Columbia University

Department of Economics
Discussion Paper Series

The Knob of the Discord

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Discussion Paper No.: 0405-14

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December 2004

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December 5, 2004

Abstract

For (S, Σ) a measurable space, let $\mathcal{C}_1$ and $\mathcal{C}_2$ be convex, weak* closed sets of probability measures on Σ. We show that if $\mathcal{C}_1 \cup \mathcal{C}_2$ satisfies the Lyapunov property then there exists a set $A \in \Sigma$ such that $\min_{\mu_1 \in \mathcal{C}_1} \mu_1(A) > \max_{\mu_2 \in \mathcal{C}_2} \mu_2(A)$. We give applications to Maxmin Expected Utility and to the core of a lower probability.

1 Main result

If μ_1 and μ_2 are two probability measures on a σ-algebra Σ, then (by definition) $\mu_1 \neq \mu_2$ means that there exists a set $A \in \Sigma$ such that $\mu_1(A) > \mu_2(A)$. Equivalently, the two disjoint sets $\{\mu_1\}$ and $\{\mu_2\}$ can be separated by means of a linear functional having an especially simple form, namely one that is defined by an indicator function. Here, we are concerned with extending this property to sets of measures which are not singletons. Our motivation stems from some questions arising in the theory of decision making under uncertainty. We offer a simple application to this area after we prove our separation result. However, given its nature, we expect our result to be widely applicable in areas different from the one we consider.

Let (S, Σ) be a measurable space and let $\Delta(\Sigma)$ denote the set of all (countably additive) probability measures on Σ. $\Delta(\Sigma)$ is a subset of the norm dual of the Banach space $B(\Sigma)$ of bounded, Σ-measurable functions.

Definition 1 *Let $\mathcal{C} = \{\mu_i\}_{i \in I} \subset \Delta(\Sigma)$. We say that $\mathcal{C}$ has the Lyapunov property if the range of the vector measure $(\mu_i)_{i \in I}$ on E is a convex and compact subset of $\mathbb{R}^I$ (equipped with the product topology), for all $E \in \Sigma$.*

Notice that if $\mathcal{C}$ has the Lyapunov property, then so does any subset of $\mathcal{C}$. Sets of measures with the Lyapunov property have special importance in the theory of decision making under uncertainty. For a decision maker described by a set of priors like in [8] or in [7], the Lyapunov property corresponds to the demand that the class of unambiguous events in the sense of [12] or [7] be "rich" (see Section 2).

Theorem 2 *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be convex, weak* closed subsets of $\Delta(\Sigma)$ such that $\mathcal{C}_1 \cup \mathcal{C}_2$ has the Lyapunov property. Then $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$ if and only if there exists A in Σ such that*

$$\min_{\mu_1 \in \mathcal{C}_1} \mu_1(A) > \max_{\mu_2 \in \mathcal{C}_2} \mu_2(A).$$

Proof. Since each $\mathcal{C}_i$ is weak* compact, then it is weak compact, and convexity of $\mathcal{C}_i$ implies that there exist a measure $\lambda_i \in \mathcal{C}_i$ such that $\mu_i \ll \lambda_i$ for all $\mu_i \in \mathcal{C}_i$, $i = 1, 2$ (see, for instance [3]). Hence, all the measures in $\mathcal{C}_1 \cup \mathcal{C}_2$ are absolutely continuous with respect to $\lambda = \frac{1}{2}\lambda_1 + \frac{1}{2}\lambda_2$, and the sets $\mathcal{C}_1'$ and $\mathcal{C}_2'$ of all Radon-Nikodym derivatives of elements of $\mathcal{C}_1$ and $\mathcal{C}_2$ are disjoint, weakly compact, and convex subsets of $\mathcal{L}^1(\lambda)$. The Separating Hyperplane Theorem (see, [4], V.2.10) guarantees that there exist $g_0 \in \mathcal{L}^\infty(\lambda) - \{0\}$, $\beta \in \mathbb{R}$, such that

$$\min_{f_1 \in \mathcal{C}_1'} \int g_0 f_1 d\lambda > \max_{f_2 \in \mathcal{C}_2'} \int g_0 f_2 d\lambda. \tag{1}$$

W.l.o.g. $0 \leq g_0(s) \leq 1$ for λ-almost all $s \in S$ (otherwise take $\frac{g_0 - \operatorname{essinf} g_0}{\|g_0 - \operatorname{essinf} g_0\|_\infty}$).

