Developments in the Theory and Applications of the Variability Response Function Concept

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ABSTRACT

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Uncertainty quantification in Civil Engineering applications is crucial to the decision making process in the analysis, design, and retrofitting of infrastructure. The consensus amongst researchers is that deterministic approaches to problem solving can lead to very misleading results, and the assessment of infrastructure performance needs to be addressed within a probabilistic framework. As a result, there is great demand to identify and acquire probabilistic information about uncertain system parameters which affect the performance of a structure. Unfortunately, it is difficult to obtain a full probabilistic description of uncertain system parameters, specifically their spatial correlation structures.

In response to this limitation, researchers have sought a means to circumventing the need for a full probabilistic description of system uncertainties in determining structural response statistics. One approach is the Variability Response Function (VRF) concept, introduced by Shinozuka, which decomposes the variability of a response quantity into a deterministic function that is solely dependent on the deterministic components of the structure and the Spectral Density Function (SDF) of the uncertain system parameters modeled as a homogeneous random field. The deterministic function is called the VRF and is analogous to the Green’s function of a differential equation.

This dissertation explores the limits of the applicability of the VRF concept in Structural Mechanics problems. The VRF concept is applied to nonlinear statically determinate and indeterminate beams as well as plane stress structures where the flexibility is considered to be a random field. In the latter part of the dissertation the VRF concept is applied to the problem of stochastic characterization of homogenized effective properties through an equivalent energy based homogenization technique. The final chapter of this dissertation
presents a novel methodology to rapidly generate sample microstructures for random two phase materials.
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Chapter 1

Introduction

1.1 Introduction

This dissertation contributes to the theoretical development and extends the applicability of the Variability Response Function (VRF) concept. The VRF is a means to assess the variability of the response of a structure with very limited information of the probabilistic characteristics of uncertain system parameters. There are many uncertainties that exist in Civil Engineering systems which profoundly affect the systems’ performance. Uncertainties often times exist in the following system parameters: distribution and magnitude of applied loading, geometry of a structure, boundary conditions, material properties, morphology of heterogeneous structures, data from sensors, and assumptions in computational models. The failure of a structure is strongly dependent on the realization of specific threshold quantities of the uncertain system parameters. Thus, an important area of study is the quantification of the uncertainties in Civil Engineering systems in order to establish probabilistic information of critical response quantities that identify failure modes.

Probabilistic Engineering Mechanics is a general area of research dedicated to analyzing the effects of uncertainties in Civil Engineering within the framework of probability theory. Two sub-topics are Stochastic Finite Element Analysis, which deals with incorporating probabilistic
analysis into Computational Mechanics, and Structural Reliability, which specifically deals with obtaining the probability of failure of structures given a specified exceedance threshold. Seminal texts on these subjects are references [2, 3, 4, 5, 6, 7].

The major difficulties in quantifying uncertainty are a lack of available data and the inability to accurately simulate complex random fields. The lack of available information is due to expensive costs of detailed instrumentation, time limitations of collecting data, and infrequency of occurrence of the event being recorded. Consider, for example, recording earthquake ground motion using a seismometer. In order to acquire spatial correlation data of ground motion near a particular fault, many seismometers need to be located in the nearby region. Since ground motion is strongly dependent on soil properties, the data recorded is site and event specific and may result in misguided conclusions if used for another location. Furthermore, the infrequency of occurrence of large earthquakes at a site makes it difficult to ascertain distributional and spectral characteristics for ground motion in that region.

Brute force Monte Carlo simulation is the most universal approach to acquiring probabilistic information of a response quantity. This requires the ability to simulate random fields describing the uncertain system parameters. The accuracy of the results depends on the accuracy of the simulation algorithm of the random field considered. Although there are a number of random field models which can be accurately simulated, recorded data suggests that real random fields are oftentimes more complicated than the available mathematical models. The simulation of random fields, particularly non-Gaussian and inhomogeneous fields, is an active and ongoing research area.

In light of the aforementioned limitations, the VRF concept, introduced by Shinozuka [8], provides information of the variability of a structure’s response with no information of the spatial/temporal correlation structure of the uncertain system parameters. The VRF is a deterministic function dependent on the structure, its boundary conditions, and loading. It is independent of the distributional and spectral characteristics of the uncertain system parameters. It identifies the sensitivity of the correlation structure of the uncertain parameters
on the variability of the response. With only knowledge of the variance of the uncertain system parameters, the VRF can be used to determine the supremum of the response variability.

There are two major drawbacks to the VRF concept. First, the VRF concept is only applicable for statistically homogeneous random fields. No theoretical developments have been made with regards to establishing VRFs for structures with statistically inhomogeneous uncertainties. The second drawback is that the existence of VRFs has only been derived for statically determinate linear structures. A recent methodology, termed the Generalized Variability Response Function (GVRF) Methodology \[9, 10, 11\], is a Monte Carlo based approach that has been used to establish approximate VRFs for statically indeterminate, linear structures where the stochasticity is modeled as a one-dimensional random field. The aim of this dissertation is to expand the applicability of the VRF concept to more general conditions in Structural Engineering. This thesis specifically addresses the latter drawback mentioned above. In this thesis, applications of the VRF concept are expanded to include nonlinear statically determinate and indeterminate beams as well as two-dimensional structures.

According to \[2\] stochasticity in the operator (structure) is more complicated than stochasticity in the input (loading). In the studies of this dissertation, the compliance or flexibility is considered to be a homogeneous random field.

\[1.2\] Dissertation Outline

There are five main accomplishments of this dissertation:

i The existence of the VRF for statically determinate structures of materials having a specific class of nonlinear constitutive laws is analytical derived.

ii The GVRF methodology is used to establish approximate VRFs for a statically indeterminate beam of a material having one of the nonlinear constitutive laws mentioned above.
iii The GVRF methodology is formulated when the stochasticity of the structure is modeled as a two-dimensional field. In a numerical example, approximate VRFs are established for a plane stress linear structure.

iv The VRF concept is applied to the problem of stochastic upscaling of material properties. In [12], the existence of the VRF for homogenized effective properties is proven for linear statically determinate structures. The GVRF methodology is employed to established approximate VRFs for homogenized effective properties for statically indeterminate beams and plane stress problems.

v A novel methodology for the simulation of two phase random heterogeneous morphologies is developed. Two numerical examples are presented.

In chapter 2, the VRF concept is introduced. An up to date literature review on the theory and applications of the VRF is given. The original derivation of the VRF for statically determinate beams with flexibility modeled as a random field is presented [13]. The chapter concludes with a numerical example. In chapter 3, the existence of the VRF is proven for a class of nonlinear statically determinate structures. In two numerical examples, closed form expressions for the VRF for a cantilever having a square root constitutive law is analytically derived. It should be noted that this is the first analytical development of the VRF concept since it was first introduced. All research prior to this dissertation focuses on numerical techniques, such as first order perturbation or Monte Carlo simulation, to establish approximate VRFs.

In chapter 4, the GVRF methodology developed in [9, 10, 11] is formulated in detail for statically indeterminate beams with a square root constitutive law. The translation field theory and its analogous associated field theory is presented as well as algorithms to simulate Gaussian, U-beta, and translated fields. Chapter 5 contains two numerical examples. The GVRF methodology is used to reproduce the exact VRF for the statically determinate beam in an example in chapter 3 and then is used to establish approximate VRFs for a fixed-pinned
nonlinear beam.

Chapter 6 formulates the GVRF methodology for linear structures where the stochasticity is modeled by a two-dimensional random field. It is a straightforward extension from the one-dimensional case. In a numerical example, approximate VRFs are established for the displacement response of a plane stress problem.

In Chapter 7, the VRF concept is applied to determining the variance of homogenized material properties. The homogenized material properties are considered to be the response quantity of interest analogous to that of the displacement response. The derivation of the VRF for homogenized material properties for linear, statically determinate linear structures, first shown in [12], is presented. Two numerical examples are presented using the GVRF methodology: approximate VRFs are established for the homogenized flexibility of a statically indeterminate beam and for the homogenized compliance of a plane stress structure.

In Chapter 8, a novel approach to generating virtual samples of microstructures of two phase random media is developed. Typically the morphology of two-phase heterogeneous materials is modeled by the indicator function and a specified two-point correlation function. In this methodology, an underlying Filtered Poisson Field is translated into a Binary random field by performing a level-cut. This theory is developed by Grigoriu in [14,15]. The novelty of the approach presented in this dissertation is that the parameters of this model are optimized to match pre-specified two-point correlation functions using Genetic Algorithms. Two numerical examples are presented.

The dissertation concludes with a discussion on the limitations and accomplishments of the VRF concept in a theoretical and application sense. The work presented in this dissertation is summarized and the potential direction of future research is contemplated.
Chapter 2

Variability Response Function

Concept

2.1 Introduction

The Variability Response Function (VRF) is a deterministic function that identifies the sensitivity of a response quantity to the correlation structure (or spectral contents), of a property of the structure that is modeled as a homogeneous random field. The VRF is defined as

\[
\text{Var}[u(x)] = \int_{-\infty}^{\infty} S_f(\kappa) \text{VRF}(x,\kappa) d\kappa, \tag{2.1}
\]

where \( S_f(\kappa) \) is the spectral density function (SDF) of the homogeneous random field \( f(x) \) modeling the system stochasticity. The VRF concept is analogous to the Green's function for differential equations. For a differential equation, with solution \( u(x) \) and forcing function \( f(x) \), defined as \( \mathcal{L}[u(x)] = f(x) \), the solution is given by

\[
u(x) = \int_{0}^{x} G(x,s)f(s)ds. \tag{2.2}\]
The Greens function is a function of the domain variable $x$ and a dummy variable of integration $s$ whereas the VRF is a function of the domain variable and the frequency domain. The SDF is analogous to the forcing function and the variance of the response is analogous to the solution. More conceptually, analogies to the VRF are often drawn from the Frequency Response Function encountered in Structural Dynamics. The Frequency Response Function is a transfer function which expresses the structural response (displacement, velocity, or acceleration) due to an applied dynamic force \[ f(x) \]. The most important quality of the VRF is its independence of the SDF and PDF of the random field, $f(x)$. It only depends on the structure and its boundary conditions (ie displacement and loading), and the response quantity of interest. Thus, once the VRF is known for a structure, performing only a simple integration gives the variance of the desired response quantity. The VRF also provides a “spectral distribution-free” upper and lower bound on the variance of the response. Given a variance of the input stochasticity, an SDF defined by the delta function at the peak of the VRF provides the supremum of the response variance. Figure 2.1 provides graphical insight into the significance of the VRF. Consider two different potential SDFs describing the spatial variation of a structure’s stochasticity: it can be determined just by visual inspection that SDF$_1$ produces a larger variance than SDF$_2$.

![](image)

**Figure 2.1:** Visualization of the VRF: Example of VRF with Hypothetical SDFs

An excellent review of the literature can be found in the PhD. thesis of Manuel Miranda
The concept of the VRF has been introduced by Shinozuka [8] and later developed by Deodatis and Shinozuka [13]. The VRF is been established for statically determinate beams and trusses with the flexibility and compliance modeled as a random fields, respectively. The VRF only has a closed-form and exact analytical solution for statically determinate structures. However, Deodatis [17] developed the VRF within a finite element framework by performing a first-order Taylor expansion of the stiffness matrix with respect to a stochastic random field. The stiffness matrix is a summation of a deterministic stiffness matrix, consisting of mean value of the random field, and the stochastic stiffness matrix, which can be efficiently calculated using the Weighted Integral Method [18, 19, 20, 21]. Following this methodology, first-order approximations of the VRF has been established for plane stress/strain models, trusses, and frames for stochastic material properties, geometrical imperfections, and loading [22, 23, 24, 25, 26, 27]. It is important to note that these results using first-order expansion are approximate and only sufficient for small variances of the stochastic system properties.

2.2 Random Field Model for Beam Flexibility

Although the stiffness formulation is commonly used in stochastic mechanics, in some cases it is advantageous to consider the flexibility or compliance in the equilibrium equations. The compliance of a material can be measured as easily as its elastic modulus. It has been found that closed form expressions are only possible by modeling the beam flexibility in determining response variability of statically determinate trusses and frames [8, 20, 13, 17, 28]. The homogeneous random field model for the flexibility takes the following form

\[
\frac{1}{EI(x)} = \frac{(1 + f(x))}{EI_0}, \tag{2.3}
\]

where \(\frac{1}{EI_0}\) is the average flexibility and \(f(x)\) is a zero-mean, homogeneous random field describing the fluctuations about its mean value. In order for the problem to be well-posed,
the bounds for the flexibility are

\[ 0 < \frac{1}{EI_{\max}} \leq \frac{1}{EI(x)} \leq \frac{1}{EI_{\min}} < \infty. \]  

(2.4)

The strictly positive and bounded requirements ensure that the problem is well posed, following from the Babuska-Lax-Milgram theorem [29, 30, 31]. It should be noted that in [29, 30, 31], the theory proving the well-posed conditions of the problem corresponds to a stiffness formulation. Thus, the upper bound in equation (2.4) corresponds to the stiffness being strictly positive, and the lower bound corresponds to the stiffness being bounded. The bounds also ensure that all realizations of the flexibility are physically meaningful.

It is important to note that the Lognormal distribution violates these conditions. However, the Lognormal distribution can be used in numerical studies with caution. Since simulation techniques cannot simulate a realization of infinity, the sample functions will always be bounded. Attention must be paid to whether the correlation functions/statistical moments affecting the response statistics of interest are accurately represented by the simulation algorithm.

2.3 Exact VRF for Statically Determinate Beams

Consider the statically determinate beam in figure 2.2 with deterministic distributed load, \( q(x) \), and heterogeneous, random flexibility, \( \frac{1}{EI(x)} \) defined in equation (2.3). The governing differential equation for the displacement field, \( u(x) \), for an Euler-Bernoulli beam is

\[ \frac{d^2}{dx^2} \left( EI(x) \frac{d^2 u}{dx^2} \right) = q(x). \]  

(2.5)

Since the moment distribution, \( M(x) \), is only dependent on the boundary conditions and loading for a statistically determinate beam, it is therefore deterministic. The differential
Figure 2.2: Statically determinate, pinned-pinned, beam.

The equation reduces to

\[ \frac{d^2 u}{dx^2} = \frac{M(x)}{EI(x)}. \]  

(2.6)

This allows \( u(x) \) to be described by a Green’s function independent of \( M(x) \) and \( EI(x) \),

\[
u(x) = \int_0^x \frac{1}{EI(x')} G(x, x') M(x') dx'
\]

\[
= \int_0^x \frac{1}{EI_0} (1 + f(x')) G(x, x') M(x') dx'.
\]

(2.7)

Taking expectation of displacement gives

\[
\mathbb{E} [u(x)] = \int_0^x \frac{1}{EI_0} (1 + \mathbb{E} [f(x')]) G(x, x') M(x') dx'
\]

\[
= \int_0^x \frac{1}{EI_0} G(x, x') M(x') dx',
\]

(2.8)
and the mean squared

\[ \mathbb{E} \left[ u(x)^2 \right] = \frac{1}{EI_0^2} \int_0^x \int_0^x \mathbb{E} \left[ (1 + f(x'_1))(1 + f(x'_2)) \right] G(x, x'_1)G(x, x'_2) \]

\[ M(x'_1)M(x'_2)dx'_1dx'_2 \]

\[ = \frac{1}{EI_0^2} \int_0^x \int_0^x [1 + \mathbb{E} [f(x'_1)f(x'_2)]] G(x, x'_1)G(x, x'_2) \]

\[ M(x'_1)M(x'_2)dx'_1dx'_2. \] (2.9)

The variance is then given by

\[ \text{Var} \left[ u(x) \right] = \frac{1}{EI_0^2} \int_0^x \int_0^x \mathbb{E} \left[ f(x'_1)f(x'_2) \right] G(x, x'_1)G(x, x'_2) \]

\[ M(x'_1)M(x'_2)dx'_1dx'_2. \] (2.10)

Utilizing the fact that \( f(x) \) is statistically homogeneous, the Wiener-Khinchin transformation is applied

\[ \text{Var} \left[ u(x) \right] = \frac{1}{EI_0^2} \int_{-\infty}^{\infty} \int_0^x \int_0^x \int S_f(\kappa) \exp \left( i\kappa (x'_2 - x'_1) \right) G(x, x'_1)G(x, x'_2) \]

\[ M(x'_1)M(x'_2)dx'_1dx'_2d\kappa. \] (2.11)

Defining \( V(x, \kappa) \) as

\[ V(x, \kappa) = \frac{1}{EI_0} \int_0^x G(x, x')M(x') \exp \left( i\kappa x' \right) dx', \] (2.12)
then

\[
\text{Var } [u(x)] = \int_{-\infty}^{\infty} S_f(\kappa)V(x, \kappa)V^*(x, \kappa) d\kappa
\]

\[
= \int_{-\infty}^{\infty} S_f(\kappa)|V(x, \kappa)|^2 d\kappa
\]

\[
= \int_{-\infty}^{\infty} S_f(\kappa)\text{VRF}(x, \kappa) d\kappa,
\]

with the VRF defined as

\[
\text{VRF}(x, \kappa) = |V(x, \kappa)|^2
\]

\[
= \frac{1}{EI_0^2} \int_{0}^{x} \int_{0}^{x} G(x, x')G(x, x'')M(x_1')M(x_2') \exp (i\kappa(x_2' - x_1')) dx_1' dx_2'.
\]

It is important to reiterate that the closed form solution of the VRF is only possible because the moment distribution is independent of the stiffness of the structure, enabling a simplification of the beam equation from equations (2.5) to (2.6) such that the ratio of the moment to stiffness is the forcing function of the differential equation. This condition cannot be obtained for statically indeterminate structures.

2.3.1 Numerical Example

Consider the cantilever shown below with the following properties: \(EI_0 = 1.5E6\), \(q(x) = q(L - x)/L\), \(M_0 = 10\), \(L = 20\). Since the structure is statically determinate, equation (2.6) can be used to determine the displacement. The Green’s function can be derived from its definition, continuity constraints, and boundary conditions, as outlined in reference [32].
Figure 2.3: Statically determinate cantilever beam.

A general first order linear ODE with homogeneous boundary conditions takes the form

\[
\frac{d^n u}{dx^n} + a_1(x) \frac{d^{n-1} u}{dx^{n-1}} + \ldots + a_{n-1}(x) \frac{du}{dx} + a_n(x)u \equiv L u = f(x)
\]

\( B u = 0, \) \hspace{1cm} (2.15)

and its Green’s function satisfies the differential equation

\[
L_x G(x, y) = \delta(x - y)
\]

\( B G(x, y) = 0 \)

(2.16)

with the subscript \( x \) denoting the variable of differentiation. The following \( n \) equations are needed to satisfy continuity

\[
\lim_{\epsilon \to 0} \frac{d^{n-1} G(x, y)}{dx^{n-1}} \bigg|_{x = y - \epsilon}^{x = y + \epsilon} = 1
\]

\[
\lim_{\epsilon \to 0} \frac{d^k G(x, y)}{dx^k} \bigg|_{x = y - \epsilon}^{x = y + \epsilon} = 0, \quad (k = 0, 1, 2, \ldots, n - 2).
\]

(2.17)

A common approach to solve for the Green’s function is to split the ODE into two conditions, \( x < y \) and \( x > y \), resulting in two homogeneous ODEs from which a piece-wise solution can
be obtained

\[
G(x, y) : \begin{cases} 
\mathcal{L}G^+ = 0, & x > y \\
\mathcal{L}G^- = 0, & x < y.
\end{cases}
\] (2.18)

The continuity constraints of equation (2.17) are be applied to solve for the constants of integration

\[
\lim_{\epsilon \to 0} \frac{d^{n-1}G^+(y + \epsilon, y)}{dx^{n-1}} - \frac{d^{n-1}G^-(y - \epsilon, y)}{dx^{n-1}} = 1
\]

\[
\lim_{\epsilon \to 0} \frac{d^kG^+(y + \epsilon, y)}{dx^k} - \frac{d^kG^-(y - \epsilon, y)}{dx^k} = 0, (k = 0, 1, 2, ..., n - 2),
\] (2.19)

and the boundary conditions are applied to provide the additional \(n\) equations needed for a unique solution.

For the cantilever shown in 2.3, the governing equation for displacement is

\[
\frac{d^2u}{dx^2} = \frac{M(x)}{EI_0} (1 + f(x))
\]

\[
u(0) = \frac{du}{dx} \bigg|_{x=0} = 0,
\] (2.20)

with \(M(x) = M_0 - \frac{qL^2}{6} + \frac{qLx}{2} + \frac{qx^2}{2} + \frac{qx^3}{3L}\). Defining the Green’s function as

\[
\frac{d^2G(x, y)}{dx^2} = \delta(x - y)
\]

\[
G(0, y) = \frac{dG(x, y)}{dx} \bigg|_{x=0} = 0,
\] (2.21)

the aforementioned method can be applied to get

\[
G(x, y) : \begin{cases} 
\frac{d^2G^+(x,y)}{dx^2} = 0 \Rightarrow G^+(x, y) = A(y)x + B(y), & x > y \\
\frac{d^2G^-(x,y)}{dx^2} = 0 \Rightarrow G^-(x, y) = C(y)x + D(y), & x < y.
\end{cases}
\] (2.22)

Observe that \(C(y)\), and \(D(y)\) must equal zero to satisfy the boundary conditions. Applying
the continuity equations give

\[
\lim_{\epsilon \to 0} \left. \frac{dG^+(x, y)}{dx} \right|_{x=y+\epsilon} = 1 \Rightarrow A(y) = 1
\]

\[
\lim_{\epsilon \to 0} G(x, y) \bigg|_{x=y+\epsilon} = 0 \Rightarrow B(y) = -y.
\]

Thus the Green’s function is

\[
G(x, y) = \begin{cases} 
  x - y, & x > y \\
  0, & x < y.
\end{cases}
\]

From equations (2.12) and (2.14), the VRF can be solved for. The function \( V(x, k) \) is

\[
V(x, k) = \int_0^x (x - s)(M_0 - \frac{qL^2}{6} + \frac{qLx}{2} + \frac{qx^2}{2} + \frac{qx^3}{3L})\exp(iks)ds
\]

\[= \frac{1}{EI_0 L k^5} \left\{ -8 \sin(kx) q - 2 kx q + 3 kq L - 1/2 qx L^2 k^3 - 1/6 qL^3 k^3 
+ M_0 L k^3 + L^2 \sin(kx) qk^2 - 3 \cos(kx) kq L + 1/2 \cos(kx) k^3 qx^2 L 
+ 6 \cos(kx) kx q - 1/3 \cos(kx) k^3 qx^3 + 1/6 \cos(kx) qL^3 k^3 
- \cos(kx) M_0 L k^3 - 1/2 \cos(kx) qx L^2 k^3 + 2 \sin(kx) x^2 qk^2 
- 2 \sin(kx) qx L k^2 \right\} + \frac{i}{EI_0 L k^5} \left\{ k^4 x M_0 L + k^2 qx L + 8 \cos(kx) q 
- L^2 \cos(kx) qk^2 + k^2 qL^2 - 3 \sin(kx) kq L + 1/2 \sin(kx) k^3 qx^2 L 
+ 6 \sin(kx) kx q - 1/3 \sin(kx) k^3 qx^3 + 1/6 \sin(kx) qL^3 k^3 
- \sin(kx) M_0 L k^3 - 1/6 k^4 xq L^3 - 1/2 \sin(kx) qx L^2 k^3 
- 8 q - 2 \cos(kx) x^2 qk^2 + 2 \cos(kx) qx L k^2 \right\}.
\]
The VRF is

\[
VRF(x, k) = \frac{-1}{36EI_0^2L^2k^3} \left\{ 24k^7x^4M_0L\sin(kx)q - 24k^7xM_0L^4\sin(kx)q \\
+72k^7xM_0^2L^2\sin(kx) - 504\sin(kx)q^2xL^2k^3 + 36k^7x^2M_0L^3\sin(kx)q \\
+24k^5q^2x^4L\sin(kx) - 288k^2xq^2\cos(kx)L + 144k^4x^3q^2\cos(kx)L \\
+72k^4xq^2\cos(kx)L^3 - 864k^4xq\cos(kx)M_0L + 180k^4x^2q^2\cos(kx)L^2 \\
-288k^3x^2q^2\sin(kx)L + 432k^4qL^2\cos(kx)M_0 - 10q^2x^3L^3k^6\cos(kx) \\
-12q^2x^3L^2k^6\cos(kx) - 12q^2xL^5k^6\cos(kx) + 12q^2x^2L^4k^6\cos(kx) \\
+72qxL^3k^6\cos(kx)M_0 + 60q^2x^3L^2k^5\sin(kx) + 24q^2x^2L^3k^5\sin(kx) \\
-24qL^4k^6\cos(kx)M_0 - 180M_0L^2k^6\cos(kx)qx^2 + 168M_0Lk^6\cos(kx)qx^3 \\
+432M_0L^2k^5\sin(kx)qx - 4608q^2 + 4608\cos(kx)q^2 + 4608\sin(kx)q^2kx \\
-2016k^2x^2q^2\cos(kx) + 48k^4x^4q^2\cos(kx) - 480k^3x^3q^2\sin(kx) \\
-504k^2q^2L^2\cos(kx) + 2q^2L^6k^6\cos(kx) + 72M_0^2L^2k^6\cos(kx) \\
-4\sin(kx)k^7q^2x^4L^3 + 2\sin(kx)q^2L^6k^7x + 6\sin(kx)k^7q^2x^3L^4 \\
-6k^7x^2q^2L^5\sin(kx) - 576M_0Lk^5\sin(kx)x^2q - 36k^7x^3M_0L^2\sin(kx)q \\
-36k^6x^2M_0L^2q - 288k^2x^2q^2 + 504k^2q^2L^2 - 2q^2L^6k^6 - 72M_0^2L^2k^6 \\
+288k^2xq^2L + 72k^4x^2q^2L^2 + 12k^8x^2M_0L^4q + 864k^4xqM_0L \\
-72qxL^3k^6M_0 - 72k^4x^2q^2L^3 - 432k^4qL^2M_0 - 12q^2x^2L^4k^6 + 12q^2xL^5k^6 \\
+24qL^4k^6M_0 - k^8x^2q^2L^6 - 21k^6q^2x^4L^2 + 22k^6q^2x^3L^3 - 24k^6qx^3M_0L \\
+12k^6q^2x^5L - 4k^6q^2x^6 - 36k^8x^2M_0^2L^2 \right\} .
\]

A few observations that can be seen from figure 2.4 that are worth noting:

- The displacement over the length is a non-stationary random field despite the input random field being stationary, even for statically determinate structures. This fact makes the VRF more complex than the frequency response function encountered in structural dynamics, from which analogies are often drawn.
Figure 2.4: VRF for example above evaluated at $x=0.25L$, $x=0.5L$, $x=0.8L$, and $x=L$

- There are certain loading conditions that produce a VRF with its peak away from $\kappa = 0$. This motivates the need to consider power of $S_f(\kappa)$ over the full range of wave numbers.
Chapter 3

Exact VRF for a Class of Statically Determinate Nonlinear Beams

The VRF for statically determinate beams following a class of nonlinear constitutive laws is derived in this chapter. The integral expression defining the VRF contains the SDF as well as higher power terms of the SDF. The VRF is strongly dependent on the constitutive law, but closed formed expressions can only be derived for only certain simple constitutive laws. This chapter begins with a derivation of the VRF. After which an example compares the VRFs for identical structures where one has a linear constitutive law and the other nonlinear. The chapter concludes with a discussion on the difficulties encountered when trying to establish a VRF for an arbitrary constitutive law.
3.1 Expression for the Displacement of Statically Determinate Beams With Power Law Constitutive Relations

The following derivation follows the formalism of the derivation found in reference [33].

