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Abstract

This paper studies resource-allocation mechanisms by using a reduced-form notion of mechanism. We formulate a mechanism by specifying the state space of the mechanism, the set of outcomes that agents can induce in a given state, and the set of admissible outcomes in each state. This notion of mechanism includes the Walrasian mechanism and majority voting as well as all game forms. With this notion, monotonicity is not only necessary but sufficient for a social choice correspondence to be implementable. Our main result is that in the context of exchange economies, if a mechanism implements a sub-correspondence of the Pareto correspondence and satisfies localness (one's "budget set" in a given state is independent of other agents' endowments), then the mechanism necessarily implements a sub-correspondence of the core correspondence. If the mechanism also satisfies anonymity, then it actually implements a sub-correspondence of the Walrasian equilibrium correspondence.

Journal of Economic Literature Classification Numbers: C71, C72, D41, D51, D71, D89, H41.

Keywords: implementation, social choice, Walrasian equilibrium, core, axiomatic approach.

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1 Introduction

This paper investigates economic mechanisms or institutions by using a generalized notion of mechanism. The notion of mechanism proposed in this paper includes all game forms but it also includes the Walrasian mechanism and majority voting. It is a reduce-form notion in that it does not necessarily give a complete specification of how the mechanism works. On the other hand, this approach enables us to formulate dynamic institutions in a simple and static way and provides a convenient tool for axiomatic analysis of institutions. We characterize implementable social choice correspondences using our notion of mechanism and prove axiomatic characterizations of the core and Walrasian correspondences.

In the literature of mechanism design and implementation, mechanisms are usually described as game forms. The advantage of the notion of game form is that a game form can give a complete specification of an institution and leaves no ambiguity about how it works. This gives us a compelling reason to use game forms for describing mechanisms.

However, game forms are not very convenient for axiomatic analysis of institutions, particularly when the goal is to gain insight into real-life institutions. Institutions in real world are usually dynamic and their detailed specification requires extensive game forms. For example, markets involve a dynamic process of price formation, and the work on the foundation of markets usually proposes a dynamic model of markets (e.g., Osborne and Rubinstein, 1990; Gale, 2000). Unfortunately, dynamic models are not very tractable for axiomatic analysis, whose objective is to characterize institutions that satisfy a list of axioms. Extensive game forms may be suitable when one has already chosen a small class of institutions to examine, but not to *identify* important institutions. This makes it sensible to consider a static notion of mechanism that can describe dynamic institutions in a reasonable way. Static models omit some details of institutions and would not be completely satisfactory, but they offer convenient tools for axiomatic analysis.

This paper proposes a general reduced-form notion of mechanism, which is a rather straightforward generalization of the Walrasian mechanism. We define a mechanism by specifying the set of states of the mechanism, the "budget set" of each agent in a given state, and the set of admissible outcomes in a given state. Specifically, a mechanism is defined as a triple (Z, E, H). The first entry Z is an arbitrary non-empty set of possible states of the mechanism. The second entry E specifies the set of outcomes that each agent can induce in a given state. The set $E(i, z)$ is called the *effectivity set* of agent i in state z. The final component of a mechanism is H, which specifies the set of feasible outcomes that are admissible in a given state, i.e., the outcome has to belong to $H(z)$ when the state is z. The triple (Z, E, H) is called an *effectivity form*.

Effectivity forms evidently generalize the Walrasian mechanism. In the Walrasian mechanism, the state is a price vector, the effectivity set is the usual budget set, and all feasible allocations are admissible.

Effectivity forms also generalize game forms. A game form is an effectivity form in which the state is a strategy profile, one's effectivity set at a strategy profile consists of what he can induce given the other agents' strategies, and the only admissible outcome for a strategy profile is what is assigned by the outcome function.

We can generalize the notion of mechanism even further by specifying the effectivity set for each *coalition*. This generalization would be important when groups of agents often act jointly. With this generalization, we can, for example, formulate majority rule, for which a coalition in the majority can induce any alternative. Generally, we can use effectivity forms to formulate institutions that underlie the core in various settings.

A *noncooperative equilibrium* of an effectivity form is a pair (x, z) of an allocation x and a state z such that in state z, allocation x is admissible and no one has a preferred bundle in his effectivity set. We can also define *cooperative* equilibrium analogously. The noncooperative equilibrium of the market mechanism is identical to the Walrasian equilibrium. The cooperative equilibria of majority voting are equivalent to the Condorcet winners. For game forms, the noncooperative (resp. cooperative) equilibrium reduces to the Nash (resp. strong) equilibrium.

An obvious drawback of the notion of effectivity form is that it does not necessarily specify how the state is determined, which makes it unclear how the mechanism actually works. Our implicit assumption is that the state is determined by the action of agents in a dynamic process, but we do not make the process explicit for the sake of simplicity. This simplification makes our notion somewhat unsatisfactory and is problematic particularly when it comes to a practical implementation of a mechanism. However, there exists a trade-off between completeness and simplicity. While an effectivity form is not a complete model of an institution, it provides a simple and convenient way of describing an institution that may involve a complex dynamic process.

To gain intuition of the above argument, consider the dynamics of a conversation. The state of a conversation would be the topic. Anyone in the conversation can change the topic, but certain changes are considered inappropriate (e.g., from serious topics to vulgar ones). The function E can be considered as a moral rule specifying socially correct changes of topics. The equilibrium then describes the steady state of the conversation, which would depend crucially on admissible changes of topics. An academic topic would not be a steady state when the change from an academic topic to gossip is admissible and everyone prefers gossip. We do not deny a possibility of modeling conversations by means of extensive game forms, but more static reduced-form modeling would be practical.

It turns out that, for our reduced-form notion of mechanism, the standard monotonicity condition of Maskin (1999) is not only necessary but sufficient for the implementability of a social choice correspondence. This holds for general social choice problems, even for non-economic environments and for two-person case.[1] Furthermore, the result holds whether the equilibrium concept is noncooperative or cooperative.

A drawback of this characterization is that it relies on the assumption that *any* mechanism can be used. In practice, however, not all mechanisms can be used. Some mechanisms are practical and others are not. Thus, it would be more interesting to characterize social choice correspondences that can be implemented by "practical" or "reasonable" mechanisms. Fortunately, the notion of effectivity form suggests a number of interesting axioms that are not very meaningful in the traditional framework. We propose a few axioms of reasonable mechanisms, and characterize implementability when mechanisms are required to satisfy the axioms. For this, we focus on exchange economies.

We consider the following axioms. The first axiom is *anonymity*, which states that the effectivity set does not depend on the names of the agents. That is, two agents with the same endowments are given the same effectivity set. The second axiom, called *localness*, states that one's effectivity set in a given state does not depend on the other agents' endowments. The market mechanism satisfies localness because a consumer at a grocery store does not need to know other consumers' endowments. Our last axiom, called *non-exclusivity*, states that the mechanism does not prohibit any feasible allocation in any state. In other words, equilibrium only requires feasibility and individual (or coalitional) optimality, and nothing more.

We first prove that the last two axioms characterize the mechanism that underlies the notion of core. Specifically, we show that for any mechanism that satisfies localness and non-exclusivity (but not necessarily anonymity), if its equilibrium allocations are Pareto efficient and individually rational for all profiles of preferences, then the equilibrium allocations are necessarily in the core. This holds whether the equilibrium concept is noncooperative or cooperative.

[1] For game-form implementation, monotonicity is necessary but not sufficient (Maskin, 1999; Saijo, 1988). Moreover, the two-person case needs to be dealt with separately (Moore and Repullo, 1990; Dutta and Sen, 1991b).

It turns out that if we add anonymity to our list of axioms, then the axioms characterize the Walrasian mechanism, provided that the equilibrium concept is noncooperative. That is, we show that for any mechanism that satisfies anonymity, localness, and non-exclusivity, if its noncooperative equilibrium allocations are Pareto efficient for all profiles of preferences, then the equilibrium allocations are necessarily Walrasian equilibrium allocations.

We also conduct a similar axiomatic analysis in the context of public good economies. It turns out that an analogous characterization does *not* hold for public-good economies. We first show that anonymity alone is incompatible with Pareto efficiency. That is, there exists no anonymous mechanism that implements a sub-correspondence of the Pareto correspondence. The Lindahl mechanism uses personalized prices, and it is intuitively clear that a personalized system is necessary to achieve efficiency in public-good economies. Our result confirms the intuition.

We also show that the other axioms (i.e., localness and non-exclusivity) do not characterize the Lindahl mechanism. That is, there exists a local and non-exclusive mechanism whose equilibrium allocations are always Pareto efficient but are not necessarily Lindahl allocations.

These results for public-good economies are interesting given the considerable difference between Walrasian and Lindahl equilibria in terms of practical importance. The literature has explored the difference between the two equilibrium concepts, e.g., in terms of incentives for preference revelation (Roberts, 1976) and core convergence (Muench, 1972). On the other hand, for many of the axiomatic characterizations of Walrasian equilibrium in the literature, there are parallel characterizations of Lindahl equilibrium.

Walrasian equilibrium has been characterized by a number of studies. We mention only a few studies that are particularly relevant to our study.

One of the most influential characterizations of Walrasian equilibrium is the limit theorem of Debreu and Scarf (1963). Interestingly, we use the limit theorem in our proof. It should be noted, however, that while we use the limit theorem, our result is not about the limit of a sequence of economies. Our result says that the equilibrium allocations of the effectivity form that satisfy our axioms are Walrasian for *all* economies.

A similar implementation-theoretic characterization of Walrasian equilibrium is obtained by Hurwicz (1979a). Hurwicz shows that if a game form implements a sub-correspondence of the Pareto correspondence in Nash equilibrium and satisfies the property that the attainable set (i.e., the effectivity set associated with the game form) is a convex set, then the implemented correspondence is a sub-correspondence of the Walrasian correspondence.[2]

A number of studies characterize Walrasian equilibrium based on the concept of no-envy in net trades (e.g., Schmeidler and Vind, 1972; Varian, 1976; Mas-Colell, 1987). No-envy in net trades is based on the idea that, in the underlying mechanism, each agent is effective for the same set of net trades. This concept is considerably stronger than our concept of anonymity. Our anonymity axiom states that two agents with identical endowments are effective for the same set of net trades, but it allows for a mechanism in which agents with different endowments are treated differently.[3]

2 The Model

Let $\mathbb{N} = \{1, 2, \ldots\}$ be the set of *potential* agents. Let $\mathcal{N}$ be a collection of non-empty *finite* subsets $N \subseteq \mathbb{N}$ such that $|N| \geq 2$. The set of agents in an economy is a set in $\mathcal{N}$. This formulation allows for variable population as well as fixed population. If the set of agents is fixed, then $\mathcal{N}$ is a singleton.

