Localization and Heegaard Floer Homology

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ABSTRACT

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In this thesis we use Seidel-Smith localization for Lagrangian Floer cohomology to study invariants of cyclic branched covers of three-manifolds and symmetry groups of knots by constructing localization spectral sequences in Heegaard Floer homology.

Our first application is to double branched covers of knots and links in the three-sphere. Given a link K in S^3, let $\Sigma(K)$ be the double branched cover of S^3 over L. We show that there is a spectral sequence whose E^1 page is $\widehat{HFK}(\Sigma(K), K) \otimes H_*(T_n) \otimes \mathbb{Z}_2((\theta))$ and whose E^∞ page is isomorphic to $\widehat{HFK}(S^3, K) \otimes H_*(T_n) \otimes \mathbb{Z}_2((\theta))$, as $\mathbb{Z}_2((\theta))$-modules. As a consequence, we deduce a rank inequality between the knot Floer homologies $\widehat{HFK}(\Sigma(K), K)$ and $\widehat{HFK}(S^3, K)$. We also prove the analogous theorem for link Floer homology. This result was recently published by *Algebraic & Geometric Topology* [18].

Our second application is to doubly-periodic knots in the three-sphere. A knot $\widetilde{K} \subset S^3$ is q-periodic if there is a $\mathbb{Z}_q$-action preserving $\widetilde{K}$ whose fixed set is an unknot U. The quotient of $\widetilde{K}$ under the action is a second knot K. We construct equivariant Heegaard diagrams for q-periodic knots, and show that Murasugi's classical condition on the Alexander polynomials of periodic knots is a quick consequence of these diagrams. For $\widetilde{K}$ a two-periodic knot, we show that there is a spectral sequence whose E^1 page is $\widehat{HFL}(S^3, \widetilde{K} \cup U) \otimes H_*(T_{2n+1}) \otimes \mathbb{Z}_2((\theta))$ and whose E^∞ pages is isomorphic to $\widehat{HFL}(S^3, K \cup U) \otimes H_*(T_n) \otimes \mathbb{Z}_2((\theta))$, as $\mathbb{Z}_2((\theta))$-modules, and a related spectral sequence whose E^1 page is $\widehat{HFK}(S^3, \widetilde{K}) \otimes H_*(T_{2n+1}) \otimes H_*(S^0)) \otimes \mathbb{Z}_2((\theta))$ and whose E^∞ page is isomorphic to $\widehat{HFK}(S^3, K) \otimes H_*(T_n) \otimes H_*(S^0)) \otimes \mathbb{Z}_2((\theta))$. We use these spectral sequences to recover a lower bound of Edmonds on the genus of $\widetilde{K}$, orginally proved using minimal surface theory, along with a weak version of a fibredness result of Edmonds and

Livingston. These results may also be found in the preprint [17].

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Chapter 1

Introduction

The study of cyclic group actions on low-dimensional manifolds is a source of many important constructions in low-dimensional topology and knot theory. One example of classical and continuing interest is cyclic branched covers of knots in the three-sphere; another notable case is *periodic knots*, knots in the three-sphere which are preserved by a $\mathbb{Z}_q$ action whose fixed set is an unknot disjoint from the knot. In this thesis we study these two examples by applying a modern version of classical localization spectral sequences to invariants from gauge theory and symplectic topology.

Localization is a technique for studying actions of a compact Lie group G on a finite-dimensional CW complex X which was introduced by Armand Borel [5, 4] and elaborated by Michael Atiyah, Raoul Bott, Graeme Segal, and Daniel Quillen in the 1960s [1, 35, 3]. When $G = \mathbb{Z}_p$, their work gives a reformulation of work done by Paul Smith in 1938 [42]. In the simplest setting, given a $\mathbb{Z}_2$–action τ on X with fixed set X^{inv}, we obtain an action $\tau^{\#}$ on $C^*(X; \mathbb{Z}_2)$. Consider the spectral sequence of the double complex $(C^*(X; \mathbb{Z}_2) \otimes \mathbb{Z}_2[[\theta]], \partial + (1 + \tau^{\#})\theta)$, which has E^1 page $H_*(X; \mathbb{Z}_2) \otimes \mathbb{Z}_2[[\theta]]$ and the final page is the *equivariant cohomology* $H^*_{\mathbb{Z}_2}(X)$. There is then a localization map

$$H^*_{\mathbb{Z}_2}(X) \to H^*(X^{\text{inv}}; \mathbb{Z}_2) \otimes \mathbb{Z}_2[[\theta]]$$

which becomes an isomorphism after tensoring with θ^{-1}. This implies a dimension inequality between $H^*(X;\mathbb{Z}_2)$ and $H^*(X^{\text{inv}};\mathbb{Z}_2)$.

Lagrangian Floer cohomology was introduced by Floer [9, 10, 11] in 1988 to study classical problems in differential and symplectic geometry and Hamiltonian dynamics. It studies the geometry of two Lagrangian submanifolds in a symplectic manifold M by applying Morse theoretic techniques to its symplectic action functional. The theory deals with moduli spaces of holomorphic curves in M with boundary on the Lagrangians L_0 and L_1; it is by no means obvious that a finite group action on M that preserves L_0 and L_1 should give rise to an analog of Smith localization for the Lagrangian Floer cohomology. In 2010, Paul Seidel and Ivan Smith drew on parallels between a Morse theory proof of classical localization and the Lagrangian Floer construction to establish a bundle-theoretic condition, which they called *stable normal trivialization*, under which a symplectic $\mathbb{Z}_2$–action on M preserving the Lagrangians induces a localization spectral sequence for Floer cohomology.

Heegaard Floer link homology is an invariant of a nulhomologous link L in a three-manifold Y due to Peter Ozsváth and Zoltán Szabó [29], and independently in the case of knots to Jacob Rasmussen [36] which associates to (Y, L) a multigraded $\mathbb{Z}_2$-vector space $\widehat{HFL}(Y, L)$. Among its other properties, the link Floer homology of a link L in the three-sphere categorifies the multivariable Alexander polynomial of the link [32] and detects the the Thurston norm of the link complement [33] (or the genus in the case of a knot [28]), and fibredness [13, 25].

Link Floer homology is computed as a slight variation on Lagrangian Floer cohomology. In this thesis, we use Seidel and Smith's localization result to study the interaction of Heegaard Floer homology and two branched covering constructions, branched double covers of links and periodic knots.

1.1 Double Branched Covers of Links in the Three-Sphere

Our first application is a comparison between the link Floer homology of a link in the three-sphere and of the lift of the link in its double branched cover. If L is a knot in S^3, let $\Sigma(L)$ be the double branched cover of S^3 over L. The preimage of L under the branched cover map

$\pi\colon\ \Sigma(L) \to S^3$ is a nullhomologous link $\widetilde{L}$ in $\Sigma(L)$. The relationship between the knot Floer homology groups $\widehat{HFK}(S^3, L)$ and $\widehat{HFK}(\Sigma(L), \widetilde{L})$ has been investigated by Grigsby [14] and Levine [19, 20].

We prove the following theorem, conjectured by Levine [20, Conjecture 4.5] after being proved in the case of two-bridge knots by Grigsby [14, Theorem 4.3]. Let n be the bridge number of L, or equivalently the minimum number of pairs of basepoints on some Heegaard diagram $\mathcal{D}$ for (Y, L) whose underlying surface is S^2. Let $\widetilde{\mathcal{D}}$ be a double branched cover of $\mathcal{D}$ which is an n-pointed Heegaard diagram for $(\Sigma(L), L)$. (We will introduce this construction more explicitly in Chapter 3). We will work with a variant of knot Floer homology, $\widetilde{HFL}(\widetilde{\mathcal{D}})$, which is dependent on n and equal to $\widehat{HFL}(Y, L) \otimes V^{\otimes(n-1)}$. Here V is a a dimension 2 vector space with several possible gradings, which we will discuss further in Chapter 2.

Theorem 1.1.1. *There is a spectral sequence whose E^1 page is $(\widehat{HFL}(\Sigma(L), L) \otimes V^{\otimes(n-1)}) \otimes \mathbb{Z}_2((\theta))$ and whose E^∞ page is isomorphic to $(\widehat{HFL}(S^3, L) \otimes V^{\otimes(n-1)}) \otimes \mathbb{Z}_2((\theta))$ as $\mathbb{Z}_2((\theta))$-modules.*

Here $\mathbb{Z}_2((\theta))$ denotes the ring $\mathbb{Z}_2[[\theta]][\theta^{-1}]$ of Laurent series in the variable θ. In particular, we have the following rank inequality.

Corollary 1.1.2. *Given L a link in S^3 and $\Sigma(L)$ the double branched cover of S^3 over L, the following rank inequality holds:*

$$rk(\widehat{HFL}(S^3, L)) \leq rk(\widehat{HFL}(\Sigma(L), \widetilde{L})).$$

Link Floer homology admits two gradings, the Maslov or homological grading and the Alexander multi-grading. We will see that the spectral sequence of Theorem 1.1.1 is generated by a double complex whose two differentials each preserve the Alexander gradings on the basepoint-dependent invariant $\widetilde{HFK}(\mathcal{D})$, inducing a splitting of the spectral sequence along relative Alexander multi-gradings which the isomorphism of Theorem 1.1.1 fails to disrupt. Link Floer homology also splits along the spin^c structures $\mathfrak{s}$ of Y, such that we have

$$\widehat{HFK}(Y, L) = \oplus_{\mathfrak{s}} \widehat{HFK}(Y, L, \mathfrak{s}).$$

The extra factors of V in $\widetilde{HFK}(D)$ respect this splitting, such that

$$\widetilde{HFK}(\mathcal{D}, \mathfrak{s}) = \widehat{HFK}(Y, L, \mathfrak{s}) \otimes V^{\otimes(n-1)}.$$

The differentials on the spectral sequence of Theorem 1.1.1 interchange conjugate spin^c structures on $\Sigma(L)$. In the event that $H_1(\Sigma(L))$ has no two-torsion – which is automatic if L is in fact a knot – these differentials preserve a single *canonical spin^c structure* $\mathfrak{s}_0$ on $\Sigma(L)$, which is moreover the only spin^c structure to survive to the E^∞ page of the spectral sequence. Therefore we can sharpen Corollary 1.1.2 to the following.

Corollary 1.1.3. *Given a link L in S^3 such that $H_1(\Sigma(L))$ has no two-torsion, let $\mathfrak{s}_0$ be the canonical spin^c structure on its double branched cover $\Sigma(L)$. Then we have the rank inequality*

$$rk\left(\widehat{HFK}(\Sigma(L), \widetilde{L}, \mathfrak{s}_0)\right) \geq rk\left(\widehat{HFK}(S^3, L)\right).$$

In the case that $L = K$ is a knot, we also have the following.

Corollary 1.1.4. *Given a knot K in S^3 and $\mathfrak{s}_0$ the canonical spin^c structure on its double branched cover $\Sigma(K)$, let F be a Seifert surface for K and $\widetilde{F}$ its lift to $\Sigma(K)$. Then there is a rank inequality*

$$rk\left(\widetilde{HFK}(\widetilde{\mathcal{D}}, \mathfrak{s}_0, i)\right) \geq rk\left(\widetilde{HFK}(\mathcal{D}, i)\right)$$

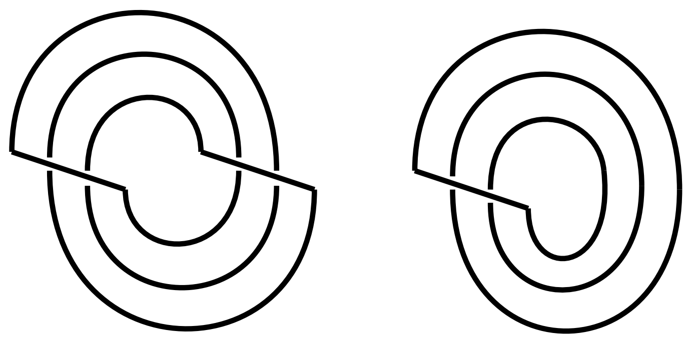
for i any Alexander grading, with Alexander gradings on the left computed relative to $\widetilde{F}$. In particular, for g the top Alexander grading such that $\widehat{HFK}(\Sigma(K), K, \mathfrak{s}_0, i)$ is nonzero, this is an inequality of the hat invariant:

$$rk\left(\widehat{HFK}(\Sigma(K), \widetilde{K}, \mathfrak{s}_0, g)\right) \geq rk\left(\widehat{HFK}(S^3, K, g)\right).$$

1.2 Periodic Knots

Our second application is to periodic knots in the three-sphere. We say $\widetilde{K} \subset S^3$ is a *periodic knot* if there is a $\mathbb{Z}_q$-action on $(S^3, \widetilde{K})$ which preserves $\widetilde{K}$ and whose fixed set is an unknot U

Figure 1: A doubly-periodic diagram for the trefoil, and its quotient knot (an unknot) under the $\mathbb{Z}_2$ action.



disjoint from $\widetilde{K}$. Let τ be a generator for the action. An important special case is $\tau^2 = 1$, in which case $\widetilde{K}$ is said to be *doubly-periodic*.

The quotient of $(S^3, \widetilde{K})$ under the group action is a second knot (S^3, K) such that the map $(S^3, \widetilde{K}) \to (S^3, K)$ is an q-fold branched cover over U. The knot K is said to be the q-fold *quotient knot* of $\widetilde{K}$.

We will first construct Heegaard diagrams for $(S^3, \widetilde{K} \cup U)$ which are preserved by the action of $\mathbb{Z}_q$ and whose quotients under the action are Heegaard diagrams for (S^3, K). These periodic Heegaard diagrams will allow us to give a simple Heegaard Floer reproof of one of Murasugi's conditions for the Alexander polynomial of a periodic knot in the case that $q = p^r$ for some prime p. Let $\lambda = \ell k(\widetilde{K}, U) = \ell k(K, U)$.

Theorem 1.2.1. *[24, Corollary 1]* $\Delta_{\widetilde{K}}(t) \equiv t^{\pm i}(1 + t + \cdots + t^{\lambda-1})^{q-1}(\Delta_K(t))^q$ *modulo* p.

Restricting to the case that $\widetilde{K}$ is doubly periodic, we will proceed to prove the following localization theorem.

Theorem 1.2.2. *There is an integer* n_1 *less than half the number of crossings of a periodic diagram* D *for* $\widetilde{K}$ *such that there is a spectral sequence whose* E^1 *page is*

$$\left(\widehat{HFL}(S^3, \widetilde{K} \cup U) \otimes V^{\otimes(2n_1-1)}\right) \otimes \mathbb{Z}_2((\theta))$$

and whose E^∞ page is isomorphic to

$$\left(\widehat{HFL}(S^3, K \cup U) \otimes V^{\otimes(n_1-1)}\right) \otimes \mathbb{Z}_2((\theta))$$

as $\mathbb{Z}_2((\theta))$-modules.

We shall see that this spectral sequence splits along the Alexander multigradings of the theory $\widehat{HFL}(S^3, \widetilde{K} \cup U) \otimes V^{\otimes(2n_1-1)}$, which will have a simple relationship to the Alexander multigrading of $\widehat{HFL}(S^3, K \cup U) \otimes V^{\otimes(n_1-1)}$.

We may also reduce the spectral sequence of Theorem 1.2.2 to contain only the information of knot Floer homology of $\widetilde{K}$ and K. Below, V and W are both two-dimensional vector spaces over $\mathbb{F}_2$, which will later be distinguished by their gradings.

Theorem 1.2.3. *There is an integer n_1 less than half the number of crossings of a periodic diagram D for $\widetilde{K}$ such that there is a spectral sequence whose E^1 page is*

$$\left(\widehat{HFK}(S^3, \widetilde{K}) \otimes V^{\otimes(2n_1-1)} \otimes W\right) \otimes \mathbb{Z}_2((\theta))$$

and whose E^∞ page is isomorphic to

$$\left(\widehat{HFK}(S^3, K) \otimes V^{\otimes(n_1-1)} \otimes W\right) \otimes \mathbb{Z}_2((\theta))$$

as $\mathbb{Z}_2((\theta))$–modules.

This spectral sequence splits along Alexander gradings of $\widehat{HFK}(S^3, \widetilde{K}) \otimes V^{\otimes(2n_1-1)} \otimes W$, which are related to the Alexander gradings of $\widehat{HFK}(S^3, \widetilde{K}) \otimes V^{\otimes(n_1-1)} \otimes W$ by division by two and an overall grading shift. Analysis of the behavior of this grading yields a reproof of a classical result, proven by Alan Edmonds using minimal surface theory.

Corollary 1.2.4. *[7, Theorem 4] Let $\widetilde{K}$ be a doubly-periodic knot in S^3 and K be its quotient knot. Then*

$$g(\widetilde{K}) \geq 2g(K) + \frac{\lambda - 1}{2}.$$

We also observe from the spectral sequence a proof of the following corollary.

Corollary 1.2.5. *Let $\widetilde{K}$ be a doubly-periodic knot in S^3 and K its quotient knot. If Edmonds' condition is sharp and $\widetilde{K}$ is fibered, K is fibered.*

This is a weaker version of a theorem proved by Livingston and Edmonds (which does *not* follow from the work in this thesis).

Theorem 1.2.6. *[8, Prop. 6.1] Let $\widetilde{K}$ be a doubly-periodic knot in S^3 and K its quotient knot. If $\widetilde{K}$ is fibered, K is fibered.*

1.3 Further generalization of the main theorem

In order to provide proofs Theorems 1.1.1, 1.2.2, and 1.2.3, we will in fact prove a slightly more general statement concerning Heegaard diagrams on the sphere and their branched double covers. As we will see in Chapter 2, given a Heegaard diagram for a link L in a three manifold Y, we may compute a theory $\widetilde{HFL}_{L'}(\mathcal{D})$ for any sublink L', equal to $\widehat{HFL}(Y, L')$ tensored with an appropriate number of copies of a two-dimensional vector space $\mathbb{Z}_2$ with varying gradings.

Let L be a link in S^3 and let L', L'' be two (possibly overlapping) sublinks. Let $\mathcal{D}$ be a Heegaard diagram on the sphere for a link $L \subset S^3$. (We will define Heegaard diagrams in Chapter 2.) Let $\widetilde{\mathcal{D}}$ be a Heegaard diagram for the branched double cover $(\Sigma(L'), \widetilde{L})$ for the lift $\widetilde{L}$ to the branched double cover $\Sigma(L')$ of S^3 over L'. We impose some mild conditions on the arrangement of curves in $\mathcal{D}$ to ensure that $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ are a *localizable* diagram pair. (For a full description of these conditions, see Definition 3.3.1.) Let $\widetilde{L''}$ be the lift of L'' in $\Sigma(L')$.

Theorem 1.3.1. *There is a spectral sequence with E^1 page $\widetilde{HFL}_{\widetilde{L''}}(\widetilde{\mathcal{D}}) \otimes \mathbb{Z}_2((\theta))$ and with E^∞ page $\mathbb{Z}_2((\theta))$-isomorphic to $\widetilde{HFL}_{L''}(\mathcal{D})$.*

In particular, specializing to $L = L'$ implies that the rank inequality of Corollary 1.1.2 applies not only to the ranks of $\widehat{HFL}(\Sigma(L), \widetilde{L})$ and $\widehat{HFL}(S^3, L)$ but also, for any sublink L'' of L, to the ranks of $\widehat{HFL}(\Sigma(L), \widetilde{L''})$ and $\widehat{HFL}(S^3, L'')$.

This reformulation was partly motivated by a question asked by Kent Baker. While it is possible to describe the behaviour of the Alexander grading under an arbitrary spectral sequence for localizable Heegaard diagrams, the general form is somewhat complicated and has no immediate applications, so we omit it here.

1.4 Seidel–Smith Localization: A Key Technical Tool

Our strategy for proving Theorems 1.1.1, 1.2.2, and 1.2.3 rests on a result of Seidel and Smith concerning equivariant Floer cohomology. Let M be an exact symplectic manifold, convex at infinity, containing exact Lagrangians L_0 and L_1 and equipped with an involution τ preserving (M, L_0, L_1). Let $(M^{\text{inv}}, L_0^{\text{inv}}, L_1^{\text{inv}})$ be the submanifolds of each space fixed by τ. Then under certain stringent conditions on the normal bundle $N(M^{\text{inv}})$ of M^{inv} in M, there is a rank inequality between the Floer cohomology $HF(L_0, L_1)$ of the two Lagrangians L_0 and L_1 in M and the Floer cohomology $HF(L_0^{\text{inv}}, L_1^{\text{inv}})$ of L_0^{inv} and L_1^{inv} in M^{inv}. More precisely, they consider the normal bundle $N(M^{\text{inv}})$ to M^{inv} in M and its pullback $\Upsilon(M^{\text{inv}})$ to $M^{\text{inv}} \times [0,1]$. We ask that M satisfy a K-theoretic condition called *stable normal triviality* relative to two Lagrangian subbundles over $L_0^{\text{inv}} \times \{0\}$ and $L_1^{\text{inv}} \times \{1\}$. Seidel and Smith prove the following.

Theorem 1.4.1. *[40, Section 3f] If $\Upsilon(M^{\text{inv}})$ carries a stable normal trivialization, there is a spectral sequence whose E^1 page is $HF(L_0, L_1) \otimes \mathbb{Z}_2((\theta))$ and whose E^∞ page is isomorphic to $HF(L_0^{\text{inv}}, L_1^{\text{inv}}) \otimes \mathbb{Z}_2((\theta))$ as $\mathbb{Z}_2((\theta))$ modules.*

In particular, there is the following useful corollary.

Corollary 1.4.2. *[40, Thm 1] If $\Upsilon(M^{\text{inv}})$ carries a stable normal trivialization, the Floer theoretic version of the Smith inequality holds:*

$$rk(HF(L_0, L_1)) \geq rk(HF(L_0^{\text{inv}}, L_1^{\text{inv}})).$$

Chapter 2

Background on Heegaard Floer Homology

We pause to recall the construction of the link Floer homology of a nulhomologous link L in a three-manifold Y, first defined by Ozsváth and Szabó in [32]. All work is done over $\mathbb{F}_2$.

2.0.1 Heegaard diagrams and admissibility

Definition 2.0.3. A *multipointed Heegaard diagram* $\mathcal{D} = (S, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ consists of the following data.

- An oriented surface S of genus g.
- Two sets of basepoints $\mathbf{w} = (w_1, \cdots, w_n)$ and $\mathbf{z} = (z_1, \cdots, z_n)$ on S.
- Two sets of closed embedded curves $\boldsymbol{\alpha} = \{\alpha_1, \cdots, \alpha_{g+n-1}\}$ and $\boldsymbol{\beta} = \{\beta_1, \cdots, \beta_{g+n-1}\}$ on S such that each of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ spans a g-dimensional subspace of $H_1(S)$, $\alpha_i \cap \alpha_j = \emptyset = \beta_i \cap \beta_j$ for $i \neq j$, each α_i and β_j intersect transversely, and each component of $S - \cup\alpha_i$ and of $S - \cup\beta_i$ contains exactly one point of $\mathbf{w}$ and one point of $\mathbf{z}$.

