On Fourier-Mukai type functors

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ABSTRACT

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In this thesis we study functors between bounded derived categories of sheaves and how they can be expressed in a geometric way, namely whether they are isomorphic to a Fourier-Mukai transform. Specifically, we describe the behavior of a functor between derived categories of smooth projective varieties when restricted to the derived category of the generic point of the second variety, when this last variety is a curve, a point or a rational surface. We also compute in general some sheaves that play the role of the cohomology sheaves of the kernel of a Fourier-Mukai transform and are then able to exhibit a class of functors that are neither faithful nor full, that are isomorphic to a Fourier-Mukai transform.

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Chapter 1

Introduction

Derived categories were first introduced in 1963 by Verdier [Ver77] [Ver96], who carried out ideas by Grothendieck. They were initially designed with the purpose of formulating and proving an extension of Serre's duality theorem [Ser54] which was accomplished in [Gro63] and published in [Har66].

Derived categories have since established themselves as a fundamental tool in algebraic geometry as well as in a number of other disciplines, including the study of systems of partial differential equations, microlocal analysis, and representation theory of Lie algebras and algebraic groups.

Concerning algebraic geometry, classical applications include work of Beilinson [Bei78] and Berstein-Gelfand [Ber78] on relating coherent sheaves on projective space to representations of certain finite dimensional algebras, as well as work of Rickard on Morita theory [Ric89][Ric91].

Interest in recent years has been renewed by applications to birational geometry in relation to the minimal model program [Kaw05][Kaw06], and theoretical physics, in particular to string theory [KL02].

Grothendieck's key observation was that the constructions of homological algebra don't actually just yield cohomology groups, and in fact passing to cohomology means forgetting a large amount of information. What we actually obtain are complexes that are well-defined up to quasi-isomorphism, so that two complexes should be considered the same if there is a map between them inducing an isomorphism on all cohomology groups.

The derived category $D(\mathscr{A})$ of an abelian category $\mathscr{A}$ is thus obtained by taking the category $K(\mathscr{A})$ of complexes of objects of $\mathscr{A}$, where morphisms are chain maps modulo the homotopy equivalence relation, and then inverting quasi-isomorphisms. What we obtain in this way is not an abelian category, but it is a triangulated category, i.e. a category with a shift functor and a collection of exact triangles satisfying a number of axioms. Moreover, if our category $\mathscr{A}$ has enough injectives, given a left exact functor F we can define a corresponding derived functor RF on complexes of $D(\mathscr{A})$ bounded from below: this is done by first taking a complex of injectives which is quasi-isomorphic to our original complex, and then applying F to this complex of injectives. The same can be done with right exact functors and complexes of projectives.

The Fourier-Mukai transform was introduced in 1981 by Mukai, in his paper [Muk81], as a way to get an equivalence between the derived category of an abelian variety X and that of its dual $\hat{X}$. To do that, Mukai defined the "Fourier functor" to be

$$R\mathscr{S}(\cdot) := Rp_{2*}(\mathscr{P} \overset{L}{\otimes} Lp_1^*(\cdot))$$

where $\mathscr{P}$ is the Poincaré bundle on $X \times \hat{X}$ and $p_1 : X \times \hat{X} \to X$, $p_2 : X \times \hat{X} \to \hat{X}$. This duality was used as a tool to study Picard sheaves on X.

Interest sparked from this to investigate equivalences between derived categories of any scheme. This is a very interesting question to ask especially in light of the fact that, by Bondal-Orlov [BO01], if X and Y are smooth projective varieties with ample or anti-ample canonical sheaf and $D^b_{Coh}(X) \cong D^b_{Coh}(Y)$ it then follows that there exists an isomorphism $X \cong Y$.

In his seminal paper [Orl97], Orlov showed that any equivalence between the bounded derived categories of two smooth projective varieties is isomorphic to a Fourier-Mukai transform, i.e. a functor as above where instead of $\mathscr{P}$ we can now have any complex E in the bounded derived category of the product:

Theorem 1.0.1. *[Orl97] Let X and Y be smooth projective varieties over an algebraically closed field k. Consider an exact functor*

$$F : D^b_{Coh}(X) \to D^b_{Coh}(Y)$$

If F is fully faithful and has a right adjoint, then there exists an object $E \in D^b_{Coh}(X \times Y)$ such that F is isomorphic to the Fourier-Mukai transform with kernel E,

$$\Phi_E(\cdot) := Rp_{2*}(E \overset{L}{\otimes} Lp_1^*(\cdot))$$

This result has a tremendous variety of applications. First of all it can be applied to the study of moduli problems, for instance moduli spaces of stable sheaves on K3 surfaces, see for example Mukai [Muk87] and Orlov [Orl97]. More recent work has been carried out by Bridgeland and others in birational geometry [BM02], [BKR01]. Another area of application is given by the study of Bridgeland stability conditions on the bounded derived category of abelian and K3 surfaces [Yos01].

Orlov's result has been generalized by Ballard [Bal] to the case where X and Y are projective schemes over a field, and by Lunts and Orlov in [LO01] for the case of X and Y quasi compact and separated and a fully faithful functor between the unbounded derived categories of X and Y (this requires X to have enough locally free sheaves).

One can then ask what happens when the functor we are considering is not fully faithful, namely, will the functor still be isomorphic to a Fourier-Mukai transform in general? If that is the case, this would allow us to study its action on singular cohomology [Orl97] and Hochschild cohomology groups, and allow us to deform it along with the varieties [Cal03].

However, not much in known in this direction. For all the functors that can be expressed geometrically, we know that Orlov's result still holds even when said functors are not equivalences. In general, in the algebraic geometric setting there are no known examples of a functor which is not isomorphic to a Fourer-Mukai transform. The only example known by the author of an exact functor $F : D^b_{Coh}(X) \to D^b_{Coh}(Y)$ that is not of Fourier-Mukai type occurs in the non-algebraic case and is obtained extending to the derived category the equivalence $\mathrm{Coh}(X) \cong \mathrm{Coh}(Y)$ obtained in [Ver08] when X and Y are generic non-projective K3 surfaces.

On the other hand, the uniqueness of the kernel does not hold in general: in fact for every elliptic curve X over an algebraically closed field there exists $E_1, E_2 \in D^b(X \times X)$ such that $E_1 \neq E_2$ but $\Phi_{E_1} \cong \Phi_{E_2}$, see [CS10].

In his paper [Orl97], Orlov speculated that the result should actually hold for any exact functor between bounded derived categories of smooth projective varieties. Orlov's proof however makes extensive use of "ample sequences" that mostly behave well only when the functor is full. A somewhat stronger result was obtained by Canonaco and Stellari:

Theorem 1.0.2 ([CS07]). *Let X and Y be smooth projective varieties over a field. Consider an* *exact functor* $F : D^b_{Coh}(X) \to D^b_{Coh}(Y)$ *such that for any two sheaves* $\mathscr{F}$ *and* $\mathscr{G} \in \mathrm{Coh}(X)$

$$Hom_{D^b_{Coh}(Y)}(F(\mathscr{F}), F(\mathscr{G})[j]) = 0 \text{ if } j < 0 \tag{1.1}$$

Then there exists a unique object $E \in D^b_{Coh}(X \times Y)$ *such that* F *is isomorphic to the Fourier-Mukai transform with kernel* E*.*

The condition on the functor in the theorem is slightly weaker than fullness. This is however a bittersweet result: in fact in their later paper [COS11], together with Orlov, they showed that in the smooth projective case over a field of characteristic zero fullness in fact implies faithfulness. Therefore, if we want to further generalize Orlov's theorem, a whole new approach is needed.

In Chapter 2 we present a generalization of a theorem of Bondal and Van den Bergh [BVdB03] concerning the representability of a functor $D^b_{Coh}(X) \to \underline{\mathrm{mod}}_k$, where X is defined over the field k, to the case where the functor is to $\underline{\mathrm{mod}}_K$ where K is the function field of a curve or of $\mathbb{P}^2$ over k. This will allow us to describe the behavior of a functor between derived categories of smooth projective varieties when restricted to the generic point of the second variety, if the latter has dimension 0 or 1 or is a rational surface over k.

In Chapter 3 we give an explicit procedure that given a functor $F : D^b_{Coh}(X) \to D^b_{Coh}(Y)$ computes sheaves on $X \times Y$ that are equal to the cohomology sheaves of the kernel whenever the functor is a Fourier-Mukai transform. These sheaves seem to have in general good properties that indicate that the functor behaves in many ways like a Fourier-Mukai transform even when we are unable to prove it is such. By restricting to the case of $\dim(X) = 1$ we can actually exhibit a class of functors that are not full, nor faithful, nor do they satisfy condition 1.1, and for which we can still find an isomorphism to a Fourier-Mukai transform.

Chapter 2

Representability of cohomological functors over extension fields

2.1 Introduction

Let X be a projective variety over an algebraically closed field k. In this chapter we will generalize a result of Orlov and Van den Bergh on the representability of a functor $H : D^b_{Coh}(X) \to \underline{\text{mod}}_k$ to the case of an extension field $k \subset L$:

Theorem 2.1.1. *Let X be a smooth projective variety over a field k. Let L be a finitely generated separable field extension of k with $trdeg_k L \leq 1$, or a purely transcendental field extension of transcendence degree 2 over k. Consider a contravariant, cohomological, finite type functor*

$$H : D^b_{Coh}(X) \to \underline{\text{mod}}_L$$

Then H is representable by an object $E \in D^b_{Coh}(X_L)$, i.e. there exists E such that for every $C \in D^b_{Coh}(X)$ we have

$$H(C) = \text{Mor}_{D^b_{Coh}(X_L)}(j^*C, E)$$

where $j^* : X_L \to X$ is the base change morphism.

This will allow us to tackle the question of whether a functor between the bounded derived categories of two smooth projective varieties is representable by a Fourier-Mukai transform. We remind the reader of the definition of Fourier-Mukai transform:

Definition 2.1.2. *Given two smooth projective varieties X, Y and an object $E \in D^b_{Coh}(X \times Y)$, the Fourier-Mukai transform associated to E is defined as*

$$\Phi_E(\cdot) := Rp_{2*}(E \overset{L}{\otimes} Lp_1^*(\cdot))$$

where $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ are the projection morphisms.

When $\dim Y \leq 1$ or Y is a rational surface we can answer positively to the question above after restricting to the generic point of Y:

Theorem 2.1.3. *Let X, Y be a smooth projective varieties, where $\dim Y \leq 1$ or Y is a rational surface over k. Consider a covariant exact functor*

$$H : D^b_{Coh}(X) \to D^b_{Coh}(Y)$$

let $i_1 : \eta \to Y$ the inclusion of the generic point of Y. Then there exists an object $A \in D^b_{Coh}(X \times Y)$ such that

$$i_1^* \circ H = i_1^* \circ \Phi_A.$$

2.2 The Base Change Category

In what follows, an abelian category $\mathcal{A}$ does not automatically have any limits or colimits apart from the finite ones.

Given a field K, we will denote with $\underline{\text{mod}}_K$ the category of finite dimensional K-vector spaces, whereas $\underline{\text{Mod}}_K$ will denote the category of possibly infinite-dimensional K-vector spaces. $D(\mathcal{A})$ will denote the derived category of an abelian category $\mathcal{A}$.

Given an R-linear abelian category $\mathcal{A}$ and an inclusion of rings $R \hookrightarrow S$, we can define the base change category $\mathcal{A}_S$ as in [LVdB06, §4]:

Definition 2.2.1. *The category $\mathcal{A}_S$ is given by pairs (C, ρ_C) where $C \in \mathrm{Ob}(\mathcal{A})$ and $\rho_C : S \to \mathrm{Hom}_{\mathcal{A}}(C, C)$ is an R-algebra map such that the composition $R \to S \to \mathrm{Hom}_{\mathcal{A}}(C, C)$ gives back the R-algebra structure on $\mathcal{A}$. The morphisms in $\mathcal{A}_S$ are the morphisms in $\mathcal{A}$ compatible with the S-structure.*

Definition 2.2.2. *For each element $C \in \mathcal{A}$, the functor*

$$C \otimes_R - : \underline{\text{mod}}(R) \to \mathcal{A}$$

is the unique finite colimit preserving functor with $C \otimes R = C$.

This gives for each finitely presented R-algebra S a functor

$$- \otimes S : \mathcal{A} \to \mathcal{A}_S$$

to the base change category $\mathcal{A}_S$.

Proposition 2.2.3. *[LVdB06, Proposition 4.3] The functor $- \otimes S$ is left adjoint to the forgetful functor*

$$\text{forget} : \mathcal{A}_S \to \mathcal{A}$$
$$(C, \rho_C) \mapsto C$$

Whenever the context is clear, given an object $B \in \mathcal{A}_S$, we will still denote by B the corresponding object of $\mathcal{A}$ obtained via the forgetful functor.

For the purposes of this discussion we will need a more general setting for base change - specifically, we need to be able to talk about base change for a bigger category of rings and not just the ones that are finitely presented over the base. Let us extend Definition 2.2.2 as follows:

Definition 2.2.4. *Let $\mathcal{A}$ be an R-linear abelian category satisfying AB5. Using the fact that any R-module is the filtered colimit of finitely presented R-modules, we can extend definition 2.2.2 to the general case of*

$$- \otimes S : \mathcal{A} \to \mathcal{A}_S$$

for any R-algebra S.

The notion of base change category can be extended to the case of the derived category $D(\mathcal{A})$ of an abelian R-linear category $\mathcal{A}$ in the obvious way:

Definition 2.2.5. *Given an inclusion of rings $R \hookrightarrow S$, the category $D(\mathcal{A})_S$ is given by pairs (C, ρ_C) where $C \in \text{Ob}(D(\mathcal{A}))$ and $\rho_C : S \to \text{Hom}_{D(\mathcal{A})}(C, C)$ is an R-algebra map such that the composition $R \to S \to \text{Hom}_{D(\mathcal{A})}(C, C)$ gives back the R-algebra structure on $D(\mathcal{A})$. The morphisms in $D(\mathcal{A})_S$ are the morphisms in $D(\mathcal{A})$ compatible with the S-structure.*

Again, we have a notion of tensor product:

Definition 2.2.6. *Let R be a ring, let $\mathcal{A}$ be an R-linear abelian category satisfying AB5, and let $M^\bullet$ be a complex of objects in $\mathcal{A}$:*

$$M^\bullet = \ldots \to M^{i-1} \xrightarrow{d^{i-1}} M^i \xrightarrow{d^i} M^{i+1} \to \ldots$$

Let S be a ring, with a map $R \hookrightarrow S$. Then we can define $M^\bullet \otimes S$, as an object of $D(\mathscr{A}_S)$, as

$$M^\bullet \otimes S = \ldots \to M^{i-1} \otimes S \xrightarrow{d^{i-1}\otimes 1} M^i \otimes S \xrightarrow{d^i \otimes 1} M^{i+1} \otimes S \to \ldots$$

The complex $M^\bullet \otimes S$ can also be considered as an object of $D(\mathscr{A})_S$ if needed.

Remark 2.2.7. *Suppose that $\mathcal{A}$ is a k-linear abelian category satisfying AB5 and $k \subset K$ is an extension of fields. In the situation of definition 2.2.4 and 2.2.6, similarly to the case of 2.2.3, it is easy to show that again tensoring with K is left adjoint to the forgetful functor*

- *as a functor $\mathcal{A} \to \mathcal{A}_K$;*
- *as a functor $D(\mathcal{A}) \to D(\mathcal{A}_K)$;*
- *as a functor $D(\mathcal{A}) \to D(\mathcal{A})_K$.*

Remark 2.2.8. *Let R be a ring, let $\mathcal{A}$ be an R-linear abelian category satisfying AB5, and let $M^\bullet$ be a complex of objects in $\mathcal{A}$,*

$$M^\bullet = \ldots \to M^{i-1} \xrightarrow{d^{i-1}} M^i \xrightarrow{d^i} M^{i+1} \to \ldots$$

Let $S \subset R$ a multiplicative system. In this case $M^\bullet \otimes_R S^{-1}R$, as an object of $D(\mathscr{A})$, is the same as

$$\ldots \to \operatorname*{colim}_{f\in S} f^{-1}M^{i-1} \xrightarrow{d^{i-1}} \operatorname*{colim}_{f\in S} f^{-1}M^i \xrightarrow{d^i} \operatorname*{colim}_{f\in S} f^{-1}M^{i+1} \to \ldots$$

where $\operatorname{colim}_{f\in S} f^{-1}M^i$ is obtained by taking for every $f \in S$ a copy of M^i and as morphisms only the maps

$$f^{-1}\,M^i \longrightarrow (fg)^{-1}\,M^i$$

given by multiplication by $g : M^i \to M^i$.

Lemma 2.2.9. *In the situation of the remark above, if for every element $f \in S$ the multiplication by f is a quasi-isomorphism of $M^\bullet$, then the map*

$$M^\bullet \to M^\bullet \otimes_R S^{-1}R$$

is a quasi-isomorphism in $D(\mathcal{A})$.

Proof. Since taking cohomology commutes with directed colimits we have

$$H^i(M^\bullet \otimes_R S^{-1}R) = \operatorname{colim}_{f \in S} \; f^{-1} H^i(M^\bullet)$$

but since multiplication by any $g \in S$ is a quasi-isomorphism we get

$$f^{-1} H^i(M^\bullet) \xrightarrow[g]{\cong} (fg)^{-1} H^i(M^\bullet)$$

hence the cohomology of $M^\bullet \otimes_R S^{-1}R$ consists of only one copy of $H^i(M^\bullet)$, and the map $M^\bullet \to M^\bullet \otimes_R S^{-1}R$ is a quasi- isomorphism. □

2.3 A result on base change for derived categories

The purpose of this section is to analyze the functor $D(\mathcal{A}_K) \to D(\mathcal{A})_K$ that sends an object in $D(\mathcal{A}_K)$ to the same object considered as an object of $D(\mathcal{A})$, together with its K-action. Specifically, we will prove the following:

Theorem 2.3.1. *Let $\mathcal{A}$ be a k-linear abelian category satisfying AB5, where k is a field. Let $K = k(T)$ or $K = k(T, T')$. Then the functor*

$$D(\mathcal{A}_K) \to D(\mathcal{A})_K$$
$$C^\bullet \mapsto (C^\bullet, \rho_C)$$

is essentially surjective, where $\rho_C : K \to \operatorname{Aut}(C^\bullet)$ is the obvious map.