Let C be the consumption space of an agent, which is an arbitrary non-empty set and assumed to be the same for all agents. Elements of C are called *consumption bundles*.

Each agent in an economy has a complete and transitive binary (preference) relation R_i defined over C. As usual, $x_i \, R_i \, y_i$ means that x_i is at least as good as y_i for agent i. The associated strict preference and indifference relations are denoted by P_i and I_i, respectively. The set of admissible preferences is denoted by $\mathcal{R}$. A preference profile for N is denoted by $R = (R_i)_{i \in N} \in \mathcal{R}^N$. We assume that preferences are not known to the planner.

Agent i's characteristics other than his preferences are given by an element ω_i of a non-empty set Ω. The set Ω is common to all agents. For example, in the context of exchange economies, ω_i is i's endowments and $\Omega \subseteq \mathbb{R}^\ell_+$. In what follows, $\omega_i \in \Omega$ is called the **characteristics** of agent i. We assume that ω_i is observable to the planner.

Suppose that the set of agents is $N \in \mathcal{N}$ and their characteristics are $\omega \in \Omega^N$. Then the set of (feasible) **allocations** for (N, ω) is given by a non-empty set $X(N, \omega)$. The relation between an allocation and each agent's consumption is given by a function π.

[2] An interesting difference between Hurwicz's characterization and ours is that, in the context of public good economies, Hurwicz's condition of convex attainable sets does characterize Lindahl equilibrium.

[3] Other characterizations of Walrasian equilibrium include Sonnenschein (1974), Hurwicz (1977), Ostroy (1980), Thomson (1988), Nagahisa (1991), Dagan (1996), and Serrano and Volij (2000).

For all $N \in \mathcal{N}$, all $\omega \in \Omega^N$, all $x \in X(N, w)$, and all $i \in N$, $\pi(i, x) \in C$ specifies agent i's consumption in allocation x.

To summarize, an **environment** is defined by a list $(\mathcal{N}, C, X(\cdot,\cdot), \pi, \mathcal{R}, \Omega)$. An **economy** for a given environment is a list (N, R, ω) where $N \in \mathcal{N}$ is the set of agents, $R \in \mathcal{R}^N$ is the profile of preferences, and $\omega \in \Omega^N$ is the profile of characteristics. Let $\mathcal{E}$ denote the set of all economies for a given environment.

We give a few examples of environments, which are used in subsequent sections of the paper.

Example 1 (Exchange Economies). The common consumption space is $C = \mathbb{R}^\ell_+$. The set $\mathcal{R}$ is the set of all complete, transitive, continuous, strongly monotonic,[4] and strictly convex binary relations defined over $\mathbb{R}^\ell_+$. An agent's characteristics ω_i are his endowments of goods, and $\Omega = \mathbb{R}^\ell_{++}$.[5] An allocation for (N, ω) is a list $x \in C^N$ satisfying

$$\sum_{i \in N} x_i = \sum_{i \in N} \omega_i.$$

The function π is given by $\pi(i, x) = x_i$.

Example 2 (Public Good Economies). Consider economies with one public good and one private good where technology is linear. The common consumption space is $C = \mathbb{R}^2_+$. The set $\mathcal{R}$ is the set of all complete, transitive, continuous, strongly monotonic, and strictly convex binary relations defined over $\mathbb{R}^2_+$. An agent's characteristics ω_i are his endowments of the private good, and $\Omega = \mathbb{R}_{++}$. An allocation for (N, ω) is a list $(y, x) \in \mathbb{R}_+ \times \mathbb{R}^N_+$ satisfying a feasibility condition

$$y + \sum_{i \in N} x_i = \sum_{i \in N} w_i,$$

where y denotes the level of the public good, and x_i denotes agent i's consumption of the private good. The function π is given by $\pi(i, (y, x)) = (y, x_i)$.

Example 3 (Voting). Consider a standard voting setting where the set of alternatives is given by some set A. Then the common consumption space is $C = A$. The agents are identical except for their preferences, and thus $|\Omega| = 1$. An allocation is simply an alternative $x \in A$, i.e., $X(N, \omega) = A$. The function π is given by $\pi(i, x) = x$.

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[4] A preference relation R_i defined over $\mathbb{R}^\ell_+$ is *strongly monotonic* if for all $x_i, y_i \in \mathbb{R}^\ell_+$, if $x_i \geqq y_i$ (i.e., $x_{ik} \geq y_{ik}$ for all $k \in \{1, \ldots, \ell\}$) and $x_i \neq y_i$, then $x_i \; P_i \; y_i$.

[5] $\mathbb{R}_{++} = \{x \in \mathbb{R} : x > 0\}$. When $x \in \mathbb{R}^\ell_{++}$, we may write $x \gg 0$.

These examples demonstrate the generality of our framework.

3 Generalized Mechanisms

We fix an environment $(\mathcal{N}, C, X(\cdot,\cdot), \pi, \mathcal{R}, \Omega)$. We consider the following notion of mechanism.

Definition. An **effectivity form** is a list $\Gamma = (Z^{N,\omega}, E^{N,\omega}, H^{N,\omega})_{N\in\mathcal{N},\omega\in\Omega^N}$ where

1. $Z^{N,\omega}$ is a non-empty set of possible **states** of the mechanism.

2. $E^{N,\omega}$ is a correspondence that associates with each non-empty coalition $S \subseteq N$ and each state $z \in Z^{N,\omega}$ a subset $E^{N,\omega}(S,z) \subseteq C^S$. The set $E^{N,\omega}(S,z)$ is called the **effectivity set** of coalition S in state z.

3. $H^{N,\omega}$ is a correspondence that associates with each state $z \in Z$ a non-empty subset $H^{N,\omega}(z) \subseteq X(N,\omega)$ of feasible allocations. This may be called the **outcome correspondence**.

The three components of the effectivity form are all indexed by (N,ω). This comes from our assumption that the planner knows the set of agents and their characteristics (other than preferences). Note that the index is unnecessary when the set of agents is fixed (i.e., $|\mathcal{N}| = 1$) and agents differ only in their preferences (i.e., $|\Omega| = 1$). However, it is important for our analysis to allow (N,ω) to be variable.

The first component, $Z^{N,\omega}$, specifies the set of possible states of the mechanism. In the market mechanism, the state is a price vector. In general, the state space can be any non-empty set.

The second component, $E^{N,\omega}$, specifies what each decision-making group is entitled to obtain in each state. The set $E^{N,\omega}(S,z)$ is a subset of C^S and it denotes the set of consumption-bundle profiles that coalition S can obtain in state z. This set determines whether S can "block" or "improve upon" a given allocation in state z. Note that $E^{N,\omega}(S,z)$ specifies what each member of S can obtain, not the aggregate consumption that S can obtain as a whole. When the relevant equilibrium concept is noncooperative, the set $E^{N,\omega}(S,z)$ is irrelevant for all non-singleton coalitions.

The last component, $H^{N,\omega}$, specifies the set of feasible allocations that are admissible for a given state. That is, the planner prohibits or "blocks" all allocations outside of $H^{N,\omega}(z)$ when the state is z.

We do not necessarily require that the profiles of consumption bundles in $E^{N,\omega}(S,z)$ should be feasible. This formulation is consistent with general equilibrium theory. On the other hand, the effectivity set does not have to violate feasibility. If $E^{N,\omega}(S,z)$ contains only feasible bundles, we say that the effectivity form is closed. Formally, an effectivity form $\Gamma = (Z^{N,\omega}, E^{N,\omega}, H^{N,\omega})_{N\in\mathcal{N},\omega\in\Omega^N}$ is **closed** if for all $N \in \mathcal{N}$, all $\omega \in \Omega^N$, all $S \subseteq N$, all $z \in Z^{N,\omega}$, and all $y_S \in E^{N,\omega}(S,z)$, there exists $y_{N\setminus S} \in C^{N\setminus S}$ such that $(y_S, y_{N\setminus S}) \in X(N,\omega)$.

We allow $E^{N,\omega}(S,z)$ to be empty. The set $E^{N,\omega}(S,z)$ being empty means that S cannot block any allocation in z. For example, this is the case for any minority under majority rule.

As mentioned in the introduction, an important aspect of the notion of effectivity form is that it does not explicitly specify the process in which the state is determined. Our idea is that the state is determined by the agents' action in a dynamic process, but we do not make the process explicit for the sake of simplicity. This simplification makes it ambiguous how and whether a mechanism actually works, but there exists a trade-off between completeness and simplicity. While an effectivity form is not a complete model of an institution, it provides a simple and convenient way of describing an institution that may involve a complex dynamic process. By leaving some details unspecified, effectivity forms provide clean and *static* models of dynamic institutions. A practical advantage of static models is that they are convenient for axiomatic analysis.

On the other hand, to better understand an effectivity form, it is certainly desirable to study a "foundation" of the mechanism. Once an interesting mechanism is identified in reduced form, one could proceed to build a more detailed model of the mechanism with an explicit dynamic process. This two-step approach might be more practical than conducting axiomatic analysis directly on extensive game forms.

The notion of effectivity form is a variant of Wilson's (1971) notion of effectiveness relations and Rosenthal's (1972) notion of effectiveness forms. A number of similar notions have been studied in the literature; e.g., Debreu's (1952) generalized games, Ichiishi's (1981) societies, Moulin and Peleg's (1982) effectivity functions, and Greenberg's (1990) social situations.[6] However, these notions have not been used in the mechanism-design literature, where virtually all studies use game forms. The important exceptions that we are aware of are Greenberg (1990, 1994) and Ju (2001), whose work will be discussed in the next section.

