A Coupled Viscoelastic and Damage Approach for Solids with Applications to Ice and Asphalt

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ABSTRACT

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As new materials are developed and further concerns on green alternatives and serviceability arise, understanding material behavior during the entire span of their lifetime becomes crucial to engineering applications. Moreover, many problems display a significant dependence to time and loading effects which, by varying across multiple time scales, require material models that incorporate these effects into any valid characterization and prediction. This dissertation aims at proposing a new approach to analyze and predict viscoelastic materials that deteriorate during multiple loading conditions. The model is constructed from mechanical and mathematical basis while satisfying physical laws.

In this work, the proposed constitutive law is used for the analysis of the mechanical properties of ice. The mechanical behavior, biaxial envelop and multiple loading types demonstrate the validity of the model when compared to experimental results and other ice models available in the literature. A rigorous calibration scheme for the viscoelastic and damage parameters is also presented.

Moreover, as material deterioration or damage is modeled in standard Finite Elements software, it is commonly known that computational results can be dependent on the spatial discretization or mesh. That is, damage zone and energy dissipation are dependent on the selection of the mesh yielding a disappearing damage zone and energy dissipation upon refinement. This non-physical behavior is corrected by the novel regularization approach proposed in this document, which introduces a length scale of the material and produces results that are no longer sensitive to the mesh selection.
The nonlocal damage model is finally used in the analysis of asphalt concrete viscoelastic behavior and cracking prediction. As presented in the ice case, a rigorous calibration approach is presented first followed by the validation to experimental data available in the literature under different loading conditions.

The coupled viscoelastic and damage model is compared to other model and their Finite Elements implementations are highlighted in terms of computational efficiency. A nonlinear coupled system for solving this problem is programmed as a User Element in a commercial Finite Element analysis software.
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Dedication

To my beloved wife Laura Juliana and my parents, Juan de Dios and Maria Eugenia.
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Chapter 1

Introduction

1.1 A Prony-series type viscoelastic solid coupled with a continuum damage law for polar ice modeling

Global warming has a profound effect on the evolution of polar ice in Antarctica and Greenland, increasing steadily over the last century [13]. Up to fifty percent of the total mass loss of ice can be attributed to calving [14, 15], a process during which icebergs at the end of marine-terminated glaciers break off and fall into the ocean. The mechanism driving the detachment of icebergs from ice shelves is complex and involves the initiation and propagation of fractures through the ice at scales ranging from crystal-sized imperfections to kilometer long rifts [16].

Many experimental studies have reported that ice behaves as a viscoelastic material [4, 5, 17] and exhibit three different stages of creep at constant stress: primary creep with hardening of the ice, secondary creep where ice deforms at a constant strain rate and tertiary creep characterized by strain softening.

Several models have been proposed to represent the viscoelastic behavior of polycrystalline ice [18–20] but involve complex tensor and eigen-value based expressions that might be cumbersome.
CHAPTER 1. INTRODUCTION

some to implement. Another widespread choice for incorporating viscosity effects is described by the well known Glen’s law [21]. While this choice is widely used to model glacier creep flow, it also adds nonlinearity to the system and from a numerical perspective requires very small integration time steps even before damage is initiated [1, 22].

The final stage of creep deformation is induced by the nucleation of microcracks in the ice and marks the onset of creep fracture. This plays a major role in the failure processes of ice and in particular in ice shelf calving. Previous work has shown that the calving process, which varies widely upon different ice flow dynamics, is not yet well understood or modeled [23–26]. A range of models attempting to predict the behavior of glaciers and its retreat have been proposed to address this complex mechanics. These models vary from linear elastic fracture mechanics (LEFM) approaches [27] to damage mechanics [8, 28–30] and even a combination of both has recently been proposed [31] to represent ice calving.

Given that fracture time scales are orders of magnitude faster than those of ice flow dynamics, it may not be necessary to employ a nonlinear Glenn’s creep model. Therefore in this work a Prony series model is proposed to capture the viscoelastic behavior of ice while its deterioration is accounted for by a rate-form damage law. Both constitutive laws are coupled and the proposed model is shown to be thermodynamically consistent.

While a Prony series representation is equivalent to the springs and dashpots arrangement (e.g. Generalized Maxwell model), obtaining the material parameters for all the necessary terms in the Prony series is not a trivial task. The problem of fitting those parameters to experimental data has been addressed in the literature by many authors including, the Procedure X [32, 33], the collocation method [34], multidata method [35], multidata method with sign control [36], window opening [37], power law pre-smoothing [38], optimized interconversion techniques [39], harmony search algorithms [40, 41], amongst others.

Damage initiation and propagation is modeled following Murakami’s damage model [42],
1.1. A PRONY-SERIES TYPE VISCOELASTIC SOLID COUPLED WITH A CONTINUUM DAMAGE LAW FOR POLAR ICE MODELING

where the state of damage is updated at each time step based on the current values of stress. Viscoelastic behavior in the model is achieved by assuming time-dependency on the shear component of the deviatoric stress, while the volumetric part of the stress remains purely elastic. Semi-analytical integration of the convolution integral in the constitutive law allows for fast computation of stresses and damage and also provides a stable time integration scheme.

Generality of the viscoelastic-damage model proposed is highlighted when compared against experimental data of ice under different loading conditions. Time dependent values of strain and damage of polycrystalline ice slabs subjected to different tensile stresses were reported by Mahrenholtz & Wu [4] and to different compressive stresses by Mellor & Cole [5]. Fresh-water and saline ice response under compressive strain rate were reported by Schulson et al. in [7] and [6], respectively. The proposed model is validated and good agreement with respect to the experimental data is demonstrated. The model proposed in this section is also compared to the model proposed by Duddu and Waisman [1, 22] and quantitatively shows similar trends on short times but differs significantly on the longer time scales. Furthermore the proposed constitutive model is implemented in a finite element code and is used to investigate surface crevasse propagation in grounded marine-terminating glaciers. The simulations, on idealized rectangular ice slabs in contact with the ocean, examine the depth and rate of damage propagation with increasing seawater depths at the ice/ocean interface and compared with the predictions in [8]. While some minor differences in the predictions of the two models are noted, the computational efficiency of the proposed model is significantly better. Indeed the Prony Series model offers an improved numerical stability mainly due to the semi analytical integration of the viscoelastic model which allows for a larger integration time step.
1.2 An Equivalent Stress-Gradient Regularization Model for Coupled Damage-Viscoelasticity

Continuum damage mechanics has been used to describe materials deterioration when a certain force or deformation threshold is exceeded. However, for materials that display viscoelastic rate dependence such as asphalt, polymers, biological tissues and even ice, damage effects may be significantly different from their elastic rate-independent counterpart.

Adequate description of viscoelastic materials failure require accurate prediction of damage initiation, propagation and growth rate in addition to their time-dependent response. Furthermore, appropriate numerical discretization with efficient computational implementation is also necessary in order to model realistic structures and converge to the true physics.

While the theoretical formulation of viscoelastic materials can be described from an instantaneous response to their complete loss of coherence, damage induces material softening which destabilizes the solid configuration [43]. This instability is reflected in static problems by the loss of ellipticity (or loss of hyperbolicity if dynamic effects are considered) of the system which renders it ill-posedness [44]. Moreover, this change in the governing equations induces pathological mesh-sensitive results, such as zero-energy modes upon mesh refinement, which are physically unsound [45]. For this purpose, the present work introduces a model that couples both theories of viscoelasticity and damage while predicting objective results.

Theoretical development of integral and differential forms of viscoelasticity were thoroughly discussed in [46–48] and a good overview of time-dependent mechanical properties of materials was presented by Tschoegl [49]. In the present work, a model for viscoelasticity with regular-
ized damage is introduced, in which the viscoelastic behavior is motivated from the Prony series model used by Londono et al. [3]. Viscoelastic behavior is obtained by assuming a time dependent deviatoric stress while an elastic volumetric response is maintained. Moreover, deviatoric stress components are integrated in time semi-analytically, as suggested by Taylor et al. [50] and Simo & Hughes [51].

The growth of microcavities and crack coalescence that leads to fracture, are typically attributed to the early analytical work of Kachanov [52] and Rabotnov [53]. The reader is directed to [54–61] for an extensive overview of different damage mechanics models. In the present work, damage initiation and propagation is motivated from the damage-rate formulation introduced by Murakami [42].

During damage, the loss of objectivity of the results is attributed to the lack of an intrinsic length (also called internal length, length-scale or characteristic length) that limits the strain softening during damage localization. Several models have been proposed to produce consistent strain localization by incorporating integral- or differential- type operators into standard damage mechanics formulations. Nonlocal integral formulations propose that quantities driving the damage are not only influenced by values at a single point (local) but also by neighboring values within a given influence domain. Integral models are typically attributed to the work of Bažant and his co-authors [62–66] and have been applied successfully to many damage models including discrete zone models [67], cohesive zone models [68], multiscale fatigue models [69], nano-mechanics fracture models [70] and more. A detailed review on nonlocal models for damage is presented in [71] and the textbooks [60, 72]. Jirásek & Marfia [73] compared between strain-based and displacement-based non-local approaches (mainly of the integral type).
CHAPTER 1. INTRODUCTION

Nonlocal differential methods, also known as gradient methods, were initially presented as an approximation of the local strain (or local internal variable) in the integral formulation of Peerlings et al. [74] with several studies demonstrating regularized results [75–79]. Mathematically, such approximation is derived from a Taylor series expansion of the local variable along with the first order derivative (gradient) terms. The expansion also introduces a characteristic length of the material which serves as the localization limiter. A modified implicit scheme of this formulation is followed in the present work. Lasry & Belytschko [80] used the concept of an equivalent strain as a function of the local strain obtained from limiting conditions of the material (e.g. $J_2$) and Cusatis et al. [81] applied a similar concept to high order microplane (HOM) theory with generalized continuum theories.

Simone et al. [82] showed that when the coupled system is solved simultaneously, standard bi-linear shape functions can be used and the inf-sup condition or the Babuška-Brezzi condition [83–85] is automatically satisfied. Moreover, Simone et al. [86] also concluded that in either integral of differential nonlocal models, damage initiation should be governed by parameters other than the nonlocal quantities in order to obtain correct damage initiation points in Mode-I type failures.

Jirásek [87] studied the effects of different regularization parameters through the nonlocal gradient formulation in 1D. Strain, damage, specific fracture strain and inelastic stress are some of the parameters used in his localization study.

According to Geers [88], unsound results are obtained when the material characteristic length is kept constant during damage evolution, and proposed a transient length that decreases with the growth of damage. It should be noted that transient in this context refers to varying length scale with the deformation state and not dynamic transient effects. Several approaches have been proposed since to handle transient characteristic lengths [89–92]. With the characteristic length
vanishing upon complete element failure (crack formation), [90] proposed that this length be a function of either the failure state or the strain value.

The effects of both viscoelasticity and damage have been addressed in several references [93–100]. Hinterhoeltz & Schapery [101] discussed the lack of objectivity in modeling viscoelastic materials beyond the peak stress using a modified viscoelastic law from Ha & Schapery [102]. Schapery [103] introduced a damage model for viscoelasticity incorporating aging effects and addressing intense damage localization and growth of a single dominant crack.

References [45, 104] proposed to correct the unphysical results during localization by real or artificial viscous regularization which introduces an associated characteristic length. This approach has been applied to rate-dependent material models with main applications to dynamic loading. Cervera et al. [105] and Faria et al. [106] applied damage mechanics to the viscoelastic modeling of concrete structures with viscous-induced regularization. [107, 108] proposed a cohesive crack model for rate-dependent crack growth and viscoelasticity, and compared the effect of different viscous-induced characteristic lengths in the fracture propagation of concrete. Other cohesive crack models have been presented that include viscoelasticity including Schapery [109] and Knaus & Losi [110]. According to [71], since viscous-induced regularization disappears gradually with time, its application is limited to a narrow range of time delays and loading rates. Thus, other integral and gradient regularization have also been applied to viscoelastic materials. Integral formulation for semicrystalline polymers was proposed in [111] while [112] used nonlocal damage to analyze bifurcation problems. Viscoelastic behavior with high levels of microcracking was also studied for polycrystalline ice in [8, 22] by means of a nonlocal integral formulation of damage. Lyakhovsky et al. [113] used the gradient of an isotropic damage parameter to regularize his viscoelastic and fracture model. Nguyen et al. [114] proposed a viscoelastic-viscoplastic model for gradient-enhanced damage in terms of an equivalent plastic strain and used it to compute regular-
ized softening and failure damage laws for finite deformation.

In this section, a new damage regularization approach is proposed based on an equivalent stress measure concept and is applied to a Prony Series type viscoelastic solid with a Murakami damage-rate law. Viscoelastic behavior is achieved by a semi-analytical integration of the constitutive law and damage regularization is obtained by solving an additional second-order gradient equation of an equivalent stress.

1.3 A Fully Coupled Nonlocal Model for Viscoelastic Damage in Isothermal Asphalt Concrete

Being a material of extensive use in construction in the United States, full characterization of Asphalt Concrete (AC) under different loading conditions would greatly translate in many engineering and economic benefits. As the material characterization should include the entire life span of the AC, the multiple factors contributing to its deterioration should be studied. Being a time- and rate-dependent material, predicting its mechanical response becomes more critical when material deterioration is also included. Fracture initiation and development is a key factor influencing the rapid deterioration of AC affecting not only the structural capacity of the concrete but also its permeability, water content, foundation erosion and other serviceability condition such as smoothness and drivability.

The initiation and effect of different fracture mechanisms in AC have been studied by several authors [115–120]. The viscoelastic model developed by Schapery [121] was used in modeling of the viscoelastic behavior of asphaltic and other time dependent materials [122–125]. The model proposed by Darabi et al. [126] was used by You T. et al. [127] to relate the effects temperature
and loading rates on the stress-strain response in their model for AC cylinders in tension and compression. The reader is directed to the work of Abu Al-Rub et al. [128] for an overview of different finite element techniques used for predicting asphalt pavement rutting as well as [129, 130] for a thermodynamic framework and numerical application of constitutive modeling in time and rate dependent materials including asphalt.

Similarly, a continuum damage model based on the elastic strain energy developed by Kim & Little [131] and Park et al.[132] was also proposed with extension to three dimensional isotropic damage in [102] and to material healing by [133, 134]. Using a similar viscoelastic-damage law, Mun and co-authors [39–41] showed different strategies for calculating viscoelastic and damage properties for their asphalt models. The model proposed by Darabi M. et al [134] included a micro-damage healing and compared the different strain, elastic strain energy and power equivalence hypotheses in continuum damage mechanics.

Under compression, experimental results for hot-mix asphalt binder and asphalt concrete were presented by Katsuki & Gutierrez [135] for unconfined cylinders subjected to different compressive strain rates. Their results also confirms the time- and rate-dependent behavior of both asphalt binder and AC. Yun T. & Kim Y.R. [136] also proposed a model to account to aggregate interlocking in their viscoelastoplastic model under compressive loads.

As the transportation emphasis in the U.S. has changed from the construction of new facilities to the renewal and preservation of the infrastructure, overlay systems have played a significant role in the maintenance of both Portland Cement Concrete (PCC) pavement and Asphalt Concrete (AC) pavements [137]. According to [138] thin overlays demonstrated to be nearly always a cost effective alternative on AC pavements. Thin overlays address many repair and maintenance
requirements, particularly when these requirements are driven not by structural strength but by pavement serviceability. The reader is directed to [139–141] for an overview of the design and construction of thin AC overlays.

[142] proposed a cohesive zone model for modeling crack initiation and propagation in airport pavements. A tension test for mode I crack growth across the overlay thickness was studied by Sarfraz et al. [2] for AC at low temperatures. Using elastic governing equations, [143] proposed an explicit solution to study crack saturation and initiation including study of overlay thickness. [144] studied the effect of crack saturation on asphalt overlays at low-temperatures based on a Generalized Maxwell model and the fracture initiation criteria based on the comparison of the energy release rate with fracture toughness as described in [145].

The present document proposes a thermodynamically sound and coupled viscoelasticity-damage model to analyze the behavior of asphalt concrete. Viscoelastic model is introduced in the form of a Prony Series while damage propagation is modeled following Murakami’s damage model [42]. The proposed model assumes all contribution to the viscoelastic behavior be given from the deviatoric stress component while the volumetric part remains elastic. A semi-analytical time integration scheme is used for viscoelasticity while the damage propagation is done through a forward Euler scheme. The numerical solution is obtained by Finite Elements analysis and is implemented in FEAP [146] in a User Element. A nonlocal gradient type regularization recently proposed [147] is used to regularize an equivalent stress measure of the system. This approach permits an isotropic damage evolution be constructed using a scalar measure of stress only while capturing multiaxial loading effects. To the authors’ knowledge, similar gradient-enhanced regularization schemes has not been found in the literature for asphalt concrete in which mesh insensitive results are found without a definition of a crack path a priori.


1.4 Goals and Structure

This work is aimed at understanding the damage growth and propagation in viscoelastic solids and proposing a computational method that is efficient and physically sound. The work is therefore divided into three components that spans from the mechanics of damageable viscoelastic solids, damage regularization and applications to ice and asphalt concrete. These goals are summarized as follows:

- **Constitutive model**: Propose a coupled constitutive model for viscoelastic solids including rate-dependent damage, validated to experimental data while maintaining rigorous thermodynamical consistency.

- **Gradient-enhanced damage for viscoelasticity**: Propose a nonlocal damage regularization for viscoelastic solids that includes both, the time- and rate-dependent behavior of damage while providing results that are independent on the spatial discretization.

- **Validation for applications in ice and asphalt concrete**: Validate the constitutive model to experimental data under different loading conditions. Following the material calibration, compare model prediction to experimental data available in the literature.

The three goals above are introduced across three different sections named Chapter 2, Chapter 3 and Chapter 4 which are organized as follows:

In Chapter 2, section 2.1 presents the theory of the viscoelastic and damage models. An isotropic damage model is presented in section 2.1.2 as a simplification of the damage tensors described in [42, 148, 149]. In section 2.2, the solution strategy and assumptions to the numerical
implementation are presented. Thermodynamic consistency of the model is presented in section 2.3. For this purpose the second law of thermodynamics in the form of Clausius-Duhem inequality is used. Starting with the calibration to viscoelastic and damage parameters, verification and validation with experimental data and other models are presented in section 2.4. Validation tests range from creep tests in tension and compression to biaxial failure envelope constructed from strain-rate test in compression. Chapter 2 concludes with an investigation of the effect of sea-water depth on surface crevasse rate and extent of propagation of grounded marine-terminating glaciers and compares the results to similar models in the literature. Conclusions of this study are presented at the end including a brief description on the semi-analytical time integration scheme in Appendix A.3.

Chapter 3 introduces in Section 3.1 the viscoelastic and damage growth models used in the regularization approach. Along with the derivation of implicit gradient damage equations, the strong and weak formulations are presented. The final solution of the problem is reached through an iterative Newton scheme for which an analytical Jacobian is derived. Thermodynamic consistency is discussed in Section 3.2. Analytical and numerical analyses are presented also by means of the Clausius-Duhem inequality. Finally, Section 3.3 present the results of numerical examples in one and two dimensional problems. Using different mesh sizes, comparison between local and non-local results are presented for 1D bars subjected to various creep, strain-rate and relaxation loads. In two dimensions, regularized results are shown for a schematic problem of crevasse propagation in a glacier ice slab as well as for the Kalthoff-Winkler problem of impact to a pre-notched steel plate.

Chapter 4 is organized as follows. Section 4.1 presents a summary of the mechanics and special considerations for modeling of asphalt concrete (AC) including both viscoelasticity and nonlocal damage. From published experimental data, calibration and selection of material parameters are presented in section Section 4.2. A staggered optimization approach for both damage and viscoelastic parameters is described in this section. Finally, a series of Finite Element simulations are
presented in Section 4.3 to validate the proposed model to various loading conditions and geometries available in the literature followed by the conclusions and comments on future work.

Finally, in Chapter 5 the conclusions of this thesis including a summary of the main contributions and a description of future work.
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Chapter 2

A Prony-Series Type Viscoelastic Solid Coupled with a Continuum Damage Law for Polar Ice Modeling

The inputs from the co-authors of the paper Londono et al. [3] from which this chapter is reproduced are gratefully acknowledged.

2.1 Material model for ice

In this section a generalized viscoelastic Maxwell model in the form of a convolution integral is presented for modeling the behaviour of ice. Prony Series expansion is employed for the material moduli and a semi-analytical time integration scheme, proposed in [50, 51], is adopted. Generality of the Prony Series constitutive model is achieved by including as many terms as needed to predict an intended behavior. The progressive deterioration of the material preceding failure is estimated from phenomenological considerations and history values of damage at a material point. In this analysis, the material is considered a continuous solid and therefore, does not include detailed in-
CHAPTER 2. A PRONY-SERIES TYPE VISCOELASTIC SOLID COUPLED WITH A CONTINUUM DAMAGE LAW FOR POLAR ICE MODELING

formation of any particular void or microcrack. In this paper, vectors are presented with an upper bar (e.g. $\bar{u}$, $\bar{r}$, $\bar{n}$), second order tensors are in bold (e.g. $\mathbf{\sigma}$, $\mathbf{\varepsilon}$) while fourth order tensors are in blackboard bold characters (e.g. $\mathbb{E}$, $\mathbb{C}$).

2.1.1 Prony-series type viscoelastic model

The time dependent behavior of solids can be intuitively explained as a combination of common mechanical devices. The viscoelastic response of a material is taken as a combination of time independent and time dependent mechanical devices, where the former is usually represented by the elastic response provided by a spring, while the later or viscous behavior is represented by a dashpot. Defining the elastic constant of the spring as $E$ and the dashpot viscosity as $\eta$, the stress in the spring and dashpot are written as $\sigma = E \varepsilon$ and $\sigma = \eta \dot{\varepsilon}$ respectively, where $\varepsilon$ is the strain and $\dot{\varepsilon}$ is the strain rate. Nonetheless, simple combinations of these devices such as Maxwell or Kelvin-Voigt models [150] fail to acceptably predict both phases of creep and stress relaxation. A generalization of Maxwell model (Fig. 2.1) including several Maxwell branches in parallel can be used to better predict the material behavior as well as describe a wider range of viscoelastic materials.

In this generalized model, the total strain $\varepsilon$, is identical in all branches while the total stress is the sum of stresses in the $n + 1$ branches of the model. In three dimensions, the effective stress $\bar{\sigma}$ can be formulated in terms of the total strain $\varepsilon$, and the deviatoric strain rate $\dot{\varepsilon}$ as described in Eq. 2.1.

According to experimental observations, material viscous behavior is found to be driven by changes in shape [151], while the changes in volume are observed to be elastic. Thus, as shown in
2.1. MATERIAL MODEL FOR ICE

Figure 2.1: Equivalent mechanical model consisting of springs $E_i$ and dashpots $\eta_i$ for Generalized viscoelastic solid. The strain in each branch is identical and consist of elastic, $\varepsilon^{(s)}_i$, and viscous, $\varepsilon^{(d)}_i$ components (except the leftmost branch which is purely elastic, $\varepsilon_\infty = \varepsilon$).

Eq. 2.1, the bulk modulus, $K$, is time independent while only the deviatoric stress is written in the form of a convolution integral.

\[ \tilde{\sigma}(t) = \tilde{\sigma}^{vol} + \tilde{\sigma}^{dev}(t) = K \text{tr}(\varepsilon) I + 2 \int_{-\infty}^{t} G(t - \tau) \dot{\varepsilon}(\tau) d\tau \]  
\hspace{1cm} (2.1)

where $\text{tr}(\cdot)$ is the trace operator, $I$ is the identity tensor and $\varepsilon$ the total strain tensor. The elastic bulk modulus, $K$ and shear modulus, $G$ are related to the Young’s modulus, $E$ and Poisson ratio, $\nu$ by $G = \frac{E}{2(1+\nu)}$ and $K = \frac{E}{3(1-2\nu)}$. A semi-analytical integration of the convolution integral of Eq. 2.1, proposed by Taylor [50], is presented in Appendix A.3. The material modulus, $G(t)$ which represent the shear relaxation modulus is given in the Prony-series form of Eq. 4.2.

\[ G(t) = G_\infty + \sum_{i=1}^{n} G_i \exp(-t/\lambda_i) = G_0 \left( \mu_\infty + \sum_{i=1}^{n} \mu_i \exp(-t/\lambda_i) \right) \]  
\hspace{1cm} (2.2)

As described in [151], the parameters $\mu$ in the series are the normalized relaxation modulus in which, $\mu_\infty = \frac{G_\infty}{G_0}$, $\mu_i = \frac{G_i}{G_0}$ and $G_0$ is the instantaneous shear modulus at time zero. For simplicity, in the derivations hereafter, the instantaneous modulus $G_0$ will be used without the subindex. By
definition, $\mu$ requires that

$$\mu = \mu_\infty + \sum_{i=1}^{n} \mu_i = 1, \quad \forall i = 1, 2, \ldots, n \quad (2.3)$$

### 2.1.2 Continuum Damage model

Voids and microcracks nucleation and growth are commonly described in continuum mechanics using a damage state variable. For isotropic materials, a scalar parameter $D$ varying from 0.0 to 1.0 is used to represent the damage state. A value of $D$ equal to 0.0 represents the undamaged state whereas a $D$ value of 1.0 represents a fully damaged material.

The analysis of a damaged material is simplified by transforming the material into a damage free state, so called effective space, but subjected to an effective stress. In other words, the effective stress $\bar{\sigma}$ is the force per unit of undamaged area (without voids or cracks).

