}$ as desired.

Let us again assume (3). Then $N_\eta \subset \mathscr{G}_\eta[p]$ projects isomorphically onto $\mathscr{G}_\eta[p]^{ét}$. Therefore the connected-étale sequence over the generic point splits. To prove the converse, simply note that I can take the Zariski closure of the section of $\mathscr{G}_\eta[p] \to \mathscr{G}_\eta[p]^{ét}$ inside of $\mathscr{G}[p]$ to get N. □

Lemma 3.11.5. *Let S be a smooth curve in characteristic p. Suppose $\mathscr{G}$ and $\mathscr{G}'$ are height 2, dimension 1 BT groups over S that are generically versally deformed and generically ordinary. Suppose further that their Dieudonné Isocrystals are isomorphic: $\mathbb{D}(\mathscr{G}) \otimes \mathbb{Q} \cong \mathbb{D}(\mathscr{G}') \otimes \mathbb{Q}$. Then $\mathscr{G}$ and $\mathscr{G}'$ are isomorphic.*

Proof. The isocrystals being isomorphic implies that their is an isogeny $\mathbb{D}(\mathscr{G}) \to \mathbb{D}(\mathscr{G}')$. De Jong's Theorem 3.10.18 then implies that their is an associated isogeny $\phi : \mathscr{G} \to \mathscr{G}'$. By "dividing by p", we may ensure that ϕ does not restrict to 0 on $\mathscr{G}[p]$. Now suppose for contradiction that ϕ is not an isomorphism, i.e. that it has a kernel. Then $\phi|_{\mathscr{G}[p]}$ is also has a nontrivial kernel.

We have the following diagram of connected-generically étale sequences.

$$\begin{array}{ccc}
\mathscr{G}[p]^\circ & \longrightarrow & \mathscr{G}'[p]^\circ \\
\downarrow & & \downarrow \\
\mathscr{G}[p] & \longrightarrow & \mathscr{G}'[p] \\
\downarrow & & \downarrow \\
\mathscr{G}[p]^{\acute{e}t} & \longrightarrow & \mathscr{G}'[p]^{\acute{e}t}
\end{array}$$

As we have assumed $\mathscr{G}$ is generically versally deformed, the kernel of $\phi|_{\mathscr{G}[p]}$ cannot be generically étale by (3) of Lemma 3.11.4. Thus the kernel must be the connected group scheme $\mathscr{G}[p]^\circ$ because the order of $\mathscr{G}[p]$ is p^2. We therefore get a nonzero map $\mathscr{G}[p]^{\acute{e}t} \to \mathscr{G}'[p]$. Now $\mathscr{G}[p]^{\acute{e}t}$ has order p and is generically étale by definition, so by (3) of Lemma 3.11.4, $\mathscr{G}'$ is not generically versally deformed. Thus the assumption that ϕ had a kernel yields that $\mathscr{G}'$ is not generically versally deformed, contradicting our original hypothesis; ϕ must therefore be an isomorphism. □

Theorem 3.11.6. *Let C be a smooth, geometrically irreducible, complete curve over $\mathbb{F}_q$. Suppose $p^2|q$. There is a natural bijection between the following two sets.*

$$\left\{\begin{array}{l}
\overline{\mathbb{Q}_l}\text{-local systems } \mathscr{L} \text{ on } C \text{ such that} \\
\bullet\, \mathscr{L} \text{ is irreducible of rank 2} \\
\bullet\, \mathscr{L} \text{ has trivial determinant} \\
\bullet \text{ The Frobenius traces are in } \mathbb{Q} \\
\bullet\, \mathscr{L} \text{ has infinite image,} \\
\text{up to isomorphism}
\end{array}\right\}
\longleftrightarrow
\left\{\begin{array}{l}
p\text{-divisible groups } \mathscr{G} \text{ on } C \text{ such that} \\
\bullet\, \mathscr{G} \text{ has height 2 and dimension 1} \\
\bullet\, \mathscr{G} \text{ is generically versally deformed} \\
\bullet\, \mathscr{G} \text{ has all Frobenius traces in } \mathbb{Q} \\
\bullet\, \mathscr{G} \text{ has ordinary and supersingular points,} \\
\text{up to isomorphism}
\end{array}\right\}$$

such that if $\mathscr{L}$ corresponds to $\mathcal{G}$, then $\mathscr{L} \otimes \mathbb{Q}_l(-1/2)$ is compatible with the F-isocrystal $\mathbb{D}(\mathscr{G}) \otimes \mathbb{Q}$.