Once effectivity forms are defined, it is straightforward to define equilibrium concepts for them. For a given effectivity form and a given preference profile, a *noncooperative equilibrium* is a pair (x, z) such that, in state z, allocation x is admissible and

[6] In particular, Greenberg's social situations are more general than effectivity forms in that they allow agents to change the state.

is not blocked by any agent. That is, given that the state is z, no one, including the planner, blocks x. Similarly, a *cooperative equilibrium* is a pair (x, z) such that, in state z, neither the planner nor a coalition blocks x. Formally,

Definition. Let $\Gamma = (Z^{N,\omega}, E^{N,\omega}, H^{N,\omega})_{N \in \mathcal{N}, \omega \in \Omega^N}$ be an effectivity form, and let $e = (N, R, \omega) \in \mathcal{E}$ be an economy. Then a **noncooperative equilibrium** of Γ for e is a pair $(x, z) \in X(N, \omega) \times Z^{N,\omega}$ of an allocation and a state such that $x \in H^{N,\omega}(z)$ and there exists no $i \in N$ and no $y_i \in E^{N,\omega}(i, z)$ for which

$$y_i \, P_i \, \pi(i, x).$$

On the other hand, a **cooperative equilibrium** of Γ for e is a pair $(x, z) \in X(N, \omega) \times Z^{N,\omega}$ such that $x \in H^{N,\omega}(z)$ and there exist no $S \subseteq N$ and no $(y_i)_{i \in S} \in E^{N,\omega}(S, z)$ for which

$$y_i \, P_i \, \pi(i, x) \quad \text{for all } i \in S.$$

An allocation $x \in X(N, \omega)$ is a **noncooperative** (resp. **cooperative**) **equilibrium allocation** of an effectivity form Γ for an economy $e = (N, R, \omega)$ if there exists a state $z \in Z^{N,\omega}$ such that (x, z) is a noncooperative (resp. cooperative) equilibrium of Γ for e. Let $N(\Gamma, e)$ and $C(\Gamma, e)$ denote the sets of noncooperative and cooperative equilibrium allocations of Γ for e, respectively.[7]

Note that our definitions of equilibria do not require one's consumption bundle to be in his effectivity set. All we require is that one should not have a preferred bundle in his effectivity set. A situation in which an agent consumes outside of his effectivity set is not necessarily unstable because his consumption might be desirable for him compared to his effectivity set. What causes instability is that an agent can induce a preferred outcome.[8]

We give a few examples to illustrate the above definitions.

Example 4 (Market Mechanism). Consider exchange economies defined in Ex-

[7] The set $C(\Gamma, e)$ may be called the *core* of Γ for e.

[8] This formulation is standard in economics. For example, the core does not require that the aggregate consumption of a coalition be feasible for the coalition. Individual rationality does not require that each agent's utility should be equal to the individual-rationality level.

ample 1. The market mechanism can be formulated as an effectivity form given by

$$\begin{aligned} Z^{N,\omega} &= \{p \in \mathbb{R}^{\ell}_{+} : p \neq 0\}; \\ E^{N,\omega}(i,p) &= \{y_i \in \mathbb{R}^{\ell}_{+} : p \cdot y_i \leq p \cdot \omega_i\}; \\ H^{N,\omega}(p) &= X(N,\omega). \end{aligned}$$

That is, the state is a price vector, and the effectivity set of an individual is the usual budget set. The effectivity sets of non-singleton coalitions are arbitrary. The noncooperative equilibrium of the effectivity form is equivalent to the Walrasian equilibrium.

Example 5 (Game Forms). Any (strategic) game form defines an effectivity form in a straightforward way. A **game form** is a list $((M_i^{N,\omega})_{i\in N}, g^{N,\omega})_{N\in\mathcal{N},\omega\in\Omega^N}$ where $M_i^{N,\omega}$ is a non-empty set of strategies for agent i, and $g^{N,\omega}\colon \prod_{i\in N} M_i^{N,\omega} \to X(N,\omega)$ is the outcome function. Since (N,ω) is known to the planner, the strategy sets and the outcome function may depend on (N,ω). The corresponding effectivity form is $(Z^{N,\omega}, E^{N,\omega}, H^{N,\omega})_{N\in\mathcal{N},\omega\in\Omega^N}$ where

$$\begin{aligned} Z^{N,\omega} &= \prod_{i\in N} M_i^{N,\omega}; \\ E^{N,\omega}(S,m) &= \{(\pi(i, g^{N,\omega}(m'_S, m_{N\setminus S})))_{i\in S} : m'_S \in \prod_{i\in S} M_i^{N,\omega}\}; \\ H^{N,\omega}(m) &= \{g^{N,\omega}(m)\}. \end{aligned}$$

That is, the state is a strategy profile and at each strategy profile m, coalition S can induce any allocation $g^{N,\omega}(m'_S, m_{N\setminus S})$ for $m'_S \in \prod_{i\in S} M_i^{N,\omega}$. The only allocation that is admissible for strategy profile m is $g^{N,\omega}(m)$. The noncooperative (resp. cooperative) equilibrium of the effectivity form is equivalent to the Nash (resp. strong) equilibrium of the game form.

Example 6 (Core). Consider exchange economies again. The core of exchange economies is formulated as the cooperative equilibrium of the following effectivity form:

$$\begin{aligned} Z^{N,\omega} &= \text{arbitrary}; \\ E^{N,\omega}(S,z) &= X(S,\omega_S); \\ H^{N,\omega}(z) &= X(N,\omega). \end{aligned}$$

That is, the state space is arbitrary, and each coalition can redistribute the members' endowments within the coalition.

Example 7 (Majority Rule). Consider the voting environment defined in Example 3. The majority rule states that a coalition in the majority can choose any alternative, and a minority is not effective for any alternative. This can be formulated as the following effectivity form:

$$\begin{aligned} Z^N &= \text{arbitrary;} \\ E^N(S,z) &= \begin{cases} \{(a,\ldots,a) : a \in A\} & \text{if } |S| > |N|/2; \\ \emptyset & \text{otherwise;} \end{cases} \\ H^N(z) &= A, \end{aligned}$$

where A is the set of alternatives. Cooperative equilibrium outcomes of this mechanism are Condorcet winners.

These examples clearly demonstrate that many of the major institutions studied in economics can be modeled as effectivity forms.

4 Implementation

We now introduce the notion of implementation with respect to effectivity forms.

A **social choice correspondence** is a correspondence φ that associates with each economy $(N,R,\omega) \in \mathcal{E}$ a nonempty subset $\varphi(N,R,\omega) \subseteq X(N,\omega)$ of feasible allocations.

Definition. Let φ be a social choice correspondence. An effectivity form $\Gamma = (Z^{N,\omega}, E^{N,\omega}, H^{N,\omega})_{N \in \mathcal{N}, \omega \in \Omega^N}$ **implements φ in noncooperative equilibrium** if for each economy $e \in \mathcal{E}$, $N(\Gamma,e) = \varphi(e)$. The effectivity form Γ **implements φ in cooperative equilibrium** if for each economy $e \in \mathcal{E}$, $C(\Gamma,e) = \varphi(e)$.

Remark 1. According to this definition, the market mechanism (Example 4) implements the Walrasian correspondence in noncooperative equilibrium *by definition*. Since the publication of Schmeidler (1980) and Hurwicz (1979b), the Nash implementation of the Walrasian correspondence has been a major topic in implementation theory. Our definition makes the noncooperative implementation of the Walrasian correspondence straightforward because the market mechanism does it.[9]

[9]This suggests that our use of the term "implementation" may not be completely acceptable to some readers. We say that a social choice correspondence is implementable if it is the equilibrium correspondence of some institution, regardless of whether the institution is described in game form.

Remark 2. Since a game form is an effectivity form, it follows that our notion of implementation generalizes the standard notions of implementation. Implementability by a game form in Nash (resp. strong) equilibrium implies implementability by an effectivity form in noncooperative (resp. cooperative) equilibrium. The converse does not hold in general environments, as Theorem 1 shows below.

We now characterize the class of implementable social choice correspondences.

Given an economy $e = (N, R, \omega)$, the (weak) lower-contour set of R_i at allocation x is defined by

$$L(x, R_i) = \{y_i \in C : \pi(i, x)\; R_i\; y_i\}.$$

The lower-contour sets for coalitions are defined similarly. For coalition S, the lower-contour set is given by

$$L(x, R_S) = \{y_S \in C^S : y_i \in L(x, R_i) \text{ for some } i \in S\}.$$

That is, if $y_S \in L(x, R_S)$, then y_S does not (strongly) dominate x for S.

Let $L^{N,\omega}(x, R_S)$ denote the set of elements in $L(x, R_S)$ that are feasible for (N, ω); that is,

$$L^{N,\omega}(x, R_S) = \{y_S \in L(x, R_S) \colon (y_S, y_{N \setminus S}) \in X(N, \omega) \text{ for some } y_{N \setminus S} \in C^{N \setminus S}\}.$$

Definition. A social choice correspondence φ is **Gevers monotonic** (Gevers, 1986) if for all $(N, R, \omega) \in \mathcal{E}$, all $x \in \varphi(N, R, \omega)$, and all $R' \in \mathcal{R}^N$, if for all $i \in N$,

$$L(x, R_i) \subseteq L(x, R'_i), \tag{1}$$

then $x \in \varphi(N, R', \omega)$. We say that φ is **Maskin monotonic** (Maskin, 1999) if it satisfies the above condition where (1) is replaced by

$$L^{N,\omega}(x, R_i) \subseteq L^{N,\omega}(x, R'_i).$$

That is, Gevers monotonicity says that if x is φ-optimal for (N, R, ω) and another preference profile R' is obtained by expanding each agent's lower-contour set at x, then x remains φ-optimal for (N, R', ω). Maskin monotonicity is equivalent to Gevers monotonicity except that Maskin monotonicity pays attention only to the feasible bundles in lower-contour sets.

Maskin (1999) (resp. Maskin (1979)) proves that Maskin monotonicity is necessary, but not sufficient, for implementability in Nash (resp. strong) equilibrium. Maskin (1999) and Saijo (1988) prove that if there exist three or more agents, Maskin monotonicity together with a condition called *no veto power* is sufficient for implementability in Nash equilibrium. Moore and Repullo (1990) and Dutta and Sen (1991a)[10] obtain conditions that are necessary and sufficient for implementability in Nash and strong equilibria, respectively, but their conditions are considerably more complex.

It turns out that our concept of implementation permits a simple characterization.

Theorem 1. *A social choice correspondence φ is implemented by an effectivity form in (non)cooperative equilibrium if and only if φ is Gevers monotonic. A social choice correspondence φ is implemented by a closed effectivity form in (non)cooperative equilibrium if and only if φ is Maskin monotonic.*

Proof. See the Appendix. □

The theorem says that Gevers monotonicity is a necessary and sufficient condition for implementability whether the equilibrium concept is noncooperative or cooperative. If effectivity sets are required to satisfy the feasibility condition, then a necessary and sufficient condition is Maskin monotonicity.

