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WEAK CARTELS AND COLLUSION-PROOF AUCTIONS

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Abstract. We study collusion in a large class of private-value auctions by cartels whose members cannot exchange monetary transfers among themselves (i.e., weak cartels). We provide a complete characterization of outcomes that are implementable in the presence of weak cartels, and identify optimal collusion-proof auctions for symmetric value distributions. When the density is single-peaked, the optimal collusion-proof auction can be implemented by a procedure that combines a second-price auction with a sequential one-on-one negotiation.

Keywords: Weak cartels, weakly collusion-proof auctions, optimal auctions, robustly collusion-proof auctions.

JEL-Code: D44, D82.

1. Introduction

Collusion is a pervasive problem in auctions, especially in public procurement. In 2009 the UK Office of Fair Trading (OFT) fined 103 construction firms which had been found colluding on 199 tenders between 2006 and 2009. The cartel affected construction projects worth more than 200 million pounds and including schools, universities, hospitals, and various private projects. The Dutch Construction cartel, whose revelation became a TV documentary as well as one of the biggest financial scandals in the Netherlands, allegedly

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involved 3,500 rigged bids during 1986-1998.[1] In Korea, the competition authority uncovered in 2007 a cartel in the construction of Subway Line 7 in Seoul. In this case, the largest six local companies won six different sections of the subway line construction.

The vast majority of the bid-rigging cases uncovered by competition authorities fall into the category of what McAfee and McMillan (1992) labeled weak cartels, namely cartels that do not involve exchange of side payments among cartel members. Weak cartels usually operate by designating a winning bidder and suppressing competition from other cartel members. The winning bidder is designated through "market sharing" agreements (e.g., Korea), through "bid rotation" whereby firms took turns in winning contracts (e.g., some UK cartels), or through more complicated schemes (e.g., in the Dutch case). The designated bidders place bids somewhere around the reserve price, and bids from other cartel members are either altogether suppressed (the practice of "bid suppression") or submitted at non-competitive levels (the practice of "cover bidding").

Cartels have good reasons to avoid side payments: monetary transfers leave a trail of evidence that can expose a cartel and lead to its prosecution. Compensating losing bidders in money may also lure "pretenders" who join a cartel solely to collect "the loser compensation" without ever intending to win. At the same time, it is not clear how cartel may successfully operate without exchanging side-payments among its members. If transfers are not used, compensating losing bidders entails an efficiency loss and a cartel may not work. Hence, despite the abundant empirical evidence of weak cartels, it remains unclear how weak cartels operate. We thus ask the following questions: *Can weak cartels form and operate effectively? If so, under what circumstances and what auction formats? What are their effects? How should auctions be designed to deter weak cartels?*

McAfee and McMillan (1992, henceforth MM) were the first to show that weak cartels can operate successfully, even with extreme allocative inefficiencies. They showed that in a first-price auction, symmetric bidders would benefit *ex-ante* from agreeing to randomly select a single bidder to bid the reserve price (as opposed to playing the symmetric equilibrium of the auction) whenever their value distribution has the increasing hazard rate. Further, they suggest that the optimal response by the seller is to sell the good at a fixed price. To the

[1] In 2001, a TV program, Zembla, made an investigative report on fraud inquiries in Netherlands. See Doree (2004).

extent that the increasing hazard rate is a mild condition, this theory suggests that a first-price auction is "virtually always" susceptible to a weak cartel, and that in its presence the seller can never hope to realize the efficiency gain from bidding competition. This largely negative view rests on the analysis of ex ante benefit from collusion, however. Importantly, their model does not consider bidders' (interim) incentives to participate in a cartel. Even though a cartel promises to yield strictly positive surplus to its members on average, the surplus may not accrue to all bidder types so that bidders may actually be worse off from participating in the cartel, depending on the realization of their types. In practice, the lack of interest alignment is often what causes a cartel — even well-known ones such as OPEC — to break up, so the analysis would be incomplete and potentially misleading without considering the potential conflicts of interest among cartel participants. This issue is particularly relevant for a weak cartel since the gains from a cartel manipulation cannot be redistributed via side transfers.

In the current paper, we explicitly consider the bidders' interim incentive to participate in a cartel. Doing so yields a qualitatively different result on weak cartel manipulation. In particular, our characterization of weak cartel susceptibility depends crucially on the shape of bidders' type distribution. We show that a large class of standard auctions — which we call "winner-payable" (to be explained later) — is susceptible to a weak cartel *if and only if* the auction allocates the good to a bidder with non-constant probability over an interval of types where the density of value distribution is non-decreasing (Theorem 1 and 2). This means in particular that an auction (in the winner-payable class) that allocates the good efficiently is vulnerable to cartel manipulation unless all bidders' distributions are concave. In the latter case (i.e., if the density of value distribution is always decreasing), however, a weak cartel can never be effective; in particular the seller can implement the Myerson's second-best outcome even in the presences of a weak cartel.

The intuition behind our result is explained as follows. Since members of a weak cartel can never use side payments, they can only gain from altering the allocation of the good, specifically by selecting a winner at random — i.e., by letting a randomly-selected bidder win with a low bid, with the other bidders either staying out or making non-competitive losing bids. Such a manipulation entails efficiency loss (as compared with the competitive outcome), and this loss is not borne uniformly across different types. As will be shown in our analysis, the bidder with the highest valuation is affected most adversely by this efficiency loss, and the efficiency loss is larger the more likely it is for his opponents to have low valuations. The distribution of bidders' valuations thus matters: if the density of bidder valuation is increasing, the efficiency loss from random allocation is lessened from the perspective of the high value bidder, so the overall gain from a low winning bid dominates the efficiency loss even for such a bidder. The cartel manipulation is thus profitable uniformly across all types. If the density of bidder's valuation is decreasing, however, the opposite is true, and the highest type bidder will start defecting, followed by the types just below, which makes the cartel unravel.

The complete characterization of collusion-proof auctions obtained in Theorem 1 and 2 enables us to study the normative question: *How should one design an auction in the presence of a weak cartel?* Restricting attention to winner-payable auctions, we identify the optimal collusion-proof auction for the seller when the bidder's valuation is drawn from identical distribution whose density has a single-peak (Theorem 3). In such a case, the optimal collusion-proof mechanism combines features of the Myerson's optimal auction and sequential one-on-one negotiation (Corollary 5). *The seller begins with a second-price auction with a reserve price set at the maximum between the standard optimal reserve price and the peak of the density. If no bidder bids above that reserve price (so the auction yields no sale), then the seller engages in a take-it-or-leave-it negotiation with each of the bidders sequentially in a predetermined order.* This mechanism collapses to two special forms in the case the density is everywhere decreasing and in the case it is everywhere nondecreasing. In the former case, the Myerson auction is collusion-proof and thus optimal. In the latter case, the optimal collusion-proof mechanism reduces to sequential negotiation.

This result stands in contrast to the MM's theory that the seller can do no better than posting a single price in the presence of a weak cartel. It is also interesting that the sequential negotiation treats bidders asymmetrically even though they are ex ante identical. The reason for this surprising result is that since the seller cannot discriminate across *types* of a given bidder (due to the collusion proofness requirement), she finds it optimal to discriminate across *bidders*.

Considering a bidder's incentive to participate in collusion involves a conceptual issue. A bidder's incentive to join a cartel depends on the payoff he expects to receive if he refuses to join the cartel, and that payoff in turn depends on what happens when a bidder refuses to join a cartel. In particular, how the remaining bidders update their beliefs about the refusing bidder, whether they will still form a cartel among themselves, and, if so, to what extent they can credibly punish the refusing bidder, all affect that payoff. In dealing with these issues, we initially follow the weak collusion-proofness notion of Laffont and Martimort (1997, 2000) by assuming that when a bidder refuses to participate in a cartel, the cartel collapses and the remaining bidders do not update their beliefs.

In section 5, we consider a much broader set of circumstances in terms of how a cartel is formed and operated. For instance, any informed bidder(s) as well as an uninformed mediator may propose a cartel manipulation; there can be partial or multiple cartels in operation; and participants in a cartel may punish those who have refused to participate. We show that outcomes that are weakly collusion-proof can be also implemented by the auctioneer in these environments, as long as no cartel employs strategies that are weakly dominated for themselves (Theorem 5).

The current paper is related to a number of papers on collusion in auction. Seminal contributions include Robinson (1985), Graham and Marshall (1987), von Ungern-Stenberg (1988), Mailath and Zemsky (1991), and MM, who studied whether a collusive agreement can be beneficial to its members.[2] Unlike the current paper, these papers largely focus on strong cartels, where side-payments play a crucial role for achieving efficient collusion. As mentioned above, MM does consider weak cartels and show that they involve random allocation of a good, much consistent with oft-observed practice of bid rotation.[3] As highlighted above, our approach is differentiated by its explicit consideration of the bidders' incentive for participation in the cartel. This difference explains the different results we

[2]These authors, like us, abstract from the enforcement issue — how members of a cartel may sustain collusion without a legally binding contract. Several authors study enforceability of collusion through repeated interaction (see Aoyagi (2003), Athey et al. (2004), Blume and Heidhues (2004), and Skrzypacz and Hopenhayn (2004)) or via implicit collusive strategies (see Engelbrecht-Wiggans and Kahn (2005), Brusco and Lopomo (2002), Marshall and Marx (2007, 2009), Garratt et al. (2009)). If types are distributed independently over time, repeated interaction enables members of a weak cartel to use their future market shares in a way similar to monetary transfers. If the types are persistent over time, as we envision to be more realistic, however, tampering with future market shares involves severe efficiency loss (see Athey and Bagwell (2008)). The current modeling approach is justified as long as market share cannot be adjusted frictionlessly without welfare consequences.

[3]See also Condorelli (2012). This paper analyzes the optimal allocation of a single object to a number of agents when payments made to the designer are socially wasteful and cannot be redistributed. The problem addressed is analogous to that of a cartel-mediator designing an ex-ante optimal weak cartel agreement at a standard auction with no reserve price.

obtain on the susceptibility of auctions to a weak cartel and the optimal response by the seller in its presence.

Aside from the participation incentive, our model is also more general than MM in several respects. First, we consider a more general class of auctions called "winner-payable auctions." These are the auctions in which bidders can coordinate, if they so choose, so that only one bidder can pay to win the object. Winner-payable auctions include all standard auctions such as first-price sealed-bid, second-price sealed-bid, Dutch and English auctions, or any hybrid forms, and sequential negotiation. Considering such a general class of auctions helps to isolate the features of auctions that make them vulnerable to cartels. Second, we relax the monotone hazard rate and symmetry assumptions. One may view bidder symmetry as favoring the emergence of a cartel especially when the use of side payments is limited. In practice, however, bidders are unlikely to be symmetric, so it is useful to know to what extent bidder asymmetry affects the sustainability of weak cartels.

The current paper is also related to the literature that studies collusion-proof mechanism design. This literature, pioneered by Laffont and Martimort (1997, 2000) (henceforth LM) and further generalized by Che and Kim (2006, 2009) (henceforth CK), models cartel as designing an optimal mechanism for its members (given the underlying auction mechanism they face), assuming that the members have necessary wherewithal to enforce whatever agreement they make.[4] Similar to LM (1997, 2000) and CK (2006), we explicitly consider the bidders' incentives for participating the cartel. Unlike the current paper, though, their models allow a cartel to be formed only after bidders enter into the grand auction noncooperatively. This modeling assumption, while realistic in some internal organization setting, is not applicable to auction environments where the collusion often centers around the participation into auction.

CK (2009) and Pavlov (2008) do consider collusion on participation. And they show that the second-best outcome (i.e., the Myerson (1981) benchmark) can be achieved even in the presence of a strong cartel as long as the second best involves a sufficient amount of exclusion of bidders. The mechanism that accomplishes this has features not shared by the standard auctions, however. For instance, it requires losing bidders not only to pay the winning bidders but also to incur strict loss in some states, i.e., it fails ex-post individual

[4]The likely scenario of enforcement involves the threat of retaliation through future interaction, multi-market contact, or organized crime.

rationality of the bidders. Such auctions, while theoretically interesting, are never observed in practice. By contrast, the current paper restricts attention to a more realistic, albeit broad, class of auctions rules, particularly those that ensure ex-post individual rationality. Further, the results we obtain here are more in line with the casual empiricism, namely that even weak cartels can present a serious problem for auctions. These two approaches ultimately complement each other in the sense that they clarify the features of auctions that make them vulnerable to bidder collusion.

The rest of the paper is organized as follows. Section 2 introduces a broad class of auction rules and the model of collusion. Section 3 characterizes the condition for the auction rules to be susceptible to weak cartel. Section 4 characterizes the optimal weak cartel collusion-proof auctions. Section 5 presents a more robust concept of collusion-proofness. Appendices A, B, and C (together with Supplementary Appendix) contain all the proofs not presented in the main body of the paper.

2. Model

2.1. **Environment.** A risk neutral seller has a single object for sale. The seller's valuation of the object is normalized at zero. There are $n \geq 2$ risk neutral bidders and $N := \{1, ..., n\}$ denotes the set of bidders. We assume that bidder i's private valuation of the object, v_i, is drawn from the interval $\mathcal{V}_i := [\underline{v}_i, \overline{v}_i] \subset \mathbb{R}_+$ according to a strictly increasing and continuous cumulative distribution function F_i (with density f_i). We let $\mathcal{V} := \times_{i \in N} \mathcal{V}_i$ and assume that bidders' valuations are independently distributed. When a bidder does not obtain the object, makes no payment, and receives no transfer, he earns a reservation utility normalized to zero.

The object is sold via an auction (i.e. a selling mechanism). An **auction** is defined by a triplet, $A := (\mathcal{B}, \xi, \tau)$, where $\mathcal{B} := \times_{i \in N} \mathcal{B}_i$ is a profile of message spaces (one for each bidder), $\xi : \mathcal{B} \to \mathcal{Q}$ is a rule mapping a vector of messages (typically the "bids") to a (possibly random) allocation of the object in $\mathcal{Q} := \{(x_1, ..., x_n) \in [0,1]^n | \sum_{i \in N} x_i \leq 1\}$, and $\tau : \mathcal{B} \to \mathbb{R}^n$ is a rule determining expected payments as a function of the messages. We assume that the seller cannot force bidders to participate in the auction. Therefore, for each bidder, we require that the message space $\mathcal{B}_i$ includes a non-participation option, b_i^0, the exercise of which results in no winning and no payment for bidder i, $\xi_i(b_i^0, \cdot) = \tau_i(b_i^0, \cdot) = 0$.

Whether and how a cartel can operate in an auction depends crucially on the fine details of its allocation and payment rule. CK (2009) show that if the seller faces no constraints in designing the auction, any outcome that involves sufficient exclusion can be implemented even in the presence of a cartel that can use side payment and reallocate objects among its members. By effectively "selling the project" to the cartel at a fixed price, the seller removes any scope for a manipulation of the bids by the cartel. However, optimal auction of this type may require payments from losing bidders and therefore is rarely observed in practice.

Standard auctions do not often collect payments from losing bidders, so they are potentially susceptible to bidder collusion in a way not recognized by CK (2009). In the current paper, we focus on these more realistic auction formats. Specifically, we restrict attention to a set $\mathcal{A}^*$ of auction rules that are winner-payable in the following sense.

Definition 1. *An auction A is* ***winner-payable*** *if, for all $i \in N$, there exist message vectors $\underline{b}^i, \overline{b}^i \in \mathcal{B}$ such that $\xi_i(\underline{b}^i) = \xi_i(\overline{b}^i) = 1$, $\tau_j(\underline{b}^i) = \tau_j(\overline{b}^i) = 0$ for all $j \neq i$, and*

$$\tau_i(\underline{b}^i) \leq \frac{\tau_i(b)}{\xi_i(b)} \leq \tau_i(\overline{b}^i) \quad \textit{for all } b \in \mathcal{B} \textit{ such that } \xi_i(b) > 0 \textit{ and } \frac{\tau_i(b)}{\xi_i(b)} \leq \overline{v}_i.$$

In words, an auction is winner-payable if, for each bidder i, there exist two profiles of bids, $\underline{b}^i$ and $\overline{b}_i$, which both result in all bidders except for bidder i paying nothing and bidder i winning the object for sure, at a lowest per-unit price for the object (in the former) and at a highest per-unit price (in the latter) allowed by the auction rule, respectively. One can see that most of commonly observed auctions are winner-payable.[5]

- **First-Price (or Dutch) Auctions with Reserve Price**: winner-payability holds because each bidder can obtain the object for sure at any positive price above the reserve price, if he places a bid at that price and all the other bidders place lower bids or do not participate in the auction.