We use $\mathcal{D}$ to obtain an oriented 3-manifold Y by attaching two-handles to $S \times I$ along the curves $\alpha_i \times \{0\}$ and $\beta_i \times \{1\}$ and filling in $2n$ three-balls to close the resulting manifold. This yields a handlebody decomposition $Y = H_{\boldsymbol{\alpha}} \cup_S H_{\boldsymbol{\beta}}$ of Y. The Heegaard diagram $\mathcal{D}$ furthermore determines a knot or link in Y: connect the z basepoints to the w basepoints in the complement of the curves α_i, push these arcs into the handlebody $H_{\boldsymbol{\alpha}}$, then connect the w basepoints to the z basepoints in the complement of the curves β_i and push these arcs into the $H_{\boldsymbol{\beta}}$ handlebody.

Conversely, given a pair (Y, L), we may produce a Heegaard diagram $\mathcal{D}$ for (Y, K) via the following strategy. Let $f\colon (Y, L) \to [0, 3]$ be a self-indexing Morse function with n critical points each of index zero and index three, and $g + n - 1$ critical points each of index one and index two. Furthermore, insist that L is a union of flowlines between critical points of index zero and index three, and passes once through each such critical point. Then $S = f^{-1}(\frac{3}{2})$ is a surface of genus g. Draw α curves at the intersection of S with the ascending manifolds of the critical points of index one, and β curves at the intersection of S with the descending manifolds of the critical points of index two. Finally, let the w basepoints be the intersection of flowlines in L from index zero critical points to index three critical points, and the z basepoints be the intersection of flowlines in L from index three critical points to index zero critical points. This produces a $\mathcal{D} = (S, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ satisfying the conditions of Definiton 2.0.3.

We insist on numbering our basepoints such that if $L = K_1 \cup \cdots \cup K_\ell$, there are n_j pairs of basepoints on K_j, and there are integers $0 = k_0 < k_1 < \cdots < k_\ell = n$ with $k_j - k_{j-1} = n_j$ such that $w_{k_{j-1}+1}, \cdots, w_{k_j}, z_{k_{j-1}+1}, \cdots, z_{k_j}$ are the basepoints on K_j. (While this notation may seem cumbersome in the abstract, its primary relevance to our examples will be in the case of a two-component link $L = K_1 \cup K_2$ with n_1 pairs of basepoints on K_1 and one pair of basepoint on K_2, so it will not be too terrible in practice.)

There is an important collection of two-chains on the surface S on which we shall impose one more technical condition.

Definition 2.0.4. A *periodic domain* is a 2-chain P on $S\backslash\{\mathbf{w}\}$ whose boundary may be expressed as a linear combination of the α and β curves.

Note that this definition agrees with the convention of [32, Definition 3.4], in which the set of periodic domains is the set of 2-chains with boundary a linear combination of α and β curves which contain the components of $S - \boldsymbol{\alpha} - \boldsymbol{\beta}$ containing a point in $\{\mathbf{w}\}$ algebraically zero times. The set of periodic domains on S is in bijection with $\mathbb{Z}^{b_2(Y)+n-1} = H_2(Y\#(S^1 \times S^2)^{\#(n-1)})$. If we additionally puncture the surface S by removing the basepoints $\{\mathbf{z}\}$, the remaining periodic domains are in bijection with $H_2(Y - L) \cong Z^{b_2(Y)+\ell-1}$, where ℓ is the number of components of the link. We say that D is *weakly admissible* if every periodic domain on S has both positive and negative local multiplicities, and require that any Heegaard diagram we use to compute link Floer homology have this property. Note that we may always find a self-indexing Morse function $f : (Y, L) \to [0, 3]$ such that the Heegaard diagram derived from f is weakly admissible.

2.0.2 Symmetric products and generators

The construction of the link Floer homology $\widehat{HFL}(Y, L)$ makes use of the symmetric product $\mathrm{Sym}^{g+n-1}(S)$, whose points are all unordered $(g + n - 1)$-tuples of points in S. This space is the quotient of $(S)^{g+n-1}$ by the action of the symmetric group S_{g+n-1} permuting the factors of $(S)^{g+n-1}$, and its holomorphic structure is defined by insisting that the quotient map $(S)^{g+n-1} \to \mathrm{Sym}^{g+n-1}(S)$ be holomorphic. In particular, if j is a complex structure on S, there is a natural complex structure $\mathrm{Sym}^{g+n-1}(j)$ on the symmetric product. There are two transversely intersecting submanifolds of $\mathrm{Sym}^{g+n-1}(S)$ of especial interest, namely the two totally real embedded tori $\mathbb{T}_{\boldsymbol{\alpha}} = \alpha_1 \times \cdots \times \alpha_{g+n-1}$ and $\mathbb{T}_{\boldsymbol{\beta}} = \beta_1 \times \cdots \times \beta_{g+n-1}$. The chain complex $\widehat{CFL}(\mathcal{D})$ for knot Floer homology is generated by the finite set of intersection points of $\mathbb{T}_{\boldsymbol{\alpha}}$ and $\mathbb{T}_{\boldsymbol{\beta}}$. More concretely, a generator of $\widehat{CFL}(\mathcal{D})$ is a point $\mathbf{x} = (x_1 \cdots x_{g+n-1}) \in \mathrm{Sym}^{g+n-1}(S)$ such that if we regard $\mathbf{x}$ as a set of $g+n-1$ points on the surface S, each α or β curve contains a single x_i.

2.0.3 Absolute and relative spin^c structures associated to generators

Before continuing to discuss the differential on $\widehat{CFL}(\mathcal{D})$, we pause to recall Turaev's interpretation of relative spin^c structures on three-manifolds [44]. If Y is a closed, oriented 3-manifold, we say that two nowhere-vanishing vector fields v and v' on Y are *homologous* if v and v' are homotopic (through nowhere vanishing vector fields) on the complement of some ball $B \subset V$. The set of equivalence classes of such vector fields is a copy of $\text{Spin}^c(Y)$, and an affine copy of $H^2(Y)$.

Following Ozsváth and Szabó [32], we can extend this notion to *relative spin^c structures*. Let $(N, \partial N)$ be a manifold with toroidal boundary $T_1 \cup \cdots T_\ell$. The boundary of a torus contains, up to homotopy, a canonical nowhere-vanishing vector field. Therefore we consider nowhere-vanishing vector fields v on N which restrict to the canonical vector field on each boundary component of N. In this case vector fields v and v' are homologous if they are homotopic on $N - B$ for B a ball in N°. The set of such homotopy classes is the set of relative spin^c structures, and is an affine space for the relative homology $H^2(N, \partial N)$. This space is denoted $\underline{\text{Spin}}^c(N, \partial N)$.

There is an action of *conjugation* of relative spin^c structures induced by multiplying a nowhere-vanishing vector field v by -1. This yields a map

$$\underline{\text{Spin}}^c(N, \partial N) \to \underline{\text{Spin}}^c(N, \partial N)$$
$$\underline{\mathfrak{s}} \to j(\underline{\mathfrak{s}})$$

Ordinarily if $\mathfrak{s}$ is an absolute spin^c structure on a three-manifold Y without boundary, we write the conjugate spin^c structure as $j(\mathfrak{s}) = \overline{\mathfrak{s}}$. Moreover, if v is a nowhere-vanishing vector field on N with canonical restriction to the toroidal boundary ∂N, the restriction of the field of two-planes $v^\perp$ has a canonical trivialization along ∂N. Therefore there is a well-defined notion of the relative first Chern class $c_1(\underline{\mathfrak{s}})$ of a relative spin^c structure. It follows that $c_1(j(\underline{\mathfrak{s}})) = -c_1(\mathfrak{s})$.

In the case that $L = K$ is a knot, a relative spin^c structure on $Y - K, \partial(Y - K)$ can equivalently be regarded as an absolute spin^c structure on $Y_0(K)$. (If Y is not a integer homology sphere, this may first require a choice of longitude of K.)

To each generator of $\widehat{CFL}(\mathcal{D})$, we associate a relative spin^c structure as follows. If $\mathcal{D}$ is constructed from a self-indexing Morse function $f\colon (Y^3, L) \to [0, 3]$ as in Subsection 2.0.1, an intersection point $\mathbf{x}$ corresponds to a $(g + n - 1)$-tuple of flowlines connecting all index one and index two critical points. Let $\gamma_{\mathbf{x}}$ be the union of these gradient flowlines, and $\gamma_{\mathbf{w}}$ be the union of gradient flowlines passing through each $w_i \subset \mathbf{w}$. Then the $\gamma_{\mathbf{x}} \cup \gamma_{\mathbf{w}}$ is a collection of arcs with boundary all the critical points of f. Since each component connects critical points of opposite parities, we can modify the gradient vector field of f in a small neighborhood of $\gamma_{\mathbf{x}} \cup \gamma_{\mathbf{w}}$ to obtain a nowhere-vanishing vector field v on Y. We denote the associated relative spin^c structure $\underline{s}_{\mathbf{w}}(\mathbf{x})$. This construction can be shown to be well-defined [32, Section 3.3].

Finally, before moving on, observe that if $\mathcal{D}$ is a Heegaard diagram for (Y, L), then there is an identification $H_1(Y - L) \cong \frac{H_1(S - \{\mathbf{w},\mathbf{z}\})}{\langle [\alpha_1], \cdots [\alpha_{gn-1}], [\beta_1], \cdots, [\beta_{g+n-1}] \rangle} \cong \frac{H_1(\mathrm{Sym}^{g+n-1}(S \setminus \{\mathbf{w},\mathbf{z}\}))}{H_1(\mathbb{T}_{\boldsymbol{\alpha}}) \oplus H_1(\mathbb{T}_{\boldsymbol{\beta}})}$. Under this identification the set $\underline{\mathrm{Spin}}^c(N, \partial N)$ is canonically identified with the set of homotopy classes of paths $\mathcal{P}(\mathbb{T}_{\boldsymbol{\alpha}}, \mathbb{T}_{\boldsymbol{\beta}})$. We will discuss the homology and cohomology of punctured symmetric products at greater length in Chapter 5.

2.0.4 Whitney disks and the Maslov index

In its original form, link Floer homology is computed as follows: let $\mathbf{x}, \mathbf{y}$ be two intersection points in $\widehat{CFL}(\mathcal{D})$. We consider the set $\pi_2(\mathbf{x}, \mathbf{y})$ of *Whitney disks*, that is, the set of homotopy classes of topological disks $\phi\colon \mathbb{D} \to \mathrm{Sym}^{g+n-1}(S)$ from the unit disk in the complex plane to our symmetric product such that $\phi(-i) = \mathbf{x}$, $\phi(i) = \mathbf{y}$ and ϕ maps the portion of the boundary of the unit disk with positive real part into $\mathbb{T}_{\boldsymbol{\alpha}}$ and the portion with negative real part into $\mathbb{T}_{\boldsymbol{\beta}}$. The most common method of studying such maps ϕ is to use the following familiar construction of Ozsváth and Szabó to associate to any homotopy class of Whitney disks in $\pi(x, y)$ a domain in S. There is a $(g + n - 1)$-fold branched cover

$$S \times \mathrm{Sym}^{g+n-2}(S) \to \mathrm{Sym}^{g+n-1}(S)$$

The pullback of this branched cover along ϕ is a $(g + n - 1)$-fold branched cover of $B_1(0)$ which we shall denote $\Sigma(B_1(0))$. Consider the induced map on $\Sigma(B_1(0))$ formed by projecting the total space of this fibration to S.

$$\begin{array}{ccccc} \Sigma(B_1(0)) & \longrightarrow & S \times \mathrm{Sym}^{g+n-2}(S) & \longrightarrow & S \\ \downarrow & & \downarrow & & \\ B_1(0) & \xrightarrow{\phi} & \mathrm{Sym}^{g+n-1}(S) & & \end{array}$$

We associate to ϕ the image of this projection counted with multiplicities; to wit, we let $D = \Sigma a_i D_i$ where D_i are the closures of the components of $S - \cup\alpha_i - \cup\beta_i$ and a_i is the algebraic multiplicity of the intersection of the holomorphic submanifold $V_{x_i} = \{x_i\} \times \mathrm{Sym}^{g+n-2}(S)$ with $\phi(B_1(0))$ for any interior point x_i of D_i. The boundary of D consists of α arcs from points of $\mathbf{x}$ to points of $\mathbf{y}$ and β arcs from points of $\mathbf{y}$ to points of $\mathbf{x}$. If D_i contains a basepoint z_j, then we introduce some additional notation by letting $a_i = n_{z_i}(\phi)$ be the algebraic intersection number of $z_i \times \mathrm{Sym}^{g+n-2}(S)$ with the image of ϕ.

Given $\phi \in \pi_2(\mathbf{x}, \mathbf{x})$, we define the *Maslov index* as follows. Recall that a Whitney disk $\phi\colon \mathbb{D} \to \mathrm{Sym}^{g+n-1}(S)$ maps the portion of the boundary of the unit disk $\mathbb{D}$ in the right half of the complex plane $\mathbb{C} = \{u + iv\colon\ u, v \in \mathbb{R}\}$ to a loop in $\mathbb{T}_{\boldsymbol{\alpha}}$ and the portion of the boundary in the left half to $\mathbb{T}_{\boldsymbol{\beta}}$. Choose a constant trivialization of the orientable real vector bundle $\phi^*(T(\mathbb{T}_{\boldsymbol{\alpha}}))$ over $\partial\mathbb{D}|_{v\geq 0}$. We may tensor this real trivialization with $\mathbb{C}$ and extend to a complex trivialization of $\phi^*(T(\mathrm{Sym}^{g+n-1}(S))$ by pushing across the disk linearly. Relative to this trivialization, the real bundle $\phi^*(T(\mathbb{T}_{\boldsymbol{\beta}}))$ over $\partial\mathbb{D}|_{v\geq 0}$ induces a loop of real subspaces of $\mathbb{C}^{g+n-1} = \phi^*(T\mathrm{Sym}^{g+n-1}(S))$. The winding number of this loop is the Maslov index of the map ϕ. Notice that we could also have used $\phi^*(J(T(\mathbb{T}_{\boldsymbol{\alpha}})))$ and $\phi^*(T(\mathbb{T}_{\boldsymbol{\beta}})$, where J is the complex structure on the vector bundle $T\mathrm{Sym}^{g+n-1}(S)$, and obtained the same number.

The Maslov index $\mu(\phi)$ can equivalently be computed using the associated domain $\Sigma a_i D_i$ in a formula of Lipshitz's [21, Proposition 4.2]. For each domain D_i, let $e(D_i)$ be the Euler measure of D_i. In particular, if D_i has $2k$ corners, $e(D_i) = 1 - \frac{k}{2}$. Let $p_{\mathbf{x}}(D)$ be the sum of the average of the multiplicities of D at the four corners of each point in $\mathbf{x}$ and likewise for $p_{\mathbf{y}}(D)$. Then the Maslov index is

$$\mu(\phi) = \sum a_i e(D_i) + p_{\mathbf{x}}(D) + p_{\mathbf{y}}(D). \tag{2.0.1}$$

In the case that $[\phi] \in \pi_2(\mathbf{x}, \mathbf{x})$ is a domain from $\mathbf{x}$ to itself, and therefore a periodic domain, we have the following alternate interpretation of the Maslov index. Because $\phi(i) = \phi(-i)$, we see that ϕ sends $\partial\mathbb{D}|_{v\geq 0}$ to a loop in $\mathbb{T}_{\boldsymbol{\alpha}}$ and $\partial\mathbb{D}_{v\leq 0}$ to a loop in $\mathbb{T}_{\boldsymbol{\beta}}$. Therefore we may replace ϕ by a map $\widehat{\phi}\colon\ S^2 \times I \to \mathrm{Sym}^{g+n-1}(S)$ which maps $S^1 \times \{1\}$ to $\mathbb{T}_{\boldsymbol{\alpha}}$ and $S^1 \times \{0\}$ to $\mathbb{T}_{\boldsymbol{\beta}}$. We then consider the complex pullback bundle $E = \widehat{\phi}^*(T(\mathrm{Sym}^{g+n-1}(S))$ to $S^2 \times I$ and the totally real subbundles $\widehat{\phi}|^*_{S^1\times\{1\}}(T(\mathbb{T}_{\boldsymbol{\alpha}}))$ of $E_{S^1\times\{1\}}$ and $\widehat{\phi}|^*_{S^1\times\{0\}}(T(\mathbb{T}_{\boldsymbol{\beta}}))$ of $E|_{S^1\times\{0\}}$. The Maslov index is still calculated by trivializing $\widehat{\phi}|^*_{S^1\times\{1\}}(T(\mathbb{T}_{\boldsymbol{\alpha}}))$, complexifying, and computing the winding number of the loop of real-half dimensional subspaces in $\mathbb{C}^{g+n-1}$ represented by $\widehat{\phi}|^*_{S^1\times\{0\}}(T(\mathbb{T}_{\boldsymbol{\beta}}))$ with respect to the trivialization. This number classifies the bundle in the following way: complex vector bundles over the annulus whose restriction to the boundary of the annulus carries a canonical real subbundle are in bijection with maps $[(S^1 \times I, \partial(S^1 \times I)), (BU, BO)] = \langle (S^1 \times I, \partial(S^1 \times I)), (BU, BO)\rangle \cong \mathbb{Z}$, where the map to $\mathbb{Z}$ is the Maslov index $\mu(\phi) = \mu(\widehat{\phi})$ [23, Theorem C.3.7].

Now let us look at the bundle E over $S^1 \times I$ and its real subbundles over the boundary components of $S^1 \times I$ from a slightly different perspective. The real bundles $\widehat{\phi}|^*_{S^1\times\{1\}}(T(\mathbb{T}_{\boldsymbol{\alpha}}))$ and $\widehat{\phi}|^*_{S^1\times\{0\}}(T(\mathbb{T}_{\boldsymbol{\beta}}))$ are orientable, hence trivializable over the circle, so we may choose real trivializations and tensor with $\mathbb{C}$ to obtain a complex trivialization of $E|_{\partial(S^1\times I)}$. We can now regard E as a relative vector bundle E_{rel} over $(S^1 \times I, \partial(S^1 \times I))$, and consider its relative first Chern class $c_1(E|_{\mathrm{rel}})$. Equivalently, we may use this trivialization to construct a vector bundle $\widetilde{E}$ over $(S^1 \times I)/\partial(S^1 \times I) \cong S^2$ such that the pullback $q^*(\widetilde{E})$ along the quotient map is E. Then $c_1(\widetilde{E})$ is the relative first Chern class $c_1(E|_{\mathrm{rel}})$ under the identification $H^1(S^2) \simeq H^1(S^1 \times I, \partial(S^1 \times I))$. Moreover, isomorphism classes of vector bundles over S^2 are in bijection with homotopy classes of maps $[S^2, BU] \cong \langle S^2, BU\rangle = \pi_2(BU) \cong \mathbb{Z}$, where the identification with $\mathbb{Z}$ is via the first Chern class. Using the homotopy long exact sequence of the pair (BU, BO), we observe the following relationship between μ and c_1.

$$\begin{array}{ccccc}
\pi_2(BU) & \longrightarrow & \pi_2(BU, BO) & \longrightarrow & \pi_1(BO) \\
\downarrow{\scriptstyle c_1} & & \downarrow{\scriptstyle \mu} & & \downarrow \\
\mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z} & \longrightarrow & \mathbb{Z}_2
\end{array}$$

Therefore the Maslov index $\mu(\phi)$ is twice the relative first Chern class $c_1(E|_{\text{rel}})$.

2.0.5 Differentials and gradings on $\widehat{CFL}(\mathcal{D})$

We now have the tools we need to define a differential and gradings on the Heegaard Floer complex $\widehat{CFl}(\mathcal{D})$. The differential ∂ on $\widehat{CFL}(\mathcal{D})$ counts the dimension of the moduli spaces of pseudo-holomorphic curves of Maslov index one in $\pi_2(\mathbf{x}, \mathbf{y})$.

$$\partial(\mathbf{x}) = \sum_{\mathbf{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}): \\ \mu(\phi)=1 \\ n_{w_i}(\phi)=0 \\ n_{z_j}(\phi)=0}} \# \left(\frac{M(\phi)}{\mathbb{R}} \right) \mathbf{y}$$

Ozsváth and Szabó have shown [29] that this is a well-defined differential. Indeed, once we show that the homology of $\widehat{CFL}(\mathcal{D})$ with respect to ∂ can be seen as a Floer cohomology theory, this will be a special case of the well-definedness of the differential of Definition 4.0.5.

The complex $\widehat{CFL}(\mathcal{D})$ carries a (relative, for our purposes) homological grading called the Maslov grading $M(\mathbf{x})$ which takes values in $\mathbb{Z}$. Suppose $\mathbf{x}$ and $\mathbf{y}$ are connected by a Whitney disk ϕ. Then the relative Maslov grading is determined by

$$M(\mathbf{x}) - M(\mathbf{y}) = \mu(\phi) - 2 \sum_i n_{w_i}(\phi).$$

The complex also carries an additional Alexander multigrading $\mathbf{A} = (A_1, \cdots, A_\ell)$. This multigrading takes values in an affine lattice $\mathbb{H}$ over $H_1(S^3 - L; \mathbb{Z}) \cong H_1(L)$. Recall that $H_1(S^3 - L; \mathbb{Z}) \cong \mathbb{Z}^\ell$ generated by the homology classes of meridians μ_j of the component knots K_j of L. Define the lattice $\mathbb{H}$ to consist of elements

$$\sum_{i=1}^{\ell} A_i[\mu_i]$$

where $A_i \in \mathbb{Q}$ satisfies the property that $2A_i + \ell k(K_i, L - K_i)$ is an even integer. That is, the vector $\mathbf{A}$ records the coefficients of an element in $\mathbb{H}$ with respect to to the basis consisting of the homology classes of the meridians μ_i. To compute the relative Alexander multigrading in the most elementary way, recall that the basepoints $w_{k_{j-1}+1}, \cdots, w_{k_j}, z_{k_{j-1}+1}, \cdots, z_{k_j}$ lie on

K_j (with the convention that $k_0 = 0$). Then once again if ϕ is a Whitney disk connecting $\mathbf{x}$ and $\mathbf{y}$,

$$A_j(\mathbf{x}) - A_j(\mathbf{y}) = \sum_{i=k_{j-1}+1}^{k_j} n_{z_i}(\phi) - \sum_{i=k_{j-1}+1}^{k_j} n_{w_i}(\phi).$$

We can also see the relative Alexander multigrading geometrically. Let $\mathbf{x}, \mathbf{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$, we find paths

$$a\colon\ [0,1] \to \mathbb{T}_{\boldsymbol{\alpha}} \text{ and } b\colon\ [0,1] \to \mathbb{T}_{\boldsymbol{\beta}}$$

such that $\partial a = \partial b = \mathbf{x} - \mathbf{y}$. (For example, $a \cup b$ may be the boundary of a Whitney disk ϕ from $\mathbf{y}$ to $\mathbf{x}$.) View these paths as as one-chains on $S\backslash\{\mathbf{w}, \mathbf{z}\}$. Since attaching one- and two-handles to the α and β curves on S and filling in three-balls at the basepoints yields Y, we obtain a trivial one-cycle in Y. Indeed, a domain D on $\mathcal{D}$ is the shadow of a Whitney disk if and only if its boundary, viewed as a cycle on S, descends to a trivial cycle on Y. However, if we attach α and β circles to $S\backslash\{\mathbf{w}, \mathbf{z}\}$ (and no three balls) we obtain the manifold $Y - L$ and a one cycle $\epsilon(\mathbf{x}, \mathbf{y})$ in $Y - L$.

$$\underline{\epsilon}_{\mathbf{w},\mathbf{z}}\colon\ (\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}) \times (\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}) \to H_1(Y - L; \mathbb{Z})$$

We obtain the following lemma (which has only been very slightly adjusted from the original to account for the possibility of multiple pairs of basepoints on a link component).