Greenberg (1990, Theorem 10.1.2) has obtained a result that is essentially equivalent to Theorem 1 for the case when the equilibrium concept is noncooperative and mechanisms have to be closed. The proofs for the other cases are analogous to Greenberg's.

Independently of our paper, Ju (2001) proposes an analogous notion of implementation. He obtains a result similar to the noncooperative part of Theorem 1, but his result requires a certain restriction on the domain of preferences. There are at least two differences between his notion of implementation and ours. First, Ju requires all feasible outcomes to be admissible, i.e., $H^{N,\omega}(z) = X(N,\omega)$. Actually, this condition is considered in the next section, but we do not impose it at this point. Second, Ju also requires that each agent's consumption should be in his budget set while we do not as mentioned previously. All we require is that no one has a preferred consumption bundle in his budget set (see Footnote 8).

Remark 3. Theorem 1 does not mention anything about the cardinality of agents. This means, in particular, that the result holds even when two-agent economies are

[10] See also Suh (1996).

admissible. This is not the case for Nash implementation (Moore and Repullo, 1990; Dutta and Sen, 1991b).

It may not be very surprising that our concept of implementation permits a simpler characterization.[11] Complexity in the characterizations of standard implementation concepts is mostly due to the need to resolve disagreements in announcements. Typical characterization theorems use mechanisms where each agent is asked to report a profile of preferences. This raises difficulties since agents may announce different preference profiles. This difficulty does not arise in our case because agents are assumed to take the state as given and we can let the state include information about preferences.

The following is an immediate but interesting implication of Theorem 1.

Corollary 1. *A social choice correspondence is implementable in noncooperative equilibrium if and only if it is implementable in cooperative equilibrium.*

5 The Axioms

Theorem 1 is a complete characterization of social choice correspondences that can be implemented by some mechanisms. A drawback of the result (and many of the general "existence theorems" in the implementation literature) is that it relies on the assumption that *any* mechanism can be used. In practice, not all mechanisms can be used; some mechanisms are practical and others are not. Thus, it would be more interesting to characterize social choice correspondences that can be implemented by "practical" or "reasonable" mechanisms. This section introduces a few axioms of practical mechanisms and characterizes what can be implemented by effectivity forms that satisfy those axioms.[12]

The first axiom is anonymity, which says that the effectivity set does not depend on the names of the agents.

Definition. We say that an effectivity form $(Z^{N,\omega}, E^{N,\omega}, H^{N,\omega})_{N \in \mathcal{N}, \omega \in \Omega^N}$ satisfies **anonymity** if for all $N \in \mathcal{N}$, all $\omega \in \Omega^N$, and all $S, T \subseteq N$,

$$\omega_S \sim \omega_T \Longrightarrow E^{N,\omega}(S, z) = E^{N,\omega}(T, z)$$

[11] Nash implementation does permit a simple characterization in economic environments where no veto power is vacuous, but does not in general environments.

[12] For this point, see Jackson (1992). Dutta *et al.* (1995), Sjöström (1996), and Saijo *et al.* (1996), among others, also propose definitions of reasonable mechanisms (game forms).

where $\omega_S \sim \omega_T$ means that there exists a bijection $\beta\colon T \to S$ such that for all $i \in T$, $\omega_i = \omega_{\beta(i)}$.

Anonymity says that two coalitions with the same profile of characteristics (thus the same cardinality) have the same effectivity set in each state. This is satisfied by the Walrasian mechanism because two consumers with the same endowments have the same budget set. Majority rule also satisfies anonymity.[13]

Definition. An effectivity form $(Z^{N,\omega}, E^{N,\omega}, H^{N,\omega})_{N\in\mathcal{N},\omega\in\Omega^N}$ satisfies **common state space** if for all $N, N' \in \mathcal{N}$, all $\omega \in \Omega^N$, and all $\omega' \in \Omega^{N'}$,

$$Z^{N,\omega} = Z^{N',\omega'}.$$

That is, the state space is defined independently of the set of agents (N) and the profile of characteristics (ω). This condition would be desirable when the set of agents and the profile of characteristics change frequently and it is costly to change the "framework" or "language" of the institution. The market mechanism satisfies the axiom since the state space is the set of price vectors.

Remark 4. The requirement of common state space alone is not restrictive in the sense that any implementable social choice correspondence can be implemented by an effectivity form with common state space. Indeed, if a social choice correspondence φ is implemented by an effectivity form $(Z^{N,\omega}, E^{N,\omega}, H^{N,\omega})_{N\in\mathcal{N},\omega\in\Omega^N}$, then φ is implemented by an effectivity form $(Z, \hat{E}^{N,\omega}, \hat{H}^{N,\omega})_{N\in\mathcal{N},\omega\in\Omega^N}$ where Z is the set of mappings f such that $f(N,\omega) \in Z^{N,\omega}$ for all $N \in \mathcal{N}$ and all $\omega \in \Omega^N$, and

$$\begin{aligned} \hat{E}^{N,\omega}(S, f) &= E^{N,\omega}(S, f(N,\omega)); \\ \hat{H}^{N,\omega}(f) &= H^{N,\omega}(f(N,\omega)). \end{aligned}$$

It is easy to see that the set of equilibrium allocations is identical for the two effectivity forms.

Definition. Let $\Gamma = (Z, E^{N,\omega}, H^{N,\omega})_{N\in\mathcal{N},\omega\in\Omega^N}$ be an effectivity form with common state space. Then Γ is called **local** if for all $N, N' \in \mathcal{N}$, all $\omega \in \Omega^N$, all $\omega' \in \Omega^{N'}$, and all $S \subseteq N \cap N'$, if $\omega'_S = \omega_S$, then

$$E^{N,\omega}(S, z) = E^{N',\omega'}(S, z) \quad \text{for all } z \in Z.$$

[13] Anonymity is by no means innocuous. For example, an effectivity form associated with a perfectly "anonymous" game form may violate our anonymity axiom at non-diagonal strategy profiles.

That is, an effectivity form is local if the effectivity set of a coalition *in a given state* is independent of the characteristics of the agents outside of the coalition. The market mechanism is local since a consumer at a grocery does not need to know other consumers' endowments.

Localness is also satisfied by income taxation in practice in the sense that the amount of income tax that a taxpayer has to pay for a given year is independent of the other taxpayers' income level. A violation of localness means that taxpayers have to report their income level *before* receiving a tax schedule from the government.

Localness is a strong condition, but it is meaningful for real-life institutions. The condition would be desirable when the number of agents is large and it is costly or time-consuming for the planner to collect information about all agents' characteristics before announcing effectivity sets.

Note that in the definition of localness, $|N'| \neq |N|$ is allowed. This means that the effectivity set in a given state does not depend on the number of agents in the economy. This is often the case in real life; indeed, one rarely pays careful attention to the number of people involved in an institution.[14]

Localness is satisfied whenever the set of agents is fixed (i.e., $\mathcal{N}$ is a singleton) and agents are identical except for their preferences (i.e., $|\Omega| = 1$).

When common state space and localness are satisfied, we can write the effectivity set as $E(S, \omega_S, z)$, which denotes the set of profiles of consumption bundles that coalition S with characteristic profile ω_S can induce in state z regardless of the identities and characteristics of the other agents in the economy. If anonymity is also satisfied, then we can write $E(\omega_S, z)$, which denotes the set of profiles of consumption bundles that *any* coalition T with characteristics $\omega'_T \sim \omega_S$ can induce in state z. Thus, in what follows, we sometimes write E instead of $E^{N,\omega}$.

Our final axiom pertains to the correspondence $H^{N,\omega}$.

Definition. An effectivity form $(Z^{N,\omega}, E^{N,\omega}, H^{N,\omega})_{N \in \mathcal{N}, \omega \in \Omega^N}$ is **non-exclusive** if for all $N \in \mathcal{N}$ and all $\omega \in \Omega^N$,

$$H^{N,\omega}(z) = X(N, \omega) \quad \text{ for all } z \in Z^{N,\omega}.$$

That is, an effectivity form is non-exclusive if no feasible allocation is prohibited in any state. This means that a feasible allocation is an equilibrium as long as it cannot be blocked by agents. This axiom is satisfied by the Walrasian mechanism since it only

[14]Note, however, that localness is not satisfied by majority rule (Example 7) when the number of agents is variable. This is why we do count the number of people in a faculty meeting before we vote.

requires individual optimality and feasibility. On the other hand, the axiom is violated by game forms; game forms are “exclusive” in the sense that the planner blocks all allocations but one.

6 Exchange Economies

This section considers exchange economies defined in Example 1. We start with a few standard definitions.

Given an economy e, we denote by $P(e)$ and $I(e)$ the set of Pareto efficient and individually rational feasible allocations, respectively. We also denote $IP(e) = I(e) \cap P(e)$.

We denote by $W(e)$ the set of **Walrasian equilibrium allocations** for economy $e = (N, R, \omega)$, i.e., the set of allocations x in $X(N, \omega)$ for which there exists $p \in \mathbb{R}^\ell_+ \setminus \{0\}$ such that for all $i \in N$ and all $y_i \in \mathbb{R}^\ell_+$, $p \cdot y_i \leq p \cdot \omega_i$ implies $x_i \, R_i \, y_i$.

We denote by $C(e)$ the **core** of e, which is the set of cooperative equilibrium allocations of the effectivity form in Example 6.

We define the **quasi-core** as the set of feasible allocations that no coalition *with two or more members* can block by redistributing endowments within the coalition. Let $QC(e)$ denote the quasi-core of economy e. Evidently, $QC(e) \supseteq C(e)$.

A **sub-correspondence** of a social choice correspondence φ is a social choice correspondence φ' such that $\emptyset \neq \varphi'(e) \subseteq \varphi(e)$ for all e. This is denoted as $\varphi' \subseteq \varphi$.

The first main result in this section is a characterization of the quasi-core.