Moreover, if L is a finite separable extension of $K = k(T)$ with $L = K(\alpha) = K[T]/P(T)$ then we can lift an object $(C^\bullet, \rho_C) \in D(\mathscr{A})_L$ to an object $N^\bullet$ of $D(\mathscr{A}_K)$ endowed with a map $\tilde{\psi} \in \operatorname{End}(N^\bullet)$ such that $P(\tilde{\psi})$ is zero on all cohomology groups, and the action of $\tilde{\psi}$ on $N^\bullet$ corresponds to the action of α on $C^\bullet$.

This comes down to lifting the actions of T and T' on a complex $C \in D(\mathcal{A})_K$ to actions coming from morphisms in $\mathcal{A}$ that commute with each other.

Lemma 2.3.2. *Let $\mathcal{A}$ be a k-linear abelian category satisfying AB5, where k is a field. Let $L^\bullet$ be a complex in $D(\mathcal{A})$. Let $\varphi \in \mathrm{Hom}_{D(\mathcal{A})}(L^\bullet, L^\bullet)$. Then there exists a complex $M^\bullet \in D(\mathcal{A}_{k[T]})$ and a quasi-isomorphism $L^\bullet \to M^\bullet$ as objects of $D(\mathcal{A})$ such that the action of multiplication by T on $M^\bullet$ corresponds to the action of multiplication by φ on $L^\bullet$.*

Proof. The map $\varphi : L^\bullet \to L^\bullet$ in $D(\mathcal{A})$ corresponds to a diagram of complexes in $D(\mathcal{A})$

$$\begin{array}{ccc} & Q^\bullet & \\ {}^{u}\swarrow & & \searrow^{\varphi'} \\ L^\bullet & \overset{\varphi}{\dashrightarrow} & L^\bullet \end{array}$$

where u is a quasi-isomorphism.

Let $L^\bullet[T] = L^\bullet \otimes_k k[T]$ as a complex in $D(\mathcal{A}_{k[T]})$. Consider the morphism $\varphi \otimes 1 - 1 \otimes T : L^\bullet[T] \to L^\bullet[T]$ in $D(\mathcal{A}_{k[T]})$. This can be represented by actual maps of complexes

$$\begin{array}{ccc} & Q^\bullet[T] & \\ {}^{u\otimes 1}\swarrow & & \searrow^{\varphi'\otimes 1 - u\otimes T} \\ L^\bullet[T] & \overset{\varphi\otimes 1 - 1\otimes T}{\dashrightarrow} & L^\bullet[T] \end{array}$$

The map $\varphi' \otimes 1 - u \otimes T$ is injective on all cohomology objects: to prove this we need to show that $\varphi' \otimes 1 - u \otimes T : H^r(Q^\bullet[T]) \to H^r(L^\bullet[T])$ is injective for every r.

Let $\alpha \in H^r(Q^\bullet[T])$, $\alpha \neq 0$, then

$$\alpha = \sum_{i=0}^{n} \alpha_i T^i$$

where all of the α_i are different from zero in $H^r(Q^\bullet)$. If

$$0 = (\varphi' \otimes 1 - u \otimes T)\alpha = \sum_{i=0}^{n} \varphi'(\alpha_i) T^i - \sum_{i=0}^{n} u(\alpha_i) T^{i+1}$$

then the only term of degree $n+1$ in T, $u(\alpha_n)T^{n+1}$, must be zero in $H^r(L^\bullet)$, hence $u(\alpha_n) = 0$, hence $\alpha_n = 0$ since u is a quasi-isomorphism. This contradicts our assumption that $\alpha_i \neq 0\ \forall i$, and so this proves injectivity.

Now set

$$M^\bullet = \mathrm{Cone}(Q^\bullet[T] \xrightarrow{\varphi'\otimes 1 - u\otimes T} L^\bullet[T])$$

Then we have a triangle

$$Q^\bullet[T] \xrightarrow{\varphi'\otimes 1 - u\otimes T} L^\bullet[T] \longrightarrow M^\bullet \longrightarrow (Q^\bullet[T])[1] \tag{2.1}$$

and by injectivity of the map $\varphi'\otimes 1 - u\otimes T$ on the cohomology objects we get a short exact sequence in cohomology

$$0 \to H^r(Q^\bullet[T]) \xrightarrow{\varphi'\otimes 1 - u\otimes T} H^r(L^\bullet[T]) \longrightarrow H^r(M^\bullet) \to 0$$

hence we get

$$H^r(M^\bullet) = \mathrm{Coker}(H^r(Q^\bullet[T]) \xrightarrow{\varphi'\otimes 1 - u\otimes T} H^r(L^\bullet[T]))$$

for any r.

Now consider the composition

$$L^\bullet \longrightarrow L^\bullet[T] \longrightarrow M^\bullet$$

This map is a quasi-isomorphism; to prove this we just need to show that under the map above, $H^r(L^\bullet) \cong \mathrm{Coker}(H^r(Q^\bullet[T]) \xrightarrow{\varphi'\otimes 1 - u\otimes T} H^r(L^\bullet[T]))$ for every r.

Proceed as follows: first of all, considered as a sub-object of $H^r(L^\bullet[T])$ via the obvious map $L^\bullet \to L^\bullet[T]$, $H^r(L^\bullet)$ is not in the image of $\varphi' \otimes 1 - u \otimes T$, since, for any element $\alpha = \sum_{i=1}^n \alpha_i T^i$ of $H^r(Q^\bullet[T])$, its image $\sum_{i=1}^n \varphi(\alpha_i)T^i - \sum_{i=0}^n u(\alpha_i)T^{i+1}$ is either zero or has a nonzero term of positive degree. To prove that any term of positive degree $\beta = \sum_{i=1}^n \beta_i T^i$ is in the image up to an element of degree zero, notice that it can be written as an element of lower degree plus an element of the image as follows:

$$\begin{aligned}\sum_{i=0}^n \beta_i T^i &= \sum_{i=0}^n \beta_i T^i - (\varphi' \otimes 1 - u \otimes T)(u^{-1}(\beta_n)T^{n-1}) + (\varphi' \otimes 1 - u \otimes T)(u^{-1}(\beta_n)T^{n-1}) = \\ &= \sum_{i=0}^n \beta_i T^i - \varphi'(u^{-1}(\beta_n))T^{n-1} + \beta_n T^n + (\varphi' \otimes 1 - u \otimes T)(u^{-1}(\beta_n)T^{n-1}) \\ &= \sum_{i=0}^{n-1} \beta_i T^i - \varphi'(u^{-1}(\beta_n))T^{n-1} + (\varphi' \otimes 1 - u \otimes T)(u^{-1}(\beta_n)T^{n-1})\end{aligned}$$

Hence we found a complex $M^\bullet \in D(\mathcal{A}_{k[T]})$ which is quasi-isomorphic to $L^\bullet$ as an object of $D(\mathcal{A})$; moreover the action of multiplication by φ on $L^\bullet$ corresponds to the action by multiplication by T on $M^\bullet$. □

Lemma 2.3.3. *Let $\mathcal{A}$ be a k-linear abelian category satisfying AB5, where k is a field. Let $L^\bullet$ be a complex in $D(\mathcal{A})$.*

Let $\varphi \in \mathrm{Hom}_{D(\mathcal{A})}(L^\bullet, L^\bullet)$ *such that* $f(\varphi)$ *is an isomorphism for all* $f \in k[T]$ *monic. Then there exists a complex* $N^\bullet \in D(\mathcal{A}_{k(T)})$ *and a quasi-isomorphism* $L^\bullet \to N^\bullet$ *as objects of* $D(\mathcal{A})$ *such that the action of multiplication by* T *on* $N^\bullet$ *corresponds to the action by multiplication by* φ *on* $L^\bullet$*.*

Likewise, let $\varphi, \psi \in \mathrm{Hom}_{D(\mathcal{A})}(L^\bullet, L^\bullet)$ *such that* φ *and* ψ *commute with each other and such that* $f(\varphi, \psi)$ *is a quasi-isomorphisms for all* $f \in k[T, T']$ *nonzero. Then there exists a complex* $N^\bullet \in D(\mathcal{A}_{k(T,T')})$ *and a quasi-isomorphism* $j : L^\bullet \to N^\bullet$ *as objects of* $D(\mathcal{A})$ *such that the action of multiplication by* T *(resp.* T'*) on* $N^\bullet$ *corresponds to the action by multiplication by* φ *(resp.* ψ*) on* $L^\bullet$*.*

Proof. By Lemma 2.3.2 we can find a complex $M^\bullet \in \mathcal{A}_{k[T]}$ and a quasi-isomorphism $j : L^\bullet \to M^\bullet$ as objects of $D(\mathcal{A})$ such that the action of multiplication by T on $M^\bullet$ corresponds to the action by multiplication by φ on $L^\bullet$. This implies that multiplication by $f(T)$ gives a quasi-isomorphism of $M^\bullet$ for all f monic.

Now let $N^\bullet := M^\bullet \otimes_{k[T]} k(T)$ as in Definition 2.2.6 above. This is a complex in $D(\mathcal{A}_{k(T)})$ and it is quasi-isomorphic to $L^\bullet$ as objects of $D(\mathcal{A})$, by Lemma 2.2.9. The action of φ on $L^\bullet$ corresponds to the action of T on $N^\bullet$.

For the second case, again by Lemma 2.3.2 we can find a complex $M^\bullet \in \mathcal{A}_{k[T]}$ and a quasi-isomorphism $j : L^\bullet \to M^\bullet$ as objects of $D(\mathcal{A})$ such that the action of multiplication by T on $M^\bullet$ corresponds to the action by multiplication by φ on $L^\bullet$.

Moreover, we have an exact triangle

$$L^\bullet[T] \xrightarrow{\varphi\otimes 1 - 1\otimes T} L^\bullet[T] \longrightarrow M^\bullet$$

in $D(\mathcal{A}_{k[T]})$, see (2.1).

Then, since φ and ψ commute with each other, we get a commutative diagram in $D(\mathcal{A}_{k[T]})$

$$\begin{array}{ccc} L^\bullet[T] & \xrightarrow{\varphi\otimes 1 - 1\otimes T} & L^\bullet[T] \\ \downarrow{\scriptstyle \psi\otimes 1} & & \downarrow{\scriptstyle \psi\otimes 1} \\ L^\bullet[T] & \xrightarrow{\varphi\otimes 1 - 1\otimes T} & L^\bullet[T] \end{array}$$

By the axioms of the derived category we can find a map $\tilde{\psi}$ on $M^\bullet$ so that the following diagram commutes:

$$\begin{array}{ccccccc} L^\bullet[T] & \xrightarrow{\varphi\otimes 1 - 1\otimes T} & L^\bullet[T] & \longrightarrow & M^\bullet & \longrightarrow & (L^\bullet[T])[1] \\ \downarrow{\scriptstyle \psi\otimes 1} & & \downarrow{\scriptstyle \psi\otimes 1} & & \downarrow{\scriptstyle \tilde{\psi}} & & \downarrow{\scriptstyle \psi\otimes 1} \\ L^\bullet[T] & \xrightarrow{\varphi\otimes 1 - 1\otimes T} & L^\bullet[T] & \longrightarrow & M^\bullet & \longrightarrow & (L^\bullet[T])[1] \end{array}$$

As before we can then construct $P^\bullet = M^\bullet \otimes_{k[T]} k(T)$, which is quasi-isomorphic to $M^\bullet$ and hence we get a corresponding map $\tilde{\psi} : P^\bullet \to P^\bullet$.

So we are in the following situation: we have a complex $P^\bullet \in D(\mathcal{A}_{k(T)})$ and a map $\tilde{\psi} : P^\bullet \to P^\bullet$ so that $f(\tilde{\psi})$ is a quasi-isomorphism for all $f \in k(T)[T']$ monic. By Lemma 2.3.2 again, we get a complex $Q^\bullet \in D((\mathcal{A}_{k(T)})_{k(T)[T']}) = D(\mathcal{A}_{k(T)[T']})$ which is quasi-isomorphic to $P^\bullet$.

Then define

$$N^\bullet := Q^\bullet \otimes_{k(T)[T']} k(T, T')$$

By Lemma 2.2.9, since $f(T, \psi)$ is a quasi-isomorphisms for all nonzero $f \in k(T)[T']$, the complex $N^\bullet \in D(\mathcal{A}_{k(T,T')})$ is quasi isomorphic to $Q^\bullet$ as objects of $D(\mathcal{A}_{k(T)[T']})$ hence it is quasi-isomorphic to $L^\bullet$ as objects of $D(\mathcal{A})$. The action of φ and ψ correspond to the action of T and T' respectively. ☐

Lemma 2.3.4. *Let $\mathcal{A}$ be a k-linear abelian category satisfying AB5, where k is a field. Let $L^\bullet$ be a complex in $D(\mathcal{A})$.*

Let $\varphi, \psi \in \mathrm{Hom}_{D(\mathcal{A})}(L^\bullet, L^\bullet)$ such that φ and ψ commute with each other and such that $f(\varphi)$ is a quasi-isomorphisms for all $f \in k[T]$ monic and there exists an irreducible $P \in k[T, T']$ with $P(\varphi, \psi) = 0$.

Then there exists a complex $N^\bullet \in D(\mathcal{A}_{k(T)})$ and a quasi-isomorphism $j : L^\bullet \to N^\bullet$ as objects of $D(\mathcal{A})$ such that the action of multiplication by T on $N^\bullet$ corresponds to the action by multiplication by φ on $L^\bullet$. Moreover there is a morphism $\tilde{\psi} \in \mathrm{End}(N^\bullet)$ such that the action of ψ on $L^\bullet$ corresponds to the action of $\tilde{\psi}$ on $N^\bullet$ and $P(T, \tilde{\psi})$ induces the zero map on all cohomology groups of $N^\bullet$.

Proof. By Lemma 2.3.2 we can find a complex $M^\bullet \in \mathcal{A}_{k[T]}$ and a quasi-isomorphism $j : L^\bullet \to M^\bullet$ as objects of $D(\mathcal{A})$ such that the action of multiplication by T on $M^\bullet$ corresponds to the action by multiplication by φ on $L^\bullet$.

Moreover, we have an exact triangle

$$L^\bullet[T] \xrightarrow{\varphi\otimes 1 - 1\otimes T} L^\bullet[T] \longrightarrow M^\bullet$$

in $D(\mathcal{A}_{k[T]})$, see (2.1).

Then, since φ and ψ commute with each other, we get a commutative diagram in $D(\mathcal{A}_{k[T]})$

$$\begin{array}{ccc} L^\bullet[T] & \xrightarrow{\varphi\otimes 1 - 1\otimes T} & L^\bullet[T] \\ \downarrow{\scriptstyle \psi\otimes 1} & & \downarrow{\scriptstyle \psi\otimes 1} \\ L^\bullet[T] & \xrightarrow{\varphi\otimes 1 - 1\otimes T} & L^\bullet[T] \end{array}$$

By the axioms of the derived category we can find a map $\tilde{\psi}$ on $M^\bullet$ so that the following diagram commutes:

$$\begin{array}{ccccccc} L^\bullet[T] & \xrightarrow{\varphi\otimes 1 - 1\otimes T} & L^\bullet[T] & \longrightarrow & M^\bullet & \longrightarrow & (L^\bullet[T])[1] \\ \downarrow{\scriptstyle \psi\otimes 1} & & \downarrow{\scriptstyle \psi\otimes 1} & & \downarrow{\scriptstyle \tilde{\psi}} & & \downarrow{\scriptstyle \psi\otimes 1} \\ L^\bullet[T] & \xrightarrow{\varphi\otimes 1 - 1\otimes T} & L^\bullet[T] & \longrightarrow & M^\bullet & \longrightarrow & (L^\bullet[T])[1] \end{array}$$

As before we can then construct $P^\bullet = M^\bullet \otimes_{k[T]} k(T)$, which is quasi-isomorphic to $M^\bullet$ and hence we get a corresponding map $\tilde{\psi} : P^\bullet \to P^\bullet$ and the action of ψ on $L^\bullet$ corresponds to the action of $\tilde{\psi}$ on $P^\bullet$.

Finally, $P(T, \tilde{\psi}) = 0$ on cohomology since $P(T, \psi \otimes 1) = 0$ on $L^\bullet[T]$. □

Now that we have lifted the actions of the two variables T and T' we are almost done in lifting the action of the whole $k[T, T']$ because of the following lemma:

Lemma 2.3.5. *Let $e^n D(\mathcal{A})$ be the category whose*

1. *Objects are pairs $(E, \varphi_1, \ldots, \varphi_n)$ where $E \in \mathrm{Ob}(D(\mathcal{A}))$, $\varphi_i \in \mathrm{End}_{D(\mathcal{A})}(E)$ for all i, and φ_i commutes with φ_j for all i, j;*

2. *Morphisms $a : (E, \varphi_1, \ldots, \varphi_n) \to (E', \varphi'_1, \ldots, \varphi'_n)$ are elements $a \in \mathrm{Hom}_{D(\mathcal{A})}(E, E')$ such that $a \circ \varphi_i = \varphi'_i \circ a$.*

Consider the full subcategory $e^n D'(\mathcal{A}) \subset e^n D(\mathcal{A})$ whose objects consist of those pairs $(E, \varphi_1, \ldots, \varphi_n)$ such that for every nonzero $f \in k[T_1, \ldots, T_n]$ the map $f(\varphi_1, \ldots, \varphi_n) : E \to E$ is an isomorphism in $D(\mathcal{A})$.