Lemma 2.0.5. *[32, Lemma 3.10] An oriented ℓ-component link L in Y induces a map*

$$\prod H_1(Y - L) \to \mathbb{Z}^\ell$$

where $\prod_i(\gamma)$ is the linking number of γ with the ith component K_i of L. In particular, for $\mathbf{x}, \mathbf{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ and $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$, we have

$$\prod_i(\underline{\epsilon}_{\mathbf{w},\mathbf{z}}(\mathbf{x}, \mathbf{y})) = \sum_{n_{i-1}+1}^{n_i} n_{z_j}(\phi) - \sum_{n_{i-1}+1}^{n_i} n_{z_j}(\phi).$$

Proof. The proof is nearly identical to the original: ϕ induces a nulhomology of $\underline{\epsilon}(\mathbf{x}, \mathbf{y})$, which meets the ith component K_i of L with intersection number $\sum_{n_{i-1}}^{n_1} n_{z_i}(\phi) - \sum_{n_{i-1}}^{n_1} n_{w_i}(\phi)$. □

To construct the absolute Alexander grading when Y is an integer homology sphere, we must be slightly more subtle. Recall that to every generator $\mathbf{x}$ in $\widehat{CFL}(\mathcal{D})$ there is associated a relative spin^c structure $\underline{s}_{\mathbf{w}}(\mathbf{x})$. For each component K_i of the link $L = K_1 \cup \cdots \cup K_\ell$, let μ_i be a meridian of K_i. Then the Absolute grading of $\mathbf{x}$ is given by

$$c_1\left(\underline{s}_{\mathbf{w}}(\mathbf{x})\right) + \sum_{i=1}^{\ell} PD[\mu_i] = 2\sum_{i=1}^{\ell} A_i(\mathbf{x}) PD[\mu_i]$$

In the case that $L = K$ is a knot, up to a choice of Seifert surface F for K we may pin down the Alexander grading in a more direct geometric way. Let $Y_0(K)$ be the manifold obtained by zero-surgery along the preferred longitude of K induced by F and $\underline{\mathfrak{s}}$ be the spin^c structure obtained by extending the relative spin^c structure associated to $\mathbf{x}$ over $Y_0(K)$. Then if $\widehat{F}$ is the closed surface resulting from capping off F in $Y_0(K)$, $A(\mathbf{x}) = \langle c_1(\underline{\mathfrak{s}}), [\widehat{F}]\rangle$. Of course, if Y is an integer homology sphere, this construction does not depend on the choice of F.

Formulas for the absolute Maslov grading may be found in [31, Theorem 3.3], but will not be needed here.

The differential ∂ lowers the Maslov grading by one and preserves spin^c structures on Y and the Alexander multigrading (or equivalently relative spin^c structures on $Y - \nu L$). Therefore $\widehat{CFL}(\mathcal{D})$ splits along spin^c structures on Y and along the Alexander multigrading.

The homology of $\widehat{CFL}(\mathcal{D})$ with respect to the differential ∂ is very nearly the link Floer homology of (Y, L). There is, however, a slight subtlety having to do with the number of pairs of basepoints z_i and w_i on $\mathcal{D}$. Let V_i be a vector space over $\mathbb{F}_2$ with generators in gradings $(M, \mathbf{A}) = (0, \mathbf{0})$ and $(M, (A_1, \cdots, A_j, \cdots, A_\ell)) = (-1, (0, \cdots, -1, \cdots, 0))$, with the -1 in the jth component. As before, let K_j carry n_j pairs of basepoints.

Definition 2.0.6. The homology of the complex $\widehat{CFL}(\mathcal{D})$ with respect to the differential ∂ is

$$\widetilde{HFL}(\mathcal{D}) = \widehat{HFL}((Y, L)) \otimes V_1^{\otimes(n_1-1)} \otimes \cdots \otimes V_\ell^{\otimes(n_\ell-1)}.$$

For the rest of this chapter, let us concentrate purely on the case that $Y = S^3$, and discuss the relationship of the Alexander multigrading to various knot invariants and to basic spectral sequences between the Heegaard Floer homology of links and their sublinks more carefully. To begin, the theory $\widehat{HFL}(S^3, L)$ is symmetric with respect to the Alexander multigrading as follows. Let $\widehat{HFL}_d(S^3, L, \mathbf{A})$ be the summand of the link Floer homology of L in Alexander multigrading $\mathbf{A}$ and Maslov grading d.

Proposition 2.0.7. *[32, Proposition 8.2] There is an isomorphism*

$$\widehat{HFL}_d(Y, L, \mathbf{A}) \cong \widehat{HFL}_{d-\sum A_i}(Y, L, -\mathbf{A}).$$

In particular, ignoring Maslov gradings, we see that the link Floer homology is symmetric in each of its Alexander gradings.

2.0.6 Computations and grading for sublinks

Let $\mathcal{D}$ be a Heegaard diagram for (Y^3, L), where $L = K_1 \cup \cdots K_\ell$. Let us consider one addition differential on the complex $\widehat{CFL}(\mathcal{D})$. Suppose that in addition to the disks we counted previously, we also include disks passing over z basepoints on the component K_i of L. In other words, consider the differential ∂_{K_j} defined as follows.

$$\partial_{K_j}(\mathbf{x}) = \sum_{\mathbf{y}\in\mathbb{T}_{\boldsymbol\alpha}\cap\mathbb{T}_{\boldsymbol\beta}} \sum_{\substack{\phi\in\pi_2(\mathbf{x},\mathbf{y}):\\ \mu(\phi)=1\\ n_{w_i}(\phi)=0\\ n_{z_j}(\phi)=0 \text{ if } z_i\notin K_j}} \#\left(\frac{M(\phi)}{\mathbb{R}}\right)\mathbf{y}$$

This has the effect of discounting the contribution of the component K_j to the link Floer homology, but of maintaining the effect of an extra n_j pairs of basepoints on the Heegaard surface. We denote the result of the computation $\widetilde{HFL}_{L-K_j}(\mathcal{D})$. Ergo we have the following proposition. Let W be a two-dimensional vector space over $\mathbb{F}_2$ with summands in gradings $(M, (A_1, \cdots, \widehat{A_j}, \cdots, A_\ell)) = (0, (0, ..., 0))$ and $(M, (A_1, ...\widehat{A_j}, ...A_\ell)) = (-1, (0, ..., 0))$.

Proposition 2.0.8. *[32, Proposition 7.2] The homology of the complex $\widehat{CFL}(\mathcal{D})$ with respect to the differential ∂_{K_j} is isomorphic to $\widehat{HFL}(Y, L-K_j)\otimes V_1^{\otimes(n_1-1)}\otimes\cdots\otimes W^{\otimes n_j}\otimes\cdots\otimes V_\ell^{\otimes(n_\ell-1)}$.*

We may think of Proposition 2.0.8 as the assertion that there is a spectral sequence from the $\mathbb{Z}^\ell$ graded theory $\widehat{HFL}(Y^3, L)$ to the $\mathbb{Z}^{\ell-1}$ graded theory $\widehat{HFL}(Y^3, L - K_j)$ by computing all differentials that change the k_jth entry of the $\mathbb{Z}^\ell$ multigrading. This spectral sequence comes with an overall shift in relative Alexander gradings, which is computed by considering fillings of relative spin^c structures on $Y - L$ to relative spin^c structures on $Y - (L - K_j)$. The generally slightly complicated formula admits a simple expression in the case of two-component links in the three-sphere, which is the only case of interest to this thesis.

Lemma 2.0.9. *[32, Lemma 3.13] Let* $L = K_1 \cup K_2$*, and* $\lambda = \ell k(K_1, K_2)$*. Then suppose* $\mathcal{D}$ *is a Heegaard diagram for* (S^3, L)*, and* $\mathbf{x} \in \widetilde{CFL}(\mathcal{D})$*. If* $(A_1(\mathbf{x}), A_2(\mathbf{x})) = (i, j)$ *in the complex* $\widetilde{CFL}(\mathcal{D})$ *with differential* ∂*, then in the complex* $\widetilde{CFK}(\mathcal{D})$ *with differential* ∂_{K_2}*, the Alexander grading of* $\mathbf{x}$ *is* $A_1(\mathbf{x}) = i - \frac{\lambda}{2}$.

That is, forgetting one component of a two-component link has the effect of shifting Alexander gradings of the other component downward by $\frac{\lambda}{2}$. The proof comes from an analysis of filling relative spin^c structures; the effect of extending a relative spin^c structure $\underline{\mathfrak{s}}$ on $S^3 - \nu L$ to $S^3 - \nu(L - K_j)$ is to shift the Chern class $c_1(\underline{\mathfrak{s}})$ by the Poincaré dual of the homology class of K_j in $S^3 - \nu(L - K_j)$. For a two-component link this is a shift by the linking number.

We can of course repeat this construction an arbitrary number of times, obtaining a theory for any sublink of L in Y, at the cost of more factors of W.

2.1 Link Floer homology, the multivariable Alexander polynomial, and the Thurston norm

For this section, we work only with links $L \subset S^3$. Recall that the *multivariable Alexander polynomial* of an oriented link $L = K_1 \cup \cdots \cup K_\ell$ is a polynomial invariant $\Delta_L(t_1, \cdots, t_\ell)$ with one variable for each component of the link. A full construction can be found in [38, Section 7.I]. While its relationship to the Alexander polynomials of the component knots is in general slightly complicated, in the case of a two-component $L = K_1 \cup K_2$ Murasugi proved the following using Fox calculus.

Lemma 2.1.1. *[24, Proposition 4.1] Let $L = K_1 \cup K_2$ be an oriented two-component link with $\ell k(K_1, K_2) = \lambda$. If $\Delta_L(t_1, t_2)$ is the multivariable Alexander polynomial of L and $\Delta_{K_1}(t_1)$ is the ordinary Alexander polynomial of K_1, then*

$$\Delta_L(t_1, 1) = (1 + t + t^2 + \cdots + t^{\lambda-1})\Delta_{K_1}(t)$$

The Euler characteristic of link Floer homology encodes the multivariable Alexander polynomial of the link as follows.

Proposition 2.1.2. *[32, Theorem 1.3] If L is an oriented link,and $\mathcal{B}$ is a basis for $\widehat{HFL}(S^3, L)$,*

$$\sum_{[\mathbf{x}]\in\mathcal{B}} (-1)^{M(\mathbf{x})} t_1^{A_1(\mathbf{x})} \cdots t_\ell^{A_\ell(\mathbf{x})} = \begin{cases} \left(\prod_{i=1}^{\ell}(t_i^{\frac{1}{2}} - t_i^{-\frac{1}{2}})\right) \Delta_L(t_1, \cdots, t_\ell) & \ell > 1 \\ \Delta_L(t_1) & \ell = 1 \end{cases}$$

Link Floer homology also categorifies the Thurston seminorm of the link complement. Let us recall the definition of the Thurston seminorm on a three manifold with boundary.

Definition 2.1.3. Let $\gamma \in H^2(M, \partial M)$. The Thurston seminorm $x(\gamma)$ is

$$x(\gamma) = \inf\{-\chi(S)\}$$

where S is any embedded surface in $(M, \partial M)$ with $[S] = \gamma$.

An important special case occurs when M is the complement of a link $L = K_1 \cup \cdots \cup K_\ell$, that is, $M = S^3 - \nu K_1 - \cdots - \nu K_\ell$. Then $H^2(M, \partial M) \cong H_1(L)$, and computing the Thurston seminorm of the element of $H^2(M, \partial M)$ corresponding to $[\sum a_i K_i] \in H_1(L)$ is a matter of computing the minimal Euler characteristic of an embedded surface F whose intersection with a meridian μ_i of K_i is a_i for each i. In particular, $x([K_i])$ is the minimal Euler characteristic of surface F with boundary one longitude of K_i and an arbitrary number of meridians of the components of L. (For practical purposes, one may consider taking a Seifert surface F for K_i and puncturing F wherever it intersects some other component of L. However, take note that puncturing a minimal Seifert surface for K_i does not necessarily result in a Thurston-norm minimizing surface.) When L is a knot, this determines the minimal Euler characteristic of a Seifert surface for the knot, and thus determines the genus of the knot. Because $H_1(L) \cong H^1(S^3 - \nu L)$, we commonly refer to the element of $H^2(M, \partial M)$ which spans K_i as the dual to the homology class of meridian μ_i of K_i.

Thurston showed [43] that the Thurston seminorm extends to an $\mathbb{R}$-valued function of $H_2(S^3 - \nu(L)) \cong H_2(S^3, L)$:

$$x_L \colon\ H_2(S^3, L; \mathbb{R}) \to \mathbb{R}.$$

Link Floer homology yields a related function. Recall that $\mathbb{H} \subset H^2(S^3, L; \mathbb{R}) \cong H_1(L; \mathbb{R})$ is the affine lattice of real second cohomology classes $h = \sum A_i[\mu_i]$ for which $\widehat{HFL}(S^3, L, \mathbf{A})$ is defined. We have

$$y \colon\ H^1(S^3 - L; \mathbb{R}) \to \mathbb{R}$$

which is defined by

$$y(\gamma) = \max_{\{\sum A_i[\mu_i] \in \mathbb{H} \subset H_1(L;R) : \widehat{HFL}(L, \mathbf{A}) \neq 0\}} |\langle \sum A_i[\widehat{\mu_i}], \gamma \rangle|.$$

The categorification considers the case of links with no *trivial components*, that is, unknotted components unlinked with the rest of the link.

Proposition 2.1.4. *[33, Thm 1.1] Let L be an oriented link with no trivial components. Given $\gamma \in H^1(S^3 - L; \mathbb{R})$, the link Floer homology groups determine the Thurston norm of L via the relationship*

$$x_L(\mathrm{PD}[\gamma]) + \sum_{i=1}^{\ell} |\langle \gamma, \mu_i \rangle| = 2y(\gamma).$$

Here μ_i is the homology class of the meridian for the ith component of L in $H_1(S^3 - L; \mathbb{R})$, and therefore $|\langle h, \mu_i \rangle|$ is the absolute value of the Kronecker pairing of h with μ_i.

We will primarily evaluate this equality on the dual classes to the meridians μ_i themselves. As above, we will continue to use $x_L([K_i])$ for the Thurston norm of the dual to μ_i. In the case that K is a knot, so that $x_K([K]) = 2g(K) - 1$, Proposition 2.1.4 reduces to the familiar theorem of [28, Theorem 1.2] that the top Alexander grading i for which $\widehat{HFK}(S^3, K, i)$ is nontrivial is the genus of the knot. In general, observe that if we evaluate on μ_i, we obtain $x_L([K_i]) + 1 = 2y(\mu_i)$. In other words, the total breadth of the A_i Alexander grading in the link Floer homology is the Thurston norm of the dual to K_i plus one.

2.2 Link Floer homology and fibredness

Before leaving the realm of link Floer homology background, we will require one further result concerning the knot Floer homology of fibred knots.

Proposition 2.2.1. *[25, Thm 1.1], [13, Thm 1.4] Let K be a knot, and $g(K)$ its genus. Then K is fibered if and only if $\widehat{HFK}(S^3, K, g(K)) = \mathbb{Z}_2$.*

The forward direction (that if K is fibred, then the knot Floer homology in the top nontrivial Alexander grading is $\mathbb{Z}_2$) is due to Ozsváth and Szabó [30, Theorem 1.1], whereas the other direction was proved by Ghiggini [13] in the case $g = 1$ and Ni [25] in the general case.

We are now ready to consider Heegaard diagrams that respect various possible group actions on a three-manifold.

Chapter 3

Equivariant Heegaard Diagrams

In this section we discuss the construction of Heegaard diagrams compatible with various group actions, and make some basic observations concerning their link Floer homology. In Section 3.1 we construct equivariant Heegaard diagrams for double branched covers of links in the three-sphere. Assuming Theorem 1.1.1, we then prove Corollaries 1.1.2, 1.1.3, and 1.1.4. In Section 3.2 we construct equivariant Heegaard digrams for periodic knots, and show how Theorem 1.2.1 can be derived from these diagrams. Assuming Theorems 1.2.2 and 1.2.3, we then show how Corollaries 1.2.4 and 1.2.5 follow.

Some lemmas in this chapter are true of the equivariant Heegaard diagrams developed in both Section 3.1 and Section 3.2. Since we hope for each of these sections to be independently comprehensible, we have stated appropriate versions twice, although in most cases only proved one. Note of this has been made when it occurs in the text.

At the end of this chapter we list the characteristics shared by all of the pairs Heegaard diagrams constructed in this chapter which are compatible with involutions on a three-manifold pair (Y, L). We call any Heegaard diagram pair with all of these characteristics *localizable*. In the chapters that follow we draw on this list to prove lemmas concerning the behavior of any localizable pair of Heegaard diagrams.

3.1 Heegaard Diagrams for Branched Double Covers of Links

Let L be a link in S^3. Consider the double branched cover $\Sigma(L)$ of the three-sphere over L; that is, the unique manifold with an involution $\tau : \Sigma(L) \to \Sigma(L)$ such that the quotient of $\Sigma(L)$ by the action of τ is S^3, and such that if $\pi : \Sigma(L) \to S^3$ is the quotient map, $\pi^{-1}(L)$ is exactly the set of fixed points of τ. One way of constructing this manifold is to choose a Seifert surface F of L and remove a tubular neighborhood $F \times (-1, 1)$ from S^3. We then take two copies of $S^3 \backslash (F \times (-1, 1))$ and identify the positive side of the boundary in the one copy with the negative side in the other.

Recall that the first homology of $\Sigma(L)$ is of order $|\Delta_L(-1)|$, that is, the ordinary (nonmultivariable) Alexander polynomial of the link evaluated on -1. Alternately, this number is the absolute value of the multivariable Alexander polynomial of the link evaluated on $(-1, ..., -1)$, then multiplied by $(-1^{\frac{1}{2}} - (-1)^{-\frac{1}{2}})$. Here zero corresponds to a cyclic group of infinite order. Moreover, analysis of transfer maps shows that in general if $\gamma \in H_1(\Sigma(L))$, then $\gamma + \tau^*\gamma = 0$. Therefore in general τ^* acts on H_1 as multiplication by -1.

Let us know consider relationships between Heegaard diagrams for (S^3, L) and the double branched cover $(\Sigma(L), \widetilde{L})$. Suppose $f : (Y, L) \to [0, 3]$ is a self-indexing Morse function with respect to which L is a collection of flowlines between critical points of index zero and index three. Let $\mathcal{D}$ be the Heegaard surface for (Y, L) constructed from f. That is, $\mathcal{D}$ consists of the surface $S = f^{-1}(\frac{3}{2})$, curves $\alpha = \{\alpha_1, ..., \alpha_{g+n-1}\}$ at the intersection of the ascending manifolds of critical points of index one with S, curves $\beta = \{\beta_1, ..., \beta_{g+n-1}\}$ at the intersection of the descending manifolds of critical points of index two with S, and basepoints $\mathbf{w} = (w_1, ..., w_n)$ (resp. $\mathbf{z} = (z_1, ..., z_n)$) at the negatively (resp. positively) oriented points of $S \cup K$. Now consider the map $\widetilde{f} = f \circ \pi$, which is also self-indexing Morse since π is proper. From $\widetilde{f}$ we obtain a Heegaard diagram $\widetilde{\mathcal{D}}$ for $(\Sigma(L), L)$ which has surface $\widetilde{S} = \pi|_S^{-1}(S)$, the branched double cover of S over the basepoints $\{\mathbf{w}, \mathbf{z}\}$. Moreover, each α_i lifts to two closed curves α_i^1 and α_i^2, each of which is the attaching circle of a one-handle in $\Sigma(L)$, and similarly for the β curves and two-handles. Let $\widetilde{\boldsymbol{\alpha}} = (\alpha_1^1, \alpha_1^2, \cdots, \alpha_{g+n-1}^1, \alpha_{g+n-1}^2)$ and likewise for $\widetilde{\boldsymbol{\beta}}$. Let $\widetilde{\mathbf{w}}, \widetilde{\mathbf{z}}$

be the lifts of the basepoints $\mathbf{w}, \mathbf{z}$ to $\widetilde{\mathcal{D}}$. Then we have the following lemma.

Lemma 3.1.1. *If $\mathcal{D} = (S, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ is a weakly admissible Heegaard surface for (S^3, L), then $\widetilde{\mathcal{D}} = (\Sigma(S), \widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\beta}}, \widetilde{\mathbf{w}}, \widetilde{\mathbf{z}})$ is a weakly admissible Heegaard surface for $(\Sigma(L), L)$.*

Proof. The only thing left to check is admissibility. Yet if there is a two-chain F in $\Sigma(S)$ with boundary some collection of the curves in $\widetilde{\boldsymbol{\alpha}}$ and $\widetilde{\boldsymbol{\beta}}$ with only positive (or only negative) local multiplicities, then $\pi(F)$ is a two-chain in S with boundary some of the curves in $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ with only positive (or negative) local multiplicities. Hence $\widetilde{\mathcal{D}}$ is weakly admissible if $\mathcal{D}$ is. □

The generators of $\widehat{CFL}(\mathcal{D})$ have been studied by Grigsby [14] and Levine [19]; we give a quick sketch of their proofs of the following lemmas before proceeding to discuss the Heegaard diagrams we will use to prove Seidel–Smith localization for double branched covers of links.

Lemma 3.1.2. *[19, Lemma 3.1] Let $\mathbf{s} \in \widehat{CFL}(\widetilde{\mathcal{D}})$, thought of as an $2n-2$-tuple of points on $\widetilde{S}$ with one point on each α_i^k and one on each β_i^k for $k = 1, 2$. Consider its projection $\pi(\mathbf{s})$ to $2n-2$ points on $\mathcal{D}$. There is a (not canonical) way to write $\pi(\mathbf{s})$ as $\mathbf{s_1} \cup \mathbf{s_2}$ a union of generators in $\widehat{CFL}(\mathcal{D})$.*

The proof of this lemma is the direct analog of the more general proof of Lemma 3.2.3 in Section 3.2, so we refer the reader there. We will also need the following basic lemma.

Lemma 3.1.3. *The map $\tau^{\#}$ on $\widehat{CFL}(\widetilde{\mathcal{D}})$ induced by the involution τ on $\widetilde{\mathcal{D}}$ preserves Alexander and Maslov gradings.*

Again, this is analogous to a more general case dealt with in Section 3.2; we refer the reader to Lemma 3.2.1 for the structure of the proof.

Of particular interest are the generators of the form $\widetilde{\mathbf{x}} = \pi^{-1}(\mathbf{x})$ in $\widehat{CFK}(\widetilde{\mathcal{D}})$; that is, the generators which consist of all lifts of the points of a generator $\mathbf{x}$ in $\widehat{CFK}(\mathcal{D})$. These points are exactly the invariant set of the induced involution $\tau^{\#}$ on $\widehat{CFK}(\widetilde{\mathcal{D}})$.

