Knowledge-as-Theory-and-Elements

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ABSTRACT

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This dissertation will examine the Knowledge-as-Theory-and-Elements perspective on knowledge structure. The dissertation creates a set of theoretical criteria given within a template by which lesson plans can be designed to teach mathematics and the physical sciences. The dissertation also will test the Knowledge-as-Theory-and-Elements theoretical perspective by designing lesson plans to teach a branch of mathematics, graph theory, by using the new template. The dissertation will include a comparative study investigating the effectiveness of the lesson plans conforming to the new template and the lesson plans designed by the traditional theoretical perspective Knowledge-as-Elements.
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Chapter I
Introduction

A primary focus of fundamental research in mathematics and science education is how students acquire and organize scientific knowledge. Theoretical perspectives on knowledge structures are fundamental for much of the research in this area. Researchers have made significant progress on two prominent but competing broad theoretical perspectives regarding knowledge structure coherence:

1. knowledge-as theory perspectives
2. Knowledge-as-element perspectives

Essentially, the difference is this: is a student’s knowledge represented most accurately as a coherently unified framework of theory-like character or is a student’s knowledge more aptly considered as a system of quasi-independent elements (diSessa, 2006)?

Knowledge-as-theory perspective or KT hypothesizes theory-like naive knowledge structures (Ioannides & Vosniadou, 2002). This theory-like knowledge structure is hypothesized to be comprised of coherent structures grounded in persistent ontological and epistemological commitments. KT researchers advocate for a grand hierarchical conceptual structure with theory-like properties that constrains a student’s interpretation of subordinate models and ideas (Wellman & Gelman, 1992). Therefore, KT takes a top-down approach to learning and teaching. KT proponents do not argue that students’ knowledge is “theory-like” in the same manner as the knowledge of professional research scientists such as the scientists’ awareness of the nature of their scientific theories or the scientists’ ability to engage in hypothesis testing with regard to
their theories. Because novices unconsciously develop these coherent structures through collections of daily experiences, their “theories” are not available for hypothesis testing in a manner similar to scientists’ theories (Wagner, 2006). For example, one such theory-like naïve knowledge structure is the mathematical misconception that “if a collection of objects all possessing some common property converges in some sense to an object, this object must also have the same property”. The conceptual ambiguity of this naïve knowledge structure prevents it from being tested directly as a scientific hypothesis (Davis & Vinner, 1986). In certain context, a student with this knowledge structure can translate this thinking to mean if a sequence of real numbers \( \{a_n\} \) all greater than or equal to \( a \) converges to a limit \( l \), then \( l \) is also greater than or equal to \( a \) – a correct mathematical statement; on the other hand, it can also direct a student to think that if a sequence of continuous real-valued functions \( \{f_n\} \) converges, it will converge to a continuous function also – an erroneous mathematical statement (Fischbein, 2001). Although these naïve knowledge structures do not lend simply to hypothesis testing, a novice’s alternative conceptions do constrain future learning and allow consistent predictions across conceptual domains (Carey, 1999).

Knowledge-as-elements perspective or KE focuses on much more organic collections of knowledge elements such as facts, facets, narratives, and concepts at various stages of conceptual development (McCloskey, 1983). Learning occurs through a process of restructuring and reorganizing of these basic elements. These basic elements do not have the status of a theory because they are not produced or activated under an organized system like the framework theories proposed by KT perspectives (Linn et al., 2004). Novices spontaneously connect and activate these knowledge pieces according to the relevance of the situation. Therefore, KE takes a bottom-up approach to learning and teaching. During the conceptual change process, the
elements and interactions between the knowledge pieces are revised and refined through addition, elimination, and reorganization to strengthen the network (Clark, 2006). From the KE vantage point, conceptual change involves a piecemeal evolutionary process as opposed to a broad theory-replacement process advocated by KT researchers. Recent research suggests that some “theory-like” knowledge structures also undergo evolutionary changes through time (diSessa & Sherin, 1998).

Observations in years of college level mathematics education both from a student’s perspective and from an educator’s perspective show that different students even in the same class can approach the same mathematical concept from opposing schemata (Kidron, 2008). One student can approach a concept slowly, chiseling his new understanding gradually by revising elements of his previous knowledge, while another student may see one small counterexample and quickly undergo a global restructuring of the connected threads of his theories and later test his revised theory on various lower element-like schemata (Sierpinska, 1987).

Within a mathematical concept, there exist certain schemata that require different approaches even by the same student (Gray & Tall, 1994). Take isomorphism in graph theory as an example. Certain properties of a graph $G$ emerge as global invariants: diameter or connectedness, while others emerge as local invariants of the graph $G$: this vertex $v_i$ of degree 3 is adjacent to that vertex $v_j$ of degree 4. Piecing together these diverse strands of global and local schemata in order to present a robust introduction to graph isomorphisms can be a challenge. A mathematics educator must grapple with the dilemma of when to teach global and when to teach
local. The point is that it is frequently insufficient to rely on a one-sided theory in education to account for a given mathematical concept (Vinner & Dreyfus, 1989).

Another weakness of KE and KT is the lack of coordination of knowledge elements in teaching (Carey, 2009). Both theories, especially KE, when applied to classroom instruction, place little emphasis on how various seemingly unrelated grains of knowledge elements within a concept are coordinated, that is, the coordination of these knowledge elements are frequently ignored in the lesson plan (diSessa, 2006). Take a high end example: an $m$-dimensional differentiable manifold in $\mathbb{R}^n$ is a second countable Hausdorff space $\mathbb{K}$ in $\mathbb{R}^n$, locally homeomorphic to $\mathbb{R}^m$ with $m \leq n$, together with an atlas of charts such the composition of one chart with another is a diffeomorphism of $\mathbb{R}^m$. Here the atlas of charts is critical to the manifold - a different atlas even with the same second countable space can give rise to a manifold with a different differentiable structure. In teaching manifolds, when an educator gives concrete examples of a differentiable manifold, his lesson plan must stress this coordination. It is clear that in mathematics education, a sound theory of knowledge acquisition must account for the deep coordination of seemingly independent knowledge elements (Wittmann, 2002).

It is with these observations, convictions, and considerable groundwork in classroom teaching and observation that the following perspective of knowledge acquisition in mathematics education has evolved: knowledge-as-theory-and-elements or KTE. The fundamental tenets of KTE are:

1. KTE recognizes the multiplicity of the various intertwining ideas within a given mathematical concept and designs classroom instructional activities respecting the cognitive demands of each idea with its distinct epistemological obstacles. Within a given
concept, some ideas are best approached in education by re-organizing the elements of knowledge gradually, while other ideas are best approached by a radical paradigm shift, quickly challenging and revamping existing knowledge within and outside a mathematical schema. The knowledge-as-theory-and-elements or KTE perspective is a combination of the KT and KE perspectives.

2. KTE recognizes the duality of cognitive approaches in knowledge acquisition: students, or even within the one individual student, exhibit a full spectrum of learning behaviors, some students approach new knowledge in a piecemeal elemental manner; whereas others approach new knowledge using highly organized schemata better understood as theories and later proceed to test new theories on particular elements. As one’s knowledge increases, one’s organization of knowledge pieces and acquisition of new knowledge approach the theory-like end of the cognitive spectrum.

3. KTE places critical emphasis on the coordination of knowledge elements within a given concept and incorporates the coordination of distinct elements needed for a fuller understanding of that concept. As such, KTE’s design of classroom instruction includes both the atomization of a concept into its distinct elements as well as the indivisibility of certain connections that are not necessarily naive knowledge elements but are subtle relations within the concept whose recognition is imperative for a solid understanding.

Purpose of the Study

The purpose of this study is to examine the KTE theory of knowledge acquisition in mathematics, to design a template lesson plan guided by the philosophy of KTE that can be adapted to produce specific lesson plans for a given mathematical topic, and to investigate the
validity of KTE theory by comparing the effectiveness of KTE-guided lesson plans against KE-guided lesson plans in graph theory. There were two basic questions that motivated this study:

1. Can element-like and theory-like pedagogical approaches be integrated fully and successfully in classroom instruction?

2. Can a general lesson plan template that embodies the philosophy of KTE be designed, ready to be concretized into specific lesson plans for a given mathematics topic?

Procedures of the Study

Both quantitative and qualitative analyses were conducted to answer the questions posed by the study. The participants in the experimental phase of the study were chosen from two sections of an algebra class in a public university in the mid-Atlantic region on the east coast, with twenty-seven participants in the control section and twenty-six participants in the experimental section. A background check verified that the participants had no prior exposure to graph theory. The two sections were given a pre-test to measure their background mathematical knowledge related to graph theory. The pre-test also indicated no statistical significance between the two sections. Four basic topics in introductory graph theory were chosen to be the teaching material for the experimental portion of the study: basic definitions in graph theory, graph isomorphisms, degree sequence and bipartite graphs, and planarity. A complete and suitable set of examples, propositions, and theorems were pre-determined to be taught in both sections such that no one section was taught any additional examples, propositions, or theorems, even observations so as to ensure maximum equality in learning materials between the two sections. During the current study, the investigator was randomly inspected three times during
three KTE and KE lectures to ensure equality of the materials used in the study as well other measurable factors that might affect the study.

With this list of topics, complete set of examples, propositions, and theorems, two sets of lessons plans were designed. One set of lesson plans was designed by a proponent of the KE perspective who is also a graph theorist. The other set of lesson plans were designed by the investigator from the KTE perspective. A comparison of the KTE and KE lesson plans is given in chapter four. Seven lectures were given to each group, each one lasting an hour and fifty minutes. The control group was given the KE-based lesson plans. The experimental group was given the KTE-based lesson plans. During the seven lectures, two intermediate quizzes were given to monitor progress. After the lectures were completed, a final post-test was administered to measure any differences in comprehension of graph theory concepts between the two groups.

Before the first lecture, after the third lecture, and after the last lecture, interviews were conducted on subgroups of students from each section to probe deeper into their rationalizations of problems in graph theory. The results from these interviews comprised the qualitative component of the investigation of the effectiveness of KTE perspective lesson plans compared to KE perspective lesson plans.
Chapter II
Knowledge Structure Perspectives

Research into how students acquire and organize scientific knowledge has been a centerpiece of mathematics and science education (Sawyer, 2006). Although most mathematics and science educators did not begin their respective lines of research into how students acquire and organize scientific knowledge with explicit allegiance to a particular philosophical ideology, the body of literature and current trends in research do point to the gradual but real emergence of two major competing philosophical perspectives in knowledge structure (diSessa, 2006). The basic contention revolves around a familiar philosophical dichotomy within many domains throughout intellectual history: whether a top-down view where an incipient global metanarrative dictating how learners acquire and organize information has more explanatory power, or a bottom-up view where a learner’s knowledge is a set of quasi-independent elements and learning occurs through the interaction between knowledge elements without the constraints of an overarching grand structure has more explanatory power (Wellman & Gelman, 1992).

Theories in knowledge structure coherence belonging to the former philosophical perspective are collectively called knowledge-as-theory, or KT. Those in the latter philosophical perspective are called knowledge-as-elements, or KE (diSessa, 1998)

The KT Approach

The KT perspective is rooted in Piagetian learning theory (Carey, 1985). Studies of the philosophy and history of science have also influenced many of these researchers (Wiser & Carey, 1983). To explain a conceptual shift, proponents of KT perspective often present
analogies between Piaget’s concepts of assimilation and accommodation and Kuhn’s concepts of normal science and scientific revolution (Carey, 1999). While some of these researchers have explained conceptual change in terms of framework theories and mental models (Vosniadou, 1994; Vosniadou & Brewer, 1992), others have focused on higher level ontological shifts (Chi, 1992).

One of the most prominent conceptual change theories which correspond to Kuhn’s notion of a paradigm shift or Piaget’s notion of accommodation was defined by Posner, Strike, Hewson, and Gertzog (Posner et al, 1982). They proposed that if a learner’s current conception is functional and if the learner can solve problems within the existing conceptual schema, then the learner does not feel a need to change the current conception. Even when the current conception does not solve some problems successfully, the learner may only make moderate changes to his or her conceptions. This is called “conceptual capture” or “weak restructuring” (Hewson, 1981). In such cases, the assimilations continue without any need for accommodation. It is believed that the learner must be dissatisfied with an initial conception in order to abandon it and accept a scientific conception for successful conceptual change. This more radical change is called “conceptual exchange” or “radical restructuring” (Hewson, 1981; Carey, 1985). According to Posner, the scientific conception must also be intelligible, plausible, and fruitful for successful conceptual change to occur (Posner et al, 1982). “Intelligible” means that the new conception must be clear enough to make sense to the learner. “Plausible” means the new conception must be seen as plausibly true. “Fruitful” means the new conception must appear potentially productive to the learner for solving current problems (Posner et al, 1982). Posner’s perspective assumes that these cognitive conditions should be met during the learning process for a successful conceptual change (Hunt & Minstrell, 1994). The major goal is to create a cognitive
conflict to make a learner dissatisfied with his or her existing conception. The learner then may accept a normative view as intelligible, plausible, and fruitful. This view has been an influential theory to determine a learner’s specific conceptions that result from the interaction between beliefs and knowledge of the learner (Vosniadou, 1994).