The category $D(\mathcal{A})_{k(T_1,\ldots,T_n)}$ is equivalent to the category $e^n D'(\mathcal{A})$. The equivalence is given by the functor

$$D(\mathcal{A})_{k(T_1,\ldots,T_n)} \longrightarrow e^n D'(\mathcal{A}), \quad (E, \rho_E) \longmapsto (E, \rho_E(T_1), \ldots, \rho_E(T_n)).$$

Proof. The equivalence is given by the inverse functor

$$e^n D'(\mathcal{A}) \longrightarrow D(\mathcal{A})_{k(T_1,\ldots,T_n)}$$
$$(E, \varphi_1, \ldots, \varphi_n) \mapsto \left(E, \begin{array}{rcl} \rho : k(T_1, \ldots, T_n) & \to & \mathrm{Aut}(E) \\ T_i & \mapsto & \varphi_i \end{array} \right)$$

□

We are now ready to prove Theorem 2.3.1:

Proof of Theorem 2.3.1. By Lemma 2.3.5, we just need to show that the functors

$$D(\mathcal{A}_K) \to e^1 D'(\mathcal{A})$$
$$C^\bullet \mapsto (C^\bullet, \cdot T)$$

and

$$D(\mathcal{A}_K) \to e^2 D'(\mathcal{A})$$
$$C^\bullet \mapsto (C^\bullet, \cdot T, \cdot T')$$

are essentially surjective.

Let $(E, \varphi) \in e^1 D'(\mathcal{A})$. Then by Lemma 2.3.3 there exists $N^\bullet \in \mathcal{A}_{k(T)}$ such that N is quasi isomorphic to E and the action of φ on $E^\bullet$ corresponds to the action of T on $N^\bullet$. This proves the case $i = 1$.

Similarly, let $(E, \varphi, \varphi') \in e^2 D'(\mathcal{A})$. Then by Lemma 2.3.3 there exists $N^\bullet \in \mathcal{A}_{k(T,T')}$ such that N is quasi isomorphic to E and the action of φ and φ' on $E^\bullet$ correspond to the action of T and T' respectively on $N^\bullet$. This proves the case $i = 2$.

The last part follows from Lemma 2.3.4. □

Let us now apply this theorem to the case $\mathcal{A} = \mathrm{QCoh}(X)$, where X is a quasi-compact, separated scheme over a field k. This is possible since $\mathrm{QCoh}(X)$ satisfies AB5. Before we do that, however, we need to prove a technical lemma:

Lemma 2.3.6. *Let $k \subset K$ be a field extension, X a quasi-compact and separated scheme. Let $X_K \xrightarrow{j} X$ the base change morphism. Then there is an equivalence of categories*

$$D_{QCoh}(X_K) \xrightarrow{\psi} D(\mathrm{QCoh}(X)_K)$$

under this equivalence, the functors

$$Lj^*, \cdot \otimes K : D_{QCoh}(X) \to D(\mathrm{QCoh}(X_K))$$

and

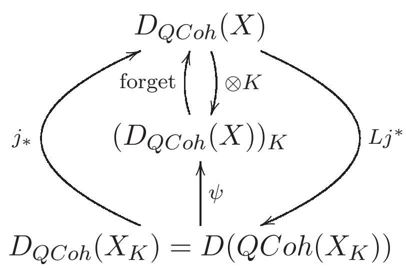
$$Rj_*, \mathrm{forget} : D_{QCoh}(X_K) \to D(\mathrm{QCoh}(X))$$

coincide.

In other words,

$$Rj_* = \mathrm{forget} \circ \psi : D(\mathrm{QCoh}(X)_K) \to D_{QCoh}(X)$$
$$\psi \circ Lj^* = - \otimes K : D_{QCoh}(X) \to (D_{QCoh}(X))_K$$

This is summarized in the following diagram:



Proof. There is an equivalence of categories induced by j_* between quasi-coherent $\mathcal{O}_{X_K}$-modules and quasi-coherent $j_*\mathcal{O}_{X_K}$-modules on X. But $j_*\mathcal{O}_{X_K} = \mathcal{O}_X \otimes K$ and an $(\mathcal{O}_X \otimes K)$-module is the same thing as an $\mathcal{O}_X$-module with a K-structure which is compatible with its k-structure.

Hence we get an equivalence

$$\psi : \mathrm{QCoh}(X_K) \to \mathrm{QCoh}(X)_K$$
$$C \mapsto (j_*C, \rho_C)$$

where ρ_C is the composition $K \to \mathcal{O}_X \otimes K \to \mathrm{End}(j_*C)$.

Under this equivalence, the two functors j_* and "forget" coincide; moreover, always under the same equivalence, both j^* and $- \otimes K$ are left adjoint to j_*, hence they also coincide.

Thus all of this also holds for the corresponding derived categories; hence the statement follows from the fact that for a quasi compact, separated scheme X we have $D_{QCoh}(X) = D(\mathrm{QCoh}(X))$. $\square$

Corollary 2.3.7. *Let X be a quasi compact, separated scheme over a field k.*

Let $K = k(T)$ or $K = k(T, T')$.

The map

$$D_{QCoh}(X_K) \longrightarrow (D_{QCoh}(X))_K$$
$$C^\bullet \mapsto (\text{forget}(C^\bullet), \rho_C)$$

is essentially surjective, where ρ_C is the obvious K-structure on C.

Moreover, if L is a finite separable extension of $K = k(T)$ with $L = K(\alpha) = K[T]/P(T)$ then we can lift an object $(C^\bullet, \rho_C) \in (D_{QCoh}(X))_L$ to an object $N^\bullet$ of $D_{QCoh}(X_K)$ endowed with a map $\tilde{\psi} \in \text{End}(N^\bullet)$ such that $P(\tilde{\psi})$ induces the zero map on all cohomology groups of $N^\bullet$.

Proof. By Lemma 2.3.6, there is an equivalence between $D_{\text{QCoh}}(X_K)$ and $D(\text{QCoh}(X)_K)$, hence it is sufficient to show that the map

$$D(QCoh(X)_K) \to (D(QCoh(X)))_K$$
$$C^\bullet \mapsto (\text{forget}(C^\bullet), \rho_C)$$

is essentially surjective.

Let $\mathcal{A} = \text{QCoh}(X)$. This category satisfies AB5, hence theorem 2.3.1 applies in this case. □

2.4 A representability theorem for derived categories

The results of the previous section will become handy to study functors from $D^b_{Coh}(X)$, where X is defined over a field k, to a vector space over a bigger field in light of the following theorem:

Theorem 2.4.1. *Let k be a field, $\mathcal{A}$ be a k-linear abelian category satisfying AB5, $\mathscr{T} = D(\mathscr{A})$, and let $k \hookrightarrow K$ an inclusion of fields.*

Consider an exact contravariant functor

$$F : \{\mathscr{T}^c\}^{op} \to \underline{\text{mod}}_K$$

Let $\mathscr{T}_K$ be the base-change category. Then there exists an $\tilde{S} \in \mathscr{T}_K$ such that

$$F(C) = \text{Mor}_{\mathscr{T}_K}(C \otimes K, \tilde{S})$$

for all $C \in \mathscr{T}^c$.

To prove this we will use the ideas from [CKN01, Lemma 2.14] where the version of this theorem with $k = K$ has been proved.

Proof of theorem 2.4.1. Let D be the functor taking a K-vector space to its dual. Then $G = D \circ F$ is exact and covariant.

Let $\tilde{G} : \mathscr{T} \to \underline{\mathrm{Mod}}_K$ be the Kan extension of G to $\mathscr{T}$. $\tilde{G}$ is exact and commutes with coproducts, hence $D \circ \tilde{G}$ is exact and takes coproducts to products. Hence by [Fra01, Theorem 3.1] the functor $D \circ \tilde{G}$ is representable, as a functor to $\underline{\mathrm{Mod}}_k$, by an object $Y \in \mathscr{T}$.

The K-action on $\underline{\mathrm{Mod}}_K$ induces a K-action $\tilde{\rho}$ on $D \circ \tilde{G} = h_Y$, hence by Yoneda we get a K-action ρ on Y, given by $K \xrightarrow{\rho} \mathrm{Nat}(h_Y, h_Y) = \mathrm{Aut}(Y)$. Therefore we obtain an object $(Y, \rho) \in \mathscr{T}_K$. We need to show that

$$D \circ \tilde{G}(C) = \mathrm{Mor}_{\mathscr{T}_K}(C \otimes K, (Y, \rho))$$

for all $C \in \mathscr{T}^c$.

To do so, first of all notice that as k-vector spaces

$$D \circ \tilde{G}(C) = \mathrm{Mor}_{\mathscr{T}}(C, Y) = \mathrm{Mor}_{\mathscr{T}_K}(C \otimes K, (Y, \rho))$$

because $K \otimes_k -$ is left adjoint to the functor forgetting the K-structure. By our definition of the K-action on $\mathrm{Mor}_{\mathscr{T}}(C, Y)$, this is the same as the K-action on $D \circ \tilde{G}(C)$; moreover the k-vector space map

$$\mathrm{Mor}_{\mathscr{T}}(C, Y) \xrightarrow{\gamma} \mathrm{Mor}_{\mathscr{T}_K}(C \otimes K, (Y, \rho))$$
$$f \mapsto f \otimes \rho$$

is compatible with the K-action since, for any $\alpha \in K$,

$$\gamma(\alpha \cdot f) = \gamma(\tilde{\rho}(\alpha) f) = \tilde{\rho}(\alpha) f \otimes \rho(\cdot) = f \otimes \rho(\alpha)\rho(\cdot) = \alpha \cdot (f \otimes \rho(\cdot))$$

hence we found that the two actions coincide and so

$$D \circ \tilde{G}(C) = Mor_{\mathscr{T}_K}(C \otimes K, (Y, \rho))$$

Let $\tilde{S} = (Y, \rho)$. Now since F is of finite type, we get

$$F(C) = (D \circ D \circ F)(C) = (D \circ G)(C) = (D \circ \tilde{G})(C) = Mor_{\mathscr{T}_K}(C \otimes K, E)$$

$\square$

Lemma 2.4.2. *Let k and K be two fields, $k \hookrightarrow K$.*

Consider the equivalence of categories

$$D^b(\underline{\text{mod}}(\Lambda)) \xrightarrow{\theta} D^b(\text{Coh}(\mathbb{P}^n_k))$$

as described in [Bei78].

Then there is also an equivalence of categories

$$D^b(\underline{\text{mod}}(\Lambda \otimes K)) \xrightarrow{\theta_K} D^b(\text{Coh}(\mathbb{P}^n_K))$$

and the diagram

$$\begin{array}{ccc} D^b(\underline{\text{mod}}(\Lambda)) & \xrightarrow{\theta} & D^b(\text{Coh}(\mathbb{P}^n_k)) \\ \downarrow & & \downarrow \\ D^b(\underline{\text{mod}}(\Lambda \otimes K)) & \xrightarrow{\theta_K} & D^b(\text{Coh}(\mathbb{P}^n_K)) \end{array}$$

is commutative.

Proof. By [Bei78], we have $\Lambda = \text{End}(\mathcal{M})$ where $\mathcal{M} = \bigoplus_{i=0}^n \mathcal{O}_{\mathbb{P}^n_k}(i)$. Set $\mathcal{M}_K = \bigoplus_{i=0}^n \mathcal{O}_{\mathbb{P}^n_K}(i)$, then

$$\begin{aligned} \text{End}_{\mathbb{P}^n_K}(\mathcal{M}_K) = \text{End}_{\mathbb{P}^n_K}\left(\bigoplus_{i=0}^n \mathcal{O}_{\mathbb{P}^n_K}(i)\right) &= \bigoplus_{i,j=0}^n \text{End}_{\mathbb{P}^n_K}\left(\mathcal{O}_{\mathbb{P}^n_K}(i), \mathcal{O}_{\mathbb{P}^n_K}(j)\right) = \\ &= \bigoplus_{i,j=0}^n K[x_0, \ldots, x_n]_{j-i} = \bigoplus_{i,j=0}^n k[x_0, \ldots, x_n]_{j-i} \otimes K \\ &= \left(\bigoplus_{i,j=0}^n k[x_0, \ldots, x_n]_{j-i}\right) \otimes K = \Lambda \otimes K \end{aligned}$$

Moreover, the equivalence θ is induced by the map

$$\underline{\text{mod}}(\Lambda) \xrightarrow{-\otimes_\Lambda \mathcal{M}} \text{Coh}(\mathbb{P}^n_k)$$

and if we let $h : \mathbb{P}^n_K \to \mathbb{P}^n_k$ be the base change morphism, we obtain the following commutative diagram:

$$\begin{array}{ccc} \underline{\text{mod}}(\Lambda) & \xrightarrow{-\otimes_\Lambda \mathcal{M}} & \text{Coh}(\mathbb{P}^n_k) \\ {\scriptstyle \otimes K}\downarrow & & \downarrow{\scriptstyle h^*} \\ \underline{\text{mod}}(\Lambda \otimes K) & \xrightarrow{-\otimes_{\Lambda \otimes K} \mathcal{M}_{\mathcal{K}}} & \text{Coh}(\mathbb{P}^n_K) \end{array}$$

this proves the last assertion. □

We are now almost ready to prove Theorem 2.1.1, but first we will prove the version of the theorem for the purely transcendental case. The following proof uses ideas from [BVdB03, Theorem A.1].

Theorem 2.4.3. *Let X be a smooth projective variety over a field k. Let $K = k(T)$ or $K = k(T, T')$. Consider a contravariant, cohomological, finite type functor*

$$H : D^b_{Coh}(X) \to \underline{\text{mod}}_K$$

Then the complex $\tilde{S}$ of Theorem 2.4.1 lifts to a complex $S \in D^b_{Coh}(X_K)$ such that H is representable by S, i.e. for every $C \in D^b_{Coh}(X)$ we have

$$H(C) = Mor_{D^b_{Coh}(X_K)}(Lj^*C, S)$$

where $j : X_K \to X$ is the base change morphism.

Proof. By Lemma 2.4.1, the functor H is representable by an element $\tilde{S} \in (D_{QCoh}(X))_K$, i.e.

$$H(C) = Mor_{(D_{QCoh}(X))_K}(C \otimes K, \tilde{S})$$

Let S be a lift of $\tilde{S}$ to $D_{QCoh}(X_K)$ (this is possible by Corollary 2.3.7). Let C be an element of $D^b_{Coh}(X)$. By applying the functors in Lemmas 2.3.1 and 2.3.6 we get a K-linear map

$$Mor_{D_{QCoh}(X_K)}(Lj^*C, S) \xrightarrow{\psi(\cdot)} Mor_{(D_{QCoh}(X))_K}(\psi \circ Lj^*C, \tilde{S})$$

and, since by Lemma 2.3.6, $\psi \circ Lj^*C = C \otimes K$, we have

$$Mor_{(D_{QCoh}(X))_K}(\psi \circ Lj^*C, \tilde{S}) = Mor_{(D_{QCoh}(X))_K}(C \otimes K, \tilde{S}) = H(C)$$

Hence to show that H is represented by S we just need to show that $\psi(\cdot)$ is an isomorphism. It suffices to show that it is an isomorphism of k-vector spaces, which follows from the following diagram of k-vector spaces:

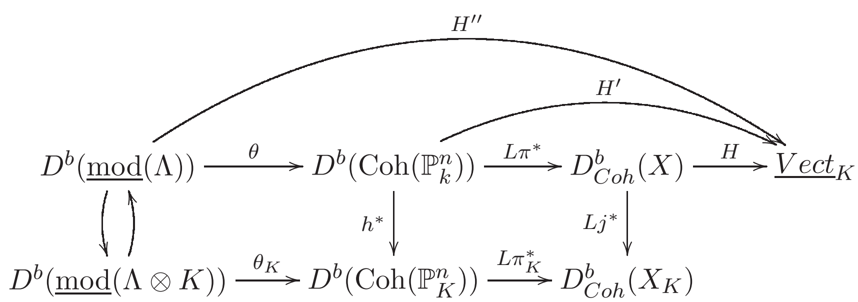
$$\begin{array}{ccc}
\text{Mor}_{D_{QCoh}(X_K)}(Lj^*C, S) & \xrightarrow{\psi(\cdot)} & \text{Mor}_{(D_{QCoh}(X))_K}(\psi \circ Lj^*C, \tilde{S}) \\
\| & & \| \\
\text{Mor}_{D_{QCoh}(X)}(C, Rj_*S) & & \text{Mor}_{(D_{QCoh}(X))_K}(C \otimes K, \tilde{S}) \\
\| & & \| \\
\text{Mor}_{D_{QCoh}(X)}(C, \text{forget}(\psi(S))) & = & \text{Mor}_{(D_{QCoh}(X))}(C, \text{forget}(\tilde{S}))
\end{array}$$

here we used the fact that $Rj_* = \text{forget} \circ \psi$, again from Lemma 2.3.6.

So $\psi(\cdot)$ is an isomorphism, and hence H is represented by $S \in D_{QCoh}(X_K)$. We still have to show that S is actually in $D^b_{Coh}(X_K)$.

Choose an embedding $\pi : X \to \mathbb{P}^n_k$. Let $H' = H \circ L\pi^*$. Let $\theta : D^b(\underline{\text{mod}}(\Lambda)) \to D^b(\text{Coh}(\mathbb{P}^n_k))$ and $\theta_K : D^b(\underline{\text{mod}}(\Lambda \otimes K)) \to D^b(\text{Coh}(\mathbb{P}^n_K))$ as defined in Lemma 2.4.2 above. Let $H'' = H' \circ \theta$. Let $h : \mathbb{P}^n_K \to \mathbb{P}^n_k$ be the base change morphism.

Consider the following diagram:



and let $A \in D^b(\text{Coh}(\mathbb{P}^n_k))$.

$$\begin{aligned} H'(A) = H(L\pi^*(A)) &= \text{Mor}_{D_{QCoh}(X_K)}(Lj^* L\pi^* A, S) = \\ &= \text{Mor}_{D_{QCoh}(X_K)}(L\pi^*_K h^* A, S) = \text{Mor}_{D_{QCoh}(\mathbb{P}^n_K)}(h^* A, R\pi_{K*} S) \end{aligned}$$

so H' is represented by $R\pi_{K*}S \in D_{QCoh}(\mathbb{P}^n_K)$.