Lemma 3.1.4. *[14, Propn 3.2] All generators of $\widehat{CFL}(\widetilde{\mathcal{D}})$ of the form $\widetilde{\mathbf{x}} = \pi^{-1}(\mathbf{x})$ are in the same spin^c structure, hereafter denoted $\mathfrak{s}_0$ and called the canonical spin^c structure on the double branched cover.*

Proof. Given two such generators $\widetilde{\mathbf{x}}$ and $\widetilde{\mathbf{y}}$, let $\gamma_{\mathbf{x},\mathbf{y}}$ be a one-cycle in S connecting $\mathbf{x}$ and $\mathbf{y}$ chosen as in Chapter 2 and $\widetilde{\gamma}_{\mathbf{x},\mathbf{y}}$ be any lift to $\Sigma(S)$. Then $\widetilde{\gamma}_{\mathbf{x},\mathbf{y}} + \tau^{\#}(\widetilde{\gamma}_{\mathbf{x},\mathbf{y}})$ is a suitable one-cycle running from $\widetilde{\mathbf{x}}$ to $\widetilde{\mathbf{y}}$. Moreover, since $\tau^{\#}$ acts by multiplication by -1 on $H_1(\Sigma(Y)) \cong \frac{H_1(\Sigma(S))}{\langle [\alpha_1^1],[\alpha_1^2)],\cdots,[\beta^1_{g+n-1}],[\beta^2_{g+n-1})]\rangle}$, the image $\epsilon(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})$ of $\widetilde{\gamma}_{\mathbf{x},\mathbf{y}} + \tau^{\#}(\widetilde{\gamma}_{\mathbf{x},\mathbf{y}})$ in $H^1(\Sigma(Y))$ is trivial. ☐

At this juncture we pause to discuss the action of the induced involution $\tau^{\#}$ on spin^c structures on $\widehat{HFL}(\Sigma(L), L)$. The spin^c structures on $\Sigma(L)$ are an affine copy of $H^2(\Sigma(L)) \cong H_1(\Sigma(L))$; setting $\mathfrak{s}_0 = 0$ removes the ambiguity of the identification between the set of spin^c structures on $\Sigma(L)$ and $H^2(\Sigma(L))$. Moreover, we shall see that for suitable choice of Heegaard diagram $\mathcal{D}$, including both the spherical bridge diagrams used later in this section and the toroidal grid diagrams of [19], and additionally any $\mathcal{D}$ which is nice in the sense of [39], $\tau^{\#}$ is a chain map. The induced involution τ^* acts by multiplication by -1 on the first homology of $\Sigma(L)$, as does conjugation of spin^c structures on $H_1(\Sigma(L))$. Ergo $\tau^*(\mathfrak{s}) = \bar{\mathfrak{s}}$. Thus the action of τ on $\widetilde{\mathcal{D}}$ induces an isomorphism

$$\widehat{HFL}(\Sigma(L), L, \mathfrak{s}) \cong \widehat{HFL}(\Sigma(L), L, \bar{\mathfrak{s}}).$$

In particular, the action of τ on $\widehat{CFL}(\widetilde{\mathcal{D}})$ can preserve only spin^c structures which are their own conjugates. In the case that $H_1(\Sigma(L))$ has no two-torsion the only spin^c structure with this property is $\mathfrak{s}_0$. Notice that this is always the case if $L = K$ is a knot, since $|H_1(\Sigma(K))|$ is cyclic of odd order. However, if $H_1(\Sigma(L))$ contains two-torsion, there may be other spin^c structures preserved by the involution.

In the case of a knot, we may also fix precisely the relationship between the Alexander gradings in $\widehat{CFK}(\widetilde{\mathcal{D}})$ and $\widehat{CFK}(\mathcal{D})$. Any choice of Seifert surface for a knot in a three-manifold gives rise to an absolute Alexander grading for $\widehat{HFK}(Y, K)$. Let F be a Seifert surface for K and $\widetilde{F}$ be its lift to a Seifert surface for $\widetilde{K}$.

Lemma 3.1.5. *If the projection $\pi(\mathrm{s})$ of a generator s in $\widehat{CFL}(\widetilde{\mathcal{D}})$ breaks up as a union $\mathbf{s_1} \cup \mathbf{s_2}$, the Alexander grading of s computed relative to $\widetilde{F}$ is the average of the Alexander gradings of $\mathbf{s_1}$ and $\mathbf{s_2}$.*

We refer the reader to [19, Lemma 3.4] for a lovely proof of this lemma.

We are now ready to consider the equivariant link Floer homology of a double branched cover of S^3 over a link. Let $\pi : (\Sigma(L), L) \to (Y, L)$ be the branched double cover map and $\tau : \Sigma(L) \to \Sigma(L)$ be the involution interchanging the two not necessarily distinct preimages of a point $x \in Y$. Assume that for our particular Heegaard diagram $\mathcal{D}$, the map $\tau^{\#}$ is a chain map on $\widehat{CFL}(\widetilde{\mathcal{D}})$.

The spectral sequence derived from the double complex

$$\begin{array}{ccccccc}
0 & \longrightarrow & \widehat{CFK}_{i+1}(\widetilde{\mathcal{D}}) & \xrightarrow{1+\tau^{\#}} & \widehat{CFK}_{i+1}(\widetilde{\mathcal{D}}) & \xrightarrow{1+\tau^{\#}} & \widehat{CFK}_{i+1}(\widetilde{\mathcal{D}}) \cdots \\
\downarrow & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial \\
0 & \longrightarrow & \widehat{CFK}_{i}(\widetilde{\mathcal{D}}) & \xrightarrow{1+\tau^{\#}} & \widehat{CFK}_{i}(\widetilde{\mathcal{D}}) & \xrightarrow{1+\tau^{\#}} & \widehat{CFK}_{i}(\widetilde{\mathcal{D}}) \cdots \\
\downarrow & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial \\
0 & \longrightarrow & \widehat{CFK}_{i-1}(\widetilde{\mathcal{D}}) & \xrightarrow{1+\tau^{\#}} & \widehat{CFK}_{i-1}(\widetilde{\mathcal{D}}) & \xrightarrow{1+\tau^{\#}} & \widehat{CFK}_{i-1}(\widetilde{\mathcal{D}}) \cdots
\end{array}$$

has been a source of interest for some time; a popular conjecture has been that its E^∞ page is isomorphic modulo torsion to $\widetilde{HFK}(\mathcal{D}) \otimes \mathbb{Z}_2[[\theta]]$. We will show a similar statement, namely Theorem 1.1.1, for a closely related spectral sequence. The E^1 page of this spectral sequence (after computing the homology of the vertical differentials) is $(\widetilde{HFK}(\widetilde{\mathcal{D}}) \otimes V^{\otimes(n-1)}) \otimes \mathbb{Z}_2[[\theta]]$, and an application of Theorem 1.4.1 for Lagrangian Floer cohomology will show that after tensoring with $\mathbb{Z}_2((\theta))$, the E^∞ page of the spectral sequence is isomorphic to $(\widetilde{HFK}(\mathcal{D}) \otimes V^{\otimes(n-1)}) \otimes \mathbb{Z}_2((\theta))$.

Let us now explain how Corollaries 1.1.3 and 1.1.4 follow from Theorem 1.1.1. The spectral sequence arising from the double complex above clearly splits along Alexander gradings and pairs of conjugate spin^c structures, since ∂ and τ preserve both. Our spectral sequence is not identical to the one described above; in particular, it may induce a different action τ^* on the complex $\widetilde{HFL}(\widetilde{\mathcal{D}})$. However, we will see that the splitting of the sequence is preserved for geometric reasons. Therefore all spin^c structures which are not their own conjugate vanish precisely at the E^2 page of the spectral sequence.

Moreover, the application of the localization maps of Theorem 1.4.1 in the case of Heegaard Floer homology will be defined by counting holomorphic disks in an appropriate symmetric product punctured along the divisors V_{w_i}, V_{z_i} and by multiplications and divisions by θ. Therefore the Alexander grading on $\widetilde{HFK}(\widetilde{\mathcal{D}})$ is preserved by the isomorphism of Theorem 1.1.1 [40, Section 2c]. Ergo it is interesting to consider not only the full spectral sequence but also its restriction to $\widehat{CFK}(\Sigma(K), K, \mathfrak{s}_0, \mathbf{A})$ for any Alexander grading $\mathbf{A}$.

In order to apply 1.4.1 to the case of (S^3, L) and its double branched cover $(\Sigma(L), L)$ we will require a Heegaard diagram D for (S^3, L) lying on the sphere S^2. Choose a bridge presentation of L in S^2; that is, a diagram of L in $S^2 = \mathbb{R}^2 \cup \{\infty\}$ such that there are a finite number of line segments $b_1, \cdots, b_n$ in the image of L in the plane with the property that at every crossing in L the overcrossing arc is a portion of the b_i and neither of the undercrossing arcs are. Distribute basepoints $\mathbf{w} = (w_1, \cdots, w_n)$ and $\mathbf{z} = (z_1, \cdots, z_n)$ along the image of L in the bridge presentation at the endpoints of the line segments b_i such that as one moves along the component K_i of L of L starting with $w_{n_{i-1}+1}$ in the direction of the orientation, beginning with an arc which is not one of the b_i, these basepoints are encountered in the order $z_{n_{i-1}+1}, w_{n_{i-1}+1}, \cdots, z_{n_i}, w_{n_i}$. For $1 \leq i \leq n-1$, let α_i be a closed curve in the plane encircling the arc of L which contains none of the bridges b_j and has endpoints z_i and w_i; let β_i be a closed curve in the plane encircling whichever bridge b_j has an endpoint w_i. Both sets of curves will be oriented counterclockwise with respect to their interiors in the plane $S^2 \backslash \{z_n\}$.

We deduce a rank inequality between $\widetilde{HFK}(\Sigma(L), L, \mathfrak{s}_0)$ and $\widetilde{HFK}(S^3, L)$. But since each of these Heegaard diagrams contains n pairs of basepoints, we obtain the rank inequality in Corollary 1.1.2. Moreover, our previous remarks concerning the splitting of the spectral sequence along spin^c structures and Alexander gradings will then imply Corollaries 1.1.3 and 1.1.4.

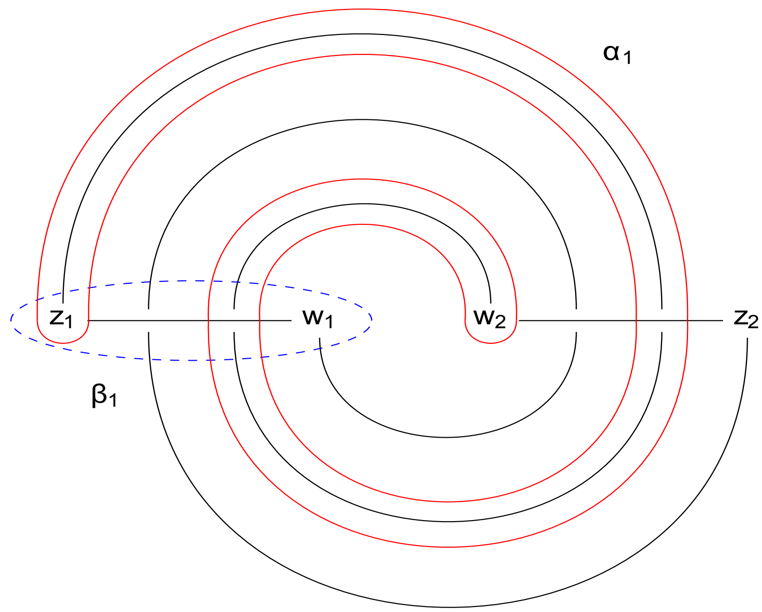


Figure 2: A Heegaard diagram on the sphere derived from a two-bridge presentation of the trefoil.

3.2 Heegaard Diagrams for Periodic Knots and Proofs of Edmonds' and Murasugi's Conditions

Let $\widetilde{K} \subset S^3$ be an oriented q-periodic knot and K its quotient knot. We will begin by constructing a Heegaard diagram for $(S^3, \widetilde{K} \cup U)$ which is preserved by the action of $\mathbb{Z}_q$ on $(S^3, \widetilde{K} \cup U)$ and whose quotient under this action is a Heegaard diagram for $(S^3, K \cup U)$.

3.2.1 Equivariant diagrams and Murasugi's Condition

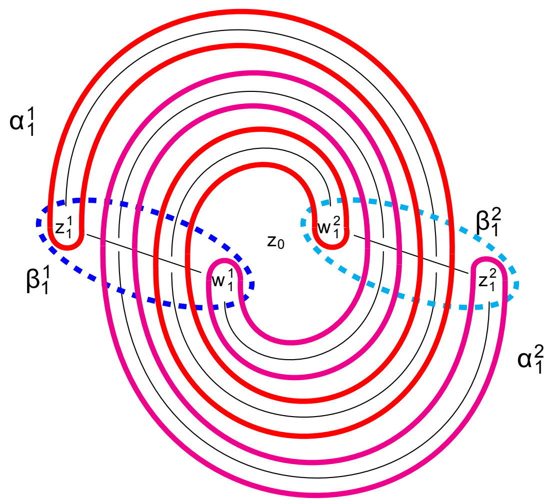
As in Section 3.1, we work with Heegaaard diagrams for $(S^3, K \cup U)$ on the sphere S^2. Regard S^3 as $\mathbb{R}^3 \cup \{\infty\}$ and arrange $\widetilde{K}$ such that the unknotted axis of periodicity U is the z-axis together with the point at infinity. Then the projection of $\widetilde{K}$ to the xy-plane together with the point at infinity is a periodic diagram $\widetilde{E}$ for $\widetilde{K}$. Taking the quotient of $(S^3, \widetilde{K})$ by the action of $\mathbb{Z}_q$ and similarly projecting to the xy-plane together with the point at infinity produces a quotient diagram E for K.

Construct a Heegaard diagram for $K \cup U$ as follows: Begin with the diagram E on $S^2 = \mathbb{R}^2 \cup \{\infty\}$. Place a basepoint w_0 at ∞ and z_0 at 0; these will be the sole basepoints on U. (This is a slight departure from the notation of Chapter 2; it will be more convenient to have the indexing start at w_0 rather than w_1 for the diagrams we construct.) Arrange basepoints $z_1, w_1, \cdots, z_{n_1}, w_{n_1}$ on K such that traversing K in the chosen orientation, one passes through the basepoints in that order. Moreover, we insist that while travelling from z_i to w_i one passes only through undercrossings and travelling from w_i to z_{i+1} or from w_{n_1} to z_1 one passes only through overcrossings. In other words, we choose basepoints so as to make E into a bridge diagram for K. Notice that n_1 is at most the number of crossings on the diagram E, or half the number of crossings on $\widetilde{E}$. Encircle the portion of the knot running from z_i to w_i with a curve α_i, oriented counterclockwise in the complement of w_0. Similarly, encircle the portion of the knot running from w_i to z_{i+1} (or from w_{n_1} to z_1) with a curve β_i, oriented counterclockwise in the complement of w_0. Notice that both α_i and β_i run counterclockwise around w_i, and moreover for each i, $S^2 \backslash \{\alpha_i, \beta_i\}$ has four components: one each containing z_i, w_i, and z_{i+1}, and one containing all other basepoints. This yields a Heegaard diagram $\mathcal{D} = (S^2, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ for $(S^3, K \cup U)$.

We may now take the branched double cover of $\mathcal{D}$ over z_0 and w_0 to produce a Heegaard diagram $\widetilde{\mathcal{D}}$ for $(S^3, \widetilde{K} \cup U)$ compatible with $\widetilde{E}$. This diagram has basepoints w_0 and z_0 for U and basepoints $z_1^1, w_1^1, \cdots, z_{n_1}^1, w_{n_1}^1, z_1^2, \cdots, w_{n_1}^2, \cdots, z_1^q, \cdots, w_{n_1}^q$ arranged in that order along the oriented knot $\widetilde{K}$. Each adjacent pair z_i^a and w_i^a is encircled by α_i^a a lift of α_i, and each adjacent pair w_i^a and z_{i+1}^a is encircled by β_i^a a lift of β_i. (Pairs $w_{n_1}^a$ and z_1^{a+1}, as well as $w_{n_1}^q$ and z_1^1, are encircled by lifts $\beta_{n_1}^a$ of β_{n_1}.) This yields a diagram $\widetilde{\mathcal{D}} = (S^2, \widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\beta}}, \widetilde{\mathbf{w}}, \widetilde{\mathbf{z}})$ with qn_1 each of α and β curves and $qn_1 + 1$ pairs of basepoints.

Our next goal will be to investigate the behavior of the relative Maslov and (particularly) Alexander gradings of generators of $\widehat{CFK}(\mathcal{D})$ and $\widehat{CFK}(\widetilde{\mathcal{D}})$. We begin with two relatively simple lemmas. As before, let τ be the involution on $(S^3, \widetilde{K})$ (and on $\widetilde{\mathcal{D}}$). Let $\tau^{\#}$ be the induced involution on $\widehat{CFK}(\widetilde{\mathcal{D}})$.

Figure 3: An equivariant Heegaard diagram $\widetilde{\mathcal{D}}$ for the trefoil together with the unknotted axis, and its quotient Heegaard diagram $\mathcal{D}$ for the Hopf link.



Lemma 3.2.1. *The induced map $\tau^{\#}$ preserves Alexander and Maslov gradings.*

Proof. Let s $\in \widehat{CFK}(\widetilde{\mathcal{D}})$. Choose a generator $\mathbf{x} \in \widehat{CFK}(\mathcal{D})$, and let $\widetilde{\mathbf{x}} = \pi^{-1}(x)$, such that $\widetilde{\mathbf{x}}$ is a generator in $\widehat{CFK}(\widetilde{\mathcal{D}})$ which is invariant under $\tau^{\#}$. Choose a Whitney disk ϕ in $\pi_2(\mathbf{s}, \widetilde{\mathbf{x}})$ and let D be its shadow on $\mathcal{D}$. Then $\tau \circ \phi$ is a Whitney disk in $\pi_2(\tau(\mathbf{s}), \widetilde{\mathbf{x}})$ with shadow $\tau(D)$. Furthermore, since w_0 and z_0 are fixed by the involution, $n_{z_0}(D) = n_{z_0}(D)$ and $n_{w_0}(D) = n_{w_0}(\tau(D))$, whereas since the remaining basepoints are interchanged by the involution, we have $\sum_{i=0}^{n_1} n_{z_i}(D) = \sum_{i=0}^{n_1} n_{z_i}(\tau(D))$ and $\sum_{i=0}^{n_1} n_{w_i}(D) = \sum_{i=0}^{n_1} n_{w_i}(\tau(D))$. These equalities imply that $M(\widetilde{\mathbf{x}}) - M(\mathbf{s}) = M(\widetilde{\mathbf{x}}) - M(\tau^{\#}(\mathbf{s}))$, and similarly for A_1 and A_2. Therefore s and $\tau^{\#}$s are in identical gradings.

□

Lemma 3.2.2. *Let $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ be a Whitney disk between generators* $\mathbf{x}$ *and* $\mathbf{y}$ *in* $\widehat{CFK}(\mathcal{D})$ *with Maslov index $\mu(\phi)$, with shadow the domain D on $\mathcal{D}$. There is a Whitney disk $\widetilde{\phi} \in \pi_2(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})$ with shadow the domain $\pi^{-1}(D)$ on $\widetilde{\mathcal{D}}$, and $\mu(\widetilde{\phi}) = q\mu(\phi) - (q-1)(n_{z_0}(\phi) + n_{w_0}(\phi))$.*

Proof. The boundary of the lift $\pi^{-1}(D)$ is trivial as a one cycle in S^3, implying that $\pi^{-1}(D)$ is the shadow of a Whitney disk $\widetilde{\phi} \in \pi_2(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})$. We will compare the Maslov index of ϕ with the

Maslov index of $\widetilde{\phi}$ using the formula 2.0.1. As in that formula, we will write D as a sum of the closures of the components of $S^2 - \boldsymbol{\alpha} - \boldsymbol{\beta}$. Say there are m such components, and label them as follows. There are two domains in $S^2 - \boldsymbol{\alpha} - \boldsymbol{\beta}$ which contain a branch point. Let these be D_1 containing z_0 and D_2 containing w_0. Let the shadow of ϕ be $a_1 D_1 + a_2 D_2 + \sum_{i=3}^m a_i D_i$. Then the Maslov index of ϕ is

$$\mu(\phi) = \sum_i a_i e(D_i) + p_{\mathbf{x}}(D) + p_{\mathbf{y}}(D)$$

Let us now consider applying the same formula to $\pi^{-1}(D) = \sum_i a_i \pi^{-1}(D_i)$. For $i \geq 3$, the lift $\pi^{-1}(D_j)$ of D_j consists of q copies of D_i, and by additivity of the Euler measure we see that $e(\widetilde{D}_i) = qe(D_i)$. For $i = 1, 2$, let $2k_i$ be the number of corners of D_i. Then $\pi^{-1}(D_i) = \widetilde{D}_i$ is a single component of $S^2 - \widetilde{\boldsymbol{\alpha}} - \widetilde{\boldsymbol{\beta}}$ with $2qk_i$ corners and Euler measure $e(\widetilde{D}_i) = 1 - \frac{qk_i}{2} = q(e(D_i)) - (q-1)$. Notice, furthermore, that $p_{\widetilde{\mathbf{x}}}(\widetilde{D}) = qp_{\mathbf{x}}(D)$ and $p_{\widetilde{\mathbf{y}}}(\widetilde{D}) = qp_{\mathbf{y}}(D)$. Therefore we compute

$$\begin{aligned}
\mu(\widetilde{\phi}) &= \sum_i a_i e(\pi^{-1}(D_i)) + p_{\widetilde{\mathbf{x}}}(\widetilde{D}) + p_{\widetilde{\mathbf{y}}}(\widetilde{D}) \\
&= a_1 e(\widetilde{D}_1) + a_2 e(\widetilde{D}_2) + q \sum_{i=3}^m e(D_i) + qp_{\mathbf{x}}(D) + qp_{\mathbf{y}}(D) \\
&= a_1(qe(D_1) - (q-1)) + a_2(qe(D_2) - (q-1)) + q \sum_{i=3}^m e(D_i) + qp_{\mathbf{x}}(D) + qp_{\mathbf{y}}(D) \\
&= q\mu(\phi) - (q-1)(k_1 + k_2).
\end{aligned}$$

Since $k_1 + k_2$ is exactly $n_{z_0}(D) + n_{w_0}(D)$ the total algebraic intersection of ϕ with the branch points, this proves the result. □

We can now construct the relationship between the Alexander gradings of the generators of $\widehat{CFL}(\widetilde{\mathcal{D}})$ and $\widehat{CFL}(\mathcal{D})$. For the case of a q-periodic knot, we will look only at the relative gradings; later in the particular case of a doubly-periodic knot we will fix the absolute gradings using symmetries of link Floer homology. Let $\pi : \widetilde{\mathcal{D}} \to \mathcal{D}$ be the restriction of the branched covering map $(S^3, \widetilde{K} \cup U) \to (S^3, K \cup U)$. We have the following, which is analogous to [19, Lemma 3.1].

Lemma 3.2.3. *Let* $\mathbf{s} \in \widehat{CFL}(\widetilde{\mathcal{D}})$*, thought of as an* qn_1*-tuple of points on* S^2 *with one point on each* α_i^k *and one on each* β_i^k*. Consider its projection* $\pi(\mathbf{s})$ *to* qn_1 *points on* $\mathcal{D}$*. There is a (not at all canonical) way to write* $\pi(\mathbf{s})$ *as* $\mathbf{s_1} \cup \mathbf{s_2} \cup \cdots \cup \mathbf{s_q}$ *a union of generators in* $\widehat{CFL}(\mathcal{D})$*.*

The proof of this lemma (pointed out by Adam Levine) is an application of the following combinatorial result of Hall [15]. Let A be a set, and $\{A_i\}_{i=1}^m$ be a collection of finite subsets (a version also exists for infinitely many A_i). A *system of distinct representatives* is a choice of elements $j_i \in A_i$ for each i such that $j_{i_1} \neq j_{i_2}$ if $i_1 = i_2$. Hall's theorem gives conditions under which a system of distinct representatives exists.