Theorem 2. *Consider an environment of exchange economies defined in Example 1 where $N \in \mathcal{N}$ implies $S \in \mathcal{N}$ for all non-empty subsets $S \subseteq N$ with $|S| \geq 2$. Suppose that a social choice correspondence $\varphi \subseteq P$ is implemented in either noncooperative or cooperative equilibrium by a local and non-exclusive effectivity form with common state space. Then $\varphi \subseteq QC$.*

Proof. Let $\Gamma = (Z, E, H^{N,\omega})_{N \in \mathcal{N}, \omega \in \Omega^N}$ be a local and non-exclusive effectivity form with common state space and suppose that it implements a sub-correspondence $\varphi \subseteq P$ in cooperative (resp. noncooperative) equilibrium. Given an economy $e = (N, R, \omega)$, suppose, by way of contradiction, that there exists an allocation $x \in \varphi(e) \setminus QC(e)$. Then x is blocked by some coalition $S \subseteq N$ with at least two members and thus there exists an allocation $y \in X(S, \omega_S)$ such that $y_i \, P_i \, x_i$ for all $i \in S$. Since $\Omega = \mathbb{R}^\ell_{++}$ and preferences are continuous and strongly monotonic, we can assume $y_i \gg 0$ for all $i \in S$.

Now, let $R'_S \in \mathcal{R}^S$ be a preference profile for S such that

$$L(x, R'_i) = L(x, R_i) \quad \text{for all } i \in S. \tag{2}$$

That is, R'_i and R_i have the same indifference curve at x_i. Let $z \in Z$ be a state such that (x, z) is a cooperative (resp. noncooperative) equilibrium of the effectivity form Γ for e. Then for all $T \subseteq S$ (resp. for all singletons $T \subseteq S$),

$$E(T, \omega_T, z) \subseteq L(x, R_T) \subseteq L(y, R'_T).$$

It then follows from non-exclusivity that (y, z) is a cooperative (resp. noncooperative) equilibrium of Γ for (S, R'_S, ω_S). But there exists $R'_S \in \mathcal{R}^S$ such that (2) holds and y is not Pareto efficient for (S, R'_S, ω_S); it suffices to perturb agent i's MRS at y_i.[15] This is in contradiction with $\varphi \subseteq P$. $\square$

Given this result, we can easily obtain a characterization of the core by adding individual rationality.

Corollary 2. *Consider an environment of exchange economies defined in Example 1 where $N \in \mathcal{N}$ implies $S \in \mathcal{N}$ for all non-empty subsets $S \subseteq N$ with $|S| \geq 2$. Suppose that a social choice correspondence $\varphi \subseteq IP$ is implemented in either noncooperative or cooperative equilibrium by a local and non-exclusive effectivity form with common state space. Then $\varphi \subseteq C$.*

In Theorem 2 and Corollary 2, anonymity is not used. It turns out that adding anonymity gives us a characterization of Walrasian equilibrium, provided that the equilibrium concept is noncooperative.

Theorem 3. *Consider the environment of exchange economies defined in Example 1 where $\mathcal{N}$ is the set of all non-empty finite subsets $N \subseteq \mathbb{N}$ with $|N| \geq 2$. Suppose that a social choice correspondence $\varphi \subseteq P$ can be implemented in noncooperative equilibrium by an anonymous, local, and non-exclusive effectivity form with common state space. Then $\varphi \subseteq W$.*

Proof. Suppose that a sub-correspondence $\varphi \subseteq P$ is implemented in noncooperative equilibrium by an anonymous, local, and non-exclusive effectivity form $\Gamma = (Z, E, H^{N,\omega})_{N \in \mathcal{N}, \omega \in \Omega^N}$ with common state space. Given an economy $e = (N, R, \omega) \in \mathcal{E}$,

[15] This step requires that S contain at least two agents. If $S = \{i\}$, then $y_i = \omega_i$ and y is trivially efficient for $(\{i\}, R'_i, \omega_i)$ since preferences are monotonic.

suppose, by way of contradiction, that there exists an allocation $x \in \varphi(e) \setminus W(e)$. For a positive integer r, let $r * x$ and $r * e$ denote the r-fold replica of x and e, respectively, in the sense of Debreu and Scarf (1963). Debreu and Scarf have proved that there exists a positive integer r such that $r * x$ is not in the core of $r * e$. Then for a sufficiently large r, $r * x$ is not in the *quasi-core* of $r * e$. Now, let $z \in Z$ be a state such that (x, z) is a noncooperative equilibrium of Γ for e. Then for all $i \in N$,

$$E(\omega_i, z) \subseteq L(x, R_i) = L(r * x, R_i).$$

This means that $(r * x, z)$ is a noncooperative equilibrium of Γ for $r * e$.[16] Thus $r * x \in \varphi(r * e)$. This is in contradiction with Theorem 2 since $r * x \notin QC(r * e)$. $\square$

We note that this theorem does not hold for cooperative equilibrium. A counter-example is the core mechanism (Example 6), which is anonymous, local, and non-exclusive. What it implements in cooperative equilibrium is the core correspondence.

7 Public Good Economies

Theorem 3 shows that Walrasian equilibrium is characterized by anonymity, localness, non-exclusivity, and Pareto efficiency. This section explores the implications of the same axioms in the context of economies with a public good. A particularly interesting question to ask is whether we obtain an analogous characterization of Lindahl equilibrium.

We first show that anonymity is not compatible with Pareto efficiency in public good economies.

Theorem 4. *In an environment of public good economies defined in Example 2, there exists no anonymous effectivity form that implements a social choice correspondence $\varphi \subseteq P$ in noncooperative equilibrium.*

Proof. Suppose that a correspondence $\varphi \subseteq P$ is implemented in noncooperative equilibrium by an anonymous effectivity form $\Gamma = (Z^{N,\omega}, E^{N,\omega}, H^{N,\omega})_{N \in \mathcal{N}, \omega \in \Omega^N}$. Consider a two-person economy $(N, R, \omega) \in \mathcal{E}$ where $N = \{1, 2\}$, $(\omega_1, \omega_2) = (10, 10)$, and the utility functions are given by

$$u_i(y, x_i) = \alpha_i \sqrt{y} + x_i \qquad \text{for all } i \in \{1, 2\}$$

[16] This step requires the equilibrium concept to be noncooperative.

where $(\alpha_1, \alpha_2) = (1, 3)$.[17] Let $(y, x) \in \varphi(N, R, \omega)$. By Pareto efficiency, the public-good level is $y = 4$. Feasibility then implies $x_1 + x_2 = 16$, and so there exists $k \in \{1, 2\}$ such that $x_k \leq 8$. Let $z \in Z^{N,\omega}$ be a state such that $((y, x), z)$ is a noncooperative equilibrium of Γ for (N, R, ω). Then $((y, x), z)$ is also a noncooperative equilibrium of Γ for (N, R', ω) where $R'_1 = R'_2 = R_k$. Agent k cannot block (y, x) since his preferences have not changed. Neither can agent $j \neq k$ since his effectivity set is the same as agent k's and $x_j \geq x_k$. However, (y, x) is not Pareto efficient for (N, R', ω), a contradiction. □

The result is intuitive given that agents should have different MRS at Pareto efficient allocations in public good economies. The Lindahl mechanism uses personalized prices and it is intuitively clear that a personalized system is necessary to attain Pareto efficiency in public good economies. Theorem 4 confirms the intuition.

We now formulate the Lindahl mechanism as an effectivity form.

Example 8 (Lindahl Mechanism).

$$
\begin{aligned}
Z &= \{p \in \mathbb{R}^{\mathbb{N}}_+ : \sum_{i \in \mathbb{N}} p_i = 1\};\\
E^{N,\omega}(i, p) &= \{(y, x_i) \in \mathbb{R}^2_+ : p_i y + x_i \leq \omega_i\};\\
H^{N,\omega}(p) &= X(N, \omega).
\end{aligned}
$$

That is, the state specifies the personalized price of the public good for each *potential* agent, with the condition that the sum of the *all* prices is one. The effectivity set is given by the usual budget set. The noncooperative equilibrium of this mechanism coincides with the Lindahl equilibrium. □

Although the Lindahl mechanism violates anonymity, it satisfies localness and non-exclusivity. Thus, in light of Theorem 3 and similarity between Walrasian and Lindahl equilibria, it makes sense to ask whether Lindahl equilibrium is characterized by localness, non-exclusivity, and Pareto efficiency. The answer turns out to be negative. As the following example shows, there exists a local and non-exclusive mechanism that implements a correspondence $\varphi \subseteq P$ but whose equilibrium outcomes are not necessarily Lindahl allocations.

Example 9 (Nonlinear Lindahl Mechanism). The example is a variant of the Lindahl mechanism where the frontier of a budget set is not necessarily straight. Let F be the set of all functions $f_i \colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $f_i(0) = 0$. Then we define an

[17] The case when two-person economies are not admissible can be proved analogously.

effectivity form by

$$Z = \{f = (f_1, f_2, \dots) \in F^{\mathbb{N}} : \sum_{i \in \mathbb{N}} f_i(y) = y \text{ for all } y \in \mathbb{R}_+\};$$

$$E^{N,\omega}(i, f) = \{(y, x_i) \in \mathbb{R}^2_+ : f_i(y) + x_i \leq \omega_i\};$$

$$H^{N,\omega}(f) = X(N, \omega).$$

Here, $f_i(y)$ is the amount that agent i is asked to pay when the public-good level is y. The condition $\sum_{i \in \mathbb{N}} f_i(y) = y$ ensures that the total payments do not exceed the cost of the public good. This effectivity form is local, non-exclusive, and individually rational (i.e., $(0, \omega_i) \in E^{N,\omega}(i, f)$ since $f_i(0) = 0$).

This mechanism has a noncooperative equilibrium in any economy. This is simply because if $((y, x), p)$ is a Lindahl equilibrium, then $((y, x), f)$ is a noncooperative equilibrium where $f_i(y) = p_i y$ for all agents present in the economy. It is easy to see that not all noncooperative equilibrium outcomes are Lindahl allocations.

The "nonlinear" Lindahl mechanism implements a correspondence $\varphi \subseteq P$ in noncooperative equilibrium. To confirm this, let $((y, x), f)$ be a noncooperative equilibrium for economy $e = (N, R, \omega)$. Suppose, by contradiction, that there exists a feasible allocation (y', x') such that $(y', x'_i)\ P_i\ (y, x_i)$ for all $i \in N$. This means that (y', x'_i) is not in i's budget set, i.e., $f_i(y') + x'_i > \omega_i$ for each i. It then follows that $\sum_{i \in N} f_i(y') + \sum_{i \in N} x'_i > \sum_{i \in N} \omega_i$. This is a contradiction since $\sum_{i \in N} f_i(y') \leq y'$. $\square$

On the other hand, the characterization of the core in exchange economies can be easily extended to public good economies. We omit the proof since it is analogous to the one for exchange economies.