By assumption, $\mathcal{C}_1 \cup \mathcal{C}_2$ has the Lyapunov property. Hence, $\mathcal{C}_1' \cup \mathcal{C}_2'$ is thin in the sense of [10]. By Lemma 1 in [10], $g_0 = \chi_A + h$ where $A \in \Sigma$ and $h \in \mathcal{L}^\infty(\lambda)$ is such that $\int h f d\lambda = 0$ for all $f \in \mathcal{C}_1' \cup \mathcal{C}_2'$. For all $\mu \in \mathcal{C}_1 \cup \mathcal{C}_2$, setting $f = d\mu/d\lambda$ we have

$$\mu(A) = \int_A f d\lambda = \int \chi_A f d\lambda = \int (\chi_A + h) f d\lambda = \int g_0 f d\lambda$$

and Eq. (1) becomes

$$\min_{\mu_1 \in \mathcal{C}_1} \mu_1(A) > \max_{\mu_2 \in \mathcal{C}_2} \mu_2(A)$$

The converse is obvious. □

Corollary 3 *Under the assumptions of Theorem 2, $\mathcal{C}_1 \subseteq \mathcal{C}_2$ if and only if $\min_{\mu_1 \in \mathcal{C}_1} \mu_1(A) \geq \min_{\mu_2 \in \mathcal{C}_2} \mu_2(A)$ for all $A \in \Sigma$.*

Proof. Let $\min_{\mu_1 \in \mathcal{C}_1} \mu_1(A) \geq \min_{\mu_2 \in \mathcal{C}_2} \mu_2(A)$ for all $A \in \Sigma$. Assume that $\mathcal{C}_1$ is not contained in $\mathcal{C}_2$. Then there exists $\bar{\mu} \in \mathcal{C}_1 - \mathcal{C}_2$. Since $\mathcal{C}_2 \cup \{\bar{\mu}\}$ is thin, Theorem 2 yields that there exists $B \in \Sigma$ such that $\bar{\mu}(B) < \mu_2(B)$ for all $\mu_2 \in \mathcal{C}_2$. Therefore $\min_{\mu_1 \in \mathcal{C}_1} \mu_1(B) \leq \bar{\mu}(B) < \min_{\mu_2 \in \mathcal{C}_2} \mu_2(B)$, which is absurd. The converse is trivial. □

We conclude this section, by proving another separation result. This extends an obvious property of two nonatomic measures: if $\mu_1 \neq \mu_2$, there exist $A, B \in \Sigma$, $A \cap B = \emptyset$ such that $\mu_1(A) > \mu_1(B)$ and $\mu_2(A) < \mu_2(B)$. Notice that this is no longer true if the nonatomicity assumption is removed. In this form, the separation theorem turns out to be a basic tool in study unambiguous events in the sense of [14] and [5] (see [2]).

Corollary 4 *Under the assumptions in Theorem 2, there exist $A, B \in \Sigma$, $A \cap B = \emptyset$, such that $\mu_1(A) - \mu_1(B) > 0 > \mu_2(A) - \mu_2(B)$ for any $\mu_1 \in \mathcal{C}_1$ and any $\mu_2 \in \mathcal{C}_2$.*

Proof. Let $A \in \Sigma$ be such that $\mu_1(A) > \mu_2(A)$ for any $\mu_i \in \mathcal{C}_i$, $i = 1, 2$. Since $\mathcal{C}_1 \cup \mathcal{C}_2$ has the Lyapunov property, the range on S of the vector measure defined by $\mathcal{C}_1 \cup \mathcal{C}_2$ is compact and convex. Hence, for any $\alpha \in [0, 1]$ there exists $B \in \Sigma$ such that $\mu_1(B) = \mu_2(B) = \alpha$ for all $\mu_i \in \mathcal{C}_i$, $i = 1, 2$. Pick one such a B so that $2\mu(B) = \min_{\mu_1 \in \mathcal{C}_1} \mu_1(A) + \max_{\mu_2 \in \mathcal{C}_2} \mu_2(A)$ for all $\mu \in \mathcal{C}_1 \cup \mathcal{C}_2$. Then, $\mu_1(A) - \mu_1(B) > 0$ and $\mu_2(A) - \mu_2(B) < 0$ for all $\mu_i \in \mathcal{C}_i$, $i = 1, 2$.