Proof. Given such an $\mathscr{L}$, we can make a BT group $\mathscr{G}$ as in Corollary 3.10.20. Xia's Lemma 3.11.1 ensures that we can modify $\mathscr{G}$ to be generically versally deformed by Frobenius "untwisting"; this process terminates because there are both supersingular and ordinary points, so the map to the universal deformation space cannot be identically 0. This BT group is unique up to (*non-unique*) isomorphism by Lemma 3.11.5.

To construct the map in the opposition direction, just reverse the procedure. Given such a $\mathscr{G}$, first form Dieudonné isocrystal $\mathbb{D}(\mathscr{G})\otimes\mathbb{Q}$. This is an absolutely irreducible F-isocrystal because there are both ordinary and supersingular points. Twisting by $\overline{\mathbb{Q}_p}(1/2)$ yields an F-isocrystal that has trivial determinant. By Abe's resolution of Deligne's Conjecture, for any $l \neq p$ there is a compatible $\mathscr{L}_l$ that is absolutely irreducible and has all Frobenius traces in $\mathbb{Q}$. This $\mathscr{L}_l$ is unique: there is only one embedding of $\mathbb{Q}$ in $\overline{\mathbb{Q}_l}$ and the Brauer-Nesbitt Theorem 3.10.3 ensures that $\mathscr{L}_l$ is uniquely determined by the Frobenius traces. Finally, $\mathscr{L}_l$ has infinite image: as $\mathscr{G}$ had both ordinary and supersingular points, it cannot be trivialized on a finite étale cover. □

Theorem 3.11.7. *Let* $X \xleftarrow{f} Z \xrightarrow{g} X$ *be a correspondence of hyperbolic curves over* $\mathbb{F}_q$. *Let* $\mathscr{L}_l$ *be an absolutely irreducible rank 2 l-adic local system on* X *with Frobenius traces in* $\mathbb{Q}$ *and infinite image. Suppose further that* $f^*\mathscr{L}_l \cong g^*\mathscr{L}_l$. *Let* $\mathscr{G}$ *be the BT group on* X *constructed in Theorem 3.11.6. Then* $f^*\mathscr{G} \cong g^*\mathscr{G}$.

Proof. This follows immediately from the uniqueness statement in Theorem 3.11.6. □

3.12. Algebraization and Finite Monodromy

In Theorem 3.11.6, the finiteness of the number of such local systems (a theorem whose only known proof goes through the Langlands correspondence) implies the finiteness of such BT groups. In general, BT groups on varieties are far from being "algebraic": for instance, over $\mathbb{F}_p$ there are uncountably many BT groups of height 2 and dimension 1 as one can see from Dieudonné theory. However, here they are constructed rather indirectly from a motive via Lafforgue's proof. All examples of such local systems that we can construct involve abelian schemes and we are very interested in the following question.

Question 3.12.1. *Let X be a smooth, connected, complete curve over $\mathbb{F}_q$ and let $\mathscr{G}$ be a BT group as in Theorem 3.11.6. Does $\mathscr{G}$ come from an "abelian motive" on X? What if we drop the assumption on the Frobenius traces being in $\mathbb{Q}$?*

Question 3.12.1 is imprecise, due primarily to the phrase "come from". We have several related conjectures that are more precise.