Theorem 3.2.4. *[15, Theorem 1] Let* $\{A_i\}_{i=1}^m$ *be finitely many subsets of a set* A*. Then a system of distinct representatives exists if and only if, for any* $1 \leq s \leq m$ *and* $1 < i_1 < ... < i_s < m$*,* $A_{i_1} \cup ... \cup A_{i_s}$ *contains at least* s *elements.*

Using this, we may prove Lemma 3.2.3

Proof of Lemma 3.2.3. Let $A = \{1, ..., q\}$. For $1 \leq i \leq n_1$, let A_i be a set of integers j, with $1 \leq j \leq n_1$, such that each j appears once in A_i for every point of $\pi(\mathbf{s}) \cap (\alpha_i \cap \beta_j)$. That is, the sets A_i record how many intersection points on α_i also lie on β_j. Notice that there are q elements in each A_i, and each j appears exactly q times in $\coprod_i A_i$. We claim the sets A_i satisfy the condition of Hall's theorem. For given $1 \leq i_1 \leq ... \leq i_m \leq n_1$, the disjoint union $\coprod_{k=1}^m A_{i_k}$ contains qm elements, and therefore must contain at least m different integers j. Therefore $\bigcup_{k=1}^m A_{i_k}$ contains at least m elements. Hence we can choose a set of distinct representatives j_i in A_i. There is a generator $\mathbf{s_1}$ consisting of points in $\pi(\mathbf{s})$ on $\alpha_i \cap \beta_{j_i}$. Remove these points from $\pi(\mathbf{s})$ (and the individual j_i from the sets A_i, producing new sets A_i') and repeat the argument, now with $q-1$ elements in each A_i' and $q-1$ appearances of each symbol j in $\coprod_i A_i'$. After q repetetions, we have broken $\mathbf{s}$ into $\mathbf{s_1} \cup ... \cup \mathbf{s_q}$, where each $\mathbf{s_i}$ is a generator for $\widehat{CFL}(\mathcal{D})$. This choice of partition is not at all unique. □

We can start by determining the relative Alexander gradings of generators of $\widehat{CFL}(\widetilde{\mathcal{D}})$ which are invariant under the action of $\mathbb{Z}_q$ on $\widetilde{\mathcal{D}}$; that is, exactly those generators which are total lifts of generators in $\widehat{CFL}(\mathcal{D})$ under the projection map π.

Lemma 3.2.5. *Let* $\mathbf{x}, \mathbf{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ *and* $\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}}$ *be their total lifts in* $\mathbb{T}_{\widetilde{\boldsymbol{\alpha}}} \cap \mathbb{T}_{\widetilde{\boldsymbol{\beta}}}$. *Then*

$$A_1(\widetilde{\mathbf{y}}) - A_1(\widetilde{\mathbf{x}}) = q\left(A_1(\mathbf{y}) - A_1(\mathbf{x})\right)$$
$$A_2(\widetilde{\mathbf{y}}) - A_2(\widetilde{\mathbf{x}}) = A_2(\mathbf{y}) - A_2(\mathbf{x}).$$

Proof. Let D be a domain from $\mathbf{x}$ to $\mathbf{y}$ on $\mathcal{D}$. Then $\pi^{-1}(D) = \widetilde{D}$ is a domain from $\mathbf{x}$ to $\mathbf{y}$ on $\widetilde{\mathcal{D}}$. Since z_0 and w_0 are branch points of π, we see that $n_{z_0}(\widetilde{D}) = n_{z_0}(D)$ and $n_{w_0}(\widetilde{D}) = n_{w_0}(D)$. Therefore

$$\begin{aligned} A_2(\widetilde{\mathbf{x}}) - A_2(\widetilde{\mathbf{y}}) &= n_{z_0}(\widetilde{D}) - n_{w_0}(\widetilde{D}) \\ &= n_{z_0}(D) - n_{w_0}(D) \\ &= A_2(\mathbf{x}) - A_2(\mathbf{y}) \end{aligned}$$

However, for $i \neq 0$, each of z_i and w_i basepoints on $\mathcal{D}$ has q preimages in $\widetilde{\mathcal{D}}$. Moreover $n_{z_i^j}(\widetilde{D}) = n_{z_i}(D)$ for all $1 \leq i \leq n_1$ and $1 \leq j \leq q$, and similarly for w_i^j, so we compute as follows.

$$\begin{aligned} A_1(\widetilde{\mathbf{x}}) - A_1(\widetilde{\mathbf{y}}) &= \sum_{i=1}^{n_1}\sum_{j=1}^{q} n_{z_i^j}(\widetilde{D}) - \sum_{i=1}^{n_1}\sum_{j=1}^{q} n_{w_i^j}(\widetilde{D}) \\ &= q\left(\sum_{i=1}^{n_1} n_{z_i}(D) - \sum_{i=1}^{n_1} n_{w_i}(D)\right) \\ &= q\left(A_1(\mathbf{x}) - A_1(\mathbf{y})\right) \end{aligned}$$

□

Finally, we can use this fact to compute the relative Alexander gradings in $\widehat{CFL}(\widetilde{\mathcal{D}})$.

Lemma 3.2.6. *Let* $\mathbf{s}, \mathbf{r}$ *be generators of* $\widehat{CFL}(\widetilde{\mathcal{D}})$, *whose projection to* $\mathcal{D}$ *can be written as* $\pi_*(\mathbf{s}) = \mathbf{s_1} \cup \cdots \cup \mathbf{s_q}$ *and* $\mathbf{r} = \mathbf{r_1} \cup \cdots \cup \mathbf{r_q}$. *Then the relative Alexander gradings between* $\mathbf{s}$ *and* $\mathbf{r}$ *is described by*

$$A_1(\mathbf{s}) - A_1(\mathbf{r}) = \sum_{j=1}^{q}\left(A_1(\mathbf{s_j}) - A_1(\mathbf{r_j})\right)$$
$$A_2(\mathbf{s}) - A_2(\mathbf{r}) = \frac{1}{q}\left(\sum_{j=1}^{q}\left(A_1(\mathbf{s_j}) - A_1(\mathbf{r_j})\right)\right).$$

Proof. The proof is quite similar to the first half of the argument of [19, Proposition 3.4]. As in Chapter 2, choose paths $a : [0,1] \to \mathbb{T}_{\widetilde{\boldsymbol{\alpha}}}$, $b : [0,1] \to \mathbb{T}_{\widetilde{\boldsymbol{\beta}}}$, with $\partial a = \partial b = \mathbf{s} - \mathbf{r}$. Then let $\epsilon(\mathbf{s},\mathbf{r}) = a - b$ be a one-cycle in $H_1(S^3 - (\widetilde{K} \cup U); \mathbb{Z})$. Consider the projection $\pi_*(\epsilon(\mathbf{s},\mathbf{r}))$ to $\mathcal{D}$. The restriction of this one-cycle to any α or β curve consists of q possibly overlapping arcs. By adding copies of the α or β circle if necessary, we may arrange that these arcs connect a point in $\mathbf{s_j}$ to a point in $\mathbf{r_j}$. That is, modulo α and β curves, which have linking number zero with the knot, $\pi_*\epsilon(\mathbf{s},\mathbf{r}) \equiv \epsilon(\mathbf{r_1},\mathbf{s_1}) \cdots \epsilon(\mathbf{r_q},\mathbf{s_q})$. Notice also that $\ell k\left(\pi_*\epsilon(\mathbf{s},\mathbf{r}), K\right) = \ell k\left(\epsilon(\mathbf{s},\mathbf{r}), \widetilde{K}\right)$, whereas $\ell k\left(\pi_*\epsilon(\mathbf{s},\mathbf{r}), U\right) = q\ell k\left(\epsilon(\mathbf{s},\mathbf{r}), U\right)$. Therefore we compute

$$\begin{aligned} A_1(\mathbf{r}) - A_1(\mathbf{s}) &= \ell k\left(\epsilon(\mathbf{s},\mathbf{r}), \widetilde{K}\right) \\ &= \ell k(\pi_*(\epsilon(\mathbf{s},\mathbf{r})), K) \\ &= \sum_{j=1}^{q} \ell k\left(\epsilon\left(\mathbf{s_j},\mathbf{r_j}\right), K\right) \\ &= \sum A_1(\mathbf{r}) - A_1(\mathbf{s}) \end{aligned}$$

and moreover

$$\begin{aligned} A_2(\mathbf{r}) - A_2(\mathbf{s}) &= \ell k\left(\epsilon(\mathbf{s},\mathbf{r}), U\right) \\ &= \frac{1}{q}\ell k\left(\pi_*(\epsilon(\mathbf{s},\mathbf{r})), U\right) \\ &= \frac{1}{q}\left(\sum_{j=1}^{q} (\ell k(\epsilon(\mathbf{s_j},\mathbf{r_j}), U)\right) \\ &= \frac{1}{q}\left(\sum_{j=1}^{q} (A_1(\mathbf{r_j}) - A_1(\mathbf{s_j}))\right) \end{aligned}$$

□

We may now give a proof of Theorem 1.2.1.

Proof of Theorem 1.2.1. Let $\widetilde{K}$ be a periodic knot with period $q = p^r$ for some prime p, and K be its quotient knot, and let $\lambda = \ell k(\widetilde{K}, U) = \ell k(K, U)$. Choose a periodic diagram $\widetilde{D}$ for $(S^3, \widetilde{K} \cup U)$ and its quotient diagram D for $(S^3, K \cup U)$ as outlined above.

Consider the Euler characteristic of $\widehat{CFL}(\widetilde{\mathcal{D}})$ computed modulo p. Let s be a generator in $\widehat{CFL}(\widetilde{\mathcal{D}})$. Either $\mathbf{s} = \widetilde{\mathbf{y}} = \pi^{-1}(\mathbf{y})$ for some $\mathbf{y} \in \widehat{CFL}(\mathcal{D})$, and thus s is invariant under the action of τ, or the order of the orbit of s under the action of τ is a multiple of p. Since the action preserves the Alexander and Maslov gradings, modulo p the terms of the Euler characteristic of $\widehat{CFL}(\widetilde{\mathcal{D}})$ corresponding to noninvariant generators sum to zero. Moreover, there is a one-to-one correspondence between generators $\mathbf{y}$ of $\widehat{CFL}(\mathcal{D})$ and their total lifts $\pi^{-1}(\mathbf{y}) = \mathbf{s}$ in $\widehat{CFL}(\widetilde{\mathcal{D}})$.

This correspondence preserves relative Alexander A_2-gradings, and multiplies Alexander A_1-gradings by a factor of q. We also claim that it preserves the parity of relative Maslov gradings if p is odd. In particular, any two generators $\mathbf{x}$ and $\mathbf{y}$ of $\widehat{CFL(\mathcal{D})}$ are joined by a domain D which does not pass over w_0. Let $\widetilde{D}$ be the lift of this domain. Then the Maslov index $\mu(\widetilde{D})$ is equal to $q\mu(D) - (q-1)n_{z_0}(D)$ by Lemma 3.2.2, and we have

$$\begin{aligned} M(\widetilde{\mathbf{x}}) - M(\widetilde{\mathbf{y}}) &= \mu(\widetilde{D}) - 2\sum_{i=1}^{n} \Big(n_{w_i^1}(\widetilde{D}) + \cdots + n_{w_i^q}(\widetilde{D})\Big) \\ &= q\mu(D) - (q-1)n_{z_0}(D) - 2q\sum_{i=1}^{n} n_{w_i}(D) \\ &\equiv \mu(D) - 2\sum_{i=1}^{n} n_{w_i}(D) \bmod 2 \\ &\equiv M(\mathbf{x}) - M(\mathbf{y}) \bmod 2 \end{aligned}$$

Here we have used the assumption that p, and therefore q, is odd, and $q-1$ is even. Ergo if p is odd, $(-1)^{M(\widetilde{\mathbf{x}})} = \pm(-1)^{M(\mathbf{x})}$ where the choice of sign is the same for all generators $\mathbf{x}$ of $\widehat{CFL}(\mathcal{D})$. (If $p = 2$ the sign is of course immaterial.)

Therefore, we capture the following equality.

$$\chi(\widehat{CFL}(\widetilde{\mathcal{D}}))(t_1, t_2) \doteq \chi(\widehat{CFL}(\widetilde{\mathcal{D}}))(t_1^q, t_2) \bmod p$$

Here $\doteq$ denotes equivalence up to an overall factor of $\pm t_1^{i_1} t_2^{i_2}$.

However, $\chi(\widehat{CFL}(\widetilde{\mathcal{D}}))$ is $\chi(\widetilde{HFL}(\widetilde{\mathcal{D}}))$ and similarly $\chi(\widehat{CFL}(\mathcal{D}))$ is $\chi(\widetilde{HFL}(\mathcal{D}))$. Hence we may express the Euler characteristics of the chain complexes as the multivariable Alexander polynomials of $\widetilde{L} = \widetilde{K} \cup U$ and $L = K \cup U$ multiplied by appropriate powers of $(1 - t_1^{-1})$ and $(1 - t_2^{-1})$ according to the number of basepoints on each component of each link. Therefore the equality above reduces to

$$\Delta_{\widetilde{L}}(t_1,t_2)(1-t_1^{-1})^{qn}(1-t_2^{-1}) \doteq \Delta_L(t_1^q,t_2)(1-(t_1^{-1})^q)^n(1-t_2^{-1}) \bmod p$$
$$\Delta_{\widetilde{L}}(t_1,t_2)(1-t_1^{-1})^{qn} \doteq \Delta_L(t_1^q,t_2)(1-(t_1^{-1})^q)^n \bmod p.$$

Recalling that q is a power of p, and that therefore $(a+b)^q \equiv a^q + b^q \bmod p$, we may reduce farther.

$$\Delta_{\widetilde{L}}(t_1,t_2)(1-t_1^{-1})^{qn} \doteq \Delta_L(t_1^q,t_2)(1-t_1^{-1})^{qn} \bmod p$$
$$\Delta_{\widetilde{L}}(t_1,t_2) \doteq \Delta_L(t_1^q,t_2) \bmod p$$

We now set $t_2 = 1$. By Lemma 2.1.1, this reduces the equality above to

$$\Delta_{\widetilde{K}}(t_1)(1+t_1+\cdots+t_1^{\lambda-1}) \doteq \Delta_K(t_1^q)(1+t_1^q+(t_1^q)^2+\cdots+((t_1)^q)^{\lambda-1} \bmod p.$$

Again using the fact that q is a power of p, we produce

$$\Delta_{\widetilde{K}}(t_1)(1+t_1+\cdots+t_1^{\lambda-1}) \doteq (\Delta_K(t_1))^q(1+t_1+t_1^2+\cdots+t_1^{(\lambda-1)})^q \bmod p$$
$$\Delta_{\widetilde{K}}(t_1) \doteq (\Delta_K(t_1))^q(1+t+\cdots+t^{\lambda-1})^{q-1} \bmod p.$$

This last is Murasugi's condition. □

3.2.2 Spectral sequences for doubly-periodic knots and Edmonds' Condition

From now on we restrict ourselves entirely to the case of $\widetilde{K}$ a doubly-periodic knot. Moreover, we insist that $\widetilde{K}$ be oriented such that $\ell k(\widetilde{K}, U) = \lambda$ is positive. Notice also that λ is necessarily odd; otherwise $\tilde{K}$ would be disconnected. We proceed to explain how Corollary 1.2.4 follows from Theorem 1.2.3. Consider the map $\tau^{\#}\colon \widehat{CFL}(\widetilde{\mathcal{D}}) \to \widehat{CFL}(\widetilde{\mathcal{D}})$ induced by the involution τ on $\widetilde{\mathcal{D}}$. In principle, we would like to consider the spectral sequence induced by a double complex of the following form.

$$
\begin{array}{ccccccccc}
 & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \\
0 & \longrightarrow & \widehat{CFL}_{i+1}(\widetilde{\mathcal{D}}) & \xrightarrow{1+\tau^{\#}} & \widehat{CFL}_{i+1}(\widetilde{\mathcal{D}}) & \xrightarrow{1+\tau^{\#}} & \widehat{CFL}_{i+1}(\widetilde{\mathcal{D}}) & \xrightarrow{1+\tau^{\#}} & \\
 & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \\
0 & \longrightarrow & \widehat{CFL}_{i}(\widetilde{\mathcal{D}}) & \xrightarrow{1+\tau^{\#}} & \widehat{CFL}_{i}(\widetilde{\mathcal{D}}) & \xrightarrow{1+\tau^{\#}} & \widehat{CFL}_{i}(\widetilde{\mathcal{D}}) & \xrightarrow{1+\tau^{\#}} & \\
 & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \\
0 & \longrightarrow & \widehat{CFL}_{i-1}(\widetilde{\mathcal{D}}) & \xrightarrow{1+\tau^{\#}} & \widehat{CFL}_{i-1}(\widetilde{\mathcal{D}}) & \xrightarrow{1+\tau^{\#}} & \widehat{CFL}_{i-1}(\widetilde{\mathcal{D}}) & \xrightarrow{1+\tau^{\#}} & \\
 & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial & &
\end{array}
$$

However, as in Section 3.1, we will need to replace $\left(\widehat{CFL}(\mathcal{D})\partial\right)$ with a chain homotopy equivalent complex with the same invariant generators but possibly different noninvariant generators, and $\tau^{\#}$ with a new involution $(\tau^{\#})'$, which has the same fixed set but is not necessarily chain homotopy equivalent. This then is the spectral sequence of 1.2.2. We will see that the map $(\tau^{\#})'$ still preserves Alexander gradings. Computing vertical differentials first, we find that the E^1 page of the spectral sequence is $\widetilde{HFL}(\widetilde{\mathcal{D}}) \otimes \mathbb{Z}_2[[\theta]]$. We will see that after tensoring with θ^{-1}, the E^{∞} page is $\mathbb{Z}_2((\theta))$-isomorphic to $\widetilde{HFL}(\mathcal{D}) \otimes \mathbb{Z}_2((\theta))$ under a map which preserves both relative Alexander gradings. Therefore the spectral sequence splits along the relative grading (A_1, A_2).

The knot Floer homology spectral sequence of Theorem 1.2.3 arises from a similar double complex. Recall that for a link $L = K_1 \cup \cdots \cup K_\ell$, the differential ∂_{K_j} corresponds to forgetting the component K_j of the link. Therefore for the link $L = \widetilde{K} \cup U$, let ∂_U be the differential corresponding to forgetting the component U of L. Again, we would hope to consider a spectral sequence of the following form.

$$\begin{array}{ccccccccc}
 & & \downarrow \partial_U & & \downarrow \partial_U & & \downarrow \partial_U & & \\
0 & \longrightarrow & \widehat{CFL}_{i+1}(\widetilde{\mathcal{D}}) & \xrightarrow{1+\tau^{\#}} & \widehat{CFL}_{i+1}(\widetilde{\mathcal{D}}) & \xrightarrow{1+\tau^{\#}} & \widehat{CFL}_{i+1}(\widetilde{\mathcal{D}}) & \xrightarrow{1+\tau^{\#}} & \\
 & & \downarrow \partial_U & & \downarrow \partial_U & & \downarrow \partial_U & & \\
0 & \longrightarrow & \widehat{CFL}_{i}(\widetilde{\mathcal{D}}) & \xrightarrow{1+\tau^{\#}} & \widehat{CFL}_{i}(\widetilde{\mathcal{D}}) & \xrightarrow{1+\tau^{\#}} & \widehat{CFL}_{i}(\widetilde{\mathcal{D}}) & \xrightarrow{1+\tau^{\#}} & \\
 & & \downarrow \partial_U & & \downarrow \partial_U & & \downarrow \partial_U & & \\
0 & \longrightarrow & \widehat{CFL}_{i-1}(\widetilde{\mathcal{D}}) & \xrightarrow{1+\tau^{\#}} & \widehat{CFL}_{i-1}(\widetilde{\mathcal{D}}) & \xrightarrow{1+\tau^{\#}} & \widehat{CFL}_{i-1}(\widetilde{\mathcal{D}}) & \xrightarrow{1+\tau^{\#}} & \\
 & & \downarrow \partial_U & & \downarrow \partial_U & & \downarrow \partial_U & &
\end{array}$$

However, we will again need to replace $\left(\widehat{CFL}(\mathcal{D})\partial_U\right)$ with a chain homotopy equivalent complex with the same invariant generators but possibly different noninvariant generators, and $\tau^{\#}$ with a new involution $(\tau^{\#})''$, which has the same fixed set but is not necessarily chain homotopy equivalent. This then is the spectral sequence of 1.2.3. Once again, although $(\tau^{\#})''$ may not be chain homotopic to $\tau^{\#}$, it does preserve the Alexander grading A_1, as does ∂_U. Therefore the spectral sequence splits along (relative) Alexander gradings with respect to the knots $\widetilde{K}$ and K.

Let us consider how we might fix the relationship between the absolute Alexander gradings of $\widetilde{HFL}(\widetilde{\mathcal{D}})$ and $\widetilde{HFL}(\mathcal{D})$ in the link Floer spectral sequence. Let $x_L([K])$ be the Thurston seminorm of the class dual to $[K]$ in $H^2(S^3-\nu(L), \partial(S^3-\nu(L)))$ and $x_{\widetilde{L}}([\widetilde{K}])$ be the Thurston seminorm of the class dual to $[\widetilde{K}]$ in $H^2(S^3-\nu(\widetilde{L}), \partial(S^3-\nu(\widetilde{L})))$. Notice that if F is a Thurston-norm minimizing surface for the class dual to $[K]$ of Euler characteristic $\chi(F) = -x_L([K])$, then the preimage $\widetilde{F} = \pi^{-1}(F)$ under the ordinary double cover $\pi : S^3 - \nu(\widetilde{L}) \to S^3 - \nu(L)$ is an embedded surface representing the class dual to $[\widetilde{K}]$ in $H^2(S^3-\nu(\widetilde{L}), \partial(S^3-\nu(\widetilde{L})))$, and $\chi(\widetilde{F}) = 2\chi(F)$. Hence $x_{\widetilde{L}}([\widetilde{K}]) \leq 2x_L([K])$.