[5]Lotteries represent a notable exception. For instance, consider a mechanism where there is a fixed number of lottery tickets, each bidder can buy a single ticket at a fixed price, the auctioneer retains the unsold tickets, and the object is assigned to the holder of a randomly selected ticket. In this mechanism $\mathcal{B}_i := \{0, 1\}$, $\xi_i(0, b_{-i}) = \tau_i(0, b_{-i}) = 0$, $\xi_i(1, b_{-i}) = 1/n$, $\tau_i(1, b_{-i}) = p$ for some $p \in \mathbb{R}$. Winner-payability fails as there is no message profile that can guarantee the object to any of the players. On the other hand, fixed-prize raffles (see Morgan (2000)) are winner-payable.

- **Second-Price (or English) Auctions with Reserve Price**: winner-payability holds because each bidder i can be guaranteed to win the good at any price above the reserve price, if another bidder bids exactly that price, i bids anything above that price, and all other bidders bid strictly lower or do not participate.
- **Sequential Take-It-or-Leave-It Offers**: Suppose the seller approaches the buyers in a given exogenous order and makes to each of them a single take-or-leave-it offer. This format is winner-payable because each bidder can win the object for sure if all other prior bidders reject their offers.[6]

More generally, winner-payability is implied by the requirement that only the winner of the auction pays for the object, if the auction is deterministic (i.e. for each profile of bids the object is assigned with probability one to only one of the bidders, whenever it is assigned), or randomization is limited to tie-breaking and it occurs with zero probability in (collusion-free) equilibrium.

2.2. **Characterization of Collusion-Free Outcomes.** An auction rule A in $\mathcal{A}^*$ induces a game of incomplete information where all bidders simultaneously submit messages (i.e. bids) to the seller. A pure strategy for player i is $\beta_i : \mathcal{V}_i \to \mathcal{B}_i$, and $\beta = (\beta_1, \cdots, \beta_n)$ denotes its profile.

Given a profile of equilibrium bidding strategies β^* of an auction, its **outcome** corresponds to a **direct mechanism** $M_A \equiv (q, t) : \mathcal{V} \to \mathcal{Q} \times \mathbb{R}^n$, where for all $v \in \mathcal{V}$, $q(v) = \xi(\beta^*(v))$ is the allocation rule for the object and $t(v) = \tau(\beta^*(v))$ is the payment rule. Given M_A, we define the interim winning probability $Q_i(v_i) = \mathbb{E}_{v_{-i}}[q_i(v_i, v_{-i})]$ and interim payment $T_i(v_i) = \mathbb{E}_{v_{-i}}[t_i(v_i, v_{-i})]$ for bidder $i \in N$ with type $v_i \in \mathcal{V}_i$. We will refer to the mapping $Q = (Q_i)_{i \in N}$ and $T = (T_i)_{i \in N}$ as interim allocation and transfer rules,

[6]More precisely, suppose the seller approaches the buyers in the order of the bidder index, say, and makes a take-it-or-leave-it offer of p_i for bidder i in his turn (i.e., when all bidders before i have rejected the seller's offers). A bid profile $b = (b_1, ..., b_n)$ in this rule may represent the highest offers bidders are willing to accept. Given this interpretation, $\xi_i(b)$ represents the probability of the event that bidder i is approached by the seller and accepts her offer of p_i, so $\xi_i(b) > 0$ means that $b_i > p_i$. Further, conditional on that event, bidder i pays p_i, so $\tau_i(b)/\xi_i(b) = p_i$ whenever $\xi_i(b) > 0$. In this case, $\underline{b}^i = \overline{b}^i$ can be set so that for $j \neq i$, $\underline{b}^i_j = \overline{b}^i_j = 0$ (so these bidders always reject the seller's offers), and $\underline{b}^i_i = \overline{b}^i_i = b_i$ (i.e, the same as the original bid for bidder i). Then, $\xi_i(\underline{b}^i) = \xi(\overline{b}^i) = 1$, and $\tau(\underline{b}^i) = \tau(\overline{b}^i) = p_i = \tau_i(b)/\xi_i(b)$.

respectively. The equilibrium payoff of player i with value v_i is then expressed as

$$U_i^{M_A}(v_i) := Q_i(v_i)v_i - T_i(v_i).$$

Any equilibrium outcome M_A must be **incentive compatible** (by definition of equilibrium) and **individually rational** (because bidders are offered the non-participation option). That is, for all $i \in N$ and $v_i \in \mathcal{V}_i$:

$$(IC) \qquad U_i^{M_A}(v_i) \geq v_i Q_i(\tilde{v}_i) - T_i(\tilde{v}_i), \text{ for all } \tilde{v}_i \in \mathcal{V}_i,$$

$$(IR) \qquad U_i^{M_A}(v_i) \geq 0$$

In the following Lemma, we characterize the set of interim allocation and transfer rules that can arise as an equilibrium outcome of some auction rule.

Lemma 1. *A profile (Q, T) is the interim allocation and transfer rules of an equilibrium outcome of an auction rule $A \in \mathcal{A}^*$ if and only if the following conditions hold:*

$$(M) \qquad Q_i \textit{ is nondecreasing, } \forall i \in N;$$

$$(Env) \qquad T_i(v_i) = v_i Q_i(v_i) - \int_{\underline{v}_i}^{v_i} Q_i(s)ds + T(\underline{v}_i) - \underline{v}_i Q_i(\underline{v}_i), \forall v_i \in \mathcal{V}_i, \forall i \in N;$$

$$(IR') \qquad U_i^{M_A}(\underline{v}_i) = \underline{v}_i Q_i(\underline{v}_i) - T(\underline{v}_i) \geq 0, \forall i \in N;$$

$$(B) \qquad \sum_{i \in N} \int_{v_i}^{\overline{v}_i} Q_i(s) dF_i(s) \leq 1 - \prod_{i \in N} F_i(v_i), \forall v \in \mathcal{V}.$$

It is well known that the conditions (M), (Env), and (IR') are necessary and sufficient for any interim rule (Q, T) to satisfy (IC) and (IR). The last condition (B), which we shall refer to as **capacity constraint**, captures the feasibility of an interim allocation rule; more precisely, the condition is necessary and sufficient for there to be an ex-post allocation rule, $q = (q_1, ..., q_n) : \mathcal{V} \to \mathcal{Q}$, that gives rise to Q as an associated interim allocation rule.[7] Combining these observations, the necessity of conditions $(M) - (B)$ is immediate. The sufficiency can be argued as follows: For any Q satisfying (B), there exists an (ex-post) allocation rule q that gives rise to Q as an associated interim allocation. We can then set

[7]See Mierendorff (2011) or Che et al. (2013). This characterization generalizes Border (1991)'s characterization derived in the symmetric bidder case to the asymmetric case.

$t_i(\theta) = \frac{T_i(\theta_i)}{Q_i(\theta_i)}$ if $Q_i(\theta_i) > 0$ and $t_i(\theta) = 0$ otherwise, to obtain the desired auction rule in Lemma 1.

2.3. **A Model of Collusion.** Our model of collusion does not involve exchanges of side-payments among cartel members. Instead, the members of a cartel can only collude by coordinating their messages in the auction. Since a non-participation message is included in the auction rule, bidders are also able to coordinate their participation decisions. As mentioned in the introduction, we abstract from the question of how a cartel can enforce an agreement among its members, but rather focus on whether there will be an incentive compatible agreement that is beneficial for all bidders.[8]

To this end, we consider an agreement by which the bidders may coordinate their bids. Formally, a **cartel agreement** is a mapping $\alpha : \mathcal{V} \to \Delta(\mathcal{B})$ that specifies a lottery over possible bid profiles in auction A for each profile of valuations for the bidders. We envision bidders in the cartel to commit to submitting their private information to the cartel (e.g., an uninformed mediator) and bidding according to its subsequent recommendation. To be precise, a cartel agreement, if unanimously accepted, leads bidders to play a game of incomplete information where each player's strategy is to report his type to the cartel and then outcomes are determined by the lottery α over bids and auction rule A. Hence, for any cartel agreement α, one can equivalently consider a direct mechanism it induces as follows.

Definition 2. *A direct mechanism $\tilde{M}_A = (\tilde{q}, \tilde{t})$ is a* ***cartel manipulation*** *of A if there exists a cartel agreement α such that*[9]

$$\tilde{q}_i(v) = \mathbb{E}_{\alpha(v)}[\xi_i(b)] \quad \text{and} \quad \tilde{t}_i(v) = \mathbb{E}_{\alpha(v)}[\tau_i(b)], \forall v \in \mathcal{V}, i \in N.$$

Since $\tilde{M}_A$ results from bidders' equilibrium play in the incomplete information game described above, it is without loss to require that $\tilde{M}_A$ be incentive compatible, i.e. satisfy (IC). Our goal is to investigate whether any auction $A \in \mathcal{A}^*$ is susceptible to some cartel manipulation $\tilde{M}_A$. To analyze this, one must know what sort of cartel manipulation will be accepted by the bidders; and this in turn requires one to analyze what happens when a bidder refuses a proposed manipulation. The latter in turn depends on the beliefs formed on

[8]This is consistent with MM and LM and most of the literature on the auction collusion.

[9]Here, $\mathbb{E}_{\alpha(v)}[\cdot]$ denotes the expectation taken by using the probability distribution $\alpha(v) \in \Delta(\mathcal{B})$.

the bidder who refuses a proposed manipulation and the abilities of the remaining bidders to punish such a bidder.

To address these issues, we initially follow the weak notion of collusion, originally developed by LM. According to this notion, a cartel manipulation takes effect when all bidders accept it, and if at least one bidder refuses a proposed manipulation, this does not trigger any revision of beliefs on the subsequent play; that is, all bidders play the auction game A non-cooperatively with their prior beliefs. Since the latter play yields the (interim) payoff of $U_i^{M_A}(v_i)$ to a bidder i with valuation v_i, for a cartel manipulation $\tilde{M}_A$ to be accepted unanimously, we must have $U_i^{\tilde{M}_A}(v_i) \geq U_i^{M_A}(v_i), \forall v_i, i$. For the manipulation to be strictly profitable, this inequality must hold strictly for some bidder type(s). Hence, the weak notion is stated as follows:

Definition 3. *Given an auction A, its collusion-free equilibrium outcome M_A is **weakly collusion-proof (or WCP)** if there exists* no *cartel manipulation $\tilde{M}_A$ of A satisfying (IC) and*

$$(C-IR) \qquad U_i^{\tilde{M}_A}(v_i) \geq U_i^{M_A}(v_i), \forall v_i, i, \text{ with strict inequality for some } v_i, i.$$

According to this definition, an auction is susceptible to bidder collusion if and only if there exists a cartel manipulation that interim Pareto dominates its collusion-free outcome.[10] This notion of collusion-proofness should be interpreted as a *necessary* requirement for an auction rule to be unsusceptible to cartel manipulation. If an auction rule fails to be weakly collusion-proof, then one should expect weak cartels to be a concern. However, it could be argued that weak cartels may still form, even when an auction is weakly collusion-proof. For example, there could be cartels that may benefit only a subset of bidders perhaps for some types, possibly at the expense of the other bidders. We address this issue in Section 5, by showing that the notion of collusion-proofness can be strengthened significantly without altering the outcome that a seller can obtain from the auction.

[10]As Holmstrom and Myerson (1983) argue, if an equilibrium outcome of an auction mechanism is interim Pareto dominated by another outcome resulting from a cartel manipulation, then there is unanimous agreement, without any communication taking place among bidders, that everyone will be better off by joining the cartel.

3. When Are Auctions Susceptible to Weak Cartels?

In this section, we study the conditions that make an auction in the class $\mathcal{A}^*$ weakly collusion-proof. First, we provide a necessary condition for an auction to be weakly collusion-proof (Theorem 1). Next we prove that this condition is also sufficient, when we consider auctions that satisfy two further natural requirements (Theorem 2).

To state the necessity result, let us define the **reserve price** faced by bidder i as:

$$r_i := \inf\{\frac{T_i(v_i)}{Q_i(v_i)} : Q_i(v_i) > 0\}. \tag{1}$$

It is straightforward to see that $Q_i(v_i) = 0$ if $v_i < r_i$ and $Q_i(v_i) > 0$ only if $v_i \geq r_i$.[11]

Theorem 1. *Suppose that an equilibrium outcome M_A of an auction rule $A \in \mathcal{A}^*$ is weakly collusion-proof. Then, each interim allocation Q_i must be constant in any interval $(a, b) \subset (r_i, \overline{v}_i]$ on which f_i is nondecreasing.*

Proof. See Appendix A (page 26). ▌

This result implies that if f_i is nondecreasing in an interval of types for bidder i, there is a scope for a profitable cartel manipulation unless the bidder's winning probability is constant in his value over that interval. To see the logic behind this result, suppose that in (collusion-free) equilibrium, bidder i's winning probability $Q_i(v_i)$ is strictly increasing in a certain region $[a, b]$ where f_i is nondecreasing (or F_i is convex). Then, one can construct a cartel manipulation, labeled $\tilde{M}_A$, that: (i) leaves unchanged the interim probability of winning and expected payments of all bidders other than bidder i and also of bidder i when his value is outside $[a, b]$ and (ii) gives the good to bidder i with a constant probability $\bar{p}$ if his value is inside $[a, b]$ where

$$\bar{p} = \frac{\int_a^b Q_i(s)dF_i(s)}{F_i(b) - F_i(a)}, \tag{2}$$

that is, $\bar{p}$ is set equal to bidder i's average winning probability over the interval $[a, b]$ in M_A.

[11] To see this, suppose that $Q_i(v_i) > 0$ for some $v_i < r_i$. Then, by definition of r_i, we have $v_i < r_i \leq \frac{T_i(v_i)}{Q_i(v_i)}$ or $v_iQ_i(v_i) - T_i(v_i) < 0$, contradicting the (IR) condition.

Our proof in Appendix A shows that (i) this cartel manipulation can be implemented by the bidders in auction A; and (ii) it is acceptable to all bidders in the sense of satisfying $(C-IR)$, thus making the auction not weakly collusion-proof.

To see (ii) (assuming that (i) is true), recall that the payoffs of all other bidders remain unchanged, and observe how bidder i's payoff is changed by the manipulation. Absent collusion, we know that that bidder i will earn the interim payoff of

$$U_i^{M_A}(v_i) = U_i^{M_A}(a) + \int_a^{v_i} Q_i(s)ds, \tag{3}$$

when his valuation is $v_i \in [a,b]$. Under the manipulation $\tilde{M}_A$, bidder i with valuation $v_i \in [a,b]$ will earn

$$U_i^{\tilde{M}_A}(v_i) = U_i^{M_A}(a) + \int_a^{v_i} \bar{p}ds = U_i^{M_A}(a) + \frac{v_i - a}{F_i(b) - F_i(a)} \int_a^b Q_i(s)dF_i(s), \tag{4}$$

where the last equality follows from (2). Note that the payoff in each of two cases rises at a speed equal to the winning probability. Since Q_i rises strictly whereas $\bar{p}$ is constant, the payoff without manipulation is strictly convex whereas the payoff under manipulation rises linearly, as depicted by Figure 1. Essentially, the manipulation speeds up the rate of payoff increase for lower value and slows down the rate for the higher value. Since F_i is convex in $[a,b]$ and since Q_i is strictly increasing,

$$\int_a^b Q_i(s)dF_i(s) \geq \frac{F_i(b) - F_i(a)}{b-a} \int_a^b Q_i(s)ds. \tag{5}$$

Substituting (5) into (4) for $v_i = b$, we get

$$U_i^{\tilde{M}_A}(b) \geq U_i^{M_A}(a) + \int_a^b Q_i(s)ds = U_i^{M_A}(b). \tag{6}$$

In other words, bidder i with valuation $v_i = b$ will be at least weakly better off from the manipulation. Given the curvatures of these two payoff functions, then the bidder will be strictly better off from the manipulation for any intermediate value $v_i \in (a,b)$ (see the left panel of Figure 1).