Conjecture 3.12.2. *Let X be a smooth, connected, complete curve over $\mathbb{F}_q$ and let $\mathscr{G}$ be a p-divisible group as in Theorem 3.11.6 (but not requiring that Frobenius traces are in $\mathbb{Q}$.) Does there exist a variety Y and a height 2 dimension 1 p-divisible group $\mathscr{H}$ on Y that is **everywhere** versally deformed such that $\mathscr{G}$ is pulled back from $\mathscr{H}$?*

$$\begin{array}{ccc} \mathscr{G} & \longrightarrow & \mathscr{H} \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

Remark 3.12.3. There are examples involving fake Hilbert modular surfaces to show that one cannot hope for Y to be a curve in Conjecture 3.12.2.

To motivate the next conjecture, recall that the modular curves are not complete. On the other hand, the universal local systems on Shimura curves parameterizing fake elliptic curves cannot all be defined over $\mathbb{Q}_l$, as we saw in Example 3.10.1. In other words, they do not form a $\mathbb{Q}$-compatible system.

Conjecture 3.12.4. *Let X be a smooth, connected, complete curve over $\mathbb{F}_q$. Suppose $\{\mathscr{L}_l\}_{l\neq p}$ is a $\mathbb{Q}$-compatible system of absolutely irreducible rank 2 local systems with trivial determinant. Then they have finite monodromy.*

Using our techniques we can prove the related Corollary 3.12.7. Note that, compared to Conjecture 3.12.4, there is exactly one additional hypothesis in Corollary 3.12.7: namely, that the p-adic companion of $\{\mathscr{L}_l\}_{l\neq p}$ has coefficients in $\mathbb{Q}_p$. Note also that Corollary 3.12.7 does not assume that X is complete. The proof of Corollary 3.12.7 is a simple consequence of the following Theorem 3.12.5.

Theorem 3.12.5. *Let X be a smooth, connected curve over $\mathbb{F}_q$ and let $\mathscr{E}$ be an rank 2 absolutely irreducible F-Isocrystal on X with trivial determinant, coefficients in $\mathbb{Q}_p$, and all Frobenius traces in $\mathbb{Q}$. Then $\mathscr{E}$ has finite monodromy.*

Proof. We claim that $\mathscr{E}$ is isoclinic at every closed point $x \in |X|$. First of all, the triviality of the determinant implies that the slopes are $(-a, a)$. As the coefficients of $\mathscr{E}$ are $\mathbb{Q}_p$, any fractional slope must appear more than once and hence $a \in \mathbb{Z}$. Then, by the Slope Bounds Theorem 3.3.5, the slopes of $\mathscr{E}_x$ differ by at most 1, forbidding slopes of the form $(-a, a)$ for $0 \neq a \in \mathbb{Z}$. Therefore, the slopes of $\mathscr{E}_x$ are $(0, 0)$.

By Abe's Lemma 3.10.7, absolutely irreducible overconvergent F-isocrystals are determined the characteristic polynomials of Frobenius elements. As there is a *unique* embedding $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$, this implies that any p-adic companion to $\mathscr{E}$ is isomorphic to $\mathscr{E}$ itself. Therefore, we may conclude by Lemma 3.10.9. □

Remark 3.12.6. Note that Theorem 3.12.5 uses Lemma 3.10.9 which critically uses Deligne's Conjecture 3.3.2. In particular, we use that $\mathscr{E}$ lives in a compatible system.

Corollary 3.12.7. *Let X be a smooth, connected curve over $\mathbb{F}_q$. Let $\{\mathscr{L}_l\}_{l \neq p}$ be a $\mathbb{Q}$-compatible system of absolutely irreducible rank 2 local systems with trivial determinant. Suppose the p-adic companion $\mathscr{E}$ has coefficients in $\mathbb{Q}_p$. Then they have finite monodromy.*

Proof. As $\{\mathscr{L}_l\}$ is a $\mathbb{Q}$-compatible system and $\mathscr{E}$ has coefficients in $\mathbb{Q}_p$, for every closed point $x \in |X|$, the Frobenius traces are in $\mathbb{Q}$. We may therefore apply Theorem 3.12.5. □

Conclusion 3.12.8. Let $\pi : X \to C$ be a smooth projective morphism where C is a smooth curve over $\mathbb{F}_q$. Let i be an even number and $c \in |C|$. Suppose there is a $\mathbb{Q}$-submotive M of "$R^i\pi_*\mathbb{Q}$" that has rank 2 and absolutely irreducible l-adic monodromy. Then M is trivialized by a finite cover of C.