Murakami et al. [42, 148, 152] based their damage law from the work of Kachanov [52] and Rabotnov [53] and proposed an anisotropic damage rate law. An isotropic damage version of the this damage law is used in this paper as included in Eq. 2.4 with $B$, $r$ and $k_\sigma$ being damage evolution parameters to be calculated from experimental data.

$$\dot{D} = B \frac{\langle \chi(\bar{\sigma}) \rangle^r}{(1 - D)^{k_\sigma}} \quad (2.4)$$

In this paper, we propose the ice damage parameter $B$ to linearly vary with respect to the internal stress, $\sigma$ as,

$$B = B_1 + B_2 |tr(\sigma)| \quad (2.5)$$
where \( \text{tr} (\cdot) \) is the trace operator. The symbols \( \langle \cdot \rangle \) denote the Macaulay bracket \([153]\) given by,

\[
\langle \chi \rangle = \begin{cases} 
\chi, & \text{if } \chi \geq 0 \\
0, & \text{if } \chi < 0,
\end{cases}
\]

(2.6)

and the parameter \( \chi \) in Eq. 2.4 is the Hayhurst’s multiaxial equivalent stress measure \([154]\) given by,

\[
\chi = \alpha \bar{\sigma}^{(1)} + \beta \sqrt{3} II \bar{\sigma}_{\text{dev}} + (1 - \alpha - \beta) I \bar{\sigma}
\]

(2.7)

where \( \bar{\sigma}^{(1)} \) is the maximum eigenvalue of the effective stress, \( \bar{\sigma} \); \( I \bar{\sigma} \) is the first invariant of \( \bar{\sigma} \) defined as \( \text{tr}(\bar{\sigma}) \) and \( II \bar{\sigma}_{\text{dev}} = \frac{1}{2} \bar{\sigma}_{\text{dev}} : \bar{\sigma}_{\text{dev}} \), is the second invariant of the deviatoric component of the effective stress, \( \bar{\sigma}_{\text{dev}} \). The parameters \( \alpha \) and \( \beta \) allow for different contribution of stress components according to the type of material. By definition \( \alpha \) and \( \beta \) are constrained to,

\[
\alpha + \beta \leq 1.0
\]

(2.8)

For brittle materials for example, the failure criteria is usually defined by the maximum (normal) stress reaching the material strength whereas for ductile materials, the Von Mises yield criterion is generally used to predict failure. According to Von Mises criterion, the yielding in the material starts when the second invariant of the deviatoric stress reaches a critical value, known as the \( J_2 \) plasticity. Moreover, note that the last term in the Hayhurst’s equation accounts for the different failure produced under compression and tension loading, introducing the hydrostatic stress measure as failure criterion.

In summary, the strong form of the problem, neglecting inertial effects, can finally be described
CHAPTER 2. A PRONY-SERIES TYPE VISCOELASTIC SOLID COUPLED WITH A CONTINUUM DAMAGE LAW FOR POLAR ICE MODELING

by Eq. 2.9-Eq. 2.13.

\[ \nabla \cdot \sigma = \bar{f}_{ext} \quad \text{in } \Omega \]  \hspace{1cm} (2.9)

\[ \sigma(t, D, \varepsilon) = M^{-1}(t, D, \varepsilon) \bar{\sigma}(t, \varepsilon) \quad \text{in } \Omega \]  \hspace{1cm} (2.10)

\[ \dot{D} - B \langle \chi(\bar{\sigma}) \rangle r (1 - D)^{k_\sigma} = 0 \]  \hspace{1cm} (2.11)

\[ \sigma \cdot \bar{n} = \bar{t}_r \quad \text{in } \partial \Omega_{tr} \]  \hspace{1cm} (2.12)

\[ \bar{u} = \bar{u}_{BC} \quad \text{in } \partial \Omega_u \]  \hspace{1cm} (2.13)

where \( \nabla \cdot \) is the divergence operator, \( \bar{f}_{ext} \) is the external force, \( \Omega \) is the physical domain of the problem and \( \bar{n} \) is outward normal vector with respect to the natural boundaries, \( \partial \Omega_{tr} \), on which the traction \( \bar{t}_r \) is applied. The prescribed displacement \( \bar{u}_{BC} \) is enforced at the essential boundaries, \( \partial \Omega_u \).

For isotropic materials, the damage variable \( M = (1 - D)^{-1} \) with \( D \) being a time dependent scalar field at a particular material point. Moreover, since \( D \) is explicitly dependent of \( \bar{u} \), Eq. 2.11 can be eliminated from the system. Material parameters \( r, k_\sigma \) and \( \chi(\bar{\sigma}) \) are obtained after material calibration to experimental data. The solution strategy therefore relies on solving the equilibrium Eq. 2.9 for the total stress \( \sigma \). The constitutive law including the effects of damage is summarized in Eq. 2.14. The solution requires the computation of the damage variable from Eq. 2.11 be applied to the effective stress \( \bar{\sigma} \) in Eq. 2.1, which yields,


\[ \sigma(t) = M^{-1} \bar{\sigma}(t) \]

\[ = \left( I - \int_0^t \bar{D}(\tau)d\tau I \right) \left[ Ktr(\epsilon)I + 2 \int_{-\infty}^t G(t-\tau)\dot{\epsilon}(\tau)d\tau \right] \]

\[ = \left( I - \int_0^t B \frac{\langle \chi(\tau) \rangle^r}{(1-D(\tau))^{\frac{r}{2}}} d\tau I \right) \left[ Ktr(\epsilon)I + 2 \int_{-\infty}^t G(t-\tau)\dot{\epsilon}(\tau)d\tau \right] \]

Note that although the damage term is applied to both, deviatoric and volumetric, stress components; a different damage approach could be used instead for each of the stress parts. Description of how the total stress is obtained from the viscoelastic constitutive law and damage growth law are presented below.

### 2.2 Solution Strategy

In this section we describe the numerical methods and assumptions used to solve the following residual equation at each time increment,

\[ \bar{R}(\bar{u}, D) = \bar{f}^{int} - \bar{f}^{ext} = \nabla \cdot \sigma - \bar{f}^{ext} = 0. \] (2.15)

Because of the nonlinear nature of the material law used, an equivalent linear representation is used (first terms in a Taylor series expansion) at each quasi-static step and solved by Newton’s iterative method as shown in Eq. 2.16.

\[ \bar{R}_m^k(\bar{u}_m^k) \approx \bar{R}_m^{k-1}(\bar{u}_m^{k-1}) + J_m^{k-1}(\bar{u}_m^{k-1}) \delta \bar{u}_m^k = 0. \] (2.16)
where the Jacobian $J^{k-1}_{m-1}(\vec{u}_m^{k-1}) = \frac{dR^{k-1}_{m-1}(\vec{u}_m^{k-1})}{d\vec{u}}$ is obtained as

$$J^{k-1}_{m-1}(\vec{u}_m^{k-1}) \delta \vec{u}_m^k = \frac{dR^{k-1}_{m-1}(\vec{u}_m^{k-1} + \epsilon \delta \vec{u}_m^k)}{d\epsilon} \bigg|_{\epsilon=0} = \nabla \cdot \left( \left[ 1 - D_m(\vec{u}_m^{k-1}) \right] \frac{d\sigma^{k-1}_{m-1}(\vec{u}_m^{k-1} + \epsilon \delta \vec{u}_m^k)}{d\epsilon} \right)_{\epsilon=0}$$

(2.17)

which require the stress and damage to be integrated in time at each material point as shown in Appendix A.2. The following are the set of assumptions made that guides the choice of the numerical methods used to solve Eq. 2.15.

1. No inertial effects are considered as indicated in Eq. 2.9. However incremental time steps similar to incremental loading steps are used to resolve the time dependence of the viscoelastic and damage laws.

2. The constitutive laws for viscoelasticity and damage are solved locally at each gauss point using internal variables, thus only the displacement field $\vec{u}$ is considered an independent global variable

$$\bar{R}(\vec{u}, D) \approx \bar{R}(\vec{u})$$

(2.18)

3. Variations of damage with respect to displacements between two consecutive time increments are assumed to be small. In other words,

$$\frac{\partial D}{\partial \vec{u}} = \frac{dD_m(\vec{u}_m^{k-1} + \epsilon \delta \vec{u}_m^k)}{d\epsilon} \bigg|_{\epsilon=0} \approx 0$$

(2.19)

where the small perturbation $\epsilon$ (not to be confused with the strain, $\epsilon$) is used to compute the Gâteaux derivative [155, 156] in the direction of $\delta \vec{u}_m^k$ at $\vec{u}_m^{k-1}$ in Eq. 2.19.
2.2. SOLUTION STRATEGY

2.2.1 Local update: Constitutive Equation update

In this section an algorithm for the constitutive update for the viscoelastic and damage internal variables is presented. First, the viscoelastic law without damage is integrated in time using the semi-analytical approach proposed by Taylor [50] and briefly described in Appendix A.3. Second, the damage parameter \( D_m(\bar{u}_m^{k-1}) \) at the current Newton iteration is integrated by solving Eq. 2.11 with a forward Euler scheme

\[
D_m(\bar{u}_m^{k-1}) = D_{m-1} + \Delta t B \left( \frac{\chi(\bar{\sigma}(\bar{u}_m^{k-1}))}{(1 - D_{m-1})^{\kappa}} \right). \tag{2.20}
\]

Selection of a Forward Euler scheme allowed a simple and efficient time integration of the damage parameter while keeping an acceptable time step size. The update of both viscoelastic and damage laws are presented in Algorithm 1.

**Algorithm 1 Gauss point constitutive update**

1: Given \( u_m^k \) calculate \( \epsilon_m^k \) and \( \epsilon_m^k \)
2: \( \Delta \epsilon_m^k = \epsilon_m^k - \epsilon_{m-1} \) (or 0 if \( m = 1 \))
3: \( \Delta \epsilon_m^k = \epsilon_m^k - \epsilon_{m-1} \) (or 0 if \( m = 1 \))

4: Calculate \( \tilde{\sigma}_{m}^{\text{dev}}, \tilde{\sigma}_{m}^{\text{vol}}, \) and \( \tilde{\sigma}_{m}^{k} \)

Semi-analytical integration of the viscoelastic law

\[ \text{if } \epsilon_m^k < \epsilon_{\text{threshold}} \text{ then} \]
5: \( \dot{D}_m^k = 0 \)
6: \( \text{else} \)
7: \( \text{Calculate } \chi_m^k \)
8: \( \text{Calculate } \dot{D}_m^k \)
9: \( \text{Update } \epsilon_{\text{threshold}} = \epsilon_m^k \)

\[ \text{end if} \]

Explicit integration of the damage law

10: \( \text{Update } D_m^k = D_{m-1} + \dot{D}_m^k \Delta t \)
11: \( \text{if } D_m^k \geq D_c \text{ then} \)
12: \( D_m^k = 1 \)
13: \( \text{end if} \)
2.3 Thermodynamic Consistency

Any proposed damage-viscoelastic model has to satisfy laws governing stored and dissipated energy. In the present section, consistency of the proposed model with respect to the laws of thermodynamics is verified analytically under arbitrary loading conditions. Moreover, the analytical analysis exposes physical constrains of the material parameters that have to be imposed in the optimization routine. From the Generalized Maxwell model the second law of thermodynamics is constructed in the form of the Clausius-Duhem inequality as defined in Eq. 2.21

\[ \sigma : \dot{\varepsilon} - \dot{\psi} \geq 0 \]  

(2.21)

where \( \sigma \) and \( \varepsilon \) are the total stress and strain in the system respectively. The Helmholtz free energy, \( \psi \) is obtained from the total elastic energy stored in all the idealized springs of the system. For a Free Energy varying with damage, the following expression is obtained.

\[ \psi(\varepsilon, D) = (1 - D) \tilde{\psi}(\varepsilon). \]  

(2.22)

Assuming isotropic damage, the internal damage variable, \( D \), is a scalar parameter and \( \tilde{\psi} \) the Free Energy stored in the effective space. The total effective Free Energy in the system is then the sum of the contributions from the purely elastic branch \( \tilde{\psi}^{(e)} \) with stiffness \( E_\infty \) plus the elastic energy of the springs from the viscous branches \( \tilde{\psi}^{(v)} \) in the mechanical model of Fig. 2.1. That is,

\[ \tilde{\psi} = \tilde{\psi}^{(e)} + \tilde{\psi}^{(v)} \]  

(2.23)

\[ = \frac{1}{2} \sigma^{(e)} \varepsilon + \sum_{i=1}^{n} \frac{1}{2} \sigma_i e_i^{(s)} \]  

(2.24)

\[ = \frac{1}{2} E_\infty \varepsilon^2 + \sum_{i=1}^{n} \frac{1}{2} E_i \left( \varepsilon - \varepsilon_i^{(d)} \right)^2. \]  

(2.25)
where the definitions of stress from the mechanical model are used. The rate of free energy $\dot{\psi}$ is derived using the chain rule with respect to the state variables as

$$
\dot{\psi}(\mathbf{e}, \dot{\mathbf{e}}, \mathbf{e}^{(d)}, \dot{\mathbf{e}}^{(d)}, D, \dot{D}) = \frac{\partial \psi}{\partial \mathbf{e}} \dot{\mathbf{e}} + \frac{\partial \psi}{\partial \mathbf{e}^{(d)}} \dot{\mathbf{e}}^{(d)} + \frac{\partial \psi}{\partial D} \dot{D}.
$$

(2.26)

Substituting the partial derivatives of $\psi$ for their expression and grouping similar terms in the Clausius-Duhem inequality yields

$$
\begin{align*}
\{ \mathbf{e} - (1 - D) \mathbf{e} + \sum_{i=1}^{n} E_i (\mathbf{e} - \mathbf{e}_i^{(d)}) \} \dot{\mathbf{e}} + & \left(1 - D \right) \sum_{i=1}^{n} E_i (\mathbf{e} - \mathbf{e}_i^{(d)}) \dot{\mathbf{e}}_i^{(d)} + \\
\left\{ \frac{1}{2} E_{\infty} \mathbf{e}^2 + \sum_{i=1}^{n} \frac{1}{2} E_i (\mathbf{e} - \mathbf{e}_i^{(d)})^2 \right\} \dot{D} & \geq 0.
\end{align*}
$$

(2.27)

The first term in Eq. 2.27 is zero from the definition of the constitutive law described in which,

$$
\sigma - (1 - D) \tilde{\sigma} = 0
$$

$$
\sigma - (1 - D) \left\{ \tilde{\sigma}^{(e)} + \tilde{\sigma}^{(v)} \right\} = 0
$$

$$
\sigma - (1 - D) \left\{ E_{\infty} \mathbf{e} + \sum_{i=1}^{n} E_i (\mathbf{e} - \mathbf{e}_i^{(d)}) \right\} = 0
$$

(2.28)

Sufficient conditions for inequality 2.27 to hold requires that the last two terms be both non-negative. From the definition of the damage variable we have: $0 \leq D \leq 1$, hence the second term in Eq. 2.27 representing the viscous rate has to satisfy inequality 2.29. Similarly, the third term in Eq. 2.27 requires that the damage rate $\dot{D}$, be non-negative. Eq. 2.30 can then be constructed from the definition of the damage rate in Eq. 2.4. Thus,
\[
\sum_{i=1}^{n} E_i (\mathbf{e}_i - \mathbf{e}_{i}^{(d)}) \dot{\mathbf{e}}_{i}^{(d)} \geq 0, \tag{2.29}
\]

\[
B \frac{\langle \chi \rangle^r}{(1 - D)^k} \geq 0. \tag{2.30}
\]

The inequality in Eq. 2.30 yields constrains on the damage parameters as

\[
B \geq 0 \quad \text{and} \quad D \leq 1 \quad \text{(2.31)}
\]

Note that the parameters \( k \) and \( r \) are not constrained. The Clausius-Duhem inequality also provides important information about the rate of work dissipated by the system. The total dissipation, obtained by the second and third parts of Eq. 2.27, defines the dissipation function, \( \mathcal{D} \), that accounts for the viscous and damage effects. Since healing or recovery is not considered in this model and including the constrains in Eq. 2.31, the damage rate is always positive.

Conveniently, the definition of the viscous stress and the effective Free Energy in Eq. 2.25 are used to demonstrate the non-negative form of the dissipation function. Separating its components into a viscous dissipation from the dashpots, \( \mathcal{D}^{(v)} \) and a dissipation due to damage, \( \mathcal{D}^{(D)} \); the total dissipation function \( \mathcal{D} \) is mathematically described by,
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\[ \mathcal{D}(\mathbf{\epsilon}, \dot{\mathbf{\epsilon}}, \mathbf{\epsilon}^{(d)}, D, \dot{D}) = \mathcal{D}^{(v)}(\mathbf{\epsilon}^{(d)}, D) + \mathcal{D}^{(D)}(\mathbf{\epsilon}, \mathbf{\epsilon}^{(d)}, \dot{D}) \] (2.32)

\[ = (1 - D) \sum_{i=1}^{n} \sigma^{(v)} i \dot{\mathbf{\epsilon}}^{(d)} + \dot{\psi} \dot{D} \] (2.33)

\[ = (1 - D) \sum_{i=1}^{n} \eta_i (\dot{\mathbf{\epsilon}}^{(d)}_i)^2 + \left\{ \frac{1}{2} E_{\infty} \dot{\mathbf{\epsilon}}^2 + \sum_{i=1}^{n} \frac{1}{2} E_i (\mathbf{\epsilon} - \mathbf{\epsilon}^{(d)}_i)^2 \right\} \dot{D} \] (2.34)

where dashpot viscosity, \( \eta_i \), and springs constants, \( E_i \) are always positive.

A numerical example is performed to verify the analytical results of the model with a single viscous branch as presented in Fig. 2.2a, with a trivial extension to more viscous branches \((n > 1)\). The model is subjected to cyclic strain with increasing amplitude (Fig. 2.2b) and changes in the damage, stress, energy and dissipation are recorded. The material parameters for this model are \( E_{\infty} = 16.5[MPa] \), \( E_1 = 38.5[MPa] \) and \( \eta_1 = 3850.0[MPa \cdot s] \) and the damage parameters in Table 2.1.

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>( B_1 )</th>
<th>( B_2 )</th>
<th>( r )</th>
<th>( k_\sigma )</th>
<th>( \varepsilon_{\text{threshold}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.0</td>
<td>( 6.025 \times 10^{-5} )</td>
<td>( 1.332 \times 10^{-5} )</td>
<td>3.239</td>
<td>2.46</td>
<td>0.0</td>
</tr>
</tbody>
</table>

The change in the value of stress as the strain and damage increase is shown in Fig. 2.3a with the simulation time highlighted in colors, as indicated by the colorbar to the right of the figure. Prior to damage initiation, the size of the ellipse in the stress-strain curve increases following the growing amplitude of the strain applied but maintaining the same axis orientation. However, as the damage grows, softening in the material produces stress drops, reducing the angle of the major axis of the ellipse until the final material collapse at \( \sim 900[s] \). Assuming the damage occurs during the tension phase only, the damage in the material increases in all the cycles in which the tensile strains
The changes in energy throughout the simulation are shown in Fig. 2.4 including the total energy and its components obtained from the elastic and viscous branches as introduced in Eq. 2.22 and Eq. 2.25. Fig. 2.4a shows the Effective Free Energy as it increases with the simulation time. The effects of damage are included in the total Free Energy calculation and presented in Fig. 2.4b. As shown in Fig. 2.4b, as the material damage grows, the total energy in the system is reduced significantly and gradually reduces its capacity to store energy until a complete collapse. Moreover, when the applied axial strain crosses the ‘x’ axis in Fig. 2.2b (which occurs every 62.5[s] with the selected frequency), the elastic energy stored in the purely elastic branch of the system, \( \psi^{(e)} \), is zero.

However, due to the effect of the dashpot in the viscous branches, the springs in those branches
Figure 2.3: Response of the model in Fig. 2.2 to cyclic loading. a Stress-Strain curve during damage process with simulation time represented by the colorbar from red ($t = 0$) to blue ($t = t_f$). b Damage growth during cyclic strain test.

experience a delay with respect to the applied strain, producing a phase lag in their calculated energy, $\psi^{(v)}$, as highlighted in Fig. 2.4b. This minor shift can be observed on both graphs of Fig. 2.4.

Assuming the amplitude of the applied strain is a function of time only, the energy dissipated by the system is found to be dependent on the strain frequency. The total energy dissipated is obtained by numerically integrating the viscous, $\mathcal{D}^{(v)}$ and damage, $\mathcal{D}^{(D)}$ dissipation functions over the time domain as shown in Eq. 2.35.

$$\psi^{(dis)} = \psi^{(dis,v)} + \psi^{(dis,D)}$$  \hspace{1cm} (2.35)

$$= \int_0^{t_f} \mathcal{D}^{(v)} \, dt + \int_0^{t_f} \mathcal{D}^{(D)} \, dt$$  \hspace{1cm} (2.36)
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Figure 2.4: Evolution of the effective and total energies in the verification example shown in Fig. 2.2. a Effective Energy in the system. b Total Energy of the system including damage effects. Note the major energy drop occurs around 760 seconds and final collapse around 900 seconds consistent with the damage growth of Fig. 2.3b.

Increasing the frequency of the applied strain in the model of Fig. 2.2a, two distinct regions of energy dissipation are found in both the dissipated effective energy, $\tilde{\psi}^{(\text{dis})}$ (with $D = \dot{D} = 0$) in Fig. 2.5a and the total dissipated energy, $\psi^{(\text{dis})}$ in Fig. 2.5b. In an initial region of Fig. 2.5a with lower frequency values, the dissipated energy increases proportionally to the increase in the applied frequency, followed by a fast transition to the second region of constant dissipated energy for higher frequencies. Note that while the transition phase is shorter and smoother when damage effects are not included, the total dissipative energy also converges to a constant value at higher frequencies.

The proposed viscoelastic-damage model is proven consistent with thermodynamic principles as shown in this section. The total dissipation function of the system, obtained from the viscous $\mathcal{S}^{(v)}$ and damage terms $\mathcal{S}^{(D)}$ in Eq. 2.32, is non-negative satisfying Clausius-Duhem inequality.
The numerical example presented in this section also verifies the theoretical results confirming its validity to produce physically admissible solutions.

Figure 2.5: Change in energy dissipated with different applied frequencies in model of figure 2.2a with $t_f = 1000\,[s]$. a Effective energy dissipated ($D = \dot{D} = 0$) and b Total energy dissipated including damage effects.

## 2.4 Verification and validation with experimental data and other models

In theoretical glaciology, ice is considered a viscoelastic, anisotropic and highly nonlinear material (Hutter [157], Gagliardini [158]). The viscoelastic behavior of ice is studied in this section and the applicability of the proposed approach to model polycrystalline ice is demonstrated through numerical examples and under different loading conditions.

While ice is considered anisotropic mainly due to the hexagonal symmetry of its crystallographic structure [158–160], at glacier scale of ice-sheets, particularly at the surface, grain orientations are more or less randomly distributed with the polycrystal response close to isotropy.
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[158, 161, 162]. Furthermore, given the limited experimental data on anisotropic ice behavior and the complexity in modeling, an isotropic model for ice deformation and damage is selected in the present work.

First, a rigorous calibration strategy is introduced to obtain the Prony series and damage parameters. Secondly, calibration results are presented for experimental data of ice in tension and compression creep tests. Their predicted behavior is finally validated against published experimental data and compared to the model introduced for polycrystalline ice by Duddu & Waisman [1, 22]. Validation tests range from creep tests in tension and compression to strain-rate test to construct a biaxial failure envelope. Furthermore, the model is implemented in a finite element code and is used to investigate surface crevasse propagation in grounded marine-terminating glaciers as compared to [8]. The simulations, on idealized rectangular ice slabs in contact with the ocean, examine the depth and rate of damage propagation with increasing seawater depth near the terminus. The results demonstrate the applicability and computational efficiency of the model to predict the viscoelastic behavior of polar ice incorporating damage effects.