Let $\tilde{G} = \theta_K^{-1}(R\pi_{K*}(S))$ so that H'' is represented by $\tilde{G}$. Then

$$H''(\Lambda) = \text{Mor}_{\Lambda \otimes K}(\Lambda \otimes K, \tilde{G})$$

and

$$\begin{aligned} \sum_n \dim H''(\Lambda[n]) &= \sum_n \dim \text{Mor}(\Lambda[n] \otimes K, \tilde{G}) = \\ &= \sum_n \dim \text{Mor}((\Lambda \otimes K)[n], \tilde{G}) < \infty \end{aligned}$$

since H'' is of finite type. Therefore $\tilde{G} \in D^b(\underline{\text{mod}}(\Lambda \otimes K))$.

This implies that $R\pi_* S \in D^b(\text{Coh}(\mathbb{P}^n_K))$ hence $S \in D^b(\text{Coh}(X_K))$. □

proof of theorem 2.1.1. The case where L is purely transcendental of degree 2 over k was treated in Theorem 2.4.3. Let L be a finitely generated separable field extension of k with $\text{trdeg}_k L \leq 1$.

There exists a field K such that K is a purely transcendental extension of k of degree less than or equal 1, and $K \subset L$ is a finite extension. Set $L = L(\alpha) = K[T]/P(T)$. Consider the composition

$$D^b_{Coh}(X) \xrightarrow{H} \underline{mod}_K \xrightarrow{\text{forget}} \underline{\text{mod}}_L, \qquad H' : D^b_{Coh}(X) \to \underline{\text{mod}}_L$$

By theorem 2.4.3, H' is representable by an object $S \in D^b_{Coh}(X_K)$. Moreover, following the proof of Corollary 2.3.7, S is endowed with a map φ such that $P(\varphi) = 0$ is zero on all the cohomology groups of S.

First of all, this implies that there exists an n such that $P(\varphi)^n = 0$. In fact, consider the good truncations $\tau_{\leq i}S$,

$$\ldots \to S^{i-1} \to Z^i \to 0$$

then since $P(\varphi)$ is zero on cohomology, it is actually zero on Z^i so that $P(\varphi) : \tau_{\leq i}S \to \tau_{\leq i}S$ factors through $\tau_{\leq i-1}S$. Hence the claim follows inductively using the fact that S is a bounded complex.

Now let $h : X_L \to X_K$ be the base change morphism, and consider the pullback $Lh^*S \in D^b_{Coh}(X_L)$. It has a $L[T]$ action induced by the morphism $Lh^*\varphi$, and $P(Lh^*\varphi)^n = 0$ so Lh^*S has in fact an $L[T]/P^n(T)$-action.

Since, over L, $P(T)$ factors as $P(T) = (T - \alpha)Q(T)$, we get that

$$L[T]/P^n(T) = L[T]/(T-\alpha)^n \times L[T]/Q^n(T)$$

This means we can find two elements e_1, e_2 of $L[T]/P^n(T)$ such that $e_1^2 = e_1$, $e_2^2 = e_2$, $e_1e_2 = 0$, $e_1 + e_2 = 1$. But since $L[T]/P^n(T)$ acts on Lh^*S, this gives two idempotent operators e_1, e_2 in $\text{Aut}_{D^b_{Coh}(X_L)}(Lh^*S)$ such that $e_1e_2 = 0$, $e_1 + e_2 = \text{id}_{Lh^*S}$.

Now since $D^b_{Coh}(X_L)$ is Karoubian by [BN93, Proposition 3.2] we have obtained that $Lh^*S = E \oplus S_2$ and $Lh^*\varphi$ acts as multiplication by α on E.

We claim that $Rh_*E = S$. Consider the map

$$S \to Rh_*Lh^*S \xrightarrow{pr_1} Rh_*E$$

Under the identification $D_{QCoh}(X_L) \xrightarrow{\psi} D(\text{QCoh}(X)_L)$ this corresponds to $S \to S \otimes L \to \text{forget}(E)$, so this is actually the identity map on S.

Then for every $C \in D^b_{Coh}(X)$ we have a map of L-vector spaces

$$\mathrm{Mor}_{D_{QCoh}(X_L)}(Lj^*C, E) \to \mathrm{Mor}_{(D_{QCoh}(X))_L}(C \otimes L, \tilde{S}) = H(C)$$

where $j : X_L \to X$ is the base change morphism, since E is a lift of S to $D^b_{Coh}(X_L)$ with the correct L-action. This map is an isomorphism because it is an isomorphism of K-vector spaces:

$$\begin{aligned}\mathrm{Mor}_{D_{QCoh}(X_L)}(Lj^*C, E) &= \mathrm{Mor}_{(D_{QCoh}(X_K))}(Li^*C, Rh_*E) \\ &= \mathrm{Mor}_{(D_{QCoh}(X_K))}(Li^*C, S) = H'(C)\end{aligned}$$

where $i : X_K \to X$ is the base change morphism. □

proof of theorem 2.1.3. Consider the composition

$$D^b_{Coh}(X) \xrightarrow{H} D^b_{Coh}(Y) \xrightarrow{i_1^*} D^b_{Coh}(\eta) \xrightarrow{H^0} \underline{\mathrm{mod}}_{k(Y)} \xrightarrow{D} \underline{\mathrm{mod}}_{k(Y)}, \qquad F : D^b_{Coh}(X) \to \underline{\mathrm{mod}}_{k(Y)}$$

where $H^0(-) = H^0(\eta, -)$ and D is the dual as $k(Y)$-vector space. F is an exact contravariant finite type functor, hence by theorem 2.1.1 it is representable by $E \in D^b_{Coh}(X_{k(Y)})$.

Now consider the following diagram:

$$\begin{array}{ccc} X_\eta & \xrightarrow{p} & \eta \\ \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle i_1} \\ X \times Y & \xrightarrow{\pi_2} & Y \\ \downarrow{\scriptstyle \pi_1} & & \\ X & & \end{array} \qquad j : X_\eta \to X$$

Let $E^\vee = \underline{\mathrm{RHom}}_{X_\eta}(E, \mathcal{O}_{X_\eta})$. Let us construct a complex $A \in D^b_{Coh}(X \times Y)$ such that $Li_2^*A = E^\vee \otimes \omega_{X_\eta}[\dim X_\eta]$.

First of all, note that i_2 is a flat map, so the derived pullback is just regular pullback in every degree. Also, i_2 is an affine map so that pushforward is also exact. Moreover, $D^b_{Coh}(X) \cong D^b(\mathrm{Coh}(X))$ and every complex here is isomorphic to a complex that is nonzero only in a finite number of degrees. Since X and Y are projective varieties, there exists a line bundle $\mathcal{L} \in \mathrm{Coh}(X \times Y)$ such that $E^\vee \otimes \omega_{X_\eta}[\dim X_\eta] \otimes i_2^*\mathcal{L}^{\otimes n}$ is generated by its global sections in each degree. Let $\{s_{i,d}\}$ be a set of generators in degree d. Consider the complex $i_{2*}(E^\vee \otimes \omega_{X_\eta}[\dim X_\eta] \otimes i_2^*\mathcal{L}^{\otimes n})$ on $D^b_{Coh}(X \times Y)$.

Then take the subcomplex generated in each degree by $\{i_{2*}s_{i,d}\} \cup \{i_{2*}ds_{i,d-1}\}$, and twist it down by $\mathcal{L}^{-n}$. This gives the desired complex $A \in D^b_{Coh}(X \times Y)$.

Then we get the following:

$$\begin{aligned}
H^0 \circ i_1^* \circ \Phi_A(C) &= H^0 i_1^* R\pi_{2*}(A \otimes \pi_1^* C) \\
&= H^0 Rp_*(i_2^* A \otimes i_2^* \pi_1^* C) \quad \text{(by flat base change)} \\
&= H^0 Rp_*(E^\vee \otimes \omega_{X_\eta}[\dim\, X_\eta] \otimes j^* C) \\
&= \mathrm{Mor}(\mathcal{O}_\eta, Rp_*(E^\vee \otimes \omega_{X_\eta}[\dim\, X_\eta] \otimes j^* C)) \\
&= \mathrm{Mor}(p^*\mathcal{O}_\eta, E^\vee \otimes \omega_{X_\eta}[\dim\, X_\eta] \otimes j^* C) \\
&= \mathrm{Mor}(\mathcal{O}_{X_\eta}, E^\vee \otimes \omega_{X_\eta}[\dim\, X_\eta] \otimes j^* C) \\
&= \mathrm{Mor}(E, \omega_{X_\eta}[\dim\, X_\eta] \otimes j^* C) \\
&= D \circ \mathrm{Mor}(j^* C, E) \\
&= D \circ F(C) \\
&= H^0 \circ i_1^* \circ H(C)
\end{aligned}$$

for every $C \in D^b_{Coh}(X)$.

Now since H is an exact functor,

$$\begin{aligned}
H^i \circ i_1^* \circ H(C) &= H^0(i_1^* \circ H(C)[i]) = H^0(i_1^* \circ H(C[i])) = \\
&= H^0(i_1^* \circ \Phi_A(C[i])) = H^0(i_1^* \circ \Phi_A(C)[i]) = \\
&= H^i \circ i_1^* \circ \Phi_A(C)
\end{aligned}$$

Hence, since all cohomology groups agree and $D^b_{Coh}(k(Y))$ is equivalent to the category of graded vector spaces over $k(Y)$, H and Φ_A agree after restricting to the generic point of Y. $\square$

Chapter 3

On the existence of Fourier-Mukai kernels

3.1 Introduction

Let X, Y be projective varieties over an algebraically closed field k. Consider an exact functor

$$F : D^b_{Coh}(X) \to D^b_{Coh}(Y)$$

In this chapter we will computes sheaves on $X \times Y$ that are equal to the cohomology sheaves of the kernel whenever the functor is a Fourier-Mukai transform.:

Theorem 3.1.1. *Let X, Y be projective varieties over an algebraically closed field k, $F : D^b_{Coh}(X) \to D^b_{Coh}(Y)$ and exact functor. There exist a sequence of sheaves $\mathscr{B}^L, \mathscr{B}^{L+1}, \dots, \mathscr{B}^N$ on $X \times Y$ and maps*

$$\mathscr{H}^i(F(\mathscr{E}(n))) \to p_{2*}(\mathscr{B}^i \otimes p_1^* \mathscr{E}(n))$$

for any coherent locally free sheaf $\mathscr{E}$, for each $i = L, \dots, N$ and for each $n \in \mathbb{Z}$ that are isomorphisms for n sufficiently high (depending on $\mathscr{E}$), with

$$\mathscr{H}^i(F(\mathscr{E}(n))) = 0$$

for $i \notin [L, N]$, $n \gg 0$. Moreover, given a map $\mathscr{E}_1 \to \mathscr{E}_2$, we have a commutative diagram

$$\begin{array}{ccc} \mathscr{H}^i(F(\mathscr{E}_1(n))) & \longrightarrow & p_{2*}(\mathscr{B}^i \otimes p_1^*\mathscr{E}_1(n)) \\ \downarrow & & \downarrow \\ \mathscr{H}^i(F(\mathscr{E}_2(n))) & \longrightarrow & p_{2*}(\mathscr{B}^i \otimes p_1^*\mathscr{E}_2(n)) \end{array}$$

We are then able, in a special case, to construct an isomorphism between a class of functors that are not full or faithful and a Fourier-Mukai transform:

Theorem 3.1.2. *Let X and Y be two smooth projective varieties of dimension one, $F : D^b_{Coh}(X) \to D^b_{Coh}(Y)$ an exact functor. Assume that the corresponding $\mathscr{B}^i = 0$ for $i \neq M$, and that $\mathscr{B}^M = \bigoplus_{i=1}^t k(p_i, q_i)$. Let Φ be the Fourier-Mukai transform associated to the sheaf $\mathscr{B}^M$ placed in degree M. Then there exists an isomorphism of functors $s : \Phi \to F$.*

Even when we don't know how to build a kernel out of the sheaves $\mathscr{B}^i$ that we construct in Theorem 3.1.1, these sheaves will turn out to have good properties in their own right. As an example, we will show that the analogue of the Cartan-Eilenberg Spectral Sequence converges when the dimension of X is one.

From now on, X and Y will be smooth projective varieties over an algebraically closed field k. $F : D^b_{Coh}(X) \to D^b_{Coh}(Y)$ is an exact functor. $\mathcal{O}_X(1)$ will be a very ample line bundle on X.

3.2 Determining the cohomology sheaves of the prospective kernel

Consider our functor $F : D^b_{Coh}(X) \to D^b_{Coh}(Y)$. If we know that F is isomorphic to a Fourier-Mukai transform Φ_E, then we are of course able to compute the cohomology sheaves $\mathscr{B}^i = \mathscr{H}^i(E)$ corresponding to E. Even if we don't know what E is, or even if it exists, we are able to compute some sheaves on $X \times Y$ such that if the functor comes from a Fourier-Mukai transform, then those sheaves will turn out to be the cohomology sheaves of the corresponding kernel.

We recall the following definition from [Har77];

Definition 3.2.1. *The graded module $\Gamma_*(\mathscr{F})$ is defined as*

$$\Gamma_*(\mathscr{F}) = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathscr{F}(n))$$

Lemma 3.2.2. *Let X, Y be smooth projective varieties over $k = \bar{k}$, $\mathscr{L}$ a very ample invertible sheaf on X. There exists an equivalence of categories between the category of coherent sheaves on $X \times Y$ and the category of graded coherent $\Gamma_*(\mathcal{O}_X) \otimes \mathcal{O}_Y$-modules $\mathscr{M} = \oplus_k \mathscr{M}_k$ such that $\oplus_{k \geq n} \mathscr{M}_k$ is finitely generated for some n, where two coherent sheaves are identified if they agree in sufficiently high degree.*

Moreover, if this correspondence associates a sheaf $\oplus \mathscr{M}_n$ on Y to a sheaf $\mathscr{B}$ on $X \times Y$, there exists a functorial map of graded $\Gamma_*(\mathcal{O}_X) \otimes \mathcal{O}_Y$-modules

$$\oplus \mathscr{M}_n \to \oplus p_{2*}(\mathscr{B} \otimes \mathscr{L}^n)$$

which is an isomorphism on the n^{th} graded piece for n sufficiently high.

Proof. Since $\mathscr{L}$ is a very ample invertible sheaf on X we have an immersion $i : X \times Y \to \mathbb{P}^m_Y$ so that $X \times Y = \mathrm{Proj}(\mathscr{S})$ with $\mathscr{S} = \Gamma_*(\mathcal{O}_X) \otimes \mathcal{O}_Y$.

By [Gro61, 3.2.4, 3.3.5, 3.4.3, 3.4.5] we have a functor

$$\left\{ \begin{array}{c} \text{Graded coherent } \Gamma_*(\mathcal{O}_X) \otimes \mathcal{O}_Y\text{-modules } \mathscr{M} = \oplus_k \mathscr{M}_k \\ \text{such that } \oplus_{k \geq n} \mathscr{M}_k \text{ is finitely generated for some } n \end{array} \right\} \Big/ \sim \; \to \{\text{Coherent } \mathcal{O}_{X \times Y}\text{-modules}\}$$

$$\mathscr{M} \mapsto \tilde{\mathscr{M}}$$

where $\oplus \mathscr{M}_k \sim \oplus \mathscr{N}_k$ if there exists an integer n such that $\mathscr{M}_k \cong \mathscr{N}_k$ for all $k \geq n$. Moreover, by [Gro61, 3.3.5.1] we know that

$$\widetilde{\Gamma_*(\mathscr{F})} \xrightarrow{\cong} \mathscr{F}$$

Hence to show that $\mathscr{M} \mapsto \tilde{\mathscr{M}}$ gives an equivalence of categories we just need to show that

$$\alpha : \mathscr{M}_k \to \Gamma_*(\tilde{\mathscr{M}}_k)$$

is an isomorphism in large enough degree, which can be checked locally and hence follows by [Ser07, §65, Proposition 5]. The last assertion follows from the fact that as we just saw we have a graded isomorphism

$$\mathscr{M} \xrightarrow{\cong} \Gamma_*(\tilde{\mathscr{M}}) = \bigoplus_{n \in \mathbb{Z}} p_{2*}(\tilde{\mathscr{M}}(n)) = \bigoplus_{n \in \mathbb{Z}} p_{2*}(\mathscr{B}(n)) = \bigoplus_{n \in \mathbb{Z}} p_{2*}(\mathscr{B} \otimes \mathscr{L}^n)$$

□

proof of Theorem 3.1.1. By [Orl97, Lemma 2.4], we can assume that F is bounded, i.e. that $F(\mathscr{E}) \in D^{[L,N]}_{Coh}(Y)$ for all coherent sheaves $\mathscr{E}$ on X, i.e. $\mathscr{H}^i(F(\mathscr{E})) = 0$ for $i \notin [L, N]$.

We will proceed by descending induction on the cohomology degree i. We can take $\mathscr{B}^{N+1} = 0$ in what follows since $\mathscr{H}^{N+1}(F(\mathscr{E})) = 0$ for all coherent sheaves $\mathscr{E}$.