Theorem 5. *Consider an environment of public good economies defined in Example 2 where $N \in \mathcal{N}$ implies $S \in \mathcal{N}$ for all non-empty subsets $S \subseteq N$ with $|S| \geq 2$. Suppose that a social choice correspondence $\varphi \subseteq IP$ is implemented in either noncooperative or cooperative equilibrium by a local and non-exclusive effectivity form with common state space. Then $\varphi \subseteq C$.*

8 General Environments

This section considers the same list of axioms in general environments. The following condition is a necessary and sufficient condition.

Definition. A social choice correspondence φ satisfies **intersection monotonic-**

ity if for all $(N, R, \omega) \in \mathcal{E}$, all $x \in \varphi(N, R, \omega)$, all $(N', R', \omega') \in \mathcal{E}$, and all $x' \in X(N', \omega')$, if for all $i \in N'$,

$$L(x', R'_i) = C \qquad \text{if } \omega'_i \notin \{\omega_j\}_{j \in N}; \tag{3}$$

$$L(x', R'_i) \supseteq \Big[\bigcap_{\substack{j \in N \text{ s.t.} \\ \omega_j = \omega'_i}} L(x, R_j) \Big] \qquad \text{otherwise,} \tag{4}$$

then $x' \in \varphi(N', R', w')$.

Condition (3) says that if $\omega'_i \neq \omega_j$ for all $j \in N$, then x' gives agent i a most preferred consumption bundle for R'_i. Such a consumption bundle does not exist in typical economic environments with monotonic preferences, in which case (3) does not hold. Condition (4) says that if $\omega'_i = \omega_j$ for some $j \in N$, then x' and R'_i are such that any allocation y that is preferred to x' for R'_i is also preferred to x for R_j for some $j \in N$ with $\omega_j = \omega'_i$.

Theorem 6. *A social choice correspondence φ is implemented by an anonymous, local, and non-exclusive effectivity form with common state space in noncooperative equilibrium if and only if φ satisfies intersection monotonicity.*

Proof. See the Appendix. □

Theorem 6 pertains to noncooperative equilibrium. For cooperative equilibrium, the following slightly weaker condition is a necessary and sufficient condition.

Definition. A social choice correspondence φ satisfies **coalitional intersection monotonicity** if for all $(N, R, \omega) \in \mathcal{E}$, all $x \in \varphi(N, R, \omega)$, all $(N', R', \omega') \in \mathcal{E}$, and all $x' \in X(N', \omega')$, if for all $S \subseteq N'$,

$$L(x', R'_S) = C^S \qquad \text{if } \omega'_S \nsim \omega_T \text{ for all } T \subseteq N; \tag{5}$$

$$L(x', R'_S) \supseteq \Big[\bigcap_{\substack{T \subseteq N \text{ s.t.} \\ \omega_T \sim \omega'_S}} L(x, R_T) \Big] \qquad \text{otherwise,} \tag{6}$$

then $x' \in \varphi(N', R', w')$.[18]

Condition (6) says that if $\omega'_S \sim \omega_T$ for some $T \subseteq N$, then x' and R'_S are such that

[18]For the definition of $\omega_T \sim \omega'_S$, see the definition of anonymity. By $\omega'_S \nsim \omega_T$, we mean that it is not the case that $\omega'_S \sim \omega_T$.

any allocation y that dominates x' for coalition S with R'_S also dominates x for R_T for some coalition $T \subseteq N$ such that $\omega_T \sim \omega'_S$.

Theorem 7. *A social choice correspondence φ is implemented by an anonymous, local, and non-exclusive effectivity form with common state space in cooperative equilibrium if and only if φ satisfies coalitional intersection monotonicity.*

Proof. See the Appendix. □

It is easy to see that intersection monotonicity implies coalitional intersection monotonicity. The converse is false as the following example shows.

Example 10 (Coalitional Intersection Monotonicity Does not Imply Intersection Monotonicity). Consider a voting environment where there are two agents (1 and 2) and two alternatives (a and b), i.e., $\mathcal{N} = \{\{1,2\}\}$ and $X(N,\omega) = \{a,b\}$. Each agent has a strict ranking over $\{a,b\}$, and so the set of admissible preferences is $\mathcal{R} = \{R^a, R^b\}$ where $a \; P^a \; b$ and $b \; P^b \; a$. There are two types of agents and we let $\Omega = \{a,b\}$. Now, consider an anonymous, local, and non-exclusive effectivity form Γ where the common state space is arbitrary and

$$E(\omega_i, z) = \emptyset;$$
$$E((\omega_1,\omega_2), z) = \begin{cases} \{\omega_1\} & \text{if } \omega_1 = \omega_2; \\ \emptyset & \text{if } \omega_1 \neq \omega_2. \end{cases}$$

That is, a coalition is effective for some alternative if and only if the coalition consists of two agents of the same type. A coalition consisting of two agents of type a (resp. b) is effective for alternative a (resp. b). The cooperative equilibrium outcomes of the effectivity form are

$$C(\Gamma,(N,R,\omega)) = \begin{cases} \{a\} & \text{if } R = (R^a,R^a) \text{ and } \omega = (a,a); \\ \{b\} & \text{if } R = (R^b,R^b) \text{ and } \omega = (b,b); \\ \{a,b\} & \text{otherwise.} \end{cases} \tag{7}$$

That is, when both agents in the economy are of type a and prefer a to b, then the grand coalition blocks b and so the unique cooperative equilibrium outcome is a. When agents differ in their types or preferences, no alternative is blocked by any coalitions.

This implies that the correspondence defined by $\varphi(\cdot) \equiv C(\Gamma,\cdot)$ satisfies coalitional intersection monotonicity (Theorem 7). But φ does not satisfy intersection monotonicity. To see this, note that

$$b \in \varphi(\{1,2\},(R^a,R^a),(a,b)). \tag{8}$$

This and intersection monotonicity imply $b \in \varphi(\{1,2\},(R^a,R^a),(a,a))$, which is in contradiction with (7).

The underlying logic is simple. If φ can be implemented in noncooperative equilibrium by an anonymous, local, and non-exclusive effectivity form, then (8) implies that no agent of type a can block b in the underlying equilibrium state. It then follows that, when both agents are of type a, alternative b is a noncooperative equilibrium outcome although b is not φ-optimal.

9 Concluding Remark

The basic idea behind our reduced-form approach to implementation is to use the notions and ideas of price theory. Given that the price theory offers the best formulation of one of the most important resource-allocation mechanisms, we find it reasonable to use some of the ideas of the theory for a more general study of mechanism design. The notion of mechanism proposed in this paper is a straightforward generalization of the Walrasian mechanism, and the ideas behind anonymity and localness are not new in price theory. We depart from price theory by treating mechanisms as variable and asking whether there exists a mechanism that achieves a given social choice correspondence in equilibrium. Our axiomatic characterizations are rather simple, but they demonstrate that our approach can generate interesting implementation-theoretic results and useful insights into resource-allocation mechanisms.

A Appendix: Proofs

A.1 Proof of Theorem 1

We give a proof only for the case when the equilibrium concept is cooperative and the mechanism is not required to be closed. The proofs for the other cases are essentially identical to the one given below.

To prove the "only if" part, let φ be a correspondence implementable in cooperative equilibrium by a mechanism $\Gamma = (Z^{N,\omega}, E^{N,\omega}, H^{N,\omega})_{N\in\mathcal{N},\omega\in\Omega^N}$. To show that φ is Gevers monotonic, let $e = (N, R, \omega) \in \mathcal{E}$, $x \in \varphi(e)$, and $R' \in \mathcal{R}^N$ be such that

$$L(x, R_i) \subseteq L(x, R'_i) \quad \text{for all } i \in N. \tag{9}$$

Since Γ implements φ in cooperative equilibrium, there exists a state $z \in Z^{N,\omega}$ such that (x, z) is a cooperative equilibrium of Γ for e. Thus,

$$\begin{aligned} x &\in H^{N,\omega}(z); \\ E^{N,\omega}(S, z) &\subseteq L(x, R_S) \quad \text{for all } S \subseteq N. \end{aligned}$$

It is easy to verify that (9) implies $L(x, R_S) \subseteq L(x, R'_S)$. Thus (x, z) is also a cooperative equilibrium of Γ for (N, R', ω). This establishes $x \in \varphi(N, R', \omega)$.

To prove the "if" part, let φ be a correspondence that is Gevers monotonic. We show that φ is implemented in cooperative equilibrium by the mechanism Γ defined by

$$\begin{aligned} Z^{N,\omega} &= \{(x, R) \in X(N, \omega) \times \mathcal{R}^N : x \in \varphi(N, R, \omega)\}; \\ E^{N,\omega}(S, (x, R)) &= L(x, R_S); \\ H^{N,\omega}(x, R) &= \{x\}. \end{aligned}$$

That is, a state for (N, ω) is a pair (x, R) such that $x \in \varphi(N, R, \omega)$. In state (x, R), each coalition is effective for any profile of consumption bundles that does not dominate x for R_S. To see that this mechanism implements φ in cooperative equilibrium, take any economy $e = (N, R, \omega) \in \mathcal{E}$.

(i) To prove $\varphi(e) \subseteq C(\Gamma, e)$, let $x \in \varphi(e)$. Then it is evident that $(x, (x, R))$ is a cooperative equilibrium of Γ for e. Thus $x \in C(\Gamma, e)$.

(ii) To prove $C(\Gamma, e) \subseteq \varphi(e)$, let $x \in C(\Gamma, e)$. Then there exists a state $(x', R') \in Z^{N,\omega}$ such that $(x, (x', R'))$ is a cooperative equilibrium of Γ for e. Since x has to be admissible in the equilibrium state, $x' = x$. Since $(x, (x, R'))$ is also a *non*cooperative equilibrium, we have

$$L(x, R_i) \supseteq E^{N,\omega}(i, (x, R')) \equiv L(x, R'_i).$$

Thus, by Gevers monotonicity, $x \in \varphi(N, R, \omega)$. $\square$

Remark 5. Obviously, the effectivity form used in the "if" part of the above proof implements φ in noncooperative as well as cooperative equilibrium.

Remark 6. The effectivity form used in the "if" part of the above proof satisfies *superadditivity*, namely, for any pair of disjoint coalitions $S, T \subseteq N$, if $y_S \in E^{N,\omega}(S, z)$ and $y_T \in E^{N,\omega}(T, z)$, then $y_{S \cup T} \in E^{N,\omega}(S \cup T, z)$. Superadditivity is reasonable although we do not require it. If superadditivity is not required, then implementability in noncooperative equilibrium trivially implies implementability in cooperative equilibrium. The above proof establishes that even if effectivity forms are required to be superadditive, implementability in noncooperative equilibrium is equivalent to implementability in cooperative equilibrium.