2.4.1 Material Calibration Strategy

Prediction of ice behavior depends not only on the model implemented but also on the proper selection of the material parameters. A calibration strategy is introduced in this section to obtain material parameters from available experimental data of polycrystalline ice. Knowledge of the model and physical intuition are pivotal in the selection of the optimization routine and its initial values. The strategy presented in this section is meant to provide a general optimization framework that can be implemented with any commercial optimization package. The optimization is done by means of the Broyden-Fletcher-Goldfarb-Shanno (BFGS) method [163–166].
With the experimental values at the \( q^{th} \) point defined as \((y_q^{(exp)}, x_q^{(exp)})\) and the simulated results of the model defined as \((y_q^{(sim)}, x_q^{(sim)})\), the optimization problem is driven by the objective function \(\Pi(a)\) defined as,

\[
\Pi(a) = \sqrt{\sum_{q=1}^{nt} [y_q^{(exp)} - y_q^{(sim)}(a)]^2}
\]

where \(nt\) is the total number of data points. The simulated results \(y_q^{(sim)}\) are function of the material parameters \(a\) and are compared at same points of the experimental data \((x_q^{(sim)} = x_q^{(exp)})\) as shown in Eq. 2.38.

\[
y_q^{(sim)}(a) = f(x_q^{(exp)}, a) \tag{2.38}
\]

with the unknown material parameters \(a\) defined in general by

\[
a = (\mu_1, \lambda_1, \ldots, \mu_n, \lambda_n, \alpha, \beta, B_1, B_2, r, k) \tag{2.39}
\]

which includes both the viscoelastic model parameters and the damage parameters. The problem is mathematically described by finding a set of constants \(a\) which minimizes \(\Pi\), thus

\[
\hat{a} = \arg \min_{a \in [\mathcal{M}, \mathcal{L}, \mathcal{A}, \mathcal{B}, \mathcal{W}, \mathcal{V}, \mathcal{R}, \mathcal{N}]} \{\Pi(a)\} \tag{2.40}
\]

where \(\hat{a}\) are the optimal viscoelastic and damage parameters. The feasible search spaces are defined for each material parameter as \([\mathcal{M}, \mathcal{L}, \mathcal{A}, \mathcal{B}, \mathcal{W}, \mathcal{V}, \mathcal{R}, \mathcal{N}]\). For the viscoelastic terms \(\mu_i\) and \(\lambda_i\), the feasible search spaces defined by \(\mathcal{M}\) and \(\mathcal{L}\) respectively are shown in Eq. 2.41.

\[
\mathcal{M} = \{\mu \in \mathbb{R}^n | 0 \leq \mu_i \leq 1.0, \forall i = 1, 2, \ldots, n\} \tag{2.41}
\]

\[
\mathcal{L} = \{\lambda \in \mathbb{R}^n | 0 \leq \lambda_i, \forall i = 1, 2, \ldots, n\}
\]
The viscous parameters $\mu_i$ must verify the inequality constraint of Eq. 2.3. Similarly, damage parameters are defined in the following feasible search spaces and subjected to the thermodynamic constrains of Eq. 2.31.

\[
\begin{align*}
\alpha \in \mathcal{A}; & \quad \mathcal{A} = \{\alpha \in \mathbb{R} | 0 \leq \alpha \leq 1.0\} \\
\beta \in \mathcal{B}; & \quad \mathcal{B} = \{\beta \in \mathbb{R} | 0 \leq \beta \leq 1.0\} \\
B_1 \in \mathcal{W}; & \quad \mathcal{W} = \{B_1 \in \mathbb{R}\} \\
B_2 \in \mathcal{V}; & \quad \mathcal{V} = \{B_2 \in \mathbb{R}\} \\
r \in \mathcal{R}; & \quad \mathcal{R} = \{r \in \mathbb{R} | 0 \leq r\} \\
k_\sigma \in \mathcal{N}; & \quad \mathcal{N} = \{k_\sigma \in \mathbb{R}\}
\end{align*}
\]  

(2.42)

with $\alpha$ and $\beta$ subjected to the inequality constraint in Eq. 2.8. Analysis of the experimental data allowed some parameters affecting the damage growth and initiation to not be included in the calibration. Damage initiation strain, $\varepsilon_{\text{threshold}}$, damage level at sudden failure, $D_c$, maximum damage rate, $\dot{D}_c$, instantaneous elastic modulus, $E$ and Poisson ratio, $\nu$ are some of the parameters excluded.

While the optimization routine could be performed directly from Eq. 2.40 and the objective function in Eq. 2.37, irregular solution space with multiple local minima prevent the optimization routines from being successful when all the material parameters are searched simultaneously. Instead, given the complexity of the problem and coupling of damage and viscoelastic laws, optimal parameters are found iteratively by a two step approach in which the unknown set of constants $a$ is grouped into viscoelastic parameters and damage parameters and optimized sequentially. That is,

\[
a = a^{\text{vis}}(\mu_1, \lambda_1, \ldots, \mu_n, \lambda_n) \cup a^{\text{dmg}}(\alpha, \beta, B_1, B_2 r, k_\sigma)
\]  

(2.43)
2.4. VERIFICATION AND VALIDATION WITH EXPERIMENTAL DATA AND OTHER MODELS

Following the approach suggested in [151], the number of terms in the Prony Series, $n$ is predefined and kept constant through the entire simulation. Moreover, the calibration of the viscoelastic parameters $\mu$ and $\lambda$ can also be optimized separately with the initial values of the relaxation time, $\lambda_i$, typically selected to be evenly spaced on a log-time scale (e.g. $\lambda_i = [t_0, 10^0, 10^1, 10^2, ..., t_f]$).

Note that material constraints are enforced at every step of the optimization problem and passed to the minimization function, $\arg\min \{ \cdot \}$. Moreover, the solution is sensitive to the initial values and several attempts were required before reaching satisfactory results.

2.4.2 Material Parameter Selection

Experimental results for ice slabs under the applied stress shown in Fig. 2.6b are presented in this section. Following the approach described in Section 2.4.1, viscoelastic and damage parameters are calibrated for polycrystalline ice using the experimental results of a single value of constant stress from each, tension or compression, creep test.

The polycrystalline ice model used for this study, is implemented as a user element in the Finite Element Analysis Program (FEAP) [146], using a four-element mesh as shown in Fig. 2.6. We employ plane-stress, four-node quadrilateral elements for the analysis. The ice slab is subjected to a constant stress on the right edge with the nodes on the left edge having roller-type boundary conditions. Comparison to experimental results is done for all stress levels, by computing the stresses and damage variables at the Gauss points and their values projected to the center node of the plate.
Experimental data

The experimental results at constant temperature were obtained by Mahrenholtz & Wu [4] and by Mellor & Cole [5] for polycrystalline ice subjected to tensile and compressive stresses, respectively. Tensile experimental results at \(-10^\circ C\) are shown in Fig. 2.7 in which rectangular ice sheets were subjected to three different stress levels: 0.64, 0.82 and 0.93 MPa, until creep rupture. At these constant stresses, the change of strain (\(\epsilon\)) and strain rate (\(\dot{\epsilon}\)) with respect to time obtained from the experimental results are shown in Fig. 2.7a and Fig. 2.7b respectively. Note that, as described in Section 2.4.2 the result of the calibration using the stress of 0.64 MPa is also included in Fig. 2.7.

Ice sheets subjected to constant stresses of 0.64, 0.82 and 0.93 MPa displayed rupture times around 200, 150 and 105 hours respectively. According to [4], polycrystalline configuration was guaranteed by properly sieving crushed ice through a 5mm mesh (average grain size of 0.86 mm) and the remaining free space filled in with water at 0\(^\circ\)C.
2.4. VERIFICATION AND VALIDATION WITH EXPERIMENTAL DATA AND OTHER MODELS

Similarly, fine-grained isotropic ice was studied in Mellor & Cole [5] showing experimental results of uniaxial compression performed at $-5^\circ C$ subjected to constant stresses of ranging from $-1.0$ to $-3.06$ MPa as shown in Fig. 2.8, which again includes the calibration results for $-1.0$ MPa. Under these stress levels, the change in the strain-rate with respect to the strain and time are recorded and compared to the model results.

Objective Functions

An acceptable prediction of the ice behavior requires the proper selection of the parameters for the viscoelastic and damage models. With the initial values of the damage parameters taken from [1], the viscoelastic parameters are optimized first using one of the stress curves only.

Under tension, experimental data for the 0.64 MPa was taken as the target of the calibration whereas the stress data of 1.00 MPa was used to optimize the compression parameters. Using the definitions in Section 2.4.1, the objective function $\Pi$ in Eq. 2.37 is specialized for viscoelastic parameters of ice data where the dependent parameter $y$ and independent parameter $x$ are,

$$y = \varepsilon_{xx}; \quad (2.44)$$
$$x = \text{time, } t \quad (2.45)$$

which defined both variables of experimental data and simulated data obtained from the model.

The objective function $\Pi(a^{vis})$ for the viscoelastic parameters is therefore defined by,

$$\Pi(a^{vis}) = \sqrt{\sum_{q=1}^{n} \left[ \varepsilon_{q}^{(exp)} - \varepsilon_{q}^{(sim)}(a^{vis}) \right]^2}$$

(2.46)
Similarly, the damage calibration is done by defining the dependent variable $y$ and independent 
variable $x$ in the objective function of Eq. 2.37 as,

\begin{align}
  y &= D; \quad (2.47) \\
  x &= \text{time}, t \quad (2.48)
\end{align}

yielding the $\Pi$ for damage as,

\[
\Pi(a^{dmg}) = \sqrt{\sum_{q=1}^{n_f} \left[ D_q^{(exp)} - D_q^{(sim)}(a^{dmg}) \right]^2} \quad (2.49)
\]

**Parameters selected**

Upon completion of the calibration, using the experimental data of 0.64 MPa tensile stress and 
1.00 MPa compressive stress, optimal material parameters are selected. Using these parameters,
the calibrated response of the ice including damage effects in tension and compression are shown
as continuous solid lines in Fig. 2.7 and Fig. 2.8.

The strain and strain-rate calibration results show a close match to the experimental results with 
the parameters selected. The optimized viscoelastic parameters and elastic moduli are presented in 
Table 2.2 with the values of Young’s modulus $E = 9500\text{[MPa]}$ and Poisson ratio $\nu = 0.35$. Note 
that in both cases the number of terms in the Prony series $n$ that started the simulation was 3, how-
ever as two of these terms yielded insignificantly low values of $\mu$, they are removed from the model.

Moreover, the high value of $\mu_1$ also suggests a *viscous* dominated behavior under tension and 
compression. While, the tension response seems to match a pure fluid model, the compression
2.4. VERIFICATION AND VALIDATION WITH EXPERIMENTAL DATA AND OTHER MODELS

Figure 2.7: Experimental results extracted from [4] for polycrystalline ice under applied tensile stresses of 0.93, 0.82 and 0.64 MPa and calibration results to 0.64 Mpa; a Strain-Time; b Strain rate change with time.

Table 2.2: Prony-series viscoelastic parameters for polycrystalline ice calibrated for the stress of 0.64 MPa in tension and 1.0 Mpa in compression. The values of Shear modulus, \( G \) and Bulk modulus, \( K \) are calculated from the values of Young’s modulus and Poisson ratio in [1].

<table>
<thead>
<tr>
<th></th>
<th>( G ) [MPa]</th>
<th>( K ) [MPa]</th>
<th>( n )</th>
<th>( \mu_1 )</th>
<th>( \lambda_1 ) [s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tension</td>
<td>3518.52</td>
<td>10.56 \times 10^3</td>
<td>1</td>
<td>9.999 \times 10^{-1}</td>
<td>900.0</td>
</tr>
<tr>
<td>Compression</td>
<td>3518.52</td>
<td>10.56 \times 10^3</td>
<td>1</td>
<td>9.700 \times 10^{-1}</td>
<td>50.0</td>
</tr>
</tbody>
</table>

response does exhibit a contribution of the elastic component and hence behaves as a combination of fluid and solid.

Starting with the damage parameters in Duddu & Waisman [1] as the initial values, optimization iterations are performed and optimal damage parameters are obtained and presented in Table 2.3.

where \( \alpha \) and \( \beta \) are constants of the Hayhurst’s equivalent stress in Eq. 2.7 and \( B, r, k_\sigma \) are damage growth parameters as described in Eq. 3.5. In polycrystalline ice, the damage parameter \( B \), depend on the level of stress as given in Eq. 2.5. The parameter \( k_\sigma \) is a scalar value for all levels.
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Figure 2.8: Experimental results extracted from [5] for polycrystalline ice under applied compressive stresses of 1.0, 1.54, 2.06 and 3.06 MPa and calibration results for 1.0 MPa; a Strain-Time; b Strain rate change with accumulated strain.

Table 2.3: Damage parameters of polycrystalline ice after optimization under tension and compression.

<table>
<thead>
<tr>
<th></th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$B_1$</th>
<th>$B_2$</th>
<th>$r$</th>
<th>$k_\sigma$</th>
<th>$\epsilon_{\text{threshold}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tension</td>
<td>0.21</td>
<td>0.63</td>
<td>$1.3955 \times 10^{-6}$</td>
<td>$4.446 \times 10^{-6}$</td>
<td>0.2</td>
<td>0.02</td>
<td>$8.0 \times 10^{-5}$</td>
</tr>
<tr>
<td>Compression</td>
<td>0.00</td>
<td>0.84</td>
<td>$-2.111 \times 10^{-5}$</td>
<td>$1.111 \times 10^{-4}$</td>
<td>1.0</td>
<td>$-2.50$</td>
<td>$1.0 \times 10^{-3}$</td>
</tr>
</tbody>
</table>

The parameter $\epsilon_{\text{threshold}}$ is the strain threshold for the damage initiation.
2.4.3 Viscoelastic Ice model without damage

In this section, the viscoelastic model in the undamaged case, is compared to the model proposed by Duddu & Waisman [1], here named Strain-decomposed (SD) model, but for a time-frame longer than the experimental period. By comparing the numerical results of both models without material deterioration, differences in their long term viscoelastic response as well as computational efficiency are highlighted and discussed. This subsection aims to highlight the differences between the SD model and the Prony-series (PS) model in two time scales: a short time scale with total time similar to the experimental duration (maximum 220 [h]) for which the model was calibrated to and a longer time scale with a duration of approximately 5000 [h]. In order to compare SD to PS viscoelastic models, the SD model is also implemented in FEAP [146]. Proper implementation is first verified by matching the published numerical results in [1] (not shown).

A numerical example identical to that described in Fig. 2.6 is used to model ice without damage effects. Fig. 2.9 shows the viscoelastic behavior of both models in the short term test. Note the linear behavior for all three stress levels in the PS model with the closest behavior obtained for the stress of 0.64 [MPa].

Extending the loading time beyond the experimental time frame, the material behavior is significantly different in each model. For a loading time reaching 5000 hours (longer test), the SD model continues to display a linear behavior as shown in Fig. 2.9, while the PS model converges to constant values of strain. This comparison provides an insight on the overall viscoelastic behavior that could be reproduced with each model. Given the structure of the model, it is easy to observe that extending the linear response of the PS model could be obtained by re-calibrating the material parameters to the desired time window.
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Figure 2.9: Undamaged behavior of Prony Series (PS) model and Strain Decomposed (SD) viscoelastic model of Duddu & Waisman [1] under plane stress tensile creep of 0.93, 0.82 and 0.64 MPa. The behavior is presented for a short time interval of $t_f = 220h$.

Figure 2.10: Undamaged behavior of Prony Series (PS) model and Strain Decomposed (SD) viscoelastic model of Duddu & Waisman [1] under plane stress tensile creep of 0.93, 0.82 and 0.64 MPa. The behavior is presented for longer simulation time with $t_f = 5000h$.

In the study of problems occurring at large time scales, such as glacier phenomena, prediction of the long term behavior is essential. To this end, accurate and efficient numerical schemes should be employed in order to integrate the model in time. In this regard, we analyze and highlight the
limitations of both models in terms of the maximum time-step size and the overall computational cost by numerical experimentation. The critical or maximum stable time-step size, $\Delta t_{cr}$ is analyzed by running the tensile creep tests of the model presented in Fig. 2.6 and finding the time step size that causes numerical instability. With a total simulation time of $5.0 \times 10^6 [s]$, Table 2.4 shows the critical time step, $\Delta t_{cr}$ for the stress values of 0.93, 0.82 and 0.64 [MPa] using both, PS and SD, models. Under these stress levels, taking $\Delta t$ beyond the critical value produced numerical instability which substantially reduces the total time than can simulated. Note that for this total simulation time, the PS model did not display any constraint in the size of $\Delta t$. This is due to the semi analytical integration scheme of the Prony Series. By imposing a limit on $\Delta t$, the SD model require a larger number of time steps to reach a given total simulation time, compared to the PS model.

While adding damage to the model will also limit the time steps in the PS model (as discussed in the following subsection), it is still a significant advantage to have no limitation on the time steps in the PS model since for example one may develop an adaptive model for ice deformation at long time scales where damage kicks in at later times.

<table>
<thead>
<tr>
<th>$\sigma$ [MPa]</th>
<th>$\Delta t_{cr}$ SD model [s]</th>
<th>$\Delta t_{cr}$ PS model [s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.93</td>
<td>305.7</td>
<td>$5.0 \times 10^6$</td>
</tr>
<tr>
<td>0.82</td>
<td>393.4</td>
<td>$5.0 \times 10^6$</td>
</tr>
<tr>
<td>0.64</td>
<td>646.4</td>
<td>$5.0 \times 10^6$</td>
</tr>
</tbody>
</table>

**2.4.4 Tensile Creep including damage**

In this subsection we analyze the behavior of the combined viscoelastic-damage model under different creep loadings.
Figures 2.11a and 2.11b show the predicted values of strain and strain-rate of the proposed model respectively. Both Figures show the predicted mechanical behavior of ice under the stresses of 0.93 and 0.82 MPa and compare them to the experimental results in [4]. The results used in the calibration ($\sigma = 0.64$ MPa) of the material parameters are also included.

According to [4], experimental results suggest ice may face sudden rupture after reaching certain level of damage or strain, thus displaying a drastic increase of strain leading to rupture. This behavior, typically included by a critical damage parameter $D_c$, is predicted directly from the model yielding smooth transition curves and acceptable failure time when compared to experimental results.

Note that when the problem is solved including the damage effects, additional Newton iterations are required to obtain the final solution. Using a time step size of $\Delta t = 60$ [s], the average number of Newton iteration under the three stress levels is 3.3 iterations per time step. The minimal number of iterations is found to be 2 at early stages of the simulation and the maximum 10
2.4. VERIFICATION AND VALIDATION WITH EXPERIMENTAL DATA AND OTHER MODELS

occurring at the time steps of final collapse.

Additionally, computational cost is measured in terms of the CPU time required to run a problem with $\Delta t < \Delta t_{cr}$. Similar mesh and boundary conditions to the ones shown in Figures 2.6a and the tension curves in 2.6b (blue lines) are used to compare the results from both models.

Under the stress levels of 0.93, 0.82 and 0.64 [MPa], a simulation is run with $\Delta t$ rounded to the largest stable time step size until complete damage is reached. The simulation is performed using FEAP [146] on an Intel Core2 Quad CPU @ 2.83 GHz×4, 4 GB memory desktop. The results of this numerical test are shown in Table 2.5.

### Table 2.5: Comparison on number of time steps $nt$ and total CPU time for model in Fig. 2.6a including damage.

<table>
<thead>
<tr>
<th>$\sigma$ [MPa]</th>
<th>$t_f$ [s]</th>
<th>$\Delta t$</th>
<th>SD model</th>
<th>PS model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$nt$</td>
<td>CPU time</td>
<td>$nt$</td>
</tr>
<tr>
<td>0.93</td>
<td>$3.672 \times 10^6$</td>
<td>153</td>
<td>2400</td>
<td>2.65</td>
</tr>
<tr>
<td>0.82</td>
<td>$4.5 \times 10^6$</td>
<td>188</td>
<td>2393</td>
<td>2.60</td>
</tr>
<tr>
<td>0.64</td>
<td>$6.48 \times 10^6$</td>
<td>320.6</td>
<td>2021</td>
<td>2.17</td>
</tr>
</tbody>
</table>

Similarly to the discussion in section 2.4.3, a critical time size was obtained for each model and the largest stable time step was selected for the simulations presented in Table 2.5. By comparing the results of both viscoelastic models including damage, PS model displays a clear advantage over the SD in terms of the numerical efficiency and computational cost. As shown in Table 2.5, the total simulated time was reached with the PS model at a fraction of the CPU time invested in the SD model.
2.4.5 Compression Creep including damage

Similarly to the tension test, the calibrated material parameters in compression were used to predict the ice response under additional compressive loads. Compressive stresses of 1.54, 2.06 and 3.06 MPa were applied to the model following the Boundary conditions in Fig. 2.6a and the red curves in Fig. 2.6b. The mechanical behavior of ice is shown in Fig. 2.12 in which the calibration stress of 1.0 MPa is also included.

As shown in Fig. 2.12 a good agreement to the experimental data is reached for the predicted values of strain-rate and strain using the Prony series ice model. From the experimental data and the mechanical prediction provided by the model, we can observe clearly the three creep stages at which the ice is subjected to during the test. A primary and secondary phase, mainly governed by the undamaged viscoelastic properties of ice, is observed by the rapid increment in strain at the beginning of the test followed by the reduction of the strain-rate to its minimum value at the secondary phase. Given the continuous deformation, the damage accumulation increases again the rate of strain, transitioning to the tertiary phase and the ice failure.

2.4.6 Biaxial failure envelope

Using the material parameters found for the compressive behavior of ice, we study the biaxial failure envelop under compression of a material point. For this validation, the experimental results presented by Schulson & Nickolayev [6] and Schulson & Bucks [7] for saline and fresh-water ice are used. To construct the failure envelope, the damage initiation is recorded for a combination of applied strains and stresses. As shown in Fig. 2.13a, the model is subjected to an applied strain-rate $\dot{\varepsilon}_{11}^0 = 1 \times 10^{-6}$ and a compressive stress $\sigma_{22}$. The magnitude of $\sigma_{22}$ is a function of the recorded stress in the 11 direction $\sigma_{11}$ in proportion to the scalar value $\phi$. The value of $\phi$ varies from 0.0 to
2.4. VERIFICATION AND VALIDATION WITH EXPERIMENTAL DATA AND OTHER MODELS

![Graph of model results and experimental data](image)

Figure 2.12: Model results and experimental data taken from Mellor & Colle [5] for polycrystalline ice under applied compressive stresses of 1.0, 1.54, 2.06 and 3.06 MPa; a Change of strain rate with time; b Strain-rate with respect to the accumulated strain.

1.0. The mathematical model can be described as,

\[
\varepsilon_{11} = \dot{\varepsilon}_{11}^0(t) \cdot t
\]

\[
\sigma_{22} = \phi \sigma_{11}
\]

From the definition of the stress tensor in Eq. 2.1 and the time integration parameters in Appendix A.3, the stress in 11 can be written as shown in Eq. 2.51. Note that for simplicity, only one
CHAPTER 2. A PRONY-SERIES TYPE VISCOELASTIC SOLID COUPLED WITH A CONTINUUM DAMAGE LAW FOR POLAR ICE MODELING

Term in the viscoelastic series, \( n \), has been used in the derivation.

\[
\sigma_{11} = \sigma_{11}^{\text{vol}} + \sigma_{11}^{\text{dev}} \\
\sigma_{11}^{\text{vol}} = K(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) \\
\sigma_{11}^{\text{dev}} = 2G \left\{ \mu_{\infty} e_{11} + \mu_{\infty} e_{11}^0 + \mu_{\infty} \exp \left( \frac{-t}{\lambda_1} \right) e_{11}^0 + \mu_1 h_{m,11}^1 \right\} \\
= 2G \left\{ \left[ \frac{2}{3} \mu_{\infty} + \frac{2}{3} \mu_1 \frac{\lambda_1}{\Delta t} \left( 1 - \exp \left( \frac{-\Delta t}{\lambda_1} \right) \right) \right] e_{11}^{\text{dev}} \\
- \left[ \frac{1}{3} \mu_{\infty} + \frac{1}{3} \mu_1 \frac{\lambda_1}{\Delta t} \left( 1 - \exp \left( \frac{-\Delta t}{\lambda_1} \right) \right) \right] (\varepsilon_{22} + \varepsilon_{33}) \right\} \\
+ \mu_1 h_{m,11}^1 \left( 1 - \exp \left( \frac{-\Delta t}{\lambda_1} \right) \right) - \mu_1 \frac{\lambda_1}{\Delta t} \left( 1 - \exp \left( \frac{-\Delta t}{\lambda_1} \right) \right) e_{11}^{m-1} \right\} \\
\text{(2.51)}
\]

where \( e_{11} \) is the deviatoric strain in the direction 11 and the indices \( m \) and \( m - 1 \) denote the quantities at the current and previous time step, respectively. A similar derivation can be constructed for \( \sigma_{22} \) yielding,

\[
\sigma_{22} = \sigma_{22}^{\text{vol}} + \sigma_{22}^{\text{dev}} \\
\sigma_{22}^{\text{vol}} = K(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) \\
\sigma_{22}^{\text{dev}} = 2G \left\{ \left[ \frac{2}{3} \mu_{\infty} + \frac{2}{3} \mu_1 \frac{\lambda_1}{\Delta t} \left( 1 - \exp \left( \frac{-\Delta t}{\lambda_1} \right) \right) \right] e_{22}^{\text{dev}} \\
- \left[ \frac{1}{3} \mu_{\infty} + \frac{1}{3} \mu_1 \frac{\lambda_1}{\Delta t} \left( 1 - \exp \left( \frac{-\Delta t}{\lambda_1} \right) \right) \right] (\varepsilon_{11} + \varepsilon_{33}) \right\} \\
+ \mu_1 h_{m,22}^1 \left( 1 - \exp \left( \frac{-\Delta t}{\lambda_1} \right) \right) - \mu_1 \frac{\lambda_1}{\Delta t} \left( 1 - \exp \left( \frac{-\Delta t}{\lambda_1} \right) \right) e_{22}^{m-1} \right\} \\
\text{(2.52)}
\]

Using the definition of the applied stress \( \sigma_{22} = \phi \sigma_{11} \) the relationship between the stress and strain can be constructed from a given values of strain as,

\[
\varepsilon_{22} = \frac{A}{B}(\varepsilon_{11} + \varepsilon_{33}) + C \\
\text{(2.53)}
\]
in which the values of $A, B$ and $C$ are time dependent parameters defined by,

$$
A = (1 - \phi)K + \frac{2}{3}G(2 + \phi) \left[ \mu_\infty + \mu_1 \frac{\lambda_1}{\Delta t} \left( 1 - \exp \left( \frac{-\Delta t}{\lambda_1} \right) \right) \right] \\
B = - (1 - \phi)K + \frac{2}{3}G(1 + 2\phi) \left[ \mu_\infty + \mu_1 \frac{\lambda_1}{\Delta t} \left( 1 - \exp \left( \frac{-\Delta t}{\lambda_1} \right) \right) \right] \\
C = 2G\mu_1 \left[ \exp \left( \frac{-\Delta t}{\lambda_1} \right) (\phi h_{m-1,11} - h_{m-1,22}) + \frac{\lambda_1}{\Delta t} \exp \left( \frac{-\Delta t}{\lambda_1} \right) (e^{m-1}_{22} - \phi e^{m-1}_{11}) \right]
$$

Using the values in Eq. 2.54 allow to compute $\varepsilon_{22}$ from Eq. 2.53 and therefore to calculate the stress components $\sigma_{11}$ and $\sigma_{22}$. The predicted parameters match reasonably well the results in both saline and fresh water tests under the applied strain-rate of $\dot{\varepsilon}_{11}^o = 1 \times 10^{-6}$ as shown in Fig. 2.13b.
2.4.7 Surface crevasse propagation in glaciers under varying sea water levels

Finally, we employ the aforementioned constitutive model to study of the propagation of water-free crevasses in grounded marine-terminating glaciers. The model is implemented in a large scale Finite Element simulation to predict the behavior of ice slabs under their own weight and different levels of hydrostatic sea pressure. The numerical results are compared to the results presented in Duddu et al. [8] using the Strain Decomposed (SD) model coupled with a non-local damage model.