Assume we found the sheaves $\mathscr{B}^N, \mathscr{B}^{N-1}, \ldots, \mathscr{B}^{i+1}$ satisfying the conclusions of the Theorem and let's compute the sheaf $\mathscr{B}^i$. To do this we will proceed in two steps: first we will construct sheaves $\mathscr{B}^i_{\mathscr{E}}$ for all coherent locally free sheaves $\mathscr{E}$ as well as maps

$$\mathscr{H}^i(F(\mathscr{E}(n))) \to p_{2*}(\mathscr{B}^i_{\mathscr{E}} \otimes p_1^*\mathcal{O}_X(n))$$

that are isomorphisms for n sufficiently high, depending on $\mathscr{E}$ and i. Then we will show that

$$\mathscr{B}^i_{\mathscr{E}} = \mathscr{B}^i_{\mathcal{O}_X} \otimes p_1^*\mathscr{E}$$

For the first step, the key is showing that the sheaf $\bigoplus_{n>n_0} \mathscr{H}^i(F(\mathscr{E}(n)))$ on Y is finitely generated for each n_0 as a $\Gamma_*(X, \mathcal{O}_X) \otimes \mathcal{O}_Y$-module. To do this, proceed as follows: let s be an integer such that we have a surjection $\mathcal{O}_X^{\oplus s} \to \mathcal{O}_X(1)$. Let $\mathscr{E}$ be a coherent locally free sheaf on X. Then by tensoring the map above with $\mathscr{E}$ and twisting by n we have a short exact sequence of locally free sheaves

$$0 \to K(n) \to \mathscr{E}^{\oplus s}(n) \to \mathscr{E}(n+1) \to 0$$

Hence

$$0 \to p_1^*(K(n)) \to p_1^*(\mathscr{E}(n)^{\oplus s}) \to p_1^*(\mathscr{E}(n+1)) \to 0$$

is also a short exact sequence of locally free sheaves, and tensoring with $\mathscr{B}^{i+1}$ will yield another short exact sequence:

$$0 \to \mathscr{B}^{i+1} \otimes p_1^*K(n) \to \mathscr{B}^{i+1} \otimes p_1^*\mathscr{E}^{\oplus s} \to \mathscr{B}^{i+1} \otimes p_1^*\mathscr{E}(n+1) \to 0$$

moreover, since $p_1^*\mathcal{O}_X$ is very ample with respect to $X \times Y \to Y$, for n high enough (depending on K) the pushforward to Y will still be exact:

$$0 \to p_{2*}(\mathscr{B}^{i+1} \otimes p_1^*K(n)) \to p_{2*}(\mathscr{B}^{i+1} \otimes p_1^*\mathcal{O}_X(n)^{\oplus s}) \to p_{2*}(\mathscr{B}^{i+1} \otimes p_1^*\mathcal{O}_X(n+1)) \to 0$$

Hence we get a diagram

$$\begin{array}{ccccccccc} & & \mathscr{H}^{i+1}(F(K(n)) & \longrightarrow & \mathscr{H}^{i+1}(F(\mathscr{E}(n)^{\oplus s})) & \longrightarrow & \mathscr{H}^{i+1}(F(\mathscr{E}(n+1))) & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & p_{2*}(\mathscr{B}^{i+1}\otimes p_1^*K(n)) & \longrightarrow & p_{2*}(\mathscr{B}^{i+1}\otimes p_1^*\mathscr{E}(n)^{\oplus s}) & \longrightarrow & p_{2*}(\mathscr{B}^{i+1}\otimes p_1^*\mathscr{E}(n+1)) & \longrightarrow & 0 \end{array}$$

and for n high enough depending on K and $\mathscr{E}$, the vertical arrows are isomorphisms by the induction hypothesis; therefore the top sequence is also exact. Hence for n sufficiently high we also get a surjection

$$\mathscr{H}^i(F(\mathscr{E}(n)))^{\oplus s} \to \mathscr{H}^i(F(\mathscr{E}(n+1))) \to 0$$

since each $\mathscr{H}^i(F(\mathscr{E}(n)))$ is coherent, this is enough to conclude that the sheaf

$$\bigoplus_{n>n_0} \mathscr{H}^i(F(\mathscr{E}(n)))$$

is finitely generated for each n_0 as a $\Gamma_*(X,\mathcal{O}_X)\otimes\mathcal{O}_Y$-module, where the $\Gamma_*(X,\mathcal{O}_X)$-action comes from the action of $\Gamma_*(X,\mathcal{O}_X)$ on $\oplus\mathscr{E}(n)$ which gives a corresponding action on $\oplus F(\mathscr{E}(n))$ and hence on $\oplus\mathscr{H}^i(F(\mathscr{E}(n)))$. By Lemma 3.2.2, this corresponds to a sheaf $\mathscr{B}^i_{\mathscr{E}}$ on $X\times Y$ such that the map

$$\mathscr{H}^i(F(\mathscr{E}(n))) \to p_{2*}(\mathscr{B}^i_{\mathscr{E}}\otimes p_1^*\mathcal{O}_X(n))$$

is an isomorphisms for n sufficiently high.

Now consider the functor

$$B : \mathrm{Coh}(X) \to \mathrm{Coh}(X\times Y)$$
$$\mathscr{E} \mapsto \mathscr{B}^i_{\mathscr{E}}$$

The functor B is additive, and it is right exact on the full subcategory of locally free sheaves on X. In fact, given two coherent sheaves $\mathscr{E}_1$ and $\mathscr{E}_2$,

$$\bigoplus_n \mathscr{H}^i(F((\mathscr{E}_1+\mathscr{E}_2)(n))) = \bigoplus_n \mathscr{H}^i(F(\mathscr{E}_1(n))) \oplus \bigoplus_n \mathscr{H}^i(F(\mathscr{E}_2(n)))$$

hence the functor is additive. Moreover, given a short exact sequence $0\to\mathscr{E}_1\to\mathscr{E}_2\to\mathscr{E}_3\to 0$ with $\mathscr{E}_j$ locally free, we get a triangle $F(\mathscr{E}_1)\to F(\mathscr{E}_2)\to F(\mathscr{E}_3)$ hence for $n\gg 0$ we have (by induction hypothesis)

$$\begin{array}{ccccccccc} & & \mathscr{H}^{i+1}(F(\mathscr{E}_1(n)) & \longrightarrow & \mathscr{H}^{i+1}(F(\mathscr{E}_2(n))) & \longrightarrow & \mathscr{H}^{i+1}(F(\mathscr{E}_3(n))) & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & p_{2*}(\mathscr{B}^{i+1}\otimes p_1^*\mathscr{E}_1(n)) & \longrightarrow & p_{2*}(\mathscr{B}^{i+1}\otimes p_1^*\mathscr{E}_2(n)) & \longrightarrow & p_{2*}(\mathscr{B}^{i+1}\otimes p_1^*\mathscr{E}_3(n)) & \longrightarrow & 0 \end{array}$$

and for n sufficiently high, all of the vertical maps are isomorphisms hence the top sequence is also exact for n high, say $n > n_0$. Note that this is the only part of this proof where we need to be dealing with locally free sheaves, because we are using that the bottom sequence is exact on the left.

Hence we get

$$\bigoplus_{n>n_0} \mathscr{H}^i(F(\mathscr{E}_1(n))) \to \bigoplus_{n>n_0} \mathscr{H}^i(F(\mathscr{E}_2(n))) \to \bigoplus_{n>n_0} \mathscr{H}^i(F(\mathscr{E}_3(n))) \to 0$$

and so (by the equivalence of categories) get

$$\mathscr{B}^i_{\mathscr{E}_1} \to \mathscr{B}^i_{\mathscr{E}_2} \to \mathscr{B}^i_{\mathscr{E}_3} \to 0$$

hence the functor is right exact on the full subcategory of locally free sheaves.

Moreover, for every n, for $m \gg 0$ (depending on n) we have

$$\mathscr{H}^i(F(\mathscr{E}(n)(m))) = p_{2*}(\mathscr{B}^i_{\mathscr{E}(n)} \otimes p_1^*\mathcal{O}_X(m))$$

but also

$$\begin{aligned}\mathscr{H}^i(F(\mathscr{E}(n)(m))) &= \mathscr{H}^i(F(\mathscr{E}(n+m))) \\ &= p_{2*}(\mathscr{B}^i_{\mathscr{E}} \otimes p_1^*\mathcal{O}_X(n+m)) \\ &= p_{2*}((\mathscr{B}^i_{\mathscr{E}} \otimes p_1^*\mathcal{O}_X(n)) \otimes p_1^*\mathcal{O}_X(m))\end{aligned}$$

hence it follows from the equivalence of categories that

$$\mathscr{B}^i_{\mathscr{E}(n)} = \mathscr{B}^i_{\mathscr{E}} \otimes \mathcal{O}_X(n)$$

Now let $\mathscr{E}$ be a coherent, locally free sheaf on X. Then there exists a sequence

$$\oplus\mathcal{O}_X(b_j) \to \oplus\mathcal{O}_X(a_k) \to \mathscr{E} \to 0$$

therefore since the functor B is exact we get

$$\mathscr{B}^i_{\oplus\mathcal{O}_X(b_j)} \to \mathscr{B}^i_{\oplus\mathcal{O}_X(a_k)} \to \mathscr{B}^i_{\mathscr{E}} \to 0$$

and since B is additive and compatible with twists we can write

$$\oplus\mathscr{B}^i_{\mathcal{O}_X} \otimes p_1^*\mathcal{O}_X(b_j) \to \oplus\mathscr{B}^i_{\mathcal{O}_X} \otimes p_1^*\mathcal{O}_X(a_k) \to \mathscr{B}^i_{\mathscr{E}} \to 0$$

hence

$$\mathscr{B}^i_{\mathscr{E}} = \mathscr{B}^i_{\mathcal{O}_X} \otimes p_1^* \mathscr{E}$$

the Proposition then follows by taking $\mathscr{B}^i = \mathscr{B}^i_{\mathcal{O}_X}$. Since there is a finite number of steps in the induction, we can find an n_0 such that for $n > n_0$ the maps above are isomorphisms for all i.

The commutative diagram in the statement of the Proposition follows from the fact that the map in 3.2.2 is functorial. □

While Theorem 3.1.1 gives a map $\mathscr{H}^i(F(\mathscr{E})) \to p_{2*}(\mathscr{B}^i \otimes p_1^*\mathscr{E})$, for all n, in general it is only an isomorphism for n sufficiently large. In the case of the first M such that $\mathscr{H}^M(F(\mathscr{E}))$ is nonzero for some locally free sheaf $\mathscr{E}$ we can actually say more:

Proposition 3.2.3. *In the situation of Theorem 3.1.1, choose, M, N such that $F(\mathscr{E}) \in D^{[M,N]}_{Coh}(Y)$ for all $\mathscr{E}$ coherent locally free sheaf on X. Then the maps*

$$\mathscr{H}^M(F(\mathscr{E})) \to p_{2*}(\mathscr{B}^M \otimes p_1^*\mathscr{E})$$

are isomorphisms for all coherent locally free sheaves $\mathscr{E}$.

Proof. Let $d = \dim X$. Choose sections $s_1, \ldots, s_{d+1}$ of $\mathcal{O}_X(1)$ such that the corresponding hyperplanes have empty intersection. Then for any $m \in \mathbb{N}$ we have short exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{(s_1^m, \ldots, s_{d+1}^m)} \mathcal{O}_X(m)^{d+1} \to K_m \to 0$$

where K_m is a locally free sheaf.

Let $\mathscr{E}$ be any coherent locally free sheaf. Then by tensoring the above short exact sequence with $\mathscr{E}$ we get

$$0 \to \mathscr{E} \to \mathscr{E}(m)^{\oplus(d+1)} \to K_m \otimes \mathscr{E} \to 0$$

and so

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathscr{H}^M(F(\mathscr{E})) & \longrightarrow & \mathscr{H}^M(F(\mathscr{E}(m)^{d+1})) & \longrightarrow & \mathscr{H}^M(F(K_m \otimes \mathscr{E})) \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \longrightarrow & p_{2*}(\mathscr{B}^M \otimes p_1^*\mathscr{E}) & \longrightarrow & p_{2*}(\mathscr{B}^M \otimes p_1^*\mathscr{E}(m)^{d+1}) & \longrightarrow & p_{2*}(\mathscr{B}^M \otimes p_1^*(K_m \otimes \mathscr{E})) \end{array}$$

Let m be high enough so that the center map is an isomorphism (this is possible by Proposition 3.1.1). Then the map on the left must be injective. Thus we showed: for every coherent, locally free sheaf $\mathscr{E}$, the map $\mathscr{H}^M(F(\mathscr{E})) \to p_{2*}(\mathscr{B}^M \otimes p_1^*\mathscr{E})$ is injective.

Now let's go back to the diagram above. By what we just showed, the map on the right $\mathscr{H}^M(F(K_m \otimes \mathscr{E})) \to p_{2*}(\mathscr{B}^M \otimes p_1^*(K_m \otimes \mathscr{E}))$ is injective. Hence we have

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathscr{H}^M(F(\mathscr{E})) & \longrightarrow & \mathscr{H}^M(F(\mathscr{E}(m)^{d+1})) & \longrightarrow & \mathscr{H}^M(F(K_m \otimes \mathscr{E})) \\ & \downarrow & & \downarrow \cong & & \downarrow \\ 0 \longrightarrow & p_{2*}(\mathscr{B}^M \otimes p_1^*\mathscr{E}) & \longrightarrow & p_{2*}(\mathscr{B}^M \otimes p_1^*\mathscr{E}(m)^{d+1}) & \longrightarrow & p_{2*}(\mathscr{B}^M \otimes p_1^*(K_m \otimes \mathscr{E})) \end{array}$$

then by the 5 Lemma the left arrow is an isomorphism, i.e.

$$\mathscr{H}^M(F(\mathscr{E})) \xrightarrow{\cong} p_{2*}(\mathscr{B}^M \otimes p_1^*\mathscr{E})$$

□

Similarly to Proposition 3.2.3, we also have a stronger result than the one in Theorem 3.1.1 for the largest N' such that $\mathscr{B}^i \neq 0$. In this case, the map $\mathscr{H}^{N'}(F(\mathscr{E}(n)) \to p_{2*}(\mathscr{B}^{N'} \otimes p_1^*\mathscr{E}(n))$ can be constructed for all coherent sheaves on X instead of just the locally free ones:

Proposition 3.2.4. *In the situation of Theorem 3.1.1, let N' be the largest i such that $\mathscr{B}^i \neq 0$. Then for all $n \in \mathbb{Z}$, for any coherent sheaf $\mathscr{F}$ we have a map*

$$\mathscr{H}^{N'}(F(\mathscr{F}(n)) \to p_{2*}(\mathscr{B}^{N'} \otimes p_1^*\mathscr{F}(n))$$

which is an isomorphism for n sufficiently high.

Proof. The proof is exactly the same as the proof of Theorem 3.1.1. In this case we don't need to ask for $\mathscr{F}$ to be locally free because since $\mathscr{B}^{N'+1} = 0$, given a short exact sequence $0 \to \mathscr{E} \to \mathscr{F}_1 \to \mathscr{F}_2 \to 0$, the sequence

$$\mathscr{H}^{N'}(F(\mathscr{E}(n))) \to \mathscr{H}^{N'}(F(\mathscr{F}_1(n))) \to \mathscr{H}^{N'}(F(\mathscr{F}_2(n))) \to 0$$

is always exact for $n \gg 0$ if $\mathscr{E}$ is locally free, and this is all we need to conclude that $\mathscr{B}_{\mathscr{F}}^{N'+1} = \mathscr{B}^{N'} \otimes p_1^*\mathscr{F}$ for any coherent sheaf $\mathscr{F}$. □

3.3 A special case

In this section we will assume that X and Y are smooth projective varieties over an algebraically closed field k, and $F : D^b_{Coh}(X) \to D^b_{Coh}(Y)$ is an exact functor. In this section we will give an example of a class of functors for which we can always find an object $E \in D^b_{Coh}(X \times Y)$ and an equivalence $F \cong \Phi_E$. The sheaves $\mathscr{B}^i$ will be the ones defined as in Theorem 3.1.1.

Proposition 3.3.1. *Let $F : D^b_{Coh}(X) \to D^b_{Coh}(Y)$, $dim(X) = 1$ and assume that the sheaves $\mathscr{B}^i$ defined as in Theorem 3.1.1 are zero for $i \neq M$. Assume also that $\mathscr{B}^M$ is a coherent sheaf supported at finitely many points of $X \times Y$.*

Then for any coherent sheaf $\mathscr{F}$ on X we have $\mathscr{H}^i(F(\mathscr{F})) = 0$ for $i \neq M, M-1$ and for any locally free sheaf $\mathscr{E}$ we have $\mathscr{H}^i(F(\mathscr{E})) = 0$ for $i \neq M$.

Moreover, for each coherent sheaf $\mathscr{F}$ on X there is a functorial isomorphism

$$\mathscr{H}^M(F(\mathscr{F})) \xrightarrow{\cong} p_{2*}(\mathscr{B}^M \otimes p_1^*\mathscr{F})$$

Proof. Consider any torsion sheaf Q. Then we have a short exact sequence $0 \to \mathscr{E}' \to \mathscr{E} \to Q \to 0$ with $\mathscr{E}, \mathscr{E}'$ locally free. Twist $\mathscr{E}$ and $\mathscr{E}'$ by $n \gg 0$ so that $\mathscr{H}^i(F(\mathscr{E}'(n))) = \mathscr{H}^i(F(\mathscr{E}(n))) = 0$ for $i \neq M$. Since $0 \to \mathscr{E}'(n) \to \mathscr{E}(n) \to Q \to 0$ is still an exact sequence, from the long exact sequence on cohomology we can conclude that $\mathscr{H}^i(F(Q)) = 0$ for all $i \neq M, M-1$.

Now consider a locally free sheaf $\mathscr{E}$ on X. Let $\bar{n}$ be large enough so that we know $\mathscr{H}^i(F(\mathscr{E}(\bar{n}))) = 0$ for all $i \neq M$. Then we have a short exact sequence $0 \to \mathscr{E}(\bar{n}-1) \to \mathscr{E}(\bar{n}) \to T \to 0$ where T is a torsion sheaf. A portion of the long exact sequence in cohomology gives

$$\mathscr{H}^{i-1}(F(T)) \to \mathscr{H}^i(F(\mathscr{E}(\bar{n}-1))) \to \mathscr{H}^i(F(\mathscr{E}(\bar{n})))$$

and $\mathscr{H}^{i-1}(F(T)) = \mathscr{H}^i(F(\mathscr{E}(\bar{n}))) = 0$ for $i \neq M, M+1$ hence $\mathscr{H}^i(F(\mathscr{E}(\bar{n}-1))) = 0$ for $i \neq M, M+1$. By descending induction on $\bar{n}$ we then obtain that $\mathscr{H}^i(F(\mathscr{E}(n))) = 0$ for all n and $i \neq M, M+1$. We will show at the end of the proof that $\mathscr{H}^{M+1}(F(\mathscr{E}) = 0$.