The same argument explains why this manipulation may not work if f_i is decreasing (or equivalently, F_i is strictly concave). In this case, the inequality of (5) is reversed. Hence, as shown in the right panel of Figure 1, bidder i will be strictly worse off from the manipulation when his valuation is $v_i \approx b$. This means that the weak cartel will not be able to induce bidders with sufficiently high valuations to join the collusive agreement since it requires

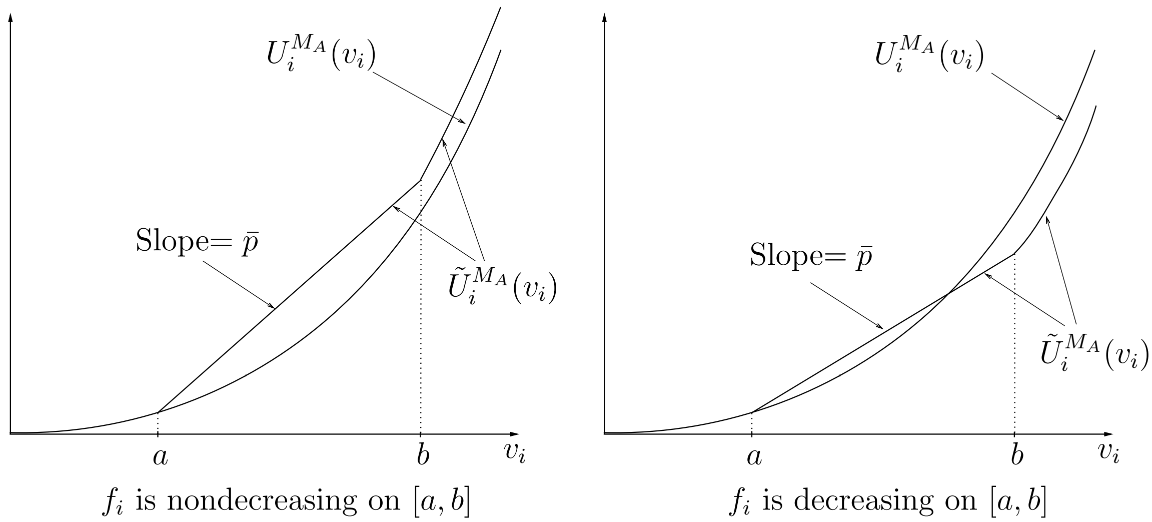

FIGURE 1. Profitability of Manipulation

them to sacrifice the probability of obtaining the object to such an extent that cannot be compensated by a lower expected payment. Although we do not formally model this, there is a sense in which the cartel unravels in this case. With the highest types dropping out, the expected payoff from manipulation falls (the straight line in the right panel rotates down), and given strict concavity of the value distributions, the next highest types also become worse off, and drop out. This process continues until the cartel unravels.

To complete our argument, we need to verify (i) above, i.e. the issue of how to implement the desired manipulation. In fact, pooling the types of bidder i in $[a, b]$ requires shifting the winning probability away from high types toward low value types of bidder i, and it is not clear whether and how such a shifting of the winning probabilities can be engineered to occur in equilibrium, especially without altering the payoffs of the other bidders.

As a first step, we observe that the interim allocation from $\tilde{M}_A$, $\tilde{Q}$, satisfies the condition (B), so it is feasible in the sense that there is an ex-post allocation rule $\tilde{q}$ that gives rise to $\tilde{Q}$ as the associated interim allocation rule. The tricky part is how to replicate the interim transfer $\tilde{T}$, which makes $\tilde{M}_A$ incentive compatible, along with the above allocation $\tilde{q}$ via a weak cartel manipulation (that does not use any side payments among the cartel members). The winner-payability plays a role here. Since the cartel can employ a randomizing device, the winner-payability allows the cartel to generate, for each profile of reported values, a distribution of bids that produces $\tilde{q}$ and $\tilde{T}$ (in expectation) for the proposed manipulation.[12]

Theorem 1 suggests that a winner-payable auction which assigns the object with higher probability to bidders with higher values is vulnerable to weak cartels unless each bidder's value distribution is strictly concave everywhere. The following three corollaries state (under certain technical qualifications) that (i) standard auctions, (ii) seller's optimal auctions (i.e. those which implement Myerson's optimal auction), and (iii) efficient auctions are all susceptible to weak cartels unless all distributions of values are strictly decreasing.

Corollary 1. *Letting* $\overline{v} := \min_{i\in N} \overline{v}_i$ *and* $\underline{v} := \max_{i\in N} \underline{v}_i$*, assume that* $\overline{v} > \underline{v}$*. Then, the collusion-free equilibrium outcomes (in weakly undominated strategies) of first-price, second-price, English, or Dutch auctions, with a reserve price* $r < \overline{v}$*, are not WCP if* $f_i(v_i)$ *is nondecreasing in* v_i *for* $v_i \in (a, b) \subset \mathcal{V}_i$*, for some* $b > r$ *and* $a \geq \underline{v}$*, for some bidder* i*.*

Proof. See Appendix A (page 29). ∎

Corollary 2. *Suppose that the virtual valuation,* $J_i(v_i) := v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$*, is strictly increasing in* v_i *for all* $i \in N$*. Suppose also that* $f_i(v_i)$ *is nondecreasing* v_i *for* $v_i \in (a, b) \subset (r_i, \overline{v}_i]$*,* $J_i(b) > 0$*, and* $\max_{j\neq i} J_j(\underline{v}_j) < J_i(b) < \max_{j\neq i} J_j(\overline{v}_j)$*, for some bidder* i*. Then, all auction rules in* $\mathcal{A}^*$ *that maximize the seller's revenue are not WCP.*

Proof. The hypotheses guarantee that there exists an interval $[b - \epsilon, b]$ with $\epsilon > 0$, where $Q_i(v_i)$ is strictly increasing in the optimal auction. The result follows from Theorem 1. ∎

Corollary 3. *Suppose that* f_i *is nondecreasing on some interval* $(a, b) \in (r_i, \overline{v}_i]$ *and* $\max_{j\neq i} \underline{v}_j < b < \max_{j\neq i} \overline{v}_j$*, for some bidder* i*. Then, all auction rules in* $\mathcal{A}^*$ *whose equilibrium outcomes are efficient are not WCP.*

Proof. The hypotheses guarantee that there exists an interval $[b - \epsilon, b]$ with $\epsilon > 0$, where $Q_i(v_i)$ is strictly increasing in any efficient auction. The result follows from Theorem 1. ∎

The next result establishes a converse of Theorem 1: a sufficient condition for an auction rule to be weakly collusion-proof. The sufficiency requires two further conditions, which roughly speaking provide minimal optimality requirements from the seller's perspective.

[12]Winner-payability is sufficient for the cartel to essentially attain any incentive compatible allocation for values above reserve prices. Therefore, focusing on set of auctions that allows for reserve prices but is larger than $\mathcal{A}^*$ would not make collusion any easier for the cartel.

THEOREM 2. *Suppose that an auction rule $A \in \mathcal{A}^*$ satisfies $\tau_i(b) \geq \xi_i(b)\underline{v}_i, \forall b \in \mathcal{B}, \forall i$ and that its collusion free outcome $M_A = (Q, T)$ satisfies condition (B) with equality at $r_1, \ldots, r_n$. If, for each $i \in N$, Q_i is constant wherever f_i is nondecreasing in some interval of $(r_i, \overline{v}_i]$, then the chosen equilibrium outcome is weakly collusion-proof.*

Proof. See Appendix A (page 29). ▌

The two conditions rule out auctions that are clearly undesirable from the seller's perspective in the sense that either it leaves the object unsold even though selling raises her revenue without altering incentives, or it sells the object to a bidder at a price below his lowest possible value.[13] Given these additional optimality conditions, the intuition behind this result is essentially the flip-side of the intuition behind Theorem 1. In other words, starting from the suggested equilibrium, any manipulation including, but not limited to, those that involve pooling of some types, must leave some bidder types strictly worse off.

Theorem 2 has the following immediate corollary, which collects in a single statement the natural counterparts to the three previous corollaries to Theorem 1.

COROLLARY 4. *If f_i is (strictly) decreasing for all $i \in N$, then the following auctions are WCP: (i) the collusion-free equilibrium equilibria of first-price, second-price, English, or Dutch auctions, with reserve price $r \geq \max_{i \in N} \underline{v}_i$ (ii) any equilibrium of any auction $\tau_i(b) \geq \xi_i(b)\underline{v}_i, \forall b \in \mathcal{B}, \forall i$ that results in an efficient allocation, and (iii) any equilibrium of any auction that maximizes the seller's revenue.*

Proof. The proof is immediate given Theorem 2 and the fact that f_i is (strictly) decreasing for all $i \in N$. ▌

Our characterization of collusion-proof auctions in Theorem 1 and 2 contrasts with that of MM, who assume the cartel can successfully form if bidders benefit ex-ante from collusion. In the symmetric environment, they show that if the hazard rate of value distribution is increasing, then a cartel will always form (all bidders will submit a bid equal to the reserve price to randomly allocate the object among them). Our results highlight that

[13] It is easy to see that without a sufficiently high reserve price when the lowest possible valuation is sufficiently high, all standard auctions are susceptible to weak cartels. Consider, for instance, a first-price auction in which the lower bound of value support, $\underline{v}$, is very high while there is no reserve price. Then, bidders will find it Pareto-improving to identically bid zero and share the object with equal probability.

ignoring the bidders' interim incentives to participate in the cartel overstates their ability to collude, to the extent that the increasing hazard rate condition is quite mild. This has an important implication on the design of the optimal collusion-proof auction. In MM, the only instrument available for the seller to cope with weak cartels is the choice of reserve price whereas in our case the problem of designing the optimal collusion-proof auction becomes nontrivial, as we show in the next section.

4. Optimal Collusion-Proof Auctions

If an auction is not weakly collusion-proof, then bidders will be able to coordinate their bidding strategies to achieve a different outcome which will make everyone better off. Therefore, if the seller has designed an auction to maximize revenue without taking into account the possibility of collusion, and the auction is not collusion-proof, then collusion will lead to lower expected revenue.

Corollary 2 shows that in a wide range of circumstances, the seller's optimal auction will not be weakly collusion-proof. Then, what is the best outcome the seller can achieve in a collusion-proof way? In this section, we look for an auction that maximizes the seller's revenue among all collusion-proof auctions. Consistent with our general approach to the problem, we require the seller to employ a winner-payable auction.

Using Lemma 1 and the necessary condition given in Theorem 1, we can write the seller's maximization problem as follows:[14]

$$[P] \qquad \max_{(Q_i)_{i\in N}} \sum_{i\in N} \int_{r_i}^{\overline{v}_i} J_i(s) Q_i(s) dF_i(s),$$

subject to (M), (B), and the collusion-proof constraint for all $i \in N$

$$(CP) \qquad Q_i \text{ is constant whenever } f_i \text{ is nondecreasing in some interval of } (r_i, \overline{v}_i].$$

The objective function is the (well-known) expression of the seller's expected revenue that is obtained by substituting condition (Env) into the original objective function for the seller. We have also used the fact that (IR') is binding at the optimum for the lowest types, i.e. for all $i \in N$, $T(\underline{v}_i) = \underline{v}_i Q_i(\underline{v}_i)$. The constraint (M) is required for the incentive compatibility. Lastly, the constraint (CP) arises from the weak collusion-proofness, namely the characterization given by Theorems 1 and 2.

[14]Recall that $J_i(s) = s - \frac{1-F_i(s)}{f_i(s)}$ is the virtual valuation function for bidder i with value s.

Unfortunately, the problem $[P]$ is not tractable to solve in full generality. Hence, we focus on a restricted class of value distributions where either bidders are symmetric with a single-peaked density or all bidders have nondecreasing (and possibly asymmetric) densities.

First, consider symmetric bidders with a single-peaked density. Specifically, it is assumed that, for each $i \in N$, we have $F_i = F$ for some distribution F with continuous density f, which is **single-peaked** in the sense that there is a value $v^* \in \mathcal{V} = [\underline{v}, \overline{v}]$ such that $f(v)$ is (weakly) increasing in v for $v < v^*$ and (weakly) decreasing in v for $v > v^*$. It is possible that $v^* = \underline{v}$ or $v^* = \overline{v}$, so single-peakedness include the cases in which f is nondecreasing or nonincreasing everywhere. Indeed, the condition is satisfied by many well known distributions, including Cauchy, Exponential, Logistic, Normal, Uniform, and Weibull. Further, we assume that the virtual valuation $J(\cdot)$ is nondecreasing. Finally, we only consider the case in which $v^* > \inf\{v \in [\underline{v}, \overline{v}] | J(v) \geq 0\}$, because otherwise Myerson's optimal auction is weakly collusion-proof according to Theorem 2 and is trivially optimal. The following theorem characterizes the optimal collusion-proof auction in $\mathcal{A}^*$.

Theorem 3. *Suppose that $F_i = F, \forall i \in N$ and the density f is single-peaked. Suppose also that $v^* > \hat{v} := \inf\{v \in [\underline{v}, \overline{v}] | J(v) \geq 0\}$. Then, the solution of $[P]$ is given by:*

$$Q_i(v) = \begin{cases} F(v)^{n-1} & \text{if } v > v^* \\ F(v^*)^{n-i} \prod_{k=1}^{i-1} F(r_k) & \text{if } v \in [r_i, v^*] \\ 0 & \text{otherwise} \end{cases}. \tag{7}$$

where $\hat{v} = r_n < r_{n-1} < \cdots < r_1 < v^*$ and

$$(r_1, \cdots, r_n) = \arg \max_{(\tilde{r}_1, \cdots, \tilde{r}_n)} \sum_{i \in N} F(v^*)^{n-i} \prod_{k=1}^{i-1} F(\tilde{r}_k) \int_{\tilde{r}_i}^{v^*} J(v) f(v) dv. \tag{8}$$

Proof. See Appendix B (page 32). ▮

The optimal auction allocates the object as follows: If there is at least one bidder with valuation above v^*, then the optimal auction allocates the object to the bidder with the highest valuation. If every bidder's valuation is below v^*, then the seller allocates the object to bidder 1 if his valuation is in $[r_1, v^*]$, to bidder 2 if $v_1 < r_1$ and $v_2 \in [r_2, v^*]$, and so on, where $r_1 > r_2 > \cdots > r_n = \hat{v}$. The seller does not allocate the object to any bidder if $v_i < r_i, \forall i$. Payments are determined to make the allocation rule incentive compatible, i.e., according to (Env).

The optimal weakly collusion-proof auction can be implemented via a (winner-payable) procedure that combines a second-price auction with individualized reserve prices and a sequential one-on-one negotiation (in case the auction does not sell the object), described more precisely as follows:

Corollary 5. *Let* $(r_1, \cdots, r_n)$ *be those defined in* (8), *and let* $(R_1, \cdots, R_n)$ *satisfy*

$$(v^* - R_i)F(v^*)^{n-1} = (v^* - r_i)F(v^*)^{n-i} \prod_{k=1}^{i-1} F(r_k). \tag{9}$$

The optimal weakly collusion-proof auction can be implemented as a Perfect Bayesian equilibrium of the following procedure. First, the seller holds a second-price (or English) auction with individualized minimum prices $(R_1, \cdots, R_n)$. *If the auction does not sell the object, then the seller approaches the bidders in the order of the bidder index, and makes take-it-or-leave offer of* r_i *to bidder* i, *in case each bidder* $j < i$ *has rejected the seller's offer of* r_j.

Proof. The left and right hand sides of (9) correspond to the payoffs that bidder i with value v^* can obtain in the second-price auction and in the sequential take-it-or-leave-it offers, respectively, provided that all other bidder j bids his value in the second-price auction if $v_j \geq v^*$ while bidding zero to accept the offer r_j afterwards if $v_j \in [r_j, v^*)$. Since bidder i with value v^* is indifferent between two payoffs, it is straightforward to see that absent collusion, the optimal strategy for bidder i is also to bid his value in the second-price auction if $v_i \geq v^*$, and to bid zero and accept the offer r_i afterwards if $v_i \in [r_i, v^*)$. Clearly, this auction rule and its equilibrium outcome satisfy the sufficient condition of Theorem 2 and is thus collusion-proof. ∎

Observe that our optimal collusion-proof auction allocates the object less frequently compared to optimal mechanism in the collusion-free environment, since each bidder $i (\neq n)$ with valuation $v_i \in [\hat{v}, r_i)$ is allocated the object in the latter, but not in our environment. This feature also appears in the optimal (collusion-proof) auction rule of MM. A novel and interesting feature of our optimal auction rule is that it treats bidders asymmetrically even though they are ex-ante symmetric. Because the common virtual valuation is increasing, the seller maximizes revenue by assigning the object efficiently, whenever bidders have $v > v^*$, and this is also collusion-proof in light of Theorem 2, since the density is decreasing in this region. However, for values below v^* our collusion-proofness constraint is binding and forces the seller to provide the good to each bidder with a constant probability *whenever the probability is positive.* This means that, for values below v^*, the seller is unable to discriminate each bidder based on his valuations. The seller can only exclude allocation to all values below a certain threshold, determined by a reserve price, and assign the good to all the remaining types between the reserve price and v^* with the same probability. Therefore, the only remaining way for the seller to price-discriminate across different valuations is by discriminating *across* ex-ante identical bidders. The seller accomplishes this by charging a higher price to a bidder she approaches earlier (who thus enjoys a higher probability of obtaining the good) than those she approaches later. This feature of the mechanism shows that the asymmetric mechanism can be a useful device by which the seller can increase the revenue against collusive bidders, when she is restricted to use winner-payable auctions.