Consider the constitutive law defined as

\[
\sigma = E(x)\left| \epsilon \right|^\frac{1}{\alpha} \text{sgn}(\epsilon) = \frac{E_0\left| \epsilon \right|^\frac{1}{\alpha}}{1 + f(x)}\text{sgn}(\epsilon),
\]

(3.1)

where \( f(x) \) is a zero mean and statistically homogeneous random field, \( \alpha \) is a positive integer, and \( \text{sgn}(...) \) is the sign function. The moment distribution along the length of the beam, \( m(x) \), which can be determined by statics, is also defined as

\[
m(x) = b \int_{-h/2}^{h/2} \sigma(x, y) y dy = b \int_{-h/2}^{h/2} E(x)\left| \epsilon \right|^\frac{1}{\alpha} \text{sgn}(\epsilon) y dy
\]

(3.2)

By utilizing the kinematic relationship

\[ y = \rho \epsilon \Rightarrow dy = \rho d\epsilon, \]

(3.3)

where \( \rho \) is the radius of curvature, the moment can be expressed by

\[
m(x) = 2\rho^2 b E(x) \int_0^{\epsilon_1} \epsilon^{\frac{1+\alpha}{\alpha}} d\epsilon
\]

\[
= 2\rho^2 b E(x) \left( \frac{\alpha}{1 + 2\alpha} \right) \epsilon_1^{\frac{1+2\alpha}{\alpha}},
\]

(3.4)

where \( \epsilon_1 \) is the strain at the extreme fiber per cross section. Substituting \( \rho = h/2\epsilon_1 \) gives

\[
m(x) = \frac{1}{2} h^2 b E(x) \left( \frac{\alpha}{1 + 2\alpha} \right) \epsilon_1^\frac{1}{\alpha}.
\]

(3.5)
Solving for $\epsilon_1$ gives
\[ \epsilon_1 = \left( \frac{2(1+2\alpha)m(x)}{\alpha h^2 b E(x)} \right)^{\alpha} \text{sgn}(m(x)). \] (3.6)

Using the kinematic relation that curvature is the second derivative of the displacement field, $u(x)$ can be solved for as
\[
\begin{align*}
\rho = \frac{h}{2} \epsilon_1 = \frac{2}{h} \left( \frac{(1+2\alpha)m(x)}{2\alpha h^2 b E(x)} \right)^{\alpha} \text{sgn}(m(x)) \Rightarrow \\
u''(x) = \frac{1}{\rho} = \frac{2\epsilon_1}{h} = \frac{2}{h} \left( \frac{(1+2\alpha)m(x)}{2\alpha h^2 b E(x)} \right)^{\alpha} \text{sgn}(m(x)) \\
u(x) = \int_0^x G(x,s) \frac{2}{h} \left( \frac{(1+2\alpha)m(s)(1+f(s))}{2\alpha h^2 b E_0} \right)^{\alpha} \text{sgn}(m(s))ds,
\end{align*}
\] (3.7)

$G(x,s)$ is the Green’s function for the structure and its boundary conditions, which can be derived for any statically determinate Euler-Bernoulli beam. From this integral expression, the variance of $u(x)$ can be computed in closed form for a given $\alpha$.

### 3.1.1 Derivation for Displacement Response for Statically Determinate Structures with a Linear Constitutive Law

In order to validate the derivation of section 3.1, the derivation for the displacement of a statically determinate linear elastic beam is given. The moment is given by
\[ m(x) = b \int_{-h/2}^{h/2} \sigma(x,y) y dy = m(x) = b \int_{-h/2}^{h/2} E(x) \epsilon y dy. \] (3.8)

Using the relations of equations (3.3), the moment is
\[ m(x) = \rho^2 b \int_{-\epsilon_1}^{\epsilon_1} E \epsilon^2 d\epsilon = \frac{2}{3} \rho^2 b E(x) \epsilon^3. \] (3.9)

From the kinematic relation $\rho = h/2\epsilon_1$, the extreme fiber strain is solved for:
\[ \epsilon_1 = \frac{6m(x)}{bE(x)h^2}, \] (3.10)
and lastly, the curvature is given by

\[ u''(x) = \frac{1}{\rho} = \frac{2\epsilon_1}{h} = \frac{12m(x)}{bE(x)h^3} = \frac{m(x)}{E(x)I} \]  

(3.11)

### 3.1.2 Variability Response Function for Statically Determinate Beams With Square-Root Constitutive Law

Let the constitutive relation for a material be defined as

\[ \sigma = E \sqrt{|\epsilon| \text{sgn}(\epsilon)} \]  

(3.12)

After some algebra following the derivation of section [3.1], the expression for the displacement field can be obtained:

\[ u(x) = \int_0^x \frac{50m^2(s)}{h^5 b^2 E_0^2} \text{sgn}(m(s)) (1 + f(s))^2 G(x,s)ds. \]  

(3.13)

The variance of \( u(x) \) is obtained by taking the expectation and expectation squared of equation (3.13). The expectation is given by

\[
\mathbb{E}[u(x)] = \int_0^x \frac{50m^2(s)}{h^5 b^2 E_0^2} \text{sgn}(m(s)) \mathbb{E} [1 + 2f(s) + f^2(s)] G(x,s)ds \\
= \int_0^x \frac{50m^2(s)}{h^5 b^2 E_0^2} \text{sgn}(m(s)) (1 + \sigma_f^2) G(x,s)ds.
\]  

(3.14)
The expectation squared is given by

\[
\mathbb{E}[u^2(x)] = \int_0^x \int_0^x \frac{50^2 m^2(s)m^2(s_1)}{h^{10}b^4 E_0^4} \text{sgn}(m(s))\text{sgn}(m(s_1))G(x,s)G(x,s_1)\times
\]

\[
\mathbb{E}[(1 + 2f(s) + f^2(s))(1 + 2f(s_1) + f^2(s_1))] \, dsds_1
\]

\[
= \int_0^x \int_0^x \frac{50^2 m^2(s)m^2(s_1)}{h^{10}b^4 E_0^4} \text{sgn}(m(s))\text{sgn}(m(s_1))G(x,s)G(x,s_1)\times
\]

\[
\mathbb{E}[1 + 2f(s) + 2f(s_1) + f^2(s) + f^2(s_1) + 4f(s)f(s_1) + 2f(s)f^2(s_1) + 2f(s_1)f^2(s) + f^2(s)f^2(s_1)] \, dsds_1
\]

(3.15)

Defining the following correlation functions as:

\[
R_1(\tau) = \mathbb{E}[f(s)f(s_1)] \tag{3.16a}
\]

\[
R_{12}(\tau) = \mathbb{E}[f(s)f^2(s_1)] = R_{21}(\tau) \tag{3.16b}
\]

\[
R_{22}(\tau) = \mathbb{E}[f^2(s)f^2(s_1)] \tag{3.16c}
\]

where \(\tau = s_1 - s\), then the expectation squared becomes

\[
\mathbb{E}[u^2(x)] = \int_0^x \int_0^x \frac{50}{h^{5}b^2 E_0^2} \left( \frac{m^2(s)m^2(s_1)}{h^{10}b^4 E_0^4} \right) \text{sgn}(m(s))\text{sgn}(m(s_1)) \times
\]

\[
G(x,s)G(x,s_1)(4R_1(\tau) + 2\sigma_f^2 + 4R_{12}(\tau) + R_{22}(\tau)) \, dsds_1. \tag{3.17}
\]

Defining the following power spectra as:

\[
S_1(\kappa) = \int_{-\infty}^{\infty} R_1(\tau)\exp(i\kappa \tau) d\tau \tag{3.18a}
\]

\[
S_{12}(\kappa) = \int_{-\infty}^{\infty} R_{12}(\tau)\exp(i\kappa \tau) d\tau \tag{3.18b}
\]

\[
S_{22}(\kappa) = \int_{-\infty}^{\infty} R_{22}(\tau)\exp(i\kappa \tau) d\tau, \tag{3.18c}
\]
then the variance of the displacement, $\text{Var} \{ u(x) \} = \mathbb{E} \{ u^2(x) \} - \mathbb{E} \{ u(x) \}^2$ becomes

$$
\text{Var} \{ u(x) \} = \int_{-\infty}^{\infty} \int_{0}^{x} \int_{0}^{x} \left( \frac{50}{h^5 b^2 E_0^2} \right)^2 m^2(s)m^2(s_1) \text{sgn}(m(s)) \text{sgn}(m(s_1)) G(x,s) \times \\
G(x,s_1) \exp(i\kappa(s_1 - s)) \left( 4S_1(\kappa) + 4S_{12}(\kappa) + S_{22}(\kappa) - \delta(\kappa) \sigma_f^4 \right) ds_1 ds d\kappa.
$$

(3.19)

The term that all spectra have in common will be denoted as $VRF^*(x,\kappa)$, that is

$$
VRF^*(x,\kappa) = \int_{0}^{x} \int_{0}^{x} 4 \left( \frac{50}{h^5 b^2 E_0^2} \right)^2 m^2(s)m^2(s_1) \text{sgn}(m(s)) \text{sgn}(m(s_1)) \times \\
G(x,s)G(x,s_1) \exp(i\kappa(s_1 - s)) ds_1 ds.
$$

(3.20)

The variance can be expressed in an integral expression analogous to the linear statically determinate case as

$$
\text{Var} \{ u(x) \} = \int_{-\infty}^{\infty} VRF^*(x,\kappa) \left[ S_1(\kappa) + S_{12}(\kappa) + \frac{1}{4} S_{22}(\kappa) - \frac{1}{4} \delta(\kappa) \sigma_f^4 \right] d\kappa
$$

(3.21)

The $S_{12}(\kappa)$ and $S_{22}(\kappa)$ terms are lines in the bispectrum and trispectrum. These are higher order spectra studied in signal processing, and they contain information about the degree of non-Gaussianity of a signal and the nonlinearity of the operator $F$. A few observations can be made from the above derivation:

- Certain nonlinear constitutive laws beget higher power correlation functions affecting the variance of the response.

- A unique and independent VRF exists for statically determinate structures for a class of nonlinear constitutive laws under monotonic loading. This is the first time such a proof is provided.

- Each constitutive law has a unique VRF and invokes different scale factors for the power spectra.

Take the following cantilever in the figure 3.1 with nonlinear constitutive law mentioned
in equation (3.12).

\[ m(x) = M - \frac{q}{2}(L - x)^2. \]

The term \( VRF^*(x, \kappa) \) is defined as

\[
VRF^*(x, \kappa) = \int_0^x \int_0^x \left( \frac{50}{b^5b^2E_0^2} \right)^2 (M - \frac{q}{2}(L - s)^2)^2 (M - \frac{q}{2}(L - s_1)^2)^2 \\
\times \text{sgn}(M - \frac{q}{2}(L - s)^2) \text{sgn}(M - \frac{q}{2}(L - s_1)^2)(x - s)(x - s_1) \\
\times \exp(i\kappa(s_1 - s)ds_1ds). \tag{3.22}
\]

In figure 3.2 the \( VRF^* \) at \( x = L \) is plotted along with the VRF for the identical structure with a linear constitutive law (see chapter 2 for VRF for linear statically determinate structures).
Figure 3.2: VRFs for Deflection at $x = L$ for Cantilever in figure 3.1

The VRFs are quite similar in shape, but this is a result of the loading. Another loading condition is shown in figure 3.3 in which the distributed loading varies linearly and is given by $q(x) = q(L - x)/L$. 
Keeping all the parameters the same as the previous example but changing the loading distribution profoundly affects the VRFs. From figure 3.4, which plots the $VRF^*$ for the nonlinear cantilever against that of the linear structure, it is clear that the VRF is quite dependent on the constitutive law.
Figure 3.4: VRFs for Deflection at $x = L$ for Cantilever in figure 3.3

One usage of the VRF is demonstrated below. Consider a random field having an SDF and higher power SDFs as shown in figure 3.3.
Utilizing only the SDFs in figure 3.5 and equation (3.21) the response variance can be determined once the $VRF^*$ term is known. For the example in figure 3.3 following the nonlinear constitutive law, the integral of equation (3.21) is computed numerically to give the response variance:

$$\text{Var} [u(L)] = 2 \sum_{j=1}^{N} VRF^*(L, \kappa_j) \left[ S_1(\kappa_j) + S_{12}(\kappa_j) + \frac{1}{4} S_{22}(\kappa_j) \right] \Delta \kappa - \frac{1}{4} VRF^*(L, 0) \sigma_f^4$$

(3.23)

$$= .1065$$
3.2 Discussion of the Displacement Response for Statistically Determinate Beams with Arbitrary Constitutive Law

The derivation for an integral expression for the displacement field in section 3.1 is followed here for an arbitrary constitutive law. Consider the constitutive law

\[ \sigma = Ef(\epsilon). \tag{3.24} \]

The moment distribution along the length of the beam, \( m(x) \), can be determined by statics. It is also defined as

\[ m(x) = b \int_{-h/2}^{h/2} \sigma(x,y)y \, dy. \tag{3.25} \]

By utilizing the kinematic relationship

\[ y = \rho \epsilon \]
\[ dy = \rho d\epsilon, \tag{3.26} \]

where \( \rho \) is the radius of curvature, and by taking a constitutive law, \( \sigma = E(x)f(\epsilon) \), the moment is expressed by

\[ m(x) = 2\rho^2 b E \int_0^{\epsilon_1} f(\epsilon)\epsilon \, d\epsilon = 2\rho^2 E \int_0^{\epsilon_1} h(\epsilon) \, d\epsilon, \tag{3.27} \]

where \( \epsilon_1 \) is the strain at the extreme fiber of the cross section, and \( h(\epsilon) = f(\epsilon)\epsilon \). The constitutive law taken in this derivation shall be constrained to one for which the loading is monotonic (i.e. no hysteresis curves). Let function \( H(\epsilon_1) = \int_0^{\epsilon_1} h(\epsilon) \, d\epsilon \). Since either only positive or only negative values are taken for \( \epsilon \) in the integration and only monotonic loading
is considered, function $H(\epsilon_1)$ is one-to-one. Utilizing the following kinematic relationship

$$\rho = \frac{h}{2\epsilon_1}, \quad (3.28)$$

the moment can be expressed as

$$m(x) = \frac{hbEH(\epsilon_1)}{\epsilon_1^2}. \quad (3.29)$$

Let $H^*(\epsilon_1) = H(\epsilon_1)/\epsilon_1^2$. If $H^*(\epsilon_1)$ is also one-to-one, which holds for a large class of constitutive laws, then its inverse, $H^{-1}$, exists. Then $\epsilon_1$ can be written as

$$\epsilon_1(x) = H^{-1} \left( \frac{m(x)}{hb} \right). \quad (3.30)$$

The constraint for which no inverse for $H^*(\epsilon_1)$ exists, requires that $\exists \epsilon_{11}, \epsilon_{12}$ s.t. $\epsilon_{12}^2 H^*(\epsilon_{11}) = \epsilon_{11}^2 H^*(\epsilon_{12})$ which is more easily avoided than obtained. Utilizing the kinematic relation that curvature is the second derivative of displacement, and equation (3.28), the displacement field can be written as

$$u''(x) = \frac{2\epsilon_1}{h} = \frac{2}{h} H^{-1} \left( \frac{m(x)}{hb} \right) \Rightarrow \quad (3.31a)$$

$$u(x) = \int_0^x \frac{2}{h} H^{-1} \left( \frac{m(s)}{hbE(x)} \right) G(x, s) ds. \quad (3.31b)$$

$G(x, s)$ is the Green’s function for the differential equation in (3.31a) with its boundary conditions.

It is difficult to establish a VRF for an arbitrary constitutive law because closed form expressions rarely exist for $H^{-1}$. As a result, the contributions of the SDF and its higher power terms are unknown. Therefore numerical procedures cannot be implemented to establish a $VRF^*$ term to the best of the current knowledge.
Chapter 4

Generalized Variability Response Function Methodology

4.1 Introduction

The existence of the VRF was formally proven for linear, statically determinate structures in the chapter 2 and for a class of nonlinear, statically determinate structures in chapter 3. For all other structures, the VRF’s existence has never been formally proven due to the condition that there must exist a Green’s function independent of the flexibility in equation (2.5). However its existence has not been disproved either. This chapter details a Monte Carlo based methodology, proposed by [11, 9, 10], that generalizes the VRF concept such that it is applicable to indeterminate linear structures. The aim of this methodology is to compute a Generalized VRF (GVRF) with properties very similar to those of the classical VRF. A Monte Carlo simulation procedure is developed to determine the validity of the following desired property: a unique VRF exists for indeterminate structures that is independent of the marginal PDF and SDF of the uncertain parameters. In references [11, 9, 10] the GVRFs are established for statically indeterminate, linear beam structures. The aim of this and the following chapter is to determine if the methodology is applicable for nonlinear beams.
Assume that equation (2.1), rewritten below for convenience, holds in general for any linear structure

\[ \exists \text{VRF}(x, \kappa) \text{ s.t. } \text{Var}[u(x)] = \int_{-\infty}^{\infty} S_f(\kappa) \text{VRF}(x, \kappa) d\kappa. \]  

(4.1)

If computing numerically, the form of the equation becomes

\[ \text{Var}[u(x)] = 2 \sum_{l=1}^{N} S_f(\kappa_l) \text{VRF}(x, \kappa_l) \Delta \kappa, \text{ } \kappa \in [0..\kappa_u], \]  

(4.2)

which equivalently can be written as

\[
\begin{bmatrix}
\text{Var}[u(x)]_1 \\
\text{Var}[u(x)]_2 \\
\vdots \\
\text{Var}[u(x)]_N
\end{bmatrix} = 2
\begin{bmatrix}
S_{f_1}(\kappa_1) & S_{f_1}(\kappa_2) & \ldots & S_{f_1}(\kappa_N) \\
S_{f_2}(\kappa_1) & S_{f_2}(\kappa_2) & \ldots & S_{f_2}(\kappa_N) \\
\vdots & \vdots & \ddots & \vdots \\
S_{f_N}(\kappa_1) & S_{f_N}(\kappa_2) & \ldots & S_{f_N}(\kappa_N)
\end{bmatrix}
\begin{bmatrix}
\text{VRF}(x, \kappa_1) \\
\text{VRF}(x, \kappa_2) \\
\vdots \\
\text{VRF}(x, \kappa_N)
\end{bmatrix}
\Delta \kappa.
\]  

(4.3)

For a given SDF and marginal PDF of \( f(x) \) in equation (2.4), the variance of \( u(x) \) can be computed through Monte Carlo simulation. If this is repeated \( N \) times for \( N \) different SDFs, then a system of linear equations emerges where the unknown quantity is the VRF. To highlight the difference between the VRF computed with that of chapter 2, this VRF is denoted the Generalized Variability Response Function (GVRF)

\[
\begin{bmatrix}
\text{Var}[u(x)]_1 \\
\text{Var}[u(x)]_2 \\
\vdots \\
\text{Var}[u(x)]_N
\end{bmatrix} = 2
\begin{bmatrix}
S_{f_1}(\kappa_1) & S_{f_1}(\kappa_2) & \ldots & S_{f_1}(\kappa_N) \\
S_{f_2}(\kappa_1) & S_{f_2}(\kappa_2) & \ldots & S_{f_2}(\kappa_N) \\
\vdots & \vdots & \ddots & \vdots \\
S_{f_N}(\kappa_1) & S_{f_N}(\kappa_2) & \ldots & S_{f_N}(\kappa_N)
\end{bmatrix}
\begin{bmatrix}
\text{GVRF}(x, \kappa_1) \\
\text{GVRF}(x, \kappa_2) \\
\vdots \\
\text{GVRF}(x, \kappa_N)
\end{bmatrix}
\Delta \kappa.
\]  

(4.4)

For nonlinear statically indeterminate structures, it is assumed that the following equation
can be satisfied
\[ \text{Var}[u(x)] = 2 \int_0^\infty \text{VRF}^*(x, \kappa) S^*(\kappa) d\kappa. \] (4.5)

The term \( S^*(\kappa) \) is specific to the constitutive law. For a square root constitutive law, \( S^*(\kappa) \) was derived in chapter 3 to be
\[ S^*(\kappa) = S_1(\kappa) + S_{12}(\kappa) + \frac{1}{4} S_{22}(\kappa) - \frac{1}{8} \delta(\kappa). \] (4.6)

Following the development for the linear case, now the system of linear equations is
\[
\begin{bmatrix}
\text{Var}[u(x)_1] & \text{Var}[u(x)_2] & \cdots & \text{Var}[u(x)_N] \\
S_{f_1}^*(\kappa_1) & S_{f_2}^*(\kappa_2) & \cdots & S_{f_N}^*(\kappa_N) \\
S_{f_1}^*(\kappa_1) & S_{f_2}^*(\kappa_2) & \cdots & S_{f_N}^*(\kappa_N) \\
\vdots & \vdots & \ddots & \vdots \\
S_{f_1}^*(\kappa_1) & S_{f_2}^*(\kappa_2) & \cdots & S_{f_N}^*(\kappa_N) \\
\end{bmatrix}
\begin{bmatrix}
\text{GVRF}^*(x, \kappa_1) \\
\text{GVRF}^*(x, \kappa_2) \\
\vdots \\
\text{GVRF}^*(x, \kappa_N) \\
\end{bmatrix}
= \Delta \kappa. \quad (4.7)
\]

It is important to note that the \( S^*(\kappa) \) in (4.6) can only be determined for a few nonlinear constitutive laws (see section 3.2 for discussion). The \( N \) SDFs comprising the matrix in equations (4.4) and (4.7) are called set or family of SDFs. They have identical shapes and are related to each other by a shift in the domain of its argument, \( \kappa \). The details are discussed in section 4.5. The GVRF obtained from solving the system of linear equations from a family of SDFs is one approximation of the VRF. In order to verify its uniqueness, independence, and existence, this process is repeated for different marginal PDFs and families of SDFs describing random field \( f(x) \).

Since the response variance is determined through Monte Carlo simulation, it is necessary to describe \( f(x) \) as a non-Gaussian random field that can be simulated. This is executed by either simulating Gaussian or U-beta random fields and transforming them to a desired distribution. The random field models of \( f(x) \) will be constrained such that \( f(x) \) is completely characterized by its second order statistics (finite variance, \( \sigma_f^2 \), and its autocorrelation function, \( R_f(\tau) \)).
The remainder of this chapter presents the methodology in detail. In section 4.2 the Translation Field Theory for transforming Gaussian fields and the Spectral Representation Method for simulating Gaussian random fields are presented. Section 4.3 describes U-beta random fields and the Associated Field Theory. The subsequent sections outline the non-Gaussian random fields and the SDFs considered for modeling the heterogeneous flexibility in the numerical examples presented in chapter 5. The chapter concludes with a summary of the GVRF, and numerical and computational issues that need to be taken into consideration.

### 4.2 Translating From An Underlying Gaussian Field

It is widely accepted that non-Gaussian random field models are necessary to accurately represent uncertain system parameters in Civil Engineering applications. There is a large number of researchers trying to efficiently simulate non-Gaussian and non-stationary random fields, often with application to ground motion simulation, however they mostly involve Monte Carlo techniques to match prescribed evolutionary power SDFs and marginal PDFs. There is one widely accepted non-Gaussian random field model whose mathematics is rigorously developed analytically, and that is the translational field model. In the translation field model, a Gaussian random field is transformed to a non-Gaussian field, and expressions can be derived for a complete description of its properties (ie joint probability density function (pdf), statistical moments, mean up-crossing rates, and distribution of extremes).

#### 4.2.1 Gaussian Random Fields

A homogeneous random field $G(x)$ is a Gaussian random field if the set of any $N$ samples, $(G(x_1) = g_1, G(x_2) = g_2, ..., G(x_N) = g_N)$, is a set of $N$ Gaussian random variables having
the $N^{th}$ order joint pdf, $\phi_N(g)$ given by

$$\phi_N(g) = \frac{1}{\sqrt{(2\pi)^N \det \rho_{GG}}} \exp\left( -\frac{1}{2} (g - \mu_G)^T \rho_{GG}^{-1} (g - \mu_G) \right),$$

(4.8)

where $\rho_{GG}$ is the covariance matrix, and $\mu_G$ is the vector of means of the $N$ samples. Its autocorrelation function defined as

$$R_G(\xi) = E[G(x_1)G(x_2)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_1 g_2 \phi_2(g) dg_1 dg_2,$$

(4.9)

depends only on the distance between the two instances, $x_1$ and $x_2$. The mean and autocorrelation function are invariant of any spatial shift. Therefore, a weakly homogeneous Gaussian field is strictly homogeneous due to the fact that any $N^{th}$ order joint-pdf is completely described by its mean and autocorrelation function. Gaussian random fields comprise the most widely used class of random fields due to the existence of closed form expressions for complete characterization, the consequences of central limit theorem, and the efficiency of their simulation [37].