By Proposition 3.2.4 we know that for any coherent sheaf $\mathscr{F}$ on X we have a functorial map

$$\mathscr{H}^M(F(\mathscr{F})) \xrightarrow{\cong} p_{2*}(\mathscr{B}^M \otimes p_1^*\mathscr{F})$$

which is an isomorphism by Proposition 3.2.3 if $\mathscr{F}$ is locally free (notice that the hypotheses of 3.2.3 are satisfied by the first part of this Proposition). Moreover we also know, again by Proposition 3.2.4, that for any coherent sheaf $\mathscr{F}$ the map

$$\mathscr{H}^M(F(\mathscr{F}(n))) \xrightarrow{\cong} p_{2*}(\mathscr{B}^M \otimes p_1^*\mathscr{F}(n))$$

is an isomorphism for n sufficiently high. But when $\mathscr{F}$ is a sheaf supported at a point twisting doesn't affect the sheaf, so we get that

$$\mathscr{H}^M(F(\mathscr{F})) \xrightarrow{\cong} p_{2*}(\mathscr{B}^M \otimes p_1^*\mathscr{F})$$

is also an isomorphism for torsion sheaves, and hence it is always an isomorphism since any coherent sheaf on X is the direct sum of a locally free part and a torsion part.

Now let's show that $\mathscr{H}^{M+1}(F(\mathscr{E})) = 0$: consider the diagram

$$\begin{array}{ccccccc}
\mathscr{H}^M(F(\mathscr{E}(\bar{n}))) & \longrightarrow & \mathscr{H}^M(F(T)) & \longrightarrow & \mathscr{H}^{M+1}(F(\mathscr{E}(\bar{n}-1))) & \longrightarrow & \mathscr{H}^{M+1}(F(\mathscr{E}(\bar{n}))) = 0 \\
\downarrow \cong & & \downarrow \cong & & \downarrow & & \\
p_{2*}(\mathscr{B}^M \otimes p_1^*\mathscr{E}(\bar{n})) & \longrightarrow & p_{2*}(\mathscr{B}^M \otimes p_1^*T) & \longrightarrow & 0 & &
\end{array}$$

where the bottom sequence is right exact because $\mathscr{B}^M$ is a flasque sheaf. From the five Lemma it follows that $\mathscr{H}^{M+1}(F(\mathscr{E}(\bar{n}-1))) = 0$. So we can again proceed by induction on $\bar{n}$. □

Proposition 3.3.2. *Let $F : D^b_{Coh}(X) \to D^b_{Coh}(Y)$, $\dim(X) = 1$ and assume $\mathscr{B}^i = 0$ for $i \neq M$. Assume also that $\mathscr{B}^M$ is a coherent sheaf supported at finitely many points of $X \times Y$. Let Φ be the Fourier-Mukai transform associated to the sheaf $\mathscr{B}^M$ placed in degree M.*

Then there is an isomorphism of δ-functors

$$\mathscr{H}^i(F(\cdot)) \xrightarrow{\cong} \mathscr{H}^i(\Phi(\cdot))$$

on the category of coherent sheaves on X, which gives an isomorphism of functors $F \to \Phi$ for the full subcategory of $D^b_{Coh}(X)$ consisting of locally free sheaves placed in degree zero.

Proof. The fact that there is a functorial isomorphism

$$\mathscr{H}^M(F(\cdot)) \xrightarrow{\cong} \mathscr{H}^M(\Phi(\cdot))$$

on the category of coherent sheaves on X follows immediately from Proposition 3.3.1 given that $\mathscr{H}^M(\Phi(\mathscr{F})) = p_{2*}(\mathscr{B}^M \otimes p_1^*\mathscr{F})$.

Moreover, for any locally free sheaf $\mathscr{E}$, since the only nonzero cohomology sheaf of $F(\mathscr{E})$ is in degree M and pushforward is exact for flasque sheaves,

$$F(\mathscr{E}) = \mathscr{H}^M(F(\mathscr{E}))[-M] \xrightarrow{\cong} \mathscr{H}^M(\Phi(\mathscr{E}))[-M] =$$
$$= p_{2*}(\mathscr{B}^M \otimes p_1^*\mathscr{E})[-M] = Rp_{2*}(\mathscr{B}^M[-M] \overset{L}{\otimes} Lp_1^*\mathscr{E}) = \Phi(\mathscr{E})$$

This gives the isomorphism of functors on the full subcategory of $D^b_{Coh}(X)$ of locally free sheaves placed in degree zero.

Let us now construct the isomorphism

$$\mathscr{H}^{M-1}(F(\cdot)) \xrightarrow{\cong} \mathscr{H}^{M-1}(\Phi(\cdot))$$

Consider a coherent sheaf Q on X which is not locally free. Then there is a short exact sequence $0 \to A' \to A \to Q \to 0$ where A' and A are locally free. Then we get a long exact sequence in cohomology

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathscr{H}^{M-1}(F(Q)) & \longrightarrow & \mathscr{H}^{M}(F(A')) & \longrightarrow & \mathscr{H}^{M}(F(A)) \\ & & & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathscr{H}^{M-1}(\Phi(Q)) & \longrightarrow & \mathscr{H}^{M}(\Phi(A')) & \longrightarrow & \mathscr{H}^{M}(\Phi(A)) \end{array}$$

so we get an isomorphism $\mathscr{H}^{M-1}(F(Q)) \to \mathscr{H}^{M-1}(\Phi(Q))$.

We still need to show that this map is functorial and that it does not depend on the choice of a short exact sequence. Consider a map $Q \to T$ of coherent sheaves. Then we can construct two short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & Q & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle id} & & \| & & \\ 0 & \longrightarrow & B' & \longrightarrow & A & \longrightarrow & T & \longrightarrow & 0 \end{array}$$

with A, A', B and B' torsion free. Then we get the following diagram on cohomology:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathscr{H}^{M-1}(\Phi(Q)) & \longrightarrow & \mathscr{H}^{M}(\Phi(A')) & \longrightarrow & \mathscr{H}^{M}(\Phi(A)) \\ 0 \to & \mathscr{H}^{M-1}(F(Q)) & \longrightarrow & \mathscr{H}^{M}(\mathscr{F}(A')) & \longrightarrow & \mathscr{H}^{M}(F(A)) & \\ 0 & \longrightarrow & \mathscr{H}^{M-1}(\Phi(T)) & \longrightarrow & \mathscr{H}^{M}(\Phi(B')) & \longrightarrow & \mathscr{H}^{M}(\Phi(B)) \\ 0 \to & \mathscr{H}^{M-1}(F(T)) & \longrightarrow & \mathscr{H}^{M}(F(B')) & \longrightarrow & \mathscr{H}^{M}(F(B)) & \end{array} \quad (3.1)$$

and since the two rightmost diagonal squares commute, the leftmost diagonal square will also commute. This shows functoriality.

To show that the maps we chose do not depend on the choice of a short exact sequence, notice that given two short exact sequences $0 \to A' \to A \to Q \to 0$ and $0 \to B' \to B \to Q \to 0$ there is a short exact sequence $0 \to C \to A \oplus B \to Q \to 0$ mapping to both of them. So we just need to prove this statement for two short exact sequences with maps between them. But then we are again in the situation of diagram (3.1), where $T = Q$ and the two rightmost maps in the diagram are the identity. So this follows again from the commutativity of the leftmost diagonal square.

Finally, we have to show that for every short exact sequence $0 \to B' \to B \to Q \to 0$ the diagram

$$\begin{array}{ccc} \mathscr{H}^{M-1}(F(Q)) & \longrightarrow & \mathscr{H}^{M}(F(B')) \\ \downarrow & & \downarrow \\ \mathscr{H}^{M-1}(\Phi(Q)) & \longrightarrow & \mathscr{H}^{M}(\Phi(B')) \end{array}$$

is commutative. This follows immediately by the construction when B' and B are locally free. Otherwise, construct a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & Q & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & Q & \longrightarrow & 0 \end{array}$$

with A, A' locally free. Then we get a diagram as in (3.1) with $T = Q$ and where everything commutes except possibly for the bottom leftmost parallelogram, but that follows immediately since the leftmost arrow is the identity. $\square$

Theorem 3.3.3. *Let X and Y be two projective varieties, dim$(X) = 1$, $F : D^b_{Coh}(X) \to D^b_{Coh}(Y)$ an exact functor. Assume that the corresponding $\mathscr{B}^i = 0$ for $i \neq M$, and that $\mathscr{B}^M$ is a skyscraper sheaf supported at a finite number of points, $\mathscr{B}^M = \bigoplus_{j=1}^t k(p_i, q_i)$. Let Φ be the Fourier-Mukai transform associated to the sheaf $\mathscr{B}^M$ placed in degree M. Restrict the two functors to the full subcategory of sheaves supported in degree 0 (here the only triangles are short exact sequences of sheaves). Then there exists an isomorphism of triangulated functors $s(\cdot) : \Phi(\cdot) \to F(\cdot)$.*

Before we prove the Theorem, let us prove two technical Lemmas that we will use in the proof.

Lemma 3.3.4. *Let X be a projective variety and $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X. Consider a surjective map $\alpha : \oplus_t \mathcal{O}_X \to Q$ where Q is torsion sheaf. Then there exists an integer h, depending on Q, such that for all $m \geq h(Q)$ and for any map $\beta : \mathcal{O}_X(-m) \to Q$ there exists a* *map* $\gamma : \mathcal{O}_X(-m) \to \oplus_t \mathcal{O}_X$ *making the following diagram commute:*

$$\begin{array}{ccc} \mathcal{O}_X(-m) & \overset{\gamma}{\dashrightarrow} & \bigoplus_t \mathcal{O}_X \\ & {\scriptstyle\beta}\searrow \quad \swarrow{\scriptstyle\alpha} & \\ & Q & \end{array}$$

Proof. We have a short exact sequence

$$0 \to \mathrm{Ker}(\alpha) \to \oplus \mathcal{O}_X \to Q \to 0$$

Twist by $\mathcal{O}_X(m)$ to get

$$0 \to \mathrm{Ker}(\alpha)(m) \to \oplus \mathcal{O}_X(m) \to Q(m) \to 0$$

A map $\beta : \mathcal{O}_X(-m) \to Q$ is the same thing as a map $\mathcal{O}_X \to Q(m)$, hence as an element $\beta(m) \in H^0(X, \mathcal{O}_X(m))$. By Serre vanishing, there exists an $h \geq 0$ such that $H^1(X, \mathrm{Ker}(\alpha)(m) = 0$ for all $m \geq h$. Hence β lifts to a section $\gamma(m)$ of $H^0(X, \mathcal{O}_X(m))$. Twist down by m to get the desired map $\gamma : \mathcal{O}_X(-m) \to \oplus \mathcal{O}_X$. □

Lemma 3.3.5. *Let X be a smooth projective variety over an algebraically closed field, let $p_1, \ldots, p_t \in X$ and let $\mathscr{E}$ be a locally free sheaf of rank r generated by global sections. Then there exist and open set $\mathscr{U}$ containing $p_1, \ldots, p_t$ and global sections $s_1, \ldots, s_r$ of $\mathscr{E}$ that generate the stalk $\mathscr{E}_p$ at each point $p \in \mathscr{U}$.*

Proof. Assume we found $s_1, \ldots, s_i \in \Gamma(X, \mathscr{E})$ that are linearly independent at each stalk at $p_1, \ldots, p_t$ so that we have

$$0 \to \mathcal{O}_X^{\oplus i} \to \mathscr{E} \xrightarrow{f} Q \to 0$$
$$e_j \mapsto s_j$$

Let's find a global section of $\mathscr{E}$ such that its image in Q doesn't vanish at $p_1, \ldots, p_t$. Let $u_i \in \Gamma(X, \mathscr{E})$ such that $f(u_i)$ doesn't vanish at p_i (we can do this because f is surjective on stalks and $\mathscr{E}$ is generated by global sections). Then $u_1, \ldots, u_t$ form a sub-vector space V of $\Gamma(X, \mathscr{E})$ of dimension l for some l and, for each i, $\dim(\{u \in V : f(u)(p_i) = 0\}) \leq l - 1$. Hence

$$\{u \in V : f(u)(p_i) = 0 \text{ for some i}\} = \bigcup_i \{u \in V : f(u)(p_i) = 0\}$$

is a union of subsets of dimension less or equal to $l-1$ and hence it is strictly contained in V since our field of definition is infinite (because it is algebraically closed). So we can find a section s_{i+1} in V that doesn't vanish at any of the p_i. Then $s_1, \ldots, s_{i+1}$ are linearly independent at each p_i as sections of $\mathscr{E}$. We can keep doing this as long as $\mathrm{rk}Q > 0$. Then the sections $s_1, \ldots, s_r$ will generate the stalk $\mathscr{E}_p$ at each point p in an open set $\mathscr{U}$ containing $p_1, \ldots, p_t$. $\square$

Proof of Theorem 3.3.3. We will first construct the isomorphism on objects, starting with the subcategory of locally free sheaves and torsion sheaves. This will a priori involve making non-canonical choices, but as it later turns out, the choices we are making are actually unique. Then we will prove that the isomorphisms are compatible with morphisms and this will allow us to define said isomorphism on a general coherent sheaf. Lastly, we will show that the given isomorphisms induce maps of triangles when applied to a short exact sequence of sheaves.

I. On the subcategory of locally free sheaves: Let $\mathscr{E}$ be a locally free sheaf on X. Then by Proposition 3.3.2 there is a functorial equivalence $s(\mathscr{E}) : \Phi_{\mathscr{B}}(\mathscr{E}) \to F(\mathscr{E})$.

II. On torsion sheaves: Consider a torsion sheaf Q on X. There exists a short exact sequence $0 \to K \to \mathcal{O}_X^{\oplus t} \xrightarrow{\alpha} Q \to 0$, with K a locally free sheaf. Then we have a diagram

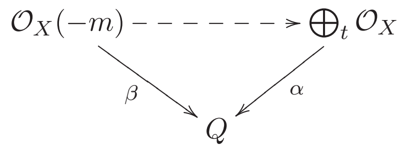
$$\begin{CD} \Phi(K) @>>> \Phi(\mathcal{O}_X^{\oplus t}) @>>> \Phi(Q) \\ @V{qis}V{s(K)}V @VV{s(\mathcal{O}_X^{\oplus t})}V @VVV \\ \mathscr{F}(K) @>>> F(\mathcal{O}_X^{\oplus t}) @>>> F(Q) \end{CD}$$

so there exists a dotted arrow $\Phi(Q) \to F(Q)$ which is a quasi-isomorphism (this dotted arrow is not necessarily unique). Choose one such arrow and call it $s(Q)$. Notice that $s(Q)$ will induce on cohomology the maps that we found in Proposition 3.3.2 because the maps induced on the M^{th} cohomology are the same as the ones in Proposition 3.3.2, and the maps $\mathscr{H}^{M-1}(\Phi(Q)) \to \mathscr{H}^M(\Phi(K))$ and $\mathscr{H}^{M-1}(F(Q)) \to \mathscr{H}^M(F(K))$ are injective.

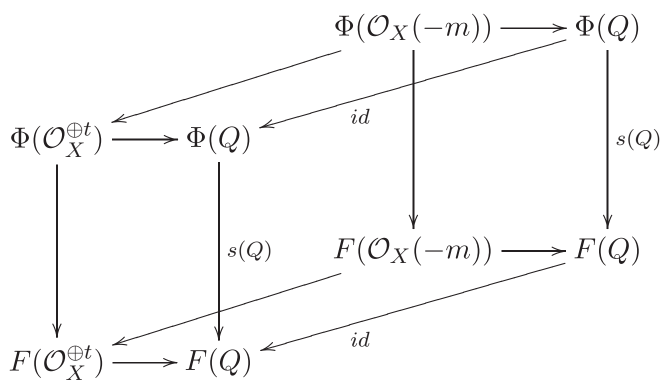
III. $s(-)$ **is compatible with maps** $\mathscr{E} \to Q$**,** $\mathscr{E}$ **locally free,** Q **torsion:** First of all we will prove the following: for any map $\mathcal{O}_X(i) \to Q$, the diagram

$$\begin{CD} \Phi(\mathcal{O}_X(i)) @>>> \Phi(Q) \\ @VV{s(\mathcal{O}_X(i))}V @VV{s(Q)}V \\ F(\mathcal{O}_X(i)) @>>> F(Q) \end{CD}$$

commutes. In fact, by Lemma 3.3.4, for every map $\beta : \mathcal{O}_X(-m) \to Q$ with $m \geq h(Q)$ we have a diagram

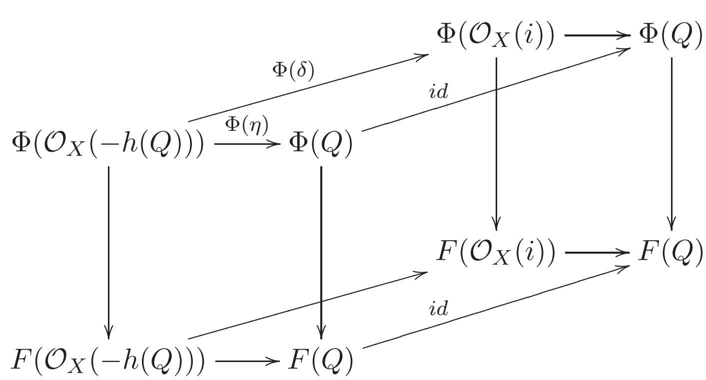


so by applying the functors F and Φ we obtain the following diagram:



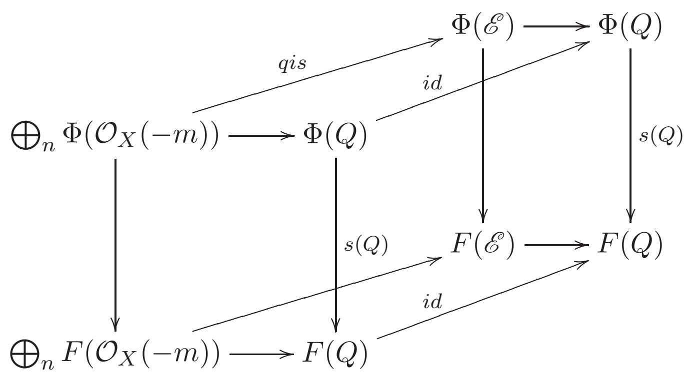
and the bottom left square commutes, hence the top right square will also commute.