Remark 1. *If, for some exogenous reason, the seller is not allowed to use an asymmetric mechanism, the optimal winner payable auction will still be a sequential procedure. It will combine an efficient assignment for values above v^*, with a subsequent simultaneous take-it-or-leave-it offer to all bidders (with a fair lottery in case of excess demand), if the object remains unsold in the auction.*

When bidders are asymmetric, computing the optimal auction in full generality becomes intractable. However, building on the insight we developed earlier, we can find the revenue maximizing mechanism when the density f_i is nondecreasing for all $i \in N$. In this case, Theorems 1 and 2 require the interim winning probability for each bidder i to be constant in the range $[r_i, \overline{v}_i]$ and to be equal to zero in $[\underline{v}_i, r_i)$. Hence, the seller's problem reduces to choosing $r_1, \ldots, r_n$ optimally, taking into account that the n constant interim winning probabilities are pinned down by binding capacity constraint (B). This problem can be reformulated as follows:

Theorem 4. *Suppose that all f_i's are nondecreasing. Then, the program $[P]$ amounts to solving*

$$\max_{\pi \in \Pi} \Big[\max_{(r_i)_{i \in N}} \sum_{i \in N} \Big(\prod_{j : \pi(j) < \pi(i)} F_j(r_j) \Big) \left(1 - F_i(r_i)\right) r_i \Big], \tag{10}$$

where Π is the set of all permutation functions $\pi : N \to N$. At the solution of (10), $r_i \geq r_j$ if $\pi(i) < \pi(j)$.

Proof. See Appendix B (page 38). ▮

This result says that an algorithm for finding the optimal mechanism is first to find an optimal profile $(r_i)_{i\in N}$ for each possible order π, and then to choose the order π that achieves the highest revenue. In terms of practical implementation, π represents the order in which the seller makes a set of sequential take-it-or-leave-it offers $(r_i)_{i\in N}$. Given an order π, the seller makes a take-it-or-leave-it offer r_i to bidder i if all bidders approached earlier have rejected the seller's offers, which occurs with probability $\prod_{j:\pi(j)<\pi(i)} F_j(r_j)$. The result also says that the seller offers a (weakly) higher price to whomever is approached earlier.

5. Strengthening the Notion of Collusion-Proofness

The weak notion of collusion-proofness presumes that a cartel will form if, and only if, all bidders benefit at least weakly from coordinating their bids. This provides a conservative test on the susceptibility of an auction to bidder collusion; if an auction fails to be weakly collusion-proof, there will be a consensus among bidders to form a cartel and manipulate the auction. At the same time, the limited scope of collusion it allows for leaves open the possibility that even weakly collusion-proof auctions may be susceptible to collusion.

In this section, we show that the optimal weakly collusion-proof mechanism identified in the previous section can be made unsusceptible to collusion in a much stronger sense. To this end, we stack the deck against the seller by taking a quite permissive approach on how cartels form and behave. First, any informed bidder(s) as well as an uninformed mediator is allowed to propose a cartel manipulation. Second, the cartel formation need not be all-inclusive; so there can be partial or multiple cartels in operation. Also, bidders need not unanimously agree to form a cartel, in the sense that after some bidders reject a cartel proposal, the remaining bidders can form an alternative cartel. Further, if a bidder refuses to participate, the remaining bidders may punish the refusing bidder. We then show that the outcome of the optimal collusion-proof auction identified in the previous section can be implemented even if cartels can form and behave as outlined above, as long as cartel members plays only **cartel-undominated strategies** — a notion which is formalized in the next paragraph.

Take an auction $A \in \mathcal{A}^*$ and let $u_i^A(b \mid v_i) = v_i\xi_i(b) - \tau_i(b)$ for any bid profile $b \in \mathcal{B}$, $i \in N$ and $v_i \in \mathcal{V}_i$. For any potential cartel $C \subset N$, let $b_C = (b_i)_{i\in C}$ and $b_{N\setminus C} = (b_i)_{i\in N\setminus C}$ denote two arbitrary bid profiles for bidders within C and bidders outside C, respectively.

Then, we say a bid profile b'_C is **cartel-dominated** at v_C if there exists b''_C such that

$$u_i^A(b'_C, \tilde{b}_{N\setminus C}|v_i) \leq u_i^A(b''_C, \tilde{b}_{N\setminus C}|v_i), \forall \tilde{b}_{N\setminus C} \text{ and } \forall i \in C$$

with strict inequality for at least one $i \in C$ and one $\tilde{b}_{N\setminus C}$. We say that a bid profile b'_C is **cartel-undominated** at v_C if there is no b''_C that cartel-dominates it.[15]

We now describe a **cartel-game** and present our notion of robust collusion-proofness. A cartel game starts after the seller has announced auction A. All bidders and uninformed third parties are allowed to propose cartel agreements to other bidders (not necessarily to all bidders). Analogous to our earlier definition of an all-inclusive cartel agreement, an agreement specifies a mapping from reports to lotteries over bids for the participating bidders. However, the agreement in this case also specifies which agreement comes into force among accepting bidders depending on the set of accepting and rejecting bidders. With all available — possibly multiple — proposals, each bidder decides which proposal, if any, to accept. If a bidder accepts a cartel proposal, then he commits to it. We assume that no bidder can accept more than one proposal and that each bidder's decision to accept or reject a proposal is observable to the proposer.[16] Following this stage, bidders update their beliefs about others' types based on the proposals — and possibly acceptance/rejection decisions — they have observed. Given the updated beliefs, they play in the subsequent auction A.

Definition 4. *An auction A with (interim) equilibrium outcome $(Q_i, T_i)_{i\in N}$ is* ***robustly collusion-proof (or RCP)*** *if there exists no equilibrium outcome of a cartel-game following A which is different from $(Q_i, T_i)_{i\in N}$ for at least one $i \in N$ and a positive measure of $v_i \in \mathcal{V}_i$, and where cartel-undominated strategies are played at any history.*

Finally, we now state the main result of this section. This result follows from a more general result (Theorem 6) provided in Appendix C, where we show that if a WCP mechanism satisfies two conditions, **monotone dominant strategy incentive compatibility (mDSIC)** and **weak non-bossiness**, then it is also robustly collusion-proof.

[15]Observe that this condition impose less restrictions on cartel behavior than requiring that every bidder plays a weakly-undominated strategy.

[16]Our result does not depend on whether or not bidders other than the proposer can observe the acceptance/rejection decision.

Theorem 5. *There exists a robustly collusion-proof auction which implements the same (interim) equilibrium outcome as the optimal WCP auction identified in Theorem 3.*

Proof. See Appendix C (page 43). ▌

In addition to the result above, the following result is also a straightforward corollary of Theorem 6 in Appendix C. The proof is omitted.

Corollary 6. *(i) If f_i is increasing for all $i \in N$ then the optimal mechanism of Theorem 4 is RCP. (ii) If f_i is (strictly) decreasing for all $i \in N$ then the Myerson optimal auction with the canonical payment rule is RCP.*[17]

6. Conclusion

The possibility of exchanging side payments facilitates the formation of profitable cartels. Cartels with side-payments can always be arranged at standard private value auctions in a way that is beneficial for all cartel members. In light of this, it would be natural to expect that in the absence of side-payments, cartels should be more difficult to sustain.

The result of McAfee and McMillan (1992) contradicts this intuition. As far as we maintain the standard assumption of increasing hazard rate, collusion is always profitable for bidders ex-ante, even in the case in which the cartel is weak. In this paper, we take an alternative approach by considering the bidders' interim incentives to form a weak cartel.

We do not limit our analysis to standard first- or second-price auction, but consider a broad class of auctions that are winner-payable. We provide a tight, sufficient and necessary, condition for an auction in this class to be weakly collusion-proof, which requires that whenever there is some bidder whose value distribution is convex in some interval, he must obtain a constant interim winning probability in that interval. Hence, for instance, when all value distributions are decreasing, the classic revenue-maximizing auction is collusion-proof. In this case, any attempt to form a cartel, even an all inclusive one facilitated by a mediator, unravels.

Our characterization of collusion-proof auctions leads us to identify an optimal auction that maximizes the revenue of the seller facing a weak cartel. For a broad class of value distributions with single-peaked density, the optimal mechanism consists of an auction with

[17]See Appendix C, equation (41), for a formal definition of the canonical payment rule.

a relatively high reserve price, followed by one-to-one negotiations with bidders in case the object is unsold at the auction. An interesting feature that emerges from this mechanism is that the auctioneer may benefit from treating bidders asymmetrically. While an increasing interim allocation helps screen *across different types* of any individual bidder, if it cannot be implemented due to collusion among bidders, the seller should rather choose to screen *across bidders* by treating them asymmetrically.

Finally, our optimal collusion-proof auctions continue to be collusion-proof under a variety of different, less restrictive, collusion formation scenarios, as far as we maintain the mild requirement that cartels always play undominated strategies.

Appendix Appendix A:. Proofs for Section 3

Proof of Theorem 1: Let us first define

$$B_i(v_i) = \begin{cases} \frac{T_i(v_i)}{Q_i(v_i)} & \text{if } v_i \in [r_i, \overline{v}_i] \\ 0 & \text{otherwise} \end{cases}.$$

It is straightforward to see that the function B_i is nondecreasing.[18]

Suppose now for a contradiction that $M_A = (q, t)$ is WCP but $Q_k(\cdot)$ is not constant in the interval $(a,b) \subset (r_k, \overline{v}_k]$ for some $k \in N$, where f_k is nondecreasing.

Let us define $\tilde{Q}(\cdot) = (\tilde{Q}_1, \cdots, \tilde{Q}_n)$ as follows:

$$\tilde{Q}_i(v_i) = \begin{cases} \bar{p} & \text{if } i = k \text{ and } v_i \in (a, b) \\ Q_i(v_i) & \text{otherwise} \end{cases}, \tag{11}$$

where $\bar{p}$ is defined to satisfy

$$\bar{p}(F_k(b) - F_k(a)) = \int_a^b Q_k(s) dF_k(s). \tag{12}$$

Observe first that $\tilde{Q}$ satisfies (M). For this, we only need to check that $Q_k(a) \le \bar{p} = \frac{\int_a^b Q_k(s)dF_k(s)}{(F_k(b) - F_k(a))} \le Q_k(b)$, which clearly holds since $Q_k(\cdot)$ is nondecreasing. The next claim shows that, in addition to (M), $\tilde{Q}$ also satisfies (B).

Claim 1. *The interim allocation rule $\tilde{Q}(\cdot)$ satisfies (B).*

Proof. Since $Q(\cdot)$ satisfies (B), it suffices to show that for all $v = (v_1, \cdots, v_n) \in \mathcal{V}$,

$$\sum_{i \in N} \int_{v_i}^{\overline{v}_i} \tilde{Q}_i(s) dF_i(s) \le \sum_{i \in N} \int_{v_i}^{\overline{v}_i} Q_i(s) dF_i(s),$$

which, given (11), will hold if for all $v_k \in [\underline{v}_k, \overline{v}_k]$,

$$\int_{v_k}^{\overline{v}_k} \tilde{Q}_k(s) dF_k(s) \le \int_{v_k}^{\overline{v}_k} Q_k(s) dF_k(s). \tag{13}$$

[18] To see it, suppose to the contrary that there are two types v_i and $\tilde{v}_i > v_i$ such that $Q_i(\tilde{v}_i) \ge Q_i(v_i) > 0$ but $B_i(\tilde{v}_i) < B_i(v_i)$. Then, $U_i^{M_A}(v_i) = Q_i(v_i)(v_i - B_i(v_i)) < Q_i(\tilde{v}_i)(v_i - B_i(\tilde{v}_i))$ so v_i finds it profitable to deviate to $\tilde{v}_i$'s strategy.

Note that (13) clearly holds for $v_k \geq b$ since $\tilde{Q}_k(s) = Q_k(s), \forall s \in [b, \overline{v}_k]$. Let us pick $v_k \in [a, b)$ and then we obtain as desired

$$\begin{aligned}\int_{v_k}^{\overline{v}_k} \tilde{Q}_k(s)dF_k(s) &= \int_{v_k}^{b} \bar{p}dF_k(s) + \int_{b}^{\overline{v}_k} Q_k(s)dF_k(s) \\ &= \left[\frac{F_k(b) - F_k(v_k)}{F_k(b) - F_k(a)}\right] \int_{a}^{b} Q_k(s)dF_k(s) + \int_{b}^{\overline{v}_k} Q_k(s)dF_k(s) \\ &\leq \int_{v_k}^{b} Q_k(s)dF_k(s) + \int_{b}^{\overline{v}_k} Q_k(s)dF_k(s) = \int_{v_k}^{\overline{v}_k} Q_k(s)dF_k(s), \qquad (14)\end{aligned}$$

where the second equality follows from the definition of $\bar{p}$, and the inequality from the fact that $Q_k(\cdot)$ is nondecreasing and thus

$$\int_{a}^{b} \frac{Q_k(s)}{F_k(b) - F_k(a)} dF_k(s) \leq \int_{v_k}^{b} \frac{Q_k(s)}{F_k(b) - F_k(v_k)} dF_k(s).$$

Also, for $v_k < a$, we have

$$\begin{aligned}\int_{v_k}^{\overline{v}_k} \tilde{Q}_k(s)dF_k(s) &= \int_{v_k}^{a} Q_k(s)dF_k(s) + \int_{a}^{\overline{v}_k} \tilde{Q}_k(s)dF_k(s) \\ &\leq \int_{v_k}^{a} Q_k(s)dF_k(s) + \int_{a}^{\overline{v}_k} Q_k(s)dF_k(s) = \int_{v_k}^{\overline{v}_k} Q_k(s)dF_k(s),\end{aligned}$$

where the inequality follows from (14). ▮

From $\tilde{Q}$, we can derive an interim transfer rule $\tilde{T}$ by fixing $\tilde{T}_i(r_i) = T_i(r_i)$ for all $i \in N$ and then using (Env). Since the profile $(\tilde{Q}, \tilde{T})$ thus defined satisfies all the conditions in Lemma 1, we can find an auction rule whose equilibrium outcome is a direct mechanism $\tilde{M}_A = (\tilde{q}, \tilde{t})$ that has $(\tilde{Q}, \tilde{T})$ as the interim rule.

Let us first show that $\tilde{M}_A$ interim Pareto dominates the original equilibrium payoff of the auction rule A (i.e. satisfies $(C - IR)$). First, it is clear that all other bidders than k will have their payoffs unaffected. Moreover, bidder k's payoff will only be affected when his value is above a. To show that $U_k^{\tilde{M}_A}(v_k) \geq U_k^{M_A}(v_k)$ for all $v_k \in [a, \overline{v}_k]$, with strict inequality for some v_k, it suffices to show that $U_k^{\tilde{M}_A}(b) \geq U_k^{M_A}(b)$, since $U_k^{\tilde{M}_A}(\cdot)$ is linear in $[a, b]$ while $U_k^{M_A}(\cdot)$ is convex but not linear, and since $\tilde{Q}_k(v_k) = Q_k(v_k)$ for all $v_k \in (b, \overline{v}_k]$ so $\tilde{U}_k(\cdot)$ and $U_k(\cdot)$ have the same slope beyond b.

To check that $U_k^{\tilde{M}_A}(\underline{v}_k) \geq U_k^{M_A}(\underline{v}_k)$, note first that

$$\frac{\bar{p}(F_k(b)-F_k(a))}{b-a} = \int_a^b \frac{Q_k(s)f_k(s)}{b-a}ds \geq \int_a^b \frac{Q_k(s)}{b-a}ds \int_a^b \frac{f_k(s)}{b-a}ds,$$

where the inequality follows since both $Q_k(\cdot)$ and $f_k(\cdot)$ are nondecreasing in the interval (a,b). Rearranging yields

$$U_k^{\tilde{M}_A}(b) - U_k^{\tilde{M}_A}(a) = \bar{p}(b-a) \geq \int_a^b Q_k(s)ds = U_k^{M_A}(b) - U_k^{M_A}(a).$$

or $U_k^{\tilde{M}_A}(b) \geq U_k^{M_A}(b)$ since $U_k^{\tilde{M}_A}(a) = U_k^{M_A}(a)$.