Under different water levels, the propagation rate and extent of a pre-existing crevasse at the surface, $d_0^p$, is studied assuming free-slip at the grounded (bottom) boundary. For the numerical comparison, a similar geometry and conditions to those reported in [8] are implemented in FEAP, as shown in Fig. 2.14a. Those elements with a damage higher than the critical value of damage, $D_c \geq 0.45$ have been removed for clarity in this figure.

The damage is induced in the ice slab by the gravity load of the saline ice with density of $\rho_{(ice)} = 910Kg/m^3$ at a temperature of $T = -10^\circ C$. Following similar assumptions as in [8], damage propagation is only induced due to tensile stresses. The effects of the water levels, $h_w$, normalized as a function of the slab height, $H$ is also studied. As shown Fig. 2.15, the final crack length is plotted as a function of the sea water levels on the glacier-sea (right) boundary. The results are compared to the simulation results in [8].

With the damage induced by tensile forces, it's clear that compressive hydrostatic forces induced by sea water level ($\rho_{(sea-water)} = 1020Kg/m^3$) have significant effect in resisting and limiting the propagation of crevasses. Both models shown in Fig. 2.15, demonstrate quantitatively this effect and even display very similar crevasse depth when water levels drops below 50% of the
2.4. VERIFICATION AND VALIDATION WITH EXPERIMENTAL DATA AND OTHER MODELS

Figure 2.14: Marine-terminating glacier used in numerical simulations. a Schematic model used for glacier crevasse propagation with varying water level height $h_w$. Dimensions $H = 500m$ and $L = 2500m$. Initial crevasse length, $d_o = 0.075H$. b Deformation shape of the ice finite element model with a scale factor of 5.0 under $h_w = 400.0m$ after 3.0 days (CPU time 6.4 hours). c Deformation shape of the ice finite element model with a scale factor of 5.0 subjected to $h_w = 250.0m$ after 3.0 days (CPU time 14.9 hours).

glacier height ($h_w/H \geq 0.50$). For water levels above 250$m$ ($h_w/H > 0.50$), the model presented by Duddu et al. [8] displays a deeper crevasse at the end of the simulation. In the Prony series
results (PS), the final crack length, $d_s$ for the water levels of $h_w = 400m$ is $129.1m$, for $h_w = 250m$ is $d_s = 381.5m$ and when there is no water pressure, ($h_w = 0.0m$) the crevasse propagates through the entire thickness of the glacier.

Note that the final break of the glacier ($d_s/H = 1.0$) using the PS model without water ($h_w/H = 0.0$) is comparable to the large crack length obtained by Duddu et al. [8] which yielded $d_s/H = 0.94$. We also study the rate of crevasse propagation as function of time in Fig. 2.16. Growth of the water-free crevasse is initially induced by the self weight of the ice until it reaches a constant value of depth at steady state conditions. The damage rate is compared to the results in [8] for different water levels.

The comparison allows to highlight the significant effect of the water sea level in the damage propagation depth and rate as well as the qualitative similarities between both ice models. Figures 2.16 and 2.15 show that at steady state conditions, the Prony series model displays a slightly larger crack depth compared to the results in [8]. Moreover, in spite of having different damage initiation times, both models display similar smooth damage-rate curves until the end of the simulation.
2.5 CONCLUSIONS

A constitutive model that captures the viscoelastic nature of polar ice including deterioration induced by damage is proposed and implemented in a finite element software. The ice model couples a generalized Maxwell viscoelastic solid with a rate-form damage law. Prony Series expansion is employed for the viscoelastic material moduli which allows for semi analytical integration in time, while material deterioration and failure is approximated by a Murakami damage law and integrated explicitly in time. The viscoelastic and damage material parameters in tension and compression
are calibrated using a gradient-based constrained optimization algorithm in a staggered scheme.

Rigorous analysis of the thermodynamic formulation, presented in the form of the Clausius-Duhem inequality, confirms the well posedness of the constitutive law and yields appropriate constraints on the damage parameters. Explicit formulation of the Dissipation Function is presented along with a numerical example under the application of cyclic loading and its dependency on the applied frequency is highlighted. It is shown that the total energy dissipated in the system increases towards a constant value with the increment of the applied loading frequency.

Verification and validation with experimental data is done after proper calibration of material parameters. Good agreement is found in predicted behavior of ice under tension creep, compression creep and biaxial failure envelope to experimental data. Moreover, a numerical investigation for surface crevasse propagation in grounded marine-terminating glaciers with increasing seawater depth is performed using 2D finite elements, highlighting the major drivers in glacier cracking and collapse mechanisms.

Computational efficiency in terms of maximum stable time step, $\Delta t_{cr}$, and total CPU time is highlighted when compared to a similar ice model in the literature. Differences in the short and long term behavior of both ice models without damage are emphasized. In summary, the prediction of damage and rupture of polycrystalline ice under several boundary and loading conditions, illustrates the advantages of the proposed ice model and highlights its flexibility over other models.
Chapter 3

An Equivalent Stress-Gradient Regularization Model for Coupled Damage-Viscoelasticity

The inputs from the co-authors of the paper Londono et al. [147] from which this chapter is reproduced are gratefully acknowledged.

3.1 Problem statement

Viscoelastic constitutive law Consider a generalized Maxwell model, similar to the one described in Londono et al. [3]. Viscoelastic behavior is achieved by assuming that the deviatoric components of stress are time dependent, as described in Eq. 3.1. That is

\[
\sigma(t) = \sigma^{vol} + \sigma^{dev}(t) = K \text{tr}(\epsilon)I + 2 \int_0^t G(t-\tau)\dot{\epsilon}(\tau) d\tau,
\]  

(3.1)
where $K$ is the Bulk modulus, $I$ is the identity tensor, $G(t)$ is the time dependent shear modulus, the deviatoric $\sigma^{\text{dev}}(t)$ and volumetric $\sigma^{\text{vol}}$ components of the stress tensor are defined in terms of the trace operator $\text{tr}(\cdot)$, the strain $\varepsilon$ is defined in terms of the symmetric gradient operator and the displacement $u$ by $\varepsilon = \nabla^s u = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$, and the deviatoric strain $\varepsilon = \varepsilon - \frac{1}{3} \text{tr}(\varepsilon) I$. Note that each time derivative is superimposed with an upper dot (e.g. $\dot{\varepsilon}$) and the time independent material moduli $K$ and $G_0$ are described in terms of the Young’s modulus $E$ and Poisson ratio $\nu$ by $K = \frac{E}{3(1-2\nu)}$ and $G_0 = \frac{E}{2(1+\nu)}$, respectively. The time dependent shear modulus, $G(t)$ is expanded in a Prony series in which,

$$G(t) = G_\infty + \sum_{i=1}^{n} G_i \exp(-t/\lambda_i) = G_0 \left( \mu_\infty + \sum_{i=1}^{n} \mu_i \exp(-t/\lambda_i) \right).$$  \hspace{1cm} (3.2)$$

Note that the non-negative parameters $\mu_i$ describe the normalized relaxation modulus as, $\mu_\infty = \frac{G_\infty}{G_0}$ and $\mu_i = \frac{G_i}{G_0}$, with $G_0$ being the instantaneous shear modulus. This normalization implies that parameters $\mu_i$ must satisfy

$$\mu_\infty + \sum_{i=1}^{n} \mu_i = 1, \hspace{0.5cm} \forall i = 1, 2, \ldots, n.$$  \hspace{1cm} (3.3)$$

where $n$ is the number of terms in the Prony series needed to represent a given material behavior.

**Continuum Damage Mechanics**  A solid medium which contains microcracks and voids can be described by an effective stress concept [57, 72] through the hypothesis of strain equivalence [167] or alternatively energy equivalence [168]. In the context of strain equivalence, Simo & Ju [169] introduced a linear fourth order transformation $M$ of the form

$$\sigma = M^{-1} : \tilde{\sigma}$$  \hspace{1cm} (3.4)$$
to transfer the stress in physical medium to the effective medium, as illustrate in Fig. 3.1. The colon operator “;” corresponds to the contraction operation defined in indicial notation as $a : b = a_{ij}b_{ij}$. For isotropic damage, $M$ is written in terms of the internal damage field $D$ as $M = (1 - D)^{-1}I$, where $I$ is the identity tensor. The value of $D$ is computed at each material point and ranges from 0.0 at the undamaged state to 1.0 at the fully damaged condition.

In this work, we employ a damage-rate model proposed by Murakami [42, 148] as shown in Eq. 3.5:

$$
\dot{D} = B \frac{<\chi(\sigma)>}{k_{\sigma}} (1 - D)^{k_{\sigma}}.
$$

Figure 3.1: Transformation from the physical space a to the effective space b by means of the damage projector $M$. with $B$, $r$ and $k_{\sigma}$ are material dependent damage evolution parameters obtained from experimental data. A linear form of the material parameter $B$ was proposed by Londono et al. [3] and is also
used here. In this case, $B$ vary linearly with respect to the internal stress $\sigma$ as

$$B = B_1 + B_2 \left| \text{tr}(\tilde{\sigma}) \right|. \quad (3.6)$$

The symbols $\langle \cdot \rangle$ are the Macaulay brackets [153] defined by,

$$\langle \chi \rangle = \begin{cases} 
0, & \text{if } \chi \leq 0, \\
\chi, & \text{if } \chi > 0
\end{cases} \quad (3.7)$$

and $\chi$ is the local Hayhurst’s multiaxial equivalent stress measure [154] defined by

$$\chi = \alpha \tilde{\sigma}^{(1)} + \beta \sqrt{3J_2} + (1 - \alpha - \beta) \text{tr}(\tilde{\sigma}), \quad (3.8)$$

where $\alpha$ and $\beta$ are non-negative material parameters that govern the stresses-driven damage growth such that $\alpha + \beta \leq 1.0$. Thus, $\chi$ accounts for complex loading conditions by including a linear combination of the largest principal effective stress $\tilde{\sigma}^{(1)}$, the first invariant of the effective stress $\text{tr}(\tilde{\sigma})$, and the second invariant of the deviatoric effective stress tensor $J_2 = \frac{1}{2} \tilde{\sigma}^{(dev)} : \tilde{\sigma}^{(dev)}$. According to the experimental results compiled by Hayhurst [154], different material microstructures generate significantly different failure mechanisms when subjected to a similar loading profile. These differences can be captured by the adequate selection of the parameters $\alpha$ and $\beta$ in Eq. 3.8 which allow a wide range of failure envelopes be constructed for a particular material and their different stress invariants.

Materials under complex loading with damage governed by isotropic ductile failure are accounted for by the measure of their Von Mises stress in the second invariant of the deviatoric stress, that is a larger value of $\beta$ (e.g. Aluminum). Moreover, materials that follow a failure envelope closely delimited by the maximum principal stress (e.g. Copper) are dominated by $\alpha$. 58
Conversely, low values of $\alpha$ and $\beta$ drive the damage propagation by the hydrostatic stress components. Further discussion on the importance and application of these material parameters and the flexibility that they provide to model a broad range of damage drivers is highlighted in Section 3.3.3.

**Stress-gradient enhanced damage**  In this work, we propose a new nonlocal gradient type regularization approach for viscoelastic materials. The key idea is to regularize the Hayhurst’s equivalent stress measure in Eq. 3.8 rather than the typical choice of strain regularization. This approach permits an isotropic damage evolution be constructed using a scalar measure of stress only while capturing multiaxial loading effects. Moreover, given the dependence of the damage law on an equivalent stress measure, a direct application of the nonlocal equivalent stress, $\bar{\chi}$ yields a naturally regularized and physically sound results.

Note that alternatively, a solution to regularize the problem could be constructed directly from the rate of damage growth $\dot{D}$, with the source term directly driven by the Murakami’s damage law. However, selecting the latter as the regularization scheme could lead to numerical problems such as constraining the damage solution within physical bounds at the end of an element life ($0 \geq D \geq 1$) as well as during the Newton iterations ($\dot{D} \geq 0$).

Following the work of Peerlings et al. [74], and as presented in Section 1.2, spurious mesh sensitivity and convergence problems in damage mechanics can be corrected by a nonlocal damage approach. In our case, it is motivated by taking a weighted average of all *local* equivalent stress
terms \( \chi \) in Eq. 3.8 at a local neighborhood defined by an internal length \( l_c \)

\[
\bar{\chi} = \frac{1}{V} \int_{\Omega} g(\xi) \chi(x+\xi)dV
\]  

(3.9)

with the weight function \( g(\xi) \) satisfying \( \frac{1}{V} \int_{\Omega} g(\xi)dV = 1 \) where the \( V \) corresponds to the volume, area or line segment of integration domain for three, two and one dimension analysis respectively. The weight function is selected to be the standard Gaussian distribution function defined in Eq. 3.10,

\[
g(\xi) = \exp \left( -\left( \frac{2|\xi|}{l_c} \right)^2 \right)
\]  

(3.10)

where \(|\xi|\) corresponds to coordinate vector which in two dimensions and Cartesian coordinates corresponds to \(|\xi| = (x^2 + y^2)^{1/2}\) with \( l_c \) being the characteristic length with the same order of magnitude of the material inhomogeneities.

Plugging a first order Taylor series expansion instead of the local variable \( \chi \), and specializing Eq. 3.9 to an area integral of the 2D domain \( \Omega \), one gets a new scalar quantity, referred as nonlocal equivalent stress \( \bar{\chi} \) given as

\[
\bar{\chi} = \chi + c\nabla^2 \chi
\]  

(3.11)

where \( c \) is the result of the weight functions \( g(\xi) \) and the averaging volume defined in terms of \( l_c \) such that \( c = \frac{l_c^2}{2} \).

As explained in [89], in the context of the Finite Element Method, the parameter \( \chi \) is subjected to second order gradients as formulated in Eq. 3.11. As the stress measures in \( \chi \) are implicit functions of strain and since the strain involves first order derivatives of displacements, a third derivative...
of the displacement is required to compute the local values of $\chi$ in Eq. 3.11. Given this formulation and following the same approach as in [74], in order to avoid the required $C^1$-continuity requirement for the displacement shape functions, a mathematical modification to Eq. 3.11 yields an implicit formulation in which the second order gradient operator is applied on a independent and continuous nonlocal equivalent stress parameters $\bar{\chi}$ as shown in Eq. 3.12,

$$\bar{\chi} - c\nabla^2 \bar{\chi} = \chi \tag{3.12}$$

The damage evolution follows Kuhn-Tucker conditions in Eq. 3.13 which describes the irreversible nature of the damage accumulation.

$$\dot{\kappa} \geq 0; \quad f(\bar{\chi}, \kappa) \leq 0; \quad \kappa f(\bar{\chi}, \kappa) = 0 \tag{3.13}$$

where $\kappa$ is a history variable governing the damage growth, with a non-zero value defined as the maximum of $\bar{\chi}$ the regularized equivalent stress and $\bar{\chi}_{\text{max}}$ the maximum previously sustained regularized equivalent stress, as shown in Eq. 3.14.

$$\kappa = \max(\bar{\chi}_{\text{max}}, \bar{\chi}) \tag{3.14}$$

Remark 1: At the beginning of the simulation, $\bar{\chi}_{\text{max}}$ is set to a non zero value ($\chi_{\text{threshold}}$) that serves as a damage initiation threshold.

The nonlocal version of the damage evolution proposed in Eq. 3.5 is therefore enforced following
the conditions of Eq. 3.15

\[
\dot{D} = \begin{cases} 
0, & \text{if } \kappa < \bar{\chi} \\
B \frac{(\bar{\chi}(\sigma))^r}{(1 - D)^k}, & \text{if } \kappa \geq \bar{\chi}
\end{cases}
\]  

(3.15)

**Strong Form** In summary, we consider a two dimensional solid \( \Omega \) with boundary \( \Gamma \). As accustomed, \( \Gamma \) is partitioned into two parts, \( \Gamma = \Gamma_x \cup \Gamma_\phi \), where the Dirichlet boundaries are denoted by \( \Gamma_x \), and the Neumann boundaries associated with the flux of \( x \) are denoted by \( \Gamma_\phi \) as shown in Fig. 3.2.

![Figure 3.2: Schematic of a viscoelastic solid material in domain \( \Omega \) with Dirichlet \( \Gamma_x \) and Neumann \( \Gamma_\phi \) boundaries. The extent of the damage to the material is characterized by the damage parameter \( d \). The parameter is 1 in the fully damaged phase and 0 in the undamaged phase. The width of the damage zone is determined by the parameter \( l_c \). Note that the problem is time dependent and the solid deforms with time.](image)

Plugging the nonlocal equivalent stress measure in Eq. 3.12 into the damage formulation of Eq. 3.15 allows to compute a nonlocal damage rate. Together, with the definition of the equivalent stress measure in Eq. 3.8, the strong form of the problem is finally described by the system of Eq. 3.16-3.22.
3.1. PROBLEM STATEMENT

\[ \nabla \cdot \sigma(t,D,u) + b = f_{\text{ext}}(t) \quad \text{in } \Omega \quad (3.16) \]

\[ \dot{\chi} - c \nabla^2 \chi = \chi(\sigma) \quad \text{in } \Omega \quad (3.17) \]

\[ \chi(\sigma) = \alpha \sigma^{(1)} + \beta \sqrt{3J_2} + (1 - \alpha - \beta)\text{tr}(\sigma) \quad (3.18) \]

\[ \sigma(t,D,u) = (1 - D)\bar{\sigma}(t,u) \]

\[ = (1 - D) \left[ K\text{tr}(\epsilon)I + 2 \int_0^t G(t - \tau)\dot{\epsilon}(\tau)d\tau \right] \text{in } \Omega \quad (3.19) \]

\[ \dot{D} = B \frac{\langle \chi \rangle^r}{(1 - D)^{k_\alpha}} \quad \text{in } \Omega \quad (3.20) \]

\[ \sigma \cdot n = t(t) \quad \text{in } \Gamma_\phi \quad (3.21) \]

\[ u = u_{\text{BC}}(t) \quad \text{in } \Gamma_x \quad (3.22) \]

\[ \epsilon = \nabla^2 u = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad \text{in } \Omega \quad (3.23) \]

\[ \nabla \dot{\chi} \cdot n = 0 \quad \text{in } \Gamma_\phi \quad (3.24) \]

\[ D(t = 0) = 0 \quad \text{in } \Omega \quad (3.25) \]

\[ \dot{\chi}(t = 0) = 0 \quad \text{in } \Omega \quad (3.26) \]

where \( \Omega \) is the physical domain of the problem and \( f_{\text{ext}}(t) \) is the external force.

The traction vector \( t \) that could be function of time \( t \) and the outward normal vector, \( n \) are both applied on the natural boundaries, \( \Gamma_\phi \). The prescribed displacements \( u_{\text{BC}} \) are enforced at the essential boundaries, \( \Gamma_x \), and a non-standard boundary condition in equation Eq. 3.24 is imposed on the nonlocal equivalent stress \( \dot{\chi} \). Since \( \dot{\chi} \) is directly related to the damage in the solid, the boundary condition in Eq. 3.24 can be interpreted as insulated conditions of damage through the boundary.
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The problem is therefore reduced to solving the coupled system for the displacement \( \mathbf{u} \) and nonlocal equivalent stress \( \bar{\chi} \). The time integration of the damage and viscoelastic parameters in Eq. 3.19 are obtained following the integration scheme proposed by Londono et al. [3]. Thus, the damage term in the first parenthesis of Eq. 3.19 is computed by a forward Euler integration scheme of Eq. 3.20. The second term in the Eq. 3.19 is obtained by a semi-analytical time integration which avoids storing the entire history of the deformation.

3.1.1 Consistent linearization

Due to the non-linear nature of the problem, Eq. 3.16 and Eq. 3.17 are written in residual form \( \mathbf{R} = [\mathbf{R}_u; \mathbf{R}_\chi] \) as shown in Eq. 3.27 and Eq. 3.28, respectively, where \( \mathbf{R}_u \) corresponds to the residual of the equilibrium equation and \( \mathbf{R}_\chi \) is the additional non local equivalent stress equation. In the current work, the system is discretized using the finite element method and a monolithic scheme is proposed to solve the coupled equations iteratively by means of a Newton method.

\[
\mathbf{R}_u = \nabla \cdot \mathbf{\sigma} + \mathbf{b} - \mathbf{f}^{\text{ext}} = 0 \quad (3.27)
\]

\[
\mathbf{R}_\chi = \bar{\chi} - c\nabla^2 \bar{\chi} - \chi(\bar{\sigma}) = 0 \quad (3.28)
\]

Multiplying the equations by trial functions \( w_u \) and \( w_\chi \) and integrating both equations over the domain \( \Omega \), the weak form in Eq. 3.29 is obtained.

\[
\mathbf{R}_u = \int_{\Omega} \nabla w_u \mathbf{\sigma} d\Omega - \int_{\Gamma_\delta} w_u \mathbf{t} d\Gamma_\delta - \int_{\Omega} w_u \mathbf{b} d\Omega = 0
\]

\[
\mathbf{R}_\chi = \int_{\Omega} (w_\chi \bar{\chi} + c(\nabla w_\chi \cdot \nabla \bar{\chi}) - w_\chi \chi) d\Omega = 0 \quad (3.29)
\]
3.1. PROBLEM STATEMENT

where the divergence theorem was used where appropriate. In the absence of body forces \( \mathbf{b} = 0 \), the linearized set of residual equations in Eq. 3.29 can be written as,

\[
\begin{bmatrix}
\delta \mathbf{u}_m^k \\
\delta \tilde{\chi}_m^k
\end{bmatrix} =
\begin{bmatrix}
\mathbf{R}_{u,m}^k \\
\mathbf{R}_{\tilde{\chi},m}^k
\end{bmatrix}
\] (3.30)

where the unknown fields \( \mathbf{u} \) and \( \tilde{\chi} \) are obtained by Newton’s method in which the current field updates \( \delta \mathbf{u}_m^k \) and \( \delta \tilde{\chi}_m^k \) at iteration \( k \) and time step \( m \) are obtained by solving Eq. 3.30. \( \mathbf{J}(\mathbf{u}_m^{k-1}, \tilde{\chi}_m^{k-1}) \) is the Jacobian of the system, updated iteratively with Newton iterations.