Posner embedded his explanation of conceptual change within a conceptual ecological perspective (Posner et al, 1982). A learner’s conceptual ecology consists of conceptions and ideas rooted in his epistemological beliefs. This conceptual ecology perspective has proven influential. Even though Posner’s primary mechanism for conceptual change has been rejected by many proponents of the knowledge-as-elements perspective, many knowledge-as-elements as well as many knowledge-as-theory proponents have adopted this larger conceptual ecology architecture into their perspectives (Linder, 1993). From a conceptual ecology perspective, the constituent ideas, ontological categories, and epistemological beliefs influence a learner’s interactions with new ideas and problems significantly (Strike & Posner, 1992). Misconceptions are therefore not only inaccurate beliefs; misconceptions organize and constrain learning in a manner similar to paradigms in science. In other words, prior conceptions are highly resistant to change because concepts are not independent from the cognitive artifacts within a learner’s conceptual ecology. Some concepts are attached to others and they generate thoughts and perceptions. Because of this web-based relationship between concepts, a revision to a concept requires revisions to others (Tao & Gunstone, 1999).

Another area of research that supports the KT perspective focuses on the notion that adults’ and children’s concepts are each coherent but incommensurable with one another (Carey, 1985). In other words, people maintain coherent theory-like understandings of concepts. According to Carey, change between concepts can be achieved through three processes:
replacement, differentiation, and coalescence (Carey, 1991). In replacement, one concept displaces another concept, where the two concepts are fundamentally different; it is an overwrite procedure. Differentiation is another process in which the initial concept splits into two or more new concepts such as dog differentiated into the more specific terms collie and terrier. Coalescence is the opposite process of differentiation; Coalescence involves two or more original concepts coalescing into a single concept, such as collie and terrier into the more general category of dog (diSessa & Sherin, 1998; Carey, 1991).

Several researchers focus on conceptual change processes in terms of mental models (Ioannides & Vosniadou, 2002; Linder, 1993; McCloskey, 1983; Smith, Blakeslee, & Anderson, 1993; Vosniadou, 1994; Vosniadou & Brewer, 1992). For example, Ioannides and Vosniadou explored spontaneous changes and instruction-based changes at the mental model level. Spontaneous change is change that occurs in young children without specific instruction through the enrichment of observations and other kinds of learning, such as language learning (Ioannides & Vosniadou, 2002). This position is very similar to Carey’s argument that even very young children develop theories and make predictions about phenomena (Carey 1999). Their causal explanations reflect ontological commitments that are subject to revision and radical change. Instruction-based change focuses on the evolution of children’s mental models through the introduction of formal scientific instruction. Instruction leads children to construct synthetic mental models that are still inconsistent with the scientific theory (Smith, Blakeslee, & Anderson, 1993). These synthetic mental models imply that students begin to synthesize the scientific theory with their initial theory. Early pioneer work by McCloskey indicates that students make changes in their beliefs based on the instruction of an authority figure, but they
still lack the full scientific theory due to their ontological and epistemological commitments (McCloskey, 1983).

This development of synthetic models reveals that ontological commitments must be changed in order to restructure a student’s framework theory fully (Anderson, 1993). Therefore akin to Carey’s opinion, Vosniadou claims that children’s generation of scientific models is constrained by their framework theories (Vosniadou, 1992). For example, elementary school students in Vosniadou and Brewer’s study consistently constructed the Earth models in a disc or a rectangular flat shape based on their everyday experience (Vosniadou & Brewer, 1992). Vosniadou and Brewer called these “initial” models because they are not affected by the scientific model of the Earth. Older students constructed synthetic Earth models, for example Dual Earth, hollow sphere, and flattened sphere that are influenced by the spherical shape of the Earth from instruction. Vosniadou suggested that in the formation of these mental models, students’ beliefs about the Earth based on their observations and cultural influences are constrained by a naïve framework of presuppositions (Vosniadou & Brewer, 1992).

According to Vosniadou, students generate misconceptions or synthetic mental models that combine aspects of the scientific model with their initial models within the constraints of their framework theories (Vosniadou, 1994). The presuppositions of the framework theory need to be revised and eventually replaced to allow for the scientific model. The framework theory perspective is consistent with Chi’s argument that conceptual change requires an ontological shift (Chi, 1992). Chi believes that the conceptual change process is difficult because either the student assigns the concept to a different ontological category from the scientific one or the student lacks an appropriate category to which the concept could be assigned (Chi, 2005). If students become aware of their ontological commitments, they can then become aware of how
the scientific theory does not fit with their existing knowledge structure. In turn, they can assign the concept into a correct category by revising their ontological commitments, categories, and presuppositions (Chi, 2005).

In summary, the KT perspective hypothesizes theory-like naïve knowledge structures (Sawyer, 2006). This theory-like knowledge involves coherent structures grounded in persistent ontological and epistemological commitments (Chi, 1992). Because novices unconsciously develop these coherent structures through collections of daily experiences, their “theories” are not available for hypothesis testing in a manner similar to scientists’ theories (Carey, 1999). However, novices’ alternative conceptions do constrain future learning and allow novices to make consistent predictions across conceptual domains. The KT perspective hypothesizes revolutionary change in knowledge structures through various mechanisms (Hewson et Gertzog, 1982). Some researchers frame their conceptual change theories in terms of Piaget’s notion of assimilation and accommodation or Kuhn’s notion of a paradigm shift (Wiser & Carey, 1983). Some researchers explain conceptual change in terms of the notion of incommensurability demonstrating the distinction between the roots of the concepts (Vosniadou & Brewer, 1992). Other researchers propose ontological shifts and the evolution of mental models (Anderson, 1993). Although these KT theories have developed in different domains, such as force and motion (McCloskey, 1983), astronomy (Vosniadou & Brewer, 1992), biology (Carey, 1999), and heat/temperature (Wiser & Carey, 1983), they all assert that learners at any given time maintain a small number of well-developed coherent naïve theories based on their everyday experiences and that these theories have explanatory power to make consistent predictions and explanations across significant domains (Sawyer, 2006).
The KE Approach

In contrast with the KT perspective, many researchers in mathematics and science education turn to the other main direction in knowledge acquisition: knowledge-as-elements. The KE perspective characterizes students’ understanding in terms of collections of multiple quasi-independent elements (Brown, 1995; diSessa, 1988, 1993; Hunt & Minstrell, 1994; Linn, Eylon, & Davis, 2004). Learning occurs through a process of restructuring and reorganizing these ideas (Linn et al, 2004). Anderson and Thagard provide relatively mechanical/mathematical examples of this perspective (Anderson, 1993; Thagard, 1992). Other researchers maintain more organic perspectives that focus on collections of elements including, but not limited to, phenomenological primitives, facts, facets, narratives, concepts, and mental models at various stages of development and sophistication (Linn et al, 2004). diSessa focuses more on the nature of the elements (diSessa et al, 2004). Hunt and Minstrell focus on the facets students use in the classroom (Hunt & Minstrell, 1994). Linn focuses on the process through which students reorganize, revise, and connect these elements (Linn et al, 2004).

diSessa’s theory is the best known theory within the KT perspective. diSessa hypothesizes that the knowledge structures of novices consist primarily of unstructured collections of many simple elements that he calls phenomenological primitives or p-prims (diSessa, 1993). P-prims are developed through a sense-of-mechanism that reflects our interactions with the physical world such as pushing, pulling, throwing, and holding. The learner only assumes that “something happens because that’s the way things are” (diSessa, 1993). These implicit presuppositions influence learners’ reasoning when they interpret the world (Ueno, 1993). P-prims do not have the status of a theory because they are not produced or activated under a highly organized system like the framework theories proposed by KT perspective
(diSessa & Sherin, 1998). P-prims are generated from a learner’s experiences, observations, and abstractions of phenomena. Individual p-prims are loosely connected into larger conceptual networks. diSessa describes cuing priority, reliability priority, and structured priorities to propose how those p-prims are recognized and activated according to context (diSessa et al, 2004).

Some recent empirical evidence supports diSessa’s argument. Southerland, Abrams, Cummins, and Anzelmod explored the nature of students’ biological knowledge structure (Abrams et al, 2001). They concluded that p-prims have more explanatory power than conceptual frameworks theory concerning the shifting nature of students’ conceptions of biological phenomena. Clark’s longitudinal study in thermodynamics suggests that students’ understanding of heat and temperature can be explained through a related elemental perspective (Clark, 2006).

In terms of specific instructional strategies, the KE perspectives suggest that conceptual change requires restructuring, editing, and organizing rather than discrete changes from one conception to another especially for complex and rich domains such as mechanics and thermodynamics (Wittman, 2002). Toward this goal, engaging students with multiple computational representations hold promise for instruction. Instruction engaging multiple representations can re-represent concepts in multiple ways to highlight specific variables within each context separately while ignoring the others (Ainsworth, 1999). The complexity of a phenomenon can be simplified to help a learner focus on the specific aspects of the phenomenon across multiple contexts.

Parnafes investigated students’ learning processes about physical phenomena through computational representation (Parnafes, 2007). Her research was grounded upon a KE perspective. The research analysis suggested that instruction engaging multiple representations
can highlight the important aspects of phenomena so that a learner can see and differentiate them and can help students identify fragmentations in their causal responses and encourage students to engage in conflict resolution and coherence building between ideas. Additionally, instructions engaging multiple computational representations can also provide interactive visual aids for the investigation of physical phenomena (Parnafes, 2007). While multiple instructional strategies can offer synergistic benefits, instruction engaging multiple computational representations seems particularly powerful from a KE perspective (Nemnirosky & Tierney, 2001; Parnafes, 2007).

In summary, the KE perspective hypothesizes that naïve knowledge structures consist of multiple conceptual elements including, but not limited to, phenomenological primitives, facts, facets, narratives, concepts, and mental models at various stages of development and sophistication (diSessa, 2006). Novices connect and activate these knowledge pieces spontaneously according to the relevance of the situation. During the conceptual change process, the elements and interactions between the elements are revised and refined through addition, elimination, and reorganization to strengthen the network. From this perspective, conceptual change involves a piecemeal evolutionary process rather than a broad theory replacement process (Wagner, 2006).

The Insufficiencies of KT and KE

Upon examining the points of contention between both perspectives closely, each of the two theoretical perspectives appears insufficient to explain the complex processes of conceptual change and naïve knowledge structure. Thinking is difficult to capture even in a controlled environment. Literature evidence suggests that the degree of the richness of a scientific domain should be an indicator to decide which theory is more useful for describing and analyzing
conceptual change, but virtually all KT and KE advocates base their research on different domains. For example, diSessa developed his KE perspective for the domain of mechanics (diSessa, 1993). Vosniadou’s knowledge-as-theory perspective was developed in astronomy in which students have far less first-hand interactions (Vosniadou, 1994). Other researchers use biology as a testing ground where the meanings of “life” or “species” hold murky denotations even to some specialists in the field (Abrams, Cummins, & Anzelmod, 2001). After years of argument between the two theoretical perspectives, a valid perspective must respect both the particular nature of a domain and the particularities of a concept since some concepts are more theory-like, while others are more elemental (Carey, 2009).

Another key issue to consider is that students’ learning processes and trajectories may involve periods and characteristics of both coherence and transition. Fewer researchers now espouse radical KT perspective (diSessa, 1993). For example Wiser and Amin suggest that conceptual change involves both revolutionary as well as evolutionary components (Wiser & Amin, 2001). Similarly, Carey suggests that both strong and weak restructuring occur and that the process takes time (Carey, 2009). Vosniadou also agrees that restructuring takes time and that students may embrace multiple “synthetic models” temporarily between stages (Vosniadou et al, 2008).

