

The Gauss curvature flow : Regularity and Asymptotic Behavior

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ABSTRACT

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This thesis contains the author's results on the evolution of convex hypersurfaces by positive powers of the Gauss curvature

$$\frac{\partial}{\partial t} F = K^\alpha \vec{n}.$$

We first establish interior estimates for strictly convex solutions by deriving lower bounds for the principal curvatures and upper bounds for the Gauss curvature. We also investigate the optimal regularity of weakly convex translating solutions. The interesting case is when the translator has flat sides. We prove the existence of such translators and show that they are of optimal class $C^{1,1}$. Finally, we classify all closed self-similar solutions of the Gauss curvature flow which is closely related to the asymptotic behavior.

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to my parents

Chapter 1

Introduction

We consider a family of complete non-compact strictly convex hypersurfaces Σ_t embedded in \mathbb{R}^{n+1} which evolve by any positive power of their Gauss curvature K .

Given a complete and strict convex hypersurface Σ_0 embedded in \mathbb{R}^{n+1} , we let $F_0 : M^n \rightarrow \mathbb{R}^{n+1}$ be an immersion with $F_0(M^n) = \Sigma_0$. For a given number $\alpha > 0$, we say that an one-parameter family of immersions $F : M^n \times (0, T) \rightarrow \mathbb{R}^{n+1}$ is a solution of the α -Gauss curvature flow, if for each $t \in (0, T)$, $F(M^n, t) = \Sigma_t$ is a complete and strictly convex hypersurface embedded in \mathbb{R}^{n+1} , and $F(\cdot, t)$ satisfies

$$(*^\alpha) \quad \begin{cases} \frac{\partial}{\partial t} F(p, t) &= K^\alpha(p, t) \vec{n}(p, t) \\ \lim_{t \rightarrow 0} F(p, t) &= F_0(p) \end{cases}$$

where $K(p, t)$ is the Gauss curvature of Σ_t at $F(p, t)$ and $\vec{n}(p, t)$ is the interior unit normal vector of Σ_t at the point $F(p, t)$.

The Gauss curvature flow was first introduced by W. Firey in [18], where he showed that a closed strictly convex and centrally symmetric solution in \mathbb{R}^3 converges to a round sphere after rescaling the solution. In [25] K. Tso established the existence of closed and strictly convex solutions in \mathbb{R}^{n+1} and showed that it converges to a point. B. Chow [12] extended Tso's result to the flow by positive powers of the Gauss curvature, namely a strictly convex closed solution to the α -Gauss curvature flow $\partial_t F = K^\alpha \vec{n}$. Schnürer [24] showed the existence of strictly convex entire graph solution to the α -Gauss curvature flow for $0 < \alpha < \frac{1}{n-1}$. The author, jointly with P. Daskalopoulos, L. Kim, and K. Lee [10], established the all-times existence of non-compact, complete, and strictly convex solution solutions to the α -Gauss curvature flow for $\alpha > 0$. In addition, J. Urbas [26], [27] showed that given an exponent $\alpha > \frac{1}{2}$ and a strictly convex open bounded domain $\Omega \subset \mathbb{R}^{n+1}$, there exists a complete smooth strictly convex graph translating by the α -power of the Gauss curvature K^α .

Regarding weakly convex solutions, B. Andrews [1] showed the optimal $C^{1,1}$ regularity of the solution to the Gauss curvature flow in the case $n = 2$, and extended to the result to $\alpha \in [\frac{1}{2}, 1]$ and $n = 2$ with X. Chen [3]. In [16] the $C^{1,\beta}$ regularity of weakly convex viscosity solutions to the α -Gauss curvature flow was established. The author, jointly with P. Daskalopoulos and K. Lee [11], established the optimal $C^{1,1}$ regularity to the translating solutions to the Gauss curvature flow in the case $n = 2$. They also showed that the translator defined on a square has flat sides so that $C^{1,1}$ is actually the optimal regularity.

In particular, if the initial closed convex hypersurface Σ_0 has a flat side, then the solution Σ_t to the α -Gauss curvature flow with $\alpha > \frac{1}{n}$ preserves the flat side for some finite time. See [14], [15], [23]. If $\alpha \leq \frac{1}{n}$, B. Andrews [2] showed that any weakly convex closed hypersurface becomes strictly convex immediately by the α -Gauss curvature flow.

The convergence of the flow to a closed self-similar solution has been widely studied in [1, 2, 3, 4, 12, 13, 19, 20, 21, 22] for $\alpha > \frac{1}{n+2}$. In the case $\alpha = \frac{1}{n+2}$, E. Calabi [7] that showed that closed self-similar solutions are ellipsoids. In the case $\alpha = \frac{1}{n}$, B. Chow [12] established the convergence to the round sphere, namely the sphere is the unique closed self-similar solution. In the two dimension case $n = 2$, B. Andrews showed the convergence to sphere for $\alpha = 1$ in [1], and for $\alpha \in (\frac{1}{2}, 1)$ with X. Chen in [3]. P. Guan and L. Ni [19] obtained the convergence of the Gauss curvature flow to a self-similar solution and in [4] they extended the same result to $\alpha \geq \frac{1}{n+2}$ jointly with B. Andrews. So, it remained to classify the closed self-similar solutions $\Sigma_t = (T - t)^{\frac{1}{1+n\alpha}} \Sigma$. Namely, it is enough to classify strictly convex closed hypersurfaces Σ satisfying

$$(**^\alpha) \quad K^\alpha(p) = -\langle F(p), \vec{n}(p) \rangle.$$

The author and P. Daskalopoulos [9] showed the uniqueness of the closed self-similar solution to the α -Gauss curvature flow for $\alpha \in [\frac{1}{n}, 1 + \frac{1}{n})$, and jointly with S. Brendle, they extended the result to $\alpha > \frac{1}{n+2}$. Moreover, in the case $\alpha = \frac{1}{n+2}$, they gave an alternating proof of the classification result of Calabi [7].

Chapter 2

Preliminaries

1 Notation

We summarize the following notation which will be frequently used in what follows.

- (i) We recall that $g_{ij} = \langle F_i, F_j \rangle$, where $F_i := \nabla_i F$. Also, we denote as usual by g^{ij} the inverse matrix of g_{ij} and $F^i = g^{ij} F_j$.
- (ii) Assume Σ_t is a strictly convex graph solution of $(*)^a$ in \mathbb{R}^{n+1} . Then, we let $\bar{u}(\cdot, t) : M^n \rightarrow \mathbb{R}$ denote the *height function* $\bar{u}(p, t) = \langle F(p, t), \vec{e}_{n+1} \rangle$.
- (iii) Given constants M and $\beta \geq 0$, we define the *cut-off function* ψ_β by

$$\psi_\beta(p, t) = (M - \beta t - \bar{u}(p, t))_+ = \max\{M - \beta t - \bar{u}(p, t), 0\}.$$

In particular, we denote $\psi_0 := (M - \bar{u}(p, t))_+$ by ψ for convenience.

Also, given a constant $R > 0$ and a point $Y \in \mathbb{R}^{n+1}$, we define the *cut-off function* $\bar{\psi}$ by

$$\bar{\psi}(p, t) = (R^2 - |F(p, t) - Y|^2)_+ = \max\{R^2 - |F(p, t) - Y|^2, 0\}$$

- (iv) Given a ball $B_R^{n+1}(Y) \subset \mathbb{R}^{n+1}$ and a complete hypersurface $\Sigma \subset \mathbb{R}^{n+1}$, we say that a compact hypersurface Σ^c with boundary $\partial\Sigma^c$ is cut off from Σ by $B_R^{n+1}(Y)$, if $\Sigma^c \subset \Sigma$ and $\partial\Sigma^c \subset \partial B_R^{n+1}(Y)$ hold. Moreover, Given an immersion $F : M^n \times [0, T] \rightarrow \mathbb{R}^{n+1}$ defining $F(M^n, t) = \Sigma_t$ and a ball $B_R^{n+1}(Y)$ cutting Σ_t , we define a cut-off function $\eta : M^n \times [0, T] \rightarrow \mathbb{R}^{n+1}$ by

$$\eta(p, t) = (|F(p, t) - Y|^2 - R^2)_+.$$

- (v) For a strictly convex smooth hypersurface Σ_t , we denote by b^{ij} the the inverse matrix $(h^{-1})^{ij}$ of its *second fundamental form* h_{ij} , namely $b^{ij}h_{jk} = \delta_k^i$. Notice that the eigenvalues of b^{ij} on an orthonormal frame are the *principal radii of curvature*.

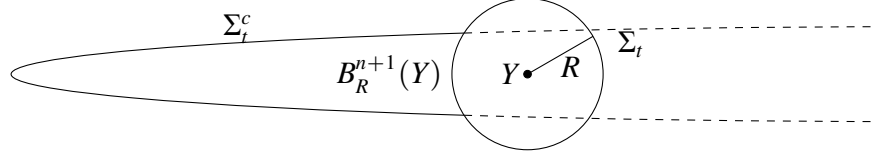


FIGURE 1. Cutting ball

(vi) We denote by \mathcal{L} the *linearized* operator

$$\mathcal{L} = \alpha K^\alpha b^{ij} \nabla_i \nabla_j$$

Furthermore, $\langle \cdot, \cdot \rangle_{\mathcal{L}}$ denotes the associated inner product $\langle \nabla f, \nabla g \rangle_{\mathcal{L}} = \alpha K^\alpha b^{ij} \nabla_i f \nabla_j g$, where f, g are differentiable functions on M^n , and $\| \cdot \|_{\mathcal{L}}$ denotes the \mathcal{L} -norm given by the inner product $\langle \cdot, \cdot \rangle_{\mathcal{L}}$

(vii) We denote by $\nu = \langle \vec{n}, \vec{e}_{n+1} \rangle^{-1}$ the *gradient function*.

(viii) H and λ_{\min} denote the *mean curvature* and the *smallest principal curvature*, respectively.

(ix) We will use in the sequel the functions $Z : M^n \rightarrow \mathbb{R}$ and $W : M^n \rightarrow \mathbb{R}$ defined by

$$Z(p) = \left(K^\alpha b^{ij} g_{ij} - \frac{n\alpha - 1}{2\alpha} |F|^2 \right)(p), \quad W(p) = \left(K^\alpha \lambda_{\min}^{-1} - \frac{n\alpha - 1}{2n\alpha} |F|^2 \right)(p).$$

2 Evolution equations

In this section we will derive some basic equations of the α -Gauss curvature flow.

PROPOSITION 2.1. *Assume that Σ_t is a complete strictly convex graph solution of $(*)^\alpha$. Then, the following hold*

$$(2.1) \quad \partial_t g_{ij} = -2K^\alpha h_{ij}$$

$$(2.2) \quad \partial_t g^{ij} = 2K^\alpha h^{ij}$$

$$(2.3) \quad \partial_t \vec{n} = -\nabla K^\alpha := -(\nabla_j K^\alpha) F^j$$

$$(2.4) \quad \partial_t \psi_\beta = \mathcal{L} \psi + (n\alpha - 1) \nu^{-1} K^\alpha - \beta$$

$$(2.5) \quad \partial_t \bar{\psi} \leq \mathcal{L} \bar{\psi} + 2(n\alpha + 1)(\lambda_{\min}^{-1} + R) K^\alpha$$

$$(2.6) \quad \partial_t h_{ij} = \mathcal{L} h_{ij} + \alpha K^\alpha (\alpha b^{kl} b^{mn} - b^{km} b^{nl}) \nabla_i h_{mn} \nabla_j h_{kl} + \alpha K^\alpha H h_{ij} - (1 + n\alpha) K^\alpha h_{ik} h_j^k$$

$$(2.7) \quad \partial_t K^\alpha = \mathcal{L} K^\alpha + \alpha K^{2\alpha} H$$

$$(2.8) \quad \partial_t b^{pq} = \mathcal{L} b^{pq} - \alpha K^\alpha b^{ip} b^{jq} (\alpha b^{kl} b^{mn} + b^{km} b^{nl}) \nabla_i h_{kl} \nabla_j h_{mn} - \alpha K^\alpha H b^{pq} + (1 + n\alpha) K^\alpha g^{pq}$$

$$(2.9) \quad \partial_t \nu = \mathcal{L} \nu - 2\nu^{-1} \|\nabla \nu\|_{\mathcal{L}}^2 - \alpha K^\alpha H \nu$$

Proof of (2.1). Observe $\partial_t g_{ij} = \langle \partial_t \nabla_i F, \nabla_j F \rangle + \langle \nabla_i F, \partial_t \nabla_j F \rangle = \langle \nabla_i \partial_t F, \nabla_j F \rangle + \langle \nabla_i F, \nabla_j \partial_t F \rangle$. Hence, $\langle \nabla_i F, \partial_t F \rangle = 0$ gives $\partial_t g_{ij} = -2\langle \partial_t F, \nabla_i \nabla_j F \rangle = -2\langle K^\alpha \vec{n}, h_{ij} \vec{n} \rangle = -2K^\alpha h_{ij}$.

Proof of (2.2). From $g^{ij} g_{jk} = \delta_k^i$, we can derive $\partial_t g^{ij} = -g^{ik} g^{jl} \partial_t g_{kl} = 2K^\alpha g^{ik} g^{jl} h_{kl} = 2K^\alpha h^{ij}$.

Proof of (2.3). $|\vec{n}|^2 = 1$ implies $\langle \partial_t \vec{n}, \vec{n} \rangle = 0$. Also, $\langle \vec{n}, \nabla_i F \rangle = 0$ leads to

$$\langle \partial_t \vec{n}, \nabla_i F \rangle = -\langle \vec{n}, \partial_t \nabla_i F \rangle = -\langle \vec{n}, \nabla_i \partial_t F \rangle = -\langle \vec{n}, \nabla_i (K^\alpha \vec{n}) \rangle = -\nabla_i K^\alpha$$

from which (2.3) readily follows.

Proof of (2.4). The definition of the linearized operator $\mathcal{L} := \alpha K^\alpha b^{ij} \nabla_i \nabla_j$ gives

$$(2.10) \quad \mathcal{L}F := \alpha K^\alpha b^{ij} \nabla_i \nabla_j F = \alpha K^\alpha b^{ij} h_{ij} \vec{n} = n\alpha K^\alpha \vec{n} = (n\alpha) \partial_t F$$

which yields that $\mathcal{L}\bar{u} = (n\alpha) \partial_t \bar{u}$. Therefore,

$$\partial_t \psi_\beta = -\beta - \partial_t \bar{u} = -\mathcal{L}\bar{u} + (n\alpha - 1) \partial_t \bar{u} - \beta$$

holds on the support of $\psi_\beta := (M - \beta t - \bar{u}(p, t))_+$. Substituting for $\partial_t \bar{u} = \langle \partial_t F, \vec{e}_{n+1} \rangle = \langle K^\alpha \vec{n}, \vec{e}_{n+1} \rangle = K^\alpha v^{-1}$, where $v := \langle \vec{n}, \vec{e}_{n+1} \rangle^{-1}$, yields (2.4).

Proof of (2.5). By (2.10), on the support of $\bar{\psi} := (R^2 - |F - Y|^2)_+$ we have

$$\begin{aligned} \partial_t \bar{\psi} &= -2\langle (\mathcal{L}F - (n\alpha) \partial_t F) + \partial_t F, F - Y \rangle = -2\langle \mathcal{L}F, F - Y \rangle + 2(n\alpha - 1)\langle K^\alpha \vec{n}, F - Y \rangle \\ &\leq \mathcal{L}\bar{\psi} + 2\|\nabla F\|_{\mathcal{L}}^2 + 2K^\alpha |n\alpha - 1| |F - Y| \leq \mathcal{L}\bar{\psi} + 2\alpha K^\alpha b^{ij} g_{ij} + 2(n\alpha + 1)RK^\alpha \\ &\leq \mathcal{L}\bar{\psi} + 2n\alpha \lambda_{\min}^{-1} K^\alpha + 2(n\alpha + 1)RK^\alpha \leq \mathcal{L}\bar{\psi} + 2(n\alpha + 1)(\lambda_{\min}^{-1} + R)K^\alpha. \end{aligned}$$

Proof of (2.6). We have

$$\begin{aligned} \partial_t h_{ij} &= \partial_t \langle \nabla_i \nabla_j F, \vec{n} \rangle = \langle \nabla_i \nabla_j \partial_t F, \vec{n} \rangle + \langle \nabla_i \nabla_j F, \partial_t \vec{n} \rangle = \langle \nabla_i \nabla_j (K^\alpha \vec{n}), \vec{n} \rangle + \langle h_{ij} \vec{n}, \partial_t \vec{n} \rangle \\ &= \nabla_i \nabla_j K^\alpha + \langle \nabla_j K^\alpha \nabla_i \vec{n}, \vec{n} \rangle + \langle \nabla_i K^\alpha \nabla_j \vec{n}, \vec{n} \rangle + K^\alpha \langle \nabla_i \nabla_j \vec{n}, \vec{n} \rangle + 0. \end{aligned}$$

By $\langle \nabla_i \vec{n}, \vec{n} \rangle = 0$ and $\langle \nabla_i \nabla_j \vec{n}, \vec{n} \rangle = -\langle \nabla_j \vec{n}, \nabla_i \vec{n} \rangle = -h_{im} h_j^m$, we deduce

$$(2.11) \quad \partial_t h_{ij} = \nabla_i \nabla_j K^\alpha - K^\alpha h_{im} h_j^m.$$

Applying $K = \det(h_{ij} g^{jk}) = \det(h_{ij}) \det(g^{kl})$, $\nabla \log \det(h_{ij}) = b^{kl} \nabla h_{kl}$ and $\partial b^{kl} / \partial h_{mn} = -b^{km} b^{ln}$ on the first term on the right hand side of (2.11), yields

$$\begin{aligned} \partial_t h_{ij} &= \nabla_i \nabla_j K^\alpha - K^\alpha h_{im} h_j^m = \nabla_i (\alpha K^\alpha b^{kl} \nabla_j h_{kl}) - K^\alpha h_{im} h_j^m \\ &= \alpha^2 K^\alpha b^{kl} b^{mn} \nabla_i h_{mn} \nabla_j h_{kl} + \alpha K^\alpha \frac{\partial b^{kl}}{\partial h_{mn}} \nabla_i h_{mn} \nabla_j h_{kl} + \alpha K^\alpha b^{kl} \nabla_i \nabla_j h_{kl} - K^\alpha h_{im} h_j^m \\ &= \alpha K^\alpha b^{kl} \nabla_i \nabla_j h_{kl} + \alpha K^\alpha (\alpha b^{kl} b^{mn} - b^{km} b^{ln}) \nabla_i h_{mn} \nabla_j h_{kl} - K^\alpha h_{im} h_j^m. \end{aligned}$$

On the other hand,

$$\begin{aligned}
\alpha K^\alpha b^{kl} \nabla_i \nabla_j h_{kl} &= \alpha K^\alpha b^{kl} \nabla_i \nabla_k h_{jl} = \alpha K^\alpha b^{kl} \nabla_k \nabla_i h_{jl} + \alpha K^\alpha b^{kl} R_{ikjm} h_l^m + \alpha K^\alpha b^{kl} R_{iklm} h_j^m \\
&= \alpha K^\alpha b^{kl} \nabla_k \nabla_l h_{ij} + \alpha K^\alpha (h_{ij} h_{km} - h_{im} h_{kj}) h_l^m b^{lk} + \alpha K^\alpha (h_{il} h_{km} b^{kl} - h_{im} h_{kl} b^{kl}) h_j^m \\
&= \mathcal{L} h_{ij} + \alpha K^\alpha (h_{ij} h_{km} - h_{im} h_{kj}) g^{mk} + \alpha K^\alpha (h_{im} - n h_{im}) h_j^m \\
&= \mathcal{L} h_{ij} + \alpha K^\alpha H h_{ij} - n \alpha K^\alpha h_{im} h_j^m
\end{aligned}$$

and (2.6) easily follows.

Proof of (2.7). From (2.11) we have

$$\mathcal{L} K^\alpha = \alpha K^\alpha b^{ij} (\partial_i h_{ij} + K^\alpha h_{im} h_j^m) = \alpha K^\alpha b^{ij} \partial_i h_{ij} + \alpha K^{2\alpha} H.$$

Also, from $K = \det(h_{ij}) \det(g^{kl})$, we derive that

$$\begin{aligned}
\partial_i K^\alpha &= \alpha K^\alpha \partial_i (\log(\det h_{ij}) + \log(\det g^{kl})) = \alpha K^\alpha b^{ij} \partial_i h_{ij} + \alpha K^\alpha g_{kl} \partial_i g^{kl} \\
&= \mathcal{L} K^\alpha - \alpha K^{2\alpha} H + \alpha K^\alpha g_{kl} \partial_i g^{kl}.
\end{aligned}$$

Aplying (2.2) on the last term yields (2.7).

Proof of (2.8). The identity $b^{ij} h_{jk} = \delta_k^i$ leads to

$$(2.12) \quad \partial_i b^{pq} = -b^{ip} b^{jq} \partial_i h_{ij}$$

$$(2.13) \quad \nabla_m b^{pq} = -b^{ip} b^{jq} \nabla_m h_{ij}$$

Therefore,

$$\begin{aligned}
\nabla_n \nabla_m b^{pq} &= -b^{jq} \nabla_n b^{ip} \nabla_m h_{ij} - b^{ip} \nabla_n b^{jq} \nabla_m h_{ij} - b^{ip} b^{jq} \nabla_n \nabla_m h_{ij} \\
&= 2b^{jq} b^{ik} b^{pl} \nabla_n h_{kl} \nabla_m h_{ij} - b^{ip} b^{jq} \nabla_n \nabla_m h_{ij}.
\end{aligned}$$

Hence, $\mathcal{L} b^{pq} := \alpha K^\alpha b^{nm} \nabla_n \nabla_m b^{pq}$ satisfies

$$\mathcal{L} b^{pq} = 2\alpha K^\alpha b^{nm} b^{jq} b^{ik} b^{pl} \nabla_n h_{kl} \nabla_m h_{ij} - b^{ip} b^{jq} \mathcal{L} h_{ij} = 2\alpha K^\alpha b^{lp} b^{jq} b^{ki} b^{nm} \nabla_l h_{kn} \nabla_j h_{im} - b^{ip} b^{jq} \mathcal{L} h_{ij}.$$

Combing the last identity with (2.6) and (2.12) yields

$$\begin{aligned}
\partial_i b^{pq} &= -b^{ip} b^{jq} (\mathcal{L} h_{ij} + \alpha K^\alpha (\alpha b^{kl} b^{mn} - b^{km} b^{nl}) \nabla_i h_{kl} \nabla_j h_{mn} + \alpha K^\alpha H h_{ij} - (1 + n\alpha) K^\alpha h_{ik} h_j^k) \\
&= \mathcal{L} b^{pq} - \alpha K^\alpha b^{ip} b^{jq} (\alpha b^{kl} b^{mn} + b^{km} b^{nl}) \nabla_i h_{kl} \nabla_j h_{mn} - \alpha K^\alpha H b^{pq} + (1 + n\alpha) K^\alpha g^{pq}
\end{aligned}$$

which gives (2.8).