Now let $i > -h(Q)$ and consider $\gamma : \mathcal{O}_X(i) \to Q$. Then pick any map $\delta : \mathcal{O}_X(-h(Q)) \to \mathcal{O}_X(i)$ such that δ is an isomorphism on an open set containing $p_1, \ldots, p_t$, and let $\eta = \gamma \circ \delta$. Then the map $\Phi(\delta) : \Phi(\mathcal{O}_X(-h(Q))) \to \Phi(\mathcal{O}_X(i))$ is an isomorphism: in fact the map $p_1^*(\delta) : p_1^*(\mathcal{O}_X(-h(Q))) \to p_1^*(\mathcal{O}_X(i))$ is an isomorphism on an open set containing $(p_1, q_1), \ldots, (p_t, q_t)$ and hence we will get an isomorphism when tensoring with a sheaf supported at $(p_1, q_1), \ldots, (p_t, q_t)$. So once again we get a diagram



since $\Phi(\delta)$ is invertible and the bottom left square is commutative, the top right square will also commute.

Now consider any map $\mathscr{E} \to Q$ with $\mathscr{E}$ locally free. Let $m \in \mathbb{Z}^+$ such that $\mathscr{E}(m)$ is generated by global sections, and let $n = \mathrm{rk}\mathscr{E}$. By Lemma 3.3.5 we can find $s_1, \ldots, s_n$ global sections of $\mathscr{E}(m)$ that are linearly independent at each stalk of an open set $\mathscr{U}$ containing $p_1, \ldots, p_t$. Then the corresponding map $\bigoplus_n \mathcal{O}_X \to \mathscr{E}(m)$ is injective and it is an isomorphism on $\mathscr{U}$. Twisting down by m we get a map $\bigoplus_n \mathcal{O}_X(-m) \to \mathscr{E}$ which is an isomorphism on $\mathscr{U}$. Hence we get again a diagram like the above one,



and since the diagonal maps are quasi isomorphisms and the bottom left square commutes, the top right square will also commute.

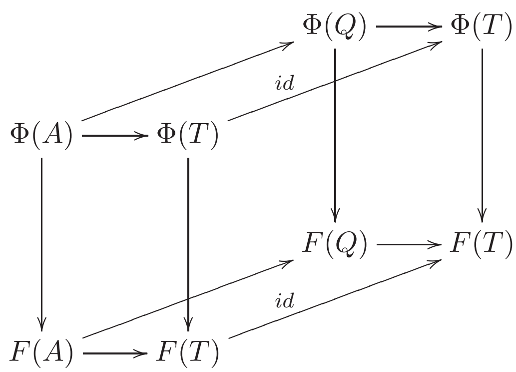
IV. $s(-)$ is compatible with maps $Q \to T$, Q and T torsion: We need to show that for any map between torsion sheaves $Q \to T$, the corresponding diagram

$$\begin{array}{ccc} \Phi(Q) & \longrightarrow & \Phi(T) \\ \downarrow{\scriptstyle s(Q)} & & \downarrow{\scriptstyle s(T)} \\ F(Q) & \longrightarrow & F(T) \end{array}$$

is commutative. To do this, consider a locally free sheaf $A = \bigoplus_r \mathcal{O}_X$ with a surjection $f : A \to Q$. For consistency we will represent this situation with a square diagram as before

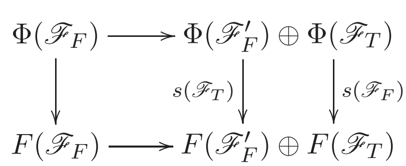
$$\begin{array}{ccc} A & \longrightarrow & T \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle id} \\ Q & \longrightarrow & T \end{array}$$

Then we get the following diagram:

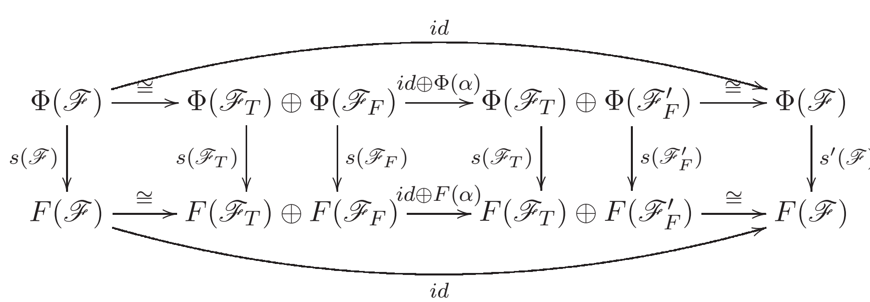


where the bottom left square commutes by III. Hence the top right square will also commute after pre-composing with the map $\Phi(A) \to \Phi(Q)$. But then we can conclude that the top right square also commutes - in fact it commutes on cohomology because of Proposition 3.3.2, so we can apply Lemma 3.3.6 below.

V. On a general coherent sheaf on X**:** Let $\mathscr{F}$ be any coherent sheaf on X. Then we have a decomposition $\mathscr{F} \cong \mathscr{F}_T \oplus \mathscr{F}_F$ where $\mathscr{F}_T$ is the canonical summand consisting of the torsion part of $\mathscr{F}$ and $\mathscr{F}_F$ corresponds to the torsion free part (this summand is not canonical). Then define $s(\mathscr{F}) = s(\mathscr{F}_T) \oplus s(\mathscr{F}_F)$. We need to show that this map doesn't depend on the choice of the decomposition. So consider two such decompositions $\mathscr{F} \cong \mathscr{F}_T \oplus \mathscr{F}_F$ and $\mathscr{F} \cong \mathscr{F}_T \oplus \mathscr{F}'_F$ and call $s(\mathscr{F})$ and $s'(\mathscr{F})$ respectively the two induced maps on $\Phi(\mathscr{F})$. Then the identity $\mathscr{F} \to \mathscr{F}$ induces a map $\alpha : \mathscr{F}_F \to \mathscr{F}'_F \oplus \mathscr{F}_T$, and by I. and III. the following diagram is commutative:



whereas the diagram for the torsion part is clearly commutative because the induced maps are just the identity, hence every square in the following diagram is commutative



Hence the external rectangle commutes, which proves precisely that $s(\mathscr{F}) = s'(\mathscr{F})$.

VI. $s(-)$ **is compatible with any maps** $A \to B$**, for** A **and** B **coherent sheaves:** Given a map $f : A \to B$, write $A = A_F \oplus A_T$ and $B = B_F \oplus B_T$. Then s will be compatible with $\Phi(f)$ and $F(f)$ because it is compatible with the maps $A_F \to B_F$, $A_F \to B_T$, and $A_T \to B_T$.

VII. $s(-)$ **is compatible with triangles of the type** $0 \to A \to B \to C \to 0$ **for** A **and** B **locally free:** The last thing to show is that given a short exact sequence of coherent sheaves on X, $0 \to A \to B \to C \to 0$, the maps $s(A)$, $s(B)$ and $s(C)$ give a morphism of triangles

$$\begin{array}{ccccccc} \Phi(A) & \longrightarrow & \Phi(B) & \longrightarrow & \Phi(C) & \longrightarrow & \Phi(A)[1] \\ \downarrow{\scriptstyle s(A)} & & \downarrow{\scriptstyle s(B)} & & \downarrow{\scriptstyle s(C)} & & \downarrow{\scriptstyle s(A)[1]} \\ F(A) & \longrightarrow & F(B) & \longrightarrow & F(C) & \longrightarrow & F(A)[1] \end{array} \tag{3.2}$$

First of all we will analyze the map $\Phi(B) \to \Phi(C)$. We know that $\Phi(B)$ is supported in degree M, whereas $\Phi(C)$ is supported in degrees M and $M-1$ hence, by [Dol60], as a complex we have $\Phi(C) \cong \mathscr{H}^M(\phi(C))[-M] \oplus \mathscr{H}^M(\phi(C))[-M+1]$ (in a non-canonical way). The situation looks as follows:

$$\begin{array}{ccccc} \mathscr{H}^M(\Phi(B))[-M] & \longrightarrow & \mathscr{H}^M(\Phi(C))[-M] & & \\ & \searrow & \uparrow{\scriptstyle 0} & \searrow & \\ & & \mathscr{H}^{M-1}(\Phi(C))[-M+1] & \longrightarrow & \mathscr{H}^M(\Phi(A))[-M+1] \end{array}$$

We will now show that the induced maps $\Phi(B) \to \mathscr{H}^{M-1}(\Phi(C))[-M+1]$ as well as $\mathscr{H}^M(\Phi(C))[-M] \to \Phi(A)[1]$ are zero in $D^b_{Coh}(Y)$ for some choice of a decomposition $\Phi(C) \cong \mathscr{H}^{M-1}(\Phi(C))[-M+1] \oplus \mathscr{H}^M(\Phi(C))[-M]$. In fact, consider a locally free resolution of $p_1^* C$, $\bar{C}_{-1} \to \bar{C}_0$. Then the map $B \to C$ induces a map of complexes

$$\begin{array}{ccc} (p_1^* B \otimes \mathscr{B}^M)[-M] & \longrightarrow & (\bar{C}_0 \otimes \mathscr{B}^M)[-M] \\ & & \uparrow \\ & & (\bar{C}_{-1} \otimes \mathscr{B}^M)[-M+1] \end{array}$$

now since the complexes are direct sums of complexes of vector spaces over $k(p_i, q_i)$, we can write the complex on the right as a direct sum of its cohomology groups and get a map of complexes

$$\begin{array}{ccc} (p_1^* B \otimes \mathscr{B}^M)[-M] & \longrightarrow & (\mathscr{H}^M(p_1^*(C) \overset{L}{\otimes} \mathscr{B}^M))[-M] \\ & & \oplus \\ & & (\mathscr{H}^{M-1}(p_1^*(C) \overset{L}{\otimes} \mathscr{B}^M))[-M+1] \end{array}$$

and by pushing forward to Y we get a map of complexes

$$\Phi(B) \longrightarrow \begin{array}{c} p_{2*}(\mathscr{H}^M(p_1^*(C) \overset{L}{\otimes} \mathscr{B}^M))[-M] \cong \mathscr{H}^M(\Phi(C))[-M] \\ \oplus \\ p_{2*}(\mathscr{H}^{M-1}(p_1^*(C) \overset{L}{\otimes} \mathscr{B}^M))[-M+1] \cong \mathscr{H}^{M-1}(\Phi(C))[-M+1] \end{array}$$

this proves precisely that the first map in question is zero (p_{2*} is exact here because the sheaves are flasque). For the second map we can reason as follows: since the map $\Phi(B) \to \mathscr{H}^{M-1}(\phi(C))[-M+1]$ is zero, it follows that the composition $\Phi(B) \to \mathscr{H}^M(\Phi(C))[-M] \to \Phi(A)[1]$ is zero, the result follows if the map

$$\mathrm{Hom}(\mathscr{H}^M(\Phi(C))[-M], \Phi(A)[1]) \to \mathrm{Hom}(\Phi(B), \Phi(A)[1])$$

is injective, i.e. the map

$$\mathrm{Ext}^1(\mathscr{H}^M(\Phi(C)), \mathscr{H}^M(\Phi(A))) \to \mathrm{Ext}^1(\mathscr{H}^M(\Phi(B)), \mathscr{H}^M(\Phi(A)))$$

is injective. A short computation shows that the map in question is

$$\bigoplus_{i=1}^{e} \bigoplus_{r} \mathscr{H}^M(\Phi(A))/\mathfrak{m}_{q_i}\mathscr{H}^M(\Phi(A)) \xrightarrow{\alpha} \bigoplus_{i=1}^{e} \bigoplus_{j_i} \mathscr{H}^M(\Phi(A))/\mathfrak{m}_{q_i}\mathscr{H}^M(\Phi(A))$$

where $j_i \leq r$ and there exists a basis such that $\alpha = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ 0 & \dots & 0 \end{pmatrix}$

where π is an equalizer at Q, hence it is injective as desired.

We're finally ready to show that

$$\begin{array}{ccc} \Phi(C) & \longrightarrow & \Phi(A)[1] \\ \downarrow & & \downarrow \\ F(C) & \longrightarrow & F(A)[1] \end{array}$$

commutes. To do this, take the same decomposition $\Phi(C) \cong \mathscr{H}^{M-1}(\Phi(C))[-M+1] \oplus \mathscr{H}^M(\Phi(C))[-M]$ as above. We will show that the two diagrams

$$\begin{array}{ccc} \mathscr{H}^M(\Phi(C))[-M] & \longrightarrow & \Phi(A)[1] \\ \downarrow & & \downarrow \\ F(C) & \longrightarrow & F(A)[1] \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathscr{H}^{M-1}(\Phi(C))[-M+1] & \longrightarrow & \Phi(A)[1] \\ \downarrow & & \downarrow \\ F(C) & \longrightarrow & F(A)[1] \end{array}$$

are both commutative. Notice that the composition $\Phi(B) \to \mathscr{H}^M(\Phi(C))[-M] \to F(C) \to F(A)[1]$ is zero, because we already know that the central square in (3.2) commutes. By the same computation as above, we get that

$$\mathrm{Hom}(\mathscr{H}^M(\Phi(C))[-M], F(A)[1]) \to \mathrm{Hom}(\Phi(B), F(A)[1])$$

is again injective hence the composition $\mathscr{H}^M(\Phi(C))[-M] \to F(C) \to F(A)[1]$ is zero. In the same way, we know that $\mathscr{H}^M(\Phi(C))[-M] \to \Phi(A) \to F(A)[1]$ is also zero. This shows that the first square commutes.

To show that the second square above is commutative, we just need to show that the square

$$\begin{CD}
\mathscr{H}^{M-1}(\Phi(C))[-M+1] @>>> \Phi(A)[1] \\
@VVV @VVV \\
\mathscr{H}^{M-1}(F(C))[-M+1] @>>> F(A)[1]
\end{CD}$$

is commutative. But this follows from Proposition 3.3.2.

VIII. $s(-)$ is compatible with triangles of the type $0 \to A \to B \to C \to 0$ for any A and B: in this situation we can find A', B' locally free and a diagram

$$\begin{CD}
0 @>>> A' @>>> B' @>>> C @>>> 0 \\
@. @VVV @VVV @| @. \\
0 @>>> A @>>> B @>>> C @>>> 0
\end{CD}$$

Then we get

$$\begin{CD}
\Phi(A) @>>> \Phi(B) @>>> \Phi(C) @>>> \Phi(A)[1] \\
@AAA @AAA @| \circlearrowleft @AAA \\
\Phi(A') @>>> \Phi(B') @>>> \Phi(C) @>>> \Phi(A')[1] \\
@VVV @VVV @VVV \circlearrowleft @VVV \circlearrowleft \quad s(A)[1] \\
F(A') @>>> F(B') @>>> F(C) @>>> F(A')[1] \\
@VVV @VVV @| \circlearrowleft @VVV \\
F(A) @>>> F(B) @>>> F(C) @>>> F(A)[1]
\end{CD}$$

where the top and bottom right squares commute because Φ and F are functors, the middle square by part VII, and the semi-circle by part III. Therefore the boundary maps commute:

$$\begin{array}{ccc} \Phi(C) & \longrightarrow & \Phi(A)[1] \\ \downarrow{\scriptstyle s(C)} & & \downarrow{\scriptstyle s(A)[1]} \\ F(C) & \longrightarrow & F(A)[1] \end{array}$$

□

Lemma 3.3.6. *In the setup of Theorem 3.3.3, let A, B be two torsion coherent sheaves on X. Consider a coherent sheaf A' on X with a surjection $A' \to A$. Consider a map*

$$\Phi(A) \to F(B)$$

that induces the zero map on all cohomology groups. If the composition $\Phi(A') \to \Phi(A) \to F(B)$ is zero, then the map is zero to begin with.

Proof. We know that

$$\Phi(A) \cong p_{2*}(\mathrm{Tor}^1(\mathscr{B}, p_1^* A))[-M+1] \oplus p_{2*}(\mathscr{B} \otimes p_1^* A)[-M]$$

(since $\mathscr{B}$ is supported at a finite number of points and hence is flasque). Moreover, we know that $F(B)$ is isomorphic to $\Phi(B)$ (even if we haven't already established an isomorphism of functors yet) so we also know that

$$F(B) \cong p_{2*}(\mathrm{Tor}^1(\mathscr{B}, p_1^* B))[-M+1] \oplus p_{2*}(\mathscr{B} \otimes p_1^* B)[-M]$$

Fix two isomorphisms as above. Now if we know that the given map $\Phi(A) \to F(B)$ is zero on cohomology, the map can be represented by a map

$$p_{2*}(\mathscr{B} \otimes p_1^* A)[-M] \to p_{2*}(\mathrm{Tor}^1(\mathscr{B}, p_1^* B))[-M+1]$$

i.e. an element of $\mathrm{Ext}^1(p_{2*}(\mathscr{B} \otimes p_1^* A), p_{2*}(\mathrm{Tor}^1(\mathscr{B}, p_1^* B)))$. Then it suffices to show that the map

$$\mathrm{Ext}^1(p_{2*}(\mathscr{B} \otimes p_1^* A), p_{2*}(\mathrm{Tor}^1(\mathscr{B}, p_1^* B))) \to \mathrm{Ext}^1(p_{2*}(\mathscr{B} \otimes p_1^* A'), p_{2*}(\mathrm{Tor}^1(\mathscr{B}, p_1^* B)))$$

is injective.