A.2 Proof of Theorem 6

("only if" part) Suppose that a correspondence φ is implemented in noncooperative equilibrium by an effectivity form $\Gamma = (Z, E, H^{N,\omega})_{N \in \mathcal{N}, \omega \in \Omega^N}$ that is anonymous, local, non-exclusive, and satisfies common state space. To show that φ is intersection monotonic, take any $e = (N, R, \omega) \in \mathcal{E}$, $e' = (N', R', \omega') \in \mathcal{E}$, $x \in \varphi(e)$, and $x' \in X(N', \omega')$ such that for all $i \in N'$, (3) and (4) holds. Since $x \in \varphi(e) = N(\Gamma, e)$, there exists a state $z \in Z$ such that

$$E(\omega_i, z) \subseteq L(x, R_i) \quad \text{for all } i \in N. \tag{10}$$

We would like to prove that for all $i \in N'$,

$$E(\omega'_i, z) \subseteq L(x', R'_i). \tag{11}$$

This holds trivially for i such that $\omega'_i \notin \{\omega_j\}_{j \in N}$ by (3). Thus consider $i \in N'$ for whom there exists $j \in N$ such that $\omega_j = \omega'_i$. Then $E(\omega'_i, z) = E(\omega_j, z) \subseteq L(x, R_j)$ by (10).

Since this holds for all $j \in N$ such that $\omega_j = \omega'_i$, we have

$$E(\omega'_i, z) \subseteq \bigcap_{\substack{j \in N \text{ s.t.} \\ \omega_j = \omega'_i}} L(x, R_j) \subseteq L(x', R'_i),$$

where the second inclusion follows from (4). Thus we conclude that (11) holds for all $i \in N'$. Since Γ is non-exclusive, this implies $x' \in N(\Gamma, (N', R', \omega')) = \varphi(N', R', \omega')$.

("if" part) Let φ be a correspondence satisfying intersection monotonicity. We now define an effectivity form that satisfies all of our axioms:

$$Z = \{(e, x) : e \in \mathcal{E} \text{ and } x \in \varphi(e)\};$$

$$E(\omega_i, (N', R', \omega', x')) = \begin{cases} C & \text{if } \omega_i \notin \{\omega'_j\}_{j \in N'}; \\ \bigcap_{\substack{j \in N' \text{ s.t.} \\ \omega'_j = \omega_i}} L(x', R'_j) & \text{otherwise;} \end{cases} \tag{12}$$

$$H^{N,\omega}(z) = X(N, \omega).$$

We show that this mechanism implements φ in noncooperative equilibrium.

Take any economy $e = (N, R, \omega) \in \mathcal{E}$. We first show that $\varphi(e) \subseteq N(\Gamma, e)$. So, let $x \in \varphi(e)$. Then $z \equiv (e, x)$ is an element of Z. It is evident that for all $i \in N$,

$$E(\omega_i, z) = \bigcap_{\substack{j \in N \text{ s.t.} \\ \omega_j = \omega_i}} L(x, R_j) \subseteq L(x, R_i).$$

Thus $(x, (e, x))$ is a noncooperative equilibrium of Γ for e. Thus $x \in N(\Gamma, e)$.

To show that $N(\Gamma, e) \subseteq \varphi(e)$, let $x \in N(\Gamma, e)$. This means that there exists $z' \equiv (N', R', \omega', x')$ in Z such that for all $i \in N$,

$$E(\omega_i, z') \subseteq L(x, R_i).$$

This and (12) mean that for all $i \in N$,

$$\begin{aligned} C &= L(x, R_i) && \text{if } \omega_i \notin \{\omega'_j\}_{j \in N'}; \\ \Big[\bigcap_{\substack{j \in N' \text{ s.t.} \\ \omega'_j = \omega_i}} L(x', R'_j)\Big] &\subseteq L(x, R_i) && \text{otherwise.} \end{aligned}$$

Hence, by intersection monotonicity, $x \in \varphi(N, R, \omega)$. $\square$

A.3 Proof of Theorem 7

("only if" part) Suppose that a correspondence φ is implemented in cooperative equilibrium by an effectivity form $\Gamma = (Z, E, H^{N,\omega})_{N \in \mathcal{N}, \omega \in \Omega^N}$ that satisfies all of our axioms. To show that φ satisfies coalitional intersection monotonicity, take any $e = (N, R, \omega) \in \mathcal{E}$, $x \in \varphi(e)$, $e' = (N', R', \omega') \in \mathcal{E}$, and $x' \in X(N', \omega')$ such that for all $S \subseteq N'$, (5) and (6) hold. Since $x \in \varphi(e) = C(\Gamma, e)$, there exists a state $z \in Z$ such that

$$E(\omega_S, z) \subseteq L(x, R_S) \quad \text{for all } S \subseteq N.$$

We would like to prove that for all $S \subseteq N'$,

$$E(\omega'_S, z) \subseteq L(x', R'_S). \tag{13}$$

This holds trivially if there exists no $T \subseteq N$ such that $\omega'_S \sim \omega_T$, by (5). Thus consider $S \subseteq N'$ such that $\omega'_S \sim \omega_T$ for some $T \subseteq N$. Then $E(\omega'_S, z) = E(\omega_T, z) \subseteq L(x, R_T)$ by (6). Since this holds for all $T \subseteq N$ such that $\omega_T \sim \omega'_S$, we have

$$E(\omega'_S, z) \subseteq \bigcap_{\substack{T \subseteq N \text{ s.t.} \\ \omega_T \sim \omega'_S}} L(x, R_T) \subseteq L(x', R'_S),$$

where the second inclusion follows from (6). Thus we conclude that (13) holds for all $S \subseteq N'$. Since Γ is non-exclusive, this implies $x' \in C(\Gamma, (N', R', \omega')) = \varphi(N', R', \omega')$.

("if" part) Let φ be a correspondence that satisfies coalitional intersection monotonicity. We define an effectivity form that satisfies all of our axioms:

$$\begin{aligned}
Z &= \{(e, x) : e \in \mathcal{E} \text{ and } x \in \varphi(e)\}; \\
E(\omega_S, (N', R', \omega', x')) &= \begin{cases} C^S & \text{if } \omega_S \nsim \omega'_T \text{ for all } T \subseteq N'; \\ \displaystyle\bigcap_{\substack{T \subseteq N' \text{ s.t.} \\ \omega'_T \sim \omega_S}} L(x', R'_T) & \text{otherwise;} \end{cases} \\
H^{N,\omega}(z) &= X(N, \omega).
\end{aligned} \tag{14}$$

We show that this mechanism implements φ in cooperative equilibrium.

Take any economy $e = (N, R, \omega) \in \mathcal{E}$. We first show that $\varphi(e) \subseteq C(\Gamma, e)$. So, let

$x \in \varphi(e)$. Then $z \equiv (e, x)$ is an element of Z. It is evident that for all $S \subseteq N$,

$$E(\omega_S, (e, x)) = \bigcap_{\substack{T \subseteq N \text{ s.t.} \\ \omega_T \sim \omega_S}} L(x, R_T) \subseteq L(x, R_S)$$

simply because $\omega_S \sim \omega_S$. Thus $(x, (e, x))$ is a cooperative equilibrium of Γ for e. Thus $x \in C(\Gamma, e)$.

To show that $C(\Gamma, e) \subseteq \varphi(e)$, let $x \in C(\Gamma, e)$. This means that there exists a state $z' \equiv (N', R', \omega', x')$ in Z such that for all $S \subseteq N$,

$$E(\omega_S, z') \subseteq L(x, R_S).$$

This and (14) mean that for all $S \subseteq N$,

$$\begin{aligned} C^S = L(x, R_S) \qquad & \text{if } \omega_S \nsim \omega'_T \text{ for all } T \subseteq N'; \\ \Big[\bigcap_{\substack{T \subseteq N' \text{ s.t.} \\ \omega'_T \sim \omega_S}} L(x', R'_T) \Big] \subseteq L(x, R_S) \qquad & \text{otherwise.} \end{aligned}$$

Hence, by coalitional intersection monotonicity, $x \in \varphi(N, R, \omega)$. $\square$

B Appendix: Independence of the Axioms

This section examines the independence of the axioms in Theorem 3. We exhibit effectivity forms that satisfy all of the axioms in the theorem except for one. To make this exercise more interesting, we have chosen mechanisms that satisfy the individual rationality condition, i.e., $\omega_i \in E^{N,\omega}(i, z)$.

Example 11 (Not Anonymous). We exhibit an effectivity form that satisfies all axioms in Theorem 3 except for anonymity. We define a state as a list $z = (k, p, \delta, R_k) \in \mathbb{N} \times \mathbb{R}^\ell_{++} \times \mathbb{R}_+ \times \mathcal{R}$. To describe the effectivity set, fix a state $z = (k, p, \delta, R_k)$ and $\omega_k \in \Omega$. Let $x^*_k \in \mathbb{R}^\ell_+$ be the most preferred bundle in

$$\{y_k \in \mathbb{R}^\ell_+ : p \cdot y_k \leq \max\{0, p \cdot \omega_k - \delta\}\}$$

for R_k. Then the effectivity set of agent k with ω_k in state (k, p, δ, R_k) is defined by

$$E(k, \omega_k, (k, p, \delta, R_k)) = \begin{cases} \{y_k \in \mathbb{R}^\ell_+ : x^*_k \, R_k \, y_k\} & \text{if } x^*_k \, R_k \, \omega_k \text{ and } \delta > 0; \\ \{y_k \in \mathbb{R}^\ell_+ : p \cdot y_k \leq p \cdot \omega_k\} & \text{otherwise.} \end{cases}$$

The effectivity set of agents $i \neq k$ is defined by

$$E(i, \omega_i, (k, p, \delta, R_k)) = \{y_k \in \mathbb{R}^\ell_+ : p \cdot y_i \leq p \cdot \omega_i + \delta\}. \tag{15}$$

Finally, we let $H^{N,\omega}(z) = X(N, \omega)$ for all states z.

This mechanism has a noncooperative equilibrium in any economy. To see this, let (x, p) be a Walrasian equilibrium of $e = (N, R, \omega)$. Then trivially, $(x, (i, p, 0, R_i))$ is a noncooperative equilibrium regardless of $i \in \mathbb{N}$.