### 4.2.2 Translation Field Theory

Let $x$ be an $n$ dimensional vector in $\mathbb{R}^n$ and a function $G(x)$, with mapping $G : \mathbb{R}^n \rightarrow \mathbb{R}$, be a homogeneous Gaussian field with mean, $\mu$, equal zero, autocorrelation, $R_G(\xi) = E[g(x + \xi)g(x)]$, and marginal PDF is $\Phi(g) = P(G(x) \leq g)$. The lag term, $\xi$, between positions $x_2$ and $x_1$ is defined as

$$\xi = \begin{cases} (x_2 - x_1, x_2 - x_1, ..., x_2 - x_1) & \text{for a general anisotropic field} \\ |x_2 - x_1| & \text{for an isotropic field} \end{cases}$$

(4.10)
Let $F(x)$, with mapping $F : \mathbb{R}^n \rightarrow \mathbb{R}$, be a random field having a monotonically increasing and continuous PDF, $P_F(f)$. Then

$$F(x) = P_F^{-1} \circ \Phi(G(x)) = T(G(x))$$

(4.11)

where $T(\cdot) = P_F^{-1} \circ \Phi(\cdot)$. The constraint that $P_F(f)$ is monotonically increasing and continuous ensures the existence of its inverse. This is called a memoryless transformation since the value of $F$ evaluated at position $x$ is dependent on the value of $G$ at position $x$ and no other position. The fact that $P_F(f)$ is the marginal PDF of $F$ can be proven as follows

$$\mathbb{P}(F(x) \leq f) = \mathbb{P}(P_F^{-1} \circ \Phi(G(x)) \leq f) = \mathbb{P}(G(x) \leq \Phi^{-1}(P_F(f)))$$

(4.12)

$$= \Phi \circ \Phi^{-1}(P_F(f)) = P_F(f)$$

If $P_F(f)$ is not strictly increasing or is piece-wise, then its inverse can be defined so that it returns the smallest value $f$ of the subset $F_p$ where $F_p$ is defined such that $f \in F_p \Leftrightarrow P_F(f) \geq p$:

$$P_F^{-1}(p) = \inf \{ f | f \in F_p \}.$$  (4.13)

The $m^{th}$ order joint PDF is

$$P_F(f_1, f_2, \ldots, f_m) = \mathbb{P}(T(G(x_1)) \leq f_1, T(G(x_2)) \leq f_2, \ldots, T(G(x_m)) \leq f_m)$$

$$= \mathbb{P}(G(x_1) \leq T^{-1}(f_1), G(x_2) \leq T^{-1}(f_2), \ldots, G(x_m) \leq T^{-1}(f_m))$$

(4.14)

$$= \Phi(G(x_1) \leq T^{-1}(f_1), G(x_2) \leq T^{-1}(f_2), \ldots, G(x_m) \leq T^{-1}(f_m); \rho)$$

where $\rho$ is the covariance matrix of the underlying Gaussian field, and $T^{-1}(\cdot) = \Phi^{-1} \circ P_F(\cdot)$.

It is observed that the joint PDF of the translated field, $F(x)$, depends on the covariance matrix of the underlying field, which is only a function of the relative distance between the coordinates of the positions. Thus the translated field, $F(x)$, is shift invariant and strictly homogeneous. This is also a consequence of the memoryless transformation.
Moments and correlation functions of the translated field can be defined in terms of the underlying field

\[ \mathbb{E}[F(x)^m] = \mathbb{E}[T(G(x))^m] = \int_{-\infty}^{\infty} T(g)^m \phi(g) dg. \]  

(4.15)

Any order correlation function can be written as

\[ \mathbb{E}[F(x_1)F(x_2)\ldots F(x_m)] = \mathbb{E}[T(G(x_1))T(G(x_2))\ldots T(G(x_m))] \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} T(g_1)T(g_2)\ldots T(g_m)\phi(g_1, g_2, \ldots, g_m; \rho) dg_1 dg_2 \ldots dg_m. \]  

(4.16)

Similar to the observations made in equation (4.14), the \( m \)th order correlation functions are dependent on the relative distance between the coordinates of the positions, meaning the translation field \( F(x) \) is homogeneous as a consequence of the homogeneity of the underlying Gaussian field.

### 4.2.3 SDF of Translated Fields

The family of \( N \) SDFs (equation (4.4)) of the translated fields must be computed before solving the system of linear equations to determine the GVRF. The \( n \)th SDF, \( S_{f_n}(\kappa) \), can be determined by taking the FFT of its autocorrelation function, \( R_{f_n}(\xi) \), which is defined as

\[ R_{f_n}(\xi) = \mathbb{E}[F(x_1)F(x_2)] = \mathbb{E}[T(G(x_1))T(G(x_2))] \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} T(g_1)T(g_2)\phi(g_1, g_2, R_{G_n}(\xi)) dg_1 dg_2. \]  

(4.17)

The SDF then is determined from the Weiner-Khinchin transformation

\[ S_{f_n}(\kappa) = \int_{-\infty}^{\infty} R_{f_n}(\xi) \exp(i\kappa \xi) d\xi. \]  

(4.18)
The higher power SDFs can be computed in a straightforward manner. Let $R^{(ij)}_{fn}$ be defined as

$$
R^{(ij)}_{fn}(\xi) = E \left[ F(x_1)^iF(x_2)^j \right] = E \left[ T(G(x_1))^iT(G(x_2))^j \right] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} T(g_1)^iT(g_2)^j \phi(g_1, g_2, R_{Gn}(\xi)) dg_1 dg_2.
$$

(4.19)

Then the higher power SDFs are determined from the Weiner-Khinchin transformation

$$
S^{(ij)}_{fn}(\kappa) = \int_{-\infty}^{\infty} R^{(ij)}_{fn}(\xi) \exp(i\kappa\xi) d\xi.
$$

(4.20)

### 4.2.4 Simulation of Homogeneous Gaussian Random Fields via Spectral Representation

Significant work outlining the simulation of Gaussian fields by means of spectral representation date back to the works in references [38, 39, 40] and later in [41, 42, 43, 43, 13]. The theory of spectral representation is developed in references [44, 45]. This section is a brief review of the simulation methodology outlined in the review article in reference [41].

Consider a zero mean homogeneous random field $f_0(x)$ with SDF $S_{f_0}(\kappa)$ such that

$$
2 \int_{0}^{\infty} S_{f_0}(\kappa) d\kappa < \infty.
$$

(4.21)

From two mutually orthogonal real fields $u(\kappa)$ and $\nu(\kappa)$, $f_0(x)$ can be defined as

$$
f_0(x) = \int_{0}^{\infty} \cos(\kappa) du(\kappa) + \sin(\kappa) d\nu(\kappa).
$$

(4.22)

The fields $u(\kappa)$ and $\nu(\kappa)$ and their corresponding increments $du(\kappa)$ and $d\nu(\kappa)$ are defined for
κ ≥ 0, and satisfy the following conditions:

\[
\begin{align*}
\mathbb{E}[u(\kappa)] &= 0, & \mathbb{E}[\nu(\kappa)] &= 0, & \mathbb{E}[du(\kappa)] &= 0, & \mathbb{E}[d\nu(\kappa)] &= 0 \\
\mathbb{E}[du(\kappa)du(\kappa')] &= \mathbb{E}[d\nu(\kappa)d\nu(\kappa')] = 0, & \kappa \neq \kappa' \\
\mathbb{E}[u(\kappa)\nu(\kappa')] &= \mathbb{E}[du(\kappa)d\nu(\kappa')] = 0, & \forall \kappa, \kappa' \\
\mathbb{E}[u(\kappa)^2] &= \mathbb{E}[\nu(\kappa)^2] = 2S_{f_0}(\kappa) \\
\mathbb{E}[du(\kappa)^2] &= \mathbb{E}[d\nu(\kappa)^2] = 2S_{f_0}(\kappa)d\kappa,
\end{align*}
\]

where it is assumed that \(S_{f_0}(\kappa)\) is the derivative of the SDF of another field, \(F_0(\kappa)\), such that

\[
\frac{dS_{f_0}(\kappa)}{d\kappa} = S_{f_0}(\kappa), \quad \text{for} \ \kappa \geq 0. \tag{4.24}
\]

By discretizing the wave number domain into discrete increments \(\Delta \kappa\), the integral in equation (4.22) becomes an infinite summation where

\[
f_0(x) = \sum_{j=1}^{\infty} \cos(\kappa_j x)du(\kappa_j) + \sin(\kappa_j x)d\nu(\kappa_j) \tag{4.25}
\]

The incremental fields can be defined as

\[
\begin{align*}
du(\kappa_j) &= 2\sqrt{S_{f_0}(\kappa_j)\Delta \kappa} \cos(\theta_j) \\
d\nu(\kappa_j) &= -2\sqrt{S_{f_0}(\kappa_j)\Delta \kappa} \sin(\theta_j), \tag{4.26}
\end{align*}
\]

and the requirements of equation (4.23) are satisfied. The phase angles \(\theta_j\) are independent, identically distributed uniform random variables between \([0, 2\pi]\). After substituting equation (4.26) into (4.22) and a little algebra, \(f_0(x)\) can be written as

\[
f_0(x) = 2\sum_{j=1}^{\infty} \sqrt{S_{f_0}(\kappa_j)\Delta \kappa} \cos(\kappa_j x + \theta_j). \tag{4.27}
\]

Function \(f_0(x)\) can be simulated by first truncating the infinite series to a finite one. An
upper cut-off wave number $\kappa_u$ is determined such that

$$\int_{\kappa_u}^{\infty} 2S_{f_0}(\kappa) d\kappa < \epsilon$$  \hfill (4.28)

for some threshold $\epsilon$. Then a simulated sample function, $f(x)$, can be written as the finite series

$$f(x) = \sum_{j=0}^{N-1} 2\sqrt{S_{f_0}(\kappa_j)} \Delta\kappa \cos(\kappa_j x + \theta_j)$$  \hfill (4.29)

where $\kappa_j = j\Delta\kappa$ and $\Delta\kappa = \kappa_u/N$. The simulated field $f(x)$ has following properties

- $f(x)$ is periodic with period $T_0 = 2\pi/\Delta\kappa$.
- its ensemble average, $\mathbb{E}[\{f(x)\}] = [f_0(x)] = 0$.
- the autocorrelation function of the ensemble average $R(f(x))(\xi) = R(f_0)(\xi)$.
- in the limit as $T_0 \to \infty$, each sample function, $f^{(i)}(x)$ is ergodic in mean and autocorrelation:

$$\lim_{T_0 \to \infty} \frac{1}{T_0 - 1} \sum_{p=1}^{T_0} f^{(i)}(x_p) dx = 0$$

$$\lim_{T_0 \to \infty} \frac{1}{T_0 - 1} \sum_{p=1}^{T_0} f^{(i)}(x_p) f^{(i)}(x_p + \xi) dx = R_{f_0}(\xi).$$  \hfill (4.30)

- in the limit as $N \to \infty$, the sample function $f(x)$ is a Gaussian field because of the central limit theorem.

The spatial domain is discretized such that $x_p = p\Delta x$, and sample functions $f^{(i)}(p\Delta x)$ can be very rapidly simulated through the use of FFT \cite{46}. Equation (4.29) is rewritten in the following form:

$$f^{(i)}(p\Delta x) = \Re \left\{ \sum_{j=0}^{N-1} 2\sqrt{S_{f_0}(j\Delta\kappa)} \Delta\kappa \exp \left[ i\theta^{(i)}_j \right] \exp \left[ i(j\Delta\kappa)(p\Delta x) \right] \right\}$$  \hfill (4.31)
where \( \Re \) indicates the real part, and index \( p = 0, 1, \ldots N - 1 \). Studies on rate of convergence and degree of Gaussianity and ergodicity can be found in references \([41, 9]\). The simulations conducted to establish the GVRFs for the problems studied in this thesis follow the algorithm of equation (4.31).

### 4.3 U-beta Random Fields

A zero mean U-beta random field is a sinusoidal field with a random phase shift, \( \theta \in \mathcal{U}[0 \ 2\pi] \), and deterministic amplitude, described as follows:

\[
U(x) = \sqrt{2} \sigma_u \cos(\kappa_\delta x + \theta). \tag{4.32}
\]

The U-beta distribution is a special case of the Beta distribution. Its marginal PDF equals

\[
P_U(u) = 1 - \frac{1}{\pi} \arccos\left( \frac{u}{\sqrt{2} \sigma_u} \right), \tag{4.33}
\]

and its marginal pdf is

\[
p_U(u) = \frac{1}{\pi \sqrt{2 \sigma_u^2 - u^2}}. \tag{4.34}
\]

The autocorrelation function can easily be determined from its definition

\[
\mathbb{E}[U(x)U(x + \xi)] = \frac{1}{2\pi} \int_0^{2\pi} \sqrt{2} \sigma_u \cos(\kappa_\delta x + \theta) \sqrt{2} \sigma_u \cos(\kappa_\delta (x + \xi) + \theta) d\theta = \sigma_u \cos(\kappa_\delta \xi) = R_U(\xi) \tag{4.35}
\]

Following from the Wiener-Khinchin transformation, the power SDF is defined by the Dirac Delta function, centered at wave number \( \kappa_\delta \),

\[
S_U(\kappa) = \frac{\sigma_u^2}{2} [\delta(\kappa - \kappa_\delta) + \delta(\kappa - \kappa_\delta)]. \tag{4.36}
\]
From the definition of the autocorrelation function, the U-beta field is shown to be weakly homogeneous. Furthermore, the shift invariance of the $n^{th}$ joint PDF is shown in reference [9], implying the U-beta random field is strictly homogeneous.

The simulation of U-beta random fields is a trivial task since it only involves generating samples of the uniformly distributed random variable, $\theta$. As a result, the ensemble averaged statistical moments of an ensemble of generated sample functions rapidly converges to the exact moments. This minimizes the number of Monte Carlo simulations necessary for convergence to the correct solution. For this reason, the computation of the GVRF from an underlying U-beta field is denoted as the Fast Monte Carlo Simulation Method. The number of Monte Carlo simulations necessary to reach the same convergence of that of an underlying Gaussian field is roughly an order of magnitude lower.

### 4.3.1 Translating from An Underlying U-beta Random Field: Associated Fields

This section details the simulation of random fields modeling $f(x)$ in equation (2.3) by translating from an underlying U-beta field. The transformation from an underlying U-beta field is identical to the transformation in the translation field theory of section 4.2. However, there does not exist a general theory completely characterizing the transformed field as in the case for the underlying Gaussian. In order to distinguish transformations from an underlying U-beta field from the translation field theory, these transformations are denoted as Associated Fields [26][27]. The transformation from U-beta to $f(x)$ can be accomplished by the following operation

$$f(x) = P_f^{-1} \circ P_U(u(x)) = A(u(x))$$

where $P_U(u(x))$ is defined in equation (4.33), $P_f(f(x))$ is the marginal PDF of random fields described in section 4.4, and $A(\cdot)$ is the notation for an associated field.
Moments of Associated Fields

An important consequence of the memoryless transformation from the underlying, strictly homogeneous, U-beta field, is that \( f(x) \) is also strictly homogeneous. Moments of the associated fields can be expressed in the following integral:

\[
\mathbb{E}[f(x_1)f(x_2)...f(x_n)] = \mathbb{E}[\mathcal{A}(\sqrt{2}\sigma_u\cos(\kappa\delta x_1 + \theta))\mathcal{A}(\sqrt{2}\sigma_u\cos(\kappa\delta x_2 + \theta))]
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{A}(\sqrt{2}\sigma_u\cos(\kappa\delta x_1 + \theta))\mathcal{A}(\sqrt{2}\sigma_u\cos(\kappa\delta x_2 + \theta))d\theta
\]

\[ (4.38) \]

4.3.2 SDF of Associated Fields

The family of \( N \) SDFs (equation (4.4)) of the associated fields must be computed before solving the system of linear equations to determine the GVRF. The process to computing the \( n^{th} \) member’s SDF, \( S_{f_n}(x) \), is given below. The derivation is generalized for an arbitrary power of the SDF, \( S^{(ij)}_{f_n}(x) \).

- Express the autocorrelation function \( R^{(ij)}_{f_n}(\xi) \) in terms of \( \mathcal{A}(u(\xi)) \) and random variable \( \theta \).

- Recognize that by a change of variables, \( R^{(ij)}_{f_n}(\xi) \) is periodic between zero and \( 2\pi \).

- Express \( R^{(ij)}_{f_n}(\xi) \) by a Fourier series representation and compute the Fourier coefficients \( a_m \).

- Apply relation between \( R^{(ij)}_{f_n}(\xi) \) and \( S^{(ij)}_{f_n}(\kappa) \) to derive an expression for \( S^{(ij)}_{f_n}(\kappa) \)

The derivation is detailed as follows:

\[
R^{(ij)}_{f_n}(\xi) = \int_0^{2\pi} \mathcal{A}^i(u(x))\mathcal{A}^j(u(x + \xi))f_\theta(\theta)d\theta
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} \mathcal{A}^i(\sqrt{2}\sigma_u\cos(\theta))\mathcal{A}^j(\sqrt{2}\sigma_u\cos(\kappa\delta\xi + \theta))d\theta.
\]

\[ (4.39) \]
Notice that the term involving $x$ has been dropped due to the shift invariance of $U(x)$. The $n^{th}$ SDF in the set, is defined uniquely by $\kappa_\delta$, where $\kappa_\delta = (n - 1)\Delta \kappa$. Due to the memoryless transformation, $A(u(x))$ is periodic with period $\frac{2\pi}{\kappa_\delta}$ implying the same for $R_{f_n}(\xi)$. This implies that $A^i(u(x))$ is periodic between $\left[0, \frac{2\pi}{\kappa_\delta}\right]$. By making the following substitution of variables

\[ \xi' = (n - 1)\Delta \kappa \xi, \tag{4.40} \]

it is observed that

- $R_{f_n}^{(ij)}(\xi')$ is independent of $n$, thus the $n$ subscript will be dropped.
- $R_{f}^{(ij)}(\xi')$ is periodic within $2\pi$.

Since $R_{f}^{(ij)}(\xi')$ is an even function, it can be expressed in the following Fourier series representation

\[ R_{f}^{(ij)}(\xi') = a_0 + \sum_{m=1}^{\infty} a_m \cos(m\xi') \tag{4.41} \]

where Fourier coefficients, $a_m$ equal

\[
\begin{align*}
a_0 &= \frac{1}{2\pi} \int_{0}^{2\pi} R_{f}^{(ij)}(\xi') d\xi' \\
&= \frac{1}{4\pi^2} \int_{0}^{2\pi} \int_{0}^{2\pi} A^i(\sqrt{2}\sigma_u \cos(\theta)) A^j(\sqrt{2}\sigma_u \cos(\xi' + \theta)) d\theta d\xi' \\
a_m &= \frac{1}{\pi} \int_{0}^{2\pi} R_{f}^{(ij)}(\xi') d\xi' \\
&= \frac{1}{2\pi^2} \int_{0}^{2\pi} \cos(m\xi') \int_{0}^{2\pi} A^i(\sqrt{2}\sigma_u \cos(\theta)) A^j(\sqrt{2}\sigma_u \cos(\xi' + \theta)) d\theta d\xi'.
\end{align*}
\tag{4.42}
\]

The Fourier coefficients can be efficiently computed using Gaussian quadrature or a similar numerical technique. The independence of $R_{f}^{(ij)}(\xi')$ of $n$ greatly reduces the computation costs since the Fourier coefficients $a_m$ apply to all $N$ autocorrelation functions. Once the Fourier coefficients are solved for, substituting $\xi$ back in for $\xi'$ gives

\[ R_{f_n}^{(ij)}(\xi) = a_0 + \sum_{m=1}^{\infty} a_m \cos(m(n - 1)\Delta \kappa \xi). \tag{4.43} \]
Finally the spectral density functions, $S_{f_n}^{(ij)}(\kappa)$ can be computed by performing an FFT on equation (4.43) to give

$$S_{f_n}^{(ij)}(\kappa) = a_0 \delta(\kappa) + \frac{a_m}{2} \left[ \delta(\kappa - m(n - 1)\Delta \kappa) + \delta(\kappa + m(n - 1)\Delta \kappa) \right].$$

\[(4.44)\]

### 4.4 Summary of Random Field Models

In order to establish the uniqueness of the computed GVRF, the non-Gaussian random fields considered must have very different distributional and spectral characteristics. For the studies conducted in this thesis, all combinations of three classes of PDFs and three SDFs are considered. Each class of PDF consists of two distributions of different variances, resulting in a total of 18 random fields considered. The three classes are shifted and scaled Lognormal(LN), truncated Gaussian(TG), and Uniform(UN) distributions. The inverses of the marginal PDFs of the target distributions are needed in order to transform from the underlying, simulated fields. Since the transformations are memoryless, and thus point-wise, the transformations in the subsequent sections will be discussed for random variables (i.e., scalar quantities) for notation purposes.

**Uniform Distribution**

Let $F$ be a random variable following the distribution of a Uniform distribution, then the inverse of its PDF is

$$P_F^{-1}(\cdot) = (a_u - a_l)(\cdot) + a_l,$$

where $a_l$ and $a_u$ are the lower and upper bounds, respectively. Translating from an underlying Gaussian variable gives

$$T(\cdot) = P_F^{-1}(\Phi(\cdot)).$$

\[(4.46)\]

The numerical computation of $\Phi(\cdot)$ is facilitated by the use of the Fortran77 package DCDFLIB \[47\]. Similarly, the associated variable, transformed from a random variable following a U-beta
distribution is

\[ A(\cdot) = P_F^{-1}(P_U(\cdot)), \quad (4.47) \]

where \( P_U(\cdot) \) is the PDF of the U-beta field defined in equation \((4.33)\). The following bounds on \( a_u \) and \( a_l \) are given as

\[ -1 < a_l < a_u < 1 \quad (4.48) \]

The lower bound ensures that the flexibility is strictly greater than zero. Figure 4.1 shows the cumulative distribution function (CDF) and pdf of a Uniform random variable.

![Figure 4.1: pdf and CDF of Scaled Uniform Random Variable](image)

**Truncated Gaussian Distribution**

Let \( F \) be a random variable following a truncated Gaussian distribution. The PDF is piece-wise and discontinuous, taking the form

\[ P_F(f) = \begin{cases} 
0, & f < a_l \\
\int_{a_l}^{a_u} \frac{1}{\sqrt{2\pi} \sigma_F} \exp\left(-\frac{(f-\mu_F)^2}{2\sigma_F^2}\right) df, & a_l \leq f \leq a_u \\
1, & a_u < f.
\end{cases} \quad (4.49) \]
Since there is no closed form expression for the inverse of a Gaussian distribution, a cumbersome numerical method is necessary to compute $P_F^{-1}(f)$ due to the jump discontinuities introduced. However, it is observed that if translating from a Gaussian random variable, $G$, $F$ can be written as

$$F = T(G) = \begin{cases} 
a_1, & sG + m < a_l \\
sG + m, & a_l \leq sG + m \leq a_u \\
a_u, & a_u < sG + m. 
\end{cases}$$

(4.50)

The parameters $a_l$ and $a_u$ are the lower and upper bounds, respectively, while $s$ and $m$ are the scale and shift factors, respectively. The same bounds of equation (4.48) are needed for the parameters of the truncated Gaussian to satisfy the conditions ensuring the problem is well-posed.

Due to the unavailability of $P_F^{-1}(f)$, the associated variable is computed in two steps. The U-beta random variable is transformed to a Gaussian random variable, which is then translated to a truncated Gaussian random variable. If $U$ is a U-beta random variable with PDF, $P_U(u)$, as defined in equation (4.33), then truncated Gaussian random variable, $F$, is given as

$$F = T(\Phi^{-1}(P_U(u))),$$

(4.51)

where $T(\cdot)$ is given in equation (4.50). $F$ must be computed numerically since there is no closed form expression for $\Phi^{-1}$. Figure 4.2 shows a typical pdf and CDF of a truncated Gaussian distribution.
Lognormal Distribution

Let $F$ be a random variable with a shifted and scaled Lognormal distribution, then the inverse PDF of $F$ is defined as

$$P_F^{-1}(\cdot) = \exp \left( m + s \Phi^{-1}(\cdot) \right) + a,$$  \hspace{1cm} (4.52)

where parameters $m$, $s$, and $a$ allow for the shifting and scaling. When translating from an underlying Gaussian random variable, $G$, $F$ is given as

$$F = \mathcal{T}(G) = \exp(m + s \Phi^{-1}(\Phi(G))) + a$$

$$= \exp(m + sG) + a.$$  \hspace{1cm} (4.53)

For a Lognormal random field, a closed form expression exists for its autocorrelation function

$$R_f(\xi) = \exp \left( 2m + s^2(1 + \rho(\xi)) \right) + a^2,$$  \hspace{1cm} (4.54)

where $\rho(\xi) = R_g(\xi)/\sigma_g$ is the normalized autocorrelation function of the Gaussian field. The value of $a$ is constrained so that the autocorrelation of the Lognormal field equals zero when
the autocorrelation function of the Gaussian field equals zero. The physical meaning of this restriction is that two positions far away are uncorrelated (recall the fields considered are zero mean). Therefore, once the parameters $m$ and $s$ are chosen, $a$ is given by

$$a = -\sqrt{\exp(2m + s^2)}.$$  

(4.55)

Transforming from a U-beta random variable, $U$, with PDF $P_U(u)$ is done by tranforming to a Gaussian, and then to a Lognormal:

$$F = T(\Phi^{-1}(U)) = \exp(m + s\Phi^{-1}(P_U(u))) + a.$$  

(4.56)

The Lognormal distribution has been extensively used to model material properties because it returns strictly positive values and is related to the Gaussian distribution by a simple transformation. Furthermore, closed form expressions exist for its joint pdf and autocorrelation function. Nonetheless, the Lognormal distribution is unbounded and thus violates the conditions in section 2.2 to guarantee a well-posed problem. The Lognormal distribution should be used with caution when modeling constitutive heterogeneity through simulation. It is important to make sure that the simulation algorithm accurately captures the statistical correlation functions of interest. For example, for the problems studied in this thesis up to the fourth order moments of the flexibility affect the response variance. Thus the simulations must be able to accurately produce the $S_{22}(\kappa)$ term. Figure 4.3 shows the pdf and CDF of a typical Lognormal distribution.
4.5 Family of SDFs

There are no closed form solutions, in general, for the SDFs of the non-Gaussian associated or translated fields. The SDFs of the non-Gaussian fields outlined in section 4.4 must be computed from the underlying fields as detailed in sections 4.2.3 and 4.3.2. Thus, the family of SDFs of the underlying fields are first decided upon, and then are transformed to the family of SDFs of $f(x)$. It is the SDFs of transformed field $f(x)$ that comprise the matrix of equation (4.4). The requirements of the structure of the family of SDFs are such that

- Each SDF in a family has its power distributed over the wave number domain uniquely relative to the others such that the effect of power concentrated at each wave number within a threshold, $\kappa_u$, is captured when considering the entire family.
- Each SDF has the same variance.
- The SDFs are organized in a way such that the condition number of the matrix is minimized.