But since A' surjects onto A, we have a surjection $p_{2*}(A' \otimes \mathscr{B}^M) \to p_{2*}(A \otimes \mathscr{B}^M)$ and both of these sheaves are supported at the points $q_1, \ldots, q_t$, hence this is a surjection of vector spaces and therefore it splits. Hence the map on Ext^1 above is injective. □

Theorem 3.3.7. *Consider a functor $F : D^b_{Coh}(X) \to D^b_{Coh}(Y)$ and assume that $dim(X) = 1$ and that there exists an isomorphism of triangulated functors $s : \Phi \to F$ on the full subcategory of coherent sheaves placed in degree zero. Then s extends to an isomorphism of triangulated functors on the whole $D^b_{Coh}(X)$.*

Proof. Consider a complex $C^\bullet \in D^b_{Coh}(X)$. Then by [Dol60] $C^\bullet \cong \oplus H^i(C^\bullet)[-i]$, in a non-canonical way. Choose one such isomorphism for each $C^\bullet$. By Theorem 3.3.3, since both functors are compatible with shifting, we immediately get an isomorphism $s(C^\bullet) : \Phi(C^\bullet) \to F(C^\bullet)$.

Now consider a map $C^\bullet \to D^\bullet$. This is the same as a map $\oplus H^i(C^\bullet)[-i] \to \oplus H^i(D^\bullet)[-i]$, and again since the two functors are compatible with shifting, and X has dimension 1, it is enough to show that $s(-)$ is compatible with maps $\mathscr{F} \to \mathscr{G}$ and $\mathscr{F} \to \mathscr{G}[1]$, where $\mathscr{F}$ and $\mathscr{G}$ are sheaves. The first case follows from the fact that s is an isomorphism of triangulated functors. A map $\alpha : \mathscr{F} \to \mathscr{G}[1]$ corresponds to an element in $\mathrm{Ext}^1(\mathscr{F}, \mathscr{G})$ so we have a short exact sequence

$$0 \to \mathscr{G} \to \mathscr{H} \to \mathscr{F} \to 0$$

and by Theorem 3.3.3 we get an isomorphism of triangles

$$\begin{array}{ccccccc}
\Phi(\mathscr{G}) & \longrightarrow & \Phi(\mathscr{H}) & \longrightarrow & \Phi(\mathscr{F}) & \xrightarrow{\Phi(\alpha)} & \Phi(\mathscr{G})[1] \\
\downarrow{\scriptstyle s(\mathscr{G})} & & \downarrow{\scriptstyle s(\mathscr{H})} & & \downarrow{\scriptstyle s(\mathscr{F})} & & \downarrow{\scriptstyle s(\mathscr{G})[1]} \\
\mathscr{F}(\mathscr{G}) & \longrightarrow & F(\mathscr{H}) & \longrightarrow & F(\mathscr{F}) & \xrightarrow{F(\alpha)} & F(\mathscr{G})[1]
\end{array}$$

hence s is compatible with α. The fact that s is compatible with triangles is immediate. □

Proof of Theorem 3.1.2. This follows immediately from Theorem 3.3.3 and Theorem 3.3.7. □

Remark 3.3.8. *Notice that any functor satisfying the hypothesis of 3.1.2 will not be full and will not satisfy*

$$Hom_{D^b_{Coh}(Y)}(F(\mathscr{F}, \mathscr{G}[j]) = 0 \text{ if } j < 0$$

for all $\mathscr{F}, \mathscr{G} \in \mathcal{O}_X$ (take for example $\mathscr{F}$ to be supported at one of the p_i's). Hence this improves the result of [CS07].

3.4 A Spectral Sequence

Even when we don't know how to build a kernel out of the sheaves $\mathscr{B}^i$ that we constructed in Theorem 3.1.1, these sheaves still satisfy some good properties. As an example, we will show that the analogue of the Cartan-Eilenberg Spectral Sequence converges when the dimension of X is one.

Consider a Fourier-Mukai functor Φ_E with $E \in D^b_{Coh}(X \times Y)$. Then for each locally free sheaf $\mathscr{E} \in \mathrm{Coh}(X)$ the Cartan-Eilenberg Spectral Sequence gives

$$E_2^{pq} = R^p p_{2*}(\mathscr{H}^q(E) \otimes p_1^*\mathscr{E}) \Rightarrow \mathscr{H}^{p+q}(\Phi_E(\mathscr{E}))$$

Now assume F is an exact functor $D^b_{Coh}(X) \to D^b_{Coh}(Y)$, and suppose we computed the cohomology sheaves $\mathscr{B}^i$ of the prospective kernel in $D^b_{Coh}(X \times Y)$ as in Theorem 3.1.1. Then we have the following:

Proposition 3.4.1. *For any X, Y smooth projective and $\mathscr{E}$ a locally free sheaf on X the following sequence is exact:*

$$0 \to R^1 p_{2*}(\mathscr{B}^M \otimes p_1^*\mathscr{E}) \to \mathscr{H}^{M+1}(F(\mathscr{E})) \to$$
$$\to p_{2*}(\mathscr{B}^{M+1} \otimes p_1^*\mathscr{E}) \to R^2 p_{2*}(\mathscr{B}^M \otimes p_1^*\mathscr{E}) \to \mathscr{H}^{M+2}(F(\mathscr{E}))$$

Proof. Assume that there is an embedding $X \to \mathbb{P}^d$. Then for every $m > 0$ we have a short exact sequence $0 \to \mathcal{O}_X \to \mathcal{O}_X(m)^{\oplus(d+1)} \to K_m \to 0$ where K_m is a locally free sheaf.

Let $\mathscr{E}$ be a locally free sheaf on X. Then by tensoring the sequence above with $\mathscr{E}$ we get a short exact sequence

$$0 \to \mathscr{E} \to \mathscr{E}(m)^{\oplus(d+1)} \to K_m \otimes \mathscr{E} \to 0$$

Choose m high enough so that $R^1 p_{2*}(\mathscr{B}^M \otimes p_1^*(\mathscr{E}(m))) = 0$. Assume that $F(\mathscr{E}) \in D^{[M,N]}_{Coh}(Y)$ for all coherent locally free sheaves $\mathscr{E}$ on X (again, we can do this by [Orl97, Lemma 2.4]). By applying the functor F and then taking cohomology we get a long exact sequence

$$0 \to \mathscr{H}^M(F(\mathscr{E})) \to \mathscr{H}^M(F(\mathscr{E}(m)))^{\oplus(d+1)} \to \mathscr{H}^M(F(K_m \otimes \mathscr{E})) \to \mathscr{H}^{M+1}(F(\mathscr{E})) \to \dots$$

By Proposition 3.2.3, for any coherent locally free sheaf $\mathscr{F}$ we have a functorial isomorphism

$$\mathscr{H}^M(F(\mathscr{F})) \xrightarrow{\cong} p_{2*}(\mathscr{B}^M \otimes p_1^*\mathscr{F})$$

Then we get the following diagram:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \mathscr{H}^M(F(\mathscr{E}(m)))^{\oplus(d+1)} & \longrightarrow & \mathscr{H}^M(F(K_m \otimes \mathscr{E})) & \longrightarrow & \mathscr{H}^{M+1}(F(\mathscr{E})) & \longrightarrow \cdots & & \\
& \downarrow & & \downarrow \cong & & & & & \\
\cdots \longrightarrow & p_{2*}(\mathscr{B}^M \otimes p_1^*\mathscr{E}(m)^{\oplus(d+1)}) & \longrightarrow & p_{2*}(\mathscr{B}^M \otimes p_1^*(K_m \otimes \mathscr{E})) & \longrightarrow & R^1p_{2*}(\mathscr{B}^M \otimes p_1^*\mathscr{E}) & \longrightarrow 0 & &
\end{array}$$

so there exists a map

$$R^1p_{2*}(\mathscr{B}^M \otimes p_1^*\mathscr{E}) \to \mathscr{H}^{M+1}(F(\mathscr{E}))$$

By Theorem 3.1.1 we also have a map $\mathscr{H}^{M+1}(F(\mathscr{E})) \to p_{2*}(\mathscr{B}^{M+1} \otimes p_1^*\mathscr{E})$. The fact that the sequence

$$0 \to R^1p_{2*}(\mathscr{B}^M \otimes p_1^*\mathscr{E}) \to \mathscr{H}^{M+1}(F(\mathscr{E})) \to p_{2*}(\mathscr{B}^{M+1} \otimes p_1^*\mathscr{E})$$

is exact follows from diagram chasing. This is the first part of our sequence.

Now since the sequence above is exact for any $\mathscr{E}$ coherent locally free sheaf on X, it will also be exact for $K_m \otimes \mathscr{E}$. So we have the following diagram:

$$\begin{array}{ccccccc}
& & & & 0 & & \\
& & & & \downarrow & & \\
& & & & R^1p_{2*}(\mathscr{B}^M \otimes p_1^*(K_m \otimes \mathscr{E})) & & \\
& & & & \downarrow & & \\
\cdots \longrightarrow & \mathscr{H}^{M+1}(F(\mathscr{E})) & \longrightarrow & \mathscr{H}^{M+1}(F(\mathscr{E}(m)))^{\oplus(d+1)} & \longrightarrow \mathscr{H}^{M+1}(F(K_m \otimes \mathscr{E})) & \longrightarrow \cdots & \\
& \downarrow & & \downarrow \cong & \downarrow & & \\
\cdots \to & p_{2*}(\mathscr{B}^{M+1} \otimes p_1^*\mathscr{E} & \to & p_{2*}(\mathscr{B}^{M+1} \otimes p_1^*\mathscr{E}(m)^{\oplus(d+1)}) & \to p_{2*}(\mathscr{B}^{M+1} \otimes p_1^*(K_m \otimes \mathscr{E})) & \to \cdots &
\end{array}$$

by diagram chasing we get a map

$$p_{2*}(\mathscr{B}^{M+1} \otimes p_1^*\mathscr{E}) \to R^1p_{2*}(\mathscr{B}^M \otimes p_1^*(K_m \otimes \mathscr{E}))$$

This has an obvious map to $\mathscr{H}^{M+2}(F(\mathscr{E}))$ given by the composition

$$R^1p_{2*}(\mathscr{B}^M \otimes p_1^*(K_m \otimes \mathscr{E})) \to \mathscr{H}^{M+1}(F(K_m \otimes \mathscr{E})) \to \mathscr{H}^{M+2}(F(\mathscr{E}))$$

but since $R^1p_{2*}(\mathscr{B}^M \otimes p_1^*\mathscr{E}(m)) = R^2p_{2*}(\mathscr{B}^M \otimes p_1^*\mathscr{E}(m)) = 0$, we know that

$$R^1p_{2*}(\mathscr{B}^M \otimes p_1^*(K_m \otimes \mathscr{E})) \cong R^2p_{2*}(\mathscr{B}^M \otimes p_1^*\mathscr{E})$$

this gives the second part of our sequence,

$$p_{2*}(\mathscr{B}^{M+1} \otimes p_1^*\mathscr{E}) \to R^2p_{2*}(\mathscr{B}^M \otimes p_1^*\mathscr{E}) \to \mathscr{H}^{M+2}(F(\mathscr{E}))$$

Exactness of the whole sequence again follows by diagram chasing. □

Proposition 3.4.2. *Let X, Y smooth projective varieties over an algebraically closed field with* $\dim(X) = 1$, $F : D^b_{Coh}(X) \to D^b_{Coh}(Y)$ *an exact functor, and consider the sheaves $\mathscr{B}^i$ as in Theorem 3.1.1. Then for all locally free sheaves $\mathscr{E}$ on X there is a spectral sequence*

$$E_2^{pq} = R^p p_{2*}(\mathscr{B}^q \otimes p_1^*\mathscr{E}) \Rightarrow \mathscr{H}^{p+q}(F(\mathscr{E}))$$

Proof. The only nonzero terms of the spectral sequence are $E_2^{0,q}$ and $E_2^{1,q}$. Therefore all the differentials are zero and to show that the SS converges we need to show:

- There exists a map $F^1H^q = E_2^{1,q-1} = R^1p_{2*}(\mathscr{B}^{q-1} \otimes p_1^*\mathscr{E}) \hookrightarrow \mathscr{H}^q(F(\mathscr{E}))$
- $E_2^{0,q} = p_{2*}(\mathscr{B}^q \otimes p_1^*\mathscr{E}) \cong \mathscr{H}^q(F(\mathscr{E}))/R^1p_{2*}(\mathscr{B}^{q-1} \otimes p_1^*\mathscr{E})$

Since $\dim X = 1$ we have $R^2p_{2*}(\mathscr{B}^q \otimes p_1^*\mathscr{E}) = 0$. Therefore the exact sequence of Proposition 3.4.1 becomes a short exact sequence

$$0 \to R^1p_{2*}(\mathscr{B}^M \otimes p_1^*\mathscr{E}) \to \mathscr{H}^{M+1}(F(\mathscr{E})) \to p_{2*}(\mathscr{B}^{M+1} \otimes p_1^*\mathscr{E}) \to 0 \tag{3.3}$$

Choose m high enough so that $R^p p_{2*}(\mathscr{B}^q \otimes p_1^*(\mathscr{E}(m))) = 0$ for all q and all $p > 0$, and such that $\mathscr{H}^i(F(\mathscr{E}(m))) \cong p_{2*}(\mathscr{B}^i \otimes p_1^*\mathscr{E}(m))$ for all i (this can be done by Theorem 3.1.1). Then using again the short exact sequence in the proof of Proposition 3.4.1

$$0 \to \mathscr{E} \to \mathscr{E}(m)^{\oplus(d+1)} \to K_m \otimes \mathscr{E} \to 0$$

we get that

$$R^1p_{2*}(\mathscr{B}^i \otimes p_1^*(\mathscr{E} \otimes K_m)) \cong R^2p_{2*}(\mathscr{B}^i \otimes p_1^*\mathscr{E}) = 0$$

for all i.

Now assume by induction that we get the same short exact sequence as (3.3) starting with $\mathscr{B}^{M+n-1}$ for any locally free sheaf $\mathscr{E}$:

$$0 \to R^1p_{2*}(\mathscr{B}^{M+n-1} \otimes p_1^*\mathscr{E}) \to \mathscr{H}^{M+n}(F(\mathscr{E})) \to p_{2*}(\mathscr{B}^{M+n} \otimes p_1^*\mathscr{E}) \to 0$$

then the same exact sequence will hold if we substitute $\mathscr{E}$ with $K_m \otimes \mathscr{E}$:

$$0 \to R^1p_{2*}(\mathscr{B}^{M+n-1} \otimes p_1^*(K_m \otimes \mathscr{E})) \to \mathscr{H}^{M+n}(F(K_m \otimes \mathscr{E})) \to p_{2*}(\mathscr{B}^{M+n} \otimes p_1^*(K_m \otimes \mathscr{E})) \to 0$$

But since $R^1p_{2*}(\mathscr{B}^i \otimes p_1^*(\mathscr{E} \otimes K_m)) = 0$ this gives an isomorphism

$$\mathscr{H}^{M+n}(F(K_m \otimes \mathscr{E})) \cong p_{2*}(\mathscr{B}^{M+n} \otimes p_1^*(K_m \otimes \mathscr{E}))$$

Hence from the diagram

$$\begin{array}{ccccc}
\cdots \to \mathscr{H}^{M+n}(F(K_m \otimes \mathscr{E})) & & \longrightarrow & & \mathscr{H}^{M+n+1}(F(\mathscr{E})) \to \cdots \\
\downarrow \cong & & & & \downarrow \\
\cdots \to p_{2*}(\mathscr{B}^{M+n} \otimes p_1^*(K_m \otimes \mathscr{E})) \to R^1p_{2*}(\mathscr{B}^{M+n} \otimes p_1^*\mathscr{E}) \to 0 & & & & 0 \to p_{2*}(\mathscr{B}^{M+n+1} \otimes p_1^*\mathscr{E}) \to \cdots
\end{array}$$

we get a sequence

$$0 \to R^1p_{2*}(\mathscr{B}^{M+n} \otimes p_1^*\mathscr{E}) \to \mathscr{H}^{M+n+1}(F(\mathscr{E})) \to p_{2*}(\mathscr{B}^{M+n+1} \otimes p_1^*\mathscr{E}) \tag{3.4}$$

which is exact by diagram chasing. Again, we also have the corresponding exact sequence for the locally free sheaf $K_m \otimes \mathscr{E}$:

$$0 \to R^1p_{2*}(\mathscr{B}^{M+n} \otimes p_1^*(K_m \otimes \mathscr{E})) \to \mathscr{H}^{M+n+1}(F(K_m \otimes \mathscr{E})) \to p_{2*}(\mathscr{B}^{M+n+1} \otimes p_1^*(K_m \otimes \mathscr{E}))$$

and the first term of the sequence is zero, i.e. the map $\mathscr{H}^{M+n+1}(F(K_m \otimes \mathscr{E})) \to p_{2*}(\mathscr{B}^{M+n+1} \otimes p_1^*(K_m \otimes \mathscr{E}))$ is injective. This is reflected in the following diagram:

$$\begin{array}{ccccc}
\cdots \to \mathscr{H}^{M+n+1}(F(\mathscr{E})) & \longrightarrow & \mathscr{H}^{M+n+1}(F(\mathscr{E}(m)))^{\oplus(d+1)} & \longrightarrow & \mathscr{H}^{M+n+1}(F(K_m \otimes \mathscr{E})) \to \cdots \\
\downarrow & & \downarrow \cong & & \downarrow \\
0 \to p_{2*}(\mathscr{B}^{M+n+1} \otimes p_1^*\mathscr{E}) & \to & p_{2*}(\mathscr{B}^{M+n+1} \otimes p_1^*\mathscr{E}(m)^{\oplus(d+1)}) & \to & p_{2*}(\mathscr{B}^{M+n+1} \otimes p_1^*(K_m \otimes \mathscr{E})) \to \cdots
\end{array}$$

By diagram chasing this tells us that the map

$$\mathscr{H}^{M+n+1}(F(\mathscr{E})) \to p_{2*}(\mathscr{B}^{M+n+1} \otimes p_1^*\mathscr{E})$$

is actually surjective, hence (3.4) becomes a short exact sequence, and this completes the proof. □

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