At the same time, diSessa and others are currently working on research on coordination classes to explain systematicities and connections between ideas rather than focusing predominantly on the quasi-independence of the various elements (diSessa, 2011). Both KE and KT are slowly retreating from their once-firm research assertions. It is now clear that a sound perspective in knowledge structure coherence should address how the various strands of knowledge elements within a concept are coordinated to form a solid understanding.
The KT perspective discussed above acknowledges that conceptual change may take extended periods of time, but it generally provides less detail about the mechanisms for these transitions. Tools for explaining and modeling transitional times are critical, because these transitional times may extend across many school grades and on into adulthood. Although there are strong arguments from the KT perspective regarding the theory-like understandings of young children, for example in terms biology (Inagaki & Hatano, 2002; Carey, 1999) or even astronomy (Vosniadou & Brewer, 1994), the arguments are weaker for older students. Ioannides and Vosniadou present compelling data concerning the coherence of younger students’ understanding of the concept of force, but that study presents less compelling data regarding the coherence of older students’ understandings, which can only be grouped into a catch-all category of “gravity and other” (Ioannides & Vosniadou, 2002). Older students’ understanding seems more transitional and fragmented. This transition has been shown to extend well into adulthood, if not permanently (Carey, 2009). A sound theoretical perspective therefore must recognize the diversity of thinking patterns among different students as well as within a student as he ages.

Both KT and KE are beginning to recognize these shortcomings in their original theories. A convergence toward the center of these perspectives appears to be needed in order to account for coherence, systematicity, and transition (diSessa, 2011).
Chapter III
KTE and KTE Lesson Plan Template

Given the insufficiencies of the two competing perspectives KT and KE, the current study proposed a centrist perspective: knowledge-as-theory-and-elements, KTE. KTE emerges from the contention between KT and KE as a moderate theoretical perspective on knowledge structure. KTE recognizes that some concepts are more theory-like in their character, for example the knowledge structure “if a set of mathematical objects all possessing some common property converges in some sense to an object, then this object also possesses the same property”. Other concepts or knowledge structures resist generalizations. A sound perspective must account for the diversity of knowledge structures within a topic in mathematics and science across the spectrum and lesson plans should be designed accordingly and judiciously.

This recognition of the diversity of knowledge structures bears important pedagogical implications in the design of lesson plans. In-class activities are chosen and designed based on the particular nature of the underlying knowledge structures. KTE made the philosophical realization that when faced with a highly innate but visually or cognitively inert theory-like knowledge structure that pervades a topic thoroughly, it is much more advantageous to make it clear at the beginning so students can learn with a purpose in their mind and generalize the knowledge structure into more complex observations. Based on this realization, KTE proposed two criteria to judge whether a theory-like knowledge structure should be introduced early in a lesson plan before the introduction of any concrete examples:

1. The complexity of the concept – simple concepts, though theoretical, can be introduced early;
2. The degree of integration of the concept into the overall understanding of the topic area depending on how deeply embedded a concept is in the overall architecture of the larger topic.

This moderate approach to the organization of knowledge structure also affects the choice of teaching materials used. Whereas KE stresses the precedence of concrete examples and multiple representations over theoretical generalizations, KT stresses the importance of general thinking models that govern concrete examples and emphasize conceptual change as an important gateway to learning and teaching (Ainsworth, 1999; Carey, 1985). KTE steers a center course between the two parent perspectives and recognizes that a student’s instantiations should be topic–specific and student–specific. Some topics can be taught more efficiently with tangible models and multiple representations, while others are taught better without any distractions from concrete examples at first that might incur additional cognitive cost on the students.

KT and KE place little emphasis on how various seemingly unrelated grains of knowledge elements within a concept are coordinated, that is, the coordination of these knowledge elements is frequently ignored in the lesson plan (diSessa, 2006). KTE places a premium on coordination of various knowledge elements in and outside of a concept. This emphasis in KTE affects the objective and the assessment of lesson plans guided by KTE. As a result, the KTE objective in teaching a particular topic in mathematics or science in general includes motivating new connections between the various strands of knowledge structures. Similarly in assessment, KTE guided lesson plans seek to include unlearned concepts to motivate deeper understanding and connections with other concepts in and outside the current topic area.
With these moderate philosophical approaches along with their respective pedagogical emphases, the following general lesson plan template appears compatible with KTE:
Table 1: KTE Lesson Plan Template

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Objectives</strong></td>
<td>To understand the topic in such a way that a student grasps the mathematical concept and be motivated to make new connections between concepts on his own.</td>
</tr>
<tr>
<td><strong>Activities</strong></td>
<td>Two criteria: simplicity &amp; permanence of the theory-like knowledge structure in the overall architecture of the topic. Yes: on both introduce the knowledge structure first, followed by concrete examples/representations. No: one or both criteria start with concrete examples/representations, then motivate students to make theoretical generalizations.</td>
</tr>
<tr>
<td><strong>Materials</strong></td>
<td>Topic specific: some concepts can be taught better with tangible models, others may resist multiple representations. Any materials should not distract students from learning the concept.</td>
</tr>
<tr>
<td><strong>Assessment</strong></td>
<td>Quizzes and tests should follow the objectives closely and include unlearned concepts so as to motivate deeper understanding and connections with other mathematical concepts in and outside the current topic area.</td>
</tr>
<tr>
<td><strong>Reflection</strong></td>
<td>Did any students make any connections that were not taught in class? If yes, did the connections make the instructor notice a pedagogical subtlety previously unknown to him and are the connections constructive to deeper understanding? How can the instructor use them in future teaching?</td>
</tr>
</tbody>
</table>
Chapter IV
Field Trials of KTE Template and KE Lesson Plans

Four topic areas were covered in this study: general introduction to graphs lasting one lecture, graph isomorphisms lasting three lectures, general graph properties and bipartite graphs lasting two lectures, and planarity lasting one lecture. In examining each topic, KTE and KE agreed upon a list of subtopics to be covered. These subtopics are components within a topic that the investigator and the graph theorist deemed essential to the understanding of a topic. For example, the topic of graph isomorphisms in the current study consisted of five subtopics: vertices and edges, cycles in graph isomorphisms, complements, connectedness, and other graph properties. For example, the subtopic of vertices and edges called for a lesson plan that showed the students how to use the numbers of vertices and edges to differentiate graphs and the subtopic of cycles in graph isomorphisms called for a lesson plan that showed the students how to use cycles to differentiate graphs.

Each subtopic had different durations in teaching. The more complex a subtopic is, the longer its duration. The duration also depended on students’ general feedback and flow of the class on a particular day. Although KTE and KE lesson plans’ different structures also contributed to different durations, this difference was minimized by the common set of examples, propositions, theorems, and observations which must be shared by both lesson plans in each subtopic and by the overall time constraint of the study.

There are certain subtopics in which the KTE and KE lesson plans shared similar structures. For example, in the first major topic area: introduction to graphs which talked about basic definitions of graphs, vertices, edges, distance, and other definitions, the treatments of the
materials were similar in both groups. The first major area in which the two groups received
differently structured lesson plans was graph isomorphisms, for example, in the subtopic of
cycles in graph isomorphisms. This chapter will present and highlight the differences by
showing the KTE lesson plan and KE lesson plan for this subtopic as it was the first time the two
lectures differed substantially and that cycles are a key in understanding graph theory. Overall,
KTE lesson plans differed from the KE lesson plan in three of the four topics: graph
isomorphisms, general graph properties and bipartite graphs, and planarity.

KTE Lesson Plans

A $k$-cycle in a graph $G$ is a sequence of $k$ vertices $v_1, v_2, ..., v_k$ such that $v_i$ and $v_{i+1}$ are
adjacent and $v_k$ and $v_1$ are adjacent. Based on the KTE lesson plan template shown in the last
chapter, KTE made a mathematical–pedagogical analysis of cycles using the 2-criteria rule in the
lesson plan template, that is, the simplicity of cycles in and of themselves and their pervasiveness
in graph theory overall. When isolated, cycles are simple to understand for students in both
groups in the current study. That is, it is simple for students to see that a graph which is also a
3-cycle is not isomorphic to a graph which is 4-cycle:

![Diagram of G1 and G2]

Figure 1: $G_1$ and $G_2$ are not isomorphic as graphs and cycles.
In general, it is relatively simple to see that “given two graphs $G_1$ and $G_2$, suppose $G_1$ is also a $j$-cycle, $G_2$ is also a $k$-cycle, then $G_1 = G_2$ if and only if $j = k$.”

As a result, when isolated, a $k$-cycle is a relatively simple theory-like knowledge structure that the students can grasp. Therefore cycles passes the simplicity criterion. It is true that arbitrary cycles embedded in a larger graph are not simple to spot, but teaching must begin with simple and isolated concepts and progress to more complex scenarios. Denote the above mentioned italicized theory-like observation as theory-like knowledge-structure 1 or TKS1.

Cycle recognition and cycle counting form a bulwark of graph theory and appear in numerous later occasions such as finding Hamiltonian circuits in arbitrary graphs. Therefore cycles are also pervasive in graph theory.

Cycles passed tests posed by the 2-criteria guideline in the KTE lesson plan template. Therefore from the KTE standpoint, it could be more advantageous for a student to learn TKS1 first – before they bury themselves in a sea of examples where they are more likely to focus on the entire graph when checking for isomorphism instead of looking at this or that particular part of a graph such as this cycle that courses through the whole graph or that cycle discreetly tucked at the corner. Thus at this juncture, KTE turned to a theory-first, example-later pedagogical approach that clarifies what is an otherwise simple principle very quickly for the students. With TKS1, KTE reasoned that it is simpler for the students to see that with graphs $G$ and $H$, one having a 5-cycle, the other not having a 5-cycle, then $G \neq H$. This observation can be formalized into a simple but effective observation to differentiate graphs:

“Given two graphs $G$ and $H$, suppose $G$ has a $k$-cycle, and $H$ does not have a $k$-cycle, then $G \neq H$.” Denote this italicized theory-like observation as TKS2.
KTE reasoned that with TKS2 introduced earlier, students are more likely to gaze at graphs afterward differently than before and some can begin to generalize this theory-like observation into a much more powerful observation:

“if two graphs $G$ and $H$ are isomorphic, then the number of $k$-cycles in $G$ must be the same as the number of $k$-cycles in $H$ for every $k.$” Denote this italicized theory-like observation as TKS3.

KTE examined cycles for their simplicity and pervasiveness, went back to the lesson plan template, and designed the lesson plan for cycles in graph isomorphisms with both a top-down approach and a bottom-up approach.

The KTE lesson plan began with a re-introduction of the definition of cycles. Then it asked the top-down theoretical question: “given two graphs $G_1$ and $G_2$, suppose $G_1$ is also a $j$-cycle, $G_2$ is also a $k$-cycle, can $G_1$ and $G_2$ be isomorphic, why or why not?” Denote this question as Q1. Therefore the KTE lesson plan posed TKS1 as a question first before any concrete examples. The lesson plan allocated 2 minutes for the students to arrive at an answer either concretely or theoretically. Only when students answered the relatively simple question in the negative with concrete and/or theoretical justifications, did the lesson plan proceed to a concrete example: are the graphs $G_1$ and $G_2$ isomorphic, why or why not?

![Figure 2: First concrete example used in cycles subtopic lecture](image_url)
In the KTE perspective, this simple example serves as an element-like knowledge structure or EKS to illustrate TKS1. Denote this simple example by EKS1. After students gave answers to EKS1, the KTE lesson plan gave another concrete example EKS2: “Are the graphs \(G_3\) and \(G_4\) isomorphic, why or why not?”

![Figure 3: second concrete example used in cycles subtopic lecture.](image)

Notice \(G_3\) has a 5-cycle and \(G_4\) has no 5-cycles.

So far, the KTE lesson plan on cycles in graph isomorphisms opened with theory-like observation posed as a question and then supported it with concrete examples. This top-down approach by KTE is consistent with its parent perspective KT’s philosophy. Thus at this juncture, the KTE lesson plan’s approach is top-down:

**TKS 1**

\[\downarrow\]

**EKS1 & EKS2**

![Figure 4: First part of KTE lesson plan on cycles in graph isomorphisms](image)
Next, the KTE lesson plan reversed the order of $G_3$ and $G_4$ in EKS2 and asked if the two graphs are isomorphic and asked the students to come up with more than one justification. Notice $G_3$ (new one) has a 6-cycle, but no 5-cycles and $G_4$ (new one) has a 5-cycle, but no 6-cycles.