The structure determined to sufficiently satisfy these conditions is one where the SDFs of the underlying fields have identical shape, and every SDF is related to its adjacent one by a shift.
of the amount $\Delta \kappa$, which is the discretization size of the domain. In order for the variance to be same for every SDF, the SDFs are defined in a circulant manner: as the shifts occur, the values that would extend beyond the threshold, $\kappa_u$, are carried over to the beginning at $\kappa = 0$. This point is clarified in figures [4.4 - 4.9] where the wave number is discretized such that $N = 128$ and $\kappa_u = 2\pi$. The general shape that all SDFs in the family have in common is termed the parent SDF and is denoted as $S_p$. Given a parent SDF, $S_p$, the $n^{th}$ SDF of the family is defined as

$$S_{f_n}(\kappa_j) = \begin{cases} S_p(\kappa_j + \kappa_u - n\Delta \kappa + \Delta \kappa), & 0 \leq \kappa_j \leq (n - 1)\Delta \kappa \\ S_p(\kappa_j - n\Delta \kappa), & n\Delta \kappa \leq \kappa_j \leq \kappa_u, \end{cases}$$

(4.57)

for $\kappa \in [0 \quad \kappa_u]$, and is symmetric about $\kappa = 0$ (ie $S_{f_n}(\kappa) = S_{f_n}(-\kappa)$). The condition number of a matrix indicates the degree to which numerical errors are encountered when it is inverted, and is defined as

$$\text{Cond}(S) = \|S\| \cdot \|S^{-1}\|,$$

(4.58)

where $\|\cdot\|$ is the matrix norm. A low condition number signifies a well behaved matrix whose inverse can be computed accurately [48].

The parent SDFs considered in establishing the GVRF should be significantly different in order to establish the GVRF’s independence of the fields’ spectral characteristics. For the numerical examples in chapters 5 and 7 the following three parent SDFs of the underlying fields are considered:

$$S_{p_1}(\kappa) = \exp(-2|\kappa|)$$

$$S_{p_2}(\kappa) = 16000\kappa^2\exp(-40|\kappa|)$$

$$S_{p_3}(\kappa) = \delta(\kappa)$$

(4.59)
The variance of the three parent SDFs equals one, so that the variances of the non-Gaussian fields shall be determined by the parameters of their PDF given in section 4.4. $S_{p_3}$ corresponds to the underlying U-beta field, while $S_{p_1}$ and $S_{p_2}$ correspond to the underlying Gaussian field. The key characteristic that distinguishes the parent SDFs, as can be seen in figures 4.4 to 4.9, is the degree to which power is spread over the wave number domain: $S_{p_1}$ is slowly decaying, $S_{p_2}$ is moderately decaying, and $S_{p_3}$ is entirely concentrated at one wave number. The correlation structure of the different non-Gaussian fields tested will range from one having no distinct dominant frequency to one where there is only one frequency.

$^1$Note for shifts of the amount $n\Delta \kappa (n \neq 0)$, the SDF, $S_{f_n} = \frac{1}{2} [\delta (\kappa - n\Delta \kappa) + \delta (\kappa + n\Delta \kappa)]$ to keep the variance equal to 1.
Figure 4.4: Selected Members of the Family SDFs for $S_{p1}$
Figure 4.5: Selected SDFs of a Truncated Gaussian Field Corresponding to the Underlying SDFs of Figure 4.4
Figure 4.6: Selected Members of the Family SDFs for $S_{p2}$
Figure 4.7: Selected SDFs of a Uniform Field corresponding to the underlying SDFs of figure 4.6.
Figure 4.8: Selected Members of the Family of SDFs for $S_{p3}$
Figure 4.9: Selected SDFs of a Lognormal Field Corresponding to the Underlying SDFs of Figure 4.8
4.6 Summary of GVRF Methodology

For the sake of the reproducibility of the examples presented in subsequent chapters, this section concisely presents the methodology for computing the GVRF for a structure. The procedure outlined below provides a detailed description of the flow chart in figure 4.10.

- Choose the parent family of SDFs, $S_{p_i}$, from which the GVRF is to be determined. At this stage, it is necessary to determine upper cutoff wave number, $\kappa_u$, and the number of discretization points, $N$. The cutoff wave number should be a value such that the variance of the response is less than a small tolerance $\epsilon$ due to an input field with dominant frequency $\kappa_u$. It was shown in reference [8] that in the limit of the frequency going to infinity, the variance of the response converges to zero.

- Choose the non-Gaussian PDF to represent the heterogeneous flexibility.

The following two items can be done in parallel:

- Compute the matrix of SDFs for random field $f(x)$ as detailed in section 4.2.3 for the translation field and section 4.3.2 for the associated field.

- Perform $N$ Monte Carlo Simulation for the $N$ different SDFs of $S_{p_i}$.
  - Simulate the underlying random field as detailed in section 4.2.4 for the translation field and section 4.3 for the associated field.
  - Transform to the desired non-Gaussian field as detailed in section 4.2 for the translation field and section 4.3.1 for the associated field.
  - Solve the deterministic Structural Mechanics problem and save the response quantities of interest.
  - After reaching a desired number of simulations to ensure convergence, compute
the variance of the response, $u(x)$, as follows:

$$
\mathbb{E} [u(x)] = \frac{1}{N_{mc} - 1} \sum_{j=1}^{N_{mc}} u(x)_j
$$

$$
\mathbb{E} [u(x)^2] = \frac{1}{N_{mc} - 1} \sum_{j=1}^{N_{mc}} u(x)_j^2
$$

$$
\text{Var} [u(x)] = \mathbb{E} [u(x)^2] - \mathbb{E} [u(x)]^2.
$$

(4.60)

- Assemble the vector of $N$ response variances for the $N$ SDFs, and solve the system of linear equations to obtain the GVRF for this particular random field.

- Determine if the chosen values for $\kappa_u$ and $N$ are acceptable. If not, change the values of $\kappa_u$ and $N$ and repeat the procedure for the same random field; else repeat the entire procedure for different PDFs and families of SDFs. Once a sufficient number of PDFs/SDFs have been tested, it should be examined whether an independent and unique VRF exists for the structure studied.
Figure 4.10: Flowchart for GVRF methodology
4.6.1 Table of Random Field Models

Table 4.1 contains information of the random fields mentioned in sections 4.4 and 4.5 as well as the notation which is adopted for the numerical examples for beam structures (chapter 5 and section 7.4.1).

<table>
<thead>
<tr>
<th>Id.</th>
<th>PDF</th>
<th>$S_{p}$</th>
<th>Parameters</th>
<th>$\sigma_f$</th>
<th>Cond($S_f$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>LN3S1</td>
<td>LN3</td>
<td>$S_{p1}$</td>
<td>$a_t \quad a_u \quad m \quad s \quad a$</td>
<td>$.60$</td>
<td>$20.4$</td>
</tr>
<tr>
<td>LN1S1</td>
<td>LN1</td>
<td>$S_{p1}$</td>
<td>$- \quad - \quad -.45 \quad \sqrt{.45} \quad -.799$</td>
<td>$.20$</td>
<td>$20.4$</td>
</tr>
<tr>
<td>LN3S2</td>
<td>LN3</td>
<td>$S_{p2}$</td>
<td>$- \quad - \quad -.45 \quad \sqrt{.45} \quad -.798$</td>
<td>$.57$</td>
<td>$1.30$</td>
</tr>
<tr>
<td>LN1S2</td>
<td>LN1</td>
<td>$S_{p2}$</td>
<td>$- \quad - \quad -.45 \quad \sqrt{.45} \quad -.799$</td>
<td>$.20$</td>
<td>$20.4$</td>
</tr>
<tr>
<td>LN3S3</td>
<td>LN3</td>
<td>$S_{p3}$</td>
<td>$- \quad - \quad -.45 \quad \sqrt{.45} \quad -.798$</td>
<td>$.57$</td>
<td>$1.30$</td>
</tr>
<tr>
<td>LN1S3</td>
<td>LN1</td>
<td>$S_{p3}$</td>
<td>$- \quad - \quad .90 \quad .90 \quad .67$</td>
<td>$.20$</td>
<td>$1.52$</td>
</tr>
<tr>
<td>TG3S1</td>
<td>TG3</td>
<td>$S_{p1}$</td>
<td>$- \quad - \quad .90 \quad .90 \quad .67$</td>
<td>$.67$</td>
<td>$22.44$</td>
</tr>
<tr>
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<td>TG1</td>
<td>$S_{p1}$</td>
<td>$- \quad - \quad .90 \quad .90 \quad .67$</td>
<td>$.67$</td>
<td>$3.96$</td>
</tr>
<tr>
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<td>TG3</td>
<td>$S_{p2}$</td>
<td>$- \quad - \quad .90 \quad .90 \quad .67$</td>
<td>$.67$</td>
<td>$3.96$</td>
</tr>
<tr>
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<td>TG1</td>
<td>$S_{p2}$</td>
<td>$- \quad - \quad .90 \quad .90 \quad .67$</td>
<td>$.67$</td>
<td>$20.4$</td>
</tr>
<tr>
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<td>TG3</td>
<td>$S_{p3}$</td>
<td>$- \quad - \quad .90 \quad .90 \quad .67$</td>
<td>$.67$</td>
<td>$1.00$</td>
</tr>
<tr>
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<td>TG1</td>
<td>$S_{p3}$</td>
<td>$- \quad - \quad .90 \quad .90 \quad .67$</td>
<td>$.67$</td>
<td>$1.00$</td>
</tr>
<tr>
<td>UN3S1</td>
<td>UN3</td>
<td>$S_{p1}$</td>
<td>$- \quad - \quad .99 \quad .99 \quad .57$</td>
<td>$.57$</td>
<td>$21.27$</td>
</tr>
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<td>UN1</td>
<td>$S_{p1}$</td>
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<td>$.46$</td>
<td>$21.27$</td>
</tr>
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<td>$S_{p2}$</td>
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<td>$3.77$</td>
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<td>UN1</td>
<td>$S_{p2}$</td>
<td>$- \quad - \quad .80 \quad .80 \quad .46$</td>
<td>$.46$</td>
<td>$3.77$</td>
</tr>
<tr>
<td>UN3S3</td>
<td>UN3</td>
<td>$S_{p3}$</td>
<td>$- \quad - \quad .99 \quad .99 \quad .57$</td>
<td>$.57$</td>
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</tr>
<tr>
<td>UN1S3</td>
<td>UN1</td>
<td>$S_{p3}$</td>
<td>$- \quad - \quad .80 \quad .80 \quad .46$</td>
<td>$.46$</td>
<td>$1.02$</td>
</tr>
</tbody>
</table>

Table 4.1: Parameters of Random Field Models

4.7 Numerical and Computation Issues

Due to the numerical nature of this problem, numerical errors cannot be avoided to some extent. This procedure is very computationally demanding since for each combination of PDF/SDF there are $N \times N_{mc}$ simulations, where $N$ is the number of discretization points in the wave number domain, and $N_{mc}$ is the number of simulations per Monte Carlo procedure. If $N$ is reduced, the condition number of the matrix of SDFs increases, which produces more
error in the solution to the system of linear equations. Furthermore, the vector containing
the variances of the response for the $N$ different SDFs is less smooth with a reduction in $N$.

It has been observed that numerical errors in the solution arise due to the circulant
structure of the matrix in combination with $N$ not being large enough. If the last value of
the computed GVRF is clearly erroneous, the GVRF can be computed using a subset of the
system of equations as follows:

\[
\begin{bmatrix}
S_{f_1}(\kappa_1) & S_{f_1}(\kappa_2) & \ldots & S_{f_1}(\kappa_{N-k}) & S_{f_1}(\kappa_{N-k+1}) & \ldots \\
S_{f_2}(\kappa_1) & S_{f_2}(\kappa_2) & \ldots & S_{f_2}(\kappa_{N-k}) & S_{f_2}(\kappa_{N-k+1}) & \ldots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
S_{f_{N-k}}(\kappa_1) & S_{f_{N-k}}(\kappa_2) & \ldots & S_{f_{N-k}}(\kappa_{N-k}) & S_{f_{N-k}}(\kappa_{N-k+1}) & \ldots \\
S_{f_{N-k+1}}(\kappa_1) & S_{f_{N-k+1}}(\kappa_2) & \ldots & S_{f_{N-k+1}}(\kappa_{N-k}) & S_{f_{N-k+1}}(\kappa_{N-k+1}) & \ldots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
S_{f_N}(\kappa_1) & S_{f_N}(\kappa_2) & \ldots & S_{f_N}(\kappa_{N-k}) & S_{f_N}(\kappa_{N-k+1}) & \ldots
\end{bmatrix}
\times
\begin{bmatrix}
\text{GVRF}(x, \kappa_1) \\
\text{GVRF}(x, \kappa_2) \\
\vdots \\
\text{GVRF}(x, \kappa_{N-k}) \\
\text{GVRF}(x, \kappa_{N-k+1}) \\
\vdots \\
\text{GVRF}(x, \kappa_N)
\end{bmatrix}
= 2\Delta\kappa
\begin{bmatrix}
\text{Var}\{u(x)_1\} \\
\text{Var}\{u(x)_2\} \\
\vdots \\
\text{Var}\{u(x)_{N-k}\} \\
\text{Var}\{u(x)_{N-k+1}\} \\
\vdots \\
\text{Var}\{u(x)_N\}
\end{bmatrix}. \tag{4.61}
\]

Solving for the system of equations enclosed in the boxes reduces errors in the computed
GVRF at higher wave numbers. Equation (4.61) is general, however taking $k$ to be 1 is
usually sufficient.
Chapter 5

Generalized Variability Response Function for Nonlinear Beams:
Numerical Examples

5.1 Introduction

In reference [9], the GVRF has been established for statically indeterminate, linear structures using the methodology outlined in chapter 4. This chapter extends upon the work in reference [9] by investigating if the GVRF methodology is applicable for indeterminate structures with a nonlinear constitutive law. The following nonlinear constitutive law is considered for the structures analyzed

\[ \sigma(x, \epsilon) = E_c(x) \sqrt{|\epsilon|} \text{sgn}(\epsilon), \]  \hspace{1cm} (5.1)

where \( E_c \) is independent of \( \epsilon \). The constitutive law of equation (5.1) is chosen because it is strongly nonlinear, and the derivative is strictly positive, ensuring convergence of the Newton-Raphson iterative scheme employed in the analysis. Also the VRF for statically determinate beams has been proven for this constitutive law in chapter 3 so the GVRF methodology
applied to the statically determinate structure is as a validation of the implementation. For nonlinear problems, the constitutive law is typically written in its incremental form:

$$d\sigma(x, \epsilon) = E_t(x, \epsilon) d\epsilon,$$  \hspace{1cm} (5.2)

and the tangent modulus, $E_t(x, \epsilon)$ is defined as

$$E_t(x, \epsilon) = \frac{d\sigma(x, \epsilon)}{d\epsilon} = \frac{E_c(x)}{2\sqrt{\epsilon}}.$$  \hspace{1cm} (5.3)

Since $E_t|_{\epsilon=0, \forall x} = \infty$ is unrealistic and incapable of being computed, the following definition of $E_t(x, \epsilon)$ is implemented in the finite element analysis:

$$E_t(x, \epsilon) = \begin{cases} 
E_c(x) \sqrt{\epsilon}, & \epsilon \leq \epsilon_c \\
\frac{E_c(x)}{2\sqrt{\epsilon}}, & \epsilon_c < \epsilon,
\end{cases}$$  \hspace{1cm} (5.4)

where $\epsilon_c$ is taken as $10^{-6}$. Figure 5.1 plots the theoretical constitutive law of equation (5.1) with that implemented numerically using the tangent modulus of equation (5.4).
After a description of the finite element formulation employed for nonlinear beam structures, two numerical examples are presented: a statically determinate cantilever (section 5.3) and a fixed-simply supported beam (section 5.4).

5.2 Finite Element Analysis of Beam Structures

The strong form of a linear, elastic Euler-Bernouli beam

\[(EI(x)u^\prime\prime\prime) = q(x)\]

\[u^\prime|_\Gamma = \ddot{u} \text{ or } (EI(x)u(x)^\prime\prime)|_\Gamma = \ddot{M} \text{ and } (u|_\Gamma = \ddot{u} \text{ or } (EI(x)u(x)^\prime)|_\Gamma = \ddot{V}),\]

where \(\Gamma\) is the boundary, is converted to the weak form by multiplying by a kinematically admissible displacement field, \(\delta w(x), (\delta w(x)|_\Gamma = 0)\) integrating over the domain, and utilizing the divergence theorem

\[\int_0^L EI(x)u''\delta w''dx + (EI(x)u'')'\delta w|_\Gamma - EIu''\delta w'|_\Gamma = \int_0^L q(x)\delta w(x)dx. \]

Following the Galerkin based finite element method, the virtual displacement field is taken to be the real displacement field, \(w(x) = u(x)\) on the interior of the structure. The energy functional \(\Pi\) is defined such that its partial with respect to \(u\) gives the weak form equations, leading to the expression:

\[\Pi = \frac{1}{2} \int_0^L EI(x)u''^2dx - \int_0^L q(x)u(x)dx.\]

The beam is discretized into \(N\) elements, and the displacement field within each element is described by a superposition of continuous shape functions scaled by finite nodal displacement
quantities representing the degrees of freedom of the system:

\[ u(x) = \sum_{j=1}^{4} N_j \cdot u_j, \text{ where} \]

\[ N_1(x) = 1 - \frac{3x^2}{L_e^2} + \frac{2x^3}{L_e^3} \]
\[ N_2(x) = x - \frac{2x^2}{L_e} + \frac{x^3}{L_e^2} \]  \hspace{1cm} (5.8)
\[ N_3(x) = \frac{3x^2}{L_e^2} - \frac{2x^3}{L_e^3} \]
\[ N_4(x) = -\frac{2x^2}{L_e} - \frac{x^3}{L_e^2} \]

The nodal displacements, \( u_j \) can be seen in figure 5.2.

The energy function can be written as

\[ \Pi = \frac{1}{2} Ku - Pu. \]  \hspace{1cm} (5.9)

The element stiffness matrix, \( k \), is given as

\[ k = \int_{0}^{L_e} B(x)^T EI(x) B(x) dx, \]  \hspace{1cm} (5.10)
where $B_j(x) = N_j(x)'$, and the element nodal load vector, $p$ is given by

$$p = \int_0^{L_e} q(x)N(x)dx.$$  \hfill (5.11)

The global load vector, $P$, and global stiffness matrix, $K$, are assembled by transforming the element nodal quantities from the element coordinate system to the global coordinate system and using the principle of superposition when applicable. Equating the first variation of the functional to zero gives the equilibrium conditions

$$\delta \Pi = Ku - P = 0.$$ \hfill (5.12)

For nonlinear constitutive laws, the strong form of equation (5.5) cannot be obtained in general. However, the energy functional can be defined, and is expressed as:

$$\Pi = b\int_0^L \int_{-h/2}^{h/2} \int_0^\epsilon \sigma(\epsilon)d\epsilon dy dx - \int_0^L q(x)u(x)dx.$$ \hfill (5.13)

The element stiffness matrices and load vectors can be assembled to obtain a nonlinear system of equations

$$K(U)U = P.$$ \hfill (5.14)

Generally the nonlinear system of equations is solved numerically. The nodal displacements are solved for incrementally by use of the tangent stiffness matrix, defined as

$$K_{t_n} = K(U_n) = \frac{\partial P}{\partial U}_{U=U_n}.$$ \hfill (5.15)

The global tangent stiffness matrix is assembled from the element tangent stiffness matrices, $k_{t_n}$

$$k_{t_n} = b\int_0^{L_e} B(x)^TB(x)\int_{-h/2}^{h/2} \frac{d\sigma}{d\epsilon} \epsilon dy dx.$$ \hfill (5.16)
The solution for $U$ is computed using the Newton-Raphson iteration procedure

$$
U_{n+1} = U_n + K_{t_n}^{-1} (P - K_{t_n} U_n).
$$

(5.17)

Details on the Newton-Raphson method, such as derivation and proofs of convergence, can be found in reference [49]. Proofs for the uniqueness of the finite element solution, convergence to the exact solution, and detailed derivations can be found in references [50, 51].

An approximation is introduced in the analysis of the following examples: the discretization is small enough so that the heterogeneous flexibility can be modeled as a piece-wise constant function and is constant for each element. The need for the approximation is because the random fields modeling the stiffness are simulated and do not have a functional form. Substituting the constitutive law of equation (5.1), the tangent element stiffness matrix becomes

$$
k_{t_n} = b \int_0^{L_e} B(x)^T B(x) \int_{-h/2}^{h/2} \frac{E_e(x)}{2\sqrt{\epsilon_n}} y dy dx
$$

(5.18a)

$$
\approx bE_e \int_0^{L_e} B(x)^T B(x) \int_{-h/2}^{h/2} \frac{1}{2\sqrt{\epsilon_n}} y dy dx
$$

(5.18b)

Convergence studies are presented for each structure for deterministic test functions of the heterogeneous stiffness.

5.3 Example 1: Nonlinear Cantilever

Consider the cantilever in figure 5.3 with constitutive law of equation (5.1), where $q(x) = 50$, $M = 3500$, and $L = 16$. Its width and height are taken as: $b = 1$, and $h = 12^{1/3}$, respectively.
The boundary conditions for this structure are

\[ u(0) = u'(0) = 0, \text{ and} \]
\[ \frac{bh^2}{5} E_c(L) \sqrt{\frac{h}{2} u''(L)} = M, \left. \frac{d}{dx} \left[ \frac{bh^2}{2} E_c(x) \sqrt{\frac{h}{2} u''(x)} \right] \right|_{x=L} = 0. \] (5.19)

The heterogeneous component of the tangent modulus is given by

\[ E_c(x) = \frac{E_0}{1 + f(x)}, \] (5.20)

where \( E_0 = 3.25 \times 10^4 \), and \( f(x) \) is a random field described in chapter 4.

In chapter 3 a closed form expression of the displacement field is derived, and given below:

\[ u(x) = \int_0^x \frac{50(x - s)(M - \frac{q}{2}(L - x)^2)^2}{b^2 h^5 E_c(s)^2} \text{sgn}(M - \frac{q}{2}(L - x)^2)ds. \] (5.21)

The finite element code used for the simulations is first validated for the homogeneous, nonlinear stiffness \( E_c \) is constant), and it is plotted with the analytical solution of equation [5.21] in figure 5.4. In order to demonstrate the degree to which the system is nonlinear, figure 5.5 shows the load versus tip deflection curve. The load is incrementally applied by
The finite element solution also needs to be validated due to the piece-wise constant approximation of the heterogeneous stiffness (see equation 5.18b). A deterministic test function is chosen to model the tangent stiffness along the length of the beam. Figure 5.6 shows the
convergence evolution of the finite element solution using a piece-wise constant approximation of the stiffness with that of the “exact” solution, obtained by integrating the continuous modulus function within the element in computing the element stiffness matrix (see equation 5.18a). The test function for the tangent modulus,

\[ E_t(x, \epsilon) = \frac{3.25 \times 10^4}{2\sqrt{\epsilon} \left(1 + .99\cos(2\pi x)\right)} \] (5.22)

is chosen so that its frequency equals the upper cut of frequency, \( \kappa_u = 2\pi \), of the random field models describing the heterogeneous stiffness.

Figure 5.6: Convergence of FE Solution for Cantilever for number of elements, \( N \), using tangent modulus of equation (5.22)

Figures for the GVRFs are located in section 5.5.1
5.4 Example 2: Nonlinear Fixed-Simply Supported Beam

The GVRF is computed for the statically indeterminate fixed-simply supported beam structure in figure 5.7, which has the following boundary conditions:

\[ u(0) = u'(0) = 0 \quad \text{and} \quad \frac{bh^2}{5} E_c(L) \sqrt{\frac{h}{2}} u''(L) = M, u(L) = 0. \] (5.23)

The values for the parameters are, \( L = 16 \), \( q(x) = 50 \), and \( M = 3500 \). \( E_c \) is defined in equation (5.20), and the constitutive law in equation (5.1) is followed.