Proof of (2.9). By (2.3) we have $\partial_i \nu = -\nu^2 \langle \vec{e}_{n+1}, \partial_i \vec{n} \rangle = \nu^2 \langle \vec{e}_{n+1}, \nabla K^\alpha \rangle$. Furthermore,

$$\begin{aligned} \mathcal{L}\nu &= -\alpha K^\alpha b^{ij} \nabla_i (\nu^2 \langle \vec{e}_{n+1}, \nabla_j \vec{n} \rangle) = -2\alpha K^\alpha b^{ij} \nu \nabla_i \nu \langle \vec{e}_{n+1}, \nabla_j \vec{n} \rangle + \alpha K^\alpha b^{ij} \nu^2 \langle \vec{e}_{n+1}, \nabla_i (h_{jk} F^k) \rangle \\ &= 2\nu^{-1} \|\nabla \nu\|_{\mathcal{L}}^2 + \alpha K^\alpha b^{ij} h_{jk} h_i^k \nu^2 \langle \vec{e}_{n+1}, \vec{n} \rangle + \nu^2 \langle \vec{e}_{n+1}, \alpha K^\alpha b^{ij} (\nabla_i h_{jk}) F^k \rangle \\ &= 2\nu^{-1} \|\nabla \nu\|_{\mathcal{L}}^2 + \alpha K^\alpha H \nu + \nu^2 \langle \vec{e}_{n+1}, \nabla K^\alpha \rangle \end{aligned}$$

Combining the above yields (2.9).

3 Equations of Self-Shrinkers

In this section we will derive basic equations of the closed self-similar solutions Σ to the α -Gauss curvature flow. Since $(T-t)^{\frac{1}{\alpha+1}} \Sigma$ is a solution to the α -Gauss curvature flow, an immersion $F : M^n \rightarrow \mathbb{R}^{n+1}$ of $F(M^n) = \Sigma$ satisfies (** $^\alpha$)

PROPOSITION 2.2. *Given a strictly convex smooth solution $F : M^n \rightarrow \mathbb{R}^{n+1}$ to (** $^\alpha$), the following hold*

$$(2.14) \quad \nabla_i b^{jk} = -b^{jl} b^{km} \nabla_i h_{lm},$$

$$(2.15) \quad \mathcal{L}|F|^2 = 2\alpha K^\alpha b^{ij} (g_{ij} - h_{ij} K^\alpha) = 2\alpha K^\alpha b^{ij} g_{ij} - 2n\alpha K^{2\alpha},$$

$$(2.16) \quad \nabla_i K^\alpha = h_{ij} \langle F, F^j \rangle,$$

$$(2.17) \quad \mathcal{L}K^\alpha = \langle F, \nabla K^\alpha \rangle + n\alpha K^\alpha - \alpha K^{2\alpha} H,$$

$$(2.18) \quad \begin{aligned} \mathcal{L}b^{pq} &= K^{-\alpha} b^{pr} b^{qs} \nabla_r K^\alpha \nabla_s K^\alpha + \alpha K^\alpha b^{pr} b^{qs} b^{ij} b^{km} \nabla_r h_{ik} \nabla_s h_{jm} \\ &\quad + \langle F, \nabla b^{pq} \rangle - b^{pq} - (n\alpha - 1) g^{pq} K^\alpha + \alpha K^\alpha H b^{pq}. \end{aligned}$$

PROOF. From $\nabla_i (b^{jk} h_{kl}) = \nabla_i \delta_l^j = 0$, we can derive $h_{kl} \nabla_i b^{jk} = -b^{jk} \nabla_i h_{kl}$. Hence, we have (2.14) by

$$\nabla_i b^{jm} = b^{lm} h_{kl} \nabla_i b^{jk} = -b^{lm} b^{jk} \nabla_i h_{kl}.$$

Also, by definition $\mathcal{L} := \alpha K^\alpha b^{ij} \nabla_i \nabla_j$ we have

$$\mathcal{L}|F|^2 = 2\alpha K^\alpha b^{ij} \langle F_i, F_j \rangle + 2\alpha K^\alpha b^{ij} \langle F, \nabla_i \nabla_j F \vec{n} \rangle = 2\alpha K^\alpha b^{ij} g_{ij} + 2\alpha K^\alpha b^{ij} \langle F, h_{ij} \vec{n} \rangle.$$

Thus, the given equation (** $^\alpha$) implies (5.5).

Equation (2.16) can be simply obtained by differentiating (** $^\alpha$)

$$\nabla_i K^\alpha = h_{ik} \langle F, F^k \rangle.$$

Differentiating the equation above again we obtain

$$\nabla_i \nabla_j K^\alpha = \nabla_i h_{jk} \langle F, F^k \rangle + h_{ij} + h_{ik} h_j^k \langle F, \vec{n} \rangle = \langle F, \nabla h_{ij} \rangle + h_{ij} - h_{ik} h_j^k K^\alpha.$$

On the other hand, (2.14) and direct differentiation yield

$$\nabla_i \nabla_j K^\alpha = \nabla_i (\alpha K^\alpha b^{pq} \nabla_j h_{pq}) = \alpha K^\alpha b^{pq} \nabla_i \nabla_j h_{pq} + \alpha^2 K^\alpha b^{rs} b^{pq} \nabla_i h_{rs} \nabla_j h_{pq} - \alpha K^\alpha b^{pr} b^{qs} \nabla_i h_{rs} \nabla_j h_{pq}.$$

Observing

$$\begin{aligned} \nabla_i \nabla_j h_{pq} &= \nabla_i \nabla_p h_{jq} = \nabla_p \nabla_i h_{jq} + R_{ipjm} h_q^m + R_{ipqm} h_j^m \\ &= \nabla_p \nabla_q h_{ij} + (h_{ij} h_{pm} - h_{im} h_{jp}) h_q^m + (h_{iq} h_{pm} - h_{im} h_{pq}) h_j^m \end{aligned}$$

we obtain

$$\alpha K^\alpha b^{pq} \nabla_i \nabla_j h_{pq} = \alpha K^\alpha b^{pq} \nabla_p \nabla_q h_{ij} + \alpha K^\alpha H h_{ij} - n \alpha K^\alpha h_{im} h_j^m = \mathcal{L} h_{ij} + \alpha K^\alpha H h_{ij} - n \alpha K^\alpha h_{im} h_j^m.$$

Combining the equations above yields

$$(2.6) \quad \begin{aligned} \mathcal{L} h_{ij} &= -\alpha^2 K^\alpha b^{rs} b^{pq} \nabla_i h_{rs} \nabla_j h_{pq} + \alpha K^\alpha b^{pr} b^{qs} \nabla_i h_{rs} \nabla_j h_{pq} \\ &\quad + \langle F, \nabla h_{ij} \rangle + h_{ij} + (n\alpha - 1) h_{ik} h_j^k K^\alpha - \alpha K^\alpha H h_{ij}. \end{aligned}$$

We now observe

$$\begin{aligned} \mathcal{L} K^\alpha &= \alpha K^\alpha b^{ij} \nabla_i (\alpha K^\alpha b^{pq} \nabla_j h_{pq}) \\ &= \alpha^3 K^{2\alpha} b^{ij} b^{pq} b^{rs} \nabla_i h_{rs} \nabla_j h_{pq} - \alpha^2 K^{2\alpha} b^{ij} b^{pr} b^{qs} \nabla_i h_{rs} \nabla_j h_{pq} + \alpha K^\alpha b^{pq} \mathcal{L} h_{pq} \end{aligned}$$

which gives (2.17), since

$$\mathcal{L} K^\alpha = \alpha K^\alpha b^{ij} (\langle F, \nabla h_{ij} \rangle + h_{ij} + (n\alpha - 1) h_{ik} h_j^k K^\alpha - \alpha K^\alpha H h_{ij}) = \langle F, \nabla K^\alpha \rangle + n \alpha K^\alpha - \alpha K^{2\alpha} H.$$

Finally, by using (2.14), we can derive

$$\mathcal{L} b^{pq} = \alpha K^\alpha b^{ij} \nabla_i (-b^{pr} b^{qs} \nabla_j h_{rs}) = 2\alpha K^\alpha b^{ij} b^{pk} b^{rm} b^{qs} \nabla_i h_{km} \nabla_j h_{rs} - b^{pr} b^{qs} \mathcal{L} h_{rs}.$$

Applying (2.6) yields

$$\begin{aligned} \mathcal{L} b^{pq} &= \alpha^2 K^\alpha b^{pr} b^{qs} b^{ij} b^{km} \nabla_r h_{ij} \nabla_s h_{km} + \alpha K^\alpha b^{pr} b^{qs} b^{ij} b^{km} \nabla_r h_{ik} \nabla_s h_{jm} \\ &\quad + \langle F, \nabla b^{pq} \rangle - b^{pq} - (n\alpha - 1) g^{pq} K^\alpha + \alpha K^\alpha H b^{pq}. \end{aligned}$$

Thus, $\nabla K^\alpha = \alpha K^\alpha b^{ij} \nabla h_{ij}$ gives the desired result. \square

4 Euler's formula

Finally, we recall Euler's formula to use Pogorelov computations.

PROPOSITION 2.3. *Let Σ be a smooth hypersurface and $F : M^n \rightarrow \mathbb{R}^{n+1}$ be a smooth immersion with $F(M^n) = \Sigma$. Then, for all $p \in M^n$ and $i \in \{1, \dots, n\}$, the following holds*

$$\frac{h_{ii}(p)}{g_{ii}(p)} \leq \lambda_{\max}(p).$$

We can simply modify Euler's formula to b^{ij} as follows:

PROPOSITION 2.4. *Let Σ be a smooth strictly convex hypersurface and $F : M^n \rightarrow \mathbb{R}^{n+1}$ be a smooth immersion with $F(M^n) = \Sigma$. Then, for all $p \in M^n$ and $i \in \{1, \dots, n\}$, the following holds*

$$\frac{b^{ii}(p)}{g^{ii}(p)} \leq \frac{1}{\lambda_{\min}(p)}.$$

PROOF. Let $\{E_1, \dots, E_n\}$ be an orthonormal basis of $T\Sigma_{F(p)}$ satisfying $L(E_j) = \lambda_j E_j$, where L is the Weingarten map and $\lambda_1, \dots, \lambda_n$ are the principal curvatures of Σ at p . Denote by $\{a_{ij}\}$ the matrix satisfying $F_i(p) := \nabla_i F(p) = a_{ij} E_j$ and by $\{c_{ij}\}$ the diagonal matrix $\text{diag}(\lambda_1, \dots, \lambda_n)$. Then, $LF_i(p) = h_{ij}(p)F^j(p)$ implies

$$b^{ij}(p)LF_j(p) = b^{ij}(p)h_{jk}(p)F^k(p) = F^i(p) = g^{ij}(p)a_{jm}E_m$$

Observing that for the sum $a_{jk}E_k = \sum_{k=1}^n a_{jk}E_k$ we have $L(a_{jk}E_k) = a_{jk}L(E_k) = a_{jk}\lambda_k E_k = a_{jk}c_{km}E_m$, it follows that

$$g^{ij}(p)a_{jm}E_m = b^{ij}(p)LF_j(p) = b^{ij}(p)L(a_{jk}E_k) = b^{ij}(p)a_{jk}c_{km}E_m$$

Hence,

$$g^{ij}(p)a_{jm} = b^{ij}(p)a_{jk}c_{km}$$

Denoting by a^{ij} and c^{ij} the inverse matrices of a_{ij} and c_{ij} respectively, we have

$$g^{ij}(p)a_{jm}c^{ml}a^{ln} = b^{ij}(p)a_{jk}c_{km}c^{ml}a^{ln} = b^{in}(p)$$

Thus, for each $i \in \{1, \dots, n\}$, the following holds

$$b^{ii}(p) = \sum_{j,l,m} g^{ij}(p)a_{jm}c^{ml}a^{li} = \sum_{j,l,m} \langle F^i(p), F^j(p) \rangle a_{jm}c^{ml}a^{li}$$

By setting $F^i(p) = \bar{a}_{ij}E_j$, we observe $a^{ji} = a^{jk}\langle F_k(p), F^i(p) \rangle = a^{jk}\langle a_{kl}E_l, \bar{a}_{im}E_m \rangle = a^{jk}a_{kl}\bar{a}_{im}\delta_{ml} = \bar{a}_{ij}$. Thus, we have $\langle F^i(p), F^j(p) \rangle = \langle a^{ki}E_k, a^{rj}E_r \rangle = a^{ki}a^{kj}$, which yields

$$\begin{aligned} b^{ii}(p) &= \sum_{j,k,l,m} a^{ki}a^{kj}a_{jm}c^{ml}a^{li} = \sum_{k,l,m} a^{ki}\delta_m^k c^{ml}a^{li} = \sum_{k,l} a^{ki}c^{kl}a^{li} = \sum_k a^{ki}a^{ki}\lambda_k^{-1} = \sum_k (a^{ki})^2 \lambda_k^{-1} \\ &\leq \sum_k (a^{ki})^2 \lambda_{\min}^{-1} = \lambda_{\min}^{-1} \sum_{k,j} \langle a^{ki}E_k, a^{jj}E_j \rangle = \lambda_{\min}^{-1} \langle F^i(p), F^i(p) \rangle = \lambda_{\min}^{-1} g^{ii}(p) \end{aligned}$$

□

PROPOSITION 2.5. Let Σ be a strictly convex smooth hypersurface and $F : M^n \rightarrow \mathbb{R}^{n+1}$ be a smooth immersion with $F(M^n) = \Sigma$. Then, for all $p \in M^n$ and $i \in \{1, \dots, n\}$, the following holds

$$\frac{b^{i1}g^{ij}b^{j1}(p)}{g_{11}(p)} \leq \frac{1}{\lambda_{\min}^2(p)}.$$

PROOF. For a fixed point $p \in M^n$, we choose an orthonormal basis $\{E_1, \dots, E_n\}$ of $T\Sigma_{F(p)}$ such that $L(E_j) = \lambda_j E_j$, where L is the Weingarten map and $\lambda_1, \dots, \lambda_n$ are the principal curvatures of Σ at p . Given a chart (φ, U) of $p \in \varphi(U) \subset M^n$, we denote by $\{a_{ij}\}$ the matrix satisfying $F_i(p) := \nabla_i F(p) = a_{ij} E_j$ and by $\{c_{ij}\}$ the diagonal matrix $\text{diag}(\lambda_1, \dots, \lambda_n)$. We also denote by $\{a^{ij}\}$ and $\{c^{ij}\}$ the inverse matrices of $\{a_{ij}\}$ and $\{c_{ij}\}$, respectively.

We observe $g_{ij}(p) = \langle F_i, F_j \rangle(p) = \langle a_{ik} E_k, a_{jl} E_l \rangle = a_{ik} a_{jk}$. Also, we can obtain $F^i(p) = a^{ji} E_j$ by $a^{ji} = a^{jk} \langle F_k(p), F^i(p) \rangle = \langle a^{jk} a_{kl} E_l, F^i(p) \rangle = \langle E_j, F^i(p) \rangle$. So, we have $g^{ij}(p) = \langle F^i, F^j \rangle(p) = a^{ki} a^{kj}$. In addition, $LF_i(p) = h_{ij}(p) F^j(p)$ implies

$$\begin{aligned} a^{mi} E_m = F^i(p) &= b^{ij}(p) h_{jk}(p) F^k(p) = b^{ij}(p) LF_j(p) \\ &= b^{ij}(p) L(a_{jk} E_k) = b^{ij}(p) a_{jk} L(E_k) = b^{ij}(p) a_{jk} \lambda_k E_k = b^{ij}(p) a_{jk} c_{km} E_m. \end{aligned}$$

Hence, we have $b^{in}(p) = b^{ij}(p) a_{jk} c_{km} c^{ml} a^{ln} = a^{mi} c^{ml} a^{ln}$, and thus the following holds

$$\begin{aligned} b^{1r} g_{rs} b^{s1}(p) &= a^{i1} c^{ij} a^{jr} a_{rk} a_{sk} a^{m1} c^{ml} a^{ls} = a^{i1} c^{ij} \delta_k^j \delta_k^l a^{m1} c^{ml} = a^{i1} c^{ij} a^{m1} c^{mj} = \sum_j (a^{j1})^2 \lambda_j^{-2} \\ &\leq \sum_j (a^{j1})^2 \lambda_{\min}^{-2} = \lambda_{\min}^{-2} \sum_{k,j} \langle a^{k1} E_k, a^{j1} E_j \rangle = \lambda_{\min}^{-2} \langle F^1(p), F^1(p) \rangle = \lambda_{\min}^{-2} g^{11}(p), \end{aligned}$$

which is the desired result for $i = 1$ and we can obtain the same result for each $i \in \{1, \dots, n\}$. \square

Chapter 3

Regularity of strictly convex solutions

In this chapter, we study the regularity of strictly convex solutions. In particular, we will establish the lower bounds for principal curvatures and the upper bounds for the Gauss curvature, which yield the upper bounds for principal curvatures. By using the curvature estimates, the existence results [10], [12] were obtained as follows:

THEOREM 3.1 (Chow). *Let Σ_0 be a closed, smooth, and strictly convex hypersurface embedded in \mathbb{R}^{n+1} . Then, given an immersion $F_0 : M^n \rightarrow \mathbb{R}^{n+1}$ such that $\Sigma_0 = F_0(M^n)$, and for any $\alpha \in (0, +\infty)$, there exists a closed smooth and strictly convex solution $\Sigma_t := F(M^n, t)$ of the α -Gauss curvature flow $(*)^\alpha$ for finite time $0 < t < T$, where Σ_t converges to a point as t approaches to T .*

THEOREM 3.2 (Choi-Daskalopoulos-Kim-Lee). *Let Σ_0 be a complete, non-compact, and strictly convex hypersurface embedded in \mathbb{R}^{n+1} . Then, given an immersion $F_0 : M^n \rightarrow \mathbb{R}^{n+1}$ such that $\Sigma_0 = F_0(M^n)$, and for any $\alpha \in (0, +\infty)$, there exists a complete, non-compact, smooth and strictly convex solution $\Sigma_t := F(M^n, t)$ of the α -Gauss curvature flow $(*)^\alpha$ which is defined for all time $0 < t < +\infty$.*

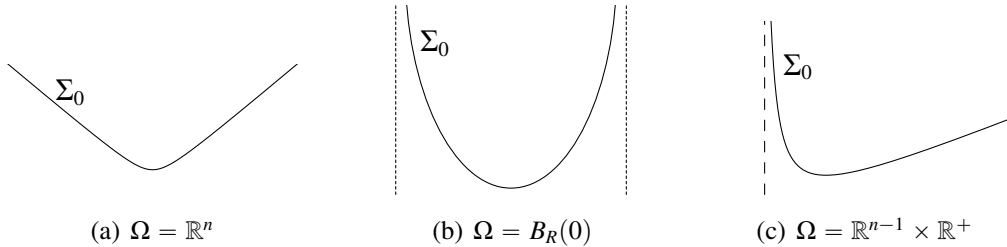


FIGURE 1. Examples of the initial graph Σ_0 defined on a domain $\Omega \subset \mathbb{R}^n$.

1 Lower bounds on the principal curvatures

In this section we will obtain lower bounds on the principal curvatures. One can easily establish the following lower bounds by combining equations (2.1) and (2.8) and applying the maximum principle.

THEOREM 3.3. *Given a closed, smooth, and strictly convex solution Σ_t to $(*)^\alpha$, the following holds*

$$b^{ij}g_{ij} \leq \max_{\Sigma_0} b^{ij}g_{ij}.$$

We will achieve the interior estimates by using Pogorelov type estimates with respect to b^{ii} . Recall that b^{ij} denotes the inverse matrix of the second fundamental form h_{ij} . Since b^{ii} depends on charts, we will find the relation between b^{ii} and the principal curvatures as [8]. We recall the definition of the cut-off function $\psi_\beta(p, t) := (M - \beta t - \bar{u}(p, t))_+$, where $\bar{u}(p, t) := \langle F(p, t), \vec{e}_{n+1} \rangle$ denotes the height function.

THEOREM 3.4 (Local lower bound for the principal curvatures). *Assume that Σ_t is a complete strictly convex smooth graph solution of $(*)^\alpha$ defined on $M^n \times [0, T]$, for some $T > 0$. Then, given constants $\beta > 0$ and $M \geq \beta$, the following holds*

$$(\psi_\beta^{-n(1+\frac{1}{\alpha})} \lambda_{\min})(p, t) \geq M^{-n(1+\frac{1}{\alpha})} \min \left\{ \inf_{Q_M} \lambda_{\min}(p, 0), \frac{\beta^{(n-1)+\frac{1}{\alpha}}}{(n^2(n+1)(n\alpha-1)_+)^{(n-1)+\frac{1}{\alpha}}} \right\}$$

where $Q_M = \{p \in M^n : \bar{u}(p, 0) < M\}$, and $(n\alpha - 1)_+^{-1} = +\infty$, if $n\alpha \leq 1$.