Figure 5: EKS3, a reversal of EKS2
This reversal by KTE had several pedagogical purposes. One, it hints at the concept that if $G \neq H$, then $H \neq G$ or isomorphism is a symmetric relation. Two, this quick and timely graph reversal prompts the mental tallying of a count: $n_k(G) = $ the number of $k$-cycles in $G$. Students in general manifest an asymmetry of a preferential reading from the left to the right. This graph reversal elicits a cognitive disturbance whose resolution can be clarified by the instantiation of the counting mechanism $n_k(G)$. After students gave their various answers, the KTE lesson plan asked: “when looking at two graphs, if one of them has a $k$-cycle, what must the other graph have in order for them to be isomorphic?” Denote this question as Q2. Allocating 2 minutes for this question, the KTE lesson plan introduced the theoretical generalization TKS2: “given two graphs $G$ and $H$, suppose $G$ has a $k$-cycle, and $H$ does not have a $k$-cycle, then $G \neq H$. “
The reversal example EKS3 not only serves as a concrete example illustrating TKS1, but it also served as a cognitive conduit to TKS2. Here the concrete example first (EKS3), theory (TKS2) later approach at this juncture is the bottom-up approach used by the KTE lesson plan is consistent with its parent perspective KE’s philosophy.

Overall, the KTE lesson plan showed two TKS and nine EKS. In additional to the above mentioned TKS1 and TKS2, it concluded by asking: “think about two isomorphic graphs, what can you say about the number of $k$-cycles in either graph for any positive integer $k$?” and deferred any answers to self-reflection because it was the purpose of this study to see if students in any group can generalize the knowledge acquired into the more powerful TKS3:

“If two graphs $G$ and $H$ are isomorphic, then the number of $k$-cycles in $G$ must be the same as the number of $k$-cycles in $H$ for every $k$.” This TKS requires the instantiation of $n_k(G)$ as a counting mechanism and it states $n_k$ as an invariant within each class of isomorphic graphs – a powerful generalization of all three TKS.

The shrewd reader may detect an overall bottom-up approach in the structure of the KTE lesson plan where the theory-like knowledge structures are increasingly more elevated in cognitive difficulty: TKS1→TKS2→TKS3. This overall structure in the KTE lesson plan is consistent with KE’s bottom-up approach where a learner progresses from simple observations to more theoretical and more complex generalizations. But the presence of an overall structure in KTE’s lesson plan is also consistent with KT’s philosophy that a grand overarching hierarchical conceptual structure guides learning and understanding. The broken arrow indicates that TKS3 was not taught but only asked in the experimental group.
As a lesson plan, KTE lesson plan, born out of the lesson plan template, is locally dualistic and cyclic: dualistic in the sense that at times it used a top-down approach (TKS1→EKS1) and at times it used a bottom-up approach (EKS3→TKS2); cyclic in the sense that the same examples can be re-used later to elicit a new concept or to give a different conceptual complexion to an older concept (E2 vs. E3). Another feature of the KTE lesson plan is that it questioned the students at particular junctures in the lecture to prime their thinking and to prompt generalizations:

\[
\begin{array}{ccc}
\text{TKS1} & \text{TKS2} & \text{TKS3} \\
\downarrow Q1 & \downarrow Q2 & \downarrow Q3 \\
\end{array}
\]

\[
\text{EKS1, EKS2} \rightarrow \text{EKS3} \rightarrow \text{TKS2} \rightarrow \cdots \text{EKS9} \rightarrow \text{TKS3}
\]

\[
\leftrightarrow \quad \leftrightarrow \quad \leftrightarrow
\]

\text{Re-Use} \quad \text{Re-Use} \quad \text{Re-Use}

Figure 6: overall structure of KTE lesson plan

In assessment, the quizzes and posttest used both concrete and theoretical questions. Concrete questions involved pairs of graphs where the number of k-cycles may be different for the two graphs and students were asked to either give an explicit isomorphism or state clear characteristics that differentiate one graph from the other. The theoretical questions asked the students to list general properties two isomorphic graphs must share. The current discussion will defer reflections to chapters VI and VII.

The entire lesson plan for cycles in graph isomorphisms occupied the second one-third of lecture 2 in the current study for both groups. The rationalization and the design of the KTE
lesson plan in the above discussion are reflected in the KTE lesson plan on the subtopic of cycles in graph isomorphisms:
Table 2: KTE Lesson Plan on Cycles

<table>
<thead>
<tr>
<th>Objectives</th>
<th>To understand cycles as a fundamental part of graphs, both as an invariant and as a tool for isomorphism recognition and other more powerful properties of graphs.</th>
</tr>
</thead>
</table>
| Activities | TKS1 ↓ Q1  
            | TKS2 ↓ Q2  
            | EKS1, EKS2 → EKS3 → TKS2 → ...EKS9 - - - → TKS3  
            | Re-Use  
            | Re-Use  
            | Re-Use  
            | 2 to 4 minutes intervals between each question or EKS. |
| Materials  | TKS, EKS, Q_k |
| Assessment | Both specific isomorphism recognition problems using cycles and general questions where \( n_k(G) \)’s invariance is tested. |
| Reflection | Diameter, an invariant previously deliberately withheld from both groups, was uncovered by an experimental student twice. To equalize the two groups, the investigator taught diameter in the KE group. See the Result section for more discussion of this phenomenon. |
KE Lesson Plans

KE views knowledge as a quasi-independent collection of facts, facets, and narratives. As a result, the KE philosophy is suspicious of the validity of general theory-like knowledge structures that both guide and constrain a student’s understanding in a particular topic area. The pedagogical implications are immense. KE stresses a bottom-up approach to teaching and learning in which concrete examples precede more general observations. Additionally, KE places a premium on contextual sensitivity as a cue to instantiation.

KE lesson plan’s overall objective on the subtopic of cycles in graph isomorphisms is for students “to look beyond vertices and edges to see the essence of a graph”. As agreed between the investigator and the graph theorist who is a KE believer and designed the KE lesson plans for the study, the KE lesson plans used the same set of examples (EKS), propositions, and theorems (TKS), and observations. Since the KE philosophy does not affirm theory-like knowledge structures, the KE lesson plan named theoretical generalizations, the TKS in KTE lesson plans, as propositions and theorems. Therefore in the KE lesson plan, T1 refers to TKS1 in the KTE lesson plan. The examples were named as E_k where the sub-index k means the kth example as opposed to EKS in the KTE lesson plans. The KE lesson plan deferred theoretical generalizations until the middle-latter or the latter portion of the lecture. KE lesson plans generally did not re-use a concrete example. Each subtopic was treated with spirit of self-sufficiency and thoroughness.

The KE lesson plan on cycles in graph isomorphism is a clear and streamlined lecture kicked off by concrete examples E1, E2 and E3. E1 in the KE lesson plan is EKS1 in the KTE
lesson plan; E2 is EKS2. E3 is a more complex example (E3 appeared later as an EKS in KTE lesson plan):

![Diagrams of G1 and G2]

Figure 7: E3 in KE lesson plan

Approximately 2 to 4 minutes were allocated for each example. Questions such as Q2 in the KTE lesson plans were generally dissuaded or deferred until the end. The KE lesson plan on cycles in graph isomorphism was a linearized progression of increasingly complex examples and propositions and theorems toward the end:

\[ E_1, E_2 \rightarrow E_3 \rightarrow T_1, \text{more } E_k \rightarrow T_2 \rightarrow \cdots E_9 \rightarrow T_3 \]

Figure 8: Overall structure of KE lesson plan

The broken arrow indicates that T3, KE’s counterpart of TKS3 in KTE lesson plan, was asked by the end of the KE lesson plan, and answers were deferred for self-reflection. This question also prompts the instantiation of the counting mechanism \( n_k(G) \). Same as the KTE lesson plan, the KE lesson plan contained two propositions T1 and T2 and nine examples, and concluded with a question hinting at T3.

Viewed from a distance, the KE lesson plan was more organized with a clear objective and the design followed a more intelligible structure from the beginning to the end. KTE lesson plans, on the other hand, contained more reflections which can bear extra cognitive costs as
students may lose their focus on the present material with more stops-and-goes such as questions interspersed in the lesson plan and the revisits of older examples.

KE’s emphasis on contextual cues was reflected in the request by the designer of the KE lesson plan to make larger representations of graphs on the chalkboard during the KE lectures. Additionally, KE also pays more attention to key words. Although most of the problems in the assessments were not word problems, key words appeared on the KE lesson plan’s write up. The KE lesson plan on cycles in graph isomorphism is given:
Table 3: KE Lesson Plan on Cycles

<table>
<thead>
<tr>
<th>Topic: Graph Isomorphisms</th>
<th>Subtopic: Cycles in Graph Isomorphisms</th>
<th>Lecture: 2nd</th>
</tr>
</thead>
</table>

**Content Objectives:**

*To look beyond vertices and edges and see the essence of a graph*

**Key Words:**

cycles, graph isomorphism

**Visual Aids:**

Drawings of graphs should be large to facilitate viewing.

**Materials:**

E1, E2, ….. Propositions T1, T2, and T3 (ask the last one, but do not prove in class)

**Activities:**

E1, E2 → E3 → T1, more E_k → T2 → … E9 - - -> T3

**Assessment:**

Since KE pays close attention to context sensitivity as a cue to learning and teaching, problems testing students’ knowledge in the “same topics” and “subtopics” were clustered together on quizzes and tests. Both group used the same tests throughout the study: same in content, same in format.
Chapter V
Materials and Methods

Subjects and Procedures

The subjects of this study were students from two sections of an algebra class in a large public university in the mid-Atlantic region. A total of 53 students participated in the investigation, twenty-seven in one section and twenty-six in the other section. No restrictions were placed on their socio-economic status.

A pretest was given to all students. An independent samples t-test was used to analyze their performances on the pretest in order to make sure the differences in their background knowledge related to graph theory between the two groups were statistically insignificant. The section with twenty-seven students was the control group and the section with twenty-six students was the experimental group. The pretest is contained in Appendix A.

Seven classroom instructional periods, each of one and a half hours in duration were given to both groups. The control group received KE guided lesson plans in graph theory. The test group received the KTE guided lesson plans on graph theory.

Before the first lecture, after the third lecture, and after the last lecture, interviews were conducted on a small group of students from each section to probe deeper into their rationalizations of problems in graph theory. The results from these interviews comprised the qualitative component of the investigation of the effectiveness of KTE perspective lesson plans compared to KE perspective lesson plans. The interview questions are contained in Appendices B, C and D.
Two intermediate quizzes were administered to monitor their progress. A post test was given to all students in both groups. An independent samples t-test was used again to analyze their performances to detect any differences in the mean performances of the two groups. The quizzes are contained in Appendices E and F. A post test was administered to detect any differences between the two groups. The post test is contained in Appendix G.

**Instruments: Measurements**

**Tests**

The pretest (see Appendix A) investigated a student’s general mathematical knowledge as related to graph theory:

1. Familiarity with connections and parity of integers, for example: “you are at a party of 7 people (including yourself). Is there a way for each of the 7 people at the party to know exactly 3 other people at the party?”
2. General graph recognition, for example: “are the following two figures the same figure but drawn differently, or are they different figure?”

Similar to the pretest, the post-test (see Appendix G) consists of questions designed to investigate a student’s understanding of graph theory:

1. Basic understanding of the structures of graphs including listing some of the properties that two isomorphic graphs must share.
2. Planarity of graphs including what properties make some graphs planar while others non-planar.
3. General questions pertaining to the properties of graphs including is it always possible for any simple graph to have at least two vertices of the same degree.

Interviews

In addition to the pre-test, intermediate quizzes, and the post test, three rounds of interviews were conducted on a total of six students: three from the control group and three from the experimental group. These students were interviewed three times: before the lessons on graph theory began, half-way through the lessons (after the 3rd lesson and before the 4th lesson), and finally after the post test was completed. Each of six students was selected based on their performances on the pre-test: two were low performing students; two were mid-level students; and two were top students. The three interviews examined students’ thinking on graph theory problems and their reasoning as well their ability to discover new and unlearned patterns in graph theory.

Statistical Analysis

An unpaired samples t-test was used both to separate the initial student population into two distinct groups based on their pretest and to detect any statistically meaningful differences in the performances of the two groups on the post-test.

The independent sample t-test analyzed two samples to see if they were from the same population by comparing their means. The samples were independent in the sense that they are separate samples containing different sets of individual subjects and the individual measures in group A are in no way linked or related to any of the individual measures in group B and vice versa. This method of analysis is ideal in two experimental settings. First, it may be used when an investigator wants to determine if two samples have equal means. This study used the
independent samples t-test to separate the students based on their pretest performance. The unpaired samples t-test was then used to ensure that the means of the samples in each of the two groups, *thus separated*, were statistically equivalent. Second, the samples t-test was also the method of choice when an investigator wants to determine if an independent variable can produce a significant result. During the current investigation, the independent variable was the new instructional method in the test group: the use of a lesson plan guided by KTE philosophies. The investigator wanted to see if this new pedagogical method produced a statistically significant effect. The samples t-test was used to detect any statistically meaningful differences in the means of the two groups, using as a variable the new pedagogical method.