The degree to which the structure’s response is nonlinear can be seen from the load versus displacement plot in figure 5.8. The displacement considered is located at \( x = \frac{5}{8}L \), and the stiffness is homogeneous.
Figure 5.8: Load Factor, $\xi$, Versus Displacement at $x = \frac{L}{2}$ for Fixed-simply supported beam with Homogeneous, Nonlinear Stiffness

Figure 5.9 shows the convergence of the finite element solution to the exact solution as the number of elements, $N$ increase. The test function heterogeneous stiffness chosen is the same as for the cantilever, defined in equation (5.22). The load factor is defined in the previous numerical example.
Figure 5.9: Convergence of FE Solution for Fixed-simply supported beam for number of elements, \( N \), using tangent modulus of equation (5.22)

Figures for the GVRFs are located in section 5.5.2

5.5 GVRF Plots

The notation for the marginal PDFs and SDFs used to model random field \( f(x) \) is described in section 4.6.1. The number of SDFs per family (see section 4.5) is \( N = 128 \), and upper cutoff wavenumber is \( \kappa_u = 2\pi \). The discretization of each SDF is \( \Delta \kappa = \kappa_u/N = .0491 \). In order to display the characteristics of the GVRFs, the plots are organized as follows:

- For each marginal PDF, GVRFs for every family of SDFs are plotted together.
- For each family of SDFs, the GVRFs for all the marginal PDFs are plotted together.
5.5.1 GVRF Plots for Nonlinear Cantilever

Figure 5.10: GVRFs For Tip Deflection of Cantilever For Each PDF Tested
Figure 5.11: GVRFs For Tip Deflection of Cantilever For Each SDF Tested
Figure 5.12: ALL GVRFs For Tip Deflection of Cantilever

Figure 5.13 contains plots of the variances of the tip deflection of the cantilever computed through Monte Carlo simulation for each SDF of the respective family. These functions in vector form comprise the left hand side of the system of linear equations of equation (4.4).
Figure 5.13: Variances of Tip Deflection of Cantilver For Each PDF Tested
5.5.2 GVRF Plots for Nonlinear Fixed-Simply Supported Structure

Figure 5.14: GVRFs For Deflection at $x = \frac{L}{2}$ For Fixed-Simply Supported Beam For Each PDF Tested
Figure 5.15: GVRFs For Deflection at $x = \frac{L}{2}$ For Fixed-Simply Supported Beam For Each SDF Tested
Figure 5.16: ALL GVRFs for Deflection at $x = \frac{L}{2}$ For Fixed-Simply Supported Beam

Figure 5.17 contains plots of the variances of the deflection of the fixed-simply supported beam at $x = \frac{5}{8}L$ computed through Monte Carlo simulation for each SDF of the respective family. These functions in vector form comprise the left hand side in equation (4.4).
Figure 5.17: Variances of Deflection at $x = \frac{L}{2}$ of Fixed-Simply Supported Beam For Each PDF Tested
5.5.3 Validation of GVRFs

In this section the validity of the $GVRF^*$s are tested. A Monte Carlo simulation is performed to compute the exact response coefficient of variation (COV) for selected marginal PDFs and a specific underlying SDF. This exact response COV is compared with the response COV that is determined by computing the integral expression defining the $GVRF^*$, that is

$$Var[u(L)] = \int_{-\kappa_u}^{\kappa_u} GVRF^*(L, \kappa) \left[ S_1(\kappa) + S_{12}(\kappa) + \frac{1}{4} S_{22}(\kappa) - \frac{1}{4} \sigma^4 \delta(\kappa) \right] d\kappa, \quad (5.24)$$

and then solving for the COV as

$$COV = \frac{\sqrt{Var[u(x)]}}{|E[u(x)]|}. \quad (5.25)$$

The SDF of the underlying Gaussian field, denoted $S_4(\kappa)$ is defined to have a variance of 1 and is given as

$$S_4(\kappa) = \frac{1}{1.023368} \exp(-3k^2). \quad (5.26)$$

Figure 5.18 plots the underlying SDF (equation (5.26)) along with the SDFs and their higher power terms for the three marginal PDFs tested (TG3, BE1, LN3). BE1 refers to a shifted and scaled beta distribution with $\alpha = 3.26$, $\beta = 323.01$, $\sigma = .6$. The lower and upper bounds are $-.99$ and $99$, respectively.
Figure 5.18: SDFs of Underlying Field and Translated Fields for Validation

In figure 5.19 the red line indicates the COV of the tip displacement of the cantilever from Monte Carlo simulation. The blue diamonds indicate the calculated COV by computing the integral in equation (5.24). Note that the COV’s are very large because the mean of the tip displacement is small due to the concentrated moment counteracting the distributed loading.
As expected, the results are close since the $GVRF^*$s are very close to the exact solution. It is safe to conclude that the errors are strictly numerical. Note that the number of elements chosen in the finite element analysis is $N = 168$ due to computational cost considerations. See figure 5.6(a) for the discrepancy associated with this solution and the exact solution.

For the fixed-simply supported beam the errors are too large to be considered only numerical. Figure 5.20 plots the results of the validation tests. The red line indicates the COV of the midspan displacement computed through Monte Carlo simulation, whereas the
blue diamonds are the COVs predicted by computing equation \((5.24)\). The underlying SDFs and marginal PDFs are the same as those used for the validation tests for the cantilever.

![Validation Plots for Fixed-Simply Supported Beam](image)

Figure 5.20: Validation Plots for Fixed-Simply Supported Beam: The red line indicates the response COV due to MCS. The blue diamonds indicate the predicted COV by the GVRFs by computing equation \((5.24)\).

### 5.5.4 Discussion of Results

The GVRF methodology is shown to be capable of reproducing the exact \(VRF^*\) for the statically determinate structure with minor numerical discrepancy. The GVRFs for each combination of SDF and marginal PDF are virtually identical in shape and slightly different in magnitude. When validating the results, there are discrepancies between the
exact COV computed through Monte Carlo simulation and the response COV determined from performing the integration of the definition of $VRF^*$. Since it is known that $VRF^*$ is exact, it is safely concluded that the errors are due to numerical issues in the implementation of the methodology. These discrepancies provide insight into the errors encountered in the validation of the statically indeterminate case.

There are convergence issues in the LN3 case which contribute to the errors in its GVRF, as can be seen in figures 5.10(f) and 5.14(f). The LN3 distribution requires an excessive number of Monte Carlo simulations for convergence due to its large variance and unbounded positive limit. In addition, since the structure is nonlinear (with decaying stiffness), the response statistics takes longer to converge than the input random field. The GVRFs are computed using 409600 deterministic analysis per Monte Carlo simulation, and the results are not as smooth as desired. In order to accurately compute a GVRF from the system of linear equations in (4.7) the response variance of the $N$ Monte Carlo simulations should be a smooth function. Only the fast Monte Carlo method is employed to establish the GVRF for the LN3 distribution since its convergence is much faster than that of an SDF having power distributed over a range of wave numbers. The number of simulations required for the latter case to converge is not practical.

However it is clear from the validation tests in figure 5.20 as well as the $GVRF^*$ plots in figures 5.14 and 5.15 that there are discrepancies that are not entirely explained by the slow convergence of the Monte Carlo simulations. It is observed that there is more of a marginal PDF dependence than a spatial/spectral dependence. In figure 5.20(c) the predicted COV of the S3LN3 case is much more accurate than the other cases. For a specific marginal PDF, the dependence of the $GVRF^*$s from the SDF increases as the higher power terms, $S_{12}(\kappa)$ and $S_{22}(\kappa)$, increase relative to $S_1(\kappa)$. The explanation of the statements above is given next. The Uniform and truncated Gaussian distributions produce a $GVRF^*$ that has properties essentially identical to a VRF. This is also the case for LN1, but errors become obvious for the LN3 case. For the Uniform, truncated Gaussian and LN1 cases, the $S_{12}(\kappa)$, $S_{22}(\kappa)$, and
\( \sigma^4 \delta(\kappa) \) terms are quite small in comparison to the \( S_1(\kappa) \) term. In the case of the Uniform and truncated Gaussian distributions, this is because these distributions are bounded between \([0, 1]\), so higher power terms are lower in magnitude than the second order term. Also since both these distributions are symmetric about their means, the term \( S_{12}(\kappa) \) is zero by definition (it is a measure of the skewness of the distribution). For the LN1 distribution, the variance is small enough such that the \( S_{12}(\kappa), S_{22}(\kappa), \text{ and } \sigma^4 \delta(\kappa) \) terms are small relative to \( S_1(\kappa) \). For these cases, the \( GVRF^* \) can approximated as

\[
\text{Var}[u(x)] \approx \int_{-\infty}^{\infty} GVRF^*(x, \kappa) S_1(\kappa) d\kappa.
\]  

(5.27)

This is the same form encountered for the linear case \([9]\).

For the LN3 distribution, the \( S_{12}(\kappa) \) and \( S_{22}(\kappa) \) terms dominate the \( S_1(\kappa) \) term. Figure 5.18 confirms this observation. In figure 5.20(c) the response variance of the LN3 case is significantly off from the predicted variances of the \( GVRF^* \)s. Furthermore the discrepancies between the predicted variance from the S3LN3 and the exact solution (figure 5.20(c)) is worse than the TG3 case (5.20(a)). It is concluded that the static indeterminacy of a nonlinear structure creates a coupling of the applied loading and the SDF/PDF of the input random field which mostly involves the higher power SDFs. Therefore, a few conclusions can be made:

i) In general, a unique VRF does not exist for statically indeterminate nonlinear structures for the sake of accurate prediction of response variances. However suitably approximate \( GVRF^* \)s can be computed with the following categorical information about the SDF/PDF of the input random field:

(a) Approximate \( GVRF^* \)s can be computed for random fields with the same marginal PDFs. It must be kept in mind that the \( GVRF^* \)s dependence of the spatial correlation increases with increasing relative magnitudes of the higher power terms with respect to the SDF.

(b) In general, approximate \( GVRF^* \)s can be computed for random fields that have
similar relative magnitudes between the SDF and higher power terms.

ii The general shape of the GVRF*’s are similar, which implies that information of the effect of correlation length scales on the response variance can be inferred in general.

iii More study is needed to draw firmer conclusions regarding the PDF dependence as well as the effect of the higher power SDFs on the computed GVRF*’s.
Chapter 6

Generalized Variability Response Function Methodology for Two-Dimensional, Linear Problems

6.1 Introduction

In this chapter, the GVRF methodology presented in chapter 4 is extended for structures whose stochasticity is defined in $\mathbb{R}^2$. Since a vast number of structures fall into this category (i.e. plane stress/strain, plate bending, or membrane structures) the applicability of the VRF concept is largely expanded. As there are very few statically determinate two-dimensional structures, numerical techniques are necessary to compute their VRFs. In references [26][27] the fast Monte Carlo method is utilized to compute VRFs for the two dimensional structures. The independence of the PDF/SDF of these computed VRFs have not been evaluated. This can be accomplished through the concepts of the GVRF methodology.

The organization of this chapter is as follows. The two dimensional extension of the GVRF methodology is described followed by a discussion of its limitations. The specific random field models chosen for the numerical example is presented. This is followed by
some mathematical details that the methodology requires: the algorithm for simulating two
dimensional Gaussian and U-beta fields is presented, as well as the expression for the SDFs of
the translated and associated fields. The chapter concludes with an example for a structure
subject to plane stress conditions. The finite element analysis is detailed, and the GVRFs
are presented.

6.2 GVRF Methodology in \( \mathbb{R}^2 \)

Consider a heterogeneous body \( \Omega \) with uncertain material properties whose geometry is
described by coordinates \( x \in \mathbb{R}^2 \). Let its displacement response under a given set of loading
and boundary conditions be \( u(x) \), then the goal is to establish a VRF defined in the following
integral expression

\[
\text{Var} \left[ u(x) \right] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} VRF(x, \kappa_1, \kappa_2)S_f(\kappa_1, \kappa_2)d\kappa_1 d\kappa_2
\] (6.1)

that is unique and independent of the distributional and spectral characteristics of random
field \( f(x) \). \( f(x) \) is a statistically homogeneous random field modeling the system stochasticity.
The domain, \( (\kappa_1, \kappa_2) \in [0 \kappa_u] \times [0 \kappa_u] \) is discretized into \( N \times N \) points, such that \( \Delta \kappa_1 = \Delta \kappa_2 = \Delta \kappa = \kappa_u/N \). The integral is computed numerically such that

\[
\text{Var} \left[ u(x) \right] = 4(\Delta \kappa)^2 \sum_{j=1}^{N} \sum_{l=1}^{N} VRF(x, \kappa_j, \kappa_l)S(\kappa_j, \kappa_l).
\] (6.2)

Analogous to the one-dimensional case, a family of SDFs is formed to create a system of linear
equations relating the variance and the VRF. The structure of the family of SDFs is such that
the effect of power concentrated at each discrete wave number \( (\kappa_j, \kappa_l) \) is considered, each SDF
in the family is equal in shape and in variance, and they are defined in a circulant manner.
See section 4.5 for more details. Given a parent SDF, \( S_p(\kappa_1, \kappa_2) \), the \( (n, m)^{th} \) member in the
family is defined as

\[
S_{nm}(\kappa_j, \kappa_l) = \begin{cases} 
S_p(\kappa_j + \kappa_u - n\Delta\kappa + \Delta\kappa, \kappa_l + \kappa_u - m\Delta\kappa + \Delta\kappa), \\
n\Delta\kappa \leq \kappa_j \leq \kappa_u, \quad 0 \leq \kappa_l \leq (m-1)\Delta\kappa \\
S_p(\kappa_j - n\Delta\kappa, \kappa_l + \kappa_u - m\Delta\kappa + \Delta\kappa), \\
0 \leq \kappa_j \leq (n-1)\Delta\kappa, \quad m\Delta\kappa \leq \kappa_l \leq \kappa_u \\
S_p(\kappa_j - n\Delta\kappa, \kappa_l - m\Delta\kappa), \\
n\Delta\kappa \leq \kappa_j \leq \kappa_u, \quad m\Delta\kappa \leq \kappa_l \leq \kappa_u 
\end{cases}
\] (6.3)

The indices \(n, m\) are integers \(\in [1, N]\). Figure 6.1 shows selected members of a family of SDFs of an underlying Gaussian random field corresponding to family \(S_{p_1}\) in the example in section 6.4.
Figure 6.1: Selected SDFs from a family of an underlying Gaussian Random Field

There are two ways to organize the matrix of $N^2$ SDFs. One way is to create a 3D matrix, $\tilde{S}$ such that

$$\tilde{S}_{jlp} = S_{nm}(\kappa_j, \kappa_l),$$  \hspace{1cm} (6.4)
where \( p = N(n - 1) + m \). The set of linear equations relating the GVRF to the variance is

\[
\text{Var}[u(x)_p] = 4\mathbf{S}_{ijp} \cdot \text{GVRF}(x, \kappa_j, \kappa_l) \cdot (\Delta \kappa)^2
\]  

(6.5)

Alternatively, the matrix can be a two-dimensional matrix defined as

\[
\mathbf{\bar{S}}_{pq} = S_{nm}(\kappa_j, \kappa_l),
\]  

(6.6)

where \( p = N(n - 1) + m \), and \( q = N(j - 1) + l \). The system of equations to determine the GVRF becomes

\[
\text{Var}[u(x)_p] = 4\mathbf{\bar{S}}_{pq} \cdot \text{GVRF}_q(\Delta \kappa)^2
\]  

(6.7)

with \( \text{GVRF}_q = \text{GVRF}(\kappa_j, \kappa_l) \).

The procedure for computing the GVRF in two-dimensions is essentially identical to that of the one-dimensional case. The non-Gaussian fields representing the uncertain system parameters are transformed from simulated Gaussian or U-beta fields. The transformation is identical to the one-dimensional case since the transformations are memoryless. The response variance is computed through Monte Carlo simulation \( N^2 \) times for each combination of marginal PDF and family of SDFs. The simulation of two-dimensional Gaussian and U-beta fields and the computation of the non-Gaussian SDFs are straightforward extensions from the one-dimensional case. The simulation of two-dimensional Gaussian fields is discussed in section 6.2.3 and the computation of the SDF for translated fields is given in section 6.2.4. The simulation of the two-dimensional U-beta field is given in section 6.2.5 and the computation of the SDF for associated fields is given in section 6.2.6.

The biggest issue encountered in the two-dimensional extension of the methodology is solving the system of linear equations (6.4) or (6.6) for the GVRF. The two matrices defined in equations (6.4) and (6.6) cannot be inverted because there are no inversion techniques available for three-dimensional matrices, and the condition number of the matrix in equation
(6.6) is too large. However, there are various solvers available that can be implemented to compute the GVRF without matrix inversion. The solution for the GVRF using the three-dimensional matrix of equation (6.4) can be determined from the MATLAB routine 'FSOLVE'.

For the example presented in section 6.4, the matrix of equation (6.6) is used to compute the GVRF. A least-squares solution technique is implemented from MATLAB’s library of functions called “LSQNONNEG”, which takes advantage of the fact that the VRF is by definition nonnegative. The convergence of both solution techniques is dependent on the number of discretization points in the wave number domain. 'FSOLVE' does not perform well for the number of points, $N = 16$, chosen in the analysis. When a larger $N$ is chosen, although computational costs rise, the performance of 'FSOLVE’ improves. The routine 'LSQNONNEG” performs much better for the chosen discretization. However, this least squares approach can lose accuracy as the number of equations becomes large. Thus the choice of $N$ is not only determined by the computational cost, but also due to the performance of the equation solver employed to compute the GVRF.

6.2.1 Computational Implementation

The GVRF methodology is quite computationally expensive for two-dimensional problems. The number of Monte Carlo simulations is $N^2$, where $N$ is the number of discretization points for each dimension. In addition, the deterministic finite element analysis is more time consuming for two-dimensional problems than one-dimensional problems as the size of the stiffness matrix can become large. For the example presented in section 6.4, the wave number domain is discretized into a $16 \times 16$ grid. Despite being coarser than desirable, it nonetheless leads to 256 Monte Carlo simulations per family of SDFs. The Monte Carlo simulations are implemented in a parallel Fortran90 code and run on a IBM Bluegene super computer owned by Brookhaven National Laboratory using 4096 processors for 102400 deterministic analyses per Monte Carlo simulation. The average time for all the Monte Carlo simulations for a
family of SDFs is about 5 hours for the linear analysis of the plane stress structure given in section 6.4.

### 6.2.2 Random Field Models Chosen

Two families of SDFs and marginal PDFs are chosen to model the random field. The two SDFs of the underlying fields are

\[
S_{p1}(\kappa_x, \kappa_y) = \frac{2}{\pi} \exp \left( -2(\kappa_x^2 + \kappa_y^2) \right) 
\]

(6.8a)

\[
S_{p3}(\kappa_x, \kappa_y) = \delta(\kappa_x, \kappa_y)^1
\]

(6.8b)

where \( \delta(\kappa_x, \kappa_y) \) is the two dimensional Dirac delta function. \( S_{p1}(\kappa_x, \kappa_y) \) and \( S_{p3}(\kappa_x, \kappa_y) \) refer to the underlying Gaussian and underlying U-beta, respectively. The two marginal PDFs are the shifted and scaled Uniform(UN) and truncated Gaussian(TG) distributions described in section 4.4. For this problem the wave number domain is discretized on a \( 16 \times 16 \) grid between \([0 \ \kappa_u] \times [0 \ \kappa_u]\), where \( \kappa_u = 1.2\pi \). Table 6.1 contains information and notation of the random fields models used for the numerical example in this chapter and the two dimensional example in section 7.4.3.

<table>
<thead>
<tr>
<th>Id.</th>
<th>PDF</th>
<th>( S_p )</th>
<th>Parameters</th>
<th>( \sigma_f )</th>
</tr>
</thead>
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<tr>
<td>TG3S1</td>
<td>TG</td>
<td>( S_{p1} )</td>
<td>- .9</td>
<td>.9</td>
</tr>
<tr>
<td>UN3S1</td>
<td>UN</td>
<td>( S_{p1} )</td>
<td>-.99</td>
<td>.99</td>
</tr>
<tr>
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<td>TG</td>
<td>( S_{p3} )</td>
<td>-.9</td>
<td>.9</td>
</tr>
<tr>
<td>UN3S3</td>
<td>UN</td>
<td>( S_{p3} )</td>
<td>-.99</td>
<td>.99</td>
</tr>
</tbody>
</table>

Table 6.1: Parameters of Random Fields Tested

---

\( ^1 \)Note for shifts of the amount \((m\Delta \kappa, n\Delta \kappa)\) for \( n \neq 0 \) and \( m \neq 0 \), the SDF, \( S_{f,m,n} = \frac{1}{2} \left[ \delta(\kappa_x - m\Delta \kappa, \kappa_y - n\Delta \kappa) + \delta(\kappa_x - m\Delta \kappa, \kappa_y - n\Delta \kappa) \right] \) to keep the variance equal to 1.
6.2.3 Simulating 2D Gaussian Fields Through Spectral Representation

The simulation of multidimensional Gaussian fields by means of the Spectral Representation Method is thoroughly described in reference [52], and is a straightforward extension from the simulation of one dimensional Gaussian fields outlined in section 4.2.4. An overview of the method is given below. Let $f_0(x_1, x_2)$ be a two-dimensional univariate homogeneous Gaussian field with mean value equal to zero and SDF $S_{f_0}(\kappa_1, \kappa_2)$, which is symmetric about $\kappa_2 = 0$. Extending the spectral representation theory of [44] [45] to two-dimensions, $f_0(x_1, x_2)$ can be written as

$$f_0(x_1, x_2) = \int_{-\infty}^{\infty} \int_{0}^{\infty} \cos(\kappa_1 x_1 + \kappa_2 x_2) du(\kappa_1, \kappa_2) + \sin(\kappa_1 x_1 + \kappa_2 x_2) d\nu(\kappa_1, \kappa_2), \quad (6.9)$$

where $u(\kappa_1, \kappa_2)$ and $\nu(\kappa_1, \kappa_2)$ are mutually orthogonal real fields along with their orthogonal increments, $du(\kappa_1, \kappa_2)$, $d\nu(\kappa_1, \kappa_2)$. These fields and their increments are defined over $\kappa_1 \in [-\infty, \infty]$, $\kappa_2 \in [0, \infty]$, and the following equations hold:

$$\mathbb{E}[u(\kappa_1, \kappa_2)] = \mathbb{E}[\nu(\kappa_1, \kappa_2)] = 0 \quad (6.10a)$$

$$\mathbb{E}[u^2(\kappa_1, \kappa_2)] = \mathbb{E}[\nu^2(\kappa_1, \kappa_2)] = 2S_{f_0}(\kappa_1, \kappa_2) \quad (6.10b)$$

$$\mathbb{E}[du(\kappa_1, \kappa_2)] = \mathbb{E}[d\nu(\kappa_1, \kappa_2)] = 0 \quad (6.10c)$$

$$\mathbb{E}[du^2(\kappa_1, \kappa_2)] = \mathbb{E}[d\nu^2(\kappa_1, \kappa_2)] = 2S_{f_0}(\kappa_1, \kappa_2)d\kappa_1d\kappa_2 \quad (6.10d)$$

$$\mathbb{E}[u(\kappa_1, \kappa_2)\nu(\kappa'_1, \kappa'_2)] = 0 \quad \text{if} \quad \kappa_1 \neq \kappa'_1 \quad \text{or} \quad \kappa_2 \neq \kappa'_2 \quad (6.10e)$$

$$\mathbb{E}[du(\kappa_1, \kappa_2)d\nu(\kappa'_1, \kappa'_2)] = 0 \quad \text{if} \quad \kappa_1 \neq \kappa'_1 \quad \text{or} \quad \kappa_2 \neq \kappa'_2 \quad (6.10f)$$

$$\mathbb{E}[du(\kappa_1, \kappa_2)du(\kappa'_1, \kappa'_2)] = 0 \quad \text{if} \quad \kappa_1 \neq \kappa'_1 \quad \text{or} \quad \kappa_2 \neq \kappa'_2 \quad (6.10g)$$

$$\mathbb{E}[d\nu(\kappa_1, \kappa_2)d\nu(\kappa'_1, \kappa'_2)] = 0 \quad \text{if} \quad \kappa_1 \neq \kappa'_1 \quad \text{or} \quad \kappa_2 \neq \kappa'_2 \quad (6.10h)$$
where it is assumed that \( f_0(x_1, x_2) \) is associated with a derivative field \( F_0(x_1, x_2) \) such that

\[
\frac{\partial^2 S_{F_0}(\kappa_1, \kappa_2)}{\partial \kappa_1 \partial \kappa_2} = S_{f_0}(\kappa_1, \kappa_2). \tag{6.11}
\]

It can be shown that \( f_0(x_1, x_2) \) defined in equation (6.9) has mean value equal to zero, and autocorrelation function \( R_{f_0}(\xi_1, \xi_2) \), defined as the inverse Fourier transform of \( S_{f_0}(\kappa_1, \kappa_2) \). The integral in (6.9) can be replaced by an infinite summation given by

\[
f_0(x_1, x_2) = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \cos(\kappa_{1m} x_1 + \kappa_{2n} x_2) du(\kappa_{1m}, \kappa_{2n}) + \sin(\kappa_{1m} x_1 + \kappa_{2n} x_2) d\nu(\kappa_{1m}, \kappa_{2n}) \tag{6.12}
\]

where \( \kappa_{1m} = m \Delta \kappa_1 \), and \( \kappa_{2n} = n \Delta \kappa_2 \). It is observed that if the incremental fields are defined as

\[
du(\kappa_{1m}, \kappa_{2n}) = 2 \sqrt{S_{f_0}(\kappa_{1m}, \kappa_{2n})} \Delta \kappa_1 \Delta \kappa_2 \cos \Phi_{mn} \tag{6.13a}
\]

\[
d\nu(\kappa_{1m}, \kappa_{2n}) = -2 \sqrt{S_{f_0}(\kappa_{1m}, \kappa_{2n})} \Delta \kappa_1 \Delta \kappa_2 \sin \Phi_{mn} \tag{6.13b}
\]

where \( \Phi_{mn} \) are independent identically distributed Uniform random phase angles between \([0, 2 \pi]\) that the requirements of equations (6.10a) to (6.10h) hold and that \( f_0(x_1, x_2) \) is zero mean with SDF \( S_{f_0}(\kappa_1, \kappa_2) \). Utilizing trigonometric identities, \( f_0(x_1, x_2) \) can be written as

\[
f_0(x_1, x_2) = 2 \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \sqrt{S_{f_0}(\kappa_{1m}, \kappa_{2n})} \cos(\kappa_{1m} x_1 + \kappa_{2n} x_2 + \Phi_{mn}) + \sqrt{S_{f_0}(\kappa_{1m}, -\kappa_{2n})} \cos(\kappa_{1m} x_1 - \kappa_{2n} x_2 + \Phi_{mn}^{(2)}) \tag{6.14}
\]

In order to simulate \( f_0(x_1, x_2) \), the infinite summation must be truncated to a finite summation. Upper wave numbers are specified such that

\[
\int_{-\infty}^{\infty} \int_{0}^{\kappa_1 u} S_{f_0}(\kappa_1, \kappa_2) d\kappa_1 d\kappa_2 - \int_{-\kappa_{1u}}^{\kappa_{1u}} \int_{0}^{\kappa_{2u}} S_{f_0}(\kappa_1, \kappa_2) d\kappa_1 d\kappa_2 < \epsilon \tag{6.15}
\]
where $\epsilon$ is an arbitrarily small number. Then $\Phi^{(1)}_{mn}$ and $\Phi^{(2)}_{mn}$ are two-dimensional arrays of length $M \times N$, and the discretized wave number domain is given by

$$\Delta \kappa_1 = \frac{\kappa_1}{M} \Rightarrow \kappa_{1m} = m \Delta \kappa_1 \quad (6.16a)$$

$$\Delta \kappa_2 = \frac{\kappa_2}{N} \Rightarrow \kappa_{1n} = n \Delta \kappa_1 \quad (6.16b)$$

The summation in equation (6.14) is truncated to give

$$f(x_1, x_2) = 2 \sum_{m=-M}^{M} \sum_{n=0}^{N} \sqrt{S_{f_0}(\kappa_{1m}, \kappa_{2n})} \cos(\kappa_{1m} x_1 + \kappa_{2n} x_2 + \Phi^{(1)}_{mn}) + \sqrt{S_{f_0}(\kappa_{1m}, -\kappa_{2n})} \cos(\kappa_{1m} x_1 - \kappa_{2n} x_2 + \Phi^{(2)}_{mn}). \quad (6.17)$$

where the simulated field is denoted as $f(x_1, x_2)$ to distinguish it from the actual field $f_0(x_1, x_2)$. The spatial domain can be discretized into an $(M \times N)$ array, where

$$x_{1u} = \frac{2\pi}{\Delta \kappa_1} \quad \text{and} \quad x_{2u} = \frac{2\pi}{\Delta \kappa_2} \quad (6.18a)$$

$$\Delta x_1 = x_{1u}/M_x \quad \text{and} \quad \Delta x_2 = x_{2u}/N_x. \quad (6.18b)$$

By taking advantage of the relationship between cosine and the exponential function, equation (6.17) can be written as

$$f(p_1 \Delta x_1, p_2 \Delta x_2) = \Re \left( 2 \sum_{m=-M}^{M} \sum_{n=0}^{N} \sqrt{S_{f_0}(\kappa_{1m}, \kappa_{2n})} \exp(i\phi^{(1)}_{mn}) \right) \right)$$

$$\exp \left( i \frac{2\pi mp_1}{M_x} + \frac{2\pi np_2}{N_x} \right) + \sqrt{S_{f_0}(\kappa_{1m}, -\kappa_{2n})} \exp(i\phi^{(2)}_{mn}) \exp \left( i \frac{2\pi mp_1}{M_x} - \frac{2\pi np_2}{N_x} \right), \quad (6.19)$$

where $\Re$ denotes the real part. The form of equation (6.19) is suitable for rapid computation using the Fast Fourier Transform technique. It can be shown that in the limit as $N, M \to \infty$, ...
each sample function \( f(p_1 \Delta x_1, p_2 \Delta x_2) \) is ergodic in the mean and autocorrelation. Also the collection of \( N_s \) sample functions is ergodic in the mean and autocorrelation as \( N_s \to \infty \).