PROOF. Consider the cut-off function $\psi_\beta := (M - \beta t - \bar{u}(p, t))_+$, where $\bar{u}(p, t) = \langle F(p, t), \vec{e}_{n+1} \rangle$ denotes the height function. Since Σ_t is a complete and strictly convex graph, ψ_β is compactly supported. Therefore, for a fixed $T \in (0, +\infty)$, the function $\psi_\beta^{n(1+\frac{1}{\alpha})} \lambda_{\min}^{-1}$ attains its maximum on $M^n \times [0, T]$ at a point (p_0, t_0) . If $t_0 = 0$, then we obtain the desired result by the bound $\psi_\beta \leq M$ and the conditions on our initial data. So, we may assume in what follows that $t_0 > 0$.

We begin by choosing a chart (U, φ) with $p_0 \in \varphi(U) \subset M^n$, on which the covariant derivatives $\{\nabla_i F(p_0, t_0) := \partial_i(F \circ \varphi)(\varphi^{-1}(p_0), t_0)\}_{i=1, \dots, n}$ form an orthonormal basis of $(T\Sigma_{t_0})_{F(p_0, t_0)}$ satisfying

$$g_{ij}(p_0, t_0) = \delta_{ij}, \quad h_{ij}(p_0, t_0) = \delta_{ij} \lambda_i(p_0, t_0), \quad \lambda_1(p_0, t_0) = \lambda_{\min}(p_0, t_0).$$

In particular, at the point (p_0, t_0) we have $b^{11}(p_0, t_0) = \lambda_{\min}^{-1}(p_0, t_0)$ and $g^{11}(p_0, t_0) = 1$. Next, we define the function $w : \varphi(U) \times [0, T] \rightarrow \mathbb{R}$ by

$$w := \psi_\beta^{n(1+\frac{1}{\alpha})} \frac{b^{11}}{g^{11}}.$$

Notice that on the chart (U, φ) , if $t \neq t_0$, then the covariant derivatives $\{\nabla_i F(p_0, t)\}_{i=1, \dots, n}$ may not form an orthonormal basis of $(T\Sigma_t)_{F(p_0, t)}$. However, since Proposition 2.4 holds for every chart and immersion, we have $w \leq \psi_\beta^{n(1+\frac{1}{\alpha})} \lambda_{\min}^{-1}$. Hence, for $(p, t) \in \varphi(U) \times [0, T]$, the following holds

$$w(p, t) \leq \psi_\beta^{n(1+\frac{1}{\alpha})} \lambda_{\min}^{-1}(p, t) \leq \psi_\beta^{n(1+\frac{1}{\alpha})} \lambda_{\min}^{-1}(p_0, t_0) = w(p_0, t_0)$$

which shows that w attains its maximum at (p_0, t_0) .

Observe next that since $\nabla g^{11} = 0$, the following holds on the support of ψ_β

$$(3.1) \quad \frac{\nabla_i w}{w} = n \left(1 + \frac{1}{\alpha}\right) \frac{\nabla_i \psi_\beta}{\psi_\beta} + \frac{\nabla_i b^{11}}{b^{11}}.$$

Let us differentiate the equation above, again.

$$\frac{\nabla_i \nabla_j w}{w} - \frac{\nabla_i w \nabla_j w}{w^2} = n \left(1 + \frac{1}{\alpha}\right) \frac{\nabla_i \nabla_j \psi_\beta}{\psi_\beta} - n \left(1 + \frac{1}{\alpha}\right) \frac{\nabla_i \psi_\beta \nabla_j \psi_\beta}{\psi_\beta^2} + \frac{\nabla_i \nabla_j b^{11}}{b^{11}} - \frac{\nabla_i b^{11} \nabla_j b^{11}}{(b^{11})^2}.$$

Multiply by $\alpha K^\alpha b^{ij}$ and sum the equations over all i, j to obtain

$$\frac{\mathcal{L}w}{w} - \frac{\|\nabla w\|_{\mathcal{L}}^2}{w^2} = n \left(1 + \frac{1}{\alpha}\right) \frac{\mathcal{L}\psi_\beta}{\psi_\beta} - n \left(1 + \frac{1}{\alpha}\right) \frac{\|\nabla \psi_\beta\|_{\mathcal{L}}^2}{\psi_\beta^2} + \frac{\mathcal{L}b^{11}}{b^{11}} - \frac{\|\nabla b^{11}\|_{\mathcal{L}}^2}{(b^{11})^2}.$$

On the other hand, on the support of ψ_β , we also have

$$\frac{\partial_t w}{w} = n \left(1 + \frac{1}{\alpha}\right) \frac{\partial_t \psi_\beta}{\psi_\beta} + \frac{\partial_t b^{11}}{b^{11}} - \frac{\partial_t g^{11}}{g^{11}}.$$

Subtract the equations above. Then, $w^{-2} \|\nabla w\|_{\mathcal{L}}^2 \geq 0$ implies the following inequality

$$\frac{\mathcal{L}w}{w} - \frac{\partial_t w}{w} \geq n \left(1 + \frac{1}{\alpha}\right) \left(\frac{\mathcal{L}\psi_\beta}{\psi_\beta} - \frac{\partial_t \psi_\beta}{\psi_\beta} \right) - n \left(1 + \frac{1}{\alpha}\right) \frac{\|\nabla \psi_\beta\|_{\mathcal{L}}^2}{\psi_\beta^2} + \left(\frac{\mathcal{L}b^{11}}{b^{11}} - \frac{\partial_t b^{11}}{b^{11}} \right) - \frac{\|\nabla b^{11}\|_{\mathcal{L}}^2}{(b^{11})^2} + \frac{\partial_t g^{11}}{g^{11}}.$$

By 2.2 and (2.4) we have $\partial_t g^{11} = 2K^\alpha h^{11}$ and $\mathcal{L}\psi_\beta - \partial_t \psi_\beta = \beta - (n\alpha - 1)v^{-1}K^\alpha$, while by (2.8)

$$\mathcal{L}b^{11} - \partial_t b^{11} = \alpha K^\alpha b^{i1} b^{j1} (\alpha b^{kl} b^{mn} + b^{km} b^{nl}) \nabla_i h_{kl} \nabla_j h_{mn} + \alpha K^\alpha H b^{11} - (1 + n\alpha) K^\alpha g^{11}.$$

Combining the above yields

$$(3.2) \quad \begin{aligned} \frac{\mathcal{L}w}{w} - \frac{\partial_t w}{w} &\geq -n \left(1 + \frac{1}{\alpha}\right) \frac{\|\nabla \psi_\beta\|_{\mathcal{L}}^2}{\psi_\beta^2} - \frac{\|\nabla b^{11}\|_{\mathcal{L}}^2}{(b^{11})^2} + \frac{\alpha K^\alpha b^{i1} b^{j1} (\alpha b^{kl} b^{mn} + b^{km} b^{nl}) \nabla_i h_{kl} \nabla_j h_{mn}}{b^{11}} \\ &\quad - n \left(1 + \frac{1}{\alpha}\right) (n\alpha - 1) \frac{K^\alpha v^{-1}}{\psi_\beta} + n \left(1 + \frac{1}{\alpha}\right) \frac{\beta}{\psi_\beta} + \alpha K^\alpha H - (1 + n\alpha) K^\alpha \frac{g^{11}}{b^{11}} + 2K^\alpha \frac{h^{11}}{g^{11}}. \end{aligned}$$

Now, at (p_0, t_0) , the following holds

$$(3.3) \quad \alpha K^\alpha H - (1 + n\alpha) K^\alpha \frac{g^{11}}{b^{11}} + 2K^\alpha \frac{h^{11}}{g^{11}} \geq n\alpha K^\alpha \lambda_{\min} - (1 + n\alpha) K^\alpha \lambda_{\min} + 2K^\alpha \lambda_{\min} = K^\alpha \lambda_{\min}.$$

In addition, if $n\alpha \geq 1$, then $H \geq n\lambda_{\min}$ gives

$$\alpha K^\alpha H = \left(\alpha - \frac{1}{n}\right)K^\alpha H + \frac{1}{n}K^\alpha H \geq (n\alpha - 1)K^\alpha \lambda_{\min} + \frac{1}{n}K^\alpha H.$$

Therefore, in the case that $n\alpha \geq 1$, we can improve (3.3) to obtain

$$(3.4) \quad \alpha K^\alpha H - (1 + n\alpha)K^\alpha \frac{g^{11}}{b^{11}} + 2K^\alpha \frac{h^{11}}{g^{11}} \geq \frac{1}{n}K^\alpha H.$$

Also, at the maximum point (p_0, t_0) of w , $\nabla w(p_0, t_0) = 0$ holds. So, (3.1) leads to

$$\frac{n(1 + \alpha)}{\alpha} \frac{\|\nabla\psi_\beta\|_{\mathcal{L}}^2}{\psi_\beta^2} + \frac{\|\nabla b^{11}\|_{\mathcal{L}}^2}{(b^{11})^2} = \left(1 + \frac{\alpha}{n(1 + \alpha)}\right) \frac{\|\nabla b^{11}\|_{\mathcal{L}}^2}{(b^{11})^2} = \left(1 + \frac{\alpha}{n(1 + \alpha)}\right) \alpha \sum_{i=1}^n \frac{b^{ii} K^\alpha |\nabla_i b^{11}|^2}{(b^{11})^2}.$$

From (2.13), we get $\nabla_i b^{11} = -b^{1k} b^{1l} \nabla_i h_{kl} = -(b^{11})^2 \nabla_i h_{11}$ at (p_0, t_0) . Hence,

$$\frac{n(1 + \alpha)}{\alpha} \frac{\|\nabla\psi_\beta\|_{\mathcal{L}}^2}{\psi_\beta^2} + \frac{\|\nabla b^{11}\|_{\mathcal{L}}^2}{(b^{11})^2} = \alpha \left(1 + \frac{\alpha}{n(1 + \alpha)}\right) \sum_{i=1}^n b^{ii} (b^{11})^2 K^\alpha |\nabla_i h_{11}|^2.$$

Defining, at (p_0, t_0) , we define

$$I_i = b^{ii} (b^{11})^2 K^\alpha |\nabla_i h_{11}|^2$$

$$J_i = b^{ii} \nabla_1 h_{ii}.$$

We may rewrite the equation above as

$$(3.5) \quad \frac{n(1 + \alpha)}{\alpha} \frac{\|\nabla\psi_\beta\|_{\mathcal{L}}^2}{\psi_\beta^2} + \frac{\|\nabla b^{11}\|_{\mathcal{L}}^2}{(b^{11})^2} = \alpha \left(1 + \frac{\alpha}{n(1 + \alpha)}\right) \sum_{i=1}^n I_i$$

and also write

$$b^{kl} b^{mn} \nabla_1 h_{kl} \nabla_1 h_{mn} = |b^{mn} \nabla_1 h_{mn}|^2 = \left| \sum_{i=1}^n b^{ii} \nabla_1 h_{ii} \right|^2 = \left| \sum_{i=1}^n J_i \right|^2$$

which gives

$$\frac{\alpha^2 K^\alpha b^{i1} b^{j1} b^{kl} b^{mn}}{b^{11}} \nabla_i h_{kl} \nabla_j h_{mn} = \alpha^2 K^\alpha b^{11} b^{kl} b^{mn} \nabla_1 h_{kl} \nabla_1 h_{mn} = \alpha^2 K^\alpha b^{11} \left| \sum_{i=1}^n J_i \right|^2.$$

Since $\alpha^2 \geq \frac{\alpha^2}{1 + \alpha}$, we conclude that

$$(3.6) \quad \frac{\alpha^2 K^\alpha b^{i1} b^{j1} b^{kl} b^{mn}}{b^{11}} \nabla_i h_{kl} \nabla_j h_{mn} \geq \frac{\alpha^2}{1 + \alpha} K^\alpha b^{11} \left| \sum_{i=1}^n J_i \right|^2.$$

Finally, at (p_0, t_0) , we also have

$$\begin{aligned}
\frac{\alpha K^\alpha b^{i1} b^{j1} b^{km} b^{nl}}{b^{11}} \nabla_i h_{kl} \nabla_j h_{mn} &= \alpha K^\alpha b^{11} b^{km} b^{nl} \nabla_1 h_{kl} \nabla_1 h_{mn} \\
&= \alpha K^\alpha b^{11} \left(\sum_{i=1}^n |b^{ii} \nabla_1 h_{ii}|^2 + \sum_{i \neq j} b^{ii} b^{jj} |\nabla_1 h_{ij}|^2 \right) \\
&\geq \alpha K^\alpha b^{11} \left(\sum_{i=1}^n |b^{ii} \nabla_1 h_{ii}|^2 + 2 \sum_{i \neq 1} b^{ii} b^{11} |\nabla_1 h_{i1}|^2 \right) \\
&= \alpha K^\alpha b^{11} \sum_{i=1}^n |J_i|^2 + 2\alpha \sum_{i \neq 1} I_i.
\end{aligned}$$

Using $\alpha \geq \frac{\alpha^2}{1+\alpha}$, we may rewrite the inequality above as

$$(3.7) \quad \frac{\alpha K^\alpha b^{i1} b^{j1} b^{km} b^{nl}}{b^{11}} \nabla_i h_{kl} \nabla_j h_{mn} \geq \frac{\alpha^2}{1+\alpha} K^\alpha b^{11} \sum_{i \neq 1} |J_i|^2 + \alpha K^\alpha b^{11} |J_1|^2 + 2\alpha \sum_{i \neq 1} I_i.$$

Adding (3.6) and (3.7), gives that at (p_0, t_0) we have

$$\frac{\alpha K^\alpha b^{i1} b^{j1} (\alpha b^{kl} b^{mn} + b^{km} b^{nl})}{b^{11}} \nabla_i h_{kl} \nabla_j h_{mn} \geq \frac{\alpha^2}{1+\alpha} K^\alpha b^{11} \left(\left| \sum_{i=1}^n J_i \right|^2 + \sum_{i \neq 1} |J_i|^2 \right) + \alpha K^\alpha b^{11} |J_1|^2 + 2\alpha \sum_{i \neq 1} I_i$$

and by the Cauchy-Schwarz inequality

$$\begin{aligned}
n \left(\left| \sum_{i=1}^n J_i \right|^2 + \sum_{i \neq 1} |J_i|^2 \right) &= (1^2 + (-1)^2 + \dots + (-1)^2) \left(\left| \sum_{i=1}^n J_i \right|^2 + \sum_{i \neq 1} |J_i|^2 \right) \\
&\geq \left| \sum_{i=1}^n J_i + \sum_{i \neq 1} -J_i \right|^2 = |J_1|^2
\end{aligned}$$

and $2\alpha \geq \alpha \left(1 + \frac{\alpha}{n(1+\alpha)} \right)$, we obtain

$$\frac{\alpha K^\alpha b^{i1} b^{j1} (\alpha b^{kl} b^{mn} + b^{km} b^{nl})}{b^{11}} \nabla_i h_{kl} \nabla_j h_{mn} \geq \alpha \left(1 + \frac{\alpha}{n(1+\alpha)} \right) \left(K^\alpha b^{11} |J_1|^2 + \sum_{i \neq 1} I_i \right).$$

Combining the last inequality with (3.5) while using that $K^\alpha b^{11} |J_1|^2 = I_1$, yields

$$(3.8) \quad -\frac{n(1+\alpha)}{\alpha} \frac{\|\nabla \psi_\beta\|_{\mathcal{L}}^2}{\psi_\beta^2} - \frac{\|\nabla b^{11}\|_{\mathcal{L}}^2}{(b^{11})^2} + \frac{\alpha K^\alpha b^{i1} b^{j1} (\alpha b^{kl} b^{mn} + b^{km} b^{nl})}{b^{11}} \nabla_i h_{kl} \nabla_j h_{mn} \geq 0.$$

We conclude by (3.2), (3.3) and (3.8) that at (p_0, t_0) the following holds

$$0 \geq \frac{\mathcal{L}w}{w} - \frac{\partial_t w}{w} \geq -n \left(1 + \frac{1}{\alpha} \right) (n\alpha - 1) \frac{K^\alpha v^{-1}}{\psi_\beta} + n \left(1 + \frac{1}{\alpha} \right) \frac{\beta}{\psi_\beta} + K^\alpha \lambda_{\min}.$$

If $n\alpha \leq 1$, then the last inequality gives $n\left(1 + \frac{1}{\alpha}\right) \leq 0$, a contradiction. Hence $t_0 = 0$, and therefore the desired result holds. If $n\alpha \geq 1$, then we use the improved inequality (3.4) instead of (3.3) by and perform the same estimates for all the other terms, so that at (p_0, t_0) , we obtain

$$0 \geq \frac{\mathcal{L}w}{w} - \frac{\partial_t w}{w} \geq -n\left(1 + \frac{1}{\alpha}\right)(n\alpha - 1) \frac{K^\alpha v^{-1}}{\psi_\beta} + n\left(1 + \frac{1}{\alpha}\right) \frac{\beta}{\psi_\beta} + \frac{1}{n} K^\alpha H.$$

Hence

$$n\left(1 + \frac{1}{\alpha}\right)(n\alpha - 1) \frac{v^{-1}}{\psi_\beta} \geq n\left(1 + \frac{1}{\alpha}\right) \frac{\beta}{\psi_\beta} K^{-\alpha} + \frac{1}{n} H.$$

Since $n\alpha \geq 1$, we have $1 + \frac{1}{\alpha} \leq 1 + n$, and using also that $v \geq 1$, $\psi_\beta \leq M$, and $M \geq \beta$, we conclude from the previous inequality that

$$n^2(1+n)(n\alpha - 1)\psi_\beta^{-1} \geq n^2\left(1 + \frac{1}{\alpha}\right) K^{-\alpha} \frac{\beta}{\psi_\beta} + H \geq \left(n^2\left(1 + \frac{1}{\alpha}\right) K^{-\alpha} + H\right) \frac{\beta}{M}$$

Next, we employ the Young's inequality

$$\frac{K^{-\alpha}}{(n-1)\alpha + 1} + \frac{(n-1)\alpha H}{(n-1)\alpha + 1} \geq K^{-\frac{\alpha}{(n-1)\alpha+1}} H^{\frac{(n-1)\alpha}{(n-1)\alpha+1}} = (K^{-1} H^{n-1})^{\frac{\alpha}{(n-1)\alpha+1}}$$

and observe the following

$$K^{-1} H^{n-1} = \frac{H^{n-1}}{\lambda_1 \lambda_2 \cdots \lambda_n} = \frac{1}{\lambda_1} \frac{H^{n-1}}{\lambda_2 \cdots \lambda_n} \geq \frac{1}{\lambda_1} = \lambda_{\min}^{-1}.$$

Combining the last three inequalities yields

$$n^2(1+n)(n\alpha - 1)M\beta^{-1}\psi_\beta^{-1} \geq n^2\left(1 + \frac{1}{\alpha}\right) K^{-\alpha} + H \geq \frac{K^{-\alpha}}{(n-1)\alpha + 1} + \frac{(n-1)\alpha H}{(n-1)\alpha + 1} \geq (\lambda_{\min}^{-1})^{\frac{\alpha}{(n-1)\alpha+1}}.$$

We conclude, that if $n\alpha \geq 1$, the following holds at (p_0, t_0)

$$\lambda_{\min}^{-1} \psi_\beta^{n(1+\frac{1}{\alpha})} \leq (n^2(n+1)(n\alpha - 1)M\beta^{-1})^{(n-1)+\frac{1}{\alpha}} \psi_\beta^{1+\frac{n-1}{\alpha}}.$$

Thus, $\psi_\beta \leq M$ gives the desired result. \square

2 Speed estimates

We establish upper bounds for the speed K^α . Andrews obtained the following result for closed solutions to applying the maximum principle to the quantity $t^{\frac{n\alpha}{n\alpha+1}} K^\alpha (2\langle F, \vec{n} \rangle - \rho)^{-1}$ in [2].

THEOREM 3.5 (Andrews). *Let Σ_t be a closed, smooth and strictly convex solution to $(*)^\alpha$ defined for $t \in (0, T]$. Assume that Σ_T encloses a $(n+1)$ -ball $B_\rho^{n+1}(0)$ of radius $\rho > 0$ and Σ_0 is enclosed by $B_R^{n+1}(0)$. Then, the following holds*

$$t^{\frac{n\alpha}{n\alpha+1}} K^\alpha \leq C(R, \rho, T, \alpha, n),$$

for some constant $C(R, \rho, T, \alpha, n)$ depending on R, ρ, T, α, n .

To establish the interior estimate, we first derive the gradient estimate. We recall the notation $v := \langle \vec{n}, \vec{e}_{n+1} \rangle^{-1}$ (gradient function) and $\psi_\beta(p, t) := (M - \beta t - \bar{u}(p, t))_+$ (cut-off function) where $\bar{u}(p, t) = \langle F(p, t), \vec{e}_{n+1} \rangle$ denotes the height function.

THEOREM 3.6 (Gradient estimate). *Assume that Σ_t is a complete strictly convex smooth graph solution of $(*)$ defined on $M^n \times [0, T]$, for some $T > 0$. Given constants $\beta > 0$ and $M \geq \beta$,*

$$v(p, t) \psi_\beta(p, t) \leq M \max \left\{ \sup_{Q_M} v(p, 0), \beta^{-1} n^{\frac{1}{n\alpha+1}} (n\alpha - 1)_+ \right\}$$

where $Q_M = \{p \in M^n : \bar{u}(p, 0) < M\}$.