However, recall that $x_L([K]) + 1$ is exactly the breadth of the Alexander A_1 grading in $\widehat{HFL}(S^3, L)$, and therefore that the breadth of the A_1 grading in $\widetilde{HFL}(\mathcal{D})$ is $x_L([K])+1+(n_1-1) = x_L([K])+n_1$. Similarly, the breadth of the A_1 grading in $\widehat{HFL}(S^3, \widetilde{L})$ is $x_{\widetilde{L}}([K])+1$, and therefore the total breadth of the A_1 grading in $\widetilde{HFL}(\widetilde{\mathcal{D}})$ is $x_{\widetilde{L}}([K])+1+2n_1-1 = x_{\widetilde{L}}([\widetilde{K}])+2n_1$. Moreover, we have seen that in the spectral sequence from $E^1 = \widetilde{HFL}(\widetilde{\mathcal{D}}) \otimes \mathbb{Z}_2((\theta))$ to $E^\infty \cong \widetilde{HFL}(\mathcal{D}) \otimes \mathbb{Z}_2((\theta))$, the relative A_1 grading of two elements on the E^1 page is twice the relative A_1 grading of their residues on the E^∞ page. Therefore the total breadth of the A_1 grading on the E^1 page of the spectral sequence is at least twice the breadth of the A_1 grading on the last page of the spectral sequence. We thus have the inequality

$$\begin{aligned} x_{\widetilde{L}}([\widetilde{K}]) + 2n_1 &\geq 2x_L([K]) + 2n_1 \\ &\geq 2(x_L([K]) + n_1) \\ \text{i.e. } x_{\widetilde{L}}([\widetilde{K}]) &\geq 2x_L([K]). \end{aligned}$$

Consequently we see directly from the spectral sequence that

$$x_{\widetilde{L}}([\widetilde{K}]) = 2x_L([K]).$$

This implies that the breadth of the A_1 grading on the E^1 page is exactly twice the breadth of the A_1 grading on the E^∞ page. Therefore the breadth of the A_1 grading cannot decrease over the course of the spectral sequence. A similar argument, sans the factors of two, shows that $x_{\widetilde{L}}([U]) = x_L([U])$ and that the breadths of the Alexander A_2 grading of the E^1 and E^∞ pages of the spectral sequence are the same. Therefore the breadth of the A_2 grading does not change either over the course of the spectral sequence.

In particular, the top A_1 grading in $\widetilde{HFL}(\widetilde{\mathcal{D}})$ is sent to the top A_1 grading in $\widetilde{HFL}(\mathcal{D})$. However, by the symmetry of $\widehat{HFL}$ and the determination of the breadth of the A_1 grading by the Thurston norm, the top A_1 grading of $\widetilde{HFL}(\widetilde{\mathcal{D}})$ is the same as the top A_1 grading of $\widehat{HFL}(S^3, \widetilde{L})$, which is $A_1 = \frac{x_{\widetilde{L}}([\widetilde{K}])+1}{2} = \frac{2x_L(K)+1}{2}$. Similarly, the top A_1 grading of $\widetilde{HFL}(\mathcal{D})$ is the top A_1 grading of $\widehat{HFL}(S^3, L)$, to wit, $A_1 = \frac{x_L([K])+1}{2}$. Therefore the spectral sequence carries the A_1 Alexander grading $\frac{2x_L([K])+1}{2}$ on the E^1 page to the A_1 grading $\frac{x_L([K])+1}{2}$ on the E^∞ page. Therefore in general the A_1 grading $\frac{2x_L([K])+1}{2} + 2a$ on the E^1 page is sent to the A_1 grading $\frac{x_L([K])+1}{2} + a$ for any integer a. Notice that $x_L([K])$ is an even number: suppose F is a Thurston-seminorm minimizing surface for K in $S^3 - L$ of genus g' with geometric intersection number $\#(F \cap U) = \Lambda$. Since the algebraic intersection number λ is odd, so is Λ. Then $x_L([K]) = 1 - 2g' - \Lambda$ is even. Therefore $\frac{x_L([K])}{2}$ is an integer, and we can take $a = \frac{x_L([K])}{2}$ to see that the A_1 grading $\frac{1}{2}$ on the E^1 page is sent to the A_1 grading $\frac{1}{2}$ on the E^∞ page.

A parallel but simpler argument for the A_2 gradings shows that the A_2 grading b on the E^1 page is sent precisely to the A_2 grading b on the E^∞ page, using the fact that relative A_2 gradings of elements on the E^1 page that survive in the spectral sequence are preserved rather than doubled on the E^∞ page.

Remark 3.2.7. The observations that $x_{\widetilde{L}}([\widetilde{K}]) = 2x_L([K])$ and $x_{\widetilde{L}}([U]) = x_L([U])$ are a special case of Gabai's theorem [12, Corollary 6.13] that the Thurston norm is multiplicative for ordinary finite covers. Indeed, by appealing to Gabai's theorem (or by constructing a $\mathbb{Z}_p$ analog of Seidel and Smith's localization spectral sequence) we could similarly fix the relationship between the absolute gradings of $\widetilde{HFL}(\widetilde{\mathcal{D}})$ and $\widetilde{HFL}(\mathcal{D})$ for p-periodic knots.

Remark 3.2.8. The reader may wonder at the asymmetry in the construction that causes the A_1 gradings $\frac{1}{2}$ and $\frac{1}{2}$ to correspond, but not the gradings $-\frac{1}{2}$ and $-\frac{1}{2}$. This is the result of the convention that the two summands of the vector space V_1 have A_1 gradings 0 and -1.

This serves to illustrate the important role of the vector space V_1 in the existence of the spectral sequence; the breadth of the A_1 grading in $\widehat{HFL}(S^3, \widetilde{L})$ is one less than twice the breadth of the A_1 grading in $\widehat{HFL}(S^3, L)$, a trouble which is corrected for by increasing the breadth upstairs by $2n_1 - 1$ and the breadth downstairs by $n_1 - 1$. While one might hope to produce a spectral sequence in which $n_1 = 1$, it seems impossible to produce a link Floer homology spectral sequence for doubly periodic knots which does not involve at least one copy of V_1 on the E^1 page.

We summarize the discussion above in the following two lemmas.

Lemma 3.2.9. *The spectral sequence from $E^1 = \widetilde{HFL}(\widetilde{\mathcal{D}}) \otimes \mathbb{Z}_2((\theta))$ to $E^\infty \cong \widetilde{HFL}(\mathcal{D}) \otimes \mathbb{Z}_2((\theta))$ splits along Alexander gradings. For $a \in \mathbb{Z}, b \in \mathbb{Z} + \frac{1}{2}$, the sequences sends the gradings $(A_1, A_2) = (\frac{1}{2} + 2a, b)$ on the E^1 page to the gradings $(A_1, A_2) = (\frac{1}{2} + a, b)$ on the*

E^∞ page and kills all other gradings. In particular there is a rank inequality

$$rk\left(\widetilde{HFL}\left(\widetilde{\mathcal{D}},\left(\frac{1}{2}+2a,b\right)\right)\right)\geq rk\left(\widetilde{HFL}\left(\mathcal{D},\left(\frac{1}{2}+a,b\right)\right)\right).$$

By Lemma 2.0.9, computing the knot Floer homology complex using ∂_U yields a downward shift in Alexander gradings by $\frac{\ell k(\widetilde{K},U)}{2}$ on the E^1 page and $\frac{\ell k(K,U)}{2}$ on the E^∞ page. Since both of these numbers are $\frac{\lambda}{2}$, we obtain an overall downward shift of $\frac{\lambda}{2}$ between that link and knot Floer homology spectral sequences, leading to the following lemma.

Lemma 3.2.10. *The spectral sequence from $E^1 = \widetilde{HFK}(\widetilde{\mathcal{D}})\otimes\mathbb{Z}_2((\theta))$ to $E^\infty\cong\widetilde{HFK}(\mathcal{D})\otimes\mathbb{Z}_2((\theta))$ splits along the Alexander grading. For $a\in\mathbb{Z}$, the sequences sends the gradings $A_1=\frac{1-\lambda}{2}+2a$ on the E^1 page to the gradings $A_1=(\frac{1-\lambda}{2}+a)$ on the E^∞ page and kills all other gradings. In particular there is a rank inequality*

$$rk\left(\widetilde{HFK}\left(\widetilde{\mathcal{D}},2a+\frac{1-\lambda}{2}\right)\right)\geq rk\left(\widetilde{HFK}\left(\mathcal{D},a+\frac{1-\lambda}{2}\right)\right).$$

Having fixed the Alexander gradings in the spectral sequence, we may provide a proof of Corollary 1.2.4 (Edmonds' Condition) from 1.2.3.

Proof of Corollary 1.2.4. Once again, let V_1 denote a two-dimensional vector space over $\mathbb{F}_2$ whose two sets of gradings $(M,(A_1,A_2))$ are $(0,(0,0))$ and $(-1,(-1,0))$, and likewise let W be a two-dimensional vector space over $\mathbb{F}_2$ whose two sets of gradings $(M,(A_1,A_2))$ are $(0,(0,0))$ and $(-1,(0,0))$. By Theorem 1.2.3, there is a spectral sequence whose E^1 page is $\widehat{HFK}(S^3,\widetilde{K})\otimes V_1^{\otimes n_1-1}\otimes W\otimes\mathbb{Z}_2((\theta))$ to a theory whose E^∞ page is $\mathbb{Z}((\theta))$-isomorphic to $\widehat{HFK}(S^3,K)\otimes V_1^{\otimes 2n_1-1}\otimes W\otimes\mathbb{Z}_2((\theta))$. Moreover, this spectral sequence splits along the A_1 grading, and by Lemma 3.2.10 the subgroup of the E^1 page in A_1 grading $\frac{1-\lambda}{2}+2a$ is carried to the subgroup of the E^∞ page in grading $\frac{1-\lambda}{2}+a$. Moreover, the top A_1 grading on the E^1 page is $g(\widetilde{K})$ and the top A_1 grading on the E^∞ page is $g(K)$. Since there must be something on the E^1 page in the A_1 grading which converges to the A_1 grading $g(K)$ on the E^∞ page, we have the following inequality.

$$g(\widetilde{K})-\frac{1-\lambda}{2}\geq 2\left(g(K)-\frac{1-\lambda}{2}\right)$$

$$\text{i.e. } g(\widetilde{K})\geq 2g(K)+\frac{\lambda-1}{2}$$

□

Finally, let us prove Corollary 1.2.5.

Proof of Corollary 1.2.5. Suppose Edmonds' condition is sharp, that is, that $g(\widetilde{K}) = 2g(K) + \frac{\lambda-1}{2}$. Recall that the spectral sequence from $E^1 \cong \widetilde{\widehat{HFK}}(\widetilde{\mathcal{D}}) \otimes \mathbb{Z}_2((\theta))$ to $E^\infty \cong \widetilde{\widehat{HFK}}(\mathcal{D}) \otimes \mathbb{Z}_2((\theta))$ in general sends the Alexander grading $2a + \frac{1-\lambda}{2}$ to the Alexander grading $a + \frac{1-\lambda}{2}$. Therefore it sends the Alexander grading $g(\widetilde{K}) = 2g(K) + \frac{\lambda-1}{2} = 2\left(g(K) + \frac{\lambda-1}{2}\right) + \frac{1-\lambda}{2}$ to the Alexander grading $g(K)$. That is, sharpness of Edmonds' condition exactly says that the top Alexander grading on the E^1 page is not killed in the spectral sequence.

Suppose now that $\widetilde{K}$ is fibered. Then the top Alexander grading of $\widetilde{\widehat{HFK}}(\widetilde{\mathcal{D}}) \otimes \mathbb{Z}_2((\theta)) = (\widehat{HFK}(S^3, L) \otimes V_1^{\otimes(2n_1-1)} \otimes W) \otimes \mathbb{Z}_2((\theta))$ has rank two as a $\mathbb{Z}_2((\theta))$ module. (The knot Floer homology of fibered knots is monic in the top Alexander grading by the forward direction of Lemma 2.2.1 , and the factor of W doubles the number of entries in each Alexander grading.) Since this Alexander grading is not killed in the spectral sequence, the top Alexander grading of $\widetilde{\widehat{HFK}}(\mathcal{D}) \otimes \mathbb{Z}_2((\theta)) = \widehat{HFK}(S^3, L) \otimes V_1^{\otimes(n_1-1)} \otimes W) \otimes \mathbb{Z}_2((\theta))$ also has rank two as a $\mathbb{Z}_2((\theta))$-module. Therefore K is also fibered. □

Remark 3.2.11. The converse of Corollary 1.2.5, that when Edmonds' condition is sharp, the quotient knot K being fibered implies $\widetilde{K}$ is fibered, is false. Consider the following counterexample: the knot $\widetilde{K} = 10_{144}$ is doubly periodic with quotient knot $K = 3_1$, the trefoil. The linking number $\lambda = \ell k(\widetilde{K}, U) = 1$. Since $g(10_{144}) = 2$ and $g(3_1) = 1$, we see that $g(\widetilde{K}) = 2 = 2g(K) + \frac{\lambda-1}{2}$. Therefore Edmonds' condition is sharp. However, the trefoil is fibered, whereas 10_{144} is not.

3.3 Essential properties of Heegaard diagrams compatible with involutions

We pause for a moment to note important similarities between the Heegaard diagrams we have constructed for double branched covers of links and those we have constructed for doubly-periodic knots. That is, we consider all of the cases above involving an involution on a Heegaard

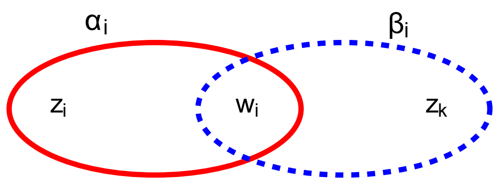


Figure 4: The arrangement of α and β curves $\mathcal{D}$. Note that if w_i is on the component K_j of L, z_k is either z_{i+1} or z_{n_j}.

diagram.

Definition 3.3.1. We say that a pair of multipointed Heegaard diagrams $(\mathcal{D}, \widetilde{\mathcal{D}})$ is *localizable* if

1. The diagram $\mathcal{D} = (S^2, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z}\}$ is a Heegaard diagram on the two-sphere for a link (S^3, L).

2. The diagram $\widetilde{\mathcal{D}} = (\Sigma(S^2), \widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\beta}}, \widetilde{\mathbf{w}}, \widetilde{\mathbf{z}})$ is the branched double cover of $\mathcal{D}$ over a subset A of $\{\mathbf{w}, \mathbf{z}\}$ consisting of m pairs of basepoints, such that $\Sigma(S^2)$ has genus $m-1$ and $\mathcal{D}$ carries $2n-m$ pairs of basepoints and $2n$ each of α and β curves.

3. If K_i is a component of L and any basepoint on K_i is in A, then all basepoints on K_i are in A.

4. On the diagram $\mathcal{D}$, the closed curves α_i and β_i are always arranged as shown in Figure 3.3. In particular, one component of $S^2 - \alpha_i - \beta_i$ is a bigon containing only the basepoint w_i. Note that Condition (4) implies that in this picture, either both z_i and z_k are branch points or neither is.

Notice that in Section 3.1, the basepoints on the diagram D we constructed were labelled $(w_1, ...w_n)$, with w_n the single w–basepoint not encircled by an α or β curve as in Figure . However, in Section 3.2 the w–basepoints are $(w_0, ..., w_{n_1})$, with w_0 the single basepoint not circled by an α and β curve as in the Figure, and $n_1 = n - 1$. For the sake of consistency in the chapters that follow, particularly Chapter 5, we will take indices modulo n; that is, we will let "w_0" indicate w_0 or w_n as appropriate.

Chapter 4

Spectral Sequences for Lagrangian Floer Cohomology

Floer cohomology is an invariant for Lagrangian submanifolds in a symplectic manifold introduced by Floer [9, 10, 11]. Many versions of the theory exist; in this section we briefly introduce Seidel and Smith's setting for Floer cohomology before stating the hypotheses and results of their main theorem for equivariant Floer cohomology. Let M be a manifold equipped with an exact symplectic form $\omega = d\theta$ and a compatible almost complex structure J. Let L_0 and L_1 be two exact Lagrangian submanifolds of M. For our purposes we can restrict to the case that L_0 and L_1 intersect transversely.

Definition 4.0.2. The Floer chain complex $CF(L_0, L_1)$ is an abelian group with generators the finite set of points $L_0 \cap L_1$.

The differential d on $CF(L_0, L_1)$ counts holomorphic disks whose boundary lies in $L_0 \cup L_1$ which run from x_- to x_+. More precisely, we choose $\mathbf{J} = J_t$ a time-dependent perturbation of

J and let $\mathcal{M}(x_-, x_+)$ be the moduli space of Floer trajectories u of the following form.

$$\begin{aligned} u\colon\ \mathbb{R}\times[0,1] &\to M \\ u(s,0)\in L_0, u(s,1) &\in L_1 \\ \partial_s u + J_t(u)\partial_t(u) &= 0 \\ \lim_{s\to\pm\infty} &= x_\pm \end{aligned}$$

This moduli space carries a natural action by $\mathbb{R}$ corresponding to translation on the coordinate s; we let the quotient by this action be $\widehat{\mathcal{M}}(x_-, x_+) = \frac{\mathcal{M}(x_-,x_+)}{\mathbb{R}}$ the set of unparametrized holomorphic curves from x_- to x_+.

Before formally defining the differential on $CF(L_0, L_1)$, we need to impose one further technical condition on M to ensure both that there are only finitely many equivalence classes of holomorphic curves between any two intersection points $x_+, x_- \in L_0 \cap L_1$ and that the image of any holomorphic curve $u\colon\ \mathbb{R}\times[0,1]\to M$ is contained in some compact set in M. We say $\phi\colon\ M\to\mathbb{R}$ is *exhausting* if it is smooth, proper, and bounded below. We consider the one-form $d^{\mathbb{C}}(\phi) = d\phi\circ J$ and the two-form $\omega_\phi = -dd^{\mathbb{C}}(\phi)$. We say that ϕ is *J-convex* or *plurisubharmonic* if ω_ϕ is compatible with the complex structure on M, that is, if $\omega_\phi(Jv, Jw) = \omega_\phi(v, w)$ and $\omega_\phi(v, Jv) > 0$ for all $v\in TM$. This ensures that ω_ϕ is a symplectic form on M. (The term plurisubharmonic indicates that the restriction of ϕ to any holomorphic curve in M is subharmonic, hence satisfies the maximum modulus principle.) A noncompact symplectic manifold M with this structure is called *convex at infinity*. If ω_ϕ is in fact the symplectic form ω on M, then M is said to be *strictly convex at infinity*.

We use an index condition to determine which strips u count for the differential. Given any $u\in\mathcal{M}(x_-, x_+)$, we can associate to u a Fredholm operator $D_{\mathbf{J}}u\colon\ \mathcal{W}^1_u\to\mathcal{W}^0_u$ from $\mathcal{W}^1_u = \{X\in W^{1,p}(u^*TM) : X(\cdot,0)\in u^*TL_0, X(\cdot,1)\in u^*TL_1\}$ to W^0_u. (Here $p>2$ is a fixed real number.) This operator describes the linearization of Floer's equation, $\partial_s u + J_t(u)\partial_t(u) = 0$, near u. We say that $\mathbf{J}$ is *regular* if $D_{\mathbf{J}}u$ is surjective for all finite energy holomorphic strips u.

Lemma 4.0.3. *If M is an exact symplectic manifold with a compatible almost convex structure J which is convex at infinity and L_0, L_1 are exact Lagrangian submanifolds also convex at infinity, then a generic choice of* $\mathbf{J}$ *perturbing J is regular.*

Floer's original proof of this result [9, Propn 2.1] and Oh's revision [26, Propn 3.2] were for compact manifolds, but, as observed in [23, Section 9.2] and indeed by Sikorav [41] in his review of Floer's paper [9], the proof carries through identically for noncompact manifolds which are convex at infinity. Choose such a generic regular $\mathbf{J}$. We let $\mathcal{M}_1(x_-, x_+)$ be the set of trajectories u in $\mathcal{M}(x_-, x_+)$ such that the Fredholm index of $D_{\mathbf{J}}u = 1$.

Lemma 4.0.4. *[9, Lemma 3.2] If* $\mathbf{J}$ *is regular, $\mathcal{M}_1(x_-, x_+)$ is a smooth, compact 1-manifold such that $\#\widehat{\mathcal{M}}_1(x_-, x_+) = n^{x_- x_+}$ is finite. Moreover, for any $x_-, x_+ \in CF(L_0, L_1)$, the sum*

$$\sum_{x \in CF(L_0, L_1)} n^{x_- x} n^{x x_+}$$

is zero modulo two.

Therefore we make the following definition.

Definition 4.0.5. [9, Defn 3.2] The Floer cohomology $HF(L_0, L_1)$ is the homology of the chain complex $CF(L_0, L_1)$ with respect to the differential

$$\delta(x_-) = \sum_{x_+ \in CF(L_0, L_1)} \#\widehat{\mathcal{M}_1}(x_-, x_+) x_+ \tag{4.0.1}$$

with respect to a regular family of almost complex structures $\mathbf{J}$ perturbing J.

Now suppose that M carries a symplectic involution τ preserving (M, L_0, L_1) and the forms ω and θ. Let the submanifold of M fixed by τ be M^{inv}, and similarly for L_i^{inv} for $i = 0, 1$. We can define the Borel (or equivariant) cohomology of (M, L_0, L_1) with respect to this involution. Seidel and Smith give a geometric description of the cochain complexes used to produce equivariant Floer cohomology; we'll content ourselves with an algebraic description, referring the reader to their paper [40, Section 3] for further geometric detail. Notice that the usual Floer chain complex $CF(L_0, L_1)$ carries an induced involution $\tau^{\#}$ which takes an intersection point

$x \in L_0 \cap L_1$ to the intersection point $\tau(x) \in L_0 \cap L_1$. This map $\tau^\#$ is not a chain map with respect to a generic family of complex structures on M. However, suppose that we are in the nice case that we can find a suitable family of complex structures $\mathbf{J}$ on M such that $\tau^\#$ commutes with the differential on $CF(L_0, L_1)$. (Part of Seidel and Smith's use of their technical conditions on the bundle $\Upsilon(M^{\mathrm{inv}})$ is to establish that, after an equivariant isotopy of L_0 and L_1 fixing the invariant sets, such a $\mathbf{J}$ exists [40, Lemma 19]. Such an isotopy can have the effect of changing what action is induced on $HF(L_0, L_1)$ by $\tau^\#$, an issue which we will address in Section 4.1.) Then $CF(L_0, L_1)$ is a chain complex over $\mathbb{F}_2[\mathbb{Z}_2] = \mathbb{F}_2[\tau^\#]/\langle(\tau^\#)^2 = 1\rangle$. Indeed, $(1+\tau^\#)^2 = 0$, so there is a chain complex

$$0 \to CF(L_0, L_1) \xrightarrow{1+\tau^\#} CF(L_0, L_1) \xrightarrow{1+\tau^\#} CF(L_0, L_1) \cdots$$

Definition 4.0.6. If $CF(L_0, L_1)$ is the Floer chain complex and $\tau^\#$ is a chain map with respect to the complex structure on M, $HF_{\mathrm{borel}}(L_0, L_1)$ is the homology of the complex $CF(L_0, L_1) \otimes \mathbb{Z}_2[[\theta]]$ with respect to the differential $\delta + (1+\tau^\#)\theta$.

Therefore the double complex

$$0 \to \overset{\delta}{\circlearrowleft} CF(L_0, L_1) \xrightarrow{1+\tau^\#} \overset{\delta}{\circlearrowleft} CF(L_0, L_1) \xrightarrow{1+\tau^\#} \overset{\delta}{\circlearrowleft} CF(L_0, L_1) \cdots$$

induces a spectral sequence whose first page is $HF(L_0, L_1) \otimes \mathbb{Z}_2[[\theta]]$ and which converges to $HF_{\mathrm{borel}}(L_0, L_1)$.

There is another more algebraic method of generating this complex. We may begin by considering the Floer *homology* complex on L_0 and L_1, which is constructed identically to the cohomology complex except in counting holomorphic strips u of Maslov index 1 with $u(s, 0) \in L_1$ and $u(s, 1) \in L_0$; that is, it is equal to the Floer cohomology $HF(L_1, L_0)$. Let $n_{x_- x_+}$ be the number of such Floer trajectories after quotienting by the translation action on $\mathbb{R}$. Let d be the Floer homology differential on $CF(L_1, L_0)$. If $\mathbf{J}$ is a time-dependent perturbation of J which is regular for Floer cohomology, it is also regular for Floer homology. Moreover, notice that $CF(L_1, L_0)$ carries an involution $\tau_\# = \tau^\#$, also not generically a chain map.