**Materials**

The software SPSS, or Statistical Package for the Social Sciences, was used throughout the study to compute all parameters in the data: means, medians, quartiles, and independent samples t-tests.
Chapter VI

Results of Trials

Pre-Test

The pre-test was given to 27 students in the control group and 26 students in the experimental group. The pre-test contained eight problems in four categories: application of graph theory in real life, basic properties of graphs in a social situation, drawing of planar and non-planar graphs, and isomorphisms of graphs through visual recognition. Each question was given ten points with two points for the correct answer, for example “Yes, these two drawings are the same” and eight points for the reasoning behind the answer. The questions required no prior understanding of graph theory and assumed no mathematical background. The pre-test is attached in the appendix A.

The mean for the control group was 33.50, with a standard deviation of 12.66; the mean for the experimental group was 34.31, with a standard deviation of 13.78. The basic statistics of these two groups are displayed in Table 4:

<table>
<thead>
<tr>
<th>Groups</th>
<th>N</th>
<th>Mean</th>
<th>Std. Deviation</th>
<th>Std. Error Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control</td>
<td>27</td>
<td>33.5</td>
<td>12.66</td>
<td>2.44</td>
</tr>
<tr>
<td>Exp.</td>
<td>26</td>
<td>34.31</td>
<td>13.78</td>
<td>2.70</td>
</tr>
</tbody>
</table>
On the Levene’s test for the equality of variances, the p-value was 0.788, hence one can assume statistically that the two groups have equal variances. Thus an independent samples t-test can be performed to compare the means of these two groups. The p-value for the independent samples t-test was 0.825. Therefore there was no meaningful statistical difference between the two groups. The statistical parameters for the comparison of these two groups are displayed in Table 5:

Table 5: Samples T-Test Comparison of Both Groups on Pre-Test

<table>
<thead>
<tr>
<th>Levene’s Test for Equality of Variances</th>
<th>t-Test for Equality of Means</th>
<th>95% Confidence Interval of the Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>Sig.</td>
<td>t</td>
</tr>
<tr>
<td>0.73</td>
<td>0.788</td>
<td>-2.22</td>
</tr>
<tr>
<td>0.73</td>
<td>0.788</td>
<td>-2.22</td>
</tr>
</tbody>
</table>

Prior to the study, both groups were asked if they had previous experience with graph theory. Although all students indicated verbally that they had no previous experience, a pre-test was necessary to ascertain background knowledge. These results from the pre-test confirmed their answers.

Interview 1

Three students from each group were interviewed before the first lecture. These interviews comprised the qualitative portion of the study since the sample of three students in the control group and three students in the experimental group was too small for a meaningful quantitative statistical comparison. On the first interview, six questions were given to students. The means for the control group and the experimental group were 18.71 and 16.92 respectively.
The data from first round of interviews correspond well with those from the pre-test. It was only with this level of equivalence between the two groups, the investigation proceeded to the first lecture.

Quiz 1

Two intermediate quizzes were given to both groups to track the progress of their understanding and identify patterns in their thinking. The first quiz was given after the third lecture and the second quiz was given before the fifth lecture.

The first intermediate quiz contained four questions falling into three categories: general properties of graphs, isomorphisms of graphs, and applications of graph theory. Each question was assigned ten points: two points for the direct answer, for example “yes, the two graphs are isomorphic” and eight points for the reasoning behind the answer.

Twenty-four students in the control group took the first intermediate quiz. The control group has a mean of 10.54 with a standard deviation of 8.62. These and other statistical parameters are given in Table 6:

<table>
<thead>
<tr>
<th>Group</th>
<th>No.</th>
<th>Range</th>
<th>Minimum</th>
<th>Maximum</th>
<th>Mean</th>
<th>Std. Deviation</th>
<th>Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control</td>
<td>24</td>
<td>34.00</td>
<td>0</td>
<td>34.00</td>
<td>10.54</td>
<td>8.62</td>
<td>74.259</td>
</tr>
</tbody>
</table>
Twenty-five students in the experimental group took the first intermediate quiz. The control group has a mean of 14.52 with a standard deviation of 9.91. These and other statistical parameters are given in Table 7:

Table 7: Result of First Intermediate Quiz of Experimental Group

<table>
<thead>
<tr>
<th>Group</th>
<th>No.</th>
<th>Range</th>
<th>Minimum</th>
<th>Maximum</th>
<th>Mean</th>
<th>Std. Deviation</th>
<th>Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exp.</td>
<td>25</td>
<td>40.00</td>
<td>.00</td>
<td>40.00</td>
<td>14.52</td>
<td>9.91</td>
<td>98.18</td>
</tr>
</tbody>
</table>

On Levene’s test for the equality of variances, the p-value was 0.689. Hence one can assume statistically that the two groups have equal variances. Thus an independent samples t-test can be performed to compare the means of these two groups. The p-value for the independent samples t-test was 0.141. Although this still implies that the means of the two groups were still statistically significant, the p-value was much smaller compared to that on the pre-test – 0.825. The statistical parameters for the comparison of these two groups are displayed in Table 8:

Table 8: Samples T-Test Comparison of Both Groups on First Intermediate Quiz

<table>
<thead>
<tr>
<th>Levene's Test for Equality of Variances</th>
<th>T-test for Equality of Means</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>Sig.</td>
</tr>
<tr>
<td>---</td>
<td>------</td>
</tr>
<tr>
<td>.163</td>
<td>.689</td>
</tr>
<tr>
<td>-1.501</td>
<td>46.558</td>
</tr>
</tbody>
</table>
The first intermediate quiz was the first time KE-guided lesson plans were compared to KTE-guided lesson plans. One topic area in which the KE and KTE used differently structured lectures was graph isomorphisms, a topic which lasted three full lectures and one which exerts a profound effect on students’ understanding of graphs in general. By the time of the first quiz, the students in both groups already went through two full lectures on graph isomorphisms. Two of the problems on the first intermediate quiz were on isomorphism recognition. The scores on these problems were not normally distributed. Restricting to these two questions, the control group has a mean of 8.72 with a median of 8; the experimental group has a mean of 9.64 with a median of 10. A Mann-Whitney U-test was performed to compare the performances. Since the minimum of the scores on these two problems was 0 in both groups, a 1 tailed Mann-Whitney U-test was used. The p-value for the Mann-Whitney U-test was 0.09. Although this p-value is greater than 0.05, it is modestly close to the required 0.05. The p-value for the 1-tailed Mann-Whitney U-test and other statistics are displayed in Table 9:

Table 9: Mann-Whitney Test on Isomorphism Problems on First Intermediate Quiz

<table>
<thead>
<tr>
<th>Statistics</th>
<th>Data</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mann-Whitney U</td>
<td>234.000</td>
</tr>
<tr>
<td>Wilcoxon W</td>
<td>534.000</td>
</tr>
<tr>
<td>Z</td>
<td>-1.332</td>
</tr>
<tr>
<td>Asymp. Sig. (1-tailed)</td>
<td>0.09</td>
</tr>
</tbody>
</table>

With these results, the investigation went back and examined the vast differences in the way in which graph isomorphisms were conceived through KE and KTE. One of the many differences
was in the introduction of graph isomorphisms using cycle recognition. Suppose \( G \) as a graph is also a \( k \)-cycle: \( G \) is a closed loop of \( k \) vertices, and let \( H \) be a graph which is also \( j \)-cycle: \( H \) is a closed loop of \( j \) vertices. If \( k \neq j \), then clearly \( G \neq H \). This is very simple to students, but without being formulated at the beginning of a graph isomorphism lecture, it is significantly more challenging for students to see that graphs \( G_1 \) and \( G_2 \), one having a 5-cycle, the other not having a 5-cycle, are not isomorphic. This observation can be formalized into a simple but extremely effective weapon to differentiate graphs:

“given two graphs \( G_1 \) and \( G_2 \), suppose \( G_1 \) has a \( k \)-cycle, and \( G_2 \) does not have a \( k \)-cycle, then \( G_1 \neq G_2 \).”

From the KTE perspective, this observation is a simple theory-like observation that underlies a critical part of graph isomorphisms and indeed all of graph theory – cycle recognition. Cycle recognition and cycle counting form a bulwark of graph theory and appear in numerous later occasions such as finding Hamiltonian circuits in arbitrary graphs. As stated before, from the KTE standpoint, it is much more advantageous for a student to learn this theorem first – before they bury themselves in a sea of examples where they are much more likely to focus on the entire graph when checking for isomorphism instead of looking at this or that particular part of a graph such as this cycle that courses through the whole graph or that cycle discreetly tucked in the corner. Thus at this juncture, KTE turns to a theory-first, example-later pedagogical approach that clarifies what is an otherwise very simple principle very quickly for the students. Armed with this, students gaze at graphs afterward very differently than before and some can even begin to generalize this theory-like observation into a much more powerful observation:
“if two graphs $G_1$ and $G_2$ are isomorphic, then the number of $k$-cycles in $G_1$ must be the same as the number of $k$-cycles in $G_2$ for every $k$.”

On the other hand, KE theory prefers to start every lecture on graph isomorphism with a wealth of examples and generalize later. This sharp difference was reflected in the design of the KTE lesson plans and that of KE lesson plans. The KE lesson plans designed by the graph theorist used in the study began every lecture in graph isomorphisms with many examples - precisely the very examples used in the KTE lesson plans as agreed upon by the investigator and the graph theorist who designed the KE lesson plans, but after theoretical priming. KTE also used many examples, but in this part of the lecture, KTE turned the table around and began with a theory-like observation, followed by relevant examples.

The essential difference is a philosophical realization made by the investigator: when faced with a highly innate but visually or cognitively inert theory-like observation that pervades deeply within a topic, it is much more advantageous to make the observation clear at the beginning so students can learn with a purpose in mind and generalize the observation into more complex observations. In teaching graph isomorphism, the KTE philosophy recognized that some areas such as cycle recognition are simple theory-like knowledge structures - these are taught more effectively with an early theoretical treatment before showcasing any examples. Moreover, these initial results also hold promise for the validity of KTE’s pedagogical implications: the 2-criteria test to judge if a concept (knowledge structure) should be introduced theoretically first in lesson plan activities. The investigator used the 2-criteria test to examine simple cycle recognitions and decided that its simplicity and pervasiveness were sufficient to be introduced theoretically first before any examples. The initial results on the two graph
isomorphism problems on the first quiz, though not full proof, do point to a sign of cautious promise for KTE as a new perspective for knowledge structure in mathematics education.

Quiz 2

The second intermediate quiz contained two questions falling into two categories: one on properties of bipartite graphs and the other on degree sequences. Each question was assigned ten points again: two points for the direct answer: “yes, this graph is bipartite” and eight points for the reasoning behind the answer. Both categories received different treatments in KE and KTE lesson plans.