Using Galerkin’s method, the displacements and nonlocal equivalent stress, are discretized by shape functions \( \mathbf{N}_u \) and \( \mathbf{N}_{\tilde{\chi}} \), respectively, as shown in Eq. 3.31

\[
\begin{align*}
\mathbf{u} &= \mathbf{N}_u \hat{\mathbf{u}}, \quad \nabla \mathbf{u} = \nabla \mathbf{N}_u \hat{\mathbf{u}} \\
\tilde{\chi} &= \mathbf{N}_{\tilde{\chi}} \hat{\tilde{\chi}}, \quad \nabla \tilde{\chi} = \nabla \mathbf{N}_{\tilde{\chi}} \hat{\tilde{\chi}}
\end{align*}
\] (3.31)

Note that the hatted terms (e.g. \( \hat{\mathbf{u}}, \hat{\tilde{\chi}} \)) are the vector of nodal values for each field in an element and \( \nabla \) is the standard gradient operator. In our implementation standard bilinear shape functions for displacements and the equivalent stress quantity are used. The Jacobian of the problem can therefore be defined by,

\[
\mathbf{J}(\mathbf{u}, \tilde{\chi}) =
\begin{bmatrix}
\frac{\partial \mathbf{R}_u}{\partial \mathbf{u}} & \frac{\partial \mathbf{R}_u}{\partial \tilde{\chi}} \\
\frac{\partial \mathbf{R}_{\tilde{\chi}}}{\partial \mathbf{u}} & \frac{\partial \mathbf{R}_{\tilde{\chi}}}{\partial \tilde{\chi}}
\end{bmatrix} =
\begin{bmatrix}
\mathbf{K}_{uu} & \mathbf{K}_{u\tilde{\chi}} \\
\mathbf{K}_{\tilde{\chi}u} & \mathbf{K}_{\tilde{\chi}\tilde{\chi}}
\end{bmatrix}
\] (3.32)

In the current work, each term of the Jacobian in Eq. 3.32 is obtained analytically using the notion of Gâteaux derivative [155, 156], defined in Eq. 3.33, in terms of the perturbation \( \epsilon \) (not to
be confused with the strain \( \varepsilon \).

\[
\frac{\partial R_{\chi}}{\partial \bar{\chi}} \delta \bar{\chi} = \lim_{\varepsilon \to 0} \frac{R_{\chi}(u, \bar{\chi} + \varepsilon \delta \bar{\chi}) - R_{\chi}(u, \bar{\chi})}{\varepsilon} = \frac{dR_{\chi}(u, \bar{\chi} + \varepsilon \delta \bar{\chi})}{d\varepsilon} \bigg|_{\varepsilon=0} (3.33)
\]

The term \( \frac{\partial R_{\chi}}{\partial \bar{\chi}} \) is presented below as an example, and all other terms are derived in Appendix B.1.

\[
\frac{\partial R_{\chi}}{\partial \bar{\chi}} \delta \bar{\chi} = \frac{d}{d\varepsilon} \int_{\Omega} \left[ w_{\chi}(\bar{\chi} + \varepsilon \delta \bar{\chi}) + c\nabla w_{\chi} \nabla (\bar{\chi} + \varepsilon \delta \bar{\chi}) - w_{\chi}\bar{\chi} \right] d\Omega \bigg|_{\varepsilon=0} (3.34)
\]

Introducing the discretization in Eq. 3.31, one gets

\[
K_{\bar{\chi}\bar{\chi}} = \int_{\Omega} \left( N_{\chi}^T N_{\chi} + c \nabla N_{\chi}^T \nabla N_{\chi} \right) d\Omega (3.35)
\]

Following the derivation of the Jacobian terms in Appendix B.1, the additional terms of the Jacobians are summarized in Eq. 3.36-3.38. Note that \( \nabla^s \) is the symmetric gradient operator used for the strain calculation and the fourth order deviatoric and volumetric projector tensors are written with blackboard bold letters \( \mathbb{I}^{(\text{dev})} \) and \( \mathbb{I}^{(\text{vol})} \) respectively. The fourth order deviatoric and volumetric projector tensors are defined in indicial notation as \( \mathbb{I}^{(\text{vol})}_{ijkl} = \frac{1}{3} \delta_{ij} \delta_{kl} \) and \( \mathbb{I}^{(\text{dev})}_{ijkl} = \mathbb{I}^{s}_{ijkl} - \mathbb{I}^{(\text{vol})}_{ijkl} \), where the double-symmetric fourth order unit tensor is defined as \( \mathbb{I}^s_{ijkl} = \frac{1}{2} (\delta_{ij} \delta_{kl} + \delta_{il} \delta_{jk}) \) and \( \delta \) Kronecker delta. Also note that the viscoelastic Lamé constants, defined as \( \lambda_{\text{vis}} \) and \( G_{\text{vis}} \), which are related to the Bulk and Viscous Shear moduli as \( \lambda_{\text{vis}} = \kappa - \frac{2}{3} G_{\text{vis}} \) are used below.
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\[
K_{uu} = \int_{\Omega} \left( \nabla^s N_u (1 - D) \left[ 3\lambda_{\text{vol}}^{\text{vis}} + 2G_{\text{vis}}^{\text{II}} \right] \nabla^s N_u \right) d\Omega \tag{3.36}
\]

\[
K_{u\bar{\chi}} = -\int_{\Omega} \left( \nabla^s N_u \bar{\sigma}(u) \Delta t B \frac{\bar{\chi}^r}{(1 - D)^k} H(\bar{\chi}) \nabla^s N_u \right) d\Omega \tag{3.37}
\]

\[
K_{\bar{\chi}u} = -\int_{\Omega} \left( N_{\chi} \left\{ \alpha \frac{\partial \bar{\sigma}^{(1)}}{\partial \bar{\sigma}} : \left[ 3\lambda_{\text{vol}}^{\text{vis}} + 2G_{\text{vis}}^{\text{II}} \right] + 3\beta \frac{G_{\text{vis}}}{\sqrt{3J_2}} \bar{\sigma}^{(\text{dev})} \right\} \nabla^s N_u \right) d\Omega \tag{3.38}
\]

where \( H(\bar{\chi}) \) in Eq. 3.37 is the Heaviside step function and the viscous shear modulus \( G_{\text{vis}} \) defined in,

\[
G_{\text{vis}} = G_0 \left[ \mu_\infty + \sum_{i=1}^{n} \mu_i \lambda_i \left( 1 - \exp \left( \frac{\Delta t}{\lambda_i} \right) \right) \right] \tag{3.39}
\]

3.2 Thermodynamic consistency

Extending the derivation in [3] to multi-dimensions, the second law of thermodynamics in the form of the Clausius-Duhem inequality is used to verify that the dissipated and stored energy of the proposed model are physically sound. Consider a damageable viscoelastic material with the following Helmholtz Free energy,

\[
\psi = \psi(\varepsilon, D) \tag{3.40}
\]
defined in terms of the strain $\varepsilon$ and internal variable $D$. Under isothermal conditions, the Clausius-Duhem inequality reads,

$$\sigma : \dot{\varepsilon} - \dot{\psi} \geq 0$$

(3.41)

where $\sigma$ is the total stress in the material. The total initial strain energy $W_0(\varepsilon)$ is split in terms of the initial elastic volumetric $U_0(\varepsilon^{vol})$ and the initial deviatoric strain energies $W_0^{\text{dev}}(\varepsilon)$ as shown in Eq. 3.42, maintaining the assumption of time dependency in the deviatoric part only.

$$W_0(\varepsilon) = U_0(\varepsilon^{vol}) + W_0^{\text{dev}}(\varepsilon(t))$$

(3.42)

In terms of the Generalized Maxwell model, the stress-like measure of the $i^{th}$ Maxwell branch, $q_i$, that links its (deviatoric) dashpot strain $\varepsilon^{(\text{dashpot})}_i$ with the spring moduli $G_i^{(\text{spring})}$ can be written as (see [3] for details on $q$),

$$q_i = \frac{\mu_i}{\lambda_i} \int_0^t \exp\left(\frac{-(t - \tau)}{\lambda_i}\right) \left(\frac{\partial W_i^{\text{dev}}(\varepsilon(\tau))}{\partial \varepsilon}\right) : \mathbb{I}^{\text{dev}} d\tau$$

(3.43)

where $\mathbb{I}^{\text{dev}}$ is the fourth order tensor projection operator into the deviatoric space. For a degraded free energy, we use the following definition,

$$\psi(\varepsilon, D) = (1 - D)\tilde{\psi}(\varepsilon)$$

(3.44)

where $\tilde{\psi}$ is the Free Energy stored in the effective space written as

$$\tilde{\psi} = W_0(\varepsilon) - \sum_{i=1}^{n} q_i : \varepsilon.$$
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\[ \sigma : \dot{\epsilon} - (1 - D) \frac{\partial \tilde{\psi}}{\partial t} + \tilde{\psi} \dot{D} \geq 0 \]  \hspace{1cm} (3.46)

Replacing the definition of the effective energy in Eq. 3.44 and grouping similar terms leads to,

\[
\sigma : \dot{\epsilon} - (1 - D) \frac{\partial W_o}{\partial \epsilon} \dot{\epsilon} + (1 - D) \sum_{i=1}^{n} (q_i : e + q_i : I^{dev} : \dot{\epsilon}) + \tilde{\psi} \dot{D} \geq 0 \hspace{1cm} (3.47)
\]

\[
\left[ \sigma - (1 - D) \left( \frac{\partial W_o}{\partial \epsilon} - \sum_{i=1}^{n} (q_i) : I^{dev} \right) \right] : \epsilon + (1 - D) \sum_{i=1}^{r} (e : q_i) + \tilde{\psi} \dot{D} \geq 0. \hspace{1cm} (3.48)
\]

Since the results have to hold for any arbitrary rate of strain \( \dot{\epsilon} \), the first term in brackets must be zero. Indeed, this term vanishes since it describes the constitutive equation in Eq. 3.19 obtained from the equivalent energy formulation as,

\[
\sigma = (1 - D) \left( \frac{\partial W_o}{\partial \epsilon} - \sum_{i=1}^{n} (q_i) : I^{dev} \right) \hspace{1cm} (3.49)
\]

\[
= (1 - D) \left( \left( \frac{\partial W_o^{dev}}{\partial \epsilon} \right) : I^{dev} + \frac{\partial U_o}{\partial \epsilon^{vol}} : I - \sum_{i=1}^{n} (q_i) : I^{dev} \right) \hspace{1cm} (3.50)
\]

which is consistent with the standard thermodynamic assumption of \( \sigma = \frac{\partial \psi}{\partial \epsilon} \). The remaining terms in inequality Eq. 3.48, namely the viscous \( \mathcal{G}^{(vis)} \) and damage \( \mathcal{G}^{(dmg)} \) energy rate terms
contribute to the non-negative dissipation rate $\mathcal{D}$, are given as,

$$\mathcal{D} = \mathcal{D}^{\text{vis}} + \mathcal{D}^{\text{dmg}}$$

$$= (1 - D) \sum_{i=1}^{n} (\mathbf{e} : \dot{\mathbf{q}}_i) + \Psi B \frac{\langle \bar{\mathcal{X}} \rangle^r}{(1 - D) \kappa \sigma}$$  \hspace{1cm} (3.51)

where the definition of the damage rate in terms of the nonlocal equivalent stress $\bar{\mathcal{X}}$ from Eq. 3.20 has been used. The remaining definition of the value of $\mathbf{q}_i$ in the first term of Eq. 3.51 is obtained from its definition in Eq. 3.43. Assuming the rate is constant within a time interval, the form of $\dot{\mathbf{q}}_i = \Delta \mathbf{q}_i / \Delta t$ is used. Since the current time $t$ is simply defined as $t = t_n + \Delta t$, the integral can be split to yield Eq. 3.52 as,

$$\Delta \mathbf{q}_i = \mathbf{q}_i(t) - \mathbf{q}_i(t_n)$$

$$= \frac{\mu_i}{\lambda_i} \int_{0}^{t} \left( \exp \left( \frac{-(t - \tau)}{\lambda_i} \right) \frac{\partial W_{\text{dev}}(\mathbf{e}(\tau))}{\partial \mathbf{e}} : \mathbb{I}_{\text{dev}} \right) d\tau$$

$$- \frac{\mu_i}{\lambda_i} \int_{0}^{t_n} \left( \exp \left( \frac{-(t - \tau)}{\lambda_i} \right) \frac{\partial W_{\text{dev}}(\mathbf{e}(\tau))}{\partial \mathbf{e}} : \mathbb{I}_{\text{dev}} \right) d\tau$$

$$= \frac{\mu_i}{\lambda_i} \int_{t_n}^{t} \left( \exp \left( \frac{-(t - \tau)}{\lambda_i} \right) \left( \frac{\partial W_{\text{dev}}(\mathbf{e}(\tau))}{\partial \mathbf{e}} \right) : \mathbb{I}_{\text{dev}} \right) d\tau$$  \hspace{1cm} (3.52)

which after integration by parts of Eq. 3.52 yields,

$$\Delta \mathbf{q}_i = \left[ \mu_i \exp \left( \frac{-(t - \tau)}{\lambda_i} \right) \left( \frac{\partial W_{\text{dev}}(\mathbf{e}(\tau))}{\partial \mathbf{e}} \right) : \mathbb{I}_{\text{dev}} \right]_{t_n}^{t}$$

$$- \mu_i \int_{t_n}^{t} \left( \exp \left( \frac{-(t - \tau)}{\lambda_i} \right) \frac{\partial}{\partial \tau} \left( \frac{\partial W_{\text{dev}}(\mathbf{e}(\tau))}{\partial \mathbf{e}} \right) : \mathbb{I}_{\text{dev}} \right) d\tau$$

$$= \mu_i \left( \frac{\partial W_{\text{dev}}(\mathbf{e}(t))}{\partial \mathbf{e}} \right) : \mathbb{I}_{\text{dev}} - \mu_i \exp \left( \frac{-(t - \tau)}{\lambda_i} \right) \left( \frac{\partial W_{\text{dev}}(\mathbf{e}(\tau))}{\partial \mathbf{e}} \right) : \mathbb{I}_{\text{dev}}$$

$$- \mu_i \frac{\partial}{\partial t} \left( \frac{\partial W_{\text{dev}}(\mathbf{e}(t))}{\partial \mathbf{e}} \right) : \mathbb{I}_{\text{dev}} \int_{t_n}^{t} \left( \exp \left( \frac{-(t - \tau)}{\lambda_i} \right) \right) d\tau$$  \hspace{1cm} (3.53)
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where the last term in Eq. 3.53 is integrated with the assumption that the time derivative \( \frac{\partial}{\partial t} \left( \frac{\partial W^\text{dev}_o}{\partial \mathbf{e}} \right) \) is constant during a time step \([t_n, t]\). Under this assumption, the constant rate term can be written as

\[
\frac{\partial}{\partial t} \left( \frac{\partial W^\text{dev}_o}{\partial \mathbf{e}}(\mathbf{e}(t)) \right) = \frac{1}{\Delta t} \left[ \left( \frac{\partial W^\text{dev}_o}{\partial \mathbf{e}}(\mathbf{e}(t)) \right) - \left( \frac{\partial W^\text{dev}_o}{\partial \mathbf{e}}(\mathbf{e}(t_n)) \right) \right]
\] (3.54)

for the time interval \([t_n, t]\) and the increment \(\Delta \mathbf{q}_i\) is therefore computed as,

\[
\Delta \mathbf{q}_i = \mu_i \left( \frac{\partial W^\text{dev}_o}{\partial \mathbf{e}}(\mathbf{e}(t)) : \mathbb{I}^\text{dev} \right) - \mu_i \exp\left( -\Delta t / \lambda_i \right) \left( \frac{\partial W^\text{dev}_o}{\partial \mathbf{e}}(\mathbf{e}(t_n)) : \mathbb{I}^\text{dev} \right)
\]

\[
- \mu_i \frac{1 - \exp\left( -\Delta t / \lambda_i \right)}{\Delta t / \lambda_i} \left[ \left( \frac{\partial W^\text{dev}_o}{\partial \mathbf{e}}(\mathbf{e}(t)) : \mathbb{I}^\text{dev} \right) - \left( \frac{\partial W^\text{dev}_o}{\partial \mathbf{e}}(\mathbf{e}(t_n)) : \mathbb{I}^\text{dev} \right) \right]
\] (3.55)

which after grouping similar terms yields,

\[
\Delta \mathbf{q}_i = \frac{\mu_i}{\Delta t / \lambda_i} \left[ \left( \Delta t / \lambda_i + \exp\left( -\Delta t / \lambda_i \right) - 1 \right) \frac{\partial W^\text{dev}_o}{\partial \mathbf{e}}(\mathbf{e}(t)) - \left( \left( \Delta t / \lambda_i - 1 \right) \exp\left( -\Delta t / \lambda_i \right) - 1 \right) \frac{\partial W^\text{dev}_o}{\partial \mathbf{e}}(\mathbf{e}(t_n)) \right] : \mathbb{I}^\text{dev}
\] (3.56)

The non-negative nature of the dissipation rate in Eq. 3.51, as a generalization of the derivation in [3] is reasonable, since the additional gradient-enhanced equation in the system, describes a non-dissipative phenomenon. Moreover, the results describe the competing mechanisms of viscous and damage dissipation processes under arbitrary loadings. Given the physical range \(D_{\text{phys}}(0.0 \leq D_{\text{phys}} \leq 1.0)\) and the assumptions of damage irreversibility \(\dot{D} \geq 0.0\), the analytical derivation confirms a non-negative energy dissipation. A numerical example is also conducted in the following section to verify the analytical derivation and the non-negative nature of the energy dissipation.
3.2.1 Damage dissipation example

A one dimensional bar shown in Fig. 3.3 with element size of 0.05m is axially subjected to pre-scribed displacements on its right end. We consider two loading cases, low- and high- frequency cyclic loads, as shown in Fig. 3.4a and Fig. 3.4c, respectively. Note that in the high frequency load of Fig. 3.4c, the simulation lasts 10s while in the low frequency case of Fig. 3.4a the cyclic load is applied for 1000s. In both loading cases three values of characteristic length $l_c$, sizes 0.1m, 0.075m and 0.05m are studied. Under the displacement control loading, the material not only suffers from permanent damage but also viscoelastic time dependent deformation. Thus, energy is dissipated due to both the viscoelastic as well as damage components of the material. Material parameters for these tests are presented in Table 3.1 and Table 3.2 with Young’s modulus $E = 9,500 [MPa]$ and Poisson ratio $\nu = 0.35$. In Table 3.2, $\alpha$ and $\beta$ are constants of the Hayhurst’s equivalent stress in Eq. 3.8 and $B, r, k_\sigma$ are damage growth parameters as described in Eq. 3.5. The damage parameter $B$, is independent of the level of stress as determined by the zero value of $B_2$ in Eq. 3.6 and the parameter $\chi_{threshold}$ is the threshold for the damage initiation used in Eq. 3.14.

Table 3.1: Prony-series viscoelastic parameters used in the 1D numerical simulations.

<table>
<thead>
<tr>
<th>$G_\alpha$ [MPa]</th>
<th>$K$ [MPa]</th>
<th>$n$</th>
<th>$\mu_1$</th>
<th>$\lambda_1$ [s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>3518.52</td>
<td>$10.56 \times 10^3$</td>
<td>1</td>
<td>$9.9986 \times 10^{-1}$</td>
<td>17.30</td>
</tr>
</tbody>
</table>

Table 3.2: Damage parameters used in the thermodynamic dissipation example.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$B_1$</th>
<th>$B_2$</th>
<th>$r$</th>
<th>$k_\sigma$</th>
<th>$\chi_{threshold}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.21</td>
<td>0.63</td>
<td>$6.550 \times 10^{-4}$</td>
<td>0.0</td>
<td>0.9</td>
<td>0.8471</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Under the low-frequency load, damage results from each characteristic length are recorded at the center of the bar (first element on the left), and presented in Fig. 3.4b with the value of $l_c = 0.05$ reaching full damage first followed by the $l_c = 0.075$. The simulation with $l_c = 0.10$ does not break
but reaches a maximum damage value just under $D = 0.8$. Similarly, under the high-frequency load shown in Fig. 3.4d, the total collapse of the bar is also reached at different points in time but with all the values of $l_c$ failing before the end of the simulation.

It is interesting to note that, under the same loading frequency, different damage results are obtained for each value of $l_c$. This is expected since the choice of the length scale $l_c$ also controls the physics of the problem. Moreover, the results in Fig. 3.5a suggest that solids that are modeled with larger values of $l_c$ yield higher total bar strength and longer failure time (see also Fig. 3.4b and Fig. 3.4d) compared to those with smaller characteristic lengths. Thus, a larger diffused damage zone is obtained with the increase in $l_c$, which in turn yield a slower failure.

With the three different values of $l_c$, viscoelastic material behavior is observed in the stress-strain curves in Fig. 3.5a with clear distinct hysteresis present from the beginning of the low-frequency simulation. However, results displaying hysteresis effects are less pronounced in the high-frequency loads of Fig. 3.5c, with hysteresis areas increasing almost exclusively during the positive loading phase that is governed by damage growth. The damage initiation point in Fig. 3.5a and Fig. 3.5c coincide for all $l_c$’s at times of around 170s and 0.4s for the low- and high-frequency loads, respectively. As the damage progresses with time, stresses and strains are affected differently for each $l_c$ until the final stress collapse.
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Figure 3.4: Numerical example of the bar in Fig. 3.3 subjected to low- and high-frequency loading. Three different values of $l_c$ of 0.1m, 0.075m and 0.05m are studied. a. Low frequency applied cyclic strain. b. Low frequency: Damage growth with time at the center of the bar. c. High frequency applied cyclic strain. d. High frequency: Damage growth with time at the center of the bar.

In both loading cases, stress drops in the stress-strain curves are not only consistent with their damage curves (major changes match the points of major damage increments) in Fig. 3.4b and Fig. 3.4d but their changes also provide information about the energy dissipation of the material. For the low frequency case, Fig. 3.5b shows the total dissipated energy $\psi^{(dis)}$ in the bar until the final time of the simulation, $t_f = 1000s$, while the dissipation energies of the high-frequency
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loading case are shown in Fig. 3.5d. Note that $\psi^{\text{dis}}$ is obtained from the viscous $\psi^{\text{vis}}$ and damage $\psi^{\text{dmg}}$ dissipation energies which, from Eq. 3.51, are related to the dissipation $\mathcal{D}$ by,

$$
\psi^{\text{dis}} = \psi^{\text{dmg}} + \psi^{\text{vis}} = \int_0^{t_f} \mathcal{D}^{\text{dmg}}(t) dt + \int_0^{t_f} \mathcal{D}^{\text{vis}}(t) dt \tag{3.57}
$$

Results in Fig. 3.5b and Fig. 3.5d, which show larger total dissipation energies for larger characteristic lengths, can be explained by analyzing the sources of energy dissipation in our model. As presented before, the energy in the model is dissipated in part by internal friction or viscous effects and part by damage.

Note that the results in Fig. 3.5b and Fig. 3.5d correspond to the energy dissipation integrated along the entire bar until the complete breaking at the center (left hand side in Fig. 3.3); that is, for the low-frequency case until 760s, 932s and 1000s for the $l_c$ values of 0.05m, 0.075m and 0.1m, respectively. In the high-frequency case, bars with the smaller characteristic lengths of $l_c = 0.05, 0.075$ break at about 7.8s while the predicted break time for $l_c = 0.10$ is about 9s. The effect of the characteristic length $l_c$ previously described are still valid, with shorter life prediction for small values of $l_c$ as shown in Fig. 3.4d.

Under low-frequency loading, the significant difference between the viscous and the damage dissipation (for all $l_c$) is explained by the fact that most damage dissipation is contributed by a small localized section of the bar while the entire bar contributes to the viscous dissipation. Since the damage grows slowly in this example viscous behaviour through the hysteresis response dominates and dissipates more energy. In the high-frequency case, damage grows quickly and viscous effects don’t have time to be effective. Hence, in this example the energy is mainly dissipated
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Figure 3.5: Stress-strain and energy dissipation curves obtained for the bar in Fig. 3.3 subjected to low- and high cyclic load with different values of $l_c$. a. Low frequency: Stress-strain results for the cyclic strain applied showing hysteresis curves and stress collapse for $l_c = 0.05m$ and $0.075m$ only. b. Low frequency: Effect of the value of $l_c$ on the total dissipated energy integrated along the bar. c. High frequency: Stress-strain results for the cyclic strain applied. Hysteresis curves mainly driven by damage increment at positive strain phases. Collapse is reached for all $l_c$. d. High frequency: Effect of the value of $l_c$ on the total dissipated energy integrated along the bar.

through damage while viscosity is negligible.

Moreover, for both low- and high- frequency loadings, the role of $l_c$ has a similar effect in that decrease in $l_c$ values results in reduction of dissipated energy. However, in the limit that $l_c \to 0$ not
only a smaller amount of energy is dissipated but also spurious mesh sensitive results and zero-energy modes, that are characteristic of local damage models, are expected.

While the characteristic length scale provides numerical regularization to the model, its value must be chosen carefully based on microstructural type experiments or theoretical micromechanics based arguments [170, 171].

In summary, the model is demonstrated to be thermodynamically consistent through analytical derivations and numerical verification studies. Different factors influencing the damage dissipation of viscoelastic materials are discussed and the effect of the characteristic length $l_c$ on the physics is highlighted. A series of numerical examples are now presented in the following section for one and two dimensional problems.

### 3.3 Numerical examples

A series of numerical examples are presented in this section to study the behavior of the proposed model under different types of boundary conditions and time effects. First, we study several one dimensional cases considering creep, relaxation and strain rate loadings which are common problems in the mechanics of viscoelastic solids. Results of the proposed nonlocal model are compared to its local-damage counterpart and their differences are highlighted. In the second example we study the propagation of crevasses in an ideal grounded ice slab [8, 22], typically observed in the polar regions. In the idealized model, the crevasse is assumed to propagate out of a surface notch and damage extent and growth time are recorded.

This section concludes with a third example, where we study an impact problem onto a notched plate, also known as the Kalthoff and Winkler problem [10, 11]. Different mesh sizes are consid-
CHAPTER 3. AN EQUIVALENT STRESS-GRADIENT REGULARIZATION MODEL FOR COUPLED DAMAGE-VISCOELASTICITY

...and the propagation of damage out of the notch tip is examined and compared with experimental results. In this example, we also study the effect different terms in the damage model have on the direction of crack propagation.

The numerical simulations were performed using the Finite Element Analysis Program - FEAP [146] along with the mathematical tools in PETSc [172, 173] on 64-bit Intel Core 2 Quad CPU 2.83 GHz ×4, with 4GB memory desktop. Data visualization in this paper was generated using the Paraview software [174] and the mesh files generated in GMSH [175].

In summary, the numerical examples demonstrate mesh-insensitive results under arbitrary loading conditions and the potential of the approach in more complex problems.

3.3.1 One dimensional tests

A one dimensional bar subjected to creep, relaxation and strain rate loading is used to compare the results of the local and nonlocal formulations using coarse, intermediate and fine mesh sizes. The central cross section of the bar is gradually reduced by 50% to trigger damage localization and investigate damage evolution. Due to symmetry, only a quarter of the bar is modeled under the boundary conditions shown in Fig. 3.3. The top right quarter of the symmetric bar is simulated by roller-type boundary conditions that constrain displacements away from the support plane while the loading is imposed on the right end. The symmetric section has a length of 0.5m and its height varies from $5.0 \times 10^{-3} m$ at the center to $1.0 \times 10^{-2} m$ at the far end with the reduced cross section increasing linearly from the center to a full cross section at 0.10m away from the bar center.

With boundary conditions shown in Fig. 3.3, different tests are performed with constant value of stresses (creep), strains (relaxation) or strain-rates applied to the bar ends. Fine, intermediate and coarse meshes are used to compare the damage regularization with element lengths of 0.02m,
0.01\textit{m} and 0.005\textit{m}, respectively. Material parameters are taken from the calibrated ice parameters for tension in \cite{3}, listed again in the Appendix B.2.