Then, the desired contradiction will follow if we show that $\tilde{M}_A$ can be implemented via a weak cartel manipulation. To this end, let $\tilde{B}_i(v_i) = \frac{\tilde{T}_i(v)}{\tilde{Q}_i(v_i)}$ if $v_i \in [r_i, \overline{v}_i]$ and $\tilde{B}_i(v_i) = 0$ otherwise.[19] We then exploit the winner-payability property to establish the following result:

Claim 2. *Given the winner payability, for any given $v_i \in [r_i, \overline{v}_i]$, there exists a randomization over bid profiles $\underline{b}^i$ and $\overline{b}^i$, denoted $\mu^i(v_i) \in [0,1]$, such that*

$$\mu^i(v_i)\tau_i(\underline{b}^i) + (1-\mu^i(v_i))\tau_i(\overline{b}^i) = \tilde{B}_i(v_i). \tag{15}$$

Proof. First, we show that

$$B_i(r_i) \leq \tilde{B}_i(v_i) \leq B_i(\overline{v}_i), \forall v_i \in [r_i, \overline{v}_i], \forall i. \tag{16}$$

This is immediate if $i \neq k$ or if $i = k$ and $v_k \in [\underline{v}_k, a]$ since in those cases, $B_i(v_i) = \tilde{B}_i(v_i)$ and $B_i(\cdot)$ is nondecreasing. Consider now $i = k$ and any $v_k \in (a, \overline{v}_k]$. The first inequality of (16) holds trivially. To prove the latter inequality, it suffices to show that $\tilde{B}_i(\overline{v}_i) \leq B_i(\overline{v}_i)$, since $\tilde{B}_i(\cdot)$ is nondecreasing. This inequality holds trivially if $\overline{v}_k = b$ since $B_k(b) \geq B_k(a) = \tilde{B}_k(a) = \tilde{B}_k(b)$. If $\overline{v}_k > b$, then $Q_k(\overline{v}_k) = \tilde{Q}_k(\overline{v}_k)$ and also

$$T(\overline{v}_k) - \tilde{T}(\overline{v}_k) = \overline{v}_k Q_k(\overline{v}_k) - \overline{v}_k \tilde{Q}_k(\overline{v}_k) + U_k^{\tilde{M}_A}(\overline{v}_k) - U_k^{M_A}(\overline{v}_k) = U_k^{\tilde{M}_A}(\overline{v}_k) - U_k^{M_A}(\overline{v}_k) \geq 0.$$

This implies $B_i(\overline{v}_i) \geq \tilde{B}_i(\overline{v}_i)$.

Next, we observe that for any $v_i \in [r_i, \overline{v}_i]$,

$$\inf\Big\{\frac{\tau_i(b)}{\xi_i(b)} \,|\, \xi_i(b) > 0, b \in \mathcal{B}\Big\} \leq B_i(v_i) \leq \sup\Big\{\frac{\tau_i(b)}{\xi_i(b)} \,|\, \xi_i(b) > 0, b \in \mathcal{B} \text{ and } \frac{\tau_i(b)}{\xi_i(b)} \leq \overline{v}_i\Big\}.$$

[19]Note that $r_i = \inf\{v_i \in \mathcal{V}_i \,|\, \tilde{Q}_i(v_i) > 0\} = \inf\{v_i \in \mathcal{V}_i \,|\, Q_i(v_i) > 0\}$.

By definition, $\tau_i(\underline{b}^i)$ and $\tau_i(\overline{b}^i)$ are equal to the LHS and RHS of the above equation, respectively. Combining this with (16) means that for each $v_i \in [r_i, \overline{v}_i]$, $\tilde{B}_i(v_i) \in [\tau_i(\underline{b}^i), \tau_i(\overline{b}^i)]$, which guarantees the existence of $\mu^i(v_i)$ as in (15). ▮

Then, if the cartel members report $v \in \mathcal{V}$ such that $v_i \geq r_i$ for some $i \in N$, let the cartel bid $\underline{b}^i$ with probability $\tilde{q}_i(v)\mu^i(v)$ and $\overline{b}^i$ with $\tilde{q}_i(v)(1 - \mu^i(v))$. So each bidder i obtains the object with probability $\tilde{q}_i(v)$ and pays $\tilde{q}_i(v)\tilde{B}_i(v_i)$ in expectation. If there is no cartel member with $v_i \geq r_i$, then let the cartel bid $(b_1^0, \cdots, b_n^0)$. It is straightforward to verify that the interim allocation and payment from this manipulation are $\tilde{Q}_i(v_i)$ and $\tilde{T}_i(v_i)$ for each bidder i with value v_i, as desired. ▮

Proof of Corollary 1: Fix a bidder k for whom f_k is nondecreasing on some interval (a, b) with $b > r$ and $a \geq \underline{v}$. We show that in any standard auction, the winning probability of bidder k is non-constant in the interval $(\max\{a, r\}, b)$, which will imply by Theorem 1 that the auction is not WCP. Consider first the second-price and English auctions where each bidder bids his value in the undominated strategy. The interim winning probability of bidder k with $v_k \in (\max\{a, r\}, b)$ is equal to $Q_k(v_k) = \prod_{i \neq k} F_i(v_k)$, which is strictly increasing in the interval $(\max\{a, r\}, b)$.

Consider next the first-price auction (or Dutch auction since the two auctions are strategically equivalent). Note first that in undominated strategy equilibrium, (i) no bidder bids more than his value and (ii) no bidder puts an atom at any bid B if B wins with positive probability. Letting $\beta_i(\cdot)$ denote bidder i's equilibrium strategy, note also that $\beta_i(\cdot)$ is nondecreasing. Given (i), we must have $Q_k(v_k) > 0$ for all $v_k \in (\max\{a, r\}, b)$ since he can always bid some amount $B \in (\max\{a, r\}, v_k)$ and enjoy a positive payoff. Next, by (ii), there must be some $v_k \in (\max\{a, r\}, b)$ such that $\beta_k(v_k) < \beta_k(b)$ since otherwise $\beta_k(b)$ would be an atom bid. For such v_k, we must have $Q_k(v_k) < Q_k(b)$ so $Q_k(\cdot)$ is non-constant in $(\max\{a, r\}, b)$. To see why, suppose to the contrary that $Q_k(v_k) = Q_k(b)$, which implies that no one else is submitting any bid between $\beta_k(v_k)$ and $\beta_k(b)$. Then, bidder k with value b can profitably deviate to lower his bid below $\beta_k(b)$ but above $\beta_k(v_k)$, a contradiction. ▮

Proof of Theorem 2: To begin, for each $i \in N$, we partition $\overline{\mathcal{V}}_i = [r_i, \overline{v}_i]$ into a two families, $\{V_i^j\}_{j \in J_i^+}$ and $\{V_i^j\}_{j \in J_i^-}$, of countably many intervals such that $f_i'(v) \geq 0$ for a.e. $v \in V_i^j$ for $j \in J_i^+$, and $f_i'(v) < 0$ for a.e. $v \in V_i^j$ for $j \in J_i^-$. In particular, each interval $V_i^j, j \in J_i^-$ can be taken to be an open interval. (One of the index sets J_i^+ and J_i^- can be empty as f_i can be everywhere nondecreasing or everywhere increasing.) This partitioning is well-defined since f_i is absolutely continuous. Let $\mathcal{V}_i^+ := \cup_{j\in J_i^+} V_i^j$ and $\mathcal{V}_i^- := \cup_{j\in J_i^-} V_i^j$.

Consider an auction A which induces an equilibrium whose interim allocation probability satisfies $Q_i'(v) = 0$ for all $v \in \mathcal{V}_i^+$ and

$$1 - \prod_{i\in N} F_i(r_i) = \sum_{i\in N} \int_{r_i}^{\overline{v}_i} Q_i(v_i) f_i(v_i) dv_i.$$

We prove that A is unsusceptible to collusion.

Define G_i to be the locally concave hull of F_i, defined as follows: For each $v \in V_i^j, \forall j \in J_i^+ \cup J_i^-$,

$$G_i(v) := \max\{sF_i(v') + (1-s)F_i(v'') | s \in [0,1], v', v'' \in V_i^j, \text{ and } sv' + (1-s)v'' = v\}.$$

In words, G_i is the lowest function such that $G_i(\cdot) \geq F_i(\cdot)$ and that it satisfies concavity in each interval V_i^j. Clearly, if $V_i^j \subset \mathcal{V}_i^-$, then $G_i(v) = F_i(v)$ for all $v \in V_i^j$, and if $V_i^j \subset \mathcal{V}_i^+$, then $G_i(v)$ is linear in v for all $v \in V_i^j$. Clearly, G_i admits density, denoted g_i, for almost every $v \in \mathcal{V}_i$. More importantly, while G_i need not be globally concave, $g_i'(v) \leq 0$ for almost every $v \in \mathcal{V}_i$.

Consider any any weak cartel manipulation $\tilde{M}_A = (\tilde{q}, \tilde{t})$ implementing an interim Pareto improvement. Since by assumption $\tau_i(b) \geq \xi_i(b)\underline{v}_i, \forall i \in N, \forall b \in \mathcal{B}$ and $\tilde{M}_A$ is a weak manipulation of A, we must have

$$U_i^{\tilde{M}_A}(\underline{v}_i) \leq \max_{b\in\mathcal{B}} \xi_i(b)\underline{v}_i - \tau_i(b) \leq \max_{b\in\mathcal{B}} \xi_i(b)\underline{v}_i - \xi_i(b)\underline{v}_i = 0$$

so due to $(C\!-\!IR)$, $U_i^{\tilde{M}_A}(\underline{v}_i) = 0$ for each $i \in N$. A similar reasoning also yields $U_i^{M_A}(\underline{v}_i) = 0$ for each $i \in N$. Then, interim Pareto domination implies that

$$X_i(v_i) := U_i^{\tilde{M}_A}(v_i) - U_i^{M_A}(v_i) = \int_{r_i}^{v_i} (\tilde{Q}_i(s) - Q_i(s)) ds \geq 0, \forall i, v_i. \tag{17}$$

Next, it follows from Lemma 1 that

$$\sum_{i\in N} \int_{r_i}^{\overline{v}_i} \tilde{Q}_i(v_i) f_i(v_i) dv_i \leq 1 - \prod_{i\in N} F_i(r_i) = \sum_{i\in N} \int_{r_i}^{\overline{v}_i} Q_i(v_i) f_i(v_i) dv_i,$$

or

$$\sum_{i\in N} \int_{r_i}^{\overline{v}_i} (\tilde{Q}_i(v_i) - Q_i(v_i)) f_i(v_i) dv_i \leq 0. \tag{18}$$

Meanwhile,

$$\begin{aligned}
&\sum_{i\in N}\int_{r_i}^{\overline{v}_i}(\tilde{Q}_i(v_i)-Q_i(v_i))f_i(v_i)dv_i \\
&=\sum_{i\in N}\int_{r_i}^{\overline{v}_i}(\tilde{Q}_i(v_i)-Q_i(v_i))g_i(v_i)dv_i-\Big(\sum_{i\in N}\int_{r_i}^{\overline{v}_i}(\tilde{Q}_i(v_i)-Q_i(v_i))[g_i(v_i)-f_i(v_i)]dv_i\Big) \\
&=\sum_{i\in N}\Big(X_i(\overline{v}_i)g_i(\overline{v}_i)-\int_{r_i}^{\overline{v}_i}X_i(v_i)g_i{}'(v_i)dv_i\Big)+\sum_{i\in N}\int_{r_i}^{\overline{v}_i}(G_i(v_i)-F_i(v_i))[\tilde{Q}_i'(v_i)-Q_i'(v_i)]dv_i \\
&\geq 0
\end{aligned}$$

where the first equality follows from the integration by parts, and the inequality holds since, for each $i\in N$, $X_i(v)\geq 0$, $g_i'(v)\leq 0$ for a.e. $v\in\mathcal{V}_i$ (by definition of g_i), and, whenever $G_i(v_i)>F_i(v_i)$, $Q_i'(v_i)=0\leq\tilde{Q}_i'(v_i)$ (by the monotonicity of $\tilde{Q}_i$).

The last inequality combined with (18) means that the inequality must hold as equality, which in turn implies that $X_i(v)=0$ for a.e. $v\in\mathcal{V}_i^-$ for each $i\in N$. We next prove that $\tilde{Q}_i(v)\geq Q_i(v)$ for a.e. $v\in\mathcal{V}_i$. To prove this, suppose to the contrary that there exists $v>r_i$ such that $\tilde{Q}_i(v)<Q_i(v)$. Then, since $X_i(\cdot)\geq 0$ implies that there is some v' (arbitrarily) close to r_i with $\tilde{Q}_i(v')\geq Q_i(v')$, by the mean value theorem, there must exist some $v''\in(v',v]$ such that $\tilde{Q}_i(v'')<Q_i(v'')$ and $\tilde{Q}_i'(v'')<Q_i'(v'')$ (and both $\tilde{Q}_i$ and Q_i are continuous at v''). It follows that $X_i'(v)=\tilde{Q}_i(v)-Q_i(v)<0$ for all $v\in(v''-\epsilon,v''+\epsilon)$, for some $\epsilon>0$. Since $\tilde{Q}'(v'')\geq 0$, $Q_i'(v'')>0$, so $v''\in\mathcal{V}_i^-$. But this means that $X_i'(v)=0$ for a.e. $v\in[v'',v''+\epsilon)$ or for a.e. $v\in(v''-\epsilon,v'']$ for some $\epsilon>0$, which is a contradiction.

Since $\tilde{Q}_i(\cdot)\geq Q_i(\cdot)$ for each i, the above equality $\sum_{i\in N}\int_{r_i}^{\overline{v}_i}(\tilde{Q}_i(v_i)-Q_i(v_i))f_i(v_i)dv_i=0$ means that $\tilde{Q}_i(v)=Q_i(v)$ for a.e. $v\in\mathcal{V}_i$ for each $i\in N$. We thus conclude that for any feasible weak-cartel manipulation, $\tilde{M}_A=(\tilde{q},\tilde{t})$

$$U_i^{\tilde{M}_A}(v_i)-U_i^{M_A}(v_i)=\int_{r_i}^{v_i}(\tilde{Q}_i(s)-Q_i(s))ds=0,\forall v_i,\forall i. \tag{19}$$

That is, auction rule A is unsusceptible to weak cartel. ∎

Appendix Appendix B:. Proofs for Section 4

Here we provide the proofs of Theorem 3 and 4. First, we establish a preliminary result in Lemma 2 below. Fix a profile of reserve prices $r = (r_i)_{i\in N}$ and define

$$\mathcal{P} := \Big\{ p = (p_1, \cdots, p_n) \in [0,1]^n \Big| \sum_{i\in S} p_i(1 - F_i(r_i)) \quad \leq 1 - \textstyle\prod_{i\in S} F_i(r_i), \forall S \subset N \Big\}. \quad (20)$$

Note that the inequalities that define $\mathcal{P}$ correspond to the constraint (B) associated with the constant interim allocation rule: $Q_i(v_i) = p_i \in [0,1]$ if $v_i \geq r_i$ and $Q_i(v_i) = 0$ otherwise. So, $\mathcal{P}$ is the set of all such allocation rules that are implementable (in the sense of being a reduced form). Clearly, $\mathcal{P}$ is a convex polytope since the inequalities defining (20) are all linear. The following lemma, whose proof is provided in the Supplementary Appendix, gives a characterization for extreme points of $\mathcal{P}$.

Lemma 2. *A vector $p = (p_1, \cdots, p_N) \in [0,1]^N$ is an extreme point of $\mathcal{P}$ defined in (20) if and only if it can be expressed as follows: For some permutation function $\pi : N \to N$,*

$$p_i = \prod_{j:\pi(j)<\pi(i)} F_j(r_j) \quad (21)$$

with $p_i = 1$ for $i = \pi^{-1}(1)$. Moreover, at each extreme point, there are exactly n sets, $S_1 \subsetneq \cdots \subsetneq S_n = N$, for which the weak inequalities in (20) are satisfied as equality.