Equation (6.19) is implemented in the generation of sample functions for the fluctuating component of the heterogeneous compliance modulus.

### 6.2.4 SDF of Translated Fields

The process of computing the SDF of the two-dimensional translated field is identical to the one-dimensional case in section 4.2.3. The \((m, n)^{th}\) autocorrelation of the translated field is given by

\[
\begin{align*}
R_{f_{mn}}(\xi_1, \xi_2) &= \mathbb{E} [f(x_1, x_2)f(x_1 + \tau_1, x_2 + \tau_2)] = \mathbb{E} [\mathcal{T}(g(x_1, x_2)) \\
\mathcal{T}(g(x_1 + \tau_1, x_2 + \tau_2))] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{T}(g_1)\mathcal{T}(g_2)\phi(g_1, g_2, R_{G_n}(\xi))dg_1dg_2.,
\end{align*}
\]

(6.20)

where \( g(x_1, x_2) \) is the underlying Gaussian field and the non-Gaussian translated field \( f(x_1, x_2) = \mathcal{T}(g(x_1, x_2)) \). Then the SDF is given by

\[
S_{f_{mn}}(\kappa_1, \kappa_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{f_{mn}}(\xi_1, \xi_2)\exp(i\kappa_1\xi_1 + i\kappa_2\xi_2).
\]

(6.21)

The autocorrelations and SDFs can be computed by any numerical technique after the integrals are truncated to a finite domain.

### 6.2.5 Simulating 2D U-beta Fields

A two-dimensional U-beta random field is a zero mean sinusoidal field with a random phase shift, \( \theta \in U[0, 2\pi] \), and deterministic amplitude, described as follows:

\[
u(x_1, x_2) = \sqrt{2}\sigma_u \cos(\kappa_1 x_1 + \kappa_2 x_1 + \theta).
\]

(6.22)
The marginal PDF and pdf are defined in section 4.3. The autocorrelation function can easily be determined from its definition

\[ E[u(x_1, x_2)u(x_1 + \xi_1, x_2 + \xi_2)] = \frac{1}{2\pi} \int_0^{2\pi} \sqrt{2\sigma_u} \cos(\kappa_{\delta_1} x_1 + \kappa_{\delta_2} x_2 + \theta) \times \sqrt{2\sigma_u} \cos(\kappa_{\delta_1}(x_1 + \xi_1) + \kappa_{\delta_2}(x_2 + \xi_2) + \theta) d\theta\]

\[ = \sigma_u \cos(\kappa_{\delta_1} \xi_1 + \kappa_{\delta_2} \xi_2) = R_{UV}(\xi_1, \xi_2). \] (6.23)

Following from the Wiener-Khinchin transformation, the SDF is defined by the Dirac Delta function, centered at wave numbers \((\kappa_{\delta_1}, \kappa_{\delta_2})\),

\[ S_u(\kappa_1, \kappa_2) = \frac{\sigma}{2} \left[ \delta(\kappa_1 - \kappa_{\delta_1}, \kappa_2 - \kappa_{\delta_2}) + \delta(\kappa_1 + \kappa_{\delta_1}, \kappa_2 + \kappa_{\delta_2}) \right]. \] (6.24)

Notice that the SDF is nonzero only in the first and fourth quadrant, and this is reflected in the sample function of equation (6.22). Similar to the one-dimensional case, the two-dimensional U-beta distribution is strictly homogeneous due to the shift-invariance of its \(n^{th}\)-order joint pdf.

The simulation of two-dimensional U-beta random fields involves generating samples of a uniformly distributed random variable, \(\theta\). There is no increase in difficulty by expanding to higher dimensions when dealing with U-beta fields. As a result, it has the same convergence properties as the one-dimensional case.

### 6.2.6 SDF of Associated Fields

The family of SDFs of the associated fields must be computed before solving the system of linear equations in (6.7) to determine the GVRF. The process is outlined in section 4.3.2. The derivation is slightly different for the two dimensional case and is detailed below. The
The autocorrelation function of the non-Gaussian associated field is

\[
R_{f,mn}(\xi_1, \xi_2) = \int_0^{2\pi} \mathcal{A}(u(x_1, x_2))\mathcal{A}(u(x_1 + \xi_1, x_2 + \xi_2))f_\Theta(\theta)d\theta
\]

(6.25)

\[
= \frac{1}{2\pi} \int_0^{2\pi} \mathcal{A}(\sqrt{2}\sigma_u \cos(\theta))\mathcal{A}(\sqrt{2}\sigma_u \cos(\kappa_\delta \xi_1 + \kappa_\delta \xi_2 + \theta))d\theta.
\]

Notice that the term involving \((x_1, x_2)\) has been dropped due to the shift invariance of \(u(x_1, x_2)\). The \((m, n)^{th}\) SDF of the underlying field, is defined uniquely by \((\kappa_\delta_1, \kappa_\delta_2)\), such that \((\kappa_\delta_1, \kappa_\delta_2) = ((m - 1)\Delta \kappa, (n - 1)\Delta \kappa)\). Due to the memoryless transformation, \(\mathcal{A}(u(x_1, x_2))\) follows the periodic structure of \(u(x_1, x_2)\) implying the same for \(R_{f,mn}(\xi_1, \xi_2)\). By making the following substitution of variables

\[
\begin{align*}
\xi'_1 &= (m - 1)\Delta \kappa \xi_1 \\
\xi'_2 &= (n - 1)\Delta \kappa \xi_2
\end{align*}
\]

(6.26)

the same observations noted in section 4.3.2 hold for \(R_f(\xi'_1, \xi'_2)\) and is expressed in the following Fourier series representation

\[
R_f(\xi'_1, \xi'_2) = a_{0,0} + \sum_{j=1}^{\infty} a_{0,j} \cos(j\xi'_2) + \sum_{j=1}^{\infty} a_{j,0} \cos(j\xi'_1) + \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} a_{j,l} \cos(j\xi'_1 + l\xi'_2),
\]

(6.27)

where Fourier coefficients, \(a_{j,l}\) equal

\[
\begin{align*}
a_{0,0} &= \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} R_f(\xi'_1, \xi'_2) d\xi'_2 d\xi'_1 \\
&= \frac{1}{8\pi^3} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \mathcal{A}(\sqrt{2}\sigma_u \cos(\theta))\mathcal{A}(\sqrt{2}\sigma_u \cos(\xi'_1 + \xi'_2 + \theta))d\theta d\xi'_2 d\xi'_1
\end{align*}
\]

(6.28a)

\[
\begin{align*}
a_{0,j} &= \frac{1}{2\pi^2} \int_0^{2\pi} \int_0^{2\pi} R_f(\xi'_1, \xi'_2) d\xi'_2 d\xi'_1 \\
&= \frac{1}{4\pi^3} \int_0^{2\pi} \int_0^{2\pi} \cos(j\xi'_2) \int_0^{2\pi} \mathcal{A}(\sqrt{2}\sigma_u \cos(\theta))\mathcal{A}(\sqrt{2}\sigma_u \cos(\xi'_1 + \xi'_2 + \theta))d\theta d\xi'_2 d\xi'_1
\end{align*}
\]

(6.28b)
\[ a_{j,l} = \frac{1}{2\pi^2} \int_0^{2\pi} \int_0^{2\pi} R_f(\xi')d\xi'd\xi'_1 \]
\[ = \frac{1}{4\pi^3} \int_0^{2\pi} \cos(j\xi_1') \int_0^{2\pi} \cos(l\xi_2') \int_0^{2\pi} A(\sqrt{2}\sigma_u\cos(\theta))A(\sqrt{2}\sigma_u) \cos(\xi_1' + \xi_2' + \theta))d\theta d\xi_2' d\xi_1' , \]

and by symmetry, \( a_{0,j} = a_{j,0} \). The Fourier coefficients can be efficiently computed using Gaussian quadrature or a similar numerical technique. \( R_f(\xi_1', \xi_2') \) is independent of \((m, n)\), and the coefficients in equation (6.28) apply to all \( N^2 \) autocorrelation functions. The SDFs can be computed as

\[ S_{f_{mn}}(\kappa_1, \kappa_2) = a_{0,0}\delta(\kappa_1, \kappa_2) + \sum_{j=1}^{\infty} a_{0,j}\delta(\kappa_1, \kappa_2 - j(m - 1)\Delta\kappa) + \]
\[ \sum_{j=1}^{\infty} a_{j,0}\delta(\kappa_1 - j(n - 1)\Delta\kappa, \kappa_2) + \]
\[ \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} a_{j,l}\delta(\kappa_1 - j(n - 1)\Delta\kappa, \kappa_2 - l(m - 1)\Delta\kappa) \] (6.29)

The infinite summation is truncated to a finite one such that wave numbers greater than \( \kappa_u \) are not considered.

### 6.3 Finite Element Analysis of Linear Elastic Plane Stress Problem

For a given body \( \Omega \), the following equilibrium, constitutive, and compatibility laws hold point-wise:

\[ \nabla \cdot \sigma + b = 0 \] (6.30a)
\[ \sigma = D : \epsilon \] (6.30b)
\[ \epsilon = \nabla u. \] (6.30c)
for stress $\sigma$, constitutive tensor $D$, strain $\epsilon$, and displacement $u$. The displacement and traction boundary conditions, $\bar{u} \in \Gamma_u$ and $\sigma \cdot n = t \in \Gamma_t$, respectively, on the boundary $\partial \Omega$ are assumed to be disjoint sets, meaning

$$\Gamma_u \cap \Gamma_t = \emptyset, \quad \text{and} \quad \Gamma_u \cup \Gamma_t = \partial \Omega. \quad (6.31)$$

If $\Omega$ is subject to plane stress conditions, the approximations below hold

$$\sigma_{zz} = \sigma_{yz} = \sigma_{xz} = 0, \quad (6.32)$$

leading to the stress, strain, and displacements vectors and constitutive tensor given by

$$\begin{align*}
\sigma &= \begin{bmatrix} \sigma_{xx} \\
\sigma_{yy} \\
\sigma_{xy} \end{bmatrix}, \quad & \epsilon &= \begin{bmatrix} \epsilon_{xx} \\
\epsilon_{yy} \\
2\epsilon_{xy} \end{bmatrix}, \quad & u &= \begin{bmatrix} u_x(x, y) \\
u_y(x, y) \end{bmatrix} \\
D &= \frac{E(x, y)}{1 - \nu^2} \begin{bmatrix} 1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & \frac{1-\nu}{2} \end{bmatrix}, \quad (6.33) \end{align*}$$

where $E(x, y)$ is the heterogeneous elastic modulus and $\nu$ is Poisson’s ratio. The strains in the $z$ direction are given by

$$\begin{align*}
\epsilon_{zz} &= \frac{-\nu}{E} (\sigma_{xx} + \sigma_{yy}) \quad (6.34a) \\
\epsilon_{yz} = \epsilon_{xz} &= 0. \quad (6.34b) \\
\end{align*}$$

The weak form of the equilibrium condition is derived by multiplying equation (6.30a) by a kinematically admissible virtual displacement field, $\mathbf{v}$, (that is $\mathbf{v} = 0 \in \Gamma_u$), integrating over the domain, and employing the divergence theorem. Assuming the body force is zero, the
weak form is derived as such

\[
\int_\Omega \nabla \cdot \sigma \cdot v d\Omega = \sigma \cdot n \cdot v \bigg|_{\Gamma_t} - \int_\Omega \sigma \cdot \nabla v d\Omega \Rightarrow \\
\int_\Omega \sigma \cdot \nabla v d\Omega - t \cdot v = 0
\]  

(6.35)

The energy functional, \( \Pi \), defined such that \( \partial \Pi / \partial v \) recovers the weak form, is given by

\[
\Pi = \frac{1}{2} \int_\Omega \nabla v : D\nabla u d\Omega - v \cdot t.
\]  

(6.36)

The domain is discretized into quadrilateral elements and the displacement field within each element is described by 4 nodal displacement quantities (degrees of freedom) for each direction and 4 \( C^0 \) shape functions. The virtual displacement field, \( v \) is taken to be the same as the real displacement field. Figure 6.2 shows the nodal displacement quantities. As can be seen in the figure, the gradient of the displacements are expressed in the natural coordinate system, \((\xi, \eta)\).

Figure 6.2: Physical Coordinate System Mapped to Natural Coordinate System for A Quadrilateral Element
The gradients of the physical and natural coordinate system relate by the Jacobian, given as
\[
\begin{bmatrix}
\frac{\partial}{\partial \xi} \\
\frac{\partial}{\partial \eta}
\end{bmatrix}
= \begin{bmatrix}
\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\
\frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial y}
\end{bmatrix}
= [J] \begin{bmatrix}
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial y}
\end{bmatrix}
\].

(6.37)

The strain is the given by
\[
\epsilon = \nabla \mathbf{u} = \mathbf{B}\tilde{\mathbf{u}} = \mathbf{A}\mathbf{J}^{-1}\mathbf{N_d}\tilde{\mathbf{u}},
\]

(6.38)

where
\[
\mathbf{A} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 1 & 0
\end{bmatrix}
\]

(6.39a)

\[
\mathbf{J}^{-1} = \begin{bmatrix}
\mathbf{J}^{-1} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \mathbf{J}^{-1}
\end{bmatrix}
\]

(6.39b)

\[
\mathbf{N_d} = \begin{bmatrix}
\frac{\partial N_1}{\partial \xi} & 0 & \frac{\partial N_2}{\partial \xi} & 0 & \frac{\partial N_3}{\partial \xi} & 0 & \frac{\partial N_4}{\partial \xi} & 0 \\
0 & \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & 0 & \frac{\partial N_3}{\partial \eta} & 0 & \frac{\partial N_4}{\partial \eta} \\
0 & 0 & \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & 0 & \frac{\partial N_3}{\partial \eta} & 0
\end{bmatrix}
\]

(6.39c)

The shape functions can be seen in figure 6.3 and are defined in the natural coordinate system as
\[
N_1(\xi, \eta) = \frac{(1 - \xi)(1 - \eta)}{4}
\]

(6.40a)

\[
N_2(\xi, \eta) = \frac{(1 + \xi)(1 - \eta)}{4}
\]

(6.40b)

\[
N_3(\xi, \eta) = \frac{(1 + \xi)(1 + \eta)}{4}
\]

(6.40c)

\[
N_4(\xi, \eta) = \frac{(1 - \xi)(1 + \eta)}{4}
\]

(6.40d)
Figure 6.3: Shape Functions for Four-Node 2D Elements

The displacement field can be defined in the natural coordinate system as

\[ \mathbf{u} = \mathbf{N} \tilde{\mathbf{u}} \]  \hspace{1cm} (6.41)

where

\[ \tilde{\mathbf{u}} = \begin{bmatrix} \tilde{u}_1 & \tilde{u}_2 & \tilde{u}_3 & \tilde{u}_4 & \tilde{u}_5 & \tilde{u}_6 & \tilde{u}_7 & \tilde{u}_8 \end{bmatrix}^T \]  \hspace{1cm} (6.42a)

\[
\mathbf{N} = \begin{bmatrix}
N_1(\xi, \eta) & 0 & N_2(\xi, \eta) & 0 & N_3(\xi, \eta) & 0 \\
0 & N_1(\xi, \eta) & 0 & N_2(\xi, \eta) & 0 & N_3(\xi, \eta) \\
N_4(\xi, \eta) & 0 \\
0 & N_4(\xi, \eta)
\end{bmatrix} \]  \hspace{1cm} (6.42b)
or can be expressed in the physical coordinate system by making use of the following relation

\[
\begin{bmatrix}
  x(\xi, \eta) \\
y(\xi, \eta)
\end{bmatrix} = \sum_{j}^{4} N_j(\xi, \eta) \begin{bmatrix}
  \bar{x}_j \\
  \bar{y}_j
\end{bmatrix},
\]

where \(\bar{x}\) and \(\bar{y}\) are the \(x, y\) nodal coordinates of the quadrilateral element. The energy functional formulated as

\[
\Pi = \frac{1}{2} \tilde{u}^T K \tilde{u} - \tilde{u}^T P,
\]

where \(K\), and \(P\) are the global stiffness matrix and load vector respectively. By taking the variation with respect to \(\tilde{u}\) and setting to zero, the system of linear equations for the solution of the nodal displacements quantities \(\tilde{u}\) is given by

\[
K \tilde{u} = P
\]

(6.45)

The global stiffness matrix is formulated by superimposing the element stiffness matrices at common degrees of freedom. The element stiffness matrix is

\[
k = \int_{-1}^{1} \int_{-1}^{1} B^T D B |J| d\xi d\eta.
\]

(6.46)

Similarly, the global load vector is formed by superimposing the element load vectors, defined as

\[
p = \int_{-1}^{1} N^T \left| \Gamma_{tn} \right| t(\partial \Gamma_{tp}/\partial \Gamma_{tn}) d\Gamma_{tn}
\]

(6.47)

where \(t\) is the traction loading. The boundary of the element in the physical coordinate system, \(\Gamma_{tp}\) such that \(\Gamma_{tp} \in \Gamma_t\), is transformed into the natural coordinate system as \(\Gamma_{tn}\).
6.4 Numerical Example

Consider the membrane shown in figure (6.4) with its loading and boundary conditions subject to plane stress conditions, where \( L_x = L_y = 10 \), \( q = 100 \), and the thickness is equal to one.

![Figure 6.4: Plane Stress Structure Analyzed](image)

The heterogeneous compliance is defined as

\[
\frac{1}{E(x,y)} = \frac{1 + f(x,y)}{E_0},
\]

where \( f(x,y) \) is a homogeneous, zero mean random field. The SDF of \( f(x,y) \) is defined over a two-dimensional wave number domain discretized such that

\[
\kappa_{xj} = (j - 1)\Delta\kappa \quad \text{(6.49a)}
\]

\[
\kappa_{yj} = (j - 1)\Delta\kappa \quad \text{(6.49b)}
\]
\[ \Delta \kappa = \frac{\kappa_u}{N}, \quad (6.49c) \]

where \( \kappa_u = 1.2\pi \) and \( N = 16 \). The implementation of the finite element code approximates the elastic modulus to be constant within each element. The convergence of the finite element code due to the piece-wise approximate of the elastic modulus is tested for an elastic modulus defined as

\[ E(x, y) = \frac{E_0}{1 + .99\sin(\kappa_u x + \kappa_u y)}. \quad (6.50) \]

Figure 6.5 contains plots of the displacement of the membrane for three different mesh sizes, 14 \( \times \) 14, 25 \( \times \) 25, and 41 \( \times \) 41. The blue mesh is the finite element solution assuming constant elastic modulus for each element, whereas the red mesh considers equation (6.50) exactly within the mesh. The displacements are scaled by a factor of 50 in order to display the differences clearly. Notice that the solution considering the exact elastic modulus differs amongst the different mesh sizes and converges to the exact solution. This is because the meshes size in figures 6.5(a) and 6.5(b) are too coarse to smoothly represent the elastic modulus of equation (6.50). By a mesh size of 41 \( \times \) 41, the exact solution converges, and the piece-wise solution converges to the exact solution.
6.4.1 Results

Referring to section 6.2.2 for notation of the random fields, the GVRFs are plotted individually in figure 6.6 for displacement at \((L_x/2, L_y)\). The variances of the response at this point are plotted in figure 6.8 for the respective random field models of \(f(x, y)\).
The GVRF plots above are plotted again in figure 6.7 at section cuts in order to better visualize the discrepancies amongst the GVRFs. They are plotted along cuts $\kappa_x = 0, \kappa_y = 0$, $\kappa_x = \kappa_y$. 

Figure 6.6: Computed GVRFs for 2D Example at $(L_x/2, L_y)$
Figure 6.7: Computed GVRFs for 2D Example at $(L_x/2, L_y)$ at Various Section Cuts

The variances of the Monte Carlo simulations are plotted below in figure 6.8
6.4.2 Validation of Results

The GVRF methodology is validated by computing the coefficient of variance (COV) of the displacement response at \( (L_x/2, L_y) \) by Monte Carlo simulation for a translated random field with an underlying SDF given by

\[
S_4(\kappa_x, \kappa_y) = \frac{1}{1.045} \exp(-3(\kappa_x^2 + \kappa_y^2)),
\]

(6.51)
and comparing the results to the COV predicted by the GVRFs through the following integration

$$\text{Var} \left[ u\left(\frac{L_x}{2}, L_y\right) \right] = \int_{-\kappa_{xu}}^{\kappa_{xu}} \int_{-\kappa_{yu}}^{\kappa_{yu}} S_4(\kappa_x, \kappa_y) \text{GVRF}(\frac{L_x}{2}, L_y, \kappa_x, \kappa_y) d\kappa_x d\kappa_y,$$

(6.52)

and then applying the definition of the COV

$$\text{COV} = \frac{\sqrt{\text{Var} \left[ u\left(\frac{L_x}{2}, L_y\right) \right]}}{\left| E \left[ u\left(\frac{L_x}{2}, L_y\right) \right] \right|}.$$  

(6.53)

The marginal PDF chosen is LN3 (defined in section 4.4) because from experience it results in the worst case scenario in terms of errors compared to other distributions tested (see section 5.5.3). Refer to section 6.2.2 for notation.

Figure 6.9: Validation Plots for example in section 6.4. The red line indicates the response COV due to MCS. The blue diamonds indicate the predicted COV by the GVRFs by computing equation (6.52).
6.5 Conclusions

It appears that the GVRF methodology is applicable to two-dimensional problems since the GVRFs computed exhibit some degree of independence from the distributional and spectral characteristics of $f(x, y)$. They are the same in shape and slightly off in magnitude. Due to the computational expense, the mesh used is too coarse. This is a likely explanation for the lack of smoothness in the results for the fields translated from an underlying Gaussian. The fast Monte Carlo approach does produce very smooth results because the SDF has its power almost entirely concentrated at one point, implying that the solution is accurate so long as the mesh includes these points. Once the GVRF methodology is deemed to be applicable for a class of structures, the fast Monte Carlo approach should be used to compute GVRFs.
Chapter 7

Stochastic Characterization of Effective Properties for Linear Structures via the Variability Response Function

7.1 Introduction

In this chapter the VRF concept is applied to the homogenization of elastic material properties. Homogenization of randomly heterogeneous material properties into effective properties occurs, often implicitly, when conducting standard tests such as tensile tests, direct shear tests, v-notch test, creep tests, and others. This is because most materials consist of random, heterogeneous morphologies at the meso- and microscales. Much work has been done to establish bounds of effective properties. The most commonly cited pioneering scientific inquiry into the bounds of effective properties are the works of Voigt and Reuss, both published in German in 1889 and 1929, respectively, from which the famous Reuss and Voigt bounds orginate. The Voigt bound is also called the isostrain average because it
gives the ratio of average stress to average strain when the material is uniformly strained. This gives the upper bound of the elastic modulus. The Reuss bound, called the isostress average, is the ratio of average stress to average strain when the material undergoes constant stress. This establishes the lower bound of the elastic modulus. There is a large body of literature published between the 1960s through the 1980s attempting to establish bounds of effective properties of heterogeneous materials, and a few seminal works are mentioned next. In references [53], a variational approach is used to determine bounds of the elastic and shear moduli of spherical inclusions for very low or high concentrations. The variational theory is further developed in [54, 55] to establish formulas for the bounds of effective elastic moduli for multiphase materials. In [56] the theory of two-phase random media is developed as well as an early explanation of the representative volume element (RVE). In [57, 58] exact infinite series solutions of the bounds of the effective elastic modulus of spherical inclusions for periodic structures are derived.

It is shown in [59] that effective properties are deterministic when the structure considered is sufficiently larger than the correlation length scale of uncertain heterogeneities modeled as ergodic random fields. The size of the structure where the effective properties become deterministic is the RVE. When the structure considered is within the RVE, the effective properties are random variables [60, 61, 62, 63, 64, 65]. With the growing interest in microscale mechanics, characterizing the probabilistic information of effective properties has become an important research topic. The development of multiscale finite element analysis has provided a means, such as through upscaling techniques, to evaluating the scale-dependence of effective properties [66, 67, 68, 69, 70, 71, 72, 73, 74, 75]. In order to quantify response stochasticity due to uncertainties at finer scales, techniques need to be developed in order to obtain probabilistic information of homogenized effective properties.