Twenty-four students in the control group took the second intermediate quiz. The control group has a mean of 11.29 with a standard deviation of 5.31. These and other statistical parameters are given in Table 10:

<table>
<thead>
<tr>
<th>Group</th>
<th>No.</th>
<th>Range</th>
<th>Minimum</th>
<th>Maximum</th>
<th>Mean</th>
<th>Std. Deviation</th>
<th>Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control</td>
<td>24</td>
<td>20</td>
<td>0</td>
<td>20</td>
<td>11.29</td>
<td>5.31</td>
<td>28.22</td>
</tr>
</tbody>
</table>

Twenty-four students in the experimental group took the second intermediate quiz. The control group has a mean of 14.54 with a standard deviation of 4.79. These and other statistical parameters are given in Table 11:
Table 11: Result of Second Intermediate Quiz of Experimental Group

<table>
<thead>
<tr>
<th>Group</th>
<th>No.</th>
<th>Range</th>
<th>Minimum</th>
<th>Maximum</th>
<th>Mean</th>
<th>Std. Deviation</th>
<th>Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exp.</td>
<td>24</td>
<td>13</td>
<td>7.00</td>
<td>20.00</td>
<td>14.54</td>
<td>4.79</td>
<td>22.96</td>
</tr>
</tbody>
</table>

On Levene’s test for the equality of variances of the two groups, the $p$-value was 0.86. Therefore a samples t-test was conducted to compare the means. The $p$-value for the independent samples t-test was 0.031. Therefore there was meaningful statistical difference between the two groups. The statistical parameters for the comparison of these two groups are displayed in Table 12:

Table 12: Samples T-Test Comparison of Both Groups on Second Intermediate Quiz

<table>
<thead>
<tr>
<th>Levene’s Test for Equality of Variances</th>
<th>Test for Equality of Means</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>$F$</td>
<td>$\text{Sig.}$</td>
</tr>
<tr>
<td>0.032</td>
<td>.886</td>
</tr>
</tbody>
</table>

One of the two problems on the second quiz was on degree sequences. The problem asked the students to give all non-isomorphic graphs with the degree sequence $(2,2,2,1,1)$. Figure 9 shows the only two graphs $G_1$ and $G_2$ exist for this degree sequence:
Students in each group either answered 0 graphs exist for this sequence, one graph, two graphs, or more if they failed to notice that some of the graphs they drew were isomorphic. Table 13 shows the distribution of the answers of both groups in the study:

<table>
<thead>
<tr>
<th>Group</th>
<th>0 graphs</th>
<th>1 graph</th>
<th>2 (correct ans.)</th>
<th>&gt;2 graphs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control Group</td>
<td>3</td>
<td>14</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>Exp. Group</td>
<td>1</td>
<td>5</td>
<td>15</td>
<td>3</td>
</tr>
</tbody>
</table>

Degree sequence is *coarser* than isomorphism: two non-isomorphic graphs can have the same degree sequence. At this point, the investigation went back and examined the KE and KTE lesson plans to search for possible answers to explain the differences in the performances between the two groups. In particular, as a new perspective, the investigation searched
especially for insufficiencies in the KTE lesson plans. In the KTE perspective, the lesson objective behind the design of lesson plans for degree sequence was three-fold:

1. to hint at the relationship between vertices and edges and to heighten a student’s awareness of any mathematical relationship between vertices and edges whenever such relationship might emerge later;

2. to develop a structural connection between degree sequence and graph isomorphism so each can hint at and deepen the other;

3. to increase a student’s ease of generating graphs by drawing, erasing, and redrawing graphs and the physical manipulation of graphs such as moving an edge by pulling, pushing, and dragging.

With these lesson plan objectives, immediately after the definition of degree sequence, the KTE lesson plan began with the question: “Can two non-isomorphic graphs have the same degree sequence and conversely can two isomorphic graphs have different degree sequence? Why or why not? Justify your answers both theoretically and elementally.” Some students played with concrete examples, others used their knowledge in graph isomorphism to tackle the question. Students came up with examples of non-isomorphic graphs with the same degree sequence. Only then did the KTE lesson plan proceed to show more examples of non-isomorphic graphs of the same degree sequence. The KTE lesson plan was designed to hint at the structural relationship between degree sequence and isomorphism class in figure 10:
In contrast, the KE lesson plan began with concrete examples of non-isomorphic graphs with the same degree sequence. It is true that eventually the KE lesson plan discussed non-isomorphic graphs with the same degree sequence, but one needs to examine the structural differences between the KE and KTE lesson plan to cast more light on the differences of the performance on the second quiz. KTE believes that different pedagogical structures can create different mental alertness and comprehension for a given mathematical topic. The KTE lesson plan on degree sequences began with a relational objective that coordinates two strands of subtly connected knowledge elements in graph theory: degree sequences and graph isomorphisms. By placing the critical question of the relation between degree sequences and graph isomorphisms at the very beginning of the lesson plan activity, KTE opened the chapter on degree sequences with an immediate mental knock of the relational differences between degree sequences and graph isomorphisms. Moreover, in clear language, the question also highlighted a learner’s awareness of graph isomorphism in and of itself much akin to teaching different numeration systems (base five, binary numbers) later in a student’s life after mastering base ten computation can motivate a student to re-inspect his understanding of base-ten arithmetic which he has been learning since
childhood and regain a deeper appreciation of reason behind base ten place values he may have taken for granted. The data in table 10 indicate that students with a clear recognition of the deeper interplay between degree sequences and graph isomorphisms cued in early in their lecture display a fuller capacity to execute searches for non-isomorphic graphs with the same degree sequence. In light of these philosophical differences with real pedagogical implications, the differences in the results of the problem on degree sequences can be explained.

The KTE objectives in degree sequence also manifested itself in the other question on the second quiz which was on bipartite graphs. The students in both groups were asked whether the graph in figure 11 is bipartite:

![Figure 11: Bipartite Question on Quiz](image)

Since both KE and KTE lesson plans agreed, before the study started, on an explicit list of concrete examples, observations, propositions, and theorems to be taught in both classes, no additional information was allowed to be given to any one group to ensure maximal equality in teaching. Precisely two concepts were taught in both classes to differentiate bipartite graphs from
non-bipartite graphs. One was the definition and the other was the proposition that a graph is bipartite if and only if every cycle is even. Most of the students in both groups used one of the two concepts. Although the results from the two groups on this particular problem were statistically equivalent, three students in the experimental group answered the bipartite problem with unexpected rationalizations that warranted closer examination. These three students twisted vertices $c$ and $d$ in such a way that the resulting graph resembled a classic bipartite graph introduced in the definition: a partition of the vertices into sets $U$ and $V$ such that any edge is between a vertex in $U$ to a vertex in $V$. Figure 12 shows the solution of one of the three students in the experimental group, notice the twisting of the vertices $c$ and $d$ in the resulting isomorphic graph drawn by the student on the left side of the figure:

![Figure 12: Unexpected Answer from an Experimental Group Student](image)

One of the explicit objectives of degree sequences KTE lesson plans was:
“to increase a student’s ease of generating graphs by drawing, erasing, and redrawing graphs and the physical manipulation of graphs such as moving an edge by pulling, pushing, and dragging.”

This coordination of understanding by “drawing” and “drawing” in order to understand began to manifest into an overall ease with graphs among some of the experimental group students taught by the KTE lesson plans. Examining the rationalization behind this bipartite graph problem by the three students in the experimental group, one can detect a general willingness to engage oneself with a graph by pulling, pushing, and dragging. The objective in this part of the study: namely the lesson plan objective on degree sequences, embodied one of the original tenets of KTE: KTE recognizes the duality of cognitive approaches in knowledge acquisition - students exhibit a full spectrum of learning behaviors, some students rationalize a problem theoretically via highly conceptualized definitions and propositions, others prefer a hands on approach where a sensory-motor activation is more conducive to instantiation. In light of this revelation, the unexpected answer from the three students in the experimental group can be explained. This unique phenomenon occurred one more time later in the study on a homework assignment on planarity. While most of the students from both groups failed to answer the problem, one group of students from the experimental section bypassed the formulaic approach involving the Euler’s formula on planar graphs and engaged the problem directly by drawing and solved the problem successfully.

Aside from these initial rudimentary promises, the KTE lesson plans also exhibited some unique drawbacks. In analyzing responses to the degree sequence problem on the second quiz, the investigator discovered a cognitive phenomenon much more pronounced in the KTE group. A closer look in table 10 revealed that although the experimental group outperformed the control
group on generation of graphs with a particular degree sequence as evidenced above and on
graph isomorphism recognition tasks on the first quiz, more students in the experimental group
answered more than 2 graphs – they failed to recognize that some of the graphs they drew were
isomorphic. This phenomenon warranted a three-fold reflection by the investigator where KE,
KT, and KTE intertwine with each other most subtly. First, it appears KE has a valid
explanation for this contradictory evidence: it maintains that a student’s instantiation is sensitive
to contextual cues; in this way, when a student is working on a isomorphism recognition
problem, he is in an “isomorphism mindset”, searching for any incompatibility between two
graphs; but when he is working on a graph generation problem with a degree sequence, he is in a
“graph creation mindset” drawing as many visually different graphs as possible – the two
different problem headings cued him differently, placing him into two different mindsets with
different mental alertness. KE’s explanatory power is felt keenly. Second, one of the primary
pieces of evidence supporting KT is the incommensurability of minds – KT researchers noticed
that many of children’s perspectives are nearly incommensurable with those of adults’, yet each
one is able to maintain its coherence well within its own ecology of conceptions. KT researchers
see these findings as evidence that adults and children possess different epistemological beliefs
that inform their perspectives differently. In the current study, the seemingly contradictory
evidence among the experimental group students taught by KTE lesson plans points to a possible
existence of two parallel instantiations which are quasi-independent such that each one is cued
differently depending on semantic or syntactical cues within a problem. Thus KT can be used to
explain this contradiction also. Third, in the original tenets of KTE, the new perspective affirms
that even the same student may approach a problem in two different ways with different
rationalizations – KTE recognizes the duality of rationalization both in and among students, thus
KT’s claim in fact supports KTE. An educator’s goal is to sharpen a student’s awareness of the mathematical meaning so as to reduce his overreliance on one particular mindset. KE’s winsome explanation points to an urge on the part of the educator to reduce the reliance on contextual cues in the long run – in the end, a good student is one who can see the common subtle pattern threading through a vast array of disparate examples despite their differences.

Post Test

There were eight problems on the post test. The eight problems fell into seven categories: one problem on properties of isomorphic graphs, three problems on recognition of isomorphic and non-isomorphic graphs, one problem on degree sequence, one problem on applications of the properties of graphs, and two problems on Hamiltonian paths and cycles which were deliberately excluded in the lectures in any of the two groups. Due to the size of the post test, two full hours were allowed to complete the entire test.

Twenty-seven students in the control group took the post test. The control group has a mean of 21.46 with a standard deviation of 11.54. These and other statistical parameters are given in Table 14:

<table>
<thead>
<tr>
<th>Group</th>
<th>No.</th>
<th>Range</th>
<th>Minimum</th>
<th>Maximum</th>
<th>Mean</th>
<th>Std. Deviation</th>
<th>Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control</td>
<td>27</td>
<td>41.00</td>
<td>.00</td>
<td>41.00</td>
<td>21.46</td>
<td>11.54</td>
<td>133.15</td>
</tr>
</tbody>
</table>
Twenty-six students in the experimental group took the post test. The experimental group has a mean of 35.13 with a standard deviation of 12.84. These and other statistical parameters are given in Table 15:

<table>
<thead>
<tr>
<th>Group</th>
<th>No.</th>
<th>Range</th>
<th>Minimum</th>
<th>Maximum</th>
<th>Mean</th>
<th>Std. Deviation</th>
<th>Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exp.</td>
<td>26</td>
<td>55.00</td>
<td>11.00</td>
<td>66.00</td>
<td>35.1346</td>
<td>12.84</td>
<td>164.99</td>
</tr>
</tbody>
</table>

On Levene’s test for the equality of variances of the two groups, the p-value is 0.908. Therefore a samples t-test was conducted to compare the means. The p-value for the independent samples t-test is 0. Thus there was meaningful statistical difference between the two groups. The statistical parameters for the comparison of these two groups are displayed in Table 16:

<table>
<thead>
<tr>
<th>Levene’s Test for Equality of Variances</th>
<th>t-Test for Equality of Means</th>
<th>95% Confidence Interval of the Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>Sig.</td>
<td>t</td>
</tr>
<tr>
<td>.013</td>
<td>.908</td>
<td>-4.080</td>
</tr>
<tr>
<td>.071</td>
<td>.494</td>
<td>-4.071</td>
</tr>
</tbody>
</table>

On the post test, three problems were on isomorphism recognition and one problem was on listing the properties two isomorphic graphs must share. On the three isomorphism recognition problems, two graphs were given and students were asked to either give an explicit isomorphism
or state a clear reason why there were non-isomorphic. On these three isomorphism recognition problems, the control group has a mean of 11.19 with a median of 12; the experimental group has a mean of 18.42 with a median of 20. The investigator also examined the possible effects to the statistical outcome of the post test by discounting any one of the three graph isomorphism recognition problems. Discounting the first of the three problems, the samples t-test statistic was $p = 0.03$, hence the outcome was significant without the first isomorphism recognition problem. Discounting the second of the three problems, the samples t-test statistic was also $p = 0.03$, hence the outcome was significant again without the second isomorphism recognition problem. Discounting the third of the three problems, the samples t-test statistic was also $p = 0.01$, hence the outcome was significant without the third isomorphism recognition problem. The results on graph isomorphism problems on the post test were consistent with those on the first quiz.