Proof of Theorem 3: First, we relax some of the constraints and consider a relaxed problem. To that end, define for any $v \in [v^*, \overline{v}]$

$$Y(v) := 1 - F(v)^n - \sum_{i\in N} \int_v^{\overline{v}} Q_i(s)f(s)ds. \quad (22)$$

The function $Y(v)$ represents the probability that the object is not assigned to a bidder whose value is at least v, even though there exists at least one such bidder. In other words, $Y(v)$ is the capacity that is *not exhausted* by the types above v. By some abuse of notation, we define $r_i = \inf\{v_i \in \mathcal{V}_i | Q_i(v_i) > 0\}$.[20] Define also $N_* := \{i \in N | r_i < v^*\}$. Given the function Y and any subset $M \subset N_*$, the constraint (B) at a value profile $v = (v_1, \cdots, v_n)$ with $v_i = r_i$ for $i \in M$ and $v_i = v^*$ for $i \in N\backslash M$, can be written as

$$0 \leq 1 - F(v^*)^{n-|M|} \prod_{i\in M} F(r_i) - \sum_{i\in N} \int_{v^*}^{\overline{v}} Q_i(v)f(v)dv - \sum_{i\in M} p_i[F(v^*) - F(r_i)]$$

[20]Later, r_i defined here will turn out to be the same as r_i defined in (1).

$$= Y(v^*) - \sum_{i \in M} p_i [F(v^*) - F(r_i)] + F(v^*)^n - F(v^*)^{n-|M|} \prod_{i \in M} F(r_i). \tag{23}$$

Since the weak collusion-proofness requires that $Q_i(\cdot)$ is constant in the interval $[r_i, v^*)$, we define $p_i = Q_i(v)$ for $v \in [r_i, v^*)$. (Note that $p_i = 0$ in case $r_i \geq v^*$.) Then, the relaxed problem, denoted $[P']$, is given as follows:

$$[P'] \qquad \max_{(Q_i)_{i \in N}} \sum_{i \in N} \int_{v^*}^{\overline{v}} J(v) Q_i(v) f(v) dv + \sum_{i \in N} p_i \cdot \int_{r_i}^{v^*} J(v) f(v) dv$$

subject to

$$Y(v) \geq 0, \forall v \in [v^*, \overline{v}] \text{ and } Y(\overline{v}) = 0, \tag{24}$$

and

$$\sum_{i \in M} p_i [F(v^*) - F(r_i)] \leq Y(v^*) + F(v^*)^n - F(v^*)^{n-|M|} \prod_{i \in M} F(r_i), \forall M \subset N_*. \tag{25}$$

Note first that the objective function is rewritten by embedding the collusion-proofness constraint (CP), so the constraint (CP) is dropped. To compare the other constraints of $[P']$ to those of $[P]$, the monotonicity constraint (M) is dropped in $[P']$. Also, the capacity constraints (B) is imposed on a much smaller set of value profiles: The inequality in (24) corresponds to the capacity constraint (B) along the diagonal where $v_i = v \in [v^*, \overline{v}], \forall i \in N$; In light of (23), the inequality (25) corresponds to the capacity constraint at the profile v where $v_i = r_i$ if $i \in M$ and $v_i = v^*$ otherwise. We aim to obtain a solution of $[P']$, which will turn out to satisfy all the constraints of $[P]$ and thus solve $[P]$ as well.

Our search for a solution of $[P']$ consists of three lemmas, Lemma 3 to 5. The first lemma shows that at the optimum, bidders who have negative virtual value must have a zero probability of obtaining the object in equilibrium. It also provides an upper bound, equal to v^*, to a set of values that can be assigned zero probability at the optimum. The proof of this result is provided in the Supplementary Appendix.

Lemma 3. *At any optimum of $[P']$, it must be that $r_i \geq \hat{v}$ for all $i \in N$. Also, it is without loss to assume that $r_i < v^*, \forall i \in N$ (i.e. $N_* = N$) at an optimal solution of $[P']$.*

The next Lemma shows that the constraint (B) must be binding at all profile $(v, \ldots, v)$ for each $v \geq v^*$.

Lemma 4. *At an optimum of* $[P']$,

$$\sum_{i\in N}\int_{v}^{\overline{v}} Q_i(s)f(s)ds = 1 - F(v)^n, \forall v \in [v^*, \overline{v}] \tag{26}$$

Proof. To solve $[P']$, let us fix r for a while at any level satisfying $r < (v^*, \cdots, v^*)$. Given $Y(v^*)$, the second term of the objective function in $[P']$ can be independently maximized as follows: Given $Y(v^*) = y$ for some $y \geq 0$,

$$[R; y] \qquad \max_{\{p_i\}_{i\in N}} \sum_{i\in N} p_i \cdot \int_{r_i}^{v^*} J(v)f(v)dv$$

subject to (25). Let $\phi(y; r)$ denote the value function obtained from solving $[R; y]$.

Claim 3. *Given any* $r < (v^*, \cdots, v^*)$*, the function* $\phi(\cdot; r)$ *is (weakly) concave and also satisfies* $\frac{\partial\phi(0;r)}{\partial y} < J(v^*)$.

Proof. To prove the concavity of ϕ, consider any y and y', and let p and p' denote the solution of $[R; y]$ and $[R; y']$, respectively. Since the constraint (25) is linear in y and p, for any $\lambda \in [0, 1]$, we have $\lambda p + (1-\lambda)p'$ satisfying the constraint of $[R; \lambda y + (1-\lambda)y']$. Given this and the linearity of the objective function in p, we must have $\phi(\lambda y + (1-\lambda)y'; r) \geq \lambda\phi(y; r) + (1-\lambda)\phi(y'; r)$, as desired.

To prove $\frac{\partial\phi(0;r)}{\partial y} < J(v^*)$, define for each $i \in N$,

$$q_i := \frac{p_i}{F(v^*)^{n-1}}\ , G(v) := \frac{F(v)}{F(v^*)},\ \text{and}\ g(v) := \frac{dG(v)}{dv}.$$

Given this, one can rewrite the program $[R; y]$ as

$$\max_{\{q_i\}_{i\in N}} F(v^*)^n\Big(\sum_{i\in N} q_i \int_{r_i}^{v^*} J(v)g(v)dv\Big) \tag{27}$$

subject to

$$\sum_{i\in M} q_i[1 - G(r_i)] \leq \frac{y}{F(v^*)^n} + 1 - \prod_{i\in M} G(r_i), \forall M \in 2^N. \tag{28}$$

Set up the Lagrangian for this problem as

$$\begin{aligned}\mathcal{L} = F(v^*)^n\Big(\sum_{i\in N} q_i \cdot \int_{r_i}^{v^*} J(v)g(v)ds\Big)\\ + \sum_{M\in 2^N}\lambda_M\Big[\frac{y}{F(v^*)^n} + 1 - \prod_{i\in M} G(r_i) - \sum_{i\in M} q_i[1 - G(r_i)]\Big],\end{aligned}$$

Letting λ_M^0 denote the Lagrangian multiplier when $y = 0$, by the envelope theorem, we have $\frac{\partial \phi(0,r_i)}{\partial y} = \frac{\partial \mathcal{L}}{\partial y} = \frac{\sum_{M \in 2^N} \lambda_M^0}{F(v^*)^n}$. The proof will be done if $\frac{\sum_{M \in 2^N} \lambda_M^0}{F(v^*)^n} < J(v^*)$.

Suppose to the contrary that $\frac{\sum_{M \in 2^N} \lambda_M^0}{F(v^*)^n} \geq J(v^*)$. To draw a contradiction, we investigate the program $[R;0]$. First of all, since the objective and constraints are all linear in $(q_1, \cdots, q_n)$, the optimum arises at one of vertices of the constraint set, which is a polytope given by the inequalities in (28). Denote that vertex by q^0. Note next that the constraints in (28) are identical to those in (20) with F and p_i being replaced by G and q_i, respectively. So, according to Lemma 2, at the vertex q^0, there are exactly n subsets of N, $M_1, \cdots, M_n$, for which (28) holds as equality, and some $j \in N$ for whom $j \in M_k, \forall k = 1, \cdots, n$. Also, letting $\mathcal{M} := \{M_1, \cdots, M_n\}$, (28) holds as strict inequality for all $M \in 2^N \backslash \mathcal{M}$, which implies that $\lambda_M^0 = 0$ for $M \in 2^N \backslash \mathcal{M}$ and thus $\sum_{M \in 2^N} \lambda_M^0 = \sum_{M \in \mathcal{M}} \lambda_M^0$. Given this and $y = 0$, we have

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial q_j} &= F(v^*)^n \int_{r_j}^{v^*} J(v) g(v) ds - \sum_{M \in \mathcal{M}} \lambda_M^0 [1 - G(r_j)] \\
&= F(v^*)^n [1 - G(r_j)] \Big[\int_{r_j}^{v^*} J(v) \frac{g(v)}{1 - G(r_j)} ds - \frac{\sum_{M \in 2^N} \lambda_M^0}{F(v^*)^n} \Big] \\
&< F(v^*)^n [1 - G(r_j)] \Big[J(v^*) - \frac{\sum_{M \in 2^N} \lambda_M^0}{F(v^*)^n} \Big] \leq 0,
\end{aligned}$$

which implies that $q_j^0 = 0$. This cannot happen, however, due to the fact that the (28) holds as equality for $M = \{j\}$ since $\{j\} \in \mathcal{M}$.▌

We now show that at the optimum, $Y(v) = 0$ for all $v \geq v^*$. Suppose that $r = (r_1, \cdots, r_n)$ is fixed at the optimal level. Then, the interim allocation rule Q that solves $[P']$ can be found as a solution of the following problem:

$$[P''] \qquad \max_{\{Q_i(\cdot)\}_{i \in N}} \sum_{i \in N} \int_{v^*}^{\overline{v}} J(v) Q_i(v) f(v) dv + \phi(Y(v^*); r)$$

subject to (24). Consider this as an optimal control problem with control variable $Q(\cdot)$, state variable $Y(\cdot)$, and salvage value $\phi(Y(v^*); r)$. Note that

$$Y'(v) = f(v) \Big[\sum_{i \in N} Q_i(v) - n F(v)^{n-1} \Big].$$

Letting γ and λ denote the costate variable and the multiplier for (24), one can write the Hamiltonian (or Lagrangian) for $[P'']$ (exclusive of the salvage value) as

$$H(v,Q,Y,\gamma,\lambda)=\sum_{i\in N}J(v)Q_i(v)f(v)+\gamma(v)f(v)\Big[\sum_{i\in N}Q_i(v)-nF(v)^{n-1}\Big]+\lambda(v)Y(v).$$

Since both objective and constraint functions are concave (in particular, $\phi(\cdot;r)$ is concave by Lemma 3), the necessary and sufficient condition for the optimum is given as follows[21]:

$$\frac{\partial H}{\partial Q_i}=(J(v)+\gamma(v))f(v)=0,\ \forall i\in N \tag{29}$$

$$\gamma'(v)=-\frac{\partial H}{\partial Y}=-\lambda(v) \tag{30}$$

$$Y(v^*)\geq 0,\ \gamma(v^*)+\frac{\partial\phi(Y(v^*);r)}{\partial y}\leq 0,\ \text{ and }\ Y(v^*)\Big[\frac{\partial\phi(Y(v^*);r)}{\partial y}+\gamma(v^*)\Big]=0 \tag{31}$$

$$Y(v),\lambda(v)\geq 0\ \text{ and }\ Y(v)\lambda(v)=0, \tag{32}$$

where $\gamma(v^*)$ is the derivative of the value function (exclusive of the salvage value) for $[P'']$. From (29), $J(v)=-\gamma(v)$ and thus using (30), we obtain $\lambda(v)=-\gamma'(v)=J'(v)>0$, which implies by (32) that $Y(v)=0,\forall v\in[v^*,\overline{v}]$. One can now verify (31) since $Y(v^*)=0$ and $\frac{\partial\phi(0;r)}{\partial y}<J(v^*)=-\gamma(v^*)$ by Lemma 3. ▌

Since the optimality condition (26) does not pin down an interim allocation rule for each individual bidder, let us set

$$Q_i(v)=F(v)^{n-1},\forall v\in(v^*,\overline{v}],\forall i\in N. \tag{33}$$

This allocation rule along with $p=(p_1,\cdots,p_n)$ defined in Lemma 5 below will satisfy the monotonicity constraint.[22]

The last lemma shows how to determine the reserve prices $(r_1,\ldots,r_n)$ and interim winning probabilities $(p_1,\ldots,p_n)$ for the types between r_i and v^*.

Lemma 5. *At an optimum of $[P']$, $r=(r_1,\cdots,r_n)$ is chosen (up to a permutation among symmetric bidders) as*

$$r=\arg\max_{\tilde{r}=(\tilde{r}_1,\cdots,\tilde{r}_n)}\sum_{i\in N}F(v^*)^{n-i}\prod_{k=1}^{i-1}F(\tilde{r}_k)\int_{\tilde{r}_i}^{v^*}J(v)f(v)dv. \tag{34}$$

[21] Refer to Leonard and Van Long (1992) or Kamien and Shwartz (1991), for instance.

[22] In fact, one can show that any interim rule satisfying (M), (B), and (26) must take the form given in (33).

Then, $p = (p_1, \cdots, p_n)$ *is given as*

$$p_i = F(v^*)^{n-i} \prod_{k=1}^{i-1} F(r_k). \tag{35}$$

Moreover, the profile $r = (r_1, \cdots, r_n)$ *in* (34) *satisfies* $\hat{v} = r_n < r_{n-1} < \cdots < r_1 < v^*$.

Proof. By Lemma 4, whatever r is chosen, we must have $Y(v^*) = 0$ at the optimum. Thus, in order for (p, r) to solve $[P']$, the pair (q, r) (where $q_i = \frac{p_i}{F(v^*)^{n-1}}$) must solve the problem in (27) with $y = 0$ and r being included as choice variable. As mentioned in the proof of Claim 3, the optimum of that problem arises at one of the vertices in the constraint set. This implies that given any vector $\tilde{r}$, we can relabel the (symmetric) bidders so that $q_i = \prod_{k=1}^{i-1} G(\tilde{r}_k) = \prod_{k=1}^{i-1} \frac{F(\tilde{r}_k)}{F(v^*)}$ or

$$p_i = F(v^*)^{n-1} q_i = F(v^*)^{n-i} \prod_{k=1}^{i-1} F(\tilde{r}_k). \tag{36}$$

Plugging this into (27) yields the objective function in (34), which can then be maximized by choosing $\tilde{r}$ optimally. Plugging the optimal $\tilde{r}$ into (36) yields (35).

We now show that $\hat{v} = r_n < r_{n-1} < \cdots < r_1 < v^*$. Define first

$$\tilde{\Pi}_n := \int_{\tilde{r}_n}^{v^*} J(v) f(v) dv, \tag{37}$$

and define recursively

$$\tilde{\Pi}_i(\tilde{r}_i, \tilde{\Pi}_{i+1}) := F(v^*)^{n-i} \int_{\tilde{r}_i}^{v^*} J(v) f(v) dv + F(\tilde{r}_i) \tilde{\Pi}_{i+1},\ i = 1, \cdots, n-1.$$

One can easily verify that the objective function in (34) is the same as the function $\tilde{\Pi}_1$ defined above. Also, the terms in the objective function that involve $\tilde{r}_i$ are all included in $\tilde{\Pi}_i$ (multiplied by some expression unrelated to $\tilde{r}_i$.). Thus, maximizing the objective function corresponds to choosing $\tilde{r}_i$ for each $i \in N$ that maximizes $\tilde{\Pi}_i$, given the maximized value of $\tilde{\Pi}_{i+1}$. Let us recursively define r_i and Π_i to be respectively the maximizer and maximized value of $\tilde{\Pi}_i$ with Π_{i+1} given. From (37), it is immediate that $r_n = \hat{v}$.