The noncooperative equilibria of the mechanism are all Pareto efficient. To see this, let (x, z) with $z = (k, p, \delta, R'_k)$ be a noncooperative equilibrium for $e = (N, R, \omega)$. If $\delta = 0$, then x is trivially Pareto efficient for e. So consider the case when $\delta > 0$. Then (15) implies that, for all $i \in N \setminus \{k\}$, we have $p \cdot x_i \geq p \cdot \omega_i + \delta > p \cdot \omega_i$. Feasibility of x then implies that k is in N and

$$p \cdot x_k < p \cdot \omega_k. \tag{16}$$

Let $x^*_k \in \mathbb{R}^\ell_+$ be the most preferred bundle in $\{y_k \in \mathbb{R}^\ell_+ : p \cdot y_k \leq \max\{0, p \cdot \omega_k - \delta\}\}$ for R'_k. Then (16) implies that $E(k, \omega_k, z) = \{y_k \in \mathbb{R}^\ell_+ : x^*_k \, R'_k \, y_k\}$. Suppose, by contradiction, that x is not Pareto efficient for e. Then there exists an allocation $y \in X(N, \omega)$ such that $y_i \, P_i \, x_i$ for all $i \in N$. This implies that $p \cdot y_k > p \cdot \omega_k - \delta$ and, for all $i \neq k$, $p \cdot y_i > p \cdot \omega_i + \delta$. Thus

$$p \cdot \sum_{i \in N} y_i > p \cdot \sum_{i \in N} \omega_i + \delta(|N| - 2),$$

which is a contradiction since $|N| \geq 2$.

An analogous argument shows that the effectivity form has a noncooperative equilibrium with $\delta > 0$ only in the case of two agents. It is easy to see that, for some economy with two agents, there exists a noncooperative equilibrium that yields a non-Walrasian allocation. □

Example 12 (Not Local). We exhibit an effectivity form that satisfies all the axioms in Theorem 3 except for localness.[19] Define a state as a pair (p, f) where p belongs to $\mathbb{R}^{\ell}_{++}$ and f is a function that associates with each $N \in \mathcal{N}$ and each $\omega \in \Omega^N$ an element $f(N, \omega) = (x, R) \in X(N, \omega) \times \mathcal{R}^N$ such that x is Pareto efficient for (N, R, ω) and $x_i \, R_i \, \omega_i$ for all $i \in N$. Let Z be the set of all pairs (p, f) of this form. The effectivity set is defined by

$$E^{N,\omega}(i, (p, f)) = \begin{cases} \{y_i \in \mathbb{R}^{\ell}_{+} : p \cdot y_i \leq p \cdot \omega_i\} & \text{if } \omega_j = \omega_k \text{ for some distinct } j, k \in N; \\ L(x, R_i) & \text{otherwise} \end{cases}$$

where $(x, R) = f(N, \omega)$. That is, when there are agents with identical endowments, f is ignored and the market mechanism is used. On the other hand, when no two agents have identical endowments, p is ignored and agent i's effectivity set is set to the lower contour set $L(x, R_i)$ where (x, R) is the one assigned by f. Finally, we set $H^{N,\omega}(p, f) = X(N, \omega)$.

This mechanism has a noncooperative equilibrium. To see this, let (x, p) be a Walrasian equilibrium of $e = (N, R, \omega)$. Then, for f such that $f(N, \omega) = (x, R)$, $(x, (p, f))$ is a noncooperative equilibrium of the mechanism for e.

The noncooperative equilibria of the mechanism are all Pareto efficient. To see this, let $(x, (p, f))$ be a noncooperative equilibrium of the mechanism for $e = (N, R, \omega)$. If $\omega_i = \omega_j$ for some $i \neq j$, then (x, p) is a Walrasian equilibrium and so x is Pareto efficient. Thus, suppose that $\omega_i \neq \omega_j$ for all $i \neq j$. Let $f(N, \omega) = (x', R')$, where x' is Pareto efficient for (N, R', ω). Let $p' \in \mathbb{R}^{\ell}_{++}$ be a supporting price vector for x', i.e., for all $y'_i \in \mathbb{R}^{\ell}_{+}$, if $y_i \, P'_i \, x'_i$, then $p' \cdot y_i > p' \cdot x'_i$. Suppose, by contradiction, that x is not Pareto efficient for (N, R, ω). Then there exists an allocation $y \in X(N, \omega)$ such that $y_i \, P_i \, x_i$ for all $i \in N$. Since agent i cannot block x_i and he prefers y_i to x_i, it follows that $y_i \notin E^{N,\omega}(i, (p, f))$. This means $y_i \notin L(x', R'_i)$, and thus $p' \cdot y_i > p' \cdot x'_i$. Since this holds for all $i \in N$, we have $p' \cdot \sum_i y_i > p' \cdot \sum_i x'_i$, a contradiction. □

Example 13 (Exclusive). We first introduce new definitions. Given a set $B \subseteq \mathbb{R}^{\ell}_{+}$ and a point $x_i \in B$, we say that a vector $p \in \mathbb{R}^{\ell}_{++}$ is **strongly normal** to B at x_i if

[19] A simpler but not individually rational example is the market mechanism operated from equal division. Specifically, let $Z = \mathbb{R}^{\ell}_{+} \setminus \{0\}$, $H^{N,\omega}(p) = X(N, \omega)$, and

$$E^{N,\omega}(i, p) = \{x_i \in \mathbb{R}^{\ell}_{+} : p \cdot x_i \leq p \cdot \sum_{j \in N} \omega_j / |N|\}.$$

That is, the planner divides the aggregate endowments equally among the agents before operating the market mechanism.

for all $R_i \in \mathcal{R}$, if $x_i \, R_i \, y_i$ for all $y_i \in B$, then for all $y_i \in \mathbb{R}^\ell_+$,

$$y_i \, P_i \, x_i \Longrightarrow p \cdot y_i > p \cdot x_i.$$

For example, if $x_i \gg 0$ and the boundary of B has a kink at x_i, then there exists no strongly normal vector to B at x_i.

We now define an effectivity form as follows. Let a state be a correspondence B that associates with each $\omega_i \in \Omega$ a set $B(\omega_i) \subseteq \mathbb{R}^\ell_+$ such that

1. $\omega_i \in B(\omega_i)$ for all $\omega_i \in \Omega$; and

2. there exists $p \in \mathbb{R}^\ell_{++}$ such that, for all $\omega_i \in \Omega$, p is strongly normal to $B(\omega_i)$ at ω_i.

Let Z be the set of all correspondences B of this form. For example, if we take a vector $p \in \mathbb{R}^\ell_{++}$ and define B by $B(\omega_i) = \{y_i \in \mathbb{R}^\ell_+ : p \cdot y_i \leq p \cdot \omega_i\}$, then $B \in Z$.

Let the effectivity set be given by $E(\omega_i, B) = B(\omega_i)$. Finally, we define $H^{N,\omega}(B)$ to be the set of all allocations $x \in X(N, \omega)$ such that

1. $x_i \in B(\omega_i)$ for all $i \in N$; and

2. there exists $p \in \mathbb{R}^\ell_{++}$ such that, for all $i \in N$, p is strongly normal to $B(\omega_i)$ at x_i.

Note that $H^{N,\omega}(B)$ is non-empty since it contains the initial allocation, ω. See Figure 1.

The mechanism has a noncooperative equilibrium in any economy. To see this, let $e = (N, R, \omega)$ be an economy and let (x, p) be a Walrasian equilibrium of the economy. Consider a state $B \in Z$ defined by $B(\omega_i) = \{y_i \in \mathbb{R}^\ell_+ : p \cdot y_i \leq p \cdot \omega_i\}$. Then it is easy to see that (x, B) is a noncooperative equilibrium of the mechanism for e.

Finally, the noncooperative equilibria of the mechanism are all Pareto efficient. To see this, let (x, B) be a noncooperative equilibrium of the mechanism for $e = (N, R, \omega)$. Since $x \in H^{N,\omega}(B)$, there exists $p \in \mathbb{R}^\ell_{++}$ such that, for all $i \in N$, p is strongly normal to $B(\omega_i)$ at x_i. Since x_i is a most preferred bundle in $B(\omega_i)$ for R_i, the definition of strong normality implies that for all $y_i \in \mathbb{R}^\ell_+$,

$$y_i \, P_i \, x_i \Longrightarrow p \cdot y_i > p \cdot x_i. \tag{17}$$

Suppose, by contradiction, that x is not Pareto efficient for e. Then there exists an allocation $y \in X(N, \omega)$ such that $y_i \, P_i \, x_i$ for all $i \in N$. Then, by (17), $p \cdot y_i > p \cdot x_i$ for all $i \in N$, and so $p \cdot \sum_i y_i > p \cdot \sum_i x_i$, a contradiction. $\square$

Example 14 (Not Pareto Efficient). Consider an effectivity form in which $E(\omega_i, z) = \{\omega_i\}$ and $H^{N,\omega}(z) = X(N, \omega)$. This mechanism is anonymous, local, and non-exclusive, but its noncooperative equilibrium allocations are not necessarily Walrasian.

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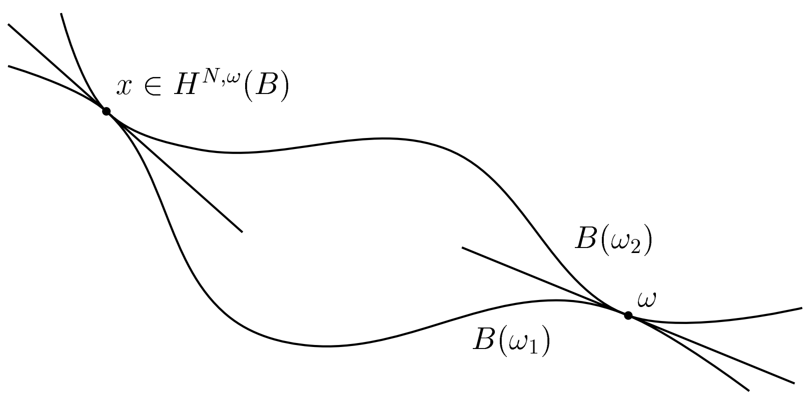

Figure 1: This is drawn in an Edgeworth box. The set $B(\omega_1)$ is the area below the lower curve. For this example, $H^{N,\omega}(B) = \{x, \omega\}$.