In a recent paper [12], the VRF is analytically derived for the effective flexibility for heterogeneous, statically determinate structures. The formulation is a straightforward extension of the derivation of the VRF for the displacement response of statically determinate structures.
The VRF for effective properties has the same properties as that for the displacement response: it is independent of the spectral and distributional characteristics of the random field, it identifies the sensitivity of the variability of the effective properties to the correlation/spectral characteristics of the random field, and it provides a least upper bound on the variance of the effective properties if the variance of the random field is known. The VRF concept for effective properties is also limited to heterogeneous structures described by statistically homogeneous random fields.

The organization of this chapter is as follows. Commonly used homogenization models are presented. Then the existence of a VRF for effective properties for statically determinate beams is derived. The GVRF methodology developed in chapters 4 and 6 is applied to linear, statically indeterminate structures in order to establish the GVRFs for effective properties of these structures. Numerical examples are provided for a fixed-simply supported beam, and a plane stress 2D structure. The chapter concludes with a discussion of the applicability of the GVRF approach for this type of problem.

7.2 Homogenization

Considering a heterogeneous body $\Omega$ described by coordinates $\mathbf{x} \in \mathbb{R}^3$, the strong form of the boundary value problem is written as

$$\sigma_{ij,j} + b_i = 0 \quad (7.1a)$$

$$\sigma_{ij} = C_{ijkl}(\mathbf{x})\epsilon_{kl} \quad (7.1b)$$

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad (7.1c)$$

$$\sigma_{ij}n_j = t_i \in \Gamma_t \quad (7.1d)$$

$$u_i = \bar{u}_i \in \Gamma_u \quad (7.1e)$$
\[ \Gamma_t \cup \Gamma_u = \partial \Omega \text{ and } \Gamma_t \cap \Gamma_u = \emptyset. \quad (7.1f) \]

The constitutive tensor, \( C(x) \), is a function of position due to random fluctuations of the elastic modulus or Poisson’s ratio. Let a homogenized counterpart of \( \Omega \) be a body where its constitutive tensor, \( \bar{C} \), is constant within \( \Omega \) but is a function of the displacement boundary conditions, traction, and an integral expression of \( C(x) \). The effective properties are determined such that the strain energy of the homogenized body equals the strain energy of \( \Omega \) under the same set of loading, that is

\[
\frac{1}{2} \int_{\Omega} \epsilon_0(x) : \bar{C} : \epsilon_0(x) dV = \frac{1}{2} \int_{\Omega} \epsilon(x) : C(x) : \epsilon(x) d\Omega
\]

\[
= \int_{\Gamma_t} u(x) \bar{t}(x) d\Gamma_t, \quad (7.2)
\]

where \( \epsilon_0(x) \) is the strain of the homogenized body. Consider the case where the Poisson’s ratio is constant and only the elastic modulus, \( E(x) \), is heterogeneous. Then the effective elastic modulus, \( \bar{E} \) is written as

\[
\bar{E} = \frac{\int_{\Gamma_t} u(x) \bar{t}(x) d\Gamma_t}{\frac{1}{2} \int_{\Omega} \epsilon_0(x) : \bar{C}' : \epsilon_0(x) dV} = \frac{\int_{\Omega} \epsilon(x) : C(x) : \epsilon(x) d\Omega}{\int_{\Omega} \epsilon(x) : \bar{C} : \epsilon(x) dV} \quad (7.3)
\]

where \( \bar{C}' = \frac{1}{\bar{E}} \bar{C} \). The effective elastic modulus is bounded by the Reuss and Voigt bounds

\[
\bar{E}_v \leq \bar{E} \leq \bar{E}_r \quad (7.4a)
\]

\[
\bar{E}_r = \frac{1}{V_{\Omega}} \left[ \int_{\Omega} E(x)^{-1} dV \right]^{-1} \quad (7.4b)
\]

\[
\bar{E}_v = \frac{1}{V_{\Omega}} \int_{\Omega} E(x) dV \quad (7.4c)
\]

The variance of the homogenized elastic modulus is

\[
\text{Var} [ \bar{E} ] = \frac{1}{\bar{C}'^2} \text{Var} \left[ \int_{\Gamma_t} u(x) \bar{t}(x) d\Gamma_t \right], \quad (7.5)
\]
where \( C_1 = \frac{1}{2} \int_{\Omega} \epsilon_0(x) : C' : \epsilon_0(x) dV \). If the fluctuations of the elastic modulus about its mean value are described by a statistically homogeneous, zero mean random field, \( f(x) \), then the goal of the VRF concept is to establish the following relationship

\[
\text{Var} [E] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_f(\kappa_1, \kappa_2, \kappa_3) VRF_E(\kappa_1, \kappa_2, \kappa_3) d\kappa_1 d\kappa_2 d\kappa_3. \tag{7.6}
\]

Only for specific cases of statically determinate structures can equation (7.6) be proven because, in general, the displacement cannot be described by a function that is separable with respect to the applied traction and the parameters of the constitutive law.

### 7.2.1 Effective Flexibility for Beam Structures

It is more convenient to describe the constitutive law in terms of the flexibility for beam structures, that is

\[
D(x) = \frac{1}{E(x)I} = \frac{1}{E_0I} (1 + f(x)). \tag{7.7}
\]

The external work due to the loading on the heterogeneous beam, \( W \), and that on the homogeneous beam, \( W_H \), under distributed load \( q(x) \), concentrated load \( P \) at \( x_p \), and concentrated moment \( M \) at \( x_M \) are given by

\[
W = \int_0^L u(x)q(x)dx + Pu(x_p) + M\theta(x_M) \tag{7.8a}
\]

\[
W_H = D \left[ \int_0^L u_0(x)q(x)dx + Pu_0(x_p) + M\theta_0(x_M) \right] = C_1 D, \tag{7.8b}
\]

where \( u_0(x) \) is the displacement of the homogenized beam with the flexibility factored out, and \( \theta_0(x) = \frac{du_0(x)}{dx} \) is the slope of the deflected shape with the flexibility factored out. The homogenized flexibility is defined as

\[
\bar{D} = \frac{1}{C_1} \left[ \int_0^L u(x)q(x)dx + Pu(x_p) + M\theta(x_M) \right]. \tag{7.9}
\]
7.3 Derivation of VRF for Effective Flexibility for Statistically Determinate Structures

The following derivation is an extension from the derivation in [12]. For a statically determinate structure, the displacement and its derivative can be written as

\[ u(x) = \int_{0}^{x} G_u(x,s) \frac{M(x)}{E_0 I} (1 + f(s)) ds \] \hfill (7.10a)

\[ \theta(x) = \int_{0}^{x} G_\theta(x,s) \frac{M(x)}{E_0 I} (1 + f(s)) ds. \] \hfill (7.10b)

The expected value of the effective flexibility is

\[
C_1 \mathbb{E} [\bar{D}] = \mathbb{E} \left[ \int_{0}^{L} q(x) \int_{0}^{x} G_u(x,s) \frac{M(s)}{E_0 I} (1 + f(s)) ds dx + P \int_{0}^{x_p} G_u(x_p,s) \frac{M(s)}{E_0 I} (1 + f(s)) ds \right.
\]

\[
- M_0 \int_{0}^{x_M} G_\theta(x_M,s) \frac{M(s)}{E_0 I} ds
\]

\[
= \int_{0}^{L} q(x) \int_{0}^{x} G_u(x,s) \frac{M(s)}{E_0 I} ds dx + P \int_{0}^{x_p} G_u(x_p,s) \frac{M(s)}{E_0 I} ds + M_0 \int_{0}^{x_M} G_\theta(x_M,s) \frac{M(s)}{E_0 I} ds,
\] \hfill (7.11)
and the expected value squared is

\[
C_2^2 \mathbb{E}[\bar{D}^2] = \int_0^L \int_0^L q(x_1)q(x_2) \int_0^{x_1} \int_0^{x_2} G_u(x_1, s_1)G_u(x_2, s_2) \frac{M(s_1)M(s_2)}{(E_0 I)^2} \\
\mathbb{E}[(1 + f(s_1))(1 + f(s_2))] ds_2 ds_1 dx_2 dx_1 \\
+ 2P \int_0^L q(x) \int_0^x \int_0^{x_P} G_u(x, s_1)G_u(x_P, s_2) \\
\frac{M(s_1)M(s_2)}{(E_0 I)^2} \mathbb{E}[(1 + f(s_1))(1 + f(s_2))] ds_2 ds_1 dx + 2M_0 \int_0^L q(x) \\
\int_0^x \int_0^{x_{M0}} G_u(x, s_1)G_\theta(x_{M0}, s_2) \frac{M(s_1)M(s_2)}{(E_0 I)^2} \\
\mathbb{E}[(1 + f(s_1))(1 + f(s_2))] ds_2 ds_1 + \\
(1 + f(s_2))^2 ds_2 ds_1 + PM_0 \int_0^{x_{M0}} \int_0^{x_{M0}} G_u(x_P, s_1)G_\theta(x_{M0}, s_2) \\
\frac{M(s_1)M(s_2)}{(E_0 I)^2} \mathbb{E}[(1 + f(s_1))(1 + f(s_2))] ds_2 ds_1 + \\
M_0^2 \int_0^{x_{M0}} \int_0^{x_{M0}} \\
\mathbb{E}[(1 + f(s_1))(1 + f(s_2))] ds_2 ds_1.
\]

(7.12)
The variance is then given by

\[
C^2_1 \text{Var} \left[ \hat{D} \right] = \int_0^L \int_0^L q(x_1)q(x_2) \int_0^{x_1} \int_0^{x_2} G_u(x_1, s_1)G_u(x_2, s_2) \frac{M(s_1)M(s_2)}{(E_0 I)^2} \nonumber \\
\mathbb{E} [f(s_1)f(s_2)] ds_2 ds_1 dx_2 dx_1 + 2P \int_0^L q(x) \int_0^x \int_0^{x_M} G_u(x, s_1) \nonumber \\
G_u(x_P, s_2) \frac{M(s_1)M(s_2)}{(E_0 I)^2} \mathbb{E} [f(s_1)f(s_2)] ds_2 ds_1 dx + 2M_0 \int_0^L q(x) \nonumber \\
\int_0^x \int_0^{x_M} G_u(x, s_1)G_\theta(x_M, s_2) \frac{M(s_1)M(s_2)}{(E_0 I)^2} \mathbb{E} [f(s_1)f(s_2)] ds_2 ds_1 dx + P \int_0^{x_P} \int_0^{x_M} G_u(x_P, s_1)G_\theta(x_M, s_2) \nonumber \\
\frac{M(s_1)M(s_2)}{(E_0 I)^2} \mathbb{E} [f(s_1)f(s_2)] ds_2 ds_1 + PM_0 \int_0^{x_P} \int_0^{x_M} G_u(x_P, s_1)G_\theta(x_M, s_2) \nonumber \\
\frac{M(s_1)M(s_2)}{(E_0 I)^2} \mathbb{E} [f(s_1)f(s_2)] ds_2 ds_1 \nonumber \\
\right]
\tag{7.13}
\]

Using the Wiener-Khinchin relation to replace the SDF with the autocorrelation function, the variance can be written as

\[
\text{Var} \left[ \hat{D} \right] = \int_{-\infty}^{\infty} VRD_\delta(\kappa)S_f(\kappa) d\kappa, \tag{7.14}
\]
where

\[
C^2 VRF_D(\kappa) = \int_0^L \int_0^L q(x_1)q(x_2) \int_0^{x_1} \int_0^{x_2} G_u(x_1, s_1)G_u(x_2, s_2) \frac{M(s_1)M(s_2)}{(E_0 I)^2} \exp(ik(s_2 - s_1)) ds_2 ds_1 dx_1 + 2P \int_0^L q(x) \int_0^x \int_0^{x^P} G_u(x, s_1) ds_2 ds_1 dx + 2M_0 \int_0^L q(x) \int_0^x \int_0^{x^M_0} G_u(x, s_1)G_\theta(x_{M_0}, s_2) \frac{M(s_1)M(s_2)}{(E_0 I)^2} \exp(ik(s_2 - s_1)) ds_2 ds_1 dx + PM_0 \int_0^L \int_0^x \int_0^{x^M_0} G_u(x, s_1)G_\theta(x_{M_0}, s_2) \frac{M(s_1)M(s_2)}{(E_0 I)^2} \exp(ik(s_2 - s_1)) ds_2 ds_1 dx.
\]

(7.15)

7.3.1 Cantilever Example

Consider the cantilever in figure 7.1 with length \(L\), distributed loading \(q(x) = q_0(L - x)/L\), average flexibility, \(\frac{1}{E_0 I}\), and the concentrated loads and moments are equal to zero with no loss of generality.

![Figure 7.1: Cantilever Analyzed](image)
The Greens function for the displacement derived in section 2.3 is

\[ G(x, s) = \begin{cases} 
  x - s, & s < x \\
  0, & s \geq x.
\end{cases} \]  

(7.16)

The moment is determined from statics to be

\[ M(x) = -\frac{q_0 L^2}{6} + \frac{q_0 L x}{2} - \frac{5q_0 x^2}{6}, \]

and the coefficient of \( C_1 \) is solved to be \( C_1 = \frac{q^2 L^5}{120} \). The variance of the homogenized flexibility can be described by equation (7.14) where the VRF is given by

\[ VRF_D = \left( \frac{120}{q^2 L^5} \right)^2 \int_0^L \int_0^L \frac{q_0 (L - x_1) q_0 (L - x_2)}{L^2} \int_0^{x_1} \int_0^{x_2} (x_1 - s_1)(x_2 - s_2) \\
\frac{(-\frac{q_0 L^2}{6} + \frac{q_0 L s_1}{2} - \frac{5q_0 s_1^2}{6})(-\frac{q_0 L^2}{6} + \frac{q_0 L s_2}{2} - \frac{5q_0 s_2^2}{6})}{(E_0 I)^2} \exp(i\kappa(s_2 - s_1)) ds_1 ds_2 dx_1 dx_2. \]  

(7.17)

The integral expression of equation (7.17) is solved for analytically with the help of MAPLE, and it is plotted in figure 7.2 using the following values for the parameters: \( L = 15, q_0 = 1000, \frac{1}{E_0 I} = 3.2 \times 10^{-6} \).

![Figure 7.2: VRF for Effective Flexibility of Cantilever in figure 7.1](image)
7.4 VRF of Effective Properties for Statically Indeterminate Structures

7.4.1 Fixed-Simply Supported Beam

The GVRF methodology outlined in chapter 4 for one dimensional problems is employed to determine the existence of the VRF for the effective flexibility for a statically indeterminate beam, meaning the validity of the following expression is tested

\[ \text{Var} \left[ \bar{D} \right] = \int_{-\infty}^{\infty} S_f(\kappa) \text{VRF} \bar{D}(\kappa) d\kappa. \] \hspace{1cm} (7.18)

For each deterministic analysis in the Monte Carlo simulations, the effective flexibility is determined by performing the computation of equation (7.9). The variance of the effective flexibility is computed and a GVRF is established by solving the linear system of equations in (4.4). Consider the fixed-simply supported beam in figure 7.3 with \( q = 1500, \ L = 16, \ E_0 I = 1.25 \times 10^7, \) and \( M = 7000. \)

![Figure 7.3: Statically Indeterminate Fixed-Simply Supported Beam Analyzed](image)

The GVRFs are plotted in figure 7.4. It is observed that the GVRFs are very similar for quite different SDFs and marginal PDFs. Refer to section 4.6.1 for notation.
The variances of the effective flexibility computed from the Monte Carlo simulations for the various tested random field models are given in figure 7.5. It should be noted that the variances are very small because the mean value is too. The largest coefficient of variation is

\[ c_v = \frac{\sigma}{|\mu|} = \frac{\sqrt{2.9 \times 10^{-15}}}{8 \times 10^{-8}} = .67. \] (7.19)
7.4.2 Validation of GVRF Methodology

The validity of the GVRFs is tested by computing the coefficient of variation (COV) of the effective flexibility by Monte Carlo simulation for a different random field model than what was used to determine the GVRFs and comparing this to the predicted COV determined by the GVRFs. The random field model chosen is a translation field, where the SDF of its underlying Gaussian field given by

\[ S_4(\kappa) = \frac{1}{1.0233678} \exp(-3\kappa^2). \]  

(7.20)

The predicted COV from the GVRFs are determined by performing the integration of the definition of the GVRF, that is

\[ \text{Var} \left[ \bar{D} \right] = \int_{-\kappa_u}^{\kappa_u} \text{GVRF}(\kappa)S_f(\kappa)d\kappa, \]  

(7.21)
where $S_f(\kappa)$ is the SDF of the translated field, and solving for the COV

$$\text{COV} = \frac{\sqrt{\text{Var}[\bar{D}]} \ | \mathbb{E}[\bar{D}] |}{\mathbb{E}[\bar{D}]}.$$  

(7.22)

The two tested marginal PDFs are LN1 and LN3. Refer to section 4.6.1 for notation of the probability distributions. Figure 7.6 plots the results of the validation. The red line is the COV of the effective flexibility computed through Monte Carlo simulation, and the blue diamonds are the COVs predicted by performing the integration of equation (7.21) for each GVRF computed.

![Figure 7.6](image)

(a) LN1  
(b) LN3

Figure 7.6: Validation Plots for Fixed-Simply Supported Beam. The red line is the exact COV determined by Monte Carlo Simulation. The blue diamonds are the predicted COVs by the GVRFs from the computation of equation (7.21)

### 7.4.3 Plane Stress Problem: Plate Tension

Consider the plane stress structure in figure 7.7 with parameters $q = 100$, $E_0 = 1.25 \times 10^7$, $L_x = L_y = 10$, and the width equals 1. The VRF for the displacement response is established in chapter 6.
The GVRF methodology for two dimensional stochastic fields is employed to establish GVRFs for the effective compliance for this structure. For this problem the wave number domain is discretized on a $16 \times 16$ grid between $[0 \ \kappa_u] \times [0 \ \kappa_u]$, where $\kappa_u = .24\pi$. The system of equations in (6.7) fails to converge when considering the underlying Gaussian cases. Only the fast Monte Carlo method (underlying U-beta) is used to computing the GVRFs. This allows an evaluation of the PDF independence only, and not the SDF independence. The validation tests, as performed in section 5.5.3 and 7.4.2, use a random field model that is translated from an underlying Gaussian field. The SDF independence can be evaluated from the validated tests.

Figure 7.8 plots the GVRFs of the effective effective compliance.
Figure 7.8: GVRFs for Effective Compliance for Plane Stress Structure in Figure 7.7

(a) S3LN3

(b) S3TG3

(c) S3UN3
The GVRF plots above are plotted again in figure 7.9 at section cuts in order to better visualize the discrepancies amongst the GVRFs. They are plotted along cuts $\kappa_x = 0$, $\kappa_y = 0$, $\kappa_x = \kappa_y$.

Figure 7.9: Computed GVRFs for 2D Example at Various Section Cuts

Figure 7.10 plots the variances computed through Monte Carlo simulation of the effective compliances.
7.4.4 Validation of GVRF Methodology for Plane Stress Structure

The validity of the GVRFs is tested by computing the coefficient of variation (COV) of the effective compliance by Monte Carlo simulation for a different random field model than what was used to determine the GVRFs and comparing this to the predicted COV determined by the GVRFs. The random field model chosen is a translation field, where the SDF of its underlying Gaussian field given by

\[ S_4(\kappa_x, \kappa_y) = 12.62626\exp(-40(\kappa_x^2 + \kappa_y^2)) \]  

(7.23)
The predicted COV from the GVRFs are determined by performing the integration of the definition of the GVRF, that is

\[
\text{Var} \left[ \bar{D} \right] = \int_{-\kappa u}^{\kappa u} \int_{-\kappa u}^{\kappa u} GVRF_D(\kappa_x, \kappa_y) S4(\kappa_x, \kappa_y) d\kappa_x d\kappa_y, \tag{7.24}
\]

where \( S_f(\kappa) \) is the SDF of the translated field, and solving for the COV

\[
\text{COV} = \frac{\sqrt{\text{Var} \left[ \bar{D} \right]}}{|\mathbb{E} \left[ \bar{D} \right]|}. \tag{7.25}
\]

The three tested marginal PDFs are LN3, TG3, and UN3. Refer to section 4.6.1 for notation of the PDFs. Figure 7.11 plots the results of the validation.
Figure 7.11: Validation Plots for Plane Stress Structure in Figure 7.7. The red line indicates the exact COV through Monte Carlo Simulation. The blue diamonds are the predicted COVs by the GVRFs through the computation of equation (7.24).

7.5 Conclusions

In this section the GVRF methodology for one and two dimensional structures is employed to compute the GVRFs for effective properties. For the one dimensional stochasticity, GVRFs for the effective flexibility are computed. For the two dimensional stochasticity, GVRFs for the effective compliance are computed. In both cases, the GVRFs are approximately equal in magnitude and shape. For the beam example, the GVRFs are essentially independent of the SDF and dependent on the PDF. The plane stress structure also shows a dependence
on the PDF. Due to numerical issues with computing the GVRFs from the underlying Gaussian random field, it is difficult to determine the degree of dependence of the SDF. However, through the validation tests, it can be concluded that the fast Monte Carlo method produces GVRFs that sufficiently predict the response variance of random field models with significantly different spectral content.
Chapter 8

Simulation of Two-Phase Microstructures Using Filtered Poisson Fields and Genetic Algorithms

*This chapter is part of a paper with Professor Eleni Chatzi published in the proceedings of the 11th International Conference on Applications of Statistics and Probability in Civil Engineering (ICASP) [76]. The establishment of VRFs for effective properties as discussed in chapter 7 considers the material properties to be smoothly varying random fields. However, a rigorous homogenization technique should first consider the scale where the individual constituents can be distinctly identified. The uncertainty in the fine scale morphological structure is one source from which uncertainty in macroscopic material properties propagate. This chapter specifically deals with the simulation of random morphologies. Future work can be to investigate establishing VRFs for effective properties at this scale, and incorporate these VRFs into an upscaling technique to bridge this scale with that where the properties are smoothly varying.
8.1 Introduction

Probabilistic characterization and simulation of morphological structures of random heterogeneous materials is an interdisciplinary research topic spanning the fields of Biology, Medicine, Structural Mechanics, Electrophysics, and many more \[77, 78\]. Once morphological structures are characterized, various physical phenomena, such as fluid and contaminant transport, structural mechanical behavior, or chemical reaction processes, can be analyzed \[79, 80, 81\]. It is often the case that random morphological structures exist at micro- and nano-scales, and attempts have been made to determine Representative Volume Elements (RVEs) through characterization of the random morphologies \[82\]. Furthermore, a field of micromechanical analyses of heterogeneous materials is emerging in order to quantify the effects that phenomena occurring at the microscale have on observed macroscopic responses, such as the propagation of microcracks \[84\]. In multi-scale finite element analysis micromechanics is implemented to determine the size of RVEs as well as the stochastic properties of homogenized material properties \[85, 71, 72, 73, 74, 75, 66, 67, 68\].

The work that has been done to generate sample microstructures that match specified correlation functions can be divided into one of two categories. One approach is to apply a perturbation method such as the Metropolis, Markov-Gibbs, Stochastic Optimization, or Simulating Annealing methods \[86, 81, 87, 77\], while the other method is to translate an underlying and easily simulated random field to a Binary field by applying a level-cut \[1, 88, 14, 15\]. The advantage to the perturbation methods is that they are quite flexible in their capability to match higher order target correlation functions since there is no reliance on an underlying random field. However the major disadvantage to this approach is that a perturbation method must be applied for each sample generated, which is computationally expensive making Monte Carlo simulation impractical. On the other hand, level-cut translation field models only need to apply a perturbation method once to identify the properties of the underlying random field that lead to the corresponding target correlation functions when translated. The underlying field is usually a Gaussian field since it is efficiently simulated,
thus yielding the method suitable for Monte Carlo simulation. The drawback to this approach is that since Gaussian fields are completely characterized by their mean and autocorrelation functions, they cannot match more than two target correlation functions. It is well known that third and higher order correlation functions are necessary to accurately characterize many heterogeneous materials [77, 80].

Grigoriu has developed a level-cut filtered Poisson field model that is capable of matching higher order correlation functions due to its ability to incorporate an arbitrary number of parameters [36, 14, 15]. In reference [14], he develops the theoretical framework for level-cut filtered Poisson fields and pseudo-analytically matches a target mean and 2-point probability function using a specific filtered Poisson field. In reference [15], he outlines a Monte Carlo based procedure to simulate homogeneous and inhomogeneous filtered Poisson fields.

This work is essentially an extension of the work of references [14, 15] as it proposes a general methodology to optimize parameters of a filtered Poisson field to match target correlation functions of a Binary random field. This is done through a Monte Carlo based methodology which avoids having to calculate the \( n^{th}\)-order PDF of the underlying filtered Poisson field since it is algebraically and numerically cumbersome to calculate. The parameters of the Filtered Poisson Field are determined using a Genetic Algorithm. Genetic Algorithms have been applied to this problem before, in which case the approach was that of the perturbation methods [89]. Genetic Algorithms are suitable for level-cut translation models since they have few parameters, and it is known that the performance of Genetic Algorithms decreases as the number of parameters increases [90].

The outline of the paper is as follows. Statistical characterization of Binary random fields will be developed followed by that of filtered Poisson fields. The modularity of filtered Poisson fields will be demonstrated. The level-cut translation field model for filtered Poisson fields will be developed. After which, a novel Monte Carlo based procedure to match target correlation functions of a Binary random field is detailed. The procedure entails translating a filtered Poisson field to a Binary field by performing a level-cut, and optimizing the parameters of
the filtered Poisson field through a Genetic Algorithm. The paper concludes with numerical examples.

8.2 Binary Random Field Model for Two-Phase Random Materials

Consider a two-phase random material occupying space \( D = V^{(1)} \cup V^{(2)} \in \mathbb{R}^d \), where \( V^{(j)} \) indicates the space occupied by phase \( j \), and \( d = 1, 2, \) or 3. Two-phase random materials can be described probabilistically by a Binary random field through the indicator function \( I^{(j)}(t) \), where \( j \) is either phase 1 or 2, such that

\[
I^{(j)}(t) = \begin{cases} 
1 & \text{if } t \in V^{(j)} \\
0 & \text{otherwise}
\end{cases}, \tag{8.1}
\]

where \( t \in V^1 \cup V^2 = D \in \mathbb{R}^d \), for \( d = 1, 2, \) or 3. Its mean and autocorrelation functions are \( \mu(t) = \mathbb{E}[I(t)] \) and \( R(t_1, t_2) = \mathbb{E}[I(t_1)I(t_2)] \), respectively dropping the superscript for convenience. For homogeneous isotropic materials mean and autocorrelation functions simplify to \( \mu(t) = \mu \) is constant and \( R(t_1, t_2) = R(|t_1 - t_2|) = R(\tau) \) where \( \tau \) is the Euclidean distance between points \( t_1 \) and \( t_2 \). Higher order correlation functions can be defined similarly although the mean and autocorrelation functions sufficiently describe a number of materials.