If $A \cap B \neq \emptyset$, write $B = (A \cap B) \cup B'$ and $A = (A \cap B) \cup A'$. Then, for any $\mu_i \in \mathcal{C}_i$, $i = 1, 2$,

$$\begin{aligned}
\mu_1(A) - \mu_1(B) &= \mu_1(A \cap B) + \mu_1(A') - \mu_1(A \cap B) - \mu_1(B') = \mu_1(A') - \mu_1(B') \\
\mu_2(A) - \mu_2(B) &= \mu_2(A') - \mu_2(B')
\end{aligned}$$

and A' and B' do the job. □

2 Application: MEU preferences and lower probabilities

In the theory of decision making under uncertainty, one is concerned with a decision maker ranking the elements of a set $\mathcal{A}$ of mappings $a : S \to X$, where S is the state space and X the prize space. For the sake of simplicity, let X be a convex subset of a vector space and $\mathcal{A}$ be the set of all simple and measurable functions from S to X. The decision maker's ranking $\succeq$, is said to satisfy the Maxmin Expected Utility (MEU) criterion if and only if for $a, b \in \mathcal{A}$

$$a \succeq b \Leftrightarrow \min_{\mu \in \mathcal{C}} \int (u \circ a)\, d\mu \geq \min_{\mu \in \mathcal{C}} \int (u \circ b)\, d\mu,$$

where $u : X \to \mathbb{R}$ is a nonconstant and affine utility function on the prize space, and $\mathcal{C}$ is a weak* closed and convex set of finitely additive probability measures on (S, Σ). The willingness to bet of a MEU decision maker is the lower probability

$$\rho(A) = \min_{\mu \in \mathcal{C}} \mu(A), \quad \forall A \in \Sigma.$$

Lower (and upper) probabilities are of central importance also in quasi-Bayesian statistics (see, for instance [13]). The core of a lower probability ρ is the set *core* (ρ) of all finitely additive probability measures ν on (S, Σ) such that $\nu \geq \rho$.

Preferences satisfying the MEU criterion have been axiomatized in [8]. In [11] and [3] necessary and sufficient conditions on $\succeq$ are given that guarantee that all the measures in $\mathcal{C}$ be countably additive. An event $A \in \Sigma$ is unambiguous in the sense of Nehring [12] or Ghirardato, Maccheroni and Marinacci [7] if $\mu(A) = \mu'(A)$ for all $\mu, \mu' \in \mathcal{C}$. In [1] (Proposition 4), it was shown that (i) the class of unambiguous events is "rich", that is there exist unambiguous events of measure α for every $\alpha \in [0,1]$, and (ii) there exists a countably additive, nonatomic probability measure on the class of unambiguous events if and only if $\mathcal{C}$ has the Lyapunov Property.

In the context of Maxmin Expected Utility, a natural question is whether or not two MEU preferences with the same utility on the prize space and the same willingness to bet are necessarily the same preference. A related question in the theory of lower probabilities is whether or not the weak* closed and convex set $\mathcal{C}$ defining a lower probability ρ coincides with its core. The following example, due to Huber and Strassen [9], answers negatively to both questions.

Example 5 *Let* $S = \{1,2,3\}$, $X = \mathbb{R}$, $\mu = (\frac{1}{2}, \frac{1}{2}, 0)$, $\nu = (\frac{4}{6}, \frac{1}{6}, \frac{1}{6})$. *Consider two MEU preferences,* $\succsim_1$ *and* $\succsim_2$*, with* $u_1(x) = u_2(x) = x$ *for any* $x \in \mathbb{R}$ *and sets of priors*

$$\mathcal{C}_1 = co\{\mu, \nu\} \quad \text{and} \quad \mathcal{C}_2 = \left\{ \left(\frac{3+t}{6}, \frac{3-t-s}{6}, \frac{s}{6} \right) : 0 \le s, t \le 1 \right\}.$$

It is readily checked that:

- $\rho_1(A) = \min_{\mu_1 \in \mathcal{C}_1} \mu_1(A) = \min_{\mu_2 \in \mathcal{C}_2} \mu_2(A) = \rho_2(A)$ *for all* $A \subset S$*, but* $\succeq_1$ *is different from* $\succeq_2$*;*
- $\mathcal{C}_1$ *is a weak* closed and convex set defining the lower probability* ρ_1*, and it is strictly included in core* (ρ_1) *(which coincides with* $\mathcal{C}_2$*).*

Both conclusions are reverted under the assumptions of Theorem 2 as the next two corollaries (both building on Corollary 3) show. In reading Corollary 6, notice that point 1. amounts to say that $\succeq_1$ is more ambiguity averse than $\succeq_2$ (see Ghirardato and Marinacci [6]), and remember that xAy is the mapping from S to X taking value x on A and y on A^c.

Corollary 6 *Let* $\succeq_1$ *and* $\succeq_2$ *be two MEU preferences with (weak* closed and convex) sets of priors* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *contained in* $\Delta(\Sigma)$ *and such that* $\mathcal{C}_1 \cup \mathcal{C}_2$ *has the Lyapunov property. Then the following conditions are equivalent:*

1. *For all* $a \in \mathcal{A}$ *and* $x \in X$*,*
$$a \succeq_1 x \Rightarrow a \succeq_2 x. \tag{2}$$
2. *For fixed* $x \succ y$ *and all* $A \in \Sigma$*,*
$$xAy \succeq_1 z \Rightarrow xAy \succeq_2 z.$$
3. u_1 *is a positive affine transformation of* u_2 *and* $\rho_1 \le \rho_2$*.*

In particular, if $u_1 = u_2$ *and* $\rho_1 = \rho_2$*, then* $\succeq_1$ *coincides with* $\succeq_2$*.*

Corollary 7 *Let ρ be a lower probability such that core $(\rho) \subset \Delta(\Sigma)$ and core (ρ) has the Lyapunov property. Then core (ρ) is the weak* closed and convex hull of any subset $\mathcal{K}$ of $\Delta(\Sigma)$ such that*

$$\rho(A) = \inf_{\nu \in \mathcal{K}} \nu(A), \quad \forall A \in \Sigma.$$

The easy proofs are omitted.

References

[1] Amarante M. (2003), "Ambiguity, measurability and multiple priors, *Economic Theory*, forthcoming.

[2] Amarante M. and E. Filiz (2004), Ambiguous Events and Maxmin Expected Utility, mimeo.

[3] Chateaneuf A., F. Maccheroni, M. Marinacci and J-M Tallon (2003), "Monotone continuous multiple priors", *Economic Theory*, forthcoming.

[4] Dunford N. and Schwartz J. T. (1954) Linear operators, part I: general theory. Wiley-Interscience, London

[5] Epstein L.G. and J. Zhang (2001), "Subjective probabilities on subjectively unambiguous events", *Econometrica* **69**, 265-306.

[6] Ghirardato P. and M. Marinacci (2002), Ambiguity made precise: a comparative foundation, *Journal of Economic Theory* **102**, 251-289.

[7] Ghirardato P., F. Maccheroni and M. Marinacci (2003), "Differentiating ambiguity and ambiguity attitude", *Journal of Economic Theory* **118** (2004), 133-173.

[8] Gilboa I. and D. Schmeidler (1989), Maxmin expected utility with a non-unique prior, *Journal of Mathematical Economics* **18**, 141-153.

[9] Huber P.J. and V. Strassen (1973) "Minimax tests and the Neyman-Pearson lemma for capacities", *Annals of Statistics* **1**, 251-263.

[10] Kingman J.F.C. and A.P. Robertson (1968), "On a Theorem of Lyapunov", *Journal of the London Mathematical Society* **43**, 347-351.

[11] M. Marinacci (2002), Probabilistic sophistication and multiple priors, *Econometrica* **70**, 755-764.

[12] Nehring K.(1999), "Capacities and probabilistic beliefs: A precarious coexistence", *Mathematical Social Sciences* **38**, 197-213.

[13] Wasserman L. A. and Kadane J. B. (1990), "Bayes' theorem for Choquet capacities", *Annals of Statistics* **18**, 1328-1339.

[14] Zhang, J. (2002), "Subjective Ambiguity, Expected Utility and Choquet Expected Utility", *Economic Theory* **20**, 159-181.