Proof. This conclusion is not quite a rigorous mathematical statement. Nonetheless, the point is that $R^i\pi_{cris,*}\mathbb{Q}_p$ is an F-isocrystal on C with coefficients in $\mathbb{Q}_p$. As i is even, we can twist the compatible system by $\mathbb{Q}(\frac{i}{2})$ to get a $\mathbb{Q}$-compatible system with p-adic companion *defined over* $\mathbb{Q}_p$. Apply Corollary 3.12.7. □

Example 3.12.9. Suppose $\pi : X \to C$ is a smooth projective morphism where H^2 of a fiber has rank 3. Then the orthogonal complement to the ample class gives a sub-motive of rank 2 with weight 2. If this sub-motive gives an absolutely irreducible Galois representation, the corollary implies that this sub-motive has finite monodromy.

3.13. Deforming the Correspondence

In this section, we indicate how one could *canonically* deform a correspondence from characteristic p to characteristic 0 in certain auspicious circumstances. The key tool is the following lifting theorem of Xia [**Xia13b**].

Theorem 3.13.1. *Let k be an algebraically closed field of characteristic p and let X be a smooth proper hyperbolic curve over k. Let $\mathscr{G} \to X$ be a height 2, dimension 1 BT group over X. If $\mathscr{G}$ is everywhere versally deformed on X, then there is a unique curve $\tilde{X}$ over $W(k)$ which is a lift of X and admits a lift $\tilde{\mathscr{G}}$ of $\mathscr{G}$. Furthermore, the lift $\tilde{\mathscr{G}}$ is unique.*

Proof. This is Theorem 1.2 in [**Xia13b**]. □

Remark 3.13.2. This may be thought of as a generalization of the Serre-Tate canonical lift.

Example 3.13.3. Let D be a non-split quaternion algebra over $\mathbb{Q}$ that is split at ∞ and let p be a finite prime where D splits. The Shimura curve X^D exists as a smooth complete curve (in the sense of stacks) over $\mathbb{Z}[\frac{1}{2d}]$ and hence over $\mathbb{F}_p$. Abusing notation, we denote by X^D the Shimura curve over $\mathbb{F}_p$. It admits a universal abelian surface $f : \mathcal{A} \to X^D$ with multiplication by $\mathcal{O}_D$. As $D \otimes \mathbb{Q}_p \cong M_{2\times 2}(\mathbb{Q}_p)$, one can use Morita equivalence (as in Example 3.10.1), i.e. apply the idempotent

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

on the height 4 dimension 1 BT group $\mathcal{A}[p^\infty]$ to get a height 2 dimension 1 BT group $\mathscr{G}$, and in fact $\mathcal{A}[p^\infty] \cong \mathscr{G} \oplus \mathscr{G}$. Here $\mathscr{G}$ is *everywhere versally deformed* on X^D. Moreover, $\mathscr{G}$ has both supersingular and ordinary points. For similar discussion, see Section 5.1 of [**GK05**].

On the other hand, as in Example 3.11.3, let X_p^D be the Igusa cover which trivializes $\mathscr{G}[p]^{et}$ on the ordinary locus. This cover is branched precisely over the supersingular locus of $\mathscr{G}$ on X. Pulling back $\mathscr{G}$ to X_p^D yields a BT group which is *generically versally deformed* but not everywhere versally deformed.

Theorem 3.13.4. *Let $X \xleftarrow{f} Z \xrightarrow{g} X$ be an étale correspondence of projective hyperbolic curves over $\mathbb{F}_q$. Suppose $\mathscr{G}$ is a height 2 dimension 1 BT group on X that is everywhere versally deformed. Suppose further that $f^*\mathscr{G} \cong g^*\mathscr{G}$. Then the correspondence together with the BT groups lift uniquely to a correspondence over $W(\mathbb{F})$*

$$\begin{array}{ccc} & \tilde{Z} & \\ \tilde{f} \swarrow & & \searrow \tilde{g} \\ \tilde{X} & & \tilde{X} \end{array}$$

If the original correspondence did not have a core, the lifted correspondence does not have a core.