**Creep test**

Mesh-insensitive behavior is demonstrated for one dimensional problems under different constant applied stresses. Two stress values of 45 and 50 MPa are applied at the right end of the bar and kept constant throughout the simulation.

![Creep test](image)

Figure 3.6: Creep test: damage evolution at the center of the bar as function of time for two applied constant tensile stresses, $\sigma = 45$ and 50 [MPa] and three different mesh sizes with element lengths of 0.02\textit{m}, 0.01\textit{m} and 0.005\textit{m} (second value of the legend). Vertical orange line represent the time step at which the plots in Fig. 3.7a and Fig. 3.7b are generated for the local and nonlocal damage respectively.

The creep results for the local and nonlocal damage model are presented in Fig. 3.6 and Fig. 3.7. We report the response of these models considering a coarse, intermediate and fine meshes. Fig. 3.6 shows the evolution of damage at the center of the bar as function of time and Fig. 3.7 shows the distribution of the damage along the bar after $t = 5480s$ (in the local damage model) and $t = 7000s$ (in the nonlocal damage model). These points in time are depicted by the orange vertical line in Fig. 3.6.
Figure 3.7: Creep test: Damage distribution along the first 0.15\( m \) of the bar subjected to 45\( MPa \) and 50\( MPa \) at the free end at \( t = 5480s \) (local damage model) and \( t = 7000s \) (nonlocal damage model) marked by the orange lines on Fig. 3.6a and Fig. 3.6b respectively. Three mesh sizes are considered for each stress level.

We plot the damage distribution in Fig. 3.7 in the first 0.15\( m \) from the bar center where most of the damage has already developed. While mesh dependence can be clearly seen in the results of the local damage model in Fig. 3.6a and Fig. 3.7a (especially pronounced at the peak damage on the left end of the bar), nearly perfect mesh independence is obtained by the proposed nonlocal model, as shown in Fig. 3.6b and Fig. 3.7b.

**Relaxation test**

Similarly, a relaxation test is performed in which a constant displacement is applied and the internal stress is measured through the simulation time. Two displacement values of 6.0 \( \times \) 10\(^{-4} \) and 5.0 \( \times \) 10\(^{-4} \) are applied on the right end of the bar and kept fixed throughout the simulation.

The results for the local and nonlocal damage of these relaxation tests are shown in Fig. 3.8 and Fig. 3.9. The same three coarse, intermediate and fine meshes are used in these simulations.
3.3. NUMERICAL EXAMPLES

Figure 3.8: Relaxation test: Damage evolution at the center of the bar as a function of time for two applied constant tensile strains, $\varepsilon = 6.0 \times 10^{-4}$ and $5.0 \times 10^{-4}$ (first value in the legend) and three different mesh sizes (element lengths of $0.02m$, $0.01m$ and $0.005m$). Vertical orange lines correspond to time steps of $t = 2500s$ for both local and nonlocal damage at which the results in Fig. 3.9a and Fig. 3.9a are generated respectively.

Fig. 3.8 records the damage evolution at the center of the bar as function of time while Fig. 3.9 shows the damage along the first $0.15m$ of the bar at $t = 2500s$ for both the local and nonlocal damage models. The times at which the plots in Fig. 3.9 are generated are highlighted by the orange vertical lines in Fig. 3.8.

Under the relaxation tests, we observe also the spurious mesh sensitive results as the damage in the bar progresses in Fig. 3.8a and along the bar in Fig. 3.9a. On the other hand, regularized and nearly mesh insensitive damage results are observed throughout the simulation in Fig. 3.8b as well as along the bar in Fig. 3.9b.
CHAPTER 3. AN EQUIVALENT STRESS-GRADIENT REGULARIZATION MODEL FOR COUPLED DAMAGE-VISCOELASTICITY

Figure 3.9: Relaxation test: Damage evolution at the center of the bar in Fig. 3.3 under two applied strains, $\varepsilon = 6.0 \times 10^{-4}$ and $5.0 \times 10^{-4}$ (first value in the legend) and three different mesh lengths of $0.02m$, $0.01m$ and $0.005m$ (second value in the legend). The results in a and in b are obtained at $t = 2500s$, both points depicted by a vertical orange lines in Fig. 3.8a and Fig. 3.8b respectively.

**Strain-rate test**

The 1D numerical example concludes with a strain-rate test. Two constant strain rates of $\dot{\varepsilon} = 1.0 \times 10^{-4}$ and $5.0 \times 10^{-5}$ [1/s] are applied to the right end of the bar using three different mesh sizes.

Fig. 3.11 shows the damage evolution as a function of time at the center of the bar and Fig. 3.11 shows the damage along the first $0.15m$ of the bar at $t = 78s$ (local damage) and $t = 85s$ (nonlocal damage). Orange vertical lines in Fig. 3.10 correspond to these time steps at which the results of Fig. 3.11a and Fig. 3.11b are obtained.

While the local damage results in Fig. 3.11a show minor mesh sensitive results, the internal damage values at the center of the bar shown in Fig. 3.10a demonstrates clear mesh dependence of the local damage approach. Mesh independent results are obtained with the proposed nonlocal
Figure 3.10: Strain rate test: Damage increment with time for two applied constant strain rates, $\dot{\varepsilon}_i = 1.0 \times 10^{-4}$ and $5.0 \times 10^{-5}$ and three different mesh sizes (element lengths of 0.02m, 0.01m and 0.005m). Vertical orange lines represents the time steps $t = 740s$ (local damage) and $t = 850s$ (nonlocal damage) at which the results of Fig. 3.11a and Fig. 3.11b respectively are generated.

Figure 3.11: Strain rate test: Damage distribution along the first 0.15m of the bar are generated subjected for two constant strain-rates of $\dot{\varepsilon} = 1.0 \times 10^{-4}$ and $5.0 \times 10^{-5}$ [1/s] and three different meshes (element lengths of 0.02m, 0.01m and 0.005m). Damage results are obtained at $t = 740s$ (local damage model) and $t = 850s$ (nonlocal damage model) marked by the orange lines on Fig. 3.10a and Fig. 3.10b respectively.

model as shown in Fig. 3.10b and Fig. 3.11.
3.3.2 Ice Slab under tensile strain-rate

A two dimensional problem representing an idealized ice slab subjected to constant strain rate is studied in this section to compare the results of the local and nonlocal damage models. Viscoelastic parameters, provided in Appendix B.2 as well as the results for the local damage in this section were obtained using the model in [3].

The ice slab is of height, $H = 250m$ and length, $L = 500m$ with a pre-existing notch at the top center with initial depth, $d^0_s = 10$ and width, $w^0_s = 10.0$. The slab is assumed to be in a plane stress state and is subject to a constant strain rate $\dot{\varepsilon} = 5.0 \times 10^{-3}[1/s]$ on the right edge while the bottom and left boundaries have roller-type boundary conditions as shown in Fig. 3.12.

![Figure 3.12: Idealized ice slab subjected to a constant strain rate used for glacier crevasse propagation. The slab has dimensions of $H = 250m$ and $L = 500m$. Initial crevasse depth, $d^0_s = 10.0m$ and width $w^0_s = 10.0m$ is assumed to be dry at the top surface with similar boundary boundary conditions to the ones used in [8, 9].](image)

Plane stress quadrilateral elements with bilinear shape functions are used and two mesh sizes named, Coarse and Fine are generated to compare the results of the local and nonlocal models. The Coarse mesh has 3,968 elements and the Fine mesh has 17,851 elements. Results for the local and nonlocal damage models are presented in Fig. 3.13 and Fig. 3.14, respectively.
3.3. NUMERICAL EXAMPLES

The results of the local model highlight some of the non physical features of this class of models in that different damage path results may be obtained upon mesh resentment. Fig. 3.13a show damage concentration at the notch tips with its propagation being restricted to a single column of vertical elements. Thus, the damage zone is controlled by the element size below the initial notch in the slab. A similar effect is observed in Fig. 3.13b for the Fine mesh in which an even narrower damage zone is obtained given the smaller element size. As expected in both cases, the damage starts propagating at the stress concentration points below the two corners of the initial notch but is later restricted to single side of the notch. Moreover, damage zone and the rate of damage in both models are also different. In the local model, the Coarse mesh results show a wider damage zone that reaches a full damage across the slab at $t = 9390s$, while the Fine mesh slab reaches full damage at $t = 9350s$ generating a narrower zone of damage.

Under the same conditions, the proposed nonlocal model with a $l_c = 2.0m$ is used to obtain the results for the Coarse mesh in Fig. 3.14a as well as the Fine mesh in Fig. 3.14b. We observe the results for both meshes not only have similar damage zones but also closer breaking time in both meshes with $t = 9105s$ (Coarse mesh) and $t = 9075s$ (Fine mesh).

Fig. 3.15 show the average stress $\sigma_{xx}$ and strain $\varepsilon_{xx}$ values across the center of the plate during the crevasse propagation simulation. One can clearly observe the spurious mesh sensitive results of the local model after the peak stress while the nonlocal model show consistent regularized results during strain and damage increments.

The comparison of the local and nonlocal damage models in this section confirms both the limitation of the local damage model and the physically sound results provided by the nonlocal model proposed.
3.3.3 Impact into a notched steel plate

Finally, a two dimensional problem based on the experimental work of Kalthoff and Winkler [10, 11] named here the KW problem, is investigated in order to validate and verify the proposed damage regularization approach.
3.3. NUMERICAL EXAMPLES

Figure 3.14: Nonlocal damage results of polar ice slab subjected to a constant strain rate of \( \dot{\varepsilon} = [1/s] \) at \( t = 9045s \). Note that mesh independent results are obtained.

The KW problem is an impact problem into a pre-notched steel plate which under certain impact velocity regimes may lead to either a brittle fracture with a crack propagating upwards from the notch tip, a ductile failure known as a shear band that is curving downward below the notch tip or both at the same time [176, 177]. While in this study we mainly demonstrate the reliability of the model in accurately predicting the propagation of the crack upwards, we also
Figure 3.15: Average stress-strain curves of local and nonlocal models, for Coarse and Fine meshes, recorded across the center of the ice slab subjected to constant strain-rate. Mesh sensitivity can clearly be observed in the local model.

show that a different calibration of the damage model can be used to predict initiation of shear bands.

As shown in Fig. 3.16a, the plate is fixed at the top and bottom edges. Due to the symmetry of the problem, only half of the plate is modeled with the main parameters shown in Fig. 3.16b. The red line emanating from the notch tip illustrates the propagation of the crack at an angle $\theta$ of approximately $70^0$.

For the numerical tests described in this section, Fig. 3.16b shows the boundary conditions of the symmetric plate with the geometry and applied velocity given in Table 3.3 during a total time of 5.0s. We validate and verify the applicability of the model by studying crack propagation upward radiating out of the notch tip.
3.3. NUMERICAL EXAMPLES

Figure 3.16: Impact onto a pre-notched steel plate, known as the Kalthoff-Winkler problem, [10, 11] with the experimental setup shown in a and numerical model assuming symmetric boundary conditions shown in b. Red hatched area at the notch tip in a corresponds to the area at which the damage initiation is studied in Section 3.3.3.

Table 3.3: Geometry and boundary parameters in KW problem in Fig. 3.16b.

<table>
<thead>
<tr>
<th>$h_p$</th>
<th>$n_l$</th>
<th>$n_w$</th>
<th>$v_0$</th>
<th>$l_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[m]</td>
<td>[m]</td>
<td>[m]</td>
<td>[m/s]</td>
<td>[m]</td>
</tr>
<tr>
<td>$5.0 \times 10^{-3}$</td>
<td>$2.5 \times 10^{-3}$</td>
<td>$2.5 \times 10^{-4}$</td>
<td>$1.0 \times 10^{-4}$</td>
<td>$1.0 \times 10^{-3}$</td>
</tr>
</tbody>
</table>

Model validation

The material parameters selected for the KW problem considering a steel plate under plane strain conditions are presented in Table 3.4 with the values of Young’s modulus $E = 200 \text{GPa}$ and Poisson ratio $\nu = 0.33$. Damage parameters are also presented in Table 3.5.

Note that, while steel is normally modeled as a linear elastic material at small strains and low temperatures, small viscoelastic effects have been reported even at room temperature [150]. The viscoelastic effects of metals including steel have been experimentally reported by Zener [178, 179] and may play significant roles in detailed analysis, for strains values greater than 2% or at high tem-
temperatures. The degree of energy dissipation is normally observed in the material hysteresis curves, which also represent the lag between the applied force and the observed deformation. The reader is directed to [150] for a comparison of the viscoelastic properties of different types of materials including metals and to [178, 179] for experimental results of various metal alloys at different temperatures and load frequencies.

Moreover, experimental results show that the viscoelastic effects become predominant as the internal temperature in metals increase, also observed in shearband formation mechanisms. Thus, in order to include some viscoelastic effect, small Prony series parameters are selected for the constitutive law of the material, as listed in Table 3.4.

<table>
<thead>
<tr>
<th>(G_0) [MPa]</th>
<th>(K) [MPa]</th>
<th>(n)</th>
<th>(\mu_1)</th>
<th>(\lambda_1) [s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>(7.519 \times 10^4)</td>
<td>(1.961 \times 10^5)</td>
<td>1</td>
<td>(1.0 \times 10^{-4})</td>
<td>(1.0 \times 10^{-10})</td>
</tr>
</tbody>
</table>

Table 3.4: Prony-series viscoelastic parameters used in the KW problem.

Table 3.5: Damage parameters used in the KW problem simulation.

<table>
<thead>
<tr>
<th>(\alpha)</th>
<th>(\beta)</th>
<th>(B_1)</th>
<th>(B_2)</th>
<th>(r)</th>
<th>(k)</th>
<th>(\chi_{\text{threshold}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.65</td>
<td>0.00</td>
<td>(0.5 \times 10^{-2})</td>
<td>0.0</td>
<td>0.9</td>
<td>1.847</td>
<td>650.0</td>
</tr>
</tbody>
</table>

As discussed in Section 3.1, the selection of the parameters \(\alpha\) and \(\beta\) in Eq. 3.8 depend on the failure mechanisms that dominate the material response. Selection of high values of \(\alpha\) represent materials whose damage envelope follows a principal stress, tension dominated failure criterion while high values of \(\beta\) correspond to materials that display \(J_2\)-type or shear dominated failure envelope.
Similarly, selection of small values of $\alpha$ and $\beta$ describe materials whose failure is governed by the maximum hydrostatic stress criterion. In the present paper, the steel plate is observed to have a brittle failure governed by tensile stresses forming an upwards crack from the initial notch. This tensile influence in the damage is captured by the parameters of $\alpha$ and $\beta$ listed in Table 3.5.

Under the given boundary conditions and materials selected, Von Mises stress $\sigma_{VM}$ and crack propagation throughout the simulation is shown in Fig. 3.17. A series of snapshots depicting crack propagation at an angle $\theta$ of about 70° are recorded at the time steps 100, 130, 160 and 190 in Fig. 3.17a, Fig. 3.17b, Fig. 3.17c and Fig. 3.17d, respectively.

The results in Fig. 3.17 display the deformed shape and stress of the KW problem in which the elements that reached a damage of 0.96 were removed from the plots. The deformed shape is scaled by a factor of 10.0 in these plots.

In addition to the direction of crack propagation, the length of the crack is also recorded as it grows during the simulation. Starting from the crack initiation point in the pre-existent notch, crack length is measured along the dominant direction of propagation in the damage zone accounting for elements with damage of at least 0.96. Fig. 3.18 shows the crack length with respect to time for the simulation and mesh shown in Fig. 3.17. Moreover, the length of the crack shown in the snapshots Fig. 3.17a-Fig. 3.17c are highlighted by orange vertical lines from left to right in Fig. 3.17 at their respective time steps.
CHAPTER 3. AN EQUIVALENT STRESS-GRADIENT REGULARIZATION MODEL FOR COUPLED DAMAGE-VISCOELASTICITY

Figure 3.17: Von mises stress and crack propagation at $\theta \sim 70^\circ$. Deformation shape with a scale factor of 10 and elements with damage greater than 0.96 were removed from the mesh. Mesh with 7269 elements and time step size, $\Delta t = 25.0\, ms$. 

(a) After 100 time steps
(b) After 130 time steps
(c) After 160 time steps
(d) After 190 time steps
Figure 3.18: Crack length as a function of time for the KW problem with damage initiating at around 2.38s and time step size of $\Delta t = 0.025s$. The four orange vertical lines correspond to the time steps 100, 130, 160 and 190 at which the snapshots in Fig. 3.17 were taken.

**Model verification**

We verify the results under the boundary conditions in Fig. 3.16b and the parameters in Table 3.3, with a Finite Elements Analysis done using Coarse and Fine meshes. It is shown that the regularized damage approach yields mesh insensitive results under the same material parameters and test configuration.

The same time in the simulation corresponding to fine and coarse results of damage distribution are shown in Fig. 3.19. Damage results of the fine mesh with a total number of 7269 elements are shown in Fig. 3.19a and for the coarse mesh with a total number of elements of 3492 is in Fig. 3.19b.

Mesh insensitive results are obtained when comparing crack width, length and direction in Fig. 3.19a and Fig. 3.19b. Moreover, the results are not only displaying mesh insensitive results but also provide similar crack propagation path when compared to the experimental results in [10, 11].
Figure 3.19: Results of the KW problem with $\alpha = 0.0$ and $\beta = 0.0$. Elements with damage greater than 96% were removed from the mesh. a Damage results of the Fine mesh (7269 elements). b Damage results of the Coarse mesh (3492 elements).

A crack direction of approximately $\theta = +68^\circ$ measured from the horizontal axis in Fig. 3.16b is obtained, which is consistent with the experiment observation of $70^\circ$.

Additionally, crack growth with respect to time for both meshes show nearly identical results, as observed in Fig. 3.20 for the Coarse and Fine meshes.

**Initiation of shear bands and cracks**

Finally, we study the effect of parameters $\alpha$ and $\beta$ in the expression of $\chi$ in Eq. 3.8. While damage propagation is governed by material dependent parameters, the proposed model provides the computational framework to describe materials with a range of failure mechanisms. Namely, materials that display brittle damage can be modeled differently than those governed by their ductile strength. In addition to the standard damage approach for compression or tension, the aforementioned model allows to drive the damage propagation in terms of physically sound measures as the
maximum principal stress $\sigma^{(1)}$, second invariant of the deviatoric stress $J_2$ or the volumetric stress $\text{tr}(\sigma)$ weighted by the parameters $\alpha$ and $\beta$. Therefore, these weights can control the location and rates of damage initiation and predict different failure modes or crack directions.

The effect of parameters $\alpha$ and $\beta$ on the damage initiation and direction of damage propagation is reported in Fig. 3.21 showing only the initial damage stage at a mesh section equivalent to the red hatched area at notch tip depicted in Fig. 3.16a. Maintaining the value of $\alpha = 0$ and identical material properties such as boundary conditions in Table 3.3, the KW problem is solved for the values of $\beta$ ranging from 0.0 to 1.0. Fig. 3.21a, Fig. 3.21b, Fig. 3.21c, Fig. 3.21d show the results of the material properties for $\beta$ values of 0.0, 0.80, 0.95 and 1.0 respectively.

Note that Fig. 3.21a corresponds to the standard tensile dominated crack with an angle of $\theta \approx 68^\circ$. As we increase the value of $\beta$, the angle of the damage propagation $\theta$ slowly decreases until a sudden instability in the crack direction around $\beta = 0.94$ is encountered. At this point a distinct branch in the damage propagation quickly moves downwards at a crack angle of $\theta \approx -64^\circ$ as the value of $\beta$ reaches its maximum value of 1.0. Thus, we conclude that the parameters $\alpha$ and
Figure 3.21: Effect of the material parameters on damage initiation and propagation studied on the KW problem with $\alpha = 0.0$ and identical mesh with 5700 elements. A zoom to the notched tip highlighted by the hatched red area in Fig. 3.16a is shown. 3.21a corresponds to a $\beta = 0.0$. 3.21b to a $\beta = 0.8$, 3.21c to a $\beta = 0.95$ and 3.21d to a $\beta = 1.0$.

$\beta$ can be used to control damage propagation with an isotropic damage model and one does not need to employ more expensive anisotropic models in this case.

### 3.4 Conclusions

A nonlocal gradient-enhanced damage model is proposed for viscoelastic solids by introducing a coupled PDE system for displacement and an equivalent stress measure. Thermodynamic consistency is derived analytically and numerically verified on an illustrative benchmark example under low- and high-frequency cyclic loading, in which the non-negative values are obtained for thermal and damage dissipation energies. The effect of the characteristic length on the underlying physics and energy parameters is demonstrated under both loading conditions. The proposed model not only leads to mesh-independent results but also show that, while the proportion of viscous and damage dissipation may vary with the different loading conditions, the total dissipated energy in
the system increases with an increment in the characteristic length of the material.

Several numerical examples are presented. First, a symmetric one dimensional bar is subjected to a series of creep, relaxation and strain rate loadings and the results for local and nonlocal damage are compared. Under these loading conditions, superior mesh independent results are obtained by the proposed nonlocal damage model when the time to fracture and the internal damage along the bar are compared. In two dimensions, we first consider an illustrative example from glaciology in which a crevasse is propagating in an ice slab. Similarly, while the local model is mesh sensitive the proposed nonlocal model display mesh insensitive results.

Model validation and verification is concluded by simulating the Kalthoff-Winkler experiment of a projectile impacting a notched steel plate. A crack direction of approximately $70^\circ$ is observed, which is validated by the experiment, on two different mesh sizes. Results are verified by obtaining similar damage zone and rate of propagation for two different meshes under the same boundary conditions and material parameters. Moreover, the model demonstrates great flexibility to account for different damage mechanisms. Selection of material parameters in the source term of the gradient equation (equivalent stress parameters $\alpha$ and $\beta$) allows different crack paths that can be calibrated to represent a broad range of damage drivers and initiation mechanisms.

In summary, the example problems throughout this paper, demonstrate that the proposed nonlocal damage model is mesh insensitive, can be controlled by the different parameters of the damage law and hence is appropriate for real complex applications in mechanics of materials.
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Chapter 4

A Fully Coupled Nonlocal Model for Viscoelastic Damage in Isothermal Asphalt Concrete

4.1 Problem statement

Anisotropic behavior and damage of Asphalt concrete is studied by several authors [180–183] which analyzed the effects of crack direction and fracture energy under different loading conditions.

This sections starts by introducing the materials laws that describe the viscoelastic and damage growth in asphalt concrete (AC). While most of the constitutive equations have already been mentioned, final version of those laws are written here for completeness including special considerations to AC when needed.

*Viscoelastic constitutive law*  A similar viscoelastic model as described in Londono et al. [3] is used also for AC analysis in which only the deviatoric components of stress are time dependent, as
described in Eq. 4.1.

\[ \sigma(t) = \sigma^{vol} + \sigma^{dev}(t) = K \text{tr}(\mathbf{e})I + 2 \int_0^t G(t - \tau) \dot{e}(\tau) d\tau, \]  

(4.1)

where \( K \) is the Bulk modulus, \( I \) is the identity tensor, \( G(t) \) is the time dependent shear modulus, the deviatoric \( \sigma^{dev}(t) \) and volumetric \( \sigma^{vol} \) components of the stress tensor are defined in terms of the trace operator \( \text{tr}(\cdot) \), the strain \( \mathbf{e} \) is defined in terms of the symmetric gradient operator and the displacement \( u \) by \( \mathbf{e} = \nabla^s u = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \), and the deviatoric strain \( \mathbf{e} = \mathbf{e} - \frac{1}{3} \text{tr}(\mathbf{e})I \). Note that each time derivative is superimposed with an upper dot (e.g. \( \dot{\mathbf{e}} \)) and the time independent bulk modulus \( K \) and instantaneous shear modulus \( G_0 \) are described in terms of the Young’s modulus \( E \) and Poisson ratio \( \nu \) by \( K = \frac{E}{3(1-2\nu)} \) and \( G_0 = \frac{E}{2(1+\nu)} \), respectively. The time dependent shear modulus, \( G(t) \) is expanded in a Prony series in which,

\[ G(t) = G_\infty + \sum_{i=1}^{n} G_i \exp(-t/\lambda_i) = G_0 \left( \mu_\infty + \sum_{i=1}^{n} \mu_i \exp(-t/\lambda_i) \right). \]  

(4.2)

Again, the non-negative parameters \( \mu_i \) describe the normalized relaxation modulus as, \( \mu_\infty = \frac{G_\infty}{G_0} \) and \( \mu_i = \frac{G_i}{G_0} \) which satisfy \( \mu_\infty + \sum_{i=1}^{n} \mu_i = 1 \) for the \( n \) terms in the Prony series.