8.3 Filtered Poisson Fields

This section closely follows the work of Grigoriu [14]. A filtered Poisson field is a summation of deterministic functions centered at arrival locations of a Poisson process and scaled by a
random variable, and is of the form

\[ X(t) = \sum_{i=1}^{N(D')} X_i h(R_i(t - \Gamma_i)), \quad t \in D \subset D' \subset \mathbb{R}^d, \]  

(8.2)

where \( N, \lambda > 0 \in \mathbb{R} \) is the number of arrivals and the rate of a Poisson process, respectively, with arrival locations \( \Gamma_i \subset D' \). \( \{X_i\} \subset \mathbb{R} \) is a set of independent identically distributed random variables, and \( \{R_i\} \subset \mathbb{R}^d \) is a set of independent identically distributed rotation matrices. The function \( h : \mathbb{R}^d \rightarrow \mathbb{R} \) is a deterministic function that is bounded and thus compact. If the random variable \( X_i \) is in \( L_2 \) and \( h \) is square integrable in \( \mathbb{R}^d \), then the mean and covariance functions of \( X(t) \) are

\[
\mathbb{E}[X(t)] = \lambda \mathbb{E}[X_1] \int_{D'} \mathbb{E}[h(R_1(t - \xi))] d\xi
\]

\[
\text{Cov}[X(t_1), X(t_2)] = \lambda \mathbb{E}[X_1^2] \int_{D'} \mathbb{E}[h(R_1(t_1 - \xi))h(R_1(t_2 - \xi))] d\xi,
\]

(8.3)

which, since \( h \) is compact, can be rewritten as

\[
\mathbb{E}[X(t)] = \lambda \mathbb{E}[X_1] \int_{\mathbb{R}^d} \mathbb{E}[h(R_1(t - \xi))] d\xi
\]

\[
\text{Cov}[X(t_1), X(t_2)] = \lambda \mathbb{E}[X_1^2] \int_{\mathbb{R}^d} \mathbb{E}[h(R_1(t_1 - \xi))h(R_1(t_2 - \xi))] d\xi
\]

(8.4)

\[ = \lambda \mathbb{E}[X_1^2] \int_{\mathbb{R}^d} \mathbb{E}[h(R_1(\alpha)h(R_1(\alpha + \tau))] d\xi. \]

With the change of variables \( \alpha = t_1 - \xi \) and \( \tau = t_2 - t_1 \), it is clear that \( X(t) \) is weakly homogeneous due to \( h \) being compact (i.e. \( \mathbb{E}[X(t)] \)) does not depend on \( t \), and \( \text{Cov}[X(t_1)X(t_2)] \) depends on \( \tau \). For the purposes of this study, the rotation matrix \( R \) is not necessary and will be dropped from the definition of \( X(t) \). The \( n^{th} \)-order PDF

\[ f_{X(t_1)X(t_2)\ldots X(t_n)}(x_1, x_2, \ldots x_n) \]  

(8.5)
of $X(t)$ is needed to establish the $n^{th}$-order statistical moments of the translated field (i.e. Binary field) and will be discussed in the following two sections.

### 8.4 Level-cut Filtered Poisson Field

A filtered Poisson field can be translated to a Binary field by applying a level-cut, meaning values above a threshold, $a$, are assigned to one phase and values below $a$ to the other. The indicator function can be described as

$$
E[I(t)] = P(I(t) = 1) = P(X(t) \geq a) = \int_a^\infty f_{X(t)}(x) \, dx
$$

$$
E[I(t_1)I(t_2)] = P(I(t_1) = 1 \cap I(t_2) = 1) = P(X(t_1) \geq a \cap X(t_2) \geq a) = \int_a^\infty \int_a^\infty f_{X(t_1)X(t_2)}(x_1, x_2) \, dx_1 \, dx_2.
$$

Higher order correlation functions can be derived similarly.

### 8.5 Probability Density Function of Filtered Poisson Fields

One can calculate the PDF of the filtered Poisson field through recursive conditional probability, the characteristic function, or brute force Monte Carlo Simulation.

#### 8.5.1 Recursive Calculation of CDF of Filtered Poisson Field

Equation (8.6) shows that one needs $P(X(t) \geq a)$ to get the average of the Binary field. Using the definition of the filtered Poisson field in equation (8.2) then
\[ P(\mathbf{X}(t) \geq a) = P(\mathbf{X}(t) \geq a | N = n) P(N = n) \]
\[ = P(\mathcal{X}_1 h(t - \Gamma_1) + \mathcal{X}_2 h(t - \Gamma_2) + ... + \mathcal{X}_n h(t - \Gamma_n) \geq a | N = n) P(N = n) \]
\[ = P(\mathcal{X}_1 \geq (a - x_2 h(t - \gamma_2) - ... - x_n h(t - \gamma_n))/h(t - \gamma_1) \]
\[ | N = n, \Gamma_1 = \gamma_1, \Gamma_2 = \gamma_2, ... \Gamma_n = \gamma_n, \mathcal{X}_2 = x_2, ... \mathcal{X}_n = x_n) P(N = n) \] \hspace{1cm} (8.7)
\[ P(\Gamma_1 = \gamma_1) P(\Gamma_2 = \gamma_1) ... P(\Gamma_n = \gamma_n) P(\mathcal{X}_2 = x_2) ... P(\mathcal{X}_n = x_n) \]
\[ \forall \ n, \gamma_1, \gamma_2, ..., \gamma_n, x_2, ..., x_n \]
\[ = \sum_{n=0}^{\infty} \frac{n e^{-\lambda} \lambda^n}{n!} \int ... \int_A f_{\mathcal{X}}(x_1)f_{\mathcal{X}}(x_2)...f_{\mathcal{X}}(x_n) dx_1 dx_2 ... dx_n d\gamma_1 d\gamma_2 ... d\gamma_n, \]
where
\[ A = a - x_2 h(t - \gamma_2) - ... - x_n h(t - \gamma_n)/h(t - \gamma_1), \] \hspace{1cm} (8.8)
and \( f_{\mathcal{X}}(x) \) is the pdf of the scaling random variable \( \mathcal{X} \) and \( v_{D'} \) is the volume of the domain \( D' \) in equation (8.2). The objective is to find the value \( a \) that corresponds to a target mean value for the Binary field. The computation of this equation involves \( 2^n \) integrals per value of \( n \) per location. The number of arrivals \( n \) can be in the thousands resulting in an excessive computational burden, especially when considering the autocorrelation function.

### 8.5.2 Characteristic Function

The \( n^{th} \)-order pdf of a random field can be determined by calculating the \( n^{th} \)-order characteristic function and then by taking the Fourier transform. The \( n^{th} \)-order characteristic function is defined as

\[ \varphi_{\mathbf{X}(t_1)\mathbf{X}(t_2)...\mathbf{X}(t_n)}(\mu_1, \mu_2, ..., \mu_n) = \mathbb{E} \left[ \exp \left( i \sum_{k=1}^{n} \mu_k \mathbf{X}(t_k) \right) \right], \] \hspace{1cm} (8.9)
and the $n^{th}$-order pdf

$$f_{X(t_1)X(t_2)...X(t_n)}(x_1, x_2, ..., x_n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} \varphi_{X(t_1)X(t_2)...X(t_n)}(\mu_1, \mu_2, ..., \mu_n) \times \exp \left( -i(\mu_1x_1 + \mu_2x_2 + ... + \mu_nx_n) \right) d\mu_1 d\mu_2 ... d\mu_n. \quad (8.10)$$

In general, calculating the characteristic function requires numerical computation. The characteristic function often times will have very high curvature and needs to be highly discretized to produce a numerically accurate Fourier transform, which can exceed computer memory limitations. If the scaling random variable $\mathcal{X}$ is Gaussian and the deterministic kernel function $h(\tau) = \exp \{-B\tau^2\}$ with constant $B > 0$, then the characteristic function for the filtered Poisson field is

$$\varphi_{X(t)}(\mu) = \exp \left\{ -\lambda v_D \left[ 1 - \frac{1}{\mu_D} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left[ i\mu \exp \left( -B\tau^2 \right) x \right] f_\mathcal{X}(x) dx d\tau \right] \right\}. \quad (8.11)$$

where $v_D$ is the volume of the domain $D$ in (8.2), and $f_\mathcal{X}(x)$ is the pdf of the Gaussian scaling variable. Figure 8.1 shows the pdf for different values of parameters $B$ and $\sigma_\mathcal{X}$, highlighting the numerical issues. Figure 8.1(a) shows $f_{X(t)}(x)$ for $\mathcal{X} \sim N(-3.66, 9.69), B = 1.22$, and figure 8.1(b) shows $f_{X(t)}(x)$ for $\mathcal{X} \sim N(-3.66, 1.69), B = .75$. In each case the characteristic function domain variable $\mu \in [0, 3]$ is discretized with 4096 points, the pdf domain variable $x$ is discretized with $2^{14}$ points, and the Fast Fourier Transform is performed using the Fortran IMSL library. As the variance of $\mathcal{X}$ reduces the characteristic function gets sharper and harder to accurately integrate. Integrating the pdf in figure 8.1(b) to determine the level-cut $a$ will produce incorrect values due to the unrealistic negative values of the pdf. As with the previous section, the difficulties escalate when considering higher order pdfs.
Figure 8.1: pdfs by Characteristic Function
8.5.3 Monte Carlo Simulation

Alternatively, the level-cut of the filtered Poisson field can be calculated by simulating numerous samples and taking the statistics of the sample. The computation of the pdfs of the underlying field can be avoided since the value of the level-cut threshold can be determined by

\[ \tilde{F}_{X(t)}(a) = \frac{N_a}{N_{samp}} \]  

(8.12)

where \( N_a \) is number of realizations and \( \tilde{X}(t) \geq a \ \forall t \), with \( \tilde{() \)} denoting the simulated field. \( N_{samp} \) is the number of samples simulated. Once the level-cut threshold is determined and performed, the autocorrelation of the simulated Binary field can be calculated as

\[ R(\tau) = \frac{N_{ar}}{N_{samp}}, \]  

(8.13)

where \( N_{ar} \) is the number of realizations such that \( \tilde{X}(\tau) \geq a \) with an assumption of stationarity but not isotropy (i.e. \( \tau = \{\tau_1, \tau_2, \tau_3\} \)). This approach is advantageous as it implements a rapid and low cost simulation of the underlying filtered Poisson field, it directly calculates the moments of the Binary field without having to calculate the pdfs of the underlying field, and does not restrict the underlying field to be one with minimal algebraic or numerical difficulties associated with computing its pdfs. Finally this approach is the most easily adaptable to parallel computing since it involves brute force Monte Carlo Simulation.

8.6 Novel Approach to Generate Two-Phase Media

Genetic Algorithms are an efficient tool widely used in search and optimization problems and are suitable for parallel computing. The main concept is derived from the theory of evolution and survival of the fittest. The set of parameters to be optimized are coded into one binary string which is termed a chromosome or individual. Multiple individuals are generated (i.e. multiple sets of parameters) to form a population. The fitness of each individual is tested
with a target function. A set of individuals are selected by a probability proportional to their fitness, then they are mated with each other by randomly switching elements of their strings. The new strings, termed offspring, are mutated by a random perturbation, and their fitnesses are tested. The discriminatory selection process ensures that the characteristics of the best fit individuals propagate through each generation, hence enforcing survival of the fittest.

The procedure to generate best fit parameters is outlined figure 8.2. An initial generation of parameters of the filtered Poisson field is created. Numerous realizations of the filtered Poisson field are simulated in order to calculated the value of the level-cut threshold, $a$, such that the volume fraction of the phases is satisfied. The field is again simulated numerous times and level-cut to create a Binary field. The correlation functions of the simulated Binary field are computed and compared to the target correlation functions, establishing a fitness for the associated individual. The Genetic Algorithm generates a new population as mentioned above and the process is repeated for a pre-determined number of generations.
8.7 Numerical Examples

In the following examples target autocorrelation functions are matched using the method developed above for real and hypothetical two-phase random media. Since only up to the second moment is considered, the scaling function $X$ follows a Gaussian distribution.

8.7.1 A note on computational demand

The following examples utilize David L. Carroll’s Genetic Algorithm driver simulating 200 generations each with a population size of 7. The computation was done through remote access of Brookhaven National Laboratory’s IBM Bluegene/L supercomputer using a partition of 1024 processors, each having 512 mb RAM and 700 MHz processor speed. The computation
of each example takes about 10 hours for code written in Fortran90.

8.7.2 Example 1

The following example matches a two-phase, homogeneous and isotropic, material having autocorrelation function

\[
R_t(\tau) = \phi(1 - \phi) \exp \left( -\frac{\tau}{\tau_0} \right) + \phi^2
\]

(8.14)

where \( \tau_0 = 6.67 \), and volume fraction, \( \phi = .15 \). This autocorrelation function is an approximation of the experimentally determined 2-point correlation function for Fontainebleau Sandstone by Yeong and Torquato [91]. The form of the filtered Poisson field that is evolved through the Genetic Algorithm is

\[
X(t) = \sum_{i=1}^{N} X_i \exp \left\{ -C_1(t_1 - \gamma_1) - C_2(t_2 - \gamma_2) \right\}
\]

(8.15)

with \( X \) being Gaussian. Table 8.1 shows the best fit value of the parameters where \( a \) is the level-cut threshold, \( \mu_X, \sigma_X \) are the mean and standard deviation of \( X \), \( \lambda \) is the rate of the Poisson random variable, along with the error, \( E \), defined as

\[
E = \sqrt{\frac{\sum_{i=1}^{N_x} (\tilde{R}(\tau_i) - R_t(\tau_i))^2}{\sum_{i=1}^{N_x} R_t(\tau_i)}}
\]

(8.16)

<table>
<thead>
<tr>
<th>FPP Parameters</th>
<th>Error</th>
</tr>
</thead>
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<tr>
<td>( a )</td>
<td>-1.483E-10</td>
</tr>
<tr>
<td>( \mu_X )</td>
<td>-3.747</td>
</tr>
<tr>
<td>( \sigma_X )</td>
<td>2.986</td>
</tr>
<tr>
<td>( C_1 )</td>
<td>1.9435</td>
</tr>
<tr>
<td>( C_2 )</td>
<td>1.7934</td>
</tr>
<tr>
<td>( \lambda )</td>
<td>0.008247</td>
</tr>
<tr>
<td>( E )</td>
<td>0.00049</td>
</tr>
</tbody>
</table>

Table 8.1: Example 1 Best fit Parameters
Figure 8.3: Example 1 Target Autocorrelation

Figure 8.4: Example 1 Simulated Autocorrelation
and therefore positive definite. Invoking Eq. 5, it is possible to establish the relation between the specific surface of the two-phase medium and 

\[
s_{r} = 2 - \frac{r_{0}}{r_{01}} - \frac{r_{0}}{r_{02}}\text{ for } d = 1
\]

\[
s_{r} = 3 - \frac{r_{0}}{r_{01}} - \frac{r_{0}}{r_{02}}\text{ for } d = 2
\]

\[
s_{r} = 4 - \frac{r_{0}}{r_{01}} - \frac{r_{0}}{r_{02}}\text{ for } d = 3
\]

It is interesting that the specific surface is proportional to the mean rate of zero crossings of a Gaussian process with auto-
correlation function \(r_{z}\).

Consider again the example of the Debye random medium with volume fraction 0.5 and binary autocorrelation \(R(z) = \frac{4}{3}e^{-\frac{z}{z_{0}}}/z_{0} + 1\).

The positive branch of Eq. 35, which yields positive definite Gaussian autocorrelations for every \(z_{0}\), is used to generate samples of the Debye medium here. Figs. 21 and 22 display sample realizations corresponding to \(z_{0} = 2\) and \(z_{0} = 20\), respectively, which were generated for \(1, 2, 3, 4\). Assumed to constitute a sufficient approximation to the condition for the scales of this problem.

Fig. 21 exhibits remarkable similarities with Fig. 6 which depicts a realization of the same medium obtained for \(1, 0, 1, 0\). This is indicative of the flexibility of the proposed method. It should be noted, however, that the underlying Gaussian fields for those two different cases of \(z_{0}\) have different autocorrelation functions. Hence, although the corresponding binary fields will have the same second order properties and therefore be equivalent for the purposes of this paper, their probabilistic characteristics of higher order will be different in general.

The fact that the differences between the two realizations are hardly distinguishable indicates that the autocorrelation function

\[\text{Figure 8.5: Fontainebleau Sandstone Image}\]

\[\text{Figure 8.6: Example 1 Sample Realization}\]

Figures 8.3 and 8.4 show the target and simulated autocorrelation function, and figures 8.5 and 8.6 show an image of Fontainebleau Sandstone and a sample realization of the translated field.

### 8.7.3 Example 2

The following example matches an anisotropic medium with autocorrelations function

\[
R(t, \tau) = \phi(1 - \phi)\exp\left(-\frac{\tau_{1}}{\tau_{01}} - \frac{\tau_{2}}{\tau_{02}}\right) + \phi^{2}
\]

(8.17)
where $\tau_{01} = 2.15$ and $\tau_{02} = 1$ and $\phi = .185$. The form of the evolved filtered Poisson field is
the same as equation (8.15). Table 8.2 shows the best fit value of the parameters where $a$ is
the level-cut threshold, $\mu_X, \sigma_X$ are the mean and standard deviation of $X$, $\lambda$ is the rate of the
Poisson random variable, along with the error, $E$, defined as

$$E = \sqrt{\frac{\sum_{i=1}^{N_1} \sum_{j=1}^{N_2} (\bar{R}(\tau_1, \tau_2) - R_t(\tau_1, \tau_2))^2}{\sum_{i=1}^{N_r} R_t(\tau_1, \tau_2)}}$$  (8.18)

<table>
<thead>
<tr>
<th>FPP Parameters</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>$\mu_X$</td>
</tr>
<tr>
<td>.0003194</td>
<td>3.8258</td>
</tr>
</tbody>
</table>

Table 8.2: Example 2 Best fit Parameters

Figures 8.7 and 8.8 show the target and simulated autocorrelation functions. Figure 8.10
shows a sample realization and is compared with figure 8.9, which is a sample realization
based on the methodology developed in reference [1].

Figure 8.7: Example 2 Target Autocorrelation
An alternative mapping to the one in Eqs. (7–9) will be examined in this section. As shown earlier in Eq. (9), the correlation matrix for a Gaussian field at three points, namely at $x_0, x_1, x_2$, is given by

$$
R_{ij} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_0, x_1, x_2) \, dx_0 \, dx_1 \, dx_2,
$$

where $f(x_0, x_1, x_2)$ is the probability density function of the Gaussian field.

The limiting case when $\lim_{\tau \to 0} R(\tau) = 1$, and binary autocorrelation $\rho_1$ is 0.25 and binary autocorrelation $\rho_2$ is 0.18.

For a given binary autocorrelation satisfying the above properties, it is possible to find the underlying Gaussian field at three points, namely at $x_0, x_1, x_2$, which the correlation matrix is positive definite. Fig. 24 indicates that for any value of $\rho_1$, it is possible to find the underlying Gaussian field at three points, namely at $x_0, x_1, x_2$, which the correlation matrix is positive definite. Fig. 24.

Contour lines depict different values of volume fraction $\phi$.

Fig. 24. Contour lines depict different values of volume fraction $\phi$.

The positive sign was selected in Eq. (9) for the limiting case when $\lim_{\tau \to 0} R(\tau) = 1$, and binary autocorrelation $\rho_1$ is 0.25 and binary autocorrelation $\rho_2$ is 0.18.

Fig. 8.9: Example 2 Sample Realization (see [1])

Fig. 8.10: Example 2 Simulated Realization

$\tau_1$ $\tau_2$

$0$ $5$ $10$ $15$ $20$ $25$ $30$ $35$ $40$ $45$

$0$ $0.02$ $0.04$ $0.06$ $0.08$ $0.1$ $0.12$ $0.14$ $0.16$ $0.18$ $0.2$

Figure 8.8: Example 2 Simulated Autocorrelation
8.8 Conclusion

A novel method to simulate two-phase random media that matches specified correlation functions has been developed. It was shown that brute force Monte Carlo simulation is the most feasible method to generate correlation functions since it places no restrictions on the parameters of the underlying filtered Poisson field, and it is most easily parallelized. In the examples shown, only the first two statistical moments were matched, so the scaling variable was chosen to be Gaussian. When attempting to match higher order correlation functions, non-Gaussian scaling variables and various forms of the kernel function should be investigated. The flexibility of this model is that this can be done with no additional computational or algebraic costs. Future work will be to match higher order correlation functions, and extend to three dimensions.
Chapter 9

Conclusions/Future Work

9.1 Research Contributions

The development of the VRF concept is motivated from practical considerations because it is rare for the spatial correlation structure of uncertain parameters to be available. Usually, at best, estimates of the mean and variance are available. From this information, the VRF can be utilized to compute the supremum of the response variance, which is very useful information for design purposes. The limitation to the VRF concept is that it exists for a small classes of structures. This thesis expands the applicability of the VRF concept to a much broader range of problems in Structural Mechanics than what has been established to date. In chapter 3, the existence of the VRF is proven for a class of nonlinear structures for the first time. The GVRF methodology is implemented in chapter 5 to demonstrate that approximate GVRFs for indeterminate nonlinear structures can be computed. The GVRFs exhibit a dependence on the PDF but nonetheless have the same shape in general.

The GVRF methodology is expanded to linear structures where the stochasticity is defined over two dimensions in chapter 6. The methodology developed is applicable to plate bending, plane strain, and plane stress problems. In an example, GVRFs for response displacement are computed for a plane stress structure and show good agreement. The validation procedure
conducted in section 6.4.2 demonstrates the ability of the GVRFs to predict the response variance for an arbitrary PDF/SDF combination.

In chapter 7, the VRF concept is applied to stochastic characterization of effective properties for linear structures. This concept is developed in reference [12] and is applied to statically determinate structures. This is a straightforward extension from the VRF concept for response displacement. The VRF not only identifies the effect of correlation structure on variability of effective properties but also identifies the RVE for this structure and its loading. The GVRF methodology is applied to a fixed simply supported beam to establish the effective flexibility for this structure and its loading. Then the two dimensional GVRF methodology is applied to a plane stress structure to determine the effective compliance.

Chapter 8 presents a methodology for simulating microstructures of random two-phase materials. A modular formulation is developed which utilizes Monte Carlo simulation through parallel computation to optimize parameters. Once the parameters are optimized, sample microstructures can be rapidly generated. In two numerical examples, the methodology is employed to match the autocorrelation function of two dimensional slices of microscale morphologies of real materials.

9.2 Future Work

An important finding from the derivation of the VRF for nonlinear structures is that higher power terms of the SDF enter into the integral expression that relates the VRF and the response variance. It is necessary to know the specific higher power terms and their coefficients in order to employ the GVRF methodology for statically indeterminate structures. Therefore, in order for the VRF concept to be applicable for a broad range of nonlinear, indeterminate structures, VRFs need to be established for their statically determinate counterparts. It is very likely that VRFs exist for many nonlinear statically determinate structures despite the unavailability of analytical expressions: the decomposition of the response variance into a
deterministic VRF and the SDF is due to the determinateness of the internal forces/moments, not the constitutive law. However, closed form expressions cannot be derived for many nonlinear structures because the constitutive law cannot be integrated and inverted. An interesting research direction is to develop a numerical procedure to determine the $VRF^*$ and the higher power terms of the SDF affecting response variance for statically determinate structures having widely used constitutive laws, such as a bilinear one.

The two dimensional extension of the GVRF methodology presented in this dissertation can be further developed in two ways. The first is to apply this methodology to more structures. Although it is conjectured in this thesis that the methodology applies to any case where the stochasticity is defined in $\mathbb{R}^2$, only a plane stress structure is studied. The methodology ought to be applied to plate bending problems as well as plane stress/strain structures with various loading conditions. Secondly, the main challenge to the two dimensional extension involves finding a convergent solution technique to the system of linear equations from which the GVRF is computed. The solution techniques employed in this thesis do not converge for a family of SDFs for computing the GVRF for effective compliance (section 7.4.3). Different solvers for systems of linear equations should be investigated as well as alterations of the GVRF methodology in order to improve the ability to solve the system of equations. It is anticipated that this is imperative for three dimensional problems.

Although the methodology to simulate random two phase materials presented in chapter 8 is modular and flexible, only autocorrelation functions are matched in the numerical examples. The methodology should be employed to match three point correlation functions (this will be the first time a level-cut translation field model matches three point correlation functions of two-phase materials). Once the parameters are optimized sample microstructures can be rapidly simulated. Therefore Monte Carlo simulation of Structural Mechanics problems can be performed. An example study would be to determine the variance of effective properties of a particular two phase material. Then determine GVRFs for effective properties of two phase materials, and compare the predicted variance from the GVRFs with that computed through
Monte Carlo simulation. Furthermore, other stochastic homogenization methods can be studied and compared. This will be the first time a Monte Carlo simulation is performed for a two phase random material with the purpose of identifying the most accurate homogenization model.

The majority of research today still deals with second order statistics, and very few researchers study higher order statistical moments due to a shortage of data and mathematical tools. However, the mean and variance of response quantities are generally not enough information to predict the probability of exceeding a given threshold. Consider the linear fixed-simply supported structure studied in section 7.4. Figure 9.1 shows the histogram for the displacement response at the midspan for a fairly simple random field model describing the flexibility: a Uniform PDF transformed from an underlying U-beta field with SDF, $S(\kappa) = \frac{1}{2} [\delta(\kappa - 10\pi/128) + \delta(\kappa + 10\pi/128)]$. It is clear from the histogram that the response can have a very complicated distribution due to uncomplicated system stochasticity. It is important to conduct research towards quantifying higher order statistical moments of structural response, and it is worth investigating the applicability of the VRF concept to this problem.
Figure 9.1: Example Histogram of Midspan Displacement for Fixed-Simply Supported Beam Analyzed in Section 7.4
Bibliography