PROOF. First use (2.4) and (2.9), that is

$$\begin{aligned} \partial_t \psi_\beta &= \mathcal{L} \psi_\beta + (n\alpha - 1) v^{-1} K^\alpha - \beta \\ \partial_t v &= \mathcal{L} v - 2v^{-1} \|\nabla v\|_{\mathcal{L}}^2 - \alpha K^\alpha H v \end{aligned}$$

to compute

$$\begin{aligned} \partial_t(\psi_\beta v) &= \psi_\beta \mathcal{L} v - 2\psi_\beta v^{-1} \|\nabla v\|_{\mathcal{L}}^2 - \alpha \psi_\beta K^\alpha H v + v \mathcal{L} \psi_\beta + (n\alpha - 1) K^\alpha - \beta v \\ &= \mathcal{L}(\psi_\beta v) - 2\langle \nabla \psi_\beta, \nabla v \rangle_{\mathcal{L}} - 2\psi_\beta v^{-1} \|\nabla v\|_{\mathcal{L}}^2 - \alpha \psi_\beta K^\alpha H v + (n\alpha - 1) K^\alpha - \beta v \\ &= \mathcal{L}(\psi_\beta v) - 2v^{-1} \langle \nabla(\psi_\beta v), \nabla v \rangle_{\mathcal{L}} - \alpha \psi_\beta K^\alpha H v + (n\alpha - 1) K^\alpha - \beta v. \end{aligned}$$

Since ψ_β is compactly supported, for a fixed $T \in (0, +\infty)$ the function $\psi_\beta v$ attains its maximum on $M^n \times [0, T]$ at some (p_0, t_0) . If $t_0 = 0$, then we obtain the desired result. Assume that $t_0 > 0$. Then, at (p_0, t_0) , we have

$$\partial_t(\psi_\beta v) - \mathcal{L}(\psi_\beta v) \geq 0$$

which is equivalent to

$$(n\alpha - 1) K^\alpha \geq \alpha \psi_\beta K^\alpha H v + \beta v = (\alpha \psi_\beta K^\alpha H + \beta) v.$$

Hence, an interior maximum can be achieved only if $n\alpha \geq 1$. From now on, we assume that $n\alpha \geq 1$. If we multiply the last inequality by $MK^{-\alpha}$ and use that $M \geq \beta$ we obtain

$$(n\alpha - 1)M \geq (\alpha M \psi_\beta H + \beta MK^{-\alpha})v \geq \beta(\alpha \psi_\beta H + MK^{-\alpha})v.$$

On the other hand, $M \geq \psi_\beta$ implies $\alpha \psi_\beta H + MK^{-\alpha} \geq \alpha \psi_\beta H + \psi_\beta K^{-\alpha} \geq (\alpha H + K^{-\alpha})\psi_\beta$. Hence,

$$(n\alpha - 1)M \geq \beta(\alpha H + K^{-\alpha})\psi_\beta v \geq \frac{\beta}{n} \left(\frac{n\alpha}{n\alpha + 1} H + \frac{1}{n\alpha + 1} K^{-\alpha} \right) \psi_\beta v.$$

Next, we apply the Young's inequality

$$\frac{n\alpha}{n\alpha + 1} H + \frac{1}{n\alpha + 1} K^{-\alpha} \geq H^{\frac{n\alpha}{n\alpha+1}} K^{-\frac{\alpha}{n\alpha+1}} = (HK^{-\frac{1}{n}})^{\frac{n\alpha}{n\alpha+1}}.$$

Since $H \geq n K^{1/n}$, we conclude that in the case $n\alpha \geq 1$, $t_0 > 0$, the following holds at (p_0, t_0)

$$(n\alpha - 1)_+ M \geq \beta n^{-1} n^{\frac{n\alpha}{n\alpha+1}} \psi_\beta v = \beta n^{-\frac{1}{n\alpha+1}} \psi_\beta v$$

from which the desired inequality readily follows. □

We now apply the trick developed in [6].

THEOREM 3.7 (Local speed bound). *Assume that Σ_t is a complete strictly convex smooth graph solution of $(*)$ defined on $M^n \times [0, T)$. Then, given a constant $M \geq 1$,*

$$\left(\frac{t}{1+t}\right) (\psi^2 K^{\frac{1}{n}})(p, t) \leq (4n\alpha + 1)^2 (2\theta)^{1+\frac{1}{2n\alpha}} (\theta\Lambda + M^2)$$

where θ and Λ are constants given by

$$\begin{aligned} \theta &= \sup\{v^2(p, s) : \bar{u}(p, s) < M, s \in [0, t]\}, \\ \Lambda &= \sup\{\lambda_{\min}^{-1}(p, s) : \bar{u}(p, s) < M, s \in [0, t]\}. \end{aligned}$$

PROOF. Choosing a fixed time $T_0 \in (0, T)$, we redefine θ and Λ by

$$\theta = \sup\{v^2(p, t) : \bar{u}(p, t) < M, t \in [0, T_0]\}, \quad \Lambda = \sup\{\lambda_{\min}^{-1}(p, t) : \bar{u}(p, t) < M, t \in [0, T_0]\}.$$

Also, we define $\eta : [0, T) \rightarrow \mathbb{R}$ by

$$\eta(t) = \frac{t}{1+t}$$

which will be used later in this proof.

Following the well used idea by Caffarelli, Nirenberg and Spruck in [6] (see also in [17] and [8]), we define the function $\varphi = \varphi(v^2)$, depending on v^2 , by

$$\varphi(v^2) = \frac{v^2}{2\theta - v^2}.$$

The evolution equation of v in (2.9) gives

$$\partial_t(v^2) = \mathcal{L}(v^2) - 2\alpha K^\alpha H v^2 - 6\|\nabla v\|_{\mathcal{L}}^2.$$

Then, the evolution equation of $\varphi(v^2)$ is

$$\partial_t \varphi = \varphi'(\mathcal{L}v^2 - 2\alpha K^\alpha H v^2 - 6\|\nabla v\|_{\mathcal{L}}^2) = \mathcal{L}\varphi - \varphi''\|\nabla v^2\|_{\mathcal{L}}^2 - \varphi'(2\alpha K^\alpha H v^2 + 6\|\nabla v\|_{\mathcal{L}}^2).$$

Also, the evolution equation of K^α in (2.7) leads to

$$\partial_t K^{2\alpha} = \mathcal{L}K^{2\alpha} - \frac{1}{2}K^{-2\alpha}\|\nabla K^{2\alpha}\|_{\mathcal{L}}^2 + 2\alpha K^{3\alpha} H$$

implying the following evolution equation for $K^{2\alpha}\varphi(v^2)$

$$\begin{aligned}\partial_t(K^{2\alpha}\varphi) &= \mathcal{L}(K^{2\alpha}\varphi) - 2\langle \nabla K^{2\alpha}, \nabla\varphi \rangle_{\mathcal{L}} + 2\alpha K^{3\alpha}H(\varphi - \varphi'v^2) \\ &\quad - \frac{1}{2}\varphi K^{-2\alpha}\|\nabla K^{2\alpha}\|_{\mathcal{L}}^2 - (4\varphi''v^2 + 6\varphi')K^{2\alpha}\|\nabla v\|_{\mathcal{L}}^2.\end{aligned}$$

Observe that

$$\begin{aligned}-2\langle \nabla K^{2\alpha}, \nabla\varphi \rangle_{\mathcal{L}} &= -\langle \nabla K^{2\alpha}, \nabla\varphi \rangle_{\mathcal{L}} + \varphi^{-1}K^{2\alpha}\|\nabla\varphi\|_{\mathcal{L}}^2 - \varphi^{-1}\langle \nabla\varphi, \nabla(K^{2\alpha}\varphi) \rangle_{\mathcal{L}} \\ &\leq \frac{1}{2}\varphi K^{-2\alpha}\|\nabla K^{2\alpha}\|_{\mathcal{L}}^2 + \frac{3}{2}\varphi^{-1}K^{2\alpha}\|\nabla\varphi\|_{\mathcal{L}}^2 - \varphi^{-1}\langle \nabla\varphi, \nabla(K^{2\alpha}\varphi) \rangle_{\mathcal{L}}.\end{aligned}$$

Hence, the following inequality holds

$$(3.9) \quad \begin{aligned}\partial_t(K^{2\alpha}\varphi) &\leq \mathcal{L}(K^{2\alpha}\varphi) - \varphi^{-1}\langle \nabla\varphi, \nabla(K^{2\alpha}\varphi) \rangle_{\mathcal{L}} + 2\alpha K^{3\alpha}H(\varphi - \varphi'v^2) \\ &\quad - (4\varphi''v^2 + 6\varphi' - 6\varphi^{-1}\varphi'^2v^2)K^{2\alpha}\|\nabla v\|_{\mathcal{L}}^2.\end{aligned}$$

Now, we have

$$\varphi(v^2) + 1 = \frac{2\theta}{2\theta - v^2}, \quad \varphi'(v^2) = \frac{2\theta}{(2\theta - v^2)^2}, \quad \varphi''(v^2) = \frac{4\theta}{(2\theta - v^2)^3}.$$

Therefore, by direct calculation we obtain

$$\varphi - \varphi'v^2 = \frac{v^2}{2\theta - v^2} - \frac{2\theta v^2}{(2\theta - v^2)^2} = -\frac{v^4}{(2\theta - v^2)^2} = -\varphi^2$$

and

$$\varphi^{-1}\nabla\varphi = \frac{2\theta - v^2}{v^2} \frac{4\theta v\nabla v}{(2\theta - v^2)^2} = 4\theta\varphi v^{-3}\nabla v$$

and also

$$4\varphi''v^2 + 6\varphi' - 6\varphi^{-1}\varphi'^2v^2 = \frac{16\theta v^2}{(2\theta - v^2)^3} + \frac{12\theta}{(2\theta - v^2)^2} - \frac{24\theta^2}{(2\theta - v^2)^3} = \frac{4\theta}{(2\theta - v^2)^2}\varphi.$$

Setting $f := K^{2\alpha}\varphi$ in (3.9) and using the identities above yields

$$\partial_t f \leq \mathcal{L}f - 4\theta\varphi v^{-3}\langle \nabla v, \nabla f \rangle_{\mathcal{L}} - 2\alpha f K^\alpha H\varphi - \frac{4\theta}{(2\theta - v^2)^2}f\|\nabla v\|_{\mathcal{L}}^2.$$

On the other hand, (2.4) gives

$$\partial_t \psi = \mathcal{L}\psi + (n\alpha - 1)v^{-1}K^\alpha \leq \mathcal{L}\psi + n\alpha K^\alpha.$$

Hence, on the support of ψ , we have

$$\partial_t \psi^{4n\alpha} \leq \mathcal{L}\psi^{4n\alpha} - 4n\alpha(4n\alpha - 1)\psi^{4n\alpha-2}\|\nabla\psi\|_{\mathcal{L}}^2 + 4n^2\alpha^2 K^\alpha \psi^{4n\alpha-1}.$$

Thus, on the support of ψ , the following holds

$$\begin{aligned} \partial_i(f\psi^{4n\alpha}) &\leq \mathcal{L}(f\psi^{4n\alpha}) - 2\langle \nabla\psi^{4n\alpha}, \nabla f \rangle_{\mathcal{L}} - 4\theta\varphi v^{-3}\psi^{4n\alpha}\langle \nabla v, \nabla f \rangle_{\mathcal{L}} - 2\alpha f K^\alpha H\varphi\psi^{4n\alpha} \\ &\quad - \frac{4\theta}{(2\theta - v^2)^2} f\psi^{4n\alpha}\|\nabla v\|_{\mathcal{L}}^2 - 4n\alpha(4n\alpha - 1)f\psi^{4n\alpha-2}\|\nabla\psi\|_{\mathcal{L}}^2 + 4n^2\alpha^2 K^\alpha\psi^{4n\alpha-1}. \end{aligned}$$

Next, we compute

$$\begin{aligned} -4\theta\varphi v^{-3}\psi^{4n\alpha}\langle \nabla v, \nabla f \rangle_{\mathcal{L}} &= \\ &= -4\theta\varphi v^{-3}\langle \nabla v, \nabla(f\psi^{4n\alpha}) \rangle_{\mathcal{L}} + 16n\alpha\theta\varphi v^{-3}f\psi^{4n\alpha-1}\langle \nabla v, \nabla\psi \rangle_{\mathcal{L}} \\ &\leq -4\theta\varphi v^{-3}\langle \nabla v, \nabla(f\psi^{4n\alpha}) \rangle_{\mathcal{L}} + \frac{4\theta f\psi^{4n\alpha}\|\nabla v\|_{\mathcal{L}}^2}{(2\theta - v^2)^2} + 16n^2\alpha^2\theta(2\theta - v^2)^2\varphi^2v^{-6}f\psi^{4n\alpha-2}\|\nabla\psi\|_{\mathcal{L}}^2 \\ &= -4\theta\varphi v^{-3}\langle \nabla v, \nabla(f\psi^{4n\alpha}) \rangle_{\mathcal{L}} + \frac{4\theta}{(2\theta - v^2)^2} f\psi^{4n\alpha}\|\nabla v\|_{\mathcal{L}}^2 + 16n^2\alpha^2\theta v^{-2}f\psi^{4n\alpha-2}\|\nabla\psi\|_{\mathcal{L}}^2. \end{aligned}$$

Moreover, we have

$$-2\langle \nabla\psi^{4n\alpha}, \nabla f \rangle_{\mathcal{L}} = -2\psi^{-4n\alpha}\langle \nabla\psi^{4n\alpha}, \nabla(f\psi^{4n\alpha}) \rangle_{\mathcal{L}} + 32n^2\alpha^2 f\psi^{4n\alpha-2}\|\nabla\psi\|_{\mathcal{L}}^2.$$

Combining the above gives

$$\begin{aligned} \partial_i(f\psi^{4n\alpha}) &\leq \mathcal{L}(f\psi^{4n\alpha}) - \langle 2\psi^{-4n\alpha}\nabla\psi^{4n\alpha} + 4\theta\varphi v^{-3}\nabla v, \nabla(f\psi^{4n\alpha}) \rangle_{\mathcal{L}} - 2\alpha f K^\alpha H\varphi\psi^{4n\alpha} \\ &\quad + (32n^2\alpha^2 + 16n^2\alpha^2\theta v^{-2} - 4n\alpha(4n\alpha - 1))f\psi^{4n\alpha-2}\|\nabla\psi\|_{\mathcal{L}}^2 + 4n^2\alpha^2 K^\alpha\psi^{4n\alpha-1}. \end{aligned}$$

In addition, on the support of ψ , we have $\nabla\psi = -\nabla\bar{u} = -\nabla\langle F, \vec{e}_{n+1} \rangle$ which leads to

$$\begin{aligned} \|\nabla\psi\|_{\mathcal{L}}^2 &= \|\nabla\langle F, \vec{e}_{n+1} \rangle\|_{\mathcal{L}}^2 \leq \sum_{m=1}^{n+1} \|\nabla\langle F, \vec{e}_m \rangle\|_{\mathcal{L}}^2 = \sum_{m=1}^{n+1} \alpha K^\alpha b^{ij} \langle F_i, \vec{e}_m \rangle \langle F_j, \vec{e}_m \rangle \\ &= \sum_{i=1}^n \sum_{j=1}^n \alpha K^\alpha b^{ij} \left(\sum_{m=1}^{n+1} \langle F_i, \vec{e}_m \rangle \langle F_j, \vec{e}_m \rangle \right) = \sum_{i=1}^n \sum_{j=1}^n \alpha K^\alpha b^{ij} g_{ij} \leq n\alpha K^\alpha \lambda_{\min}^{-1} \leq n\alpha \Lambda K^\alpha. \end{aligned}$$

Hence, $v \geq 1$ implies

$$(32n^2\alpha^2 + 16n^2\alpha^2\theta v^{-2} - 4n\alpha(4n\alpha - 1)) f\psi^{4n\alpha-2}\|\nabla\psi\|_{\mathcal{L}}^2 \leq n\alpha (16n^2\alpha^2(\theta+1) + 4n\alpha) f\psi^{4n\alpha-2}\Lambda K^\alpha.$$

Thus, by the inequalities $H \geq nK^{\frac{1}{n}}$ and $\varphi \geq 1/(2\theta)$, the evolution equation of $f\psi^{4n\alpha}$ can be reduced to the following

$$\begin{aligned} \partial_i(f\psi^{4n\alpha}) &\leq \mathcal{L}(f\psi^{4n\alpha}) - \langle 2\psi^{-4n\alpha}\nabla\psi^{4n\alpha} + 4\theta\varphi v^{-3}\nabla v, \nabla f\psi^{4n\alpha} \rangle_{\mathcal{L}} - n\alpha\theta^{-1}K^{\alpha+\frac{1}{n}}f\psi^{4n\alpha} \\ &\quad + 4n^2\alpha^2(4n\alpha(\theta+1) + 1)\Lambda K^\alpha f\psi^{4n\alpha-2} + 4n^2\alpha^2 K^\alpha\psi^{4n\alpha-1}. \end{aligned}$$

Involving $\eta =: t(1+t)^{-1}$ and $\partial\eta_t = (1+t)^{-2} \leq 1$ yields

$$\begin{aligned} \partial_i(\eta^{2n\alpha} f\psi^{4n\alpha}) &\leq \mathcal{L}(\eta^{2n\alpha} f\psi^{4n\alpha}) - \langle 2\psi^{-4n\alpha}\nabla\psi^{4n\alpha} + 4\theta\varphi v^{-3}\nabla v, \nabla\eta^{2n\alpha} f\psi^{4n\alpha} \rangle_{\mathcal{L}} - n\alpha\theta^{-1}K^{\alpha+\frac{1}{n}}\eta^{2n\alpha} f\psi^{4n\alpha} \\ &\quad + 4n^2\alpha^2(4n\alpha(\theta+1) + 1)\Lambda K^\alpha \eta^{2n\alpha} f\psi^{4n\alpha-2} + 4n^2\alpha^2 K^\alpha \eta^{2n\alpha} f\psi^{4n\alpha-1} + 2n\alpha\eta^{2n\alpha-1} f\psi^{4n\alpha}. \end{aligned}$$

Since ψ is compactly supported, $\eta^{2n\alpha} f \psi^{4n\alpha}$ attains its maximum in $M^n \times [0, T_0]$ at some (p_0, t_0) with $t_0 > 0$. Then, the last inequality implies that at (p_0, t_0)

$$\begin{aligned} n\alpha \theta^{-1} K^{\alpha + \frac{1}{n}} \eta^{2n\alpha} f \psi^{4n\alpha} &\leq 4n^2 \alpha^2 (4n\alpha(\theta + 1) + 1) \Lambda K^\alpha \eta^{2n\alpha} f \psi^{4n\alpha - 2} \\ &\quad + 4n^2 \alpha^2 K^\alpha \eta^{2n\alpha} f \psi^{4n\alpha - 1} + 2n\alpha \eta^{2n\alpha - 1} f \psi^{4n\alpha}. \end{aligned}$$

Multiplying by $(n\alpha)^{-1} \theta K^{-\alpha} \eta^{-2n\alpha + 1} f^{-1} \psi^{-4n\alpha + 2}$ yields the bound

$$\eta K^{\frac{1}{n}} \psi^2 \leq 4n\alpha \theta (4n\alpha(\theta + 1) + 1) \Lambda \eta + 4n\alpha \theta \eta \psi + 2\theta K^{-\alpha} \psi^2$$

and by $\theta \geq 1$, $\psi \leq M$, $1 \leq M$, and $\eta \leq 1$

$$\begin{aligned} \eta K^{\frac{1}{n}} \psi^2 &\leq 4n\alpha \theta \eta \left((4n\alpha(\theta + 1) + 1) \Lambda + \psi \right) + 2\theta \eta^{n\alpha} \psi^{2+2n\alpha} (\eta K^{\frac{1}{n}} \psi^2)^{-n\alpha} \\ &\leq 4n\alpha \theta (8n\alpha + 1) (\theta \Lambda + M) + 2\theta M^{2+2n\alpha} (\eta K^{\frac{1}{n}} \psi^2)^{-n\alpha} \\ &\leq 2\theta (16n^2 \alpha^2 + 2n\alpha + M^{2n\alpha} (\eta K^{\frac{1}{n}} \psi^2)^{-n\alpha}) (\theta \Lambda + M^2). \end{aligned}$$

Hence, in the case of $\eta K^{\frac{1}{n}} \psi^2 \geq M^2$, the last inequality yields

$$\eta K^{\frac{1}{n}} \psi^2 \leq 2\theta (16n^2 \alpha^2 + 2n\alpha + 1) (\theta \Lambda + M^2) \leq 2\theta (4n\alpha + 1)^2 (\theta \Lambda + M^2).$$

In the other case, we can simply obtain $\eta K^{\frac{1}{n}} \psi^2 \leq M^2 \leq 2\theta (4n\alpha + 1)^2 (\theta \Lambda + M^2)$. Thus, at (p_0, t_0) ,

$$\eta K^{\frac{1}{n}} \psi^2 \leq 2\theta (4n\alpha + 1)^2 (\theta \Lambda + M^2).$$

Let Ψ denote the maximum value $\eta^{2n\alpha} f \psi^{4n\alpha}(p_0, t_0) = \eta^{2n\alpha} \varphi K^{2\alpha} \psi^{4n\alpha}(p_0, t_0)$. Then, $\varphi \leq 1$ gives

$$\Psi \leq (\eta K^{\frac{1}{n}} \psi^2)^{2\alpha n}(p_0, t_0) \leq (2\theta)^{2\alpha n} (4n\alpha + 1)^{4\alpha n} (\theta \Lambda + M^2)^{2\alpha n}.$$

Using also that $(2\theta)^{-1} \leq \varphi$, we finally conclude that for all $p \in M^n$ and $t \in [0, T_0]$ the following holds

$$\frac{\eta^{2n\alpha} K^{2\alpha} \psi^{4n\alpha}(p, t)}{2\theta} \leq \eta^{2n\alpha} \varphi K^{2\alpha} \psi^{4n\alpha}(p, t) \leq \Psi \leq (2\theta)^{2\alpha n} (4n\alpha + 1)^{4\alpha n} (\theta \Lambda + M^2)^{2\alpha n}.$$

Hence, setting $t = T_0$ yields

$$(\eta K^{\frac{1}{n}} \psi^2)(p, T_0) \leq (2\theta)^{1 + \frac{1}{2n\alpha}} (4n\alpha + 1)^2 (\theta \Lambda + M^2)$$

and the desired result simply follows by substituting T_0 by t . \square

Chapter 4

Optimal regularity of translators

A solution Σ_t to the Gauss curvature flow in \mathbb{R}^{n+1} translating along a direction \vec{e}_{n+1} satisfies

$$\Sigma_t = \Sigma + ct\vec{e}_{n+1} =: \{Y + ct\vec{e}_{n+1} \in \mathbb{R}^{n+1} : Y \in \Sigma\},$$

where the *speed* c is a constant, and Σ is a complete convex hypersurface embedded in \mathbb{R}^{n+1} . We observe that there exist a convex open set $\Omega \subset \mathbb{R}^n$ and a convex function $u : \Omega \rightarrow \mathbb{R}$ satisfying

$$\Sigma \text{ is the boundary of } \{(x, t) : x \in \Omega, t \geq u(x)\}.$$

By the result in [27], the set Ω must be *bounded*. If $\mathcal{A}(\Omega)$ denotes the area of Ω , it follows that u is a smooth function satisfying

$$(4.1) \quad \begin{cases} \frac{\det D^2 u}{(1 + |Du|^2)^{\frac{n+1}{2}}} = \frac{|S^{n-1}|}{\mathcal{A}(\Omega)} & \text{in } \Omega, \\ \lim_{x \rightarrow \partial\Omega} |Du|(x) = +\infty & \text{on } \partial\Omega. \end{cases}$$

where S^n is the unit n -sphere.