On the property listing problem, the control group has a mean of 3.19 with a median of 2.0; the experimental group has a mean of 5.38 with a median of 5.5. The scoring range is too narrow for a samples t-test analysis, but the median indicates that more than half of the experimental students were able to cite at least five properties. One of the properties was the complements of the two isomorphic graphs must also be isomorphic. Of the twenty seven students in the control group, four cited complements; of the twenty six students in the experimental group, eleven cited complements. Complements of graphs were taught in both classes during the same lecture: the third lecture which was on graph isomorphisms. KE and KTE selected different approaches to complements. Both the KE and KTE lesson plans began with the definition, showed complements of several graphs, and asked the students to produce the complements of some given graphs, these are the same graphs the KTE lesson plan used as agreed. But early in the KTE lesson plan, a pair of the agreed upon graphs which are not
isomorphic was chosen; the lesson plan introduced the definition of complements of graphs and immediately asked the students if the two original graphs were isomorphic. The two chosen graphs were graphs with 13 vertices with complex patterns, the features that differentiated them were not visually distinct. The juxtaposition of the introduction of the complement and the two graphs cued the students into using complements as a viable way to examine the problem. Most of the students still searched for similarities. Five students in the experimental group drew the complements carefully and arrived at the non-isomorphism solution successfully. In the next lecture which concluded the portion on graph isomorphisms, the investigator inquired when it is more suitable to use complement as a way to spot graph isomorphism. When no answer was given, a pair of graphs was presented whose non-isomorphism was visually distinct without using complements. One of the original students who used complements in the previous lecture remarked that with visually indistinct graphs with many vertices and high connectivity (high degrees for many of the vertices), it is advantageous to examine the complements of the graphs to simplify the search. In organizing the knowledge structures of complementarity, the KTE template treated the concept of complements as a theory-like knowledge structure and used the 2-criteria in the template lesson plan to judge its position in the in-class activities. Complement is a simple theory-like knowledge structure with manageable cognitive cost in drawing and execution. Its pervasiveness in graph theory is modest but deeply powerful on suitable occasions. Therefore the KTE lesson plans decided to deploy the use of complements immediately after the definition to cue the students - with one cautionary caveat: the lesson plans need to allow the students to reflect upon the visual cues that occasion the instantiation of complement, namely the presence of vertices of high degrees and higher number of vertices. Therefore the KTE lesson plan waited until the next lecture which was the concluding lecture on
graph isomorphisms to loop back to complements again and elicit the visual cues from the students. In the original general template lesson plan of KTE, its objective was to motivate students to connect this knowledge structure to that knowledge structure. This objective on this part of the lesson plan was embodied precisely by the timely question when it is more suitable to use complement as a way to spot graph isomorphism – a general question. The listing property problem on the post test did not use any specific graphs. The results showed that eleven students in the experimental group generalized the concept of complementarity into an abstract graph property that any class of isomorphic graphs must share.

The same information along with the sets of graphs on complements was incorporated into the KE lesson plan also. The KE lesson plan’s treatment of complements used the same examples, began with definition of complements and made generalizations later. The main drawback of this traditional KE approach originates in its philosophy. In the KE philosophy, knowledge structures are conceived as elemental – each knowledge structure is partitioned from the next. Namely KE is skeptical of the structure within a body of knowledge. KTE, being the fallout perspective of KE and KT, recognizes the immense structure within various domains such as mathematics, physics, and biology. Reflecting on the KE lesson plan on complements used in the current study, complements were treated as a free-standing concept inserted into the graph isomorphism lectures. At best, the control group students saw complements more as a means to differentiate two non-isomorphic graphs rather than a component in the rich tapestry of related structures in graph theory that can used both to differentiate two non-isomorphic graphs and as a property that two isomorphic graphs must share.
Degree sequence was assessed again on the post test. The degree sequence 
(3,2,2,2,1,1,1) was given and students were asked to give all non-isomorphic graphs with the 
given sequence. There are nine non-isomorphic graphs with the given degree sequence. The 
statistics of the students’ answers are given in Table 17:

<table>
<thead>
<tr>
<th>Group</th>
<th>0 graphs</th>
<th>1 graphs</th>
<th>2 graphs</th>
<th>3 graphs</th>
<th>4 graphs</th>
<th>5 graphs</th>
<th>6 graphs</th>
<th>7 graphs</th>
<th>8 graphs</th>
<th>9 graphs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>6</td>
<td>7</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Exp.</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>4</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

The median for the control group was 3 and the median for the experimental group was 6. This 
result was consistent with the result on the second quiz. No students answered more than nine 
graphs. Nonetheless the experimental group still exhibited more incidence of isomorphic graphs 
than the control group, for example answering 7 graphs when two of the 7 graphs were 
isomorphic. The study deducted half a point for each pair of isomorphic graphs on this problem. 
In this case, the sheer number of non-isomorphic graphs and the complexity of the graphs with 
the given degree sequence make this problems more prone to this particular mistake than the 
degree sequence problem on the second quiz.
Interviews 2

The last round of interviews was conducted within two days after the post test. Unlike the previous interviews, this interview was designed to probe students’ thinking on unlearned concepts. The investigator presented a definition of trees in graph theory and asked the interviewees to give an alternative definition or additional properties of trees. Five of the six interviewees began by drawing trees. One of the control group students gave an alternative definition but failed to mention connectedness. The other two control group students were not able to give an alternative definition or any additional properties. In the experimental group, one student was able to give an alternative definition. Another student gave what she referred to as a circular definition: “when you separate a tree, you have 2 trees” – this was in fact a property of trees. The third student was not able to give any definitions or properties.

The sequence of rationalizations during the interview: drawing trees first, attempting to generalize later was expected by the investigator. The experimental group maintained a slight edge in the last interview: two students were able to name a property/definition whereas in the control group only one student was successful.

Another phenomenon which surfaced during the interview was that both groups failed to complete the applied problem on trees given at the end of the interview. The applied problem was given in the form of a word problem: a single-elimination tennis tournament has 64 players. No one can advance to the next round without winning the current round. How many matches will be played? None of the interviewees were able to solve this problem. This result was consistent with the result on the applied problem on the post test. Unlike the applied problem on the post test, the difficulty level of the current problem is modest. One possible explanation is
that the first part of the interview relied exclusively on theoretical rationalizations such that when faced with an applied problem, students required more time for a mental priming. Another possible explanation is that there is a disconnect between deriving an alternative definition or a property of trees and recognizing the tennis match play schedule as a tree. How to bridge this cognitive disconnect between theoretical knowledge and the recognition of a theoretical pattern in a real life application is an area for future research.
Chapter VII
Summary, Conclusions, & Recommendations

Summary

The purpose of this study was to examine the KTE theory of knowledge acquisition in mathematics, to design a template lesson plan guided by the philosophy of KTE that can be adapted to produce specific lesson plans for a given mathematical topic, and to investigate the KTE theory by comparing the effectiveness of KTE-guided lesson plans against KE-guided lesson plans in graph theory.

Two sets of lesson plans were created to teach an introduction to graph theory. One set of lesson plans was guided by KTE; the other was guided by KE. A total of 53 subjects participated in the study. The subjects were separated into a control group of 27 students and an experimental group of 26 students based on their performance on a comprehensive pre-test so that there were no statistical differences in their backgrounds between the two groups.

The control group was taught using the KE-guided lesson plans on graph theory and the experimental group was taught using the KTE-guided lesson plans. A total of seven lectures were given to both groups. Two intermediate quizzes were given to track their progress: one after the third lecture and one before the fifth lecture. Homework problems were assigned at the end of each lecture. A comprehensive post test was administered to all 53 participants to detect any differences between the two groups. Three rounds of interviews were conducted on six students, three from each group based on their performance on the pre-test: one interviewee was
in the upper one-third of the score distribution; one was in the middle one-third of the score distribution; one was in the lower one-third of the score distribution.

**Conclusions**

No statistically significant difference was discovered on the first intermediate quiz; the p-value was 0.141 with the experimental outperforming the control group, lower than that of the pre-test \( p = 0.825 \), suggesting that the two groups were beginning to drift away from each other. On the second intermediate quiz, the p-value on the samples t-test was 0.031 with the experimental group outperforming the control group. On the comprehensive post test given after the seventh lecture with review, the p-value was 0 with a mean of 21.46 for the control group and a mean of 35.13 for the experimental group.

Evidence for differentiation appeared approximately half way through the study. Certain participants in the experimental group discerned global graph properties such as diameter on their own or exhibited more flexible thinking in solving problems on the quizzes or in the homework problems. The six participants in the clinical interviews exhibited varying degrees of aptitude in graph theory.

Introductory graph theory has a key characteristic that a mathematics educator must take into account when designing lesson plans regardless of his theoretical perspective: many properties of graphs are acquired through studying graph isomorphisms. With these grand general theoretical observations and concrete field observations, the study now answers the two original questions posed at the beginning.
**Question 1:** Can the use of both element-like and theory-like pedagogical approaches be integrated fully and concretized successfully into classroom instructions for a given mathematical concept?

Based on the results of the current study, KTE gives a cautious yes – if the concept admits a dualistic pedagogical approach. Looking back on the KE-guided and KTE-guided lesson plans, there were several concepts where KTE’s careful integration of both theory-like and element-like approaches optimized learning. For example, in the lesson plan on complements, KTE began with the definition of complements with no specific examples, then immediately the KTE lesson plan followed with two visually non-distinct complex graphs each with 13 vertices and high degrees – a concrete juxtaposition, and posed the critical question: “are these two graphs isomorphic?” Five students in the experimental group drew the complements which are more visually distinct than their originals. In the end of the lecture series, the experimental students were able to generalize complements as a property to classify graphs on the property listing problem on the post test with higher rate of occurrence than their KE counterparts. Therefore tactically, KTE’s answer to this question in fact reflects KTE’s thinking in the design of the template: topic specific – the answer depends on the topic. KTE is vigilant against the miscalculations of its parent perspectives KT and KE in making broad overgeneralizations across vastly different domains. Complements are a concept in which this integration worked; other concepts will require a mathematics educator to examine the concepts’ cognitive demand, the ease of presentation, and other variables as the current investigator did for this study.
A strategic reflection concerning the philosophical underpinning of KTE as a moderate perspective on knowledge structure is in order. KTE is sensitive to the subtle connections between thinking, teaching, and learning. This question is asking in the long term, does a moderate centrist approach hold more promise than a bottom-up or top-down approach. Based on the results and evidence in the current study, the answer is yes in certain cases and mixed in other cases. There were observations captured during the study, on tests or in direct classroom observations where evidence of this dualistic approach in teaching resulted in better understanding. For example, the property listing problem on the post test is a comprehensive survey problem that probed a student’s understanding of graph theory in general as opposed to a specific topic problem. The question required a student to reflect on all the information acquired through the three and a half weeks of lectures: from basic definitions to planarity. The student must synthesize his knowledge on properties that govern graph classifications akin to finding topological invariants because these properties are preserved by graph isomorphisms. The KE group had a median of 2.0 and the KTE group had a median of 5.5. The results do suggest a qualitatively deeper comprehension in the KTE group. But other results also indicate that in certain parts the two groups were equivalent. For example, in solving the applied problem during the last interview, none of the students were able to solve it, even the students who derived alternative definitions or additional properties. On some of the problems, the KTE group showed signs of lower performance than the KE group. For example, on the degree sequence problem on the second quiz, more KTE students drew isomorphic graphs and overestimated the number of isomorphic graphs with the given degree sequence. The same phenomenon occurred again on the degree sequence problem on the post test. As discussed in the results chapter, the investigator posits that at a particular level of comprehension of a topic, there is a corresponding
double cognitive interplay between the comprehension of two concepts and the structural
proximity between them. On the one hand, the understanding of one concept can reinforce the
understanding of the other. On the other hand, the understanding of one idea can also color the
understanding of the other adversely. Intellectual history is replete with such conflations. To
phrase this epistemological phenomenon witnessed in the current study in proverbial lingo, one
must approach the fire to be assured of warmth, but not too close to be consumed by the flames.
As a centrist perspective, KTE steers the middle course, but the line between comfort and safety
is often blurry. Overall, KTE’s integration of the two pedagogical approaches was successful,
but more research is needed to analyze these anomalies.

Question 2: Can a general template lesson plan that embodies the philosophy of KTE be
designed, ready to be concretized into specific lesson plans for a given mathematics topic?

This study included a general lesson plan template with specific theoretical guidelines to
optimize classroom learning. The template lesson plan detailed in chapter three provides a
schematic to specific lesson plans. In-class activities are chosen based on two criteria:

1. The complexity of the concept – simple concepts, though theoretical, can be introduced
   early;
2. The degree of integration of the concept into the overall understanding of the topic area
   – this criterion concerns how deeply embedded a concept is in the overall structure of
   the larger topic.