In order to complete the proof, we adopt the induction argument to show the followings: For all $i = 1, \cdots, n-1$, (i) $\Pi_{i+1} < F(v^*)^{n-i} J(v^*)$; (ii) $r_{i+1} < r_i < v^*$. Consider $i = n-1$ for the initial step. Then, $\Pi_n = \int_{\hat{v}}^{v^*} J(s) f(s) ds < F(v^*) J(v^*)$ so (i) is satisfied. For (ii), note that $\frac{\partial \tilde{\Pi}_{n-1}(\tilde{r}_{n-1}, \Pi_n)}{\partial \tilde{r}_{n-1}} = f(\tilde{r}_{n-1})[\Pi_n - F(v^*) J(\tilde{r}_{n-1})]$, which is negative if $\tilde{r}_{n-1} = v^*$ and positive if $\tilde{r}_{n-1} = \hat{v} = r_n$. So we must have $r_n < r_{n-1} < v^*$. Let us now assume that (i) and (ii) hold for some $k > 1$, that is $\Pi_{k+1} < F(v^*)^{n-k}J(v^*)$ and $r_{k+1} < r_k < v^*$. We show that $\Pi_k < F(v^*)^{n-k+1}J(v^*)$ and $r_k < r_{k-1} < v^*$. First, the fact that $\Pi_k < F(v^*)^{n-k+1}J(v^*)$ follows from

$$\Pi_k - F(v^*)^{n-k+1}J(v^*) = F(r_k)\Pi_{k+1} - F(v^*)^{n-k+1}J(v^*) + F(v^*)^{n-k}\int_{r_k}^{v^*} J(v)f(v)dv$$
$$< F(v^*)^{n-k}\Big[J(v^*)(F(r_k) - F(v^*)) + \int_{r_k}^{v^*} J(v)f(v)dv\Big] < 0, \quad (38)$$

where the first inequality is due to the inductive assumption and the second inequality holds since $v^* > r_k$ and $J(v^*) > J(v), \forall v < v^*$. To show $r_{k-1} < v^*$, note that by (38), $\frac{\partial \tilde{\Pi}_{k-1}(v^*,\Pi_k)}{\partial \tilde{r}_{k-1}} = f(v^*)[\Pi_k - F(v^*)^{n-k+1}J(v^*)] < 0$, which implies $r_{k-1} < v^*$. To show lastly that $r_k < r_{k-1}$, notice that the first-order conditions w.r.t. r_k and r_{k-1} yield $J(r_k) = \frac{\Pi_{k+1}}{F(v^*)^{n-k}}$ and $J(r_{k-1}) = \frac{\Pi_k}{F(v^*)^{n-k+1}}$, respectively. So the result will follow if $\frac{\Pi_{k+1}}{F(v^*)^{n-k}} < \frac{\Pi_k}{F(v^*)^{n-k+1}}$ or $F(v^*)\Pi_{k+1} < \Pi_k$, which is true since $\Pi_k = \tilde{\Pi}_k(r_k, \Pi_{k+1}) > \tilde{\Pi}_k(v^*, \Pi_{k+1}) = F(v^*)\Pi_{k+1}$. ▌

Combining (33), (34), and (35) gives a solution to the problem $[P']$. It remains to check that it satisfies (M) and (B) and thus solves $[P]$ as well. It is straightforward to check that (M) is satisfied. To show that (B) is satisfied, it suffices to construct an ex-post allocation rule that generates the interim rule in (33) and (35):

$$q_i(v) = \begin{cases} 1 & \text{if either } v_i > \max\{v^*, \max_{j\neq i} v_j\} \\ & \quad \text{or } v_i \in [r_i, v^*), \max_{j \neq i} v_j < v^*, \text{ and } v_j < r_j, \forall j < i \,. \\ 0 & \text{otherwise} \end{cases} \quad (39)$$

It is straightforward to check that q specified above generates the desired interim rule. ▌

Proof of Theorem 4: Given the interim winning probability profile $p = (p_1, \cdots, p_n)$, the capacity constraint (B) is the same as requiring $p \in \mathcal{P}$ defined in (20). Then, p can be expressed as a convex combination of extreme points of $\mathcal{P}$ given in (21): letting Π denote the set of all permutation functions, and for each $\pi \in \Pi$, letting $p_i^\pi := \prod_{j:\pi(j)<\pi(i)} F_i(r_i)$ and $p^\pi = (p_1^\pi, \cdots, p_n^\pi)$, there is some vector $(\lambda^\pi)_{\pi\in\Pi} \in \Delta\Pi$ for each $p \in \mathcal{P}$ such that

$p = \sum_{\pi\in\Pi} \lambda^\pi p^\pi$. Given this, the objective can be rewritten as

$$\begin{aligned}
&\max_{(\lambda^\pi)_{\pi\in\Pi},(r_i)_{i\in N}} \sum_{\pi\in\Pi} \lambda^\pi \sum_{i\in N} p_i^\pi r_i(1-F_i(r_i)) \\
&= \max_{(\lambda^\pi)_{\pi\in\Pi}} \sum_{\pi\in\Pi} \lambda^\pi \Big[\max_{(r_i)_{i\in N}} \sum_{i\in N} p_i^\pi r_i(1-F_i(r_i))\Big] \\
&= \max_{\pi\in\Pi} \Big[\max_{(r_i)_{i\in N}} \sum_{i\in N} p_i^\pi r_i(1-F_i(r_i))\Big]
\end{aligned}$$

as desired, where the second equality follows since the objective function is linear with respect to the weighting vector $(\lambda^\pi)_{\pi\in\Pi}$ so the entire weight can be be assigned to a function π that maximizes the term in the square bracket.

Next, we show that $r_i \geq r_j$ if $\pi(i) < \pi(j)$. Suppose to the contrary that it is not true. Then, there must be some $i, j \in N$ such that $\pi(i) = \pi(j) - 1$ (i.e. i immediately precedes j) and $r_i < r_j$. Consider now an alternative mechanism (r', π') which is the same as (r, π), except that the orders of i and j are reversed: i.e. $\pi'(i) = \pi(j)$ and $\pi'(j) = \pi(i)$ while $\pi'(k) = \pi(k), \forall k \neq i, j$ and $r'_k = r_k, \forall k \in N$. Note that this does not affect the revenue from other bidders than i and j, while the revenue from i and j changes from

$$\left(\Pi_{k:\pi(k)<\pi(i)} F_k(r_k)\right) \left[(1-F_i(r_i))r_i + F_i(r_i)(1-F_j(r_j))r_j\right]$$

to

$$\left(\Pi_{k:\pi(k)<\pi(i)} F_k(r_k)\right) \left[(1-F_j(r_j))r_j + F_j(r_j)(1-F_i(r_i))r_i\right].$$

Subtracting the former from the latter and rearranging yield

$$\left(\Pi_{k:\pi(k)<\pi(i)} F_k(r_k)\right) (1-F_i(r_i))(1-F_j(r_j))(r_j - r_i) > 0,$$

so the revenue is higher with the alternative mechanism, a contradiction. ▮

Appendix Appendix C:. Proofs for Section 5

In this section it will be sufficient to focus on the case in which the seller offers a direct mechanism M. Let $\tilde{M} = (\tilde{q}, \tilde{t})$ denote an equilibrium outcome that results from a cartel game following M. Since $\tilde{M}$ is an equilibrium outcome, it must be incentive compatible. We next show that there is a lower bound for the payoff that each bidder enjoys in that equilibrium, assuming that all bidders play cartel-undominated strategies (on and off the equilibrium path).

To this end, fix any bidder i, and let $\pi^i = \{C^1, \ldots, C^k\}$ denote an arbitrary partition of $N\backslash\{i\}$, with the interpretation that each group $C^\ell, \ell = 1, \ldots, k$, forms a cartel, in case bidder i chooses not to join any cartel. Let Π^i denote the set of all such partitions. Finally, let $\Omega(v_C)$ be the set of cartel-undominated strategies at v_C.

LEMMA 6. *In any equilibrium outcome $\tilde{M}$, the interim payoff of each bidder i with value v_i must be at least*

$$\overline{U}^M(v_i) := \sup_{v_i'} \mathbb{E}_{v_{-i}}\Big[\inf\Big\{u_i^M(v_i', v_{C^1}', \ldots, v_{C^k}'|v_i)\Big| v_{C^\ell}' \in \Omega(v_{C^\ell}), \forall C^\ell \in \pi^i, \text{ and } \pi^i \in \Pi^i\Big\}\Big].$$

Proof. For any type profile (v_i, v_{-i}) and bidder i's report v_i', define

$$\overline{u}_i^M(v_i'|v_i, v_{-i}) = \inf\Big\{u_i^M(v_i', v_{C^1}', \ldots, v_{C^k}'|v_i)\Big| v_{C^\ell}' \in \Omega(v_{C^\ell}), \forall C^\ell \in \pi^i, \text{ and } \pi^i \in \Pi^i\Big\}.$$

Let $\mu_i(h_i) \in \Delta(\mathcal{V}_{-i})$ denote the bidder i's updated belief (under Bayes' rule) at the end of the cartel game following M, given that he has observed a (private) history h_i. Let H_i denote the set of all history that bidder i can observe with a positive probability at equilibrium. For each $h_i \in H_i$, let $\tau_i(h_i) \in [0, 1]$ denote the probability with which h_i arises at equilibrium. We now classify H_i into two categories depending on whether or not bidder i is a member of a cartel, and argue that after any history $h_i \in H_i$, the expected payoff of bidder i with value v_i is at least

$$\sup_{v_i' \in \mathcal{V}_i} \mathbb{E}_{\mu_i(h_i)}[\overline{u}_i^M(v_i'|v_i, v_{-i})]. \tag{40}$$

Note first that this is the lowest payoff bidder i can get when he is not a member of any cartel, given that all cartels employ cartel-undominated strategies and bidder i with belief $\mu_i(h_i)$ optimally responds to that. Clearly, his equilibrium payoff after history h_i where he did not join any cartel, cannot fall below (40). The same is true after history h_i where he joins some cartel, since his payoff from deviating to reject all cartel proposals is at least (40) so the payoff from having accepted some proposal cannot fall below (40). Thus, bidder i's interim payoff is at least

$$\begin{aligned}\mathbb{E}_{\tau_i}[\sup_{v_i' \in \mathcal{V}_i} \mathbb{E}_{\mu_i(h_i)}[\overline{u}_i^M(v_i'|v_i, v_{-i})]] &\geq \sup_{v_i' \in \mathcal{V}_i} \mathbb{E}_{\tau_i}[\mathbb{E}_{\mu_i(h_i)}[\overline{u}_i^M(v_i'|v_i, v_{-i})]] \\ &= \sup_{v_i' \in \mathcal{V}_i} \mathbb{E}_{v_{-i}}[\overline{u}_i^M(v_i'|v_i, v_{-i})] = \overline{U}_i^M(v_i),\end{aligned}$$

where the first equality follows from the fact that $\mathbb{E}_{\tau_i}[\mathbb{E}_{\mu_i(h_i)}[\cdot]] = \mathbb{E}_{v_{-i}}[\cdot]$ and the second equality from the definition of $\overline{u}_i^M$ and $\overline{U}_i^M$. ▮

In order to prove Theorem 5, we introduce a couple of general properties and show in Theorem 6 that they are sufficient for a WCP mechanism to be robustly collusion-proof.

DEFINITION 5. *A direct auction mechanism $M = (q, t)$ is **mDSIC** if (i) q_i is nondecreasing in v_i and nonincreasing in v_{-i}, and (ii)*

$$t_i(v_i, v_{-i}) = q_i(v_i, v_{-i})v_i - \int_{\underline{v}_i}^{v_i} q_i(s, v_{-i})ds; \tag{41}$$

Note that this property is slightly stronger than the usual dominant-strategy incentive compatibility since it requires q_i to be non-increasing with v_{-i}. To state the second property, given a direct mechanism $M = (q, t)$, let $u_i^M(v|v_i) = v_i q_i(v) - t_i(v)$ for any $v_i \in \mathcal{V}_i$ and $v \in \mathcal{V}$. Also, for any $S \subset N$, let $\mathcal{V}_S = \times_{i \in S}\mathcal{V}_i$.

DEFINITION 6. *A direct auction mechanism $M = (q, t)$ is **weakly non-bossy** if the following holds for any $C \subsetneq N$ and almost every $v_C \in \mathcal{V}_C$: for any two strategy profiles v'_C and $v''_C \le v'_C \in \mathcal{V}_C$ satisfying $u_i^M(v'_C, v_{N\setminus C}|v_i) = u_i^M(v''_C, v_{N\setminus C}|v_i), \forall i \in C, \forall v_{N\setminus C} \in \mathcal{V}_{N\setminus C}$, we must have $u_i^M(v'_C, v_{N\setminus C}|v_i) = u_i^M(v''_C, v_{N\setminus C}|v_i), \forall i \in N\setminus C, \forall v_{N\setminus C} \in \mathcal{V}_{N\setminus C}$.*

The weak non-bossiness requires no group of bidders to affect others' payoffs without changing their own payoffs. Note that this requirement is very weak since it only applies to two strategy profiles, v'_C and v''_C, that satisfy $v'_C \ge v''_C$ and also yield the same payoffs for bidders in C *irrespective of* strategies, $v_{N\setminus C}$, played by bidders outside C. All commonly known auction mechanisms are weakly non-bossy.

THEOREM 6. *If a direct auction mechanism $M = (q, t)$ is mDSIC, weakly non-bossy, and WCP, then it is robustly collusion-proof.*

Proof. Let us first make a couple observation from the fact that M is mDSIC. First, given (41), M is dominant-strategy incentive compatible. Second, if bidder i with value v_i reports truthfully, and others report any arbitrary v_{-i}, then he earns

$$u_i^M(v_i, v_{-i}|v_i) = v_i q_i(v_i, v_{-i}) - t_i(v_i, v_{-i}) = \int_{\underline{v}_i}^{v_i} q_i(s, v_{-i})ds,$$

which means that his payoff is decreasing in v_{-i} since q_i is decreasing in v_{-i}.

Now fix any coalition C of bidders and its value profile v_C. Consider any strategy profile of that coalition v'_C. Letting $v''_C = v_C \wedge v'_C$, i.e. $v''_i = \min\{v'_i, v_i\}, \forall i \in C$, we show that

$$u_i^M(v'_C, v_{N\setminus C}|v_i) \le u_i^M(v''_C, v_{N\setminus C}|v_i), \forall v_{N\setminus C}, \forall i \in C. \tag{42}$$

To do so, change the strategy of any bidder $j \in C$ from v'_j to v''_j and observe that

$$u_i^M(v'_C, v_{N\backslash C}|v_i) \leq u_i^M(v''_j, v'_{C\backslash\{j\}}, v_{N\backslash C}|v_i), \forall v_{N\backslash C}, \forall i \in C, \tag{43}$$

since the dominant-strategy incentive compatibility of M for bidder j means that

$$u_j^M(v'_C, v_{N\backslash C}|v_j) \leq u_j^M(v''_j, v'_{C\backslash\{j\}}, v_{N\backslash C}|v_j), \forall v_{N\backslash C},$$

and since, for $i \in C \setminus \{j\}$, the mDSIC property with $v''_j \leq v'_j$ implies that:

$$u_i^M(v'_C, v_{N\backslash C}|v_i) \leq u_i^M(v''_j, v'_{C\backslash\{j\}}, v_{N\backslash C}|v_i), \forall v_{N\backslash C}.$$

Now start from the strategy profile $(v''_j, v'_{C\backslash\{j\}})$ and change the strategy of another bidder $j' \in C\backslash\{j\}$ from $v'_{j'}$ to $v''_{j'}$, which (weakly) increases the payoffs of bidders in C in a way analogous to (43). The inequality (42) will then follow from repeating the same argument one by one for all bidders in C.

According to the above argument so far, we can have v'_C being C-undominated by v''_C–namely $v'_C \in \Omega(v_C)$–only if (42) holds as equality. Then, the weak non-bossiness requires that for almost every v_C, any bidder outside C must also be indifferent between $(v'_C, v_{N\backslash C})$ and $(v''_C, v_{N\backslash C})$. Thus, for almost every v_C and any $v'_C \in \Omega(v_C)$,

$$u_i^M(v'_C, v_{N\backslash C}|v_i) = u_i^M(v''_C, v_{N\backslash C}|v_i) \geq v_i^M(v_C, v_{N\backslash C}|v_i), \forall i \in N\backslash C, \forall v_{N\backslash C}, \tag{44}$$

where the inequality follows from $v''_C = v_C \wedge v'_C \leq v_C$.

Now fix any bidder $i \in N$ and consider $\pi^i = \{C^1, ..., C^k\} \in \Pi^i$. Repeatedly applying (44) from cartel C^1 to C^k, one can obtain that for any v_i and almost every $v_{-i} \in \mathcal{V}_{-i}$,

$$u_i^M(v_i, v'_{C^1}, ..., v'_{C^k}|v_i) \geq u_i^M(v_i, v_{-i}|v_i), \forall v'_{C^\ell} \in \Omega(v_{C^\ell}), \ell = 1, \cdots, k$$

which results in

$$\overline{U}_i^M(v_i) \geq \mathbb{E}_{v_{-i}}[u_i^M(v_i, v_{-i}|v_i)] = U_i^M(v_i). \tag{45}$$

Consider now any mechanism $\tilde{M}$ that satisfies (IC) and $(RC - IR)$. Combining $(RC - IR)$ and (45), we obtain $U_i^{\tilde{M}}(v_i) \geq \overline{U}_i^M(v_i) \geq U_i^M(v_i), \forall v_i, \forall i$. The inequality here must hold as equality due to the weak collusion-proofness of M, which in turn implies (by the standard revenue equivalence argument) that $Q_i(v_i) = \tilde{Q}_i(v_i)$ and $T_i(v_i) = \tilde{T}_i(v_i)$ for all $i \in N$ and almost every $v_i \in \mathcal{V}_i$. ▌

We now prove Theorem 5 using Theorem 6.

Proof of Theorem 5: Let us begin by observing that the ex-post allocation rule given in (39) satisfies Part (i) of mDSIC property. Using this allocation rule and (41), we obtain

$$t_i(v) = \begin{cases} \max_{j\neq i} v_j & \text{if } v_i \geq \max_{j\neq i} v_j \geq v^* \\ v^* & \text{if } v_i \geq v^* > \max_{j\neq i} v_j \text{ and } v_j \geq r_j \text{ for some } j < i \\ r_i & \text{if } v_i \geq r_i, v^* > \max_{j\neq i} v_j, \text{ and } v_j < r_j, \forall j < i \\ 0 & \text{otherwise} \end{cases}. \tag{46}$$

We now claim that the mechanism $M = (q, t)$ specified in (39) and (46) is RCP. By Theorem 6, it suffices to show that M is weakly non-bossy.