The following relationship between the two theories is well-known.

Lemma 4.0.7. *The Floer cohomology complex* $CF(L_0, L_1)$ *is canonically isomorphic to the complex* $\mathrm{Hom}_{\mathbb{Z}_2}(CF(L_1, L_0), \mathbb{Z}_2)$ *with the dual differential* $\mathrm{d}^\dagger$ *as chain complexes.*

Proof. Since the group $CF(L_0, L_1) = CF(L_1, L_0)$ has a canonical set of generators in the intersection points of L_0 and L_1, the group $\mathrm{Hom}(CF(L_1, L_0), \mathbb{Z}_2)$ is canonically isomorphic to $CF(L_0, L_1)$ as abelian groups. (Indeed, in some moral sense the space of maps ought to be the chain complex for Floer cohomology.) It remains to be shown that $d^\dagger = \delta$. First observe that $n_{x_- x_+} = n^{x_+ x_-}$: if $u\colon \mathbb{R} \times [0,1] \to M$ is a Floer trajectory of index 1 from x_- to x_+ which counts for the differential δ, then $v\colon \mathbb{R} \times [0,1] \to M$ defined by $v(s,t) = u(-s, 1-t)$ is a Floer trajectory from x_+ to x_- which counts for the differential δ. Let x be an intersection point of L_0 and L_1, and x^* its dual in $\mathrm{Hom}(CF(L_1, L_0), \mathbb{Z}_2)$. Then if y is another intersection point, we have

$$\begin{aligned}
\langle d^\dagger x^*, y\rangle &= \langle x^*, dy^*\rangle \\
&= \langle x^*, \sum_{z \in L_1 \cap L_0} n_{yz} z\rangle \\
&= n_{yx} \\
&= n^{xy}
\end{aligned}$$

So y^* appears in $d^\dagger x^*$ with coefficient n^{xy}. Since y appears in δx with coefficient n^{xy}, the two chain complexes are isomorphic, as promised. □

A similar argument applies to $CF(L_1, L_0)$ and $\mathrm{Hom}_{\mathbb{Z}_2}(CF(L_0, L_1), \mathbb{Z}_2)$. In particular, if $\mathbf{J}$ is a perturbation of J with respect to which $\tau^\#$ is a chain map, there is a chain map $(\tau^\#)^\dagger$ on $\mathrm{Hom}_{\mathbb{Z}_2}(CF(L_0, L_1), \mathbb{Z}_2)$ which is identified with $\tau_\#$ with respect to the isomorphism. Ergo $\tau_\#$ is also a chain map with respect to $\mathbf{J}$.

This leads to a more algebraic definition of equivariant Floer cohomology.

Lemma 4.0.8. *The equivariant Floer cohomology* $HF_{\mathrm{borel}}(L_0, L_1)$ *is isomorphic to*

$$Ext_{\mathbb{F}_2[\mathbb{Z}_2]}(CF(L_1, L_0), \mathbb{F}_2).$$

Here we regard $\mathbb{F}_2$ as the trivial module over $\mathbb{F}_2[\mathbb{Z}_2]$.

Proof. We will show that the double complex that computes $Ext_{\mathbb{F}_2[\mathbb{Z}_2]}(CF(L_1, L_0), \mathbb{F}_2)$ is isomorphic to the double complex from which our spectral sequence arises. Consider the following free resolution of $\mathbb{F}_2$ over $\mathbb{F}_2[\mathbb{Z}_2]$.

$$\cdots \xrightarrow{1+\tau_\#} \mathbb{F}_2[\mathbb{Z}_2] \xrightarrow{1+\tau_\#} \mathbb{F}_2[\mathbb{Z}_2] \longrightarrow 0$$

We may obtain a free resolution of $CF(L_1, L_0)$ by tensoring it with the chain complex above over $\mathbb{F}_2$. This produces a double complex

$$\cdots \xrightarrow{1\otimes(1+\tau_\#)} CF(L_1, L_0) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\mathbb{Z}_2] \xrightarrow{1\otimes(1+\tau_\#)} CF(L_1, L_0) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\mathbb{Z}_2] \longrightarrow 0$$

To compute $Ext_{\mathbb{F}_2[\mathbb{Z}_2]}(CF(L_0, L_1), \mathbb{F}_2)$, we must take the homology of the double complex $\mathrm{Hom}_{\mathbb{F}_2[\mathbb{Z}_2]}(CF(L_1, L_0) \otimes_{\mathbb{Z}_2} \mathbb{F}_2[\mathbb{Z}_2], \mathbb{F}_2)$ with respect to the duals of the maps d and $1 + \tau_\#$.

However, suppose $\phi \in \mathrm{Hom}_{\mathbb{F}_2[\mathbb{Z}_2]}(CF(L_1, L_0) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\mathbb{Z}_2], \mathbb{F}_2)$. Then since ϕ is equivariant with respect to the action of $\tau_\#$ on $CF(L_1, L_0) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\mathbb{Z}_2]$, we see $\phi(x \otimes \tau_\#) = \phi(\tau_\# x \otimes 1)$, that is, ϕ is determined by its behavior as a $\mathbb{F}_2$-linear map on $CF(L_1, L_0) \otimes \{1\}$. Hence there is a canonical isomorphism

$$\mathrm{Hom}_{\mathbb{F}_2[\mathbb{Z}_2]}(CF(L_1, L_0) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\mathbb{Z}_2], \mathbb{F}_2) \cong \mathrm{Hom}_{\mathbb{F}_2}(CF(L_1, L_0), \mathbb{F}_2)$$

Since this isomorphism is natural, we can compute $Ext_{\mathbb{F}_2[\mathbb{Z}_2]}(CF(L_1, L_0), \mathbb{F}_2)$ from the double complex below.

$$\cdots \xrightarrow{(1+\tau_\#)^\dagger} \mathrm{Hom}_{\mathbb{F}_2}(CF(L_1, L_0), \mathbb{F}_2) \xrightarrow{(1+\tau_\#)^\dagger} \mathrm{Hom}_{\mathbb{F}_2}(CF(L_1, L_0), \mathbb{F}_2) \xrightarrow{(1+\tau_\#)^\dagger} 0$$

We saw in the proof of Lemma 4.0.8 that $d^\dagger = \delta$; moreover, since $\tau_\#$ and $\tau^\#$ are in point of fact the same map on the generators of $CF(L_0, L_1)$, $(\tau_\#)^\dagger = \tau^\#$. Therefore this is precisely the double complex we used to define equivariant Floer cohomology. $\square$

Seidel and Smith's result concerns the existence of a localization map from $HF_{\text{borel}}(L_0, L_1)$ to $HF(L_0^{\text{inv}}, L_1^{\text{inv}})$, where the second space is the Floer cohomology of the two Lagrangians L_0^{inv} and L_1^{inv} in M^{inv}. The main goal is to produce a family of τ-invariant complex structures on M such that, for $u\colon \mathbb{R} \times [0,1] \to M^{\text{inv}}$, the index of the operator $D_{\mathbf{J}}u$ of u with respect to **J** in M differs from the index of the operator $D_{\mathbf{J}^{\text{inv}}}$ of u with respect to $\mathbf{J}^{\text{inv}}$ in M^{inv} by a constant.

Consider the normal bundle $N(M^{\text{inv}})$ to M^{inv} in M and its Lagrangian subbundles $N(L_i^{\text{inv}})$ the normal bundles to each L_i^{inv} in L_i. The construction requires one additional degree of freedom, achieved by pulling back the bundle $N(M^{\text{inv}})$ along the projection map $M^{\text{inv}} \times [0,1] \to M^{\text{inv}}$. Call this pullback $\Upsilon(M^{\text{inv}})$. This bundle is constant with respect to the interval $[0,1]$. Its restriction to each $M^{\text{inv}} \times \{t\}$ is a copy of $N(M^{\text{inv}})$ which will occasionally, by a slight abuse of notation, be called $N(M^{\text{inv}}) \times \{t\}$; similarly, for $i = 0, 1$ the copy of $N(L_i^{\text{inv}})$ above $L_i^{\text{inv}} \times \{t\}$ will be referred to as $N(L_i^{\text{inv}}) \times \{t\}$.

We make a note here of the correspondence between our notation and Seidel and Smith's original usage. Our bundle $\Upsilon(M^{\text{inv}})$ is their TM^{anti}; while our $N(L_0^{\text{inv}}) \times \{0\}$ is their TL_0^{inv} and our $N(L_1^{\text{inv}}) \times \{1\}$ is their TL_1^{anti}. (The name TL_1^{anti} is also used for the bundle that we denote $N(L_1^{\text{inv}}) \times \{0\}$, using the obvious isomorphism between the bundles.)

We are now ready to introduce the notion of a stable normal trivialization of $\Upsilon(M^{\text{inv}})$. We denote the trivial bundle $X \times \mathbb{C}^n \to X$ by $\mathbb{C}^n$, whenever the base space X is clear from context, and similarly for $\mathbb{R}^n$.

Definition 4.0.9. [40, Defn 18] A stable normal trivialization of the vector bundle $\Upsilon(M^{\text{inv}})$ over $M^{\text{inv}} \times [0,1]$ consists of the following data.

- A stable trivialization of unitary vector bundles $\phi\colon\ \Upsilon(M^{\text{inv}}) \oplus \mathbb{C}^K \to \mathbb{C}^{k_{\text{anti}}+K}$ for some K.
- A Lagrangian subbundle $\Lambda_0 \subset (\Upsilon(M^{\text{inv}}))|_{[0,1]\times L_0^{\text{inv}}}$ such that $\Lambda_0|_{\{0\}\times L_0^{\text{inv}}} = (N(L_0^{\text{inv}}) \times \{0\}) \oplus \mathbb{R}^K$ and $\phi(\Lambda_0|_{\{1\}\times L_0^{\text{inv}}}) = \mathbb{R}^{k_{\text{anti}}+K}$.

- A Lagrangian subbundle $\Lambda_1 \subset (\Upsilon(M^{\mathrm{inv}}))|_{[0,1]\times L_1^{\mathrm{inv}}}$ such that $\Lambda_1|_{\{0\}\times L_1^{\mathrm{inv}}} = (N(L_1^{\mathrm{inv}}) \times \{0\}) \oplus \mathbb{R}^K$ and $\phi(\Lambda_1|_{\{1\}\times L_1^{\mathrm{inv}}}) = i\mathbb{R}^{k_{\mathrm{anti}}+K}$.

The crucial theorem of [40], proved through extensive geometric analysis and comparison with the Morse theoretic case, is as follows.

Theorem 4.0.10. *[40, Thm 20] If $\Upsilon(M^{\mathrm{inv}})$ carries a stable normal trivialization, then after an equivariant isotopy which replaces L_0 and L_1 with $\widetilde{L}_0$ and $\widetilde{L}_1$, $HF_{\mathrm{borel}}(\widetilde{L}_0, \widetilde{L}_1)$ is well-defined and there are localization maps*

$$\Delta^{(m)}\colon\ HF_{\mathrm{borel}}(\widetilde{L}_0, \widetilde{L}_1) \to HF(L_0^{\mathrm{inv}}, L_1^{\mathrm{inv}})[[\theta]]$$

defined for $m >> 0$ and satisfying $\Delta^{(m+1)} = \theta\Delta^{(m)}$. Moreover, after tensoring over $\mathbb{Z}_2[[\theta]]$ with $\mathbb{Z}_2((\theta))$ these maps are isomorphisms.

This implies Theorem 1.4.1. Because the localization maps are well-behaved with respect to the $\mathbb{Z}_2((\theta))$ module structures, we also deduce Corollary 1.4.2. Notice while the isotopy of L_0 and L_1 to $\widetilde{L}_0$ and $\widetilde{L}_1$ does not change the first page of the spectral sequence (that is, $\widehat{HF}(L_0, L_1) \cong \widehat{HF}(\widetilde{L}_0, \widetilde{L}_1)$, it may change the action τ^* of the involution τ on $\widehat{HF}(L_0, L_1)$. That is, τ^* is not necessarily the map induced by the map $\tau^{\#}$ on the original complex. This in general makes the higher differentials of the spectral sequence difficult to compute.

When checking that a stable normal trivialization exists, we can in fact dispense with the symplectic structure and work on the level of the complex normal bundle $\Upsilon(M^{\mathrm{inv}})$ with its totally real subbundles $NL_0^{\mathrm{inv}} \times \{1\}$ and $J(NL_1^{\mathrm{inv}} \times \{1\})$. The following lemma is mentioned in [40, Section 3d]; we give a detailed proof here.

Lemma 4.0.11. *The existence of a stable normal trivialization of (M, L_0, L_1) is implied by the existence of a nullhomotopy of the map*

$$(M \times [0,1], (L_0 \times \{0\}) \cup (L_1 \times \{1\})) \to (BU, BO)$$

which classifies the complex normal bundle $\Upsilon(M^{\mathrm{inv}}) = NM^{\mathrm{inv}} \times [0,1]$ and its totally real subbundles $NL_0^{\mathrm{inv}} \times \{0\}$ over $L_0^{\mathrm{inv}} \times \{0\}$ and $J(NL_1^{\mathrm{inv}} \times \{1\})$ over $L_1^{\mathrm{inv}} \times \{1\}$.

Proof. This is largely the statement that we can dispense with the symplectic structure involved in a stable normal trivialization and argue purely in terms of complex vector bundles. Recall that any symplectic vector bundle can be equipped with a compatible complex structure which is unique up to homotopy, and any complex vector bundle similarly admits a symplectification. Moreover, two symplectic vector bundles are isomorphic if and only if their underlying complex vector bundles are isomorphic, and isomorphisms of symplectic vector bundles map Lagrangian subbundles to Lagrangian subbundles [22, Theorem 2.62]. Let ω_M be the natural symplectic structure on $N(M^{\text{inv}})$ coming from the symplectic structure on TM.

Suppose that $g\colon N(M^{\text{inv}}) \to BU$ is a classifying map of $N(M^{\text{inv}})$ thought of as a unitary vector bundle, so that the image of g lies inside $BU_{k_{\text{anti}}}$. Let $\zeta_{k_{\text{anti}}}\colon EU_{k_{\text{anti}}} \to BU_{k_{\text{anti}}}$ be the complex k_{anti}-dimensional universal bundle, and similarly let $\eta_{k_{\text{anti}}}\colon EO_{k_{\text{anti}}} \to BO_{k_{\text{anti}}}$ be the real k_{anti}-dimensional universal bundle. Equip $EU_{k_{\text{anti}}}$ with a symplectic structure ω_ζ such that $\eta_{k_{\text{anti}}} \subset \zeta_{k_{\text{anti}}}$ is a Lagrangian subbundle. Then the bundles $(N(M^{\text{inv}}), \omega_M)$ and $(N(M^{\text{inv}}), g^*(\omega_\zeta))$ are isomorphic (indeed, equal) as complex vector bundles, so there is a symplectic vector bundle isomorphism $\chi\colon (N(M^{\text{inv}}), \omega_M) \to (N(M^{\text{inv}}), g^*(\omega_\zeta))$. This extends to a symplectic vector bundle isomorphism $\widetilde{\chi}\colon (\Upsilon(M^{\text{inv}}), \widetilde{\omega}_M) \to (\Upsilon(M^{\text{inv}}), \widetilde{g}^*(\omega_\zeta) = f^*(\omega_M))$, where in both cases the symplectic forms are the pullbacks of the original symplectic forms on $N(M^{\text{inv}})$ to $\Upsilon(M^{\text{inv}})$, and therefore constant with respect to the interval $[0,1]$, as is the map $\widetilde{\chi}$. From now on, we assume that we have first applied an isomorphism of this form to $\Upsilon(M^{\text{inv}})$ so that the map $g\colon \Upsilon(M^{\text{inv}}) \to BU$ is in fact a symplectic classifying map. We can if necessary precompose the resulting stable normal trivialization with $\widetilde{\chi}$.

Consider a nullhomotopy H of f.

$$\begin{aligned} H\colon \big(M^{\text{inv}} \times [0,1], L_0^{\text{inv}} \times \{0\} \cup L_1^{\text{inv}} \times \{1\}\big) \times [0,1] &\to (BU, BO) \\ (x,t,s) &\mapsto h_s(x,t) \end{aligned}$$

Here the map h_0 is equal to f and the map h_1 is constant.

Since M^{inv} is homotopy equivalent to a compact subspace of itself, we may assume there is some $K > 0$ such that if $s = k_{\text{anti}} + K$, the image of H lies inside (BU_s, BO_s). Let $\zeta_s\colon EU_s \to$

BU_s be the complex s-dimensional universal bundle with subbundle $\eta_s\colon\ EO_s \to BO_s$ the real s-dimensional universal bundle. Then the pullbacks of ζ_s and η_s along h_1 and h_0 are certain bundles of great interest to our investigation.

$$\begin{aligned} h_0^*(\zeta_s) &= (\Upsilon(M^{\text{inv}}) \oplus \mathbb{C}^K, h_0^*\omega_\zeta) \\ h_1^*(\zeta_s) &= (\mathbb{C}^s, h_1^*\omega_\zeta) \\ (h_0|_{L_0^{\text{inv}}\times\{0\}})^*(\eta_s) &= (N(L_0^{\text{inv}}) \times \{0\}) \oplus \mathbb{R}^K \\ (h_0|_{L_1^{\text{inv}}\times\{1\}})^*(\eta_s) &= (J(N(L_1^{\text{inv}})) \times \{1\}) \oplus \mathbb{R}^K \\ (h_1|_{L_i^{\text{inv}}\times\{i\}})^*(\eta_s) &= \mathbb{R}^s \text{ for } i = 0, 1 \end{aligned}$$

For clarity's sake it should be borne in mind that $\mathbb{R}^K$ and $\mathbb{R}^s$ refer to the canonical real subspaces in $\mathbb{C}^K$ and $\mathbb{C}^s$.

Since H is a nullhomotopy, it induces a stable trivialization ψ of $\Upsilon(M^{\text{inv}})$. Write an arbitrary vector in $\Upsilon(M^{\text{inv}})$ as (x, t, v) where $(x, t) \in M^{\text{inv}} \times [0, 1]$ and v is an element of the fiber over (x, t).

$$\begin{aligned} \psi\colon\ \Upsilon(M^{\text{inv}}) \oplus \mathbb{C}^K = h_0^*(\zeta_s) &\xrightarrow{\sim} h_1^*(\zeta_s) = \mathbb{C}^s \\ (x, t, v) &\mapsto \psi(x, t, v) \end{aligned}$$

The restrictions of ψ to $(N(L_0^{\text{inv}}) \times \{0\}) \oplus \mathbb{R}^K$ and to $(J(N(L_1^{\text{inv}}) \times \{1\})) \oplus \mathbb{R}^K$ are real stable trivializations of these two bundles.

$$\begin{aligned} \psi|_{(N(L_0^{\text{inv}}\times\{0\})\oplus\mathbb{R}^K)}\colon\ (N(L_0^{\text{inv}}) \times \{0\}) \oplus \mathbb{R}^K &\to \mathbb{R}^s \\ \psi|_{(J(N(L_1^{\text{inv}})\times\{1\})\oplus\mathbb{R}^K)}\colon\ (J(N(L_1^{\text{inv}})) \times \{1\}) \oplus \mathbb{R}^K &\to \mathbb{R}^s \end{aligned}$$

Since $\Upsilon(M^{\text{inv}})$ is the pullback of $N(M^{\text{inv}})$ to $M^{\text{inv}} \times [0, 1]$, the map $h_0 = f$ is constant with respect to the interval $[0, 1]$. That is, $h_0(x, t_1) = h_0(x, t_2)$ for all $x \in M^{\text{inv}}$ and $t_1, t_2 \in [0, 1]$. Then each $\psi_t = \psi|_{M^{\text{inv}}\times\{t\}}$ is a stable trivialization of $N(M^{\text{inv}}) \times \{t\} = N(M^{\text{inv}})$. More concretely, we have symplectic trivializations

$$\begin{aligned} \psi_t \colon\ N(M^{\text{inv}}) &\to \mathbb{C}^k \\ (x, v) &\mapsto \psi(x, t, v). \end{aligned}$$

We will use the family of isomorphisms ψ_t to produce a stable normal trivialization of $\Upsilon(M^{\text{inv}})$. Consider a map ϕ defined by applying ψ_0 to each $M^{\text{inv}} \times \{t\} \subset M^{\text{inv}} \times [0, 1]$.

$$\begin{aligned} \phi \colon\ \Upsilon(M^{\text{inv}}) &\to \mathbb{C}^k \\ (x, t, v) &\mapsto \psi_0((x, v)) = \psi(x, 0, v). \end{aligned}$$

This is a stable trivialization of $\Upsilon(M^{\text{inv}})$. Because the symplectic structure on $\Upsilon(M^{\text{inv}})$ is constant with respect to the interval $[0, 1]$, it is in fact a symplectic isomorphism of vector bundles. We next need to produce two Lagrangian subbundles Λ_0 and Λ_1 satisfying the conditions outlined in Definition 4.0.9. Consider the following candidates.

$$\begin{aligned} \Lambda_0|_{L_0^{\text{inv}} \times \{t\}} &= (N(L_0^{\text{inv}}) \times \{t\}) \oplus \mathbb{R}^K \\ \Lambda_1|_{L_1^{\text{inv}} \times \{t\}} &= \psi_0^{-1} \circ \psi_t(N(L_1^{\text{inv}}) \times \{t\} \oplus i\mathbb{R}^K). \end{aligned}$$

Since the maps ψ_t form a homotopy, Λ_1 is a smooth subbundle as desired. Both subbundles are Lagrangian since their restriction to each $L_i^{\text{inv}} \times \{t\}$ is Lagrangian. The next thing to check is the restriction of Λ_i to $L_i^{\text{inv}} \times \{0\}$ for $i = 0, 1$.

$$\begin{aligned} \Lambda_0|_{L_0^{\text{inv}} \times \{0\}} &= (N(L_0^{\text{inv}}) \times \{0\}) \oplus \mathbb{R}^K \\ \Lambda_1|_{L_1^{\text{inv}} \times \{0\}} &= \psi_0^{-1} \circ \psi_0((N(L_1^{\text{inv}}) \times \{0\}) \oplus i\mathbb{R}^K) \\ &= (N(L_1^{\text{inv}}) \times \{0\}) \oplus i\mathbb{R}^K. \end{aligned}$$

This is exactly as desired. The other condition to check is that $\phi(\Lambda_0|_{L_0^{\text{inv}} \times \{1\}})$ is $\mathbb{R}^s \subset \mathbb{C}^s$

and $\phi(\Lambda_1|_{L_1^{\text{inv}}\times\{1\}})$ is $i\mathbb{R}^s \subset \mathbb{C}^s$.

$$\begin{aligned}
\phi(\Lambda_0|_{L_0^{\text{inv}}\times\{1\}}) &= \psi_0((N(L_0^{\text{inv}})\times\{0\})\oplus\mathbb{R}^K)\\
&= \psi((N(L_0^{\text{inv}})\times\{0\})\oplus\mathbb{R}^K)\\
&= \mathbb{R}^s\\
\phi(\Lambda_1|_{L_1^{\text{inv}}\times\{1\}}) &= \psi_0(\psi_0^{-1}\circ\psi_1((N(L_1^{\text{inv}})\times\{0\})\oplus i\mathbb{R}^K)\\
&= \psi_1(J(J((N(L_1^{\text{inv}})\times\{0\})\oplus i\mathbb{R}^K)))\\
&= J(\psi_1(J(N(L_1^{\text{inv}})\times\{0\})\oplus\mathbb{R}^k))\\
&= i(\mathbb{R}^s)
\end{aligned}$$

Ergo the stable trivialization ϕ of $\Upsilon(M^{\text{inv}})$ together with the subbundles Λ_0 and Λ_1 constitutes a stable normal trivialization of $\Upsilon(M^{\text{inv}})$. □

4.1 Interpretation in the context of Heegaard Floer homology

Let $\mathcal{D} = (S^2, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ and $\widetilde{\mathcal{D}} = (\Sigma(S^2), \widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\beta}}, \widetilde{\mathbf{w}}, \widetilde{\mathbf{z}})$ be a pair of Heegaard diagrams which is localizable in the sense of Definition 3.3.1. Given x a point on D, let x^1, x^2 be its two (not necessarily distinct) lifts to $\widetilde{\mathcal{D}}$ in some order. There is a natural map

$$\begin{aligned}
\iota\colon\ &\text{Sym}^{n-1}(S^2) \to \text{Sym}^{2n-2}(\Sigma(S^2))\\
&(x_1\cdots x_{n-1}) \mapsto (x_1^1 x_1^2\cdots x_{n-1}^1 x_{n-1}^2).
\end{aligned}$$

This map is a holomorphic embedding; for a proof, see Appendix 1. Moreover, consider the induced involution on $\text{Sym}^{2n-2}(\Sigma(S^2))$, which through a slight abuse of notation we will also call τ. The fixed set of τ is exactly our embedded copy of $\text{Sym}^{n-1}(S^2)$; moreover, τ preserves the two tori $\mathbb{T}_{\widetilde{\boldsymbol{\alpha}}}$ and $\mathbb{T}_{\widetilde{\boldsymbol{\beta}}}$, with fixed sets $T_{\widetilde{\boldsymbol{\alpha}}}^{\text{inv}} = \mathbb{T}_{\boldsymbol{\alpha}}$ and $T_{\widetilde{\boldsymbol{\beta}}}^{\text{inv}} = \mathbb{T}_{\boldsymbol{\beta}}$.