Proof. By Xia's Lifting Theorem 3.13.1, there is a canonical lift $\tilde{X}$ such that $\mathscr{G}$ also lifts to a BT group $\tilde{\mathscr{G}}$. As f and g are étale, there are unique lifts $\tilde{f} : \tilde{Z}_f \to \tilde{X}$ and $\tilde{g} : \tilde{Z}_g \to \tilde{X}$ lifting f and g respectively. Note that $\tilde{f}^*\tilde{\mathscr{G}}$ and $\tilde{g}^*\tilde{\mathscr{G}}$ are BT groups lifting $f^*\mathscr{G}$ and $g^*\mathscr{G}$. On the other hand, f and g being étale means that $f^*\mathscr{G} \cong g^*\mathscr{G}$ is everywhere versally deformed. Therefore, the uniqueness in Theorem 3.13.1 applied to the pair $(Z, f^*\mathscr{G} \cong g^*\mathscr{G})$ implies that $\tilde{Z}_f \cong \tilde{Z}_g$ and the correspondence lifts as desired.

If the lifted correspondence had a core, then the original correspondence has a core by Corollary 2.2.14. □

Theorem 3.13.5. *Let $X \xleftarrow{f} Z \xrightarrow{g} X$ be an étale correspondence of projective hyperbolic curves without a core over $\mathbb{F}_q$. Let $\mathscr{G}$ is a height 2 dimension 1 BT group on X that is everywhere versally deformed. Suppose further that $f^*\mathscr{G} \cong g^*\mathscr{G}$. Then X and Z are the reductions mod p of Shimura curves.*

Proof. Use Theorem 3.13.4 to lift the correspondence to characteristic 0 and apply Mochizuki's Theorem 2.1.8. □

Remark 3.13.6. We comment on the likelihood of being the situation of Theorem 3.13.4. We have already discussed the likelihood of Assumption 3.2.1 in Remark 3.2.2. Using Conclusion 3.4.1 (which is not rigorous mathematics, as we have not yet carefully written up the proof), we get an étale correspondence of projective hyperbolic curves without a core over $\mathbb{F}_q$

$$\begin{array}{ccc} & Z & \\ {\scriptstyle f}\swarrow & & \searrow{\scriptstyle g} \\ X & & X \end{array}$$

together with $\mathscr{L}_2$, an absolutely irreducible rank 2 local system with coefficients in $\mathbb{Q}_2$ such that $f^*\mathscr{L}_2 \cong g^*\mathscr{L}_2$. If we further assume that Frobenius traces are in $\mathbb{Q}$, we may use Theorem 3.11.6 to construct $\mathscr{G}$ on X that is height 2, dimension 1, and generically versally deformed. Moreover, Theorem 3.11.7 implies that $f^*\mathscr{G} \cong g^*\mathscr{G}$. If $\mathscr{G}$ were generically versally deformed, then we could directly apply Theorem 3.13.4. Unfortunately, we have seen examples (e.g. the Igusa Example 3.13.3) where $\mathscr{G}$ is not everywhere versally deformed. In particular, these examples really do occur in nature. In these cases, however, the BT groups are pulled back from correspondences where the BT groups are everywhere versally deformed. We hope that an affirmative answer to Conjecture 3.12.2 would allow us to prove this in general: namely, that there exists an étale correspondence

$$\begin{array}{ccc} & Z_{vers} & \\ \swarrow & & \searrow \\ X_{vers} & & X_{vers} \end{array}$$

together with a BT group $\mathscr{G}_{vers}$ that is *everywhere versally deformed* on X_{vers} such that $\mathscr{G}$ is pulled back from $\mathscr{G}_{vers}$ via a map of the correspondences. We do not yet know how to prove this from Conjecture 3.12.2.

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