Conversely, given an open bounded convex set $\Omega \subset \mathbb{R}^n$, there exists a solution $u : \Omega \rightarrow \mathbb{R}$ of (4.1), and any two solutions differ by a constant. (See [27] and Theorem 4.8 in [26]). Hence, given a translator Σ in \mathbb{R}^{n+1} of the Gauss curvature flow, there exists an open bounded convex set $\Omega \subset \mathbb{R}^n$ such that Σ converges to the cylinder $\partial\Omega \times \mathbb{R}$, and the immersion $F : M^n \rightarrow \mathbb{R}^{n+1}$ of $F(M^n) = \Sigma$ satisfies

$$(4.2) \quad K(p) = \frac{|S^{n-1}|}{\mathcal{A}(\Omega)} \langle \vec{n}(p), \vec{e}_{n+1} \rangle.$$

We recall the result of John Urbas in [27].

THEOREM 4.1 (Urbas). *Given an open bounded convex domain $\Omega \subset \mathbb{R}^2$, there exists a convex solution $u : \Omega \rightarrow \mathbb{R}$ satisfying (4.1), and it is unique up to addition by a constant. In particular,*

if for each $x_0 \in \partial\Omega$, there exists a ball $B \subset \mathbb{R}^2$ satisfying $\Omega \subset B$ and $x_0 \in \partial B$, then the solution u is a smooth function satisfying

$$\lim_{x \rightarrow \partial\Omega} u(x) = +\infty.$$

This result guarantees that there exists a unique C^1 translator $\Sigma = \partial\{(x, t) : x \in \Omega, t \geq u(x)\}$ for any open bounded convex domain Ω . Also, if Ω is a *uniformly convex* domain, then Σ is *strictly convex*, and thus C^∞ smooth by standard estimates. However, if Ω is weakly convex, then Σ may not be strictly convex on the boundary of Ω . Richard Hamilton conjectured that if Ω is a *square*, then Σ has *flat sides* on the boundary of Ω . This is shown in the next picture.

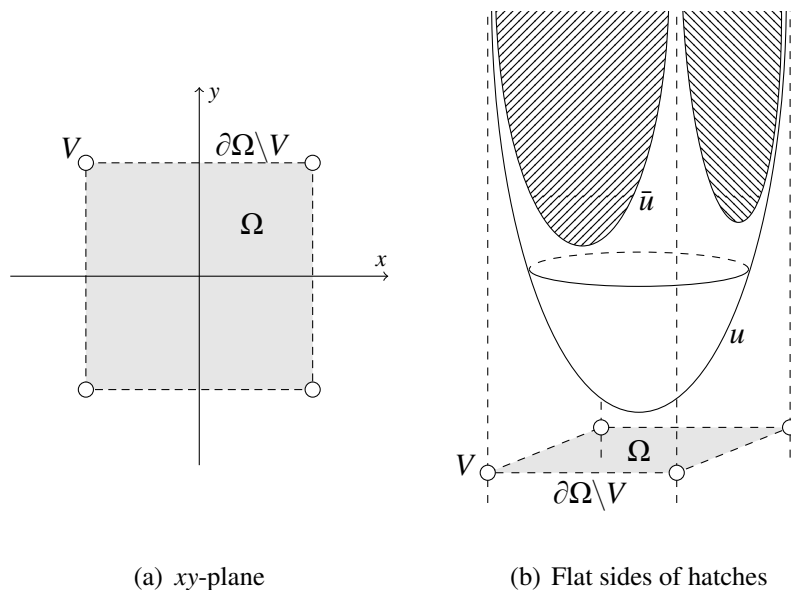


FIGURE 1. Translator Σ on a square

The Hamilton's conjecture and the *optimal regularity* of the translator was shown in [11].

THEOREM 4.2 (Choi-Daskalopoulos-Lee). *Let $\Omega := (-1, 1) \times (-1, 1) \subset \mathbb{R}^2$ be the open square and V be the set of the vertices $(\pm 1, \pm 1)$ of $\partial\Omega$. Assume $u : \Omega \rightarrow \mathbb{R}$ is a convex smooth solution of (4.1), and let Σ denote the boundary of $\{(x, t) : x \in \Omega, t \geq u(x)\}$. Then, Σ is a complete convex hypersurface of class $C_{loc}^{1,1}$, and there exists a smooth function $\bar{u} : (\partial\Omega \setminus V) \rightarrow \mathbb{R}$ satisfying*

$$\lim_{x \rightarrow y} u(x) = \bar{u}(y), \quad \lim_{x \rightarrow V} u(x) = \lim_{y \rightarrow V} \bar{u}(y) = +\infty.$$

REMARK 4.3 (Optimal regularity). Let us show that if there exists a flat side on a translator Σ , then Σ has at most $C^{1,1}$ regularity. Assume that the graph $(x, h(x, z), z)$ of a function $h(x, z)$ is a part of a translator. Then, we have

$$h_{vv}h_{\tau\tau} \geq \frac{\det D^2h}{(1 + |Dh|^2)^{\frac{3}{2}}} \geq -\frac{2\pi}{\mathcal{A}}|Dh|\langle v, e_2 \rangle$$

where $v(x, z)$ is the outward normal and $\tau(x, z)$ is a tangential direction of the level set of $h(x, z)$ at a point (x, z) . We denote by L_r the level set $\{(x, z) : h(x, z) = r\}$, and denote by $\kappa(x, z)$ the curvature of $L_{h(x,z)}$ at (x, z) . Since we have $h_{\tau\tau} = |Dh|\kappa$, the inequality above gives $h_{vv}\kappa \geq -2\pi\mathcal{A}^{-1}\langle v, e_2 \rangle$. We will establish the local lower bound for $-\langle v, e_2 \rangle$ in section 3, which guarantees that

$$h_{vv}\kappa \geq c.$$

We choose a neighborhood U of a point (x_0, z_0) on the free boundary Γ , namely $(x_0, z_0) \in \Gamma =: \partial L_{-1}$. Since the level sets L_r monotonically converge to Γ , there exists a constant c such that $\int_{L_r \cap U} ds \geq c$ for r close enough to -1 , where s is the arc length parameter. Hence, the following holds

$$2\pi \geq \int_{L_r} \kappa ds \geq \int_{L_r \cap U} \kappa ds \geq \frac{c}{\max_{L_r \cap U} h_{vv}}.$$

Thus, $\max_{L_r \cap U} h_{vv} \geq c$ holds for some uniform constant c . However, we have $D^2h = 0$ on L_{-1} . Therefore, D^2h is not a continuous function.

1 Optimal $C^{1,1}$ regularity

In this section, we will establish a local curvature estimate for smooth strictly convex complete solutions of equation (4.2). In the last section we will use this estimate to obtain the optimal $C^{1,1}$ regularity for a weakly convex solution of (4.2) in the degenerate case. We recall that a solution of (4.2) has an immersion $F : M^2 \rightarrow \mathbb{R}^3$ of $F(M^2) = \Sigma$. Given a ball $B_R(Y)$ we define the associated cut-off function η by $\eta(p) = (|F(p) - Y|^2 - R^2)_+$. We have the following result.

THEOREM 4.4 (Curvature bound). *Let Σ be a smooth strictly convex complete solution of (4.2). Let Σ^c be the cut off from Σ by a ball $B_R(Y) \subset \mathbb{R}^{n+1}$. Then, for any $p \in M^2$ with $F(p) \in \Sigma^c$, the maximum principal curvature $\Lambda(p) := \max\{\lambda_1(p), \lambda_2(p)\}$ satisfies*

$$\eta\Lambda(p) \leq \frac{9\pi}{\mathcal{A}(\Omega)} \sup_{F(q) \in \Sigma^c} |F(q) - Y|^3.$$

PROOF. We may assume, without loss of generality, that $Y = 0$. Recall the definition of the cutting ball. The continuous function $\eta\Lambda$ attains its maximum on the compact set Σ^c at some point $F(p_0) \in \Sigma^c$,

$$\eta\Lambda(p_0) = \max_{F(p) \in \Sigma^c} \eta\Lambda(p).$$

Then, because we have $\eta = 0$ on $\partial\Sigma_c$, $F(p_0)$ is an interior point of Σ_c . Thus, $\eta\Lambda$ attains a local maximum at p_0 . Moreover, we can choose an open chart (U, φ) with $p_0 \in \varphi(U)$ and $F(\varphi(U)) \subset \Sigma_c$ such that the covariant derivatives $\{\nabla_1 F(p_0), \nabla_2 F(p_0)\}$ form an orthonormal basis of $T\Sigma_{F(p_0)}$ satisfying

$$g_{ij}(p_0) = \delta_{ij}, \quad h_{ij}(p_0) = \delta_{ij}\lambda_i(p_0), \quad \lambda_1(p_0) = \Lambda(p_0).$$

Next, we define the function $w : U \rightarrow \mathbb{R}$ by

$$w = \eta \frac{h_{11}}{g_{11}}.$$

Then, the Euler formula, Proposition 2.3 guarantees $w \leq \eta\Lambda$. Therefore, for all $p \in U$, the following holds

$$w(p) \leq \eta\Lambda(p) \leq \eta\Lambda(p_0) = w(p_0).$$

Thus, w also attains its maximum at p_0 .

Now, we consider the derivative of w . Then, $\nabla g_{11} = 0$ gives

$$(4.3) \quad \frac{\nabla_i w}{w} = \frac{\nabla_i h_{11}}{h_{11}} + \frac{\nabla_i \eta}{\eta}.$$

Differentiating the equation above yields

$$\frac{\nabla_i \nabla_j w}{w} - \frac{\nabla_i w \nabla_j w}{w^2} = \frac{\nabla_i \nabla_j h_{11}}{h_{11}} - \frac{\nabla_i h_{11} \nabla_j h_{11}}{(h_{11})^2} + \frac{\nabla_i \nabla_j \eta}{\eta} - \frac{\nabla_i \eta \nabla_j \eta}{\eta^2}$$

and multiplying by Kb^{ij} , we obtain

$$(4.4) \quad \frac{\mathcal{L}w}{w} - \frac{\|\nabla w\|_{\mathcal{L}}^2}{w^2} = \frac{\mathcal{L}h_{11}}{h_{11}} - \frac{\|\nabla h_{11}\|_{\mathcal{L}}^2}{(h_{11})^2} + \frac{\mathcal{L}\eta}{\eta} - \frac{\|\nabla \eta\|_{\mathcal{L}}^2}{\eta^2}.$$

Observing next that $\mathcal{L}F := Kb^{ij} \nabla_j \nabla_j F = Kb^{ij} h_{ij} \vec{n} = 2K \vec{n}$, we compute $\mathcal{L}\eta$ on the support of η as follows:

$$\mathcal{L}\eta = \mathcal{L}|F|^2 = 2\langle F, \mathcal{L}F \rangle + 2\langle \nabla F, \nabla F \rangle_{\mathcal{L}} = 4K\langle F, \vec{n} \rangle + 2Kb^{ij} g_{ij} = 4K\langle F, \vec{n} \rangle + 2H.$$

Thus, $4K = 8\pi\mathcal{A}^{-1}\langle \vec{e}_3, \vec{n} \rangle \leq 8\pi\mathcal{A}^{-1}$ and $2H \geq 2\Lambda$ imply

$$(4.5) \quad \mathcal{L}\eta \geq -8\pi\mathcal{A}^{-1}|F| + 2\Lambda.$$

Since w attains its maximum at p_0 , we have $\nabla w(p_0) = 0$, and thus (4.3) gives

$$\frac{\|\nabla h_{11}\|_{\mathcal{L}}^2}{(h_{11})^2}(p_0) = \frac{\|\nabla \eta\|_{\mathcal{L}}^2}{\eta^2}(p_0).$$

Hence, combining $\mathcal{L}w(p_0) \leq 0$, (4.4), (4.5) and the equation above yields the following at p_0

$$(4.6) \quad 0 \geq \frac{\mathcal{L}h_{11}}{h_{11}} + \frac{2\Lambda}{\eta} - \frac{8\pi|F|}{\mathcal{A}\eta} - \frac{2\|\nabla h_{11}\|_{\mathcal{L}}^2}{(h_{11})^2}.$$

To compute $\mathcal{L}h_{11}$, we begin by differentiating K ,

$$(4.7) \quad \nabla_1 K = Kb^{ij}\nabla_1 h_{ij}.$$

By differentiating the equation above again, we obtain

$$(4.8) \quad \nabla_1 \nabla_1 K = Kb^{ij}\nabla_1 \nabla_1 h_{ij} + Kb^{ij}b^{kl}\nabla_1 h_{ij}\nabla_1 h_{kl} - Kb^{ik}b^{jl}\nabla_1 h_{ij}\nabla_1 h_{kl}.$$

We can derive $\mathcal{L}h_{11}$ from the first term $Kb^{ij}\nabla_1 \nabla_1 h_{ij}$ as follows

$$(4.9) \quad \begin{aligned} Kb^{ij}\nabla_1 \nabla_1 h_{ij} &= Kb^{ij}\nabla_1 \nabla_1 h_{j1} = Kb^{ij}(\nabla_i \nabla_1 h_{j1} + R_{1ijk}h_1^k + R_{1i1k}h_j^k) \\ &= Kb^{ij}\nabla_i \nabla_j h_{11} + Kb^{ij}(h_{1j}h_{ik} - h_{1k}h_{ij})h_1^k + Kb^{ij}(h_{11}h_{ik} - h_{1k}h_{i1})h_j^k \\ &= \mathcal{L}h_{11} - 2Kh_{1k}h_1^k + KHh_{11}. \end{aligned}$$

On the other hand, differentiating (4.2) yields

$$(4.10) \quad \nabla_1 K = \frac{2\pi}{\mathcal{A}} \langle \nabla_1 \vec{n}, \vec{e}_3 \rangle = -\frac{2\pi}{\mathcal{A}} h_{1k} \langle F^k, \vec{e}_3 \rangle.$$

To get the right hand side of (4.8), we differentiate the equation above,

$$(4.11) \quad \nabla_1 \nabla_1 K = -\frac{2\pi}{\mathcal{A}} \nabla_1 h_{1k} \langle F^k, \vec{e}_3 \rangle - \frac{2\pi}{\mathcal{A}} h_{1k} h_1^k \langle \vec{n}, \vec{e}_3 \rangle = -\frac{2\pi}{\mathcal{A}} \nabla_k h_{11} \langle F^k, \vec{e}_3 \rangle - Kh_{1k} h_1^k.$$

Combining (4.8), (4.9), and (4.11), we obtain the following at p_0

$$\mathcal{L}h_{11} = 2|\nabla_2 h_{11}|^2 - 2\nabla_1 h_{11} \nabla_1 h_{22} - \frac{2\pi}{\mathcal{A}} \nabla_k h_{11} \langle F^k, \vec{e}_3 \rangle - K^2.$$

Hence, at p_0 , applying the equation above to (4.6) and the definition of the norm $\|\cdot\|_{\mathcal{L}}$ yield

$$\begin{aligned} 0 &\geq \frac{1}{h_{11}} (2|\nabla_2 h_{11}|^2 - 2\nabla_1 h_{11} \nabla_1 h_{22} - \frac{2\pi}{\mathcal{A}} \nabla_k h_{11} \langle F^k, \vec{e}_3 \rangle - K^2) \\ &\quad - \frac{2h_{22}|\nabla_1 h_{11}|^2 + 2h_{11}|\nabla_2 h_{11}|^2}{(h_{11})^2} + \frac{2\Lambda}{\eta} - \frac{8\pi|F|}{\mathcal{A}\eta} \\ &= -\frac{2\nabla_1 h_{11}(h_{22}\nabla_1 h_{11} + h_{11}\nabla_1 h_{22})}{(h_{11})^2} + \frac{1}{h_{11}} \left(-\frac{2\pi}{\mathcal{A}} \nabla_k h_{11} \langle F^k, \vec{e}_3 \rangle - K^2 \right) + \frac{2\Lambda}{\eta} - \frac{8\pi|F|}{\mathcal{A}\eta}. \end{aligned}$$

However, (4.7) and (4.10) imply the following at p_0

$$h_{22}\nabla_1 h_{11} + h_{11}\nabla_1 h_{22} = \nabla_1 K = -\frac{2\pi}{\mathcal{A}} h_{11} \langle F^1, \vec{e}_3 \rangle.$$

Therefore, the last inequality can be reduced to

$$0 \geq -\frac{2\pi}{\mathcal{A}} \frac{\nabla_2 h_{11}}{h_{11}} \langle F^2, \vec{e}_3 \rangle + \frac{2\pi}{\mathcal{A}} \frac{\nabla_1 h_{11}}{h_{11}} \langle F^1, \vec{e}_3 \rangle - \frac{K^2}{h_{11}} + \frac{2\Lambda}{\eta} - \frac{8\pi|F|}{\mathcal{A}\eta}.$$

Observing $(h_{11})^{-1} \nabla_i h_{11}(p_0) = -\eta^{-1} \nabla_i \eta(p_0)$ by (4.3) and $\nabla w(p_0) = 0$, we have

$$0 \geq \frac{2\pi}{\mathcal{A}} \frac{\nabla_2 \eta}{\eta} \langle F^2, \vec{e}_3 \rangle - \frac{2\pi}{\mathcal{A}} \frac{\nabla_1 \eta}{\eta} \langle F^1, \vec{e}_3 \rangle - \frac{K^2}{h_{11}} + \frac{2\Lambda}{\eta} - \frac{8\pi|F|}{\mathcal{A}\eta}.$$

Applying $h_{11}(p_0) = \Lambda(p_0)$ and (4.2) to the inequality above, we obtain

$$0 \geq \frac{4\pi}{\mathcal{A}\eta} \langle F_2, F \rangle \langle F^2, \vec{e}_3 \rangle - \frac{4\pi}{\mathcal{A}\eta} \langle F_1, F \rangle \langle F^1, \vec{e}_3 \rangle - \frac{4\pi^2 |\langle \vec{n}, \vec{e}_3 \rangle|^2}{\mathcal{A}^2 \Lambda} + \frac{2\Lambda}{\eta} - \frac{8\pi|F|}{\mathcal{A}\eta}.$$

We next multiply by $\eta\Lambda$ the last inequality and apply $|\langle F_i, F \rangle \langle F^i, \vec{e}_3 \rangle| \leq |F|$ and $\langle \vec{n}, \vec{e}_3 \rangle \leq 1$. Then, by also using the definition $\eta = (|F|^2 - R^2)$, we obtain

$$0 \geq -\frac{16\pi}{\mathcal{A}} |F| \Lambda - \frac{4\pi^2 \eta}{\mathcal{A}^2} + 2\Lambda^2 \geq 2\Lambda^2 - \frac{16\pi}{\mathcal{A}} |F| \Lambda - \frac{4\pi^2}{\mathcal{A}^2} |F|^2.$$

Solving the quadratic inequality of Λ , we obtain an upper bound of Λ at p_0 ,

$$\Lambda \leq \frac{(4 + 3\sqrt{2})\pi|F|}{\mathcal{A}} \leq \frac{9\pi}{\mathcal{A}} |F|.$$

Therefore, multiplying by $\eta \leq |F|^2$ yields the desired result,

$$\eta\Lambda(p) \leq \eta\Lambda(p_0) \leq \frac{9\pi}{\mathcal{A}} |F|^3(p_0) \leq \frac{9\pi}{\mathcal{A}} \sup_{F(q) \in \Sigma_c} |F|^3(q).$$

□

2 Partial derivative bound

DEFINITION 4.5 (Axial symmetry). We say that a surface $\Sigma \subset \mathbb{R}^3$ has *axial symmetry*, if $(x, y, z) \in \Sigma$ guarantees $(-x, y, z), (x, -y, z) \in \Sigma$. Similarly, a set $\Omega \subset \mathbb{R}^2$ has *axial symmetry*, if $(x, y) \in \Omega$ guarantees $(-x, y), (x, -y) \in \Omega$.

NOTATION 4.6 (To be used in sections 4 and 5). Also, we summarize some further notation.

- (i) Given a set $A \subset \mathbb{R}^3$ and a constant s , we denote the $x = s$ level set by $L_s^x(A) = \{(s, y, z) \in A\}$. Similarly, we denote $y = s$ and $z = s$ level set by $L_s^y(A)$ and $L_s^z(A)$, respectively.
- (ii) Given a constant s and a function $f : \Omega \rightarrow \mathbb{R}$ with $\Omega \subset \mathbb{R}^2$, we denote by $L_s(f)$ the s -level set $\{(x, y) \in \Omega : f(x, y) = s\}$.
- (iii) We let e_1 and e_2 the unit vectors $(1, 0)$ and $(0, 1)$, respectively.

(iv) For a complete and convex curve $\Gamma \subset \mathbb{R}^2$, its convex hull $\text{Conv}(\Gamma)$ is given by

$$\text{Conv}(\Gamma) = \{(tx + (1-t)y) : x, y \in \Gamma, t \in [0, 1]\}.$$

If A is a subset of $\text{Conv}(\Gamma)$, then we say A is enclosed by Γ and use the notation

$$A < \Gamma.$$

(v) Given a set $A \subset \mathbb{R}^2$, $\text{cl}(A)$ and $\text{Int}(A)$ mean the closure and the interior of A , respectively.