This two-criterion test in the template is closer in philosophy to KE than to KT. The KE
philosophy places a critical emphasis on simple and elemental knowledge structures. The two-
criterion test examines the natures of a knowledge structure within a topic and prescribes a scale
to judge theoretical concepts based on their simplicity and pervasiveness. The simpler and the more pervasive a concept is, the more it is elevated to the beginning of the lesson – this is in keeping with the bottom-up philosophy of KE as a parent perspective of KTE where a lesson begins with simple observations and build up to more complex observations.

In objective and assessment, the KTE lesson plan template stresses the motivation of connections between concepts within a topic. In particular, in designing assessments, the template emphasizes new and unlearned problems to deepen a student’s understanding. In this respect, the KTE template is mindful of the overall structure within a domain, this is in keeping with the top-down philosophy of KT as a parent perspective of KTE.

The current study used the lesson plan template and concretized it to specific lesson plans to teach introductory graph theory. The design of the template was guided by the philosophy of KTE. It is true that in the current study, the KTE-guided lesson plans hold research promise, but the strength of the template and in the broader scope and the validity of the KTE philosophy as a synthesis of KT and KE require more research.

In real pedagogical practice, the separation between KT and KE may not be as clear-cut. Instructors usually use a variety of teaching techniques that embody both the elements of KT and KE. KT and KE have evolved into mature theoretical perspectives on knowledge structural coherence by 2003 and as such KTE as a theoretical perspective is a perspective borne out of KT and KE. But it is conceivable that educators in the past contemplated the possibility of a theoretical perspective on knowledge structure coherence that combines certain aspects of KT and KE. This study acknowledges such possibility and minimizes the notion that KTE is an entirely new creation by the investigator.
Recommendations

A new perspective in knowledge structure may raise more questions than it answers. During the current study, the investigator was randomly inspected three times during three KTE and KE lectures. It was brought to his attention that during two of the inspected KTE lectures when students showed an “ingenious” answer, the investigator displayed positive emotions. Upon the report of the second occurrence, the investigator curtailed his countenance and began to respond to any correct answer, ingenious or otherwise, with a nod. A researcher should and needs to be conscientious and adhere to the highest standards of objectivity in his research to ensure quality. Future graduate students working on KTE or indeed any research should reflect on objectivity before, during, and after the investigation.

Since KTE is a new perspective on knowledge structure, graduate students interested in doing research on KTE should stay within traditional areas in mathematics or physics and avoid topic areas with many epistemological traps such as different orders of infinities in abstract logic and model theory.

Excellent topic areas for a graduate student interested in conducting research on a new theoretical perspective should be well-documented topic areas with sufficient background research that serves as a support net for the graduate student to consult. The topic area should also afford sufficient examples to buttress the theory-like knowledge structures within it. The current investigation attests to the importance of a dualistic approach – both concrete examples and theoretical generalizations are needed.

A good example of a topic area is introductory group theory. The field is deep, rich in structures, but also affords numerous examples and applications, both simple and complex. A
comparative study or a qualitative analytic study can include subtopics on sets and maps, cosets and permutations, subgroups and the Lagrange theorem, normal subgroups and isomorphisms, Sylow groups and Sylow theorems, and conclude with solvable groups. Another excellent topic area is Newtonian mechanics. It has a relatively small number of deeply meaningful and interrelated formulae such as \( W = F \cdot d \) and \( F = F \cdot a \) that lend readily to verbal, mathematical, and pictorial representations. Real life applications and examples are abundant. A short term study can focus on one subtopic such as kinematics. A long term study can broaden the scope and include both kinematics and dynamics. The current investigator reasons that for a graduate student to work on KTE, it is advisedly safer to conduct an investigation on a well-established domain with moderate theoretical difficulties and assessable applications and examples for support.

Educators can use this investigation as a starting point and test the field for possible topics where dualistic approaches similar to the ones described in this study can optimize learning for students. The current study can also be seen as a moderate solution to resolve the conflicts between the two camps in the math wars. Both concrete examples and structures are important in mathematics and science education, but they fulfill different roles in teaching. Structures are barren without examples; examples are chaotic and orderless without structures. This study proposed a new perspective and tested a combined approach with some success. Educators can do the same by reflecting on the strengths of both approaches and design a moderate approach based on the cognitive demand of the topic and the capacity of his or her students.
Future research possibilities abounds. Based on the most recent work of Wiser, Amin, Carey, Vosniadou, and diSessa, there is ample evidence for a convergence toward the center of the KE and KT perspectives. The current study is the first study to propose a centrist perspective, to design a template lesson plan, and one that tested the template lesson plan guided by the newly proposed perspective in a comparative study with some success. One area for future research is how does the innate structure within a domain affect coordination classes, namely how does the overall architecture of a domain affect the ways in which various strands of knowledge structures within a domain are connected and coordinated? Currently diSessa himself is working on coordination classes in kinematics. Two missing links in knowledge structure research include:

1. how does conceptual change originate and how does it affect coordination classes?
2. how does the innate structure of a domain affect coordination classes?

Extension of this study can contribute to knowledge of either or both “missing links”.
References


Appendix A

Pre-Test

Name:________________________________________ Section:________________

1. Five friends: Aden, Brian, Cathy, Donna, and Emily are looking for jobs. They were interviewed by five companies and were offered some positions at some of these companies. Let us list these five companies as 1\textsuperscript{st}, 2\textsuperscript{nd}, 3\textsuperscript{rd}, 4\textsuperscript{th}, 5\textsuperscript{th}. Aden got 2 offers: 3\textsuperscript{rd} and 4\textsuperscript{th} companies; Brian got only one offer: 3\textsuperscript{rd} company; Cathy got 3 offers: 1\textsuperscript{st}, 2\textsuperscript{nd}, and 5\textsuperscript{th}; Donna got 2 offers: 3\textsuperscript{rd} and 4\textsuperscript{th}; lastly Emily got 2 offers: 2\textsuperscript{nd} and 5\textsuperscript{th}.

Is there a way for the five friends to choose from their offers such that each person has one company to work for? Justify yourself.

2. You are at a party of 7 people (including yourself). Is there a way for each of the 7 people at the party to know exactly 3 other people at the party? Justify yourself.
3. Take a look at the following picture:

![Diagram](image)

This picture has four points in it: \{1,2,3,4\} and these points are connected by edges: for example, 1 and 2 are connected the horizontal edge at the top, 2 and 4 are connected the diagonal edge between them. The problem is that two of the edges crossed: 1-3 and 2-4. I can redraw this picture in such a way that it is preserved meaning I don’t add any points, I don’t take away any points, I don’t take away an existing edge between two points and I don’t add an edge which does not exist now, and in the end, I can uncross the edges 1-3 and 2-4, take a look:

![Diagram](image)
Sure, I curved the edge between 1 and 3, but this is allowed.

Take a look at the following picture.

![Graph with edges crossing]

There are six points in the picture: 1, 2, 3, 4, 5, 6. The points are linked by edges. The problem again is that some of the edges connecting these points crossed each other: look at the edge between 5 and 2 and the edge between 4 and 3, these two edges crossed each other at the center of the picture.

I would like for you to preserve this picture like I did previously: there are still six points and the edges that are there now are still there – that is you don’t take away an existing edge between two points and you don’t add an edge which does not exist now, but at the same time re-draw this picture in a way such that the edges no longer cross each other. Do you think you can uncross the edges for me? Justify yourself (meaning if it can be done, you need to tell me how, draw a picture with the edges uncrossed; if you don’t think it can be done, tell me why)

What about this picture? Justify yourself.
4. Are-They-The-Same-Picture? Justify yourself.

a.

b.

c.
d.
Appendix B

1st Interview

1. Welcome

2. There are 5 people at a party; everybody shakes hands with everybody else. How many handshakes are there?

3. There are 13 people at a party; everybody shakes hands with everybody else. How many handshakes are there?

4. Look at these figures:
Are they the same figure but drawn differently or are they really different figures? What are your reasons?
Look at these figures:

Are they the same figure but drawn differently or are they really different figures? What are your reasons?

5. So now you have examined some figures, you tell me, what makes 2 figures the same, what makes them different? Give me a list and give me the reasons for that list.

6. At a party, is it possible that there will always be at least 2 people who know the same number of people? What are the reasons for your belief?
Appendix C

2nd Interview

1. How many different (non-isomorphic) graphs are there with 3 vertices a, b, and c?

2. Are the following pairs of graphs isomorphic? If they are, give me an isomorphism; if they are not, give me reasons why not.
3. Graph Theory Patterns

a. Is the number of people at a party who do not know an odd number of other people always even?

b. At a party, will there always be at least 2 people who know the same number of other people at the party?
Appendix D

3rd Interview

1 Welcome to the last interview.

We are going to talk about only 1 thing today: trees.

2. A tree is a connected graph with NO cycles. Remember what connected means? Recall what cycles mean? So a tree is connected and it has no cycles what so ever.

Can you draw a tree for me?
Is this a tree?

- a
- b

Is this a tree?

- c

3. Trees have many critical applications. Take the trees you draw (if it is big enough), what do notice about a tree? What are some of the things you can say about a tree now that you had graph theory? Knowing all you’ve learned about graphs: vertices, edges, cycles, path, degree, etc. connectedness, non-connectedness, and other properties, can you give an alternative definition for trees or derive any additional properties of trees?
5. What is the largest possible number of vertices in a graph with 19 edges and all vertices of degree at least 3? Are you thinking concretely or theoretically for this problem?

6. Determine if the following graphs are isomorphic. If they are, give an explicit isomorphism $f$; if not, give a reason why. Be sure to indicate in your reasoning, are you thinking theoretically or concretely?

a)
7. There used to be 26 teams in the NFL with 13 teams in each of the 2 conferences. An old NFL guideline said that each team had to play 14 games per season which include 11 games against teams within its own conference and 3 games against teams in the other conference. The question is: if this schedule is possible, please give me a playing schedule; if it is not, give a reason why this is impossible. Are you thinking concretely or theoretically for this problem?
Appendix F

Graph Theory Intermediate Quiz 2

Name:__________________________________________________
Section:________________

8. Is this graph bipartite? After you answer no or yes, give a reason to defend your answer. Are you thinking concretely or theoretically for this problem?

9. How many different graphs are there with the degree sequence (2,2,2,1,1). Indicate your reason. Are you thinking concretely or theoretically for this problem? In your thinking, which part of your thinking of this problem is concrete (or theoretical)?
Appendix G

Graph Theory Post Test

Name: ________________________________ Section: __________________

#1. a) For 2 graphs to be isomorphic, they must share some common properties. List as many of these properties as you can.

b) For the above problem, did you approach it concretely or theoretically? (or both)?

#2. a) Are the following graphs isomorphic? If yes, give an explicit isomorphism; if no, give a reason.
b) For the above problem, did you approach it concretely or theoretically? (or both)?

#3. a) Are the following graphs isomorphic? If yes, give an explicit isomorphism; if no, give a reason.

b) For the above problem, did you approach it concretely or theoretically? (or both)?

#4. a) Are the following graphs isomorphic? If yes, give an explicit isomorphism; if no, give a reason.
b) For the above problem, did you approach it concretely or theoretically? (or both)?

#5. a) How many graphs are there (non-isomorphic) with the degree sequence $(3,2,2,2,1,1,1)$?
b) For the above problem, did you approach it concretely or theoretically? (or both)?

#6. a) There are seven committees at a college in charge of different school related tasks. The members of these committees are students. Each pair of committees shares precisely one student in common and each student is on two committees. How many students are serving on these seven committees?

b) For the above problem, did you approach it concretely or theoretically? (or both)?

#7. a) A Hamiltonian path in a graph is a path that visits each vertex precisely once. A Hamiltonian path does not have to visit each edge. Likewise, a Hamiltonian cycle is a cycle (meaning it loops back to the same vertex where it begins) that visits each vertex precisely once (except for the last vertex which happens to be the first vertex also). A Hamiltonian cycle does not have to visit each edge. In class, we learned bipartite graph, meaning a graph G where the vertices can be separated into 2 sets U and V such that any edge is an edge between a vertex in U and a vertex in V.

Suppose a bipartite graph G is connected and G has a Hamiltonian path, what can you say about the number of vertices in U and the number of vertices in V? Give a reason for your conclusion.
b) For the above problem, did you approach it concretely or theoretically? (or both)?

c) Suppose a bipartite graph $G$ is connected and $G$ has a Hamiltonian circuit, what can you say about the number of vertices in $U$ and the number of vertices in $V$? Give a reason for your conclusion.

d) For the above problem, did you approach it concretely or theoretically? (or both)?