**Continuum Damage Mechanics**  A nonlocal damage approach based on the equivalent stress measure, \( \bar{\chi} \) is used as introduced previously. The continuum damage mechanics approach and the strain equivalence, the total stress \( \sigma \) in the material is computed from the effective stress measure \( \bar{\sigma} \) as defined by Simo & Ju [169] and related in terms of the transformation \( \mathbf{M} \) shown in Eq. 4.3.

\[ \sigma = \mathbf{M}^{-1} : \bar{\sigma} \]  

(4.3)

where the colon operator “:\;” corresponds to the contraction operation defined in indicial notation as \( a : b = a_{ij}b_{ij} \). For isotropic damage, \( \mathbf{M} \) is written in terms of the internal damage field \( D \) as
\[ M = (1 - D)^{-1} I, \] where \( I \) is the identity tensor. As expected the value of \( D \) is computed at each material point and ranges from 0.0 at the undamaged state to 1.0 at the fully damaged condition. As introduced in the previous section, a modified damage-rate model proposed by Murakami [42, 148] as shown used and shown in Eq. 4.4 in which the damage rate is now a function of the nonlocal equivalent stress \( \bar{\chi} \).

\[
\dot{D} = \begin{cases} 
0, & \text{if } \kappa < \bar{\chi} \\
B \frac{\langle \chi(\bar{\sigma}) \rangle^r}{(1 - D)^{k_{\sigma}}}, & \text{if } \kappa \geq \bar{\chi}
\end{cases}
\]  

(4.4)

where \( \kappa \) is a history variable governing the damage growth defined in Eq. 4.6 and subjected to the Kuhn-Tucker conditions in Eq. 4.5 and \( B, r \) and \( k_{\sigma} \) are material dependent damage evolution parameters obtained from experimental data as described in Londono et al. [3].

\[
\kappa \geq 0; \quad f(\bar{\chi}, \kappa) \leq 0; \quad \dot{\kappa} f(\bar{\chi}, \kappa) = 0
\]  

(4.5)

\[
\kappa = \max(\bar{\chi}_{\text{max}}, \bar{\chi})
\]  

(4.6)

The symbols \( \langle \cdot \rangle \) are the Macaulay brackets [153] and \( \bar{\chi} \) is the nonlocal version of the Hayhurst’s multiaxial equivalent stress measure [154] obtained by solving the gradient-enhanced equation Eq. 4.7.

\[
\bar{\chi} - c \nabla^2 \bar{\chi} = \chi
\]  

(4.7)

with \( c \) related to the square of the average internal length in the model \( l_i^2 \) such that \( c = \frac{l_i^2}{2} \), corresponds to the characteristic length with the same order of magnitude of the material inhomogeneities. The right hand side of Eq. 4.7 corresponds to the source term in terms of the local
equivalent stress $\chi$ defined in Eq. 4.8.

$$\chi = \alpha \tilde{\sigma}^{(1)} + \beta \sqrt{3J_2} + (1 - \alpha - \beta)\text{tr}(\tilde{\sigma}),$$

(4.8)

where $\alpha$ and $\beta$ limited to $\alpha + \beta \leq 1.0$ and which accounts for complex loading conditions by including a linear combination of the largest principal effective stress $\tilde{\sigma}^{(1)}$, the first invariant of the effective stress $\text{tr}(\tilde{\sigma})$, and the second invariant of the deviatoric effective stress tensor $J_2 = \frac{1}{2} \tilde{\sigma}^{(dev)} : \tilde{\sigma}^{(dev)}$.

The time integration of the damage and viscoelastic parameters in Eq. 3.19 are obtained following the integration scheme proposed by Londono et al. [3]. In summary, the problem is described mathematically by the system of Eq. 3.16-3.22 and it is obtained by solving the coupled displacement $u$ and nonlocal equivalent stress $\chi$ written in residual form $R = [R_u, R_\chi]$ as shown in Eq. 4.9 and Eq. 4.10 respectively

$$R_u = \nabla \cdot \sigma + b - f^{ext} = 0$$

(4.9)

$$R_\chi = \chi - c \nabla^2 \chi - \chi(\tilde{\sigma}) = 0$$

(4.10)

where $R_u$ corresponds to the residual of the equilibrium equation and $R_\chi$ is the additional non local equivalent stress equation. Similarly, the system is discretized using the finite element method and the numerical solution is obtained iteratively by a Newton Method implementation as described in Section 3.1.1.
4.2 Material Calibration

Following a similar calibration strategy used in [3], we employ the Prony series model with Murakami type damage to study the behavior of asphalt concrete. Material parameters for both viscoelastic and damage models were calibrated using the experimental data from Katsuki & Gutierrez [135]. At a temperature of $22 \pm 1^\circ C$, the target of calibration are the experimental results obtained for a strain rate of $1.7 \times 10^{-4} [s^{-1}]$ of unconfined asphalt concrete cylinders. AC cylinders are obtained to have an average diameter of $50.9 [mm]$, length of $102 [mm]$ and a density of $2.35 [g/cm^3]$. Additionally, three different strain rates of $4.2 \times 10^{-6}$, $1.7 \times 10^{-5}$ and $1.7 \times 10^{-3} [s^{-1}]$ are used to predict the stress and damage response with the calibrated parameters. Applied strains at the boundary for all four cases (one calibration and three predictions) are shown in Fig. 4.1b up to a time of $1000 [s]$. A schematic loading conditions of this test is presented in Fig. 4.1a with the applied strains, $\varepsilon_{BC}$ being calculated from the initial length of the cylinder, $l_0$ and the total displacement in the axial direction only.

Material calibration involves the selection of the viscoelastic parameters ($\mu_i, \lambda_i$) as well as the damage parameters ($\alpha, \beta, B_1, B_2, r, k_1$ and $k_2$) that yield the closest material prediction to the experimental results. Experimental data and calibration details are presented below.

4.2.1 Objective Function:

Using the experimental results of the strain rate, $\dot{\varepsilon} = 1.7 \times 10^{-4} [s^{-1}]$, material parameters are calibrated following following objective function defined in the previous section.

$$\Pi(a) = \sqrt{\sum_{q=1}^{m} \left[ y_q^{(exp)} - y_q^{(sim)}(a) \right]^2}$$ (4.11)
where $m$ is the total number of data points. The simulated results $y_q^{(\text{sim})}$ are function of the material parameters $\mathbf{a}$ and are compared at same points of the experimental data ($x_q^{(\text{sim})} = x_q^{(\text{exp})}$) as shown in Eq. 4.12.

$$
y_q^{(\text{sim})}(\mathbf{a}) = f(x_q^{(\text{exp})}, \mathbf{a}) \quad (4.12)
$$

with the unknown material parameters $\mathbf{a}$ being split into the viscoelastic and damage parameter sets as proposed in [3] and defined by

$$
\mathbf{a} = a^{\text{vis}}(\mu_1, \lambda_1, \ldots, \mu_n, \lambda_n) \cup a^{\text{dmg}}(\alpha, \beta, B_1, B_2, r, k_\sigma) \quad (4.13)
$$

In the optimization of the viscoelastic parameters in $a^{\text{vis}}$, the objective function $\Pi$ in Eq. 4.11 is specialized for the asphalt test in which the dependent parameter $y = \sigma$ and independent parameter $x = \varepsilon$ are used for both experimental and simulated variables. The objective function $\Pi(a^{\text{vis}})$ is
therefore defined by,

\[
\Pi(a^{vis}) = \sqrt{\sum_{q=1}^{nt} \left[ \sigma_{q}^{(exp)} - \sigma_{q}^{(sim)}(a^{vis}) \right]^2}
\]  \hspace{1cm} (4.14)

Again, the problem is mathematically described by finding a set of viscoelastic constants \(a^{vis}\) which minimizes \(\Pi\), thus

\[
\hat{a}^{vis} = \arg \min_{a^{vis} \in [\mathcal{M} \cdot \mathcal{L}]} \{ \Pi(a^{vis}) \}
\]  \hspace{1cm} (4.15)

where \(\hat{a}\) are the optimal viscoelastic parameters defined in the feasible search spaces by \(\mathcal{M}\) and \(\mathcal{L}\) respectively are shown in Eq. 4.16.

\[
\mathcal{M} = \{ \mu \in \mathbb{R}^n | 0 \leq \mu_i \leq 1.0, \forall i = 1,2,...,n \} \\
\mathcal{L} = \{ \lambda \in \mathbb{R}^n | 0 \leq \lambda_i, \forall i = 1,2,...,n \}
\]  \hspace{1cm} (4.16)

The viscous parameters \(\mu_i\) must verify the inequality constraint of \(\mu_\infty + \sum_{i=1}^{n} \mu_i = 1\). As suggested in [151], the number of terms in the Prony Series, \(n\) is predefined and kept constant through the entire simulation with the relaxation time, \(\lambda_i\), initially selected evenly spaced on a log-time scale. Similarly, the damage optimization is done specializing Eq. 4.11 to \(y = D\) and \(x = \epsilon\) and defining the objective function \(\Pi(a^{dmg})\) as,

\[
\Pi(a^{dmg}) = \sqrt{\sum_{q=1}^{nt} \left[ (1-D)_{q}^{(exp)} - (1-D)_{q}^{(sim)}(a^{dmg}) \right]^2}
\]  \hspace{1cm} (4.17)

and the optimization of the damage parameters defined by Eq. 4.18.

\[
\hat{a}^{dmg} = \arg \min_{a^{dmg} \in [\mathcal{A} \cdot \mathcal{B} \cdot \mathcal{W} \cdot \mathcal{V} \cdot \mathcal{R} \cdot \mathcal{N}]} \{ \Pi(a^{dmg}) \}
\]  \hspace{1cm} (4.18)
with feasible search spaces,

\[ \alpha \in \mathcal{A}; \quad \mathcal{A} = \{ \alpha \in \mathbb{R} | 0 \leq \alpha \leq 1.0 \} \]

\[ \beta \in \mathcal{B}; \quad \mathcal{B} = \{ \beta \in \mathbb{R} | 0 \leq \beta \leq 1.0 \} \]

\[ B_1 \in \mathcal{W}; \quad \mathcal{W} = \{ B_1 \in \mathbb{R} \} \]

\[ B_2 \in \mathcal{V}; \quad \mathcal{V} = \{ B_2 \in \mathbb{R} \} \]

\[ r \in \mathcal{R}; \quad \mathcal{R} = \{ r \in \mathbb{R} | 0 \leq r \} \]

\[ k_\sigma \in \mathcal{N}; \quad \mathcal{N} = \{ k_\sigma \in \mathbb{R} \} \]

(4.19)

Note that with \( \alpha \) and \( \beta \) are subjected to the inequality constraint \( \alpha + \beta \leq 1 \). In this approach, damage initiation value, \( \chi_{\text{threshold}} \), damage level at sudden failure, \( D_c \), maximum damage rate, \( \dot{D}_c \), instantaneous elastic modulus, \( E \) and Poisson ratio, \( \nu \) are obtained from the information provided in [135] and from simple simulation tests. Given the complexity of the entire optimization routine, careful selection of the initial parameters demonstrated to be critical in the convergence of the optimization algorithm.

### 4.2.2 Parameters selected:

The calibration phase is done using the experimental results of \( \dot{\varepsilon} = 1.7 \times 10^{-4}[s^1] \). After the calibration, the material parameters of the viscoelastic and damage models were obtained and the final calibration curves are presented in Fig. 4.2.

A close match to the experimental results is obtained upon calibration for the axial stress and damage as shown in Fig. 4.2a and Fig. 4.2b respectively. Optimal damage parameters are reported in Table 4.1 and the viscoelastic parameter in Table 4.2.

From the experimental results, a linear dependency of the damage parameters \( k_\sigma \) an \( B \) is ob-
4.2. MATERIAL CALIBRATION

Figure 4.2: Results of calibration process for strain rate of $1.7 \times 10^{-4} [1/s]$ applied to asphalt concrete cylinders: a) Stress-Strain results and b) Damage growth with strain.

Table 4.1: Damage parameters of asphalt concrete obtained by calibration

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$B_1$</th>
<th>$B_2$</th>
<th>$r$</th>
<th>$k_1$</th>
<th>$k_2$</th>
<th>$\chi_{\text{threshold}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.0</td>
<td>$1.14 \times 10^{-4}$</td>
<td>$-8.655 \times 10^{-3}$</td>
<td>3.239</td>
<td>-10.986</td>
<td>3.309</td>
<td>0.0</td>
</tr>
</tbody>
</table>

 Obtained from the levels of strain and strain-rates applied to the specimen respectively. This linear behavior is captured by the parameters $k_1$ and $k_2$ in $k_\sigma = k_1 + k_2|\varepsilon|$ and by the parameters $B_1$ and $B_2$ in $B = B_1 + B_2|\dot{\varepsilon}|$.

Table 4.2: Prony-series viscoelastic parameters for asphalt concrete obtained by calibration.

<table>
<thead>
<tr>
<th>Prony Series term</th>
<th>$\mu_i$</th>
<th>$\lambda_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1.965 \times 10^{-2}$</td>
<td>4050</td>
</tr>
<tr>
<td>2</td>
<td>$1.236 \times 10^{-1}$</td>
<td>4200</td>
</tr>
<tr>
<td>3</td>
<td>$4.0 \times 10^{-1}$</td>
<td>7000</td>
</tr>
</tbody>
</table>

The values of Young’s modulus were taken directly from the published results in Katsuki & Gutierrez [135], which were obtained experimentally from the linear part ($\varepsilon \leq 1.0 \times 10^{-2}$) of the stress-strain curves. The values of the Young’s Modulus are 55, 80, 140 and 340 MPa for the strain
rates of $4.2 \times 10^{-6}$, $1.7 \times 10^{-5}$, $1.7 \times 10^{-4}$ and $1.7 \times 10^{-3}$ [s$^{-1}$] respectively and the Poisson ratio $\nu = 0.35$.

4.3 Numerical results

Upon the definition of the material model and the parameter selection, a series of Finite Element simulation tests are used to demonstrate the applicability of the model to simulate asphalt concrete. This section starts with the simulation of the unconfined asphalt cylinders in compression in a similar experimental setting as used in the calibration section but subjected to different applied strain rates. Mode I fracture of an AC thin layer commonly used for pavement repair is studied and the results are compared to laboratory experimental data. Additionally, a mixed-mode fracture propagation is studied in the case of a three-point bending AC beam. In all cases, finite element results yield close match to the experimental data found in the literature.

In all cases, a FEM User Element is for coupled viscoelasticity and damage is implemented as described in [3] in the commercial Finite Element Analysis Program - FEAP [146]. Quadrilateral elements with linear shape functions are used in the 2D simulations with all mesh files generated in the mesh generator software GMSH [175] and postprocessing visualization in Paraview [174].

4.3.1 AC cylinder at various strain-rates

Material parameters obtained in the previous section are used in the Prone Series model to predict the response of unconfined asphalt cylinders at multiple compressive strain-rates. Experimental data of stress and damage for the strain rates of $4.2 \times 10^{-6}$, $1.7 \times 10^{-5}$ and $1.7 \times 10^{-3}$ [s$^{-1}$] are used and compared to the FEM implementation under similar boundary conditions.
4.3. NUMERICAL RESULTS

Change of stress with respect to the applied strain is presented in Fig. 4.3a using the optimized material parameters. Damage growth for the same strain rates is also calculated and compared to the experimental results as shown in Fig. 4.3. In both cases, the calibrated curves for $1.7 \times 10^{-4}$ are also included for completeness.

![Graph](image)

Figure 4.3: Results of numerical implementation of compressive uniaxial strain-rates applied to asphalt concrete cylinders. a Stress-Strain results and b Damage growth with strain.

The PS-damage model predicted a similar damage growth and strain-stress response as the experimental data published in [135], and therefore demonstrates that can be also used for predicting the behavior of asphalt concrete.

4.3.2 Thin Asphalt Overlay

Thin asphalt overlays have been successfully used on the maintenance of PCC pavements increasing the life cycle of PCC with cracks or defective expansion joints. Fig. 4.4 show schematically different pavement structures commonly used in highways or airports with AC overlays. Fig. 4.4a
show the rehabilitation treatment on cracked PCC. Fig. 4.4b is presented as a treatment to reflective cracking from PCC joints with or without previous AC overlays and Fig. 4.4c shows the treatment over an \textit{old} AC pavement. The word \textit{old} is mentioned in this context to described AC pavements that have reached their serviceability limits and not necessarily related to the age of the AC pavement.

Figure 4.4: Schematic of typical pavement structures with thin AC overlays. AC overlays are placed here over (a) Cracked Portland Cement Concrete (PCC) pavement, (b) PCC pavement joint with/without \textit{old} asphalt concrete (AC) layer, (c) an \textit{Old} AC pavement structure. (d) Shows an schematic of the Compact tension C(T) section used for collecting experimental data and FEM simulation.

In an effort to characterize the crack growth through the thickness of AC pavements, a tension test for mode I crack growth is studied following the proposed Compact Tension, C(T) in [2] at a controlled temperature of 12°C. Fig. 4.4d show the schematic of the asphalt C(T) model proposed by Ahmed et.al [2].

Experimental results presented in [2] using the C(T) section of Fig. 4.4d is used to validate the suitability of the model. A Finite Element mesh shown in Fig. 4.5 with 15,796 quadrilateral elements is in assumed to be under plane strain condition. The mesh has a preexistent notch at the center with a constant outward velocity of $v_0 = 0.017 \text{[mm/s]}$ being applied on each side of the circular wholes.
Figure 4.5: Compact tension C(T) test used in numerical simulation with applied displacement of $v_0 = 0.017\,[mm/s]$ and a pre-existent notch of length $L = 50\,[mm]$ measured from the bottom edge. Mesh has 15,796 quadrilateral elements. Diameter of the wholes of applied displacement, $\phi = 25\,[mm]$ as originally proposed in [2].

Given the viscoelastic parameters reported in [2] using a Young’s modulus in the form of a generalized maxwell model, viscoelastic parameters for the Prony Series model are easily transferred and normalized to satisfy the proposed model. The viscoelastic material parameters used to compute the time dependent shear modulus are presented in Table 4.3, with a Young’s modulus, $E = 70,263\,[MPa]$ and Poison ratio, $\nu = 0.3$.

Table 4.3: Normalized Prony-series viscoelastic parameters for AC calculated from [2].

<table>
<thead>
<tr>
<th>Prony Series term</th>
<th>$\mu_i$</th>
<th>$\lambda_i$ [s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.283</td>
<td>8.9</td>
</tr>
<tr>
<td>2</td>
<td>0.147</td>
<td>121.4</td>
</tr>
<tr>
<td>3</td>
<td>0.238</td>
<td>1117.0</td>
</tr>
<tr>
<td>4</td>
<td>0.173</td>
<td>10915.7</td>
</tr>
<tr>
<td>5</td>
<td>0.160</td>
<td>374487.4</td>
</tr>
</tbody>
</table>

Note that the damage parameters from the calibration section are also used here with the pa-
Parameters $B$ and $k$ being updated to represent the properties of the new mix of the experimental data. In this case, parameters are taken as $B = 2.6 \times 10^{-5}$ and $k = -3.7$, damage initiation threshold is governed by $\chi_{\text{threshold}} = 12.0$ and the characteristic length, $l_c = 0.7 \text{[mm]}$.

Three different experimental results are reported for this boundary conditions with similar peak force and softening behavior. A FEM simulation is performed under the given geometry and material parameters and the results are compared to the experimental data in Fig. 4.6 where the applied force is compared with respect to the Crack Mouth Opening Displacement (CMOD) of the specimen. The FEM simulation demonstrates a close match to the experimental data.

![Figure 4.6: Applied force and Crack Mouth Opening Displacement (CMOD) of experimental results from [2] for C(T) data and the predicted results of the FEM simulation](image)

As the simulation progresses and the displacement increases in the thin overlay, a damage zone at the zone of mode I crack propagation with the different damage increment being highlighted in the snapshots in Fig. 4.7.
4.3. NUMERICAL RESULTS

Figure 4.7: Snapshots of the Compact Tension C(T) test proposed in [2] for the damage zone developing above the preexistent notch. Deformed shape configuration is shown with elements that reached a damage of 0.99989 being removed from for plotting only.

Note that the mode I damage propagation is properly captured as the damage zone grows above the preexistent notch tip as shown in the sequence of Fig. 4.7. Moreover, the experimental results of three different test specimens reported by Sarfraz et al [2] are also acceptably matched for the model proposed yielding a similar peak force and crack mouth opening displacement.

4.3.3 Mixed-Mode Crack Propagation

An additional cracking problem is presented in this section to validate the model for mixed-mode fracture propagation. A three-point bending problem of an asphalt concrete beam is represented following the work of Song et al [12]. A simply supported asphalt concrete beam of length 376mm and height of 100mm is subjected to midspan load. An initial crack of 19mm long from the bottom
surface is assumed at 123mm from the left edge, as shown in Fig. 4.8. With the applied load at the center of the beam driving the problem and with plane-strain conditions, a mixed-mode crack propagation trajectory is expected.

![Figure 4.8: Idealized simply supported asphalt concrete beam subjected to an applied load as presented in [12]. Total height of the beam is H=100mm and total length of L=376mm. At the bottom of the beam a thin notch of 19mm at 123mm from the left edge is assumed inducing mixed-mode crack propagation.](image)

The same asphalt properties used in the previous section are assumed in this test, with the Prony Series parameters described in Table 4.3. For this test, a total of 7,894 quadrilateral-elements are used. The mesh above the prescribed crack is refined to guarantee consistent damage regularization with the selected characteristic length of 0.7mm. Fig. 4.9 shows the quadrilateral-element mesh use for this problem. Note that the proposed model does not require to pre-define an enriched crack path or element orientation, as done in [12], with the mesh refinement above the notch introduced for better resolution and and similar order of magnitude to the characteristic length.

Given the asymmetry of the initial notch, a mixed-mode cracking develops upwards from the top of the initial notch towards the location of the applied load. Different snapshots of the crack propagating at across the beam section are shown in Fig. 4.10, where the damage zone ahead the crack top is highlighted in the damage contours. Elements that reached a damage value of 0.99986
4.3. NUMERICAL RESULTS

Figure 4.9: Selected mesh for the three-point bending beam problem with 7,894 quadrilateral elements.  

have been removed after the completion of the simulation. Crack length are shown for 1,000, 2,000, 3,000 and 4,000 time steps at Fig. 4.10a - Fig. 4.10d respectively.

Figure 4.10: Snapshots of the mixed-mode three-point bending test as proposed in [12] for the damage zone developing above the eccentric preexistent crack at time steps 1,000 to 4,000 from a to d respectively. Elements that reached a damage of 0.99986 were removed for illustration purposes only.

Final crack propagation path is also compared to the the experimental results in [12]. Fig. 4.11 show the final crack trajectory of the experimental results and the finite element simulation at
time step 6,000. Both, crack initiation and direction were very well predicted with the nonlocal damage model which confirms the applicability of the model for modeling asphalt under mixed-mode cracking conditions.

![Figure 4.11: Final comparison of mixed-mode crack propagation through a simply supported beam of asphalt concrete. a Experimental results presented in [12]. b Predicted deformed shape and crack propagation with damaged elements removed for illustration purposes.](image)

4.4 Conclusions

The nonlocal gradient-enhanced damage model proposed in [147] for viscoelastic solids is used to verify and validate the behavior of Asphalt concrete under various loading conditions. The system of equations that describe asphalt viscoelasticity and damage is solved and consistent mesh insensitive results are obtained for asphalt samples subjected to tensile and compressive loadings. The
model is first calibrated using a staggered approach in which the viscoelastic and damage parameters are calibrated in different stages to reduced the size of the feasible domain. Upon calibration, different tests are carried out to validate the results to experimental data in the literature. Finite Elements results for different compressive strain rates returned a close match to the experimental data when comparing both internal stress and damage.

Moreover, the model is also used to replicate the experimental results of a thin asphalt overlay commonly used in pavement repair and maintenance. A close match of the overall applied force and crack opening is obtained for the FEM simulation as well as the peak applied force and final crack displacement. The proposed model and calibrated material parameters are also compared to experimental results of mixed-mode crack propagation. The predicted crack initiation and trajectory of an asphalt concrete beam under three-point bending display a close match to the reported experimental results.

In summary, the example problems throughout this paper, demonstrate that suitability of the nonlocal model to reproduce the viscoelastic and damage behavior of Asphalt Concrete under compressive and tensile loading while yielding mesh insensitive results.
CHAPTER 4. A FULLY COUPLED NONLOCAL MODEL FOR VISCOELASTIC DAMAGE IN ISOTHERMAL ASPHALT CONCRETE

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Chapter 5

Conclusion

5.1 Scope and Contribution of the Thesis

This work is aimed at understanding the damage growth and propagation in viscoelastic solids with applications to polar ice and asphalt materials. To this end, a computational method that is efficient and physically sound is proposed. Given the goals outlined in Section 1.4, the contributions of this dissertation can be listed as follow.

- **Constitutive model:** A constitutive model for polar ice (Chapter 2) and asphalt (Chapter 4) is proposed that captures the viscoelastic nature of these materials including their rate dependence damage behavior. In Chapter 2, the model is introduced as a generalized Maxwell viscoelastic solid with a rate-form damage law. The model definition allows for the viscoelastic model to semi-analytically integrated in time while the damage is integrated in an explicit scheme. A rigorous thermodynamical analysis, presented in the form of Clasius-Duhem inequality confirms consistency of the model with positive energy dissipation. The model is implemented within a finite element framework and its computational efficiency is confirms in terms of maximum stable time step size, $\Delta t_{cr}$ and total CPU when compared to
a similar model in the literature.

- **Gradient-enhanced damage for viscoelasticity:** For viscoelastic solids, a novel nonlocal damage regularization is proposed in Chapter 3 by introducing an additional second-order gradient equation to the system. By solving the equilibrium equation for displacement coupled with the gradient-enhanced equation, a nonlocal equivalent stress measure is obtained and used to compute the damage at a material point, a nonlocal damage.

  Again, thermodynamic consistency is verified under high and low frequency loadings. Effect of characteristic length $l_c$ in the underlying physics and energy dissipation is highlighted. Under several numerical examples in one and two dimensions, the model demonstrated results independent on the mesh selection. Benchmark problems of crack propagation in impact loading validated the results under real complex applications.