Assume that Ω is an open bounded strictly convex and smooth domain of \mathbb{R}^2 and assume in addition that Ω is axially symmetric. Let u be the unique solution of (4.1) on Ω which defines the surface Σ . The following simple property readily follows from the uniqueness of solutions.

PROPOSITION 4.7 (Symmetry of solutions). *Let Ω be an open bounded strictly convex and smooth subset of \mathbb{R}^2 which is axially symmetric. Then, a solution u of (4.1) also is axially symmetric.*

The symmetry of u implies that the half surface $\{(x, y, z) \in \Sigma : y \leq 0\}$ is the graph of a function $h : \Omega_y \rightarrow \mathbb{R}$, that is

$$\{(x, y, z) \in \Sigma : y \leq 0\} = \{(x, h(x, z), z) : (x, z) \in \Omega_y\}, \quad \text{where } \Omega_y = \{(x, z) : (x, y, z) \in \Sigma\}.$$

Moreover, the function h satisfies the following equation

$$(4.12) \quad \frac{\det D^2 h}{(1 + |Dh|^2)^{\frac{3}{2}}} = K (1 + |Dh|^2)^{\frac{1}{2}} = \frac{2\pi}{\mathcal{A}(\Omega)} \langle \vec{e}_3, \vec{n} \rangle (1 + |Dh|^2)^{\frac{1}{2}} = -\frac{2\pi}{\mathcal{A}(\Omega)} h_z.$$

The right hand side of the equation above can be written as $(2\pi/\mathcal{A})h_v \langle -e_2, v \rangle$, where v is the outward normal direction of the level set of h . Thus, the degenerate Monge-Ampere equation (4.12) has two degenerating factors $h_v = |Dh|$ and $\langle -e_2, v \rangle$. In this section, we study the lower bound for $\langle -e_2, v \rangle$ which corresponds to the upper bound of $|\partial_x u|$, the *partial derivative bound*. Notice that $|\partial_x u|$ is bounded even on the flat sides.

To obtain the lower bound for $\langle -e_2, v \rangle$, we will construct an one parameter family of very wide but short supersolutions φ_α of equation (4.12) with $\mathcal{A}(\Omega) < 8$. We will then cut the graph of φ_α so that each of them is contained in a narrow cylinder. By sliding φ_α along the z -axis we will estimate the partial derivative $\partial_x u$ at a touching point which will lead to the desired upper bound for $\partial_x u$ as stated in Theorem 4.10.

DEFINITION 4.8 (Barrier construction). Given a constant $\alpha \in (0, 1/6)$, denote by Δ_α the convex set

$$\Delta_\alpha = \left\{ (x, z) \in \mathbb{R}^2 : x \in \left(-\frac{2}{\alpha} + 1 - 3\alpha, 1 - 3\alpha \right), z \geq -\frac{2}{\alpha\pi} \log \cos \left(\frac{\alpha\pi}{2} \left(x - 1 + 3\alpha + \frac{1}{\alpha} \right) \right) \right\}.$$

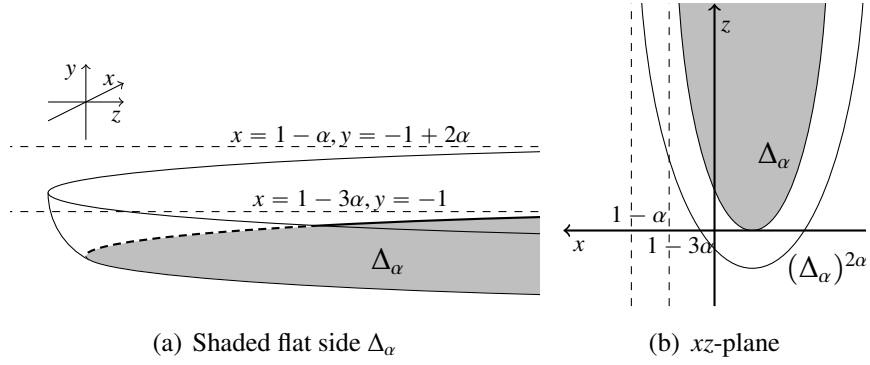


FIGURE 2. Supersolution φ_α

We denote by $d_{\Delta_\alpha}(x, z)$ the distance function $d((x, z), \Delta_\alpha)$. In particular, if $(x, z) \in \Delta_\alpha$, then $d_{\Delta_\alpha}(x, z) = 0$. By using $d_{\Delta_\alpha}(x, z)$, we define the 2α -extension $(\Delta_\alpha)^{2\alpha}$ of Δ_α by

$$(\Delta_\alpha)^{2\alpha} = \{(x, z) \in \mathbb{R}^2 : d_{\Delta_\alpha}(x, z) \leq 2\alpha\}.$$

Finally, we define the function $\varphi_\alpha : \text{cl}((\Delta_\alpha)^{2\alpha} \setminus \Delta_\alpha) \rightarrow \mathbb{R}$ by

$$\varphi_\alpha(x, z) = -1 + 2\alpha - \sqrt{4\alpha^2 - d^2(\Delta_\alpha)(x, z)}.$$

This is all shown in Figure 3.

LEMMA 4.9 (Supersolution). *Given a constant $\alpha \in (0, 1/6)$, the function φ_α in Definition 4.8 is a convex function satisfying*

$$\frac{\det D^2\varphi}{(1 + |D\varphi|^2)^{\frac{3}{2}}} \leq -\frac{\pi}{4}\varphi_z.$$

PROOF. For convenience, we let φ and d denote φ_α and d_{Δ_α} respectively. For each point $p \in \mathbb{R}^2$ with $d(p) > 0$, we denote by $\tau(p)$ and $\nu(p)$ the tangential and the normal direction of a level set $L_{d(p)}(d)$ of the distance function d satisfying $\langle \tau, e_1 \rangle \geq 0$ and $\langle \nu, e_2 \rangle \leq 0$, respectively. Then, we have

$$(4.13) \quad Dd = \nu, \quad d_\nu = |Dd| = 1, \quad d_\tau = 0.$$

We observe $\nu(p) = \nu(p + \epsilon\nu(p))$ for all $\epsilon \in \mathbb{R}$ with $d(p + \epsilon\nu(p)) > 0$, which implies

$$(4.14) \quad d_{\nu\nu} = d_{\nu\tau} = 0.$$

To derive $d_{\tau\tau}(p)$, given a point p_0 , we consider the immersion $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2$ satisfying $d(\gamma(s)) = d(p_0)$ with $\gamma(0) = p_0$, where s is the arc length parameter of the level set, $\gamma(\mathbb{R}) = L_{d(p_0)}(d)$. By differentiating $d(\gamma(s)) = d(p_0)$ twice with respect to s , we obtain

$$\langle \gamma_s, (D^2d)\gamma_s \rangle + \langle Dd, \gamma_{ss} \rangle = 0$$

We can observe that $\gamma_s(0) = \tau(p_0)$ and $\gamma_{ss}(0) = -\kappa(p_0)v(p_0)$, where $\kappa(p) > 0$ is the curvature of $L_{d(p)}(d)$ at p . Hence, $Dd(\gamma(0)) = v(p_0)$ implies $\langle \tau, (D^2d)\tau \rangle + \langle -\kappa v, v \rangle = 0$ at p_0 . Thus,

$$(4.15) \quad d_{\tau\tau}(p) = \kappa(p).$$

Hence we can directly derive from (4.13), (4.14) and (4.15) the following holding at each $p \in \text{cl}((\Delta_\alpha)^{2\alpha} \setminus \Delta_\alpha)$

$$\varphi_v = d(4\alpha^2 - d^2)^{-\frac{1}{2}}, \quad \varphi_\tau = 0, \quad \varphi_{vv} = 4\alpha^2(4\alpha^2 - d^2)^{-\frac{3}{2}}, \quad \varphi_{v\tau} = 0, \quad \varphi_{\tau\tau} = \kappa \varphi_v.$$

Therefore, φ is a convex function.

Next, combining the equalities above yields

$$(4.16) \quad \frac{\det D^2\varphi}{(1 + |D\varphi|^2)^{\frac{3}{2}}} = \frac{\varphi_{vv}\varphi_{\tau\tau}}{(1 + \varphi_v^2)^{\frac{3}{2}}} = \frac{1}{2\alpha}\varphi_{\tau\tau} = \frac{1}{2\alpha}\kappa\varphi_v.$$

Now, we consider the point $p_0 = p - d(p)v(p) \in \partial\Delta_\alpha$. Then, the convexity of $\partial\Delta_\alpha$ leads to

$$(4.17) \quad \kappa(p) \leq \kappa(p_0).$$

We recall that the Grim Reaper curve $\partial\Delta_\alpha$ is the graph of the convex function $f_\alpha(x)$ defined by

$$(4.18) \quad f_\alpha(x) = -\frac{2}{\pi\alpha} \log \cos \left(\frac{\alpha\pi}{2} \left(x - 1 + 3\alpha + \frac{1}{\alpha} \right) \right).$$

Hence, at x_0 with $p_0 = (x_0, f_\alpha(x_0))$, the following holds

$$\kappa(p_0) = \frac{f_\alpha''(x_0)}{(1 + |f_\alpha'(x_0)|^2)^{\frac{3}{2}}} = \frac{\pi\alpha/2}{(1 + |f_\alpha'(x_0)|^2)^{\frac{1}{2}}} = -\frac{\pi\alpha}{2} \langle v(p_0), e_2 \rangle.$$

Thus, $v(p_0) = v(p - d(p)v(p)) = v(p)$ implies

$$(4.19) \quad \kappa(p) \leq \kappa(p_0) = -\frac{\pi\alpha}{2} \langle v(p_0), e_2 \rangle = -\frac{\pi\alpha}{2} \langle v(p), e_2 \rangle.$$

Therefore, given a point $p \in \text{cl}((\Delta_\alpha)^{2\alpha} \setminus \Delta_\alpha)$, (4.16), (4.17), and (4.19) give the desired result

$$\frac{\det D^2\varphi}{(1 + |D\varphi|^2)^{\frac{3}{2}}} = \frac{1}{2\alpha}\varphi_v\kappa \leq -\frac{\pi}{4}\varphi_v\langle v, e_2 \rangle = -\frac{\pi}{4}(\varphi_v\langle v, e_2 \rangle + \varphi_\tau\langle \tau, e_2 \rangle) = -\frac{\pi}{4}\varphi_z.$$

□

THEOREM 4.10 (Partial derivative bound). *Let Ω be an open strictly convex smooth subset of \mathbb{R}^2 which is in additionally axially symmetric and satisfies*

$$[-1, 1] \times [-1, 1] \subset \Omega \subset \left[-\frac{4}{3}, \frac{4}{3} \right] \times \left[-\frac{4}{3}, \frac{4}{3} \right].$$

Then, a solution $u : \Omega \rightarrow \mathbb{R}$ of (4.1) satisfies

$$\partial_x u(1 - 5\alpha, -1 + \epsilon) \leq 2 + 3 \cot \frac{\pi\alpha^2}{2},$$

for all $\epsilon \in (0, 1/2)$ and $\alpha \in (0, 1/6)$.

PROOF. Since the domain Ω has strictly convex and smooth boundary, Theorem 4.1 implies that the graph $\Sigma = \{(x, y, u(x, y)) : (x, y) \in \Omega\}$ of the function u is a strictly convex complete smooth solution of (4.2). In addition by Proposition 4.7 the function u is axially symmetric and we can define a convex set Ω_y and a convex function $h : \Omega_y \rightarrow \mathbb{R}$ by

$$\Omega_y = \{(x, z) : (x, y, z) \in \Sigma\}, \quad \{(x, y, z) \in \Sigma : y \leq 0\} = \{(x, h(x, z), z) : (x, z) \in \Omega_y\}.$$

Then, $(x, y, u(x, y)) = (x, h(x, z), z)$ implies that the function h satisfies (4.12). Thus, by the given condition $\mathcal{A}(\Omega) \leq (8/3)^2 < 8$ and Lemma 4.9, the function φ_α in Definition 4.8 is a supersolution for (4.12).

To construct a barrier $\Phi_{\epsilon, \alpha}^{t, \epsilon, \alpha}$, we will cut the graph of $\epsilon + \varphi_\alpha$ by $\{1 - 4\alpha\} \times \mathbb{R}^2$ (the blue section in Figure 3(a)) and slide it along z -direction until it touches Σ at a point P_0 . We will show that the contact point P_0 is contained in $\{1 - 4\alpha\} \times \mathbb{R}^2$, namely P_0 is a point on the front part of the boundary $\partial\Phi_{\epsilon, \alpha}^{t, \epsilon, \alpha}$ of the barrier. (See the blue curve Γ_F in Figure 3(a)). Then, we will estimate the partial derivative $\partial_x u$ at P_0 by comparing with the barrier $\Phi_{\epsilon, \alpha}^{t, \epsilon, \alpha}$ at P_0 . After obtaining the bound on $\partial_x u$ at P_0 , we will use the convexity of the solution Σ the barrier $\Phi_{\epsilon, \alpha}^{t, \epsilon, \alpha}$ to show the desired bound of $\partial_x u$ at $(1 - 5\alpha, -1 + \epsilon)$.

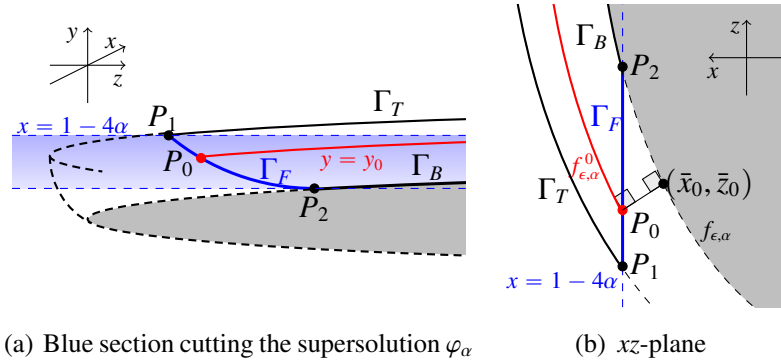


FIGURE 3. Sliding barrier

Step 1 : Sliding barrier construction. We denote by Φ_α the graph of φ_α in $[1 - 4\alpha, +\infty) \times \mathbb{R}^2$,

$$\Phi_\alpha = \{(x, \varphi_\alpha(x, z), z) : (x, z) \in \text{cl}((\Delta_\alpha)^{2\alpha} \setminus \Delta_\alpha), x \geq 1 - 4\alpha\}.$$

Then, given constants $\epsilon \in (0, 1/2)$ and $t \in \mathbb{R}$, we translate Φ_α by $\epsilon \vec{e}_2 + t \vec{e}_3$,

$$\Phi_{\epsilon, \alpha}^t = \{(x, y + \epsilon, z + t) : (x, y, z) \in \Phi_\alpha\}.$$

Notice that definition of φ_α guarantees

$$\Phi_{\epsilon, \alpha}^t \subset [1 - 4\alpha, 1 - \alpha] \times [-1 + \epsilon, -1 + \epsilon + 2\alpha] \times [-t - 2\alpha, +\infty).$$

Hence, there exists a constant $t_{\epsilon,\alpha} \in \mathbb{R}$ and a point $P_0 = (x_0, y_0, z_0) \in \mathbb{R}^3$ satisfying

$$(4.20) \quad \Sigma \cap \bigcap \Phi_{\epsilon,\alpha}^t = \emptyset \quad \text{for } t > t_{\epsilon,\alpha}, \quad P_0 \in \Sigma \cap \bigcap \Phi_{\epsilon,\alpha}^{t_{\epsilon,\alpha}}.$$

We denote by $\Delta_{\epsilon,\alpha}$ the projection of $\Phi_{\epsilon,\alpha}^{t_{\epsilon,\alpha}}$ into the xz -plane, and consider $\Phi_{\epsilon,\alpha}^{t_{\epsilon,\alpha}}$ as the graph of a function $\varphi_{\epsilon,\alpha} : \Delta_{\epsilon,\alpha} \rightarrow \mathbb{R}$

$$\Delta_{\epsilon,\alpha} = \{(x, z) : (x, y, z) \in \Phi_{\epsilon,\alpha}^{t_{\epsilon,\alpha}}\}, \quad \Phi_{\epsilon,\alpha}^{t_{\epsilon,\alpha}} = \{(x, \varphi_{\epsilon,\alpha}(x, z), z) : (x, z) \in \Delta_{\epsilon,\alpha}\}.$$

Step 2 : Position of the contact point P_0 . In this step, we will show that the contact point P_0 is contained in the front part $L_{1-4\alpha}^x(\Phi_{\epsilon,\alpha}^{t_{\epsilon,\alpha}})$ of the boundary $\partial\Phi_{\epsilon,\alpha}^{t_{\epsilon,\alpha}}$.

First of all, the contact point P_0 can not be an interior point of $\Phi_{\epsilon,\alpha}^{t_{\epsilon,\alpha}}$, because $\varphi_{\epsilon,\alpha}$ is a super-solution. Thus, P_0 is a point on the boundary $\partial\Phi_{\epsilon,\alpha}^{t_{\epsilon,\alpha}}$ of $\Phi_{\epsilon,\alpha}^{t_{\epsilon,\alpha}}$. We observe that the boundary $\partial\Phi_{\epsilon,\alpha}^{t_{\epsilon,\alpha}}$ can be decomposed into the top Γ_T , bottom Γ_B , and front Γ_F boundary as following

$$(4.21) \quad \partial\Phi_{\epsilon,\alpha}^{t_{\epsilon,\alpha}} = \Gamma_T \cup \Gamma_B \cup \Gamma_F, \quad \Gamma_T = L_{-1+\epsilon+2\alpha}^y(\Phi_{\epsilon,\alpha}^{t_{\epsilon,\alpha}}), \quad \Gamma_B = L_{-1+\epsilon}^y(\Phi_{\epsilon,\alpha}^{t_{\epsilon,\alpha}}), \quad \Gamma_F = L_{1-4\alpha}^x(\Phi_{\epsilon,\alpha}^{t_{\epsilon,\alpha}}).$$

We denote by P_1 and P_2 the end point of the top Γ_T and the bottom Γ_B boundary, respectively

$$(4.22) \quad P_1 = (x_1, y_1, z_1) = \Gamma_T \cap \Gamma_F = (1 - 4\alpha, -1 + \epsilon + 2\alpha, z_1), \\ P_2 = (x_2, y_2, z_2) = \Gamma_B \cap \Gamma_F = (1 - 4\alpha, -1 + \epsilon, z_2).$$

On the other hand (4.20) gives $\text{cl}(\Delta_{\epsilon,\alpha}) =: \Delta_{\epsilon,\alpha} \subset \text{Int}(\Omega_y)$ and $h(x, z) \leq \varphi_{\epsilon,\alpha}(x, z)$ on $\Delta_{\epsilon,\alpha}$. Also, we have $|D\varphi_{\epsilon,\alpha}| = +\infty$ on Γ_T . Hence, if $P_0 \in (\Gamma_T \setminus \{P_1\})$, then $|Dh| = +\infty$ holds at P_0 by $h \leq \varphi_{\epsilon,\alpha}$, which contradicts to $\text{cl}(\Delta_{\epsilon,\alpha}) \subset \text{Int}(\Omega_y)$. Thus,

$$P_0 \notin (\Gamma_T \setminus \{P_1\})$$

Moreover, we have $|D\varphi_{\epsilon,\alpha}| = 0$ on Γ_B . Thus, if $P_0 \in (\Gamma_B \setminus \{P_2\})$, then $|Dh| = 0$ holds at P_0 . However, Σ is a strictly convex complete surface, which means $|Dh| \neq 0$. Therefore,

$$P_0 \notin (\Gamma_B \setminus \{P_2\})$$

Hence, by (4.21) and (4.22), P_0 is a point on the front boundary Γ_F

$$(4.23) \quad P_0 = (x_0, y_0, z_0) = (1 - 4\alpha, y_0, z_0) \in \Gamma_F.$$

Step 3 : Distance between P_0 and P_2 . In this step, we will estimate $z_2 - z_0$ in terms of α .