To do so, consider any set of bidders $C \subsetneq N$ and a value profile v_C such that $v_i \neq v_j, \forall i, j \in C$ and $v_i \neq r_i, \forall i \in C$.[23] Suppose that there are two strategy profiles v'_C and $v''_C \leq v'_C$ satisfying

$$u_i^M(v'_C, v_{N\backslash C}|v_i) = u_i^M(v''_C, v_{N\backslash C}|v_i), \forall i \in C, \forall v_{N\backslash C}. \tag{47}$$

We aim to show that

$$u_i^M(v'_C, v_{N\backslash C}|v_i) = u_i^M(v''_C, v_{N\backslash C}|v_i), \forall i \in N\backslash C, \forall v_{N\backslash C}. \tag{48}$$

Let $v'_{C,k}$ and $v''_{C,k}$ denote the k-th highest value from the profile v'_C and v''_C, respectively. Let also $v_{N\backslash C,k}$ denote the k-th highest value from $v_{N\backslash C}$. Define $m' = \#\{i \in C | v'_i = v'_{C,1}\}$.

We first consider the case where $v'_{C,1} \geq v^*$, and show that (47) holds only in case $v'_{C,1} = v''_{C,1}$, which will imply (48) since only the highest report from C can affect the payoffs of bidders outside C, given the mechanism M. Suppose for a contradiction that $v''_{C,1} < v'_{C,1}$. (Note that $v''_{C,1} \leq v'_{C,1}$ since $v''_C \leq v'_C$ by assumption.) We focus on the case $v''_{C,1} \geq v^*$ since the argument in case $v''_{C,1} < v^*$ is relatively straightforward. For any $v_{N\backslash C}$ with $v_{N\backslash C,1} \in (v''_{C,1}, v'_{C,1})$, all bidders in C obtain zero payoff with the strategy profile $(v''_C, v_{N\backslash C})$, which must also be true with $(v'_C, v_{N\backslash C})$ in order for (47) to hold. Let bidder $i \in C$ be such that $v'_i = v'_{C,1}$. Suppose first $m' = 1$ and choose $v_{N\backslash C}$ such that $v_{N\backslash C,1} \in (v'_{C,2}, v'_{C,1})$. Then, bidder i's payoff with the profile $(v'_C, v_{N\backslash C})$ would be $(v_i - v_{N\backslash C,1})$, which cannot be zero if we choose $v_{N\backslash C,1} \neq v_i$, contradicting (47). Suppose alternatively $m' > 1$ and so there is another bidder $j \in C$ such that $v'_j = v'_{C,1}$. In order that bidder i obtains zero payoff with

[23] Note that it suffices to consider almost every value profile to check the weak non-bossiness.

any profile $(v'_C, v_{N\backslash C})$ satisfying $v_{N\backslash C,1} < v'_{C,1}$, we must have $v_i = v'_{C,1}$, which implies that bidder j's payoff is $\frac{1}{m'}(v_j - v'_{C,1}) \neq 0$ since $v_j \neq v_i$, contradicting (47).

Let us now consider the case where $v'_{C,1} < v^*$. There are two cases to consider depending on whether or not there is some bidder $i \in C$ such that $v'_i \in [r_i, v^*)$. The argument is straightforward when there is no such bidder, and thus omitted. In case there is at least one such bidder, let $j = \min\{i \in C | v'_i \in [r_i, v^*)\}$. Then, (48) will clearly hold if $j = n$, which is because, in that case, $v''_i \leq v'_i < r_i$ for all $i \in C\backslash\{j\}$, $v''_j \leq v'_j < v^*$, and thus the change from v'_C to v''_C cannot affect the payoff of any $k \in N\backslash C$. In case $j < n$, a change in bidder j's report from v'_j to v''_j, given that $v''_i \leq v'_i < r_i$ for all $i \in C$ preceding j, can affect the payoffs of bidders outside C only when $v''_j < r_j \leq v'_j$. In this case, however, if $v_i < r_i$ for all $i \in N\backslash C$ with $i < j$, bidder j earns $(v_j - r_j) \neq 0$ by reporting v'_j while he earns zero by reporting v''_j, which contradicts (47). ▮

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Supplementary Appendix to
Weak Cartels and Collusion-Proof Auctions

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Proof of Lemma 3: Let us denote by I_T, the inequality (20) with $S = T \subset N$. We first show that any p satisfying (21) is an extreme point of $\mathcal{P}$. Note that the allocation probability p can be achieved in a feasible way by assigning the k-th order to each agent $\pi(k)$ and then allocating the object in this order as long as the agents have values no lower than their reserve prices, which implies that p is a reduced form. Thus Lemma 1 implies that p must satisfy (20) so $p \in \mathcal{P}_N$. Suppose for a contradiction that p is not an extreme point of $\mathcal{P}$. Then, we can write $p = \lambda p' + (1-\lambda)p''$ for some $p' \neq p$ and $p'' \neq p$. Since $p_{\pi(1)} = 1$ and $p'_{\pi(1)}, p''_{\pi(1)} \leq 1$ (by $I_{\{\pi(1)\}}$), we must have $p'_{\pi(1)} = p''_{\pi(1)} = 1$. Using this and applying $I_{\{\pi(1),\pi(2)\}}$ to p' and p'', we obtain $p'_{\pi(2)}, p''_{\pi(2)} \leq F_{\pi(1)}(r_{\pi(1)}) = p_{\pi(2)}$, which implies $p'_{\pi(2)} = p''_{\pi(2)} = F_{\pi(1)}(r_{\pi(1)}) = p_{\pi(2)}$. Proceeding in this fashion, one can easily verify that $p'_{\pi(k)} = p''_{\pi(k)} = \prod_{j<k} F_{\pi(j)}(r_{\pi(j)}) = p_{\pi(k)}$ for all $k \in N$, a contradiction. To show that any extreme point $p \in \mathcal{P}$ can be expressed as in (21) for some π, we first establish the following claim.

Claim 4. *If for any two sets $S, T \subset N$, I_S and I_T hold as equality, then either $S \subset T$ or $T \subset S$.*

Proof. Suppose for a contradiction that for some $S, T \subset N$, I_S and I_T hold as equality but $S \not\subseteq T$ and $T \not\subseteq S$. Then, we must have $S \cap T \subsetneq T$ and $S \backslash T \neq \emptyset$. To draw a contradiction, let us first show

$$\sum_{i \in T\backslash S} p_i(1 - F_i(r_i)) \leq \prod_{i \in S} F_i(r_i)\Big(1 - \prod_{i \in T\backslash S} F_i(r_i)\Big). \tag{S.1}$$

For this, note that by $I_{S\cup T}$ and the assumption,

$$\sum_{i \in S\cup T} p_i(1 - F_i(r_i)) \leq 1 - \prod_{i \in S\cup T} F_i(r_i) \tag{S.2}$$

$$\sum_{i \in S} p_i(1 - F_i(r_i)) = 1 - \prod_{i \in S} F_i(r_i). \tag{S.3}$$

It is straightforward to see that (S.1) results from subtracting (S.3) from (S.2) side by side. Also, by $I_{S\cap T}$, we have

$$\sum_{i\in S\cap T} p_i(1-F_i(r_i)) \leq 1 - \prod_{i\in S\cap T} F_i(r_i).$$

Adding this inequality and (S.1) side by side yields

$$\sum_{i\in T} p_i(1-F_i(r_i)) \leq 1 + \prod_{i\in S} F_i(r_i) - \prod_{i\in S\cap T} F_i(r_i) - \prod_{i\in S\cup T} F_i(r_i) < 1 - \prod_{i\in T} F_i(r_i), \quad \text{(S.4)}$$

where the strict inequality follows since $S\cap T \subsetneq T$ and $S\backslash T \neq \emptyset$, and thus

$$\begin{aligned}\prod_{i\in S\cap T} F_i(r_i) + \prod_{i\in S\cup T} F_i(r_i) - \prod_{i\in S} F_i(r_i) - \prod_{i\in T} F_i(r_i) \\ = \Big(\prod_{i\in S\cap T} F_i(r_i) - \prod_{i\in T} F_i(r_i)\Big)\Big(1 - \prod_{i\in S\backslash T} F_i(r_i)\Big) > 0.\end{aligned}$$

However, (S.4) contradicts the assumption that I_T holds as equality. ▮

Pick any extreme point $p \in \mathcal{P}$ satisfying (20). Let us denote by $S_1, \cdots, S_m$, all subsets of N for which (20) is satisfied as equality given p. Due to the above Claim, these subsets must have a nested structure, that is $S_1 \subset S_2 \subset \cdots \subset S_m$. Since N contains n elements, m cannot be greater than n. Suppose for a contradiction that $m < n$. Then, there must be some k and $h, j \in N$ with $h \neq j$ such that $h, j \notin S_\ell$ if $\ell < k$ and $h, j \in S_\ell$ if $\ell \geq k$. We now show that p can be obtained by linearly combining some p' and $p'' \in \mathcal{P}$, contradicting that p is an extreme point. To do so, we denote by e^i, n-dimensional vector with its i-the element being 1 and all others being zero. Let $p' = p - \epsilon e^h + \delta e^j$ and $p'' = p + \epsilon e^h - \delta e^j$, where ϵ and δ are sufficiently small positive real numbers satisfying

$$\epsilon(1 - F_h(r_h)) = \delta(1 - F_j(r_j)),$$

which implies that for all $\ell < k$, $\sum_{i\in S_\ell} p'_i(1-F_i(r_i)) = \sum_{i\in S_\ell} p_i(1-F_i(r_i))$ and for all $\ell \geq k$,

$$\sum_{i\in S_\ell} p'_i(1-F_i(r_i)) = \sum_{i\in S_\ell} p_i(1-F_i(r_i)) - \epsilon(1-F_h(r_h)) + \delta(1-F_j(r_j)) = \sum_{i\in S_\ell} p_i(1-F_i(r_i))$$

and similarly for p''. From this, we see that whether p satisfies (20) as equality or strict inequality, the same is true for p' and p'', provided that ϵ and δ are sufficiently small, which means $p', p'' \in \mathcal{P}$. However, $p = \frac{1}{2}p' + \frac{1}{2}p''$, resulting in the desired contradiction. Thus, we must have $m = n$, which implies $|S_k \backslash S_{k-1}| = 1$ for all $k = 1, \cdots, n$ with $S_0 = \emptyset$.

To complete the proof, define the permutation function $\pi : N \to N$ such that $\pi(i) = k$ if $\{i\} = S_k \backslash S_{k-1}$. Then, by definition of S_1, I_{S_1} holds as equality or $p_i(1-F_i(r_i)) = 1-F_i(r_i)$ for $i = \pi^{-1}(1)$, which yields $p_i = 1$ for $i = \pi^{-1}(1)$. For an induction argument, suppose

$$p_i = \prod_{j:\pi(j)<\pi(i)} F_j(r_j) \text{ for } i = \pi^{-1}(k'), \forall k' = 1, \cdots, k-1. \tag{S.5}$$

Then, by definition of S_k and π, we must have for $i = \pi^{-1}(k)$

$$\sum_{j:\pi(j)\le\pi(i)=k} p_j[1-F_j(r_j)] = 1 - \prod_{j:\pi(j)\le\pi(i)=k} F_j(r_j),$$

which after substituting (S.5) and canceling the terms, leads us to obtain $p_i = \prod_{j:\pi(j)<\pi(i)} F_j(r_j)$, as desired. ▮

Proof of Lemma 4: We let $Q = (Q_1, \cdots, Q_n)$ denote the optimum of $[P']$. First suppose to the contrary that $r_j < \hat{v}$ for some j. Let us construct an alternative rule $\overline{Q} = (\overline{Q}_1, \cdots, \overline{Q}_n)$ as

$$\overline{Q}_i(v_i) = \begin{cases} 0 & \text{if } i = j \text{ and } v_i < \hat{v} \\ Q_i(v_i) & \text{otherwise.} \end{cases}$$

Clearly, the value of objective function is higher with $\overline{Q}$ than with Q. Also, $\overline{Q}$ satisfies the constraints (M) and (CP). So it only remains to show that (B) is satisfied, for which it suffices to construct an ex-post rule generating $\overline{Q}$.[24] To do so, let $(q_1, \cdots, q_n)$ denote the ex-post allocation rule for Q and we can construct the ex-post rule for $\overline{Q}$ as

$$\overline{q}_i(v) = \begin{cases} 0 & \text{if } i = j \text{ and } v_j < \hat{v} \\ q_i(v) & \text{otherwise.} \end{cases}$$

To prove the second statement, we first show that any optimum must have $r_i \in [\hat{v}, v^*)$ for at least one agent i. Suppose not, i.e. $r_i \ge v^*, \forall i \in N$. We draw a contradiction by constructing an alternative interim allocation rule $\tilde{Q}$ which makes the value of objective function greater than Q does, and which differs from Q only in that for an arbitrarily chosen agent k, $\tilde{Q}_k(v_k) = \tilde{p}_k = F(v^*)^{n-1}$ for $v_k \in [\hat{v}, v^*]$. It is clear that the value of objective function is greater with $\tilde{Q}$ by as much as $F(v^*)^{n-1}\int_{\hat{v}}^{v^*} J(s)dF(s) > 0$. Since $Y(v)$ for all $v \ge v^*$ is unaffected by the change from Q to $\tilde{Q}$, it only remains to check (25). Letting

[24] Recall that the condition (B) is necessary and sufficient for there to be an ex-post allocation rule that generates Q as an associated interim allocation rule.

$\tilde{r}_i = \inf\{v_i \in \mathcal{V}_i | \tilde{Q}_i(v_i) > 0\}$ and $\tilde{N}_* = \{i \in N | \tilde{r}_i < v^*\}$, we have $\tilde{N}_* = \{k\}$ so can focus on $M = \{k\}$. Then, (25) can be written as

$$F(v^*)^{n-1}[F(v^*) - F(\hat{v})] \leq Y(v^*) + F(v^*)^n - F(v^*)^{n-1}F(\hat{v}),$$

which clearly holds since $Y(v^*) \geq 0$.

Given a solution Q of $[P']$, the above argument means $N_* = \{i \in N | r_i < v^*\} \neq \emptyset$. Assuming that $N_* \neq N$, we construct an alternative solution $\tilde{Q}$ of $[P']$ such that $\tilde{r}_i = \inf\{v_i \in \mathcal{V}_i | \tilde{Q}_i(v_i) > 0\} < v^*, \forall i \in N$. Letting $N^* = N \backslash N_*$, select any agent $k \in N_*$ and then define $\tilde{Q}$ to be the same as Q, except that for each agent $i \in N^* \cup \{k\}$, $\tilde{Q}_i(v_i) = \tilde{p}_i = \frac{p_k}{|N^*|+1}$ for $v_i \in (r_k, v^*)$, where $p_k = Q_k(v_k) > 0$ for $v_k \in (r_k, v^*)$. It is clear that the value of objective function remains the same under $\tilde{Q}$. Since $Y(v)$ for all $v \geq v^*$ is unaffected, it only remains to check (25). We can focus on such M that $M \cap (N^* \cup \{k\}) \neq \emptyset$, since the allocation for each agent $i \notin N^* \cup \{k\}$ has not been changed. For any such M, note that

$$\begin{aligned}
\sum_{i \in M} \tilde{p}_i[F(v^*) - F(\tilde{r}_i)] &= \sum_{i \in M \cap (N_* \backslash \{k\})} p_i[F(v^*) - F(r_i)] + \sum_{i \in M \cap (N^* \cup \{k\})} \tfrac{p_k}{|N^*|+1}[F(v^*) - F(r_k)] \\
&\leq \sum_{i \in M \cap (N_* \backslash \{k\})} p_i[F(v^*) - F(r_i)] + p_k[F(v^*) - F(r_k)] \\
&= \sum_{i \in (M \cap N_*) \cup \{k\}} p_i[F(v^*) - F(r_i)]. \qquad \text{(S.6)}
\end{aligned}$$

Also, we have

$$\begin{aligned}
\sum_{i \in (M \cap N_*) \cup \{k\}} p_i[F(v^*) - F(r_i)] &\leq Y(v^*) + F(v^*)^n - F(v^*)^{n - |(M \cap N_*) \cup \{k\}|} \left(\prod_{i \in (M \cap N_*) \cup \{k\}} F(r_i) \right) \\
&\leq Y(v^*) + F(v^*)^n - F(v^*)^{n-|M|} \left(\prod_{i \in M} F(\tilde{r}_i) \right), \qquad \text{(S.7)}
\end{aligned}$$

where the first inequality holds since Q satisfies (25) and the second due to the facts that $M \cap (N^* \cup \{k\}) \neq \emptyset$ implies $|(M \cap N_*) \cup \{k\}| \leq |M|$ and that $\tilde{r}_i \leq \min\{r_i, v^*\}, \forall i \in N$. Combining (S.6) and (S.7), we obtain

$$\sum_{i \in M} \tilde{p}_i[F(v^*) - F(\tilde{r}_i)] \leq Y(v^*) + F(v^*)^n - F(v^*)^{n-|M|} \left(\prod_{i \in M} F(\tilde{r}_i) \right),$$

so $\tilde{Q}$ satisfies (25). ▮