- **Validation for applications in ice and asphalt concrete:** The proposed model is applied and compared to experimental and numerical results in the literature, which confirms its applicability to model these materials under for different loading conditions. For polycrystalline ice in Chapter 2, the model is compared to a series of experimental tests that included tension, compression and biaxial loading. An investigation of surface crevasse propagation in grounded marine-terminating glaciers is used to demonstrate the effect of seawater level in the damage mechanisms of the ice shelf. Chapter 4 is used to demonstrate the suitability of the model for asphalt concrete. Upon calibration, published experimental results of asphalt concrete samples under compression as well as crack propagation across a thin AC overlay are predicted yielding consistent damage and stress-strain curves that match the experiments.
5.2 Future Work

In this section, possible future work aligned with the presented framework of viscoelastic damage model solids are discussed.

- **Variable characteristic length:** When introducing the additional gradient-enhanced equation of damage, the physical quantity accounting for the length scale of the material, $l_c$, is motivated from the size of an statistically representative volume element and therefore, of fixed length in a randomly homogeneous material. Keeping this constant length has however demonstrated adverse effect in the numerical simulations as some damage propagation may be observed perpendicular to the crack path. While some authors have introduced a varying characteristic lengths, the implications of this change to the model proposed would be a next step to consider. Moreover, with authors proposing a decrease in $l_c$ with damage and others proposing an increase as the damage propagates, the physical meaning of $l_c$ to the micromechanical structure of the material is no longer obvious nor the implications to the physics represented.

- **Material healing:** Some of the materials presented here may display some degree of healing or damage recovery under certain loading conditions. Ice re-crystallization and crack closure, can be significant during the cycles of low temperatures and under compressive loads. Similarly, asphalt binder migration have been shown to play some role in the process of damage recovery in the cases of cracked asphalt concretes. While many of theses effects may be accelerated by the effects of low or high temperatures, to the knowledge of the author, the effects of material healing included in the context of gradient-enhanced damage have not been studied. Introducing these additional layer of complexity may allow for material prediction to include long and varying loading cycles. Moreover, in order to include the effects of healing by refreezing a previously melted ice section, an additional equation that accounts
CHAPTER 5. CONCLUSION

for temperature effects would also need to be added. Effects of including an additional temperature formulation is expected to influence not only the healing process (refreezing) but also the viscoelastic parameters in both, asphalt and ice.

- **Fluid driven fracture:** A common situation for both ice and asphalt concrete is the presence of water that fall through the material cracks. In the case of ice, local or transported water produced due to melting may fall into existing crevasses creating additional pressure at the crack tip. This effect is expected to play a role in the rate of damage propagation with the additional water column contributing to the crack opening. A similar effect occurs in cracked asphalt concrete sections. In this case, water affects not only crack propagation through the pavement thickness but also fills concrete voids favoring layer debonding and localized high pressure exacerbated by the dynamic loads of the vehicles above. Fluid driven fracture can therefore be significant and should be included when predicting long term damage analysis in both ice and asphalt concrete. A proposed future step could be then to include fluid driven fracture into the system of equations allowing to analyze and quantify the additional pressure and its effect in the rate of crack propagation.


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BIBLIOGRAPHY


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Appendix A:

A.1 Outer Iteration: Equilibrium Equation solution

Given our first assumption in Section 2.2, the residual equation without inertial effects is solved at each time increment. Due to the nonlinear nature of the material law used, each quasi-static step is a nonlinear problem solved using a Newton method. To do so a linearization of Eq. 2.15 is performed as follows,

\[ R^k_m(u^k_m) \approx R^{k-1}_m(u^{k-1}_m) + \frac{dR^{k-1}_m(u^{k-1}_m)}{du} \delta u^k_m = 0. \]  

(A.1)

and using a Gâteaux directional derivative to evaluate \( \frac{dR^{k-1}_m(u^{k-1}_m)}{du} \) leads to

\[
\begin{align*}
\frac{dR^{k-1}_m(u^{k-1}_m)}{du} &= \left. \frac{dR^{k-1}_m(u^{k-1}_m+\epsilon \delta u^k_m)}{d\epsilon} \right|_{\epsilon=0} = \nabla \cdot \left( \left[ 1 - D_m(u^{k-1}_m) \right] \frac{d\sigma^{k-1}_m(u^{k-1}_m+\epsilon \delta u^k_m)}{d\epsilon} \right) \bigg|_{\epsilon=0} \\
&= \nabla \cdot \left( \left[ 1 - D_m(u^{k-1}_m) \right] \frac{d\sigma^{k-1}_m(u^{k-1}_m+\epsilon \delta u^k_m)}{d\epsilon} \right) \bigg|_{\epsilon=0} \\
&= J^{k-1}_m(u^{k-1}_m)
\end{align*}
\]  

(A.2)
APPENDIX A:

Note the assumption in Eq. 2.19 has been applied in Eq. A.2, hence the Jacobian $J_m^{-1}(u_m^{k-1})$ reduces to the secant stiffness, denoted $K_{sec,m}^k$. This allows us to rewrite Eq. A.1 as

$$K_{sec,m}^k \delta u_m^k = -R_m^{-1}(u_m^{k-1}).$$  \hspace{1cm} (A.3)

Eq. A.3 is solved repeatedly until $||R_m^{-1}(u_m^k)||_2$ becomes smaller than a predefined tolerance, $tol$, as shown in Algorithm 2. Note that the term $\frac{d\sigma_m^{k-1}(u_m^{k-1}+\epsilon\delta u_m^k)}{d\epsilon}|_{\epsilon=0}$ remain to be computed and the derivation is given by the constitutive update in what follows.

Algorithm 2 Viscoelastic damage solution algorithm

1. for $m = 1 : nt$ do
   $u_0, D_0, \Delta t$ and $h_i^0$ for $i \in [1, n]$ are given and usually set to zero
   $t_m = t_{m-1} + \Delta t$
   Compute $F_{ext}^m$
   Set $normR = 2 * tol, k = 0$
   while ($normR > tol$) and ($k \leq niter$) do
      $k = k + 1$
      Constitutive update ($h_m^k, D_m^k$)
      Compute secant operator: $K_{sec,m}^k$
      Compute residual $R_m^k = \left(f_{ext}^m - f_{int}^{int,k}\right)$
      Solve the problem $\delta u_m^k = (K_{sec,m}^k)^{-1}R_m^k$
      Update solution: $u_{m+1}^k = \delta u_m^k + u_m^k$
      Update residual: $R_m^{k+1} = \left(f_{ext}^m - f_{int}^{int,k+1}\right)$
      Compute $normR = ||R_m^{k+1}||_2$
   end while
   if $k \geq k_{crit}$ then
      Set $\Delta t = \frac{\Delta t}{2}$
   end if
   Set $D_{m+1} = D_m^k$ and $h_{m+1}^i = h_m^{i,k}$
end for
A.2 Local Update: Constitutive Equation Update

To complete the description of the algorithm, the term \(1 - D_m(u_m^{k-1})\) \(\frac{d\tilde{\sigma}_m^{k-1}(u_m^{k-1} + \varepsilon \delta u_m^k)}{d\varepsilon}\) is computed in two steps. First, the viscoelastic law without damage is integrated in time using the semi-analytical approach proposed by Taylor [50]. Second, the damage law is integrated in time using a forward Euler scheme.

Let us first briefly present Taylor’s method in which, since the update are done at local (Gauss point) level, the global iteration index, \(k\), is not included. From the definition of stress in Eq. 2.1 and a chain rule of the perturbation \(\varepsilon\) with respect to the strain \(\varepsilon\), we get

\[
\frac{d\tilde{\sigma}_m^{k-1}(u_m^{k-1} + \varepsilon \delta u_m^k)}{d\varepsilon} = \left. \frac{d\tilde{\sigma}}{d\varepsilon} \frac{d\varepsilon(u_m^{k-1} + \varepsilon \delta u_m^k)}{d\varepsilon} \right|_{\varepsilon=0}
\]

the viscoelastic term in the Jacobian is obtained as,

\[
\frac{\partial \tilde{\sigma}_m}{\partial \varepsilon_m} = \frac{\partial}{\partial \varepsilon_m} \left[ 3\kappa Tr(\varepsilon_m)I + \tilde{\sigma}_m^{dev} \right] = 3\kappa(I \otimes I) + \frac{\partial \tilde{\sigma}_m^{dev}}{\partial \varepsilon_m}
\]

where \(\otimes\) is the dyadic tensor operator. From the definition of the deviatoric strain \(\varepsilon_m\) at the current time step,

\[
\frac{\partial \tilde{\sigma}_m^{dev}}{\partial \varepsilon_m} = \frac{\partial \tilde{\sigma}_m^{dev}}{\partial \varepsilon_m} \frac{\partial \varepsilon_m}{\partial \varepsilon_m}
\]

the deviatoric projection tensor is obtained in terms of the fourth order identity tensor \(\mathcal{I}\),

\[
\frac{\partial \varepsilon_m}{\partial \varepsilon_m} = \mathcal{I} - \frac{1}{3} I \otimes I
\]
APPENDIX A:

The derivation of the deviatoric effective stress is obtained from Eq. A.12 with the definition of the history strain term, $h^i_m$ in Eq. A.13, which reads

$$\frac{\partial \bar{\sigma}^{\text{dev}}_m}{\partial e_m} = 2G \left\{ \mu_\infty + \sum_{i=1}^{n} \mu_i \frac{\partial h^i_m}{\partial e_m} \right\}. \quad (A.8)$$

The derivative of $h^i_m$ at the current time is obtained from the update term of Eq. A.16 as

$$\frac{\partial h^i_m}{\partial e_m} = \frac{\lambda_i}{\Delta t} \left[ 1 - \exp \left( \frac{-\Delta t}{\lambda_i} \right) \right]. \quad (A.9)$$

and the last term in Eq. A.5 is obtained from

$$\frac{\partial \bar{\sigma}^{\text{dev}}_m}{\partial e_m} = 2G \left\{ \mu_\infty + \sum_{i=1}^{n} \mu_i \frac{\lambda_i}{\Delta t} \left[ 1 - \exp \left( \frac{-\Delta t}{\lambda_i} \right) \right] \right\} \left[ \bar{e} - \frac{1}{3} I \otimes I \right]. \quad (A.10)$$

Finally, the damage parameter $D_m(u_m^{k-1})$ at the current Newton iteration is obtained by solving Eq. 2.11 with an Euler forward scheme

$$D_m(u_m^{k-1}) = D_{m-1} + \Delta t B \left\langle \chi (\bar{\sigma}(u_m^{k-1})) \right\rangle_r \left( 1 - D_{m-1}^k \right)^{k_r} \quad (A.11)$$

The update of both viscoelastic and damage laws are presented in Algorithm 1.

A.3 Time Integration Scheme

This section describes solution process to the semi-analytical integration scheme for the viscoelastic constitutive model. According to [184], the semi-analytical approach proposed by [50] is superior to common numerical integration methods. This approach is used here for the integral in Eq. 2.1, in which the time is split into a history integral and an update integral. Following the approach in [50] and assuming the loading conditions are applied at time zero, the deviatoric stress
A.3. TIME INTEGRATION SCHEME

in Eq. 2.1 is rewritten below,

\[
\tilde{\sigma}_{\text{dev}} = 2G \left\{ \mu_\infty e(t) + \mu_\infty e_0 + \sum_{i=1}^{n} \left( \mu_i \exp \left( \frac{-t}{\lambda_i} \right) \left[ e_0 + \int_0^t \exp \left( \frac{\tau}{\lambda_i} \right) \dot{e}(\tau) d\tau \right] \right) \right\} \quad (A.12)
\]

\[
= 2G \left\{ \mu_\infty e(t) + \mu_\infty e_0 + \sum_{i=1}^{n} \mu_i \left[ \exp \left( \frac{-t}{\lambda_i} \right) e_0 + h^i_m \right] \right\}
\]

in which the integral of the last term at the right of Eq. A.12 is performed by defining a stress-like measure, \( h^i_m \) of the \( i^{th} \) Prony term at the current time step \( (t = t_m) \) as follows,

\[
h^i_m = h^i_{m-1} \exp \left( \frac{-\Delta t}{\lambda_i} \right) + \Delta h^i_m
\]  

(A.13)

where \( h^i_{m-1} \) is the history integral defined as,

\[
h^i_{m-1} = \exp \left( \frac{-t_{m-1}}{\lambda_i} \right) \int_0^{t_{m-1}} \exp \left( \frac{\tau}{\lambda_i} \right) \dot{e}(\tau) d\tau,
\]  

(A.14)

\( \Delta t \) is the time interval \( (t_m - t_{m-1}) \) and \( \Delta h^i_m \) is the update integral defined as,

\[
\Delta h^i_m = \exp \left( \frac{-t}{\lambda_i} \right) \int_{t_{m-1}}^{t} \exp \left( \frac{\tau}{\lambda_i} \right) \dot{e}(\tau) d\tau.
\]  

(A.15)

Solving the closed form integral for \( \Delta h^i_m \) and assuming \( \dot{e} \) varies linearly in each time interval, \( \Delta h^i_m \) can be written as in Eq. A.16. As pointed by [184], the approximation for the calculation of \( \Delta h^i_m \) suffers from cancellation error for small values of \( \Delta t/\lambda_i \), in which cases, the first terms of the series expansion of the exponential term in Eq. A.16 is used.

\[
\Delta h^i_m = \lambda_i \left[ 1 - \exp \left( \frac{-\Delta t}{\lambda_i} \right) \right] \frac{\Delta e}{\Delta t}
\]  

(A.16)
Finally, the constitutive law for viscoelastic solids with moduli in the form of a Prony-series is obtained by directly plugging the latter into Eq. 2.1 which defines the total stress in the material.
Appendix B:

B.1 Consistent Derivation of Jacobian Terms

The remaining terms in the analytical derivation of the Jacobian in Eq. 3.32 are presented below with the unknown displacement vector \( \mathbf{u} = [u_x; u_y] \). The notation used in this paper shows first and second order tensors in bold (e.g. \( \mathbf{u}, \mathbf{t}, \sigma, \varepsilon \)), fourth order tensors with blackboard bold letters (e.g. \( \mathbb{E}, \mathbb{C} \) and \( \nabla^s \)) is the symmetric gradient operator used for the strain calculation.

- \( \mathbf{R}_{u,u} \): Using the Gâteaux derivative in equation Eq. 3.33

\[
\left. \frac{\partial \mathbf{R}_u(\mathbf{u} + \epsilon \delta \mathbf{u}, \chi)}{\partial \epsilon} \right|_{\epsilon=0} = \left. \frac{\partial}{\partial \epsilon} \int \nabla^s \mathbf{w}_u (1 - D) \tilde{\sigma}(\mathbf{u} + \epsilon \delta \mathbf{u}) d\Omega \right|_{\epsilon=0} - \left. \frac{\partial}{\partial \epsilon} \mathbf{W}_u \right|_{\epsilon=0} \quad (B.1)
\]

\[
= \int \nabla^s \mathbf{w}_u (1 - D) \left. \frac{\partial \tilde{\sigma}(\mathbf{u} + \epsilon \delta \mathbf{u})}{\partial \epsilon} \right|_{\epsilon=0} d\Omega \quad (B.2)
\]

with the definition of the effective stress \( \tilde{\sigma} \) in Eq. 2.14, and replacing "\( \mathbf{u} \)" by "\( \mathbf{u} + \epsilon \delta \mathbf{u} \)". The derivative term can be written as a function of the volumetric and deviatoric projection tensors \( \mathbb{I}^{(vol)} \) and \( \mathbb{I}^{(dev)} \) respectively.
where the viscoelastic shear modulus $G_{vis}$ is defined by Eq. 3.39. Replacing Eq. B.4 into Eq. B.2 yields the final equation and discretization of $\nabla^s u_t$ and $\nabla^s \delta u$ can then be used to perform the numerical integration. Alternatively, Eq. B.4 can be written in terms of the viscoelastic Lamé constants, defined as $\lambda_{vis}$ and $G_{vis}$, which are related to the Bulk and Viscous Shear moduli as $\lambda_{vis} = \kappa - \frac{2}{3} G_{vis}$. Thus, Eq. B.4 can also be written in terms of the symmetric unit tensor $I^s$ as,

$$\frac{\partial \tilde{\sigma}_{ij}(u + \epsilon \delta u)}{\partial \epsilon} \bigg|_{\epsilon = 0} = 3\kappa_{vol}^s \nabla^s u + 2G_{vis}^{\text{dev}} \nabla^s \delta u$$  \hfill (B.3)

$$= \left[ 3\kappa_{vol}^s + 2G_{vis}^{\text{dev}} \right] \nabla^s \delta u$$  \hfill (B.4)

\[\text{where the viscoelastic shear modulus } \; G_{vis}\text{ is defined by Eq. } 3.39. \text{ Replacing Eq. B.4 into Eq. B.2 yields the final equation and discretization of } \nabla^s w_u \; \text{and} \; \nabla^s \delta u \text{ can then be used to perform the numerical integration. Alternatively, Eq. B.4 can be written in terms of the viscoelastic Lamé constants, defined as } \lambda_{vis} \text{ and } G_{vis}, \text{ which are related to the Bulk and Viscous Shear moduli as } \lambda_{vis} = \kappa - \frac{2}{3} G_{vis}. \text{ Thus, Eq. B.4 can also be written in terms of the symmetric unit tensor } I^s \text{ as,} \]

$$\frac{\partial \sigma(u + \epsilon \delta u)}{\partial \epsilon} \bigg|_{\epsilon = 0} = \left[ 3\lambda_{vis}^s + 2G_{vis}^{\text{vol}} \right] \nabla^s \delta u$$  \hfill (B.5)

- **$R_{u,\tilde{\chi}}$** : Using the Gâteaux derivative in equation Eq. 3.33 obtain,

$$\frac{\partial R_{u}(u, \tilde{\chi} + \epsilon \delta \tilde{\chi})}{\partial \epsilon} \bigg|_{\epsilon = 0} = \left. \frac{\partial}{\partial \epsilon} \int_{\Omega} \nabla^s w_u (1 - D(\tilde{\chi} + \epsilon \delta \tilde{\chi})) \tilde{\sigma}(u) d\Omega \right|_{\epsilon = 0} - \left. \frac{\partial}{\partial \epsilon} w_u \big|_{\epsilon = 0} \right.$$  \hfill (B.6)

$$= -\int_{\Omega} \nabla^s w_u \tilde{\sigma}(u) \frac{\partial D}{\partial \tilde{\chi}} \nabla^s \delta \tilde{\chi} d\Omega$$  \hfill (B.7)

$$= -\int_{\Omega} \nabla^s w_u \tilde{\sigma}(u) \Delta B^r \nabla^s \delta \tilde{\chi} d\Omega$$  \hfill (B.8)

\[\text{where the } H(\tilde{\chi}) \text{ is the Heaviside step function.} \]
B.1. CONSISTENT DERIVATION OF JACOBIAN TERMS

- \( R_{\tilde{\chi}, u} \): Using the Gâteaux derivative again yields,

\[
\frac{\partial R_{\tilde{\chi}}}{\partial u} \delta u = \left. \frac{\partial R_{\tilde{\chi}}(u + \epsilon \delta u, \tilde{\chi})}{\partial \epsilon} \right|_{\epsilon=0}
\]

\[
= \left. \frac{\partial}{\partial \epsilon} \left[ \int_{\Omega} w_{\chi} \chi + c \nabla w_{\chi} \nabla \chi - w_{\chi} \chi(u + \epsilon \delta u) d\Omega \right] \right|_{\epsilon=0}
\]

\[
= \left. - \int_{\Omega} w_{\chi} \frac{\partial \chi(u + \epsilon \delta u)}{\partial \epsilon} d\Omega \right|_{\epsilon=0}
\]

\[
= \left. - \int_{\Omega} w_{\chi} \frac{\partial \sigma(u + \epsilon \delta u)}{\partial \epsilon} d\Omega \right|_{\epsilon=0}
\]

using the definition of \( \chi \) for 2D in Eq. 4.8 and the definition of stress derivative in Eq. B.5 yields,

\[
\frac{\partial \sigma}{\partial \epsilon} \bigg|_{\epsilon=0} = \left[ 3 \lambda_{\text{vis}}^{(\text{vol})} + 2 G_{\text{vis}}^{\text{v}} \right] \nabla \delta u
\]

The solution for Eq. B.13 is done separately for each term. The principal stress, \( \tilde{\sigma}^{(1)} \), is obtained from largest value between the out-of-plane stress \( \tilde{\sigma}_{zz} \) or the positive term in Eq. B.15,

\[
\tilde{\sigma}^{(1,2)} = \left( \frac{\tilde{\sigma}_{xx} + \tilde{\sigma}_{yy}}{2} \right)^2 \pm \sqrt{\left( \frac{\tilde{\sigma}_{xx} - \tilde{\sigma}_{yy}}{2} \right)^2 + \tilde{\sigma}_{xy}^2}
\]
APPENDIX B:

where the derivatives of Eq. B.15 with respect to the effective stress, \( \frac{\partial \tilde{\sigma}}{\partial \tilde{\sigma}} \) yields,

\[
\frac{\alpha \partial \tilde{\sigma}^{(1)}}{\partial \tilde{\sigma}} = \alpha \begin{bmatrix}
\frac{1}{2} + \frac{4}{\sqrt{\left(\frac{\tilde{\sigma}_{xx} - \tilde{\sigma}_{yy}}{2}\right)^2 + \sigma_y^2}} & \frac{\tilde{\sigma}_{xy}}{\sqrt{\left(\frac{\tilde{\sigma}_{xx} - \tilde{\sigma}_{yy}}{2}\right)^2 + \sigma_y^2}} \\
\frac{\tilde{\sigma}_{xy}}{\sqrt{\left(\frac{\tilde{\sigma}_{xx} - \tilde{\sigma}_{yy}}{2}\right)^2 + \sigma_y^2}} & \frac{1}{2} - \frac{4}{\sqrt{\left(\frac{\tilde{\sigma}_{xx} - \tilde{\sigma}_{yy}}{2}\right)^2 + \sigma_y^2}}
\end{bmatrix} \tag{B.16}
\]

The additional terms in Eq. B.13 yields,

\[
\beta \frac{\partial \sqrt{3J_2}}{\partial \tilde{\sigma}} = \beta \frac{\partial \left(\frac{3}{2} \tilde{\sigma}^{(dev)} : \tilde{\sigma}^{(dev)}\right)^{\frac{1}{2}}}{\partial \tilde{\sigma}} \tag{B.17}
\]

\[
= \beta \frac{3}{2} \tilde{\sigma}^{(dev)} \left(\frac{3}{2} \tilde{\sigma}^{(dev)} : \tilde{\sigma}^{(dev)}\right)^{-\frac{1}{2}} \frac{\partial \tilde{\sigma}^{(dev)}}{\partial \tilde{\sigma}} \tag{B.18}
\]

\[
= \beta \frac{3}{2} \tilde{\sigma}^{(dev)} \frac{\partial \tilde{\sigma}^{(dev)}}{\partial \tilde{\sigma}} \tag{B.19}
\]

\[
= \beta \frac{3}{2} \tilde{\sigma}^{(dev)} \sqrt{3J_2} \tag{B.20}
\]

\[
= \beta \frac{3}{2} \sqrt{3J_2} \tag{B.21}
\]

\[
(1.0 - \alpha - \beta) \frac{\partial \text{tr}(\tilde{\sigma})}{\partial \tilde{\sigma}} = (1.0 - \alpha - \beta) \frac{\partial \tilde{\sigma} : \mathbb{I}}{\partial \tilde{\sigma}} \tag{B.22}
\]

\[
= (1.0 - \alpha - \beta) \mathbb{I} \tag{B.23}
\]

Finally replacing into Eq. B.12 yields,
B.2. ICE MATERIAL PARAMETERS

\[
\frac{\partial \chi}{\partial \bar{\sigma}} \frac{\partial \bar{\sigma}}{\partial e} = \alpha \frac{\partial \bar{\sigma}^{(1)}}{\partial \bar{\sigma}} : \left[ 3\lambda^{\text{vis}}_{\Pi}^{\text{vol}} + 2G^{\text{vis}}_{\Pi} \right] \\
+ 3\beta \frac{G^{\text{vis}}_{\Pi}}{\sqrt{3}J_2} \bar{\sigma}^{(\text{dev})} \\
+(3\lambda^{\text{vis}} + 2G^{\text{vis}})(1.0 - \alpha - \beta)I\nabla \delta u
\] (B.24)

B.2 Ice Material Parameters

Viscoelastic parameters used in the results section for 1D and the ice slab under constant strain-rate in 2D are presented in Table B.1 from the optimized viscoelastic parameters in [3] with Young’s modulus set to \( E = 9500MPa \) and Poisson ratio is \( \nu = 0.35 \). Damage parameters are presented in Table B.2 which are mainly obtained from [3] with updated values of \( B_2 \) and \( r \).

Table B.1: Prony-series viscoelastic parameters of polycrystalline ice used in numerical simulations from [3]

<table>
<thead>
<tr>
<th>( G_0 ) [MPa]</th>
<th>( K ) [MPa]</th>
<th>( n )</th>
<th>( \mu_1 )</th>
<th>( \lambda_1 ) [s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>3518.52</td>
<td>10.56 \times 10^4</td>
<td>1</td>
<td>9.999 \times 10^{-1}</td>
<td>900.0</td>
</tr>
</tbody>
</table>

Table B.2: Damage parameters of polycrystalline ice modified from the results in [3]

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>( B_1 )</th>
<th>( B_2 )</th>
<th>( r )</th>
<th>( k_{\sigma} )</th>
<th>( \chi_{\text{threshold}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.21</td>
<td>0.63</td>
<td>1.3955 \times 10^{-6}</td>
<td>0.0</td>
<td>0.9</td>
<td>0.02</td>
<td>8.0 \times 10^{-5}</td>
</tr>
</tbody>
</table>