We recall that the Grim reaper curve $\partial\Delta_\alpha$ is the graph of the function $f_\alpha(x)$ defined by (4.18) and $\Delta_{\epsilon,\alpha}$ is a subset of $\bar{\Delta}_{\epsilon,\alpha} =: \Delta_\alpha + t_{\epsilon,\alpha}e_z$. Hence, $\partial\bar{\Delta}_{\epsilon,\alpha}$ is the graph of the function $f_{\epsilon,\alpha}$ defined by

$$(4.24) \quad f_{\epsilon,\alpha}(x) = t_{\epsilon,\alpha} + f_\alpha(x) = t_{\epsilon,\alpha} - \frac{2}{\pi\alpha} \log \cos \left(\frac{\alpha\pi}{2} \left(x - 1 + 3\alpha + \frac{1}{\alpha} \right) \right).$$

By definition of φ_α , there exists a unique point $(\bar{x}_0, \bar{z}_0) \in \partial\Delta_{\epsilon,\alpha}$ such that

$$(4.25) \quad d((x_0, z_0), \bar{\Delta}_{\epsilon,\alpha}) = d((x_0, z_0), (\bar{x}_0, \bar{z}_0)) \leq 2\alpha.$$

We know $x_0 = x_2 = 1 - 4\alpha$ by (4.22) and (4.23). Hence, for all $x \in [\bar{x}_0, x_2]$, we can derive from (4.24) the following inequality

$$(4.26) \quad f'_{\epsilon,\alpha}(\bar{x}_0) \leq f'_{\epsilon,\alpha}(x) \leq f'_{\epsilon,\alpha}(x_2) = \tan\left(\frac{\pi\alpha}{2}\left(x_0 - 1 + 3\alpha + \frac{1}{\alpha}\right)\right) = \cot\frac{\pi\alpha^2}{2}.$$

Therefore, combining (4.25) and (4.26) yields

$$(4.27) \quad z_2 - z_0 \leq (\bar{z}_0 - z_0) + (z_2 - \bar{z}_0) \leq 2\alpha + (z_6 - \bar{z}_0) \leq 2\alpha + \int_{\bar{x}_0}^{x_2} f'_{\epsilon,\alpha}(x)dx \leq 2\alpha + 2\alpha \cot\frac{\pi\alpha^2}{2}.$$

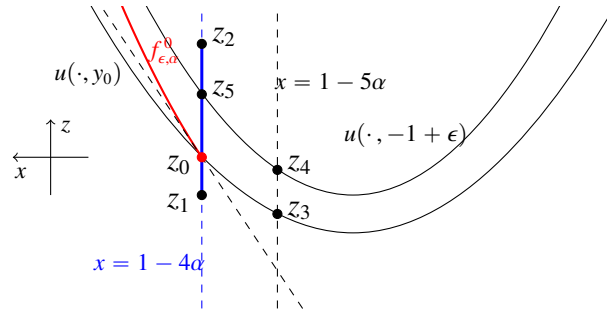


FIGURE 4. Level sets of the solution h

Step 4 : Partial derivative $\partial_x u$ bound at the contact point P_0 . We can consider the level set $L_{y_0}(\varphi_{\epsilon,\alpha})$ as the graph of a convex function $f_{\epsilon,\alpha}^0 : [1 - 4\alpha, 1 - \alpha] \rightarrow \mathbb{R}$, namely $L_{y_0}(\varphi_{\epsilon,\alpha}) = \{(x, f_{\epsilon,\alpha}^0(x)) : x \in [1 - 4\alpha, 1 - \alpha]\}$. Then, by the definitions of φ_α and (\bar{x}_0, \bar{z}_0) , we have $(f_{\epsilon,\alpha}^0)'(x_0) = f'_{\epsilon,\alpha}(\bar{x}_0)$. Thus, (4.26) yields the bound

$$(f_{\epsilon,\alpha}^0)'(x_0) \leq \cot(\pi\alpha^2/2).$$

On the other hand, (4.20) implies $L_{y_0}(\varphi_{\epsilon,\alpha}) < L_{y_0}(h)$, namely $u(x, y_0) \leq f_{\epsilon,\alpha}^0(x)$ holds for all $x \in [1 - 4\alpha, 1 - \alpha]$. Therefore,

$$(4.28) \quad \partial_x u(x_0, y_0) \leq (f_{\epsilon,\alpha}^0)'(x_0) \leq \cot(\pi\alpha^2/2).$$

Step 5 : Partial derivative $\partial_x u$ bound at the given point. We define the points P_3, P_4, P_5 on Σ by

$$\begin{aligned} P_3 &= (x_3, y_3, z_3) =: (1 - 5\alpha, y_0, u(1 - 5\alpha, y_0)), \\ P_4 &= (x_4, y_4, z_4) =: (1 - 5\alpha, -1 + \epsilon, u(1 - 5\alpha, -1 + \epsilon)), \\ P_5 &= (x_5, y_5, z_5) =: (1 - 4\alpha, -1 + \epsilon, u(1 - 4\alpha, -1 + \epsilon)). \end{aligned}$$

Since we know $P_0, P_3 \in L_{y_0}^y(\Sigma)$, the inequality (4.28) and the convexity of u give

$$z_0 - z_3 = \int_{x_3}^{x_0} \partial_x u(x, y_0) dx \leq \int_{x_3}^{x_0} \partial_x u(x_0, y_0) dx = \alpha (\partial_x u(x_0, y_0)) \leq \alpha \cot \frac{\pi \alpha^2}{2}.$$

By adding (4.27) and the inequality above, we obtain

$$(4.29) \quad z_2 - z_3 \leq 2\alpha + 3\alpha \cot(\pi \alpha^2 / 2).$$

On the other hand, (4.20) implies that $L_{-1+\epsilon}(\varphi_{\epsilon, \alpha}) < L_{-1+\epsilon}(h)$. Therefore, $x_2 = x_5 = 1 - 4\alpha$, $(x_2, z_2) \in L_{-1+\epsilon}(\varphi_{\epsilon, \alpha})$, and $(x_5, z_5) \in L_{-1+\epsilon}(h)$ guarantee

$$z_2 = \varphi_{\epsilon, \alpha}(x_2) \geq u(x_2, -1 + \epsilon) = u(x_5, -1 + \epsilon) = z_5.$$

Also, the convexity and symmetry of Σ give that

$$z_3 \leq z_4$$

Thus, subtracting the inequalities above yields $z_5 - z_4 \leq z_2 - z_3$. Applying (4.29), we have

$$z_5 - z_4 \leq 2\alpha + 3\alpha \cot(\pi \alpha^2 / 2).$$

Hence, the desired result follows by the following computation

$$z_5 - z_4 = \int_{x_4}^{x_5} \partial_x u(x, -1 + \epsilon) dx \geq \int_{x_4}^{x_5} \partial_x u(x_4, -1 + \epsilon) dx = \alpha (\partial_x u(1 - 5\alpha, -1 + \epsilon)).$$

□

3 Distance from the tip to flat sides

Let Σ be the translating solution to the Gauss curvature flow over the square Ω as in Theorem 4.2. In this final section we will show that this solution has flat sides, as stated in Theorem 4.12 below. To this end, we will study the distance from the tip of a solution Σ to each point on the *free boundary*, that is the boundary of the flat sides. To estimate this distance one needs to establish a *gradient bound* for solutions to the equation (4.1) at *a certain point* near the flat sides. Since the gradient bound depends on the global structure of Ω , we will establish an integral estimate by deriving a separation of variables structure from (4.1) as in the proof of the following Lemma.

LEMMA 4.11 (Gradient bound). *Let Ω satisfy the conditions in Theorem 4.10 and let u be a solution of (4.1) on Ω . Assume that $[a, b] \times [-1, -1 + \sigma] \subset \Omega$, for some constants $a, b, \sigma \in (0, 1)$. Then, there exists a point $x_0 \in [a, b]$ satisfying*

$$-\partial_y u(x_0, -1 + \sigma) \leq \sqrt{\frac{2M}{\pi \sigma (b - a)}},$$

where $M = \sup_{y \in (0, \sigma)} \partial_x u(b, -1 + y)$.

PROOF. Since the domain ω satisfies the assumptions of Theorem 4.10, we have $\mathcal{A} \leq 64/9$. Hence, the following inequality holds

$$\frac{u_{yy}u_{xx}}{(1+u_y^2)^{\frac{3}{2}}} \geq \frac{u_{yy}u_{xx} - u_{xy}^2}{(1+u_x^2+u_y^2)^{\frac{3}{2}}} = \frac{\det D^2u}{(1+|Du|^2)^{\frac{3}{2}}} = \frac{2\pi}{\mathcal{A}} \geq \frac{\pi}{4}.$$

which combined with Holder inequality yields

$$\left(\int_a^b \frac{u_{yy}}{(1+u_y^2)^{\frac{3}{2}}} dx \right) \left(\int_a^b u_{xx} dx \right) \geq \left(\int_a^b (\pi/4)^{\frac{1}{2}} dx \right)^2 = \frac{\pi}{4}(a-b)^2.$$

Since $a \geq 0$ and Proposition 4.7 guarantee that $u_x(\cdot, a) \geq 0$, we have

$$\int_a^b u_{xx} dx = u_x(y, b) - u_x(y, a) \leq u_x(y, b) \leq M.$$

Hence,

$$\frac{1}{b-a} \int_a^b \int_{-1}^{-1+\sigma} \frac{u_{yy}}{(1+u_y^2)^{\frac{3}{2}}} dy dx \geq \frac{\pi\sigma}{4M}(b-a).$$

Therefore, there exists a constant $x_0 \in [a, b]$ satisfying

$$\int_{-1}^{-1+\sigma} \frac{u_{yy}}{(1+u_y^2)^{\frac{3}{2}}}(x_0, \cdot) dy \geq \frac{\pi\sigma}{4M}(b-a).$$

Moreover, by $|u_y| \leq (1+u_y^2)^{\frac{1}{2}}$, we have

$$\int_{-1}^{-1+\sigma} \frac{u_{yy}}{(1+u_y^2)^{\frac{3}{2}}}(x_0, \cdot) dy = \frac{u_y}{(1+u_y^2)^{\frac{1}{2}}}(x_0, \cdot) \Big|_{-1}^{-1+\sigma} \leq \frac{u_y}{(1+u_y^2)^{\frac{1}{2}}}(x_0, -1+\sigma) + 1.$$

Hence, at $(x_0, -1+\sigma)$, the following holds

$$\frac{|u_y|}{(1+u_y^2)^{\frac{1}{2}}}(x_0, -1+\sigma) = \frac{-u_y}{(1+u_y^2)^{\frac{1}{2}}}(x_0, -1+\sigma) \leq 1 - \frac{\pi\sigma}{4M}(b-a)$$

implying the bound

$$1 + u_y^{-2}(x_0, -1+\sigma) \geq \left(1 - \frac{\pi\sigma}{4M}(b-a)\right)^{-2} \geq \left(1 + \frac{\pi\sigma}{4M}(b-a)\right)^2 \geq 1 + \frac{\pi\sigma}{2M}(b-a).$$

Solving this last inequality for $-u_y(x_0, -1+\sigma) \geq 0$ leads to the desired result. \square

The distance between the tip of the translating solution Σ over the square and its flat sides is estimated in the following result.

THEOREM 4.12 (Distance between the tip and flat sides). *Let Ω satisfy the conditions in Theorem 4.10 and let u be a solution of (4.1). Given $\alpha \in (0, 1/6)$, there exists a constant $C > 0$ satisfying*

$$u(1-6\alpha, -1) - u(0, 0) \leq 6\left(1 + \frac{1}{\alpha^2}\right).$$

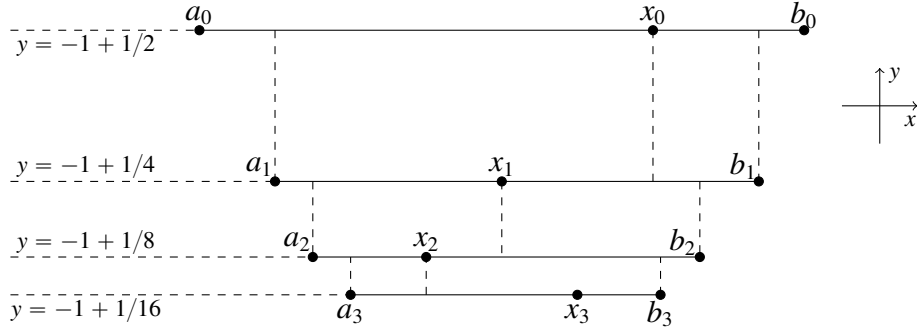


FIGURE 5. Converging points on domain Ω

PROOF. We begin by setting $a_0 = 1 - 6\alpha$, $b_0 = 1 - 5\alpha$, $\sigma_0 = \frac{1}{2}$, and $M = \sup_{y \in (0, 1/2)} \partial_x u(b_0, -1 + y)$. Then, by Lemma 4.11, there exists a point $x_0 \in [a_0, b_0]$ satisfying

$$(4.30) \quad -\partial_y u(x_0, -1 + 1/2) \leq 2(M/\pi\alpha)^{\frac{1}{2}}.$$

We choose an interval $[a_1, b_1]$ satisfying $x_0 \in [a_1, b_1] \subset [a_0, b_0]$ and $b_1 - a_1 = 2^{-1/3}\alpha$. Then, for $\sigma_1 = 2^{-2}$, we have $\sup_{y \in (0, \sigma_1)} \partial_x u(b_1, -1 + y) \leq M$. Hence, Lemma 4.11 gives a point $x_1 \in [a_1, b_1]$ satisfying

$$-\partial_y u(x_1, -1 + 1/2^2) \leq 2^{1+\frac{2}{3}}(M/\pi\alpha)^{\frac{1}{2}}.$$

By setting $\sigma_n = 2^{-1-n}$, we can inductively choose intervals $[a_n, b_n]$ satisfying $x_{n-1} \in [a_n, b_n] \subset [a_{n-1}, b_{n-1}]$ and $b_n - a_n = 2^{-n/3}\alpha$ so that we obtain a point $x_n \in [a_n, b_n]$ satisfying

$$-\partial_y u(x_n, -1 + 1/2^{n+1}) \leq 2^{1+\frac{2n}{3}}(M/\pi\alpha)^{\frac{1}{2}}.$$

Then, integrating along y yields

$$\left| u(x_n, -1 + \frac{1}{2^{n+1}}) - u(x_n, -1 + \frac{1}{2^n}) \right| \leq \int_{-1+1/2^{n+1}}^{-1+1/2^n} -\partial_y u(x_n, y) dy \leq 2^{-n/3}(M/\pi\alpha)^{\frac{1}{2}}.$$

On the other hand, $x_{n-1}, x_n \in [a_n, b_n]$ and $a_n - b_n = 2^{-n/3}\alpha$ imply

$$\left| u(x_n, -1 + \frac{1}{2^n}) - u(x_{n-1}, -1 + \frac{1}{2^n}) \right| = \left| \int_{x_{n-1}}^{x_n} \partial_x u(x, -1 + \frac{1}{2^n}) dx \right| \leq 2^{-\frac{n}{3}} M\alpha.$$

Therefore,

$$\left| u(x_n, -1 + \frac{1}{2^{n+1}}) - u(x_{n-1}, -1 + \frac{1}{2^n}) \right| \leq 2^{-\frac{n}{3}} ((M/\pi\alpha)^{\frac{1}{2}} + M\alpha).$$

By definition of x_n , the sequence $\{x_n\}_{n \in \mathbb{N}}$ converges to a point $\bar{x} \in [a_0, b_0]$. Hence, we can sum up the inequality above for all $n \in \mathbb{N}$ so that we have

$$(4.31) \quad u(\bar{x}, -1) - u(x_0, -\frac{1}{2}) \leq \sum_{n=1}^{\infty} 2^{-\frac{n}{3}} ((M/\pi\alpha)^{\frac{1}{2}} + M\alpha) \leq 4((M/\pi\alpha)^{\frac{1}{2}} + M\alpha).$$

Next, we consider the linear function

$$f(x, y) = (\partial_x u(x_0, -1/2))(x - x_0) + (\partial_y u(x_0, -1/2))(y + 1/2) + u(x_0, -1/2)$$

whose graph is the tangent hyperplane of Σ at $(x_0, -1/2, u(x_0, -1/2))$. Then, the convexity of Σ gives $f(0, 0) \leq u(0, 0)$. Hence, (4.30) and definition of M show

$$u(x_0, -\frac{1}{2}) - u(0, 0) \leq f(x_0, -\frac{1}{2}) - f(0, 0) \leq (M/\pi\alpha)^{\frac{1}{2}} + M.$$

Thus, Theorem 4.10, (4.31), and the inequality give

$$u(\bar{x}, -1) - u(0, 0) \leq 5\left(\frac{M}{\pi\alpha}\right)^{\frac{1}{2}} + 2M \leq 5\left(\frac{2}{\pi\alpha} + \frac{3}{\pi\alpha} \cot \frac{\pi\alpha^2}{2}\right)^{\frac{1}{2}} + 2\left(2 + 3 \cot \frac{\pi\alpha^2}{2}\right).$$

Applying $1/\alpha \geq 6$ and $\cot(\pi\alpha^2/2) \leq 2/(\pi\alpha^2)$, we have

$$u(\bar{x}, -1) - u(0, 0) \leq 5\left(\frac{2}{\pi\alpha} + \frac{6}{\pi^2\alpha^3}\right)^{\frac{1}{2}} + \frac{12}{\pi\alpha^2} + 4 \leq 6\left(1 + \frac{1}{\alpha^2}\right).$$

Finally, combining Proposition 4.7, the convexity of u , and $\bar{x} \geq b_0 = 1 - 6\alpha \geq 0$ yields $u(\bar{x}, -1) \geq u(1 - 6\alpha, -1)$, which leads to the desired result. \square

We will now give the proof of the main Theorem 4.2. This readily follows from the following result.

THEOREM 4.13 (Existence of flat sides). *Let $\Omega = (-1, 1) \times (-1, 1)$ and u be a solution of (4.1) on Ω . Then, there exists a function $\bar{u} : (\partial\Omega \setminus V) \rightarrow \mathbb{R}$ satisfying*

$$\bar{u}(x_0) = \lim_{x \rightarrow x_0} u(x).$$

Also, the corresponding complete solution Σ of (4.2) is a convex surface of class $C_{loc}^{1,1}$, and u satisfies

$$\lim_{x \rightarrow V} u(x) = +\infty.$$

PROOF. Let $\{\Omega_n\}_{n \in \mathbb{N}}$ be a sequence of sets satisfying the conditions in Theorem 4.10, and let $\{u_n\}_{n \in \mathbb{N}}$ and $\{\Sigma_n\}_{n \in \mathbb{N}}$ be the sequence of corresponding solutions of (4.1) with $u_n(0, 0) = 0$ and their graphs, respectively. We denote the convex hull of Σ_n by $E_n = \{tX + (1-t)Y : X, Y \in \Sigma_n, t \in [0, 1]\}$, and define a convex body E by

$$E = \bigcap_{n \in \mathbb{N}} E_n.$$

Observe that E is not an empty set, because Theorem 4.12 gives that $(1 - 6\alpha, -1, 6(1 + \alpha^{-2})) \in E_n$ which means that $(1 - 6\alpha, -1, 6(1 + \alpha^{-2})) \in E$. Now, we denote by Σ the boundary of E . Then, Σ is naturally a convex complete and non-compact surface, since $\{(0, 0, t) : t \geq 0\} \subset E_n$ implies $\{(0, 0, t) : t \geq 0\} \subset E$.

We define $u : \Omega \rightarrow \mathbb{R}$ by $(x, y, u(x, y)) \in \Sigma$. Then, the symmetry of Ω_n , the convexity of E , and $(1 - 6\alpha, -1, 6(1 + \alpha^{-2})) \in E$ guarantees that there exists a function $\bar{u} : (\partial\Omega \setminus V) \rightarrow \mathbb{R}$ satisfying

$$\bar{u}(x_0) = \lim_{x \rightarrow x_0} u(x).$$

Moreover, Theorem 4.4 shows the local $C^{1,1}$ regularity of Σ . Now, we assume that there exists a point $(1, 1, t_0)$ in E , then the mean curvature of Σ attains $+\infty$ at $(1, 1, t)$ for $t > t_0$. This contradicts to the local $C^{1,1}$ regularity of Σ . Hence, we have $\Sigma \cap (V \times \mathbb{R}) = \emptyset$, namely u satisfies

$$\lim_{x \rightarrow V} u(x) = +\infty.$$

□

Chapter 5

Asymptotic Behavior of closed solutions

In this chapter, we show the following Theorem proven in [9].

THEOREM 5.1 (Choi-Daskalopoulos). *Given $\alpha \in (\frac{1}{n}, 1 + \frac{1}{n})$, the unit n -sphere is the unique closed strictly convex smooth solution to $(**^\alpha)$.*

Notice that jointly with Brendle, the result was extended to $\alpha \geq \frac{1}{n+2}$ in [5].

THEOREM 5.2 (Brendle-Choi-Daskalopoulos). *Given $\alpha > \frac{1}{n+2}$, the unit n -sphere is the unique closed strictly convex smooth solution to $(**^\alpha)$. If $\alpha = \frac{1}{n+2}$, closed strictly convex smooth solutions to $(**^\alpha)$ are ellipsoids.*

Theorem 5.2 combined with the results in [4, 19] imply the convergence of the α -Gauss curvature flow to the round sphere (or ellipsoids), which in particular proves the higher dimensional Firey's conjecture.

THEOREM 5.3. *Let Σ_t be a strictly convex, closed and smooth solution to the α -Gauss curvature flow with $\alpha \in \frac{1}{n+2}$, $n \geq 2$. Then, there exists a finite time T at which the solution Σ_t converges after rescaling to the round sphere. If $\alpha = \frac{1}{n+2}$, the solution converges after rescaling to an ellipsoid.*

REMARK 5.4 (Pogorelov estimate on powers of a matrix). Pogorelov type estimates in this context have been frequently applied in the past by using b^{11} , the first entry of a matrix $A^{-1} := \{b^{ij}\}$. However, if one applies the Pogorelov estimate for $b^{11}K^\alpha - \frac{n\alpha-1}{2n\alpha}|F|^2$, one can obtain the result of Theorem 5.5 only for $\alpha \in (\frac{1}{n}, \frac{1}{2}]$. In this work, by using instead $(b^{1i}g_{ij}b^{j1})^{\frac{1}{2}}$, the root of the first entry of the square A^{-2} of the matrix A^{-1} , we are able to extend the result of Theorem 5.5 to the range of exponents $\alpha \in (\frac{1}{n}, 1 + \frac{1}{n})$, which includes the classical case of the Gauss curvature flow $\alpha = 1$.

In [5], a Pogorelov type estimate was developed in viscosity sense which extends the result of Theorem 5.5 for all $\alpha > \frac{1}{n}$.

1 Pogorelov type computation

We consider the function $W : M^n \rightarrow \mathbb{R}$ given by

$$W(p) := \left(K^\alpha \lambda_{\min}^{-1} - \frac{n\alpha - 1}{2n\alpha} |F|^2 \right)(p).$$

We will employ in this section a Pogorelov type computation to show that the maximum point of $W(p)$ is an umbilical point. We begin with the following standard observation which we include here for the reader's convenience.

We will now show that one of the Pogorelov type expressions of the function W plays a role as a subsolution of $(**^\alpha)$ at a given maximum point, to imply that the maximum point of $W(p)$ is an umbilical point.

THEOREM 5.5 (Pogorelov type computation). *Let Σ be a strictly convex smooth closed solution of $(**^\alpha)$ for an exponent $\alpha \in (\frac{1}{n}, 1 + \frac{1}{n})$. Assume that $F : M^n \rightarrow \mathbb{R}^{n+1}$ is a smooth immersion such that $F(M^n) = \Sigma$, and the continuous function $W(p)$ attains its maximum at a point p_0 . Then, $F(p_0)$ is an umbilical point and $\nabla|F|^2(p_0) = 0$ holds.*

PROOF. We begin by choosing a coordinate chart (U, φ) of $p_0 \in \varphi(U) \subset M^n$ such that the covariant derivatives $\{\nabla_i F(p_0) := \partial_i(F \circ \varphi)(\varphi^{-1}(p_0))\}_{i=1, \dots, n}$ form an orthonormal basis of $T\Sigma_{F(p_0)}$ satisfying

$$g_{ij}(p_0) = \delta_{ij}, \quad h_{ij}(p_0) = \delta_{ij} \lambda_i(p_0), \quad \lambda_1(p_0) = \lambda_{\min}(p_0),$$

which guarantees $b^{11}(p_0) = \lambda_{\min}^{-1}(p_0)$ and $g^{11}(p_0) = 1$. Next, we define the function $\bar{W} : \varphi(U) \rightarrow \mathbb{R}$ by

$$\bar{W}(p) := K^\alpha \left(\frac{b^{1i} g_{ij} b^{j1}}{g^{11}} \right)^{\frac{1}{2}}(p) - \frac{n\alpha - 1}{2n\alpha} |F|^2(p).$$

Then, by Proposition 2.5 we have

$$\bar{W}(p) \leq W(p) \leq W(p_0) = \bar{W}(p_0),$$

which means that \bar{W} attains its maximum at p_0 .

We will now calculate $\mathcal{L}\bar{W} := \alpha K^\alpha b^{ij} \nabla_i \nabla_j \bar{W}$ at the point p_0 . First we derive the following equation from (2.18)

$$\begin{aligned} \mathcal{L}(b_p^1 b^{p1}) &= 2\alpha K^\alpha b^{ij} \nabla_i b^{p1} \nabla_j b_p^1 + 2K^{-\alpha} b_p^1 b^{pr} b^{1s} \nabla_r K^\alpha \nabla_s K^\alpha + 2\alpha K^\alpha b_p^1 b^{pr} b^{1s} b^{ij} b^{km} \nabla_r h_{ik} \nabla_s h_{jm} \\ &\quad + \langle F, \nabla(b_p^1 b^{p1}) \rangle - 2b_p^1 b^{p1} - 2(n\alpha - 1)K^\alpha b^{11} + 2\alpha K^\alpha H b_p^1 b^{p1}. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} \mathcal{L} \left(\frac{b_p^1 b^{p1}}{g^{11}} \right)^{\frac{1}{2}} &= - \frac{\alpha K^\alpha b^{ij} \nabla_i (b_p^1 b^{1p}) \nabla_j (b_q^1 b^{1q})}{4(b_r^1 b^{r1})^{\frac{3}{2}} (g^{11})^{\frac{1}{2}}} + \frac{\alpha K^\alpha b^{ij} \nabla_i b^{p1} \nabla_j b_p^1}{(b_q^1 b^{q1} g^{11})^{\frac{1}{2}}} \\ &+ \frac{b_p^1 b^{pr} b^{1s} \nabla_r K^\alpha \nabla_s K^\alpha}{K^\alpha (b_q^1 b^{q1} g^{11})^{\frac{1}{2}}} + \frac{\alpha K^\alpha b_p^1 b^{pr} b^{1s} b^{ij} b^{km} \nabla_r h_{ik} \nabla_s h_{jm}}{(b_q^1 b^{q1} g^{11})^{\frac{1}{2}}} \\ &+ \langle F, \nabla (b_p^1 b^{p1} / g^{11})^{\frac{1}{2}} \rangle - \left(\frac{b_p^1 b^{p1}}{g^{11}} \right)^{\frac{1}{2}} - \frac{(n\alpha - 1) K^\alpha b^{11}}{(b_p^1 b^{p1} g^{11})^{\frac{1}{2}}} + \alpha K^\alpha H \left(\frac{b_p^1 b^{p1}}{g^{11}} \right)^{\frac{1}{2}}. \end{aligned}$$

Combining this with (2.17) yields

$$\begin{aligned} (5.1) \quad \mathcal{L} \bar{W} &= - \frac{n\alpha - 1}{2n\alpha} \mathcal{L} |F|^2 + 2 \left\langle \nabla K^\alpha, \nabla \left(\frac{b_p^1 b^{p1}}{g^{11}} \right)^{\frac{1}{2}} \right\rangle_{\mathcal{L}} - \frac{\alpha K^{2\alpha} b^{ij} b_p^1 b_q^1 \nabla_i b^{1p} \nabla_j b^{1q}}{(b_r^1 b^{r1})^{\frac{3}{2}} (g^{11})^{\frac{1}{2}}} \\ &+ \frac{\alpha K^{2\alpha} b^{ij} \nabla_i b^{p1} \nabla_j b_p^1}{(b_q^1 b^{q1} g^{11})^{\frac{1}{2}}} + \frac{b_p^1 b^{pr} b^{1s} \nabla_r K^\alpha \nabla_s K^\alpha}{(b_q^1 b^{q1} g^{11})^{\frac{1}{2}}} + \frac{\alpha K^{2\alpha} b_p^1 b^{pr} b^{1s} b^{ij} b^{km} \nabla_r h_{ik} \nabla_s h_{jm}}{(b_q^1 b^{q1} g^{11})^{\frac{1}{2}}} \\ &+ \left\langle F, \nabla \left(K^\alpha (b_p^1 b^{p1} / g^{11})^{\frac{1}{2}} \right) \right\rangle + (n\alpha - 1) K^\alpha \left(\frac{b_p^1 b^{p1}}{g^{11}} \right)^{\frac{1}{2}} - \frac{(n\alpha - 1) K^{2\alpha} b^{11}}{(b_p^1 b^{p1} g^{11})^{\frac{1}{2}}}. \end{aligned}$$

Observe that

$$2 \left\langle \nabla K^\alpha, \nabla \left(\frac{b_p^1 b^{p1}}{g^{11}} \right)^{\frac{1}{2}} \right\rangle_{\mathcal{L}} = 2\alpha K^\alpha b^{ij} (g^{11})^{-\frac{1}{2}} (b_q^1 b^{q1})^{-\frac{1}{2}} b_p^1 \nabla_i K^\alpha \nabla_j b^{p1},$$

and

$$\nabla \left(K^\alpha (b_p^1 b^{p1} / g^{11})^{\frac{1}{2}} \right) = \nabla \bar{W} + \frac{n\alpha - 1}{2n\alpha} \nabla |F|^2.$$

Hence, applying the equations above, (2.15) and $\nabla \bar{w}(p_0) = 0$ to (5.1) yields that the following holds at the maximum point p_0

(5.2)

$$\begin{aligned} 0 &\geq 2\alpha K^\alpha \sum_{i=1}^n b^{ii} \nabla_i K^\alpha \nabla_i b^{11} - \alpha K^{2\alpha} \sum_{i=1}^n b^{ii} h_{11} |\nabla_i b^{11}|^2 + \alpha K^{2\alpha} \sum_{j,p} b^{jj} h_{11} |\nabla_j b^{p1}|^2 + |b^{11} \nabla_1 K^\alpha|^2 \\ &+ \alpha K^{2\alpha} (b^{11})^2 \sum_{i,j} b^{ii} b^{jj} |\nabla_1 h_{ij}|^2 + \frac{n\alpha - 1}{2n\alpha} \langle F, \nabla |F|^2 \rangle + (n\alpha - 1) K^\alpha (b^{11} - \frac{1}{n} \sum_{i=1}^n b^{ii}). \end{aligned}$$

By (2.14), the second and third terms on the right hand side of the inequality above (5.2) satisfy

$$\begin{aligned} &- \sum_{i=1}^n b^{ii} h_{11} |\nabla_i b^{11}|^2 + \sum_{j,p} b^{jj} h_{11} |\nabla_j b^{p1}|^2 = - \sum_{i=1}^n b^{ii} (b^{11})^3 |\nabla_i h_{11}|^2 + \sum_{j,p} b^{jj} b^{11} (b^{pp})^2 |\nabla_j h_{p1}|^2 \\ &= \sum_{j=1}^n \sum_{p \neq 1} b^{jj} b^{11} (b^{pp})^2 |\nabla_j h_{p1}|^2 \geq \sum_{p \neq 1} (b^{11} b^{pp})^2 |\nabla_p h_{11}|^2 = \sum_{p \neq 1} (b^{pp} h_{11})^2 |\nabla_p b^{11}|^2. \end{aligned}$$

Also, by (2.14) the fifth term on the right hand side of (5.2) satisfies

$$(b^{11})^2 \sum_{i,j} b^{ii} b^{jj} |\nabla_1 h_{ij}|^2 \geq (b^{11})^4 |\nabla_1 h_{11}|^2 + 2 \sum_{i \neq 1} (b^{11})^3 b^{ii} |\nabla_i h_{11}|^2 = |\nabla_1 b^{11}|^2 + 2 \sum_{i \neq 1} b^{ii} h_{11} |\nabla_i b^{11}|^2.$$

Furthermore, we have

$$\alpha K^{2\alpha} |\nabla_1 b^{11}|^2 + 2\alpha K^\alpha b^{11} \nabla_1 K^\alpha \nabla_1 b^{11} \geq -\alpha |b^{11} \nabla_1 K^\alpha|^2.$$

Hence, by applying the inequalities above, we can reduce (5.2) to

$$(5.3) \quad 0 \geq 2\alpha \sum_{i \neq 1} b^{ii} \nabla_i K^\alpha (K^\alpha \nabla_i b^{11}) + \alpha \sum_{p \neq 1} (b^{pp} h_{11})^2 |K^\alpha \nabla_p b^{11}|^2 + 2\alpha \sum_{i \neq 1} b^{ii} h_{11} |K^\alpha \nabla_i b^{11}|^2 \\ + (1 - \alpha) |b^{11} \nabla_1 K^\alpha|^2 + \frac{n\alpha - 1}{2n\alpha} \langle F, \nabla |F|^2 \rangle + (n\alpha - 1) K^\alpha (b^{11} - \frac{1}{n} \sum_{i=1}^n b^{ii}).$$

We now employ (2.16) to obtain the following at the point p_0

$$(5.4) \quad b^{ii} \nabla_i K^\alpha = b^{ii} h_{ii} \langle F, F^i \rangle = \langle F, F^i \rangle.$$

In addition, at the point p_0 , $\nabla_i \bar{W}(p_0) = 0$ yields

$$K^\alpha \nabla_i b^{11} = -b^{11} \nabla_i K^\alpha + \frac{n\alpha - 1}{2n\alpha} \nabla_i |F|^2 = -b^{11} h_{ii} \langle F, F^i \rangle + \frac{n\alpha - 1}{n\alpha} \langle F, F_i \rangle = (\beta - \theta_i) \langle F, F_i \rangle,$$

where $\theta_i = b^{11} h_{ii}(p_0)$ and $\beta = \frac{n\alpha - 1}{n\alpha}$. We also have

$$(5.5) \quad \langle F, \nabla |F|^2 \rangle := \langle F, (\nabla_i |F|^2) F^i \rangle = \langle F, F^i \rangle \langle \nabla_i |F|^2 \rangle = 2 \langle F, F_i \rangle \langle F, F^i \rangle.$$

Hence, we can rewrite (5.3) as

$$(5.6) \quad 0 \geq \sum_{i \neq 1} \langle F, F_i \rangle^2 J_i + \langle F, F_1 \rangle^2 I_1 + (n\alpha - 1) K^\alpha (b^{11} - \frac{1}{n} \sum_{i=1}^n b^{ii}),$$

where

$$I_1 = \frac{n\alpha - 1}{n\alpha} + 1 - \alpha, \quad J_i = 2\alpha (\beta - \theta_i) + \alpha (\theta_i^{-2} + 2\theta_i^{-1}) (\beta - \theta_i)^2 + \beta.$$

We observe that $I_1 > 0$ holds, and also J_i satisfies

$$J_i = 2\alpha\beta - 2\alpha\theta_i + \alpha\beta^2\theta_i^{-2} + 2\alpha\beta^2\theta_i^{-1} - 2\alpha\beta\theta_i^{-1} - 4\alpha\beta + \alpha + 2\alpha\theta_i + \beta \\ = \alpha(1 - \beta) + \beta(1 - \alpha) + 2\alpha\beta(\beta - 1)\theta_i^{-1} + \alpha\beta^2\theta_i^{-2} = \frac{1}{n} + \beta(1 - \alpha) - \frac{2\beta}{n}\theta_i^{-1} + \alpha\beta^2\theta_i^{-2} \\ = \beta(1 - \alpha) + \frac{1}{n} + \alpha \left(\beta\theta_i^{-1} - \frac{1}{n\alpha} \right)^2 - \frac{1}{n^2\alpha} \geq \beta(1 - \alpha) + \frac{1}{n} \left(\frac{n\alpha - 1}{n\alpha} \right) = \beta(1 - \alpha + \frac{1}{n}) > 0.$$

Since we have $b^{11}(p_0) = \lambda_{\min}^{-1}(p_0) \geq \lambda_i^{-1}(p_0) \geq b^{ii}(p_0)$ and $\langle F, F_i \rangle^2(p_0) \geq 0$ for all $i \in \{1, \dots, n\}$, the inequality (5.6) and $I_1, J_i > 0$ give the desired result. \square

2 Strong maximum principle

In this section, we will show how Theorem 5.5 can be modified to give us the proof of our main result, Theorem 5.1. To this end, we will introduce the new geometric, chart-independent quantity

$$Z(p) = \left(K^\alpha b^{ij} g_{ij} - \frac{n\alpha - 1}{2\alpha} |F|^2 \right)(p)$$

and apply the strong maximum principle. If we use $W(p)$, h_{ij} can be diagonalized only at one given point. However, if we employ $Z(p)$, we can diagonalize h_{ij} at each point. We begin with the following observation which simply follows from Theorem 5.5.

PROPOSITION 5.6. *Let Σ be a strictly convex smooth closed solution of $(**^\alpha)$ for an exponent $\alpha \in (\frac{1}{n}, 1 + \frac{1}{n})$. Assume that $F : M^n \rightarrow \mathbb{R}^{n+1}$ is a smooth immersion such that $F(M^n) = \Sigma$, and the continuous function $Z(p)$ attains its maximum at a point p_0 . Then, $F(p_0)$ is an umbilical point and $\nabla|F|^2(p_0) = 0$ holds.*

PROOF. We observe $b^{ij} g_{ij}(p) = \sum_{i=1}^n \lambda_i^{-1}(p)$, where $\lambda_1(p), \dots, \lambda_n(p)$ are the principal curvatures of Σ at $F(p)$. Therefore, we have $Z(p) \leq n W(p)$. However, if $W(p_0) = \max_{p \in M^n} W(p)$, then $Z(p_0) = n w(p_0)$ holds, because $F(p_0)$ is an umbilical point by Theorem 5.5. Hence, we have

$$Z(p) \leq n W(p) \leq \max_{p \in M^n} n W(p) = \max_{p \in M^n} Z(p).$$

Thus, if Z attains its maximum at a point p_0 , then W also attains its maximum at p_0 , and thus we can obtain the desired result by Theorem 5.5. \square

We will now employ the strong maximum principle to prove Theorem 5.1.

PROOF OF THEOREM 5.1. We define a set $M_Z \subset M^n$ by

$$M_Z = \{p \in M^n : Z(p) = \max_{M^n} Z\}.$$

Since $Z(p)$ is a continuous function defined on a closed manifold M^n , Z attains its maximum, and thus M_Z is not an empty set. We now define the continuous function $\Lambda : M^n \rightarrow \mathbb{R}$ and the open set $V \subset M^n$ by

$$\Lambda(p) = \sum_{i,j} \left(\frac{\lambda_i}{\lambda_j} - \frac{\lambda_j}{\lambda_i} \right)^2(p), \quad V = \left\{ p \in M^n : \Lambda(p) < \left(\frac{10}{9} - \frac{9}{10} \right)^2 \right\}.$$

We now begin by combining (2.17) and (2.18) to obtain

$$\begin{aligned} \mathcal{L}(K^\alpha b^{pq}) &= 2 \langle \nabla K^\alpha, \nabla b^{pq} \rangle_{\mathcal{L}} + b^{pr} b^{qs} \nabla_r K^\alpha \nabla_s K^\alpha + \alpha K^{2\alpha} b^{pr} b^{qs} b^{ij} b^{km} \nabla_r h_{ik} \nabla_s h_{jm} \\ &\quad + \langle F, \nabla(K^\alpha b^{pq}) \rangle + (n\alpha - 1) K^\alpha (b^{pq} - g^{pq} K^\alpha). \end{aligned}$$

Therefore, we can derive the following from (2.15) and $\nabla g_{pq} = 0$

$$\mathcal{L}Z = 2g_{pq}\langle \nabla K^\alpha, \nabla b^{pq} \rangle_{\mathcal{L}} + b^{pr}b_p^s \nabla_r K^\alpha \nabla_s K^\alpha + \alpha K^{2\alpha} b^{pr} b_p^s b^{ij} b^{km} \nabla_r h_{ik} \nabla_s h_{jm} + \langle F, \nabla(K^\alpha b^{pq} g_{pq}) \rangle.$$

By using (5.5), we can obtain

$$\langle F, \nabla(K^\alpha b^{pq} g_{pq}) \rangle = \langle F, \nabla Z \rangle + \frac{n\alpha - 1}{2\alpha} \langle F, \nabla |F|^2 \rangle = \langle F, \nabla Z \rangle + (n - \alpha^{-1}) \langle F, F_i \rangle \langle F, F^i \rangle.$$

Hence, we have

$$(5.7) \quad \begin{aligned} \mathcal{L}Z - \langle F, \nabla Z \rangle &= 2\alpha (b^{ij} \nabla_i K^\alpha) (K^\alpha g_{pq} \nabla_j b^{pq}) + b^{pr} b_p^s \nabla_r K^\alpha \nabla_s K^\alpha \\ &\quad + \alpha K^{2\alpha} b^{pr} b_p^s b^{ij} b^{km} \nabla_r h_{ik} \nabla_s h_{jm} + (n - \alpha^{-1}) \langle F, F_i \rangle \langle F, F^i \rangle. \end{aligned}$$

Given a fixed point $p_0 \in V$, we choose an orthonormal frame at $F(p_0)$ satisfying

$$g_{ij}(p_0) = \delta_{ij}, \quad h_{ij}(p_0) = \lambda_i(p_0) \delta_{ij}.$$

Then, at the point p_0 , we can rewrite (5.7) as

$$(5.8) \quad \begin{aligned} \mathcal{L}Z - \langle F, \nabla Z \rangle &= 2\alpha \sum_{i,j} (b^{ii} \nabla_i K^\alpha) (K^\alpha \nabla_i b^{jj}) + \sum_i |b^{ii} \nabla_i K^\alpha|^2 \\ &\quad + \alpha K^{2\alpha} \sum_{i,j,k} (b^{ii})^2 b^{jj} b^{kk} |\nabla_i h_{jk}|^2 + (n - \alpha^{-1}) \sum_i \langle F, F_i \rangle^2. \end{aligned}$$

Since $p_0 \in V$ and the definition of V guarantees that $b^{ii} h_{jj}(p_0) \geq \frac{9}{10}$, by using (2.14) we can derive

$$\begin{aligned} \alpha K^{2\alpha} \sum_{i,j,k} (b^{ii})^2 b^{jj} b^{kk} |\nabla_i h_{jk}|^2 &\geq \alpha \sum_i |K^\alpha \nabla_i b^{ii}|^2 + 2\alpha \sum_{i \neq j} b^{jj} h_{ii} |K^\alpha \nabla_j b^{ii}|^2 + \alpha \sum_{i \neq j} (b^{ii} h_{jj})^2 |K^\alpha \nabla_i b^{jj}|^2 \\ &\geq \alpha \sum_i |K^\alpha \nabla_i b^{ii}|^2 + \frac{5}{2} \alpha \sum_{i \neq j} |K^\alpha \nabla_i b^{jj}|^2. \end{aligned}$$

We also have

$$\alpha \sum_i |K^\alpha \nabla_i b^{ii}|^2 + 2\alpha \sum_i (b^{ii} \nabla_i K^\alpha) (K^\alpha \nabla_i b^{ii}) \geq -\alpha \sum_i |b^{ii} \nabla_i K^\alpha|^2$$

and

$$\frac{5}{2} \alpha \sum_{i \neq j} |K^\alpha \nabla_i b^{jj}|^2 + 2\alpha \sum_{i \neq j} (b^{ii} \nabla_i K^\alpha) (K^\alpha \nabla_i b^{jj}) \geq -\frac{2}{5} \alpha \sum_{i \neq j} |b^{ii} \nabla_i K^\alpha|^2 = -\frac{2}{5} \alpha (n-1) \sum_i |b^{ii} \nabla_i K^\alpha|^2.$$

Applying the inequalities above and (5.4) to (5.8) yields

$$\begin{aligned} \mathcal{L}Z - \langle F, \nabla Z \rangle &\geq \left((1 - \alpha) - \frac{2}{5} \alpha (n-1) + (n - \alpha^{-1}) \right) \sum_i \langle F, F_i \rangle^2 \\ &= \frac{1}{5\alpha} \left(-(2n+3)\alpha^2 + 5(n+1)\alpha - 5 \right) \sum_i \langle F, F_i \rangle^2. \end{aligned}$$

Let us consider the function $y(\alpha) = -(2n + 3)\alpha^2 + 5(n + 1)\alpha - 5$. Then, we have

$$y(1 + 1/n) = 3n - 2 - (3/n) - (3/n^2) \geq 0, \quad y(1/n) = (3/n) - (3/n^2) \geq 0,$$

which implies $y(\alpha) \geq 0$ for $\alpha \in [\frac{1}{n}, 1 + \frac{1}{n}]$. Therefore, on V the following holds

$$\mathcal{L}Z - \langle F, \nabla Z \rangle \geq 0.$$

Notice that $\mathcal{L}Z - \langle F, \nabla Z \rangle$ is a chart-independent function. Hence, the Hopf maximum principle and $M_Z \subset V$ show that $M_Z = V$. However, M_Z is a closed set and V is an open set by the continuity of Z and Λ , respectively. So, we conclude that $M_Z = M^n$, and thus Proposition 5.6 gives the desired result. \square

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