Perutz has shown that for an arbitrary Heegaard diagram $D = (S, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$, there is a symplectic form ω on $\text{Sym}^{g+n-1}(S)$ which is compatible with the complex structure induced by a complex structure on S, and with respect to which the submanifolds $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ are in fact Lagrangian and the various Heegaard Floer homology theories are their Lagrangian Floer cohomologies [34, Thm 1.2]. In particular, the knot Floer homology is the Floer cohomology of these two tori in the ambient space $\mathrm{Sym}^{g+n-1}(S\backslash\{\mathbf{z},\mathbf{w}\})$, where the removal of the basepoints accounts for the restriction that holomorphic curves not be permitted to intersect the submanifolds V_{w_i} and V_{z_j} of the symmetric product.

Proposition 4.1.1. *There is a symplectic structure on* $\mathrm{Sym}^{g-n-1}(S\backslash\{\mathbf{w},\mathbf{z}\})$ *with respect to which the submanifolds* $\mathbb{T}_{\boldsymbol{\alpha}}$ *and* $\mathbb{T}_{\boldsymbol{\beta}}$ *are Lagrangian and*

$$\widetilde{HFL}(D) \cong \widehat{HFL}((Y,L)) \otimes V_1^{\otimes(n_1-1)} \otimes \cdots \otimes V_\ell^{\otimes(n_\ell-1)} \cong HF(\mathbb{T}_{\boldsymbol{\beta}}, \mathbb{T}_{\boldsymbol{\alpha}}).$$

This is essentially Theorem 1.2 of [34]; the proof is identical save for trivial notation adjustments for Heegaard diagrams with multiple basepoints. From now on we will work with this method of computing knot Floer homology; in Chapter 5 we will show that the symplectic form produced by Perutz's construction meets the requirements of Seidel and Smith's theorem.

Let us now consider the application of Theorem 4.0.10 to double branched covers of links in the three sphere and to periodic knots. We begin with the case that $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ are Heegaard diagrams for (S^3, L) and the double branched cover $(\Sigma(L), \widetilde{L})$ as in Section 3.1. We make the following suggestive choices of notation.

$$\begin{aligned} M_1 &= \mathrm{Sym}^{2n-2}(\Sigma(S^2)\backslash\{\widetilde{\mathbf{w}},\widetilde{\mathbf{z}}\}) & \qquad M_1^{\mathrm{inv}} &= \mathrm{Sym}^{n-1}(S^2\backslash\{\mathbf{w},\mathbf{z}\}) \\ L_0 &= \mathbb{T}_{\widetilde{\boldsymbol{\beta}}} & L_0^{\mathrm{inv}} &= \mathbb{T}_{\boldsymbol{\beta}} \\ L_1 &= \mathbb{T}_{\widetilde{\boldsymbol{\alpha}}} & L_1^{\mathrm{inv}} &= \mathbb{T}_{\boldsymbol{\alpha}} \end{aligned}$$

Then the following is immediate from the definitions.

Lemma 4.1.2. *With respect to our choice of symplectic manifold* M *and Lagrangians* L_0 *and* L_1*, we have the following Floer cohomology groups.*

$$HF(L_0, L_1) = HF(\mathbb{T}_{\widetilde{\boldsymbol{\beta}}}, \mathbb{T}_{\widetilde{\boldsymbol{\alpha}}}) = \widehat{HFL}(\Sigma(L), \widetilde{L}) \otimes V^{\otimes n_1 - 1} \otimes \cdots \otimes V^{\otimes n_\ell - 1}.$$

$$HF(L_0^{\mathrm{inv}}, L_1^{\mathrm{inv}}) = HF(\mathbb{T}_{\boldsymbol{\beta}}, \mathbb{T}_{\boldsymbol{\alpha}}) = \widehat{HFL}(S^3, L) \otimes V^{\otimes n_1 - 1} \otimes \cdots \otimes V^{\otimes n_\ell - 1}.$$

Similarly, given Heegaard diagrams $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ as in 3.2.1 for a periodic knot together with an unknotted axis and its quotient knot, we consider the spaces

$$M_2 = \mathrm{Sym}^{2n_1}(S^2\backslash\{\widetilde{\mathbf{w}}, \widetilde{\mathbf{z}} - z_0\}) \qquad M_2^{\mathrm{inv}} = \mathrm{Sym}^{n_1}(S^2\backslash\{\mathbf{w}, \mathbf{z} - z_0\})$$
$$M_3 = \mathrm{Sym}^{2n_1}(S^2\backslash\{\widetilde{\mathbf{w}}, \widetilde{\mathbf{z}}\}) \qquad M_3^{\mathrm{inv}} = \mathrm{Sym}^{n_1}(S^2\backslash\{\mathbf{w}, \mathbf{z}\})$$

Recall, for clarity's sake, that the total number of basepoints on the periodic Heegaard diagram $\mathcal{D}$ is $n = n_1 + 1$. Here again the Lagrangians and their invariant sets are as follows.

$$L_0 = \mathbb{T}_{\widetilde{\boldsymbol{\beta}}} \qquad L_0^{\mathrm{inv}} = \mathbb{T}_{\boldsymbol{\beta}}$$
$$L_1 = \mathbb{T}_{\widetilde{\boldsymbol{\alpha}}} \qquad L_1^{\mathrm{inv}} = \mathbb{T}_{\boldsymbol{\alpha}}$$

The following is immediate from the definitions.

Lemma 4.1.3. *With respect to our choice of symplectic manifolds M_i for $i = 1, 2$ and Lagrangians L_0 and L_1, we have the following Floer cohomology groups.*
In M_2,

$$HF(L_0, L_1) = HF(\mathbb{T}_{\widetilde{\boldsymbol{\beta}}}, \mathbb{T}_{\widetilde{\boldsymbol{\alpha}}}) = \widehat{HFL}(S^3, \widetilde{K} \cup U) \otimes V^{\otimes(2n_1-1)}.$$
$$HF(L_0^{\mathrm{inv}}, L_1^{\mathrm{inv}}) = HF(\mathbb{T}_{\boldsymbol{\beta}}, \mathbb{T}_{\boldsymbol{\alpha}}) = \widehat{HFL}(S^3, K \cup U) \otimes V^{\otimes(n_1-1)}.$$

In M_3,

$$HF(L_0, L_1) = HF(\mathbb{T}_{\widetilde{\boldsymbol{\beta}}}, \mathbb{T}_{\widetilde{\boldsymbol{\alpha}}}) = \widehat{HFK}(S^3, \widetilde{K}) \otimes V^{\otimes(2n_1-1)} \otimes W.$$
$$HF(L_0^{\mathrm{inv}}, L_1^{\mathrm{inv}}) = HF(\mathbb{T}_{\boldsymbol{\beta}}, \mathbb{T}_{\boldsymbol{\alpha}}) = \widehat{HFK}(S^3, K) \otimes V^{\otimes(n_1-1)} \otimes W.$$

Observe that in all cases we work with a symmetric products of Heegaard surfaces $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ which have been punctured along some subset $\{\mathbf{r}\}$ of $\{\mathbf{w}, \mathbf{z}\}$ containing $\{\mathbf{w}\}$. It will therefore suffice to show the following two lemmas.

Lemma 4.1.4. *Let $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ be a localizable Heegaard diagram pair, and $\{\mathbf{r}\}$ be a subset of $\{\mathbf{w}, \mathbf{z}\}$. Then if we set*

$$M = \mathrm{Sym}^{2n-2}(\Sigma(S^2)\backslash\{\widetilde{\mathbf{r}}\}) \qquad M^{\mathrm{inv}} = \mathrm{Sym}^{n-1}(S^2\backslash\{\mathbf{r}\})$$

with the Lagrangians

$$L_0 = \mathbb{T}_{\widetilde{\boldsymbol{\beta}}} \qquad L_0^{\text{inv}} = \mathbb{T}_{\boldsymbol{\beta}}$$
$$L_1 = \mathbb{T}_{\widetilde{\boldsymbol{\alpha}}} \qquad L_1^{\text{inv}} = \mathbb{T}_{\boldsymbol{\alpha}}$$

we may equip M with an exact symplectic form ω with respect to which L_i is exact for $i = 0, 1$ and τ is a symplectic involution. In addition we may give M an ω-compatible complex structure j which is convex at infinity.

To show that such a manifold M necessarily satisfies the conditions of Lemma 4.0.11 it suffices to check that a slightly larger manifold M_0 does.

Lemma 4.1.5. *Let $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ be a localizable Heegaard diagram pair, and let*

$$M_0 = \text{Sym}^{2n-2}(\Sigma(S^2)\backslash\{\widetilde{\mathbf{w}}\}) \qquad M_0^{\text{inv}} = \text{Sym}^{n-1}(S^2\backslash\{\mathbf{w}\})$$

with the Lagrangians

$$L_0 = \mathbb{T}_{\widetilde{\boldsymbol{\beta}}} \qquad L_0^{\text{inv}} = \mathbb{T}_{\boldsymbol{\beta}}$$
$$L_1 = \mathbb{T}_{\widetilde{\boldsymbol{\alpha}}} \qquad L_1^{\text{inv}} = \mathbb{T}_{\boldsymbol{\alpha}}$$

Then (M_0, L_0, L_1) satisfies the conditions of Lemma 4.0.11.

Since a relative trivialization of the complex relative normal bundle on the level of Lemma 4.0.11 descends to any full-dimensional submanifold of M_0 which contains the Lagrangians L_0 and L_1, we immediately have the following theorem.

Theorem 4.1.6. *Each of the trios (M_1, L_0, L_1), (M_2, L_0, L_1), and (M_3, L_0, L_1) carries a stable normal trivialization.*

In combination with the symplectic geometry conditions checked in Chapter 5, this implies Theorems 1.1.1, 1.2.2, and 1.2.3.

In some moral sense, this is the correct level of generality: the most important feature of our punctured Heegaard diagram $\mathcal{D}$ is that no periodic domain has nonzero index, and in all our examples $\{\mathbf{w}\}$ (or $\{\mathbf{z}\}$) is the smallest set of points at which we may puncture S^2 and produce a diagram for which this property holds. Moreover, M_0^{inv} conveniently deformation retracts onto each of $\mathbb{T}_{\boldsymbol{\alpha}}$ and $\mathbb{T}_{\boldsymbol{\beta}}$, making certain cohomology computations in Chapters 5 and 7 cleaner than they might otherwise be.

We have yet to address one important issue promised in Chapter 3. Since the involution τ^* on $\widetilde{HFL}(\widetilde{\mathcal{D}})$ may not be induced by the involution $\tau^{\#}$ on $\widehat{CFL}(\widetilde{\mathcal{D}})$ arising in the obvious way from the Heegaard diagram, why is our claim that τ^* preserves spin$^{\mathrm{c}}$ structures valid? To see this, recall that a spin$^{\mathrm{c}}$ structure is a homotopy class of paths between $\mathbb{T}_{\widetilde{\boldsymbol{\alpha}}}$ and $\mathbb{T}_{\widetilde{\boldsymbol{\beta}}}$. As we perform an equivariant isotopy of the Lagrangians $T_{\widetilde{\boldsymbol{\alpha}}}$ and $\mathbb{T}_{\widetilde{\boldsymbol{\beta}}}$, such homotopy classes are preserved, as is which pairs of homotopy classes are exchanged under the involution τ on M. Since our final choice of complex structure for M is equivariant, the action of τ determines conjugacy of spin$^{\mathrm{c}}$ structures, which is therefore also preserved. The argument for Alexander gradings – which are equivalent to relative spin$^{\mathrm{c}}$ structures on $Y^3 - \nu L$ by Subsection 2.0.5 proceeds similarly.

An important note is that the localization map of Theorem 4.0.10 is entirely constructed by counting pseudoholomorphic disks of varying index and by multiplication and division by powers of θ. In particular, the localization isomorphism preserves the Alexander multigrading on the E^∞ page, because no flowlines can pass over the missing basepoint divisors. In general, however, we should expect that these isomorphisms will not preserve the data of the Maslov grading.

Chapter 5

The Geometry of Symmetric Products of Punctured Heegaard Surfaces

In this chapter we consider the geometry of a symmetric product of a punctured Heegaard diagram.

5.1 Symplectic Geometry of Symmectric Products of Punctured Heegaard Surfaces

It remains to be verified that given a pair of Heegaard diagrams $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ which are localizable in the sense of Definition 3.3.1 and a set of punctures $\mathbf{r}$ on $\mathcal{D}$ such that $\{\mathbf{w}\} \subset \{\mathbf{r}\} \subset \{\mathbf{w}, \mathbf{z}\}$, the manifold $M = \mathrm{Sym}^{2n-2}(\Sigma(S)\backslash\{\widetilde{\mathbf{r}}\})$ satisfies the basic symplectic structural requirements of Seidel and Smith's theory. To wit, we must see that M can be equipped with a triple (j, ω, θ) such that $\omega = d\theta$, the tori $\mathbb{T}_{\widetilde{\alpha}}$ and $\mathbb{T}_{\widetilde{\beta}}$ are exact Lagrangians, the involution τ on M is symplectic, and moreover the complex structure j is convex at infinity.

Recall that $\mathcal{D}$ is a Heegaard diagram with underlying surface S^2. Let f be an exhausting function on $S^2\backslash\{\mathbf{r}\}$ as follows. Identify $S^2\backslash\{w_0\}$ with $\mathbb{C}$, and let $x = u + iv$ be a complex number.

$$f\colon\ S^2\backslash\{\mathbf{r}\} \to \mathbb{R}$$
$$x \mapsto C|x|^2 + \sum_{r_j\in\{\mathbf{r}-w_0\}} \frac{1}{|x-r_j|^2}$$

Here $C > 0$ is a constant determined as follows. The exact symplectic form associated to f is

$$\omega_f = -dd^{\mathbb{C}}(f) = \left(4C + \sum_{r_j\in\{\mathbf{r}-w_0\}} \frac{4}{|x-r_i|^4}\right) du \wedge dv.$$

The α and β curves on S^2 are certainly Lagrangian with respect to the symplectic form ω_f, but we require them to moreover be exact; that is, we wish to ensure that $\int_{\alpha_j} -d^{\mathbb{C}}(f) = 0$ for all α_j on $S^2\backslash\{\mathbf{r}\}$. We choose C large enough that the integral $\int_{\alpha_j} -d^{\mathbb{C}} f > 0$ for $1 \le j \le n-1$, and similarly for each β_j. Then we may isotope the α and β curves inward toward the punctures $\mathbf{w}$ (around which the curves are oriented counterclockwise) until we have a new, isotopic set of curves for which $\int_{\alpha_j} -d^{\mathbb{C}} f = 0 = \int_{\alpha_j} -d^{\mathbb{C}} f$. Hence we have arranged that the α and β curves are exact Lagrangians with respect to ω_f.

We now lift f to an exhausting function $\widetilde{f}$ on $\Sigma(S^2)\backslash\{\widetilde{\mathbf{r}}\}$. Our previous manipulations of the α and β curves ensure that each α_j^k and β_j^k, where $k = 1, 2$ is exact with respect to the symplectic form induced by $\widetilde{f}$.

On the product space $(\Sigma(S^2)\backslash\{\widetilde{\mathbf{r}}\})^{2n-2}$ there is a corresponding exhausting function defined as follows. Let p_m be the projection of the product to its mth factor, and let $\widehat{f} = \widetilde{f} \circ p_1 + \cdots + \widetilde{f} \circ p_{2n-2}$. Notice that $\mathrm{Sym}^{2n-2}(j)$-convexity, properness, and boundedness of $\widehat{f}$ follow from the corresponding properties of $\widetilde{f}$. The symplectic form we obtain from $\widehat{f}$ is $\omega_{\widehat{f}} = \omega_{\widetilde{f}} \otimes 1 \cdots \otimes 1 + 1 \otimes \omega_{\widetilde{f}} \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes \omega_{\widetilde{f}}$. Therefore any of the $(2n-2)!$ lifts of $\mathbb{T}_{\widetilde{\boldsymbol{\alpha}}}$ and $\mathbb{T}_{\widetilde{\boldsymbol{\beta}}}$ is an exact Lagrangian submanifold with respect to $\omega_{\widehat{f}}$.

Finally, consider the symmetric product $\mathrm{Sym}^{2n-2}(\Sigma(S^2)\backslash\{\widetilde{\mathbf{r}}\})$. There is a (possibly singular) continuous function

$$\psi\colon \operatorname{Sym}^{2n-2}(\Sigma(S^2)\backslash\{\widetilde{\mathbf{r}}\}) \to \mathbb{R}$$
$$(x_1 \cdots x_{2n-2}) \mapsto \sum_{\sigma \in S_{2n-2}} \widetilde{f}(x_{\sigma(1)}, \cdots, x_{\sigma(2n-2)})$$

which is smooth outside a neighborhood of $\{(x_1 \cdots x_{2n-2}) \in \operatorname{Sym}^{2n-2}(\Sigma(S^2)\backslash\{\widetilde{\mathbf{r}}\}) : x_k = x_j \text{ for some } k \neq j\}$, the fat diagonal. Perutz observes [34] that this function is strictly plurisubharmonic in the sense of non-smooth functions, that is, that the two-current $-dd^{\mathbb{C}}\psi$ is strictly positive. This gives us a continuous exhausting function on $\operatorname{Sym}^{2n-2}(\Sigma(S^2)\backslash\{\widetilde{\mathbf{r}}\})$. We may apply the following lemma of Richberg [37], quoted in [6, Lemma 3.10].

Lemma 5.1.1. *Let ψ be a continuous J-convex function on an integrable complex manifold V. Then for every positive function $h\colon V \to \mathbb{R}_+$, there exists a smooth J-convex function ψ' such that $|\psi(x) - \psi'(x)| < h(x)$. If ϕ is already smooth on a neighborhood of a compact subset A, then we can achieve $\phi = \phi'$ on A.*

In particular, we may take $h\colon \operatorname{Sym}^{2n-2}(\Sigma(S^2)\backslash\{\widetilde{\mathbf{r}}\}) \to \mathbb{R}_+$ to be a constant function $h(x) = \epsilon$, and apply the lemma to our map ψ and h. We produce $\psi'\colon \operatorname{Sym}^{2n-2}(\Sigma(S^2\backslash\{\widetilde{\mathbf{r}}\}) \to \mathbb{R}$ such that $|\psi'(x) - \psi(x)| < \epsilon$ and ψ' is smooth and $\operatorname{Sym}^{2n-2}(j)$-convex. Moreover, since ψ is bounded below and proper, ψ' is also, since the two real valued functions differ by at most ϵ. Therefore ψ' is an exhausting function on M, and M is convex at infinity.

We have yet to produce an appropriate symplectic form on M, which we will do using work of Perutz [34]. We begin with a definition.

Definition 5.1.2. [34, Defn 7.3] Let X be a complex manifold with complex structure J. A *Kähler cocycle* on X is a collection $(U_k, \phi_k)_{k\in K}$, where $(U_k)_{k\in K}$ is an open cover of X and $\phi_k\colon U_k \to \mathbb{R}$ is an upper semicontinuous function such that

- ϕ_k is strictly plurisubharmonic
- $\phi_k - \phi_\ell$ is pluriharmonic

If a Kähler cocycle $(U_k, \phi_k)_{k \in K}$ is smooth then we can associate to it the symplectic form ω which is $-dd^{\mathbb{C}}\phi_k$ on each U_k. Notice, for example, that a Kähler cocycle can consist of a single smooth plurisubharmonic function on all of X, as in the case of the smooth Kähler cocycle $((\Sigma(S^2)\backslash\{\widetilde{\mathbf{r}}\})^{2n-2}, \widehat{f})$ on $(\Sigma(S^2)\backslash\{\widetilde{\mathbf{r}}\})^{2n-2}$ and the singular Kähler cocycle (M, ψ) on M.

Perutz proves the following technical result.

Lemma 5.1.3. *[34, Lemma 7.4] Let (U_k, ϕ_k) be a continuous Kähler cocycle on a complex manifold X. Suppose that $X = X_1 \cup X_2$ such that X_1 and X_2 are open and the functions $\phi_k|_{U_k \cap X_1}$ are smooth. Then there exists a continuous function*

$$\chi\colon\ X \to \mathbb{R},\ Supp(\chi) \subset X_2$$

and a locally finite refinement

$$V_\ell \subset U_{k(\ell)}$$

such that the family $(V_\ell, \phi_{k(\ell)}|_{V_\ell} + \chi|_{V_\ell})$ is a smooth Kähler cocycle.

Notice that if (U_k, ϕ_k) happened to be the Kähler cocycle associated to a single J-convex function ϕ on X, then $(V_\ell, \phi_{\ell(k)}|_{V_\ell} + \chi|_{V_\ell})$ is the Kähler cocycle associated to the smooth plurisubharmonic function $\phi + \chi$.

In our case we take X to be $\mathrm{Sym}^{2n-2}(\Sigma(S)\backslash\{\widetilde{\mathbf{r}}\})$, X_1 to be the complement of the main diagonal in this symmetric product, and X_2 to be a small neighborhood of the main diagonal with no intersection with $\mathbb{T}_{\widetilde{\boldsymbol{\alpha}}}$ and $\mathbb{T}_{\widetilde{\boldsymbol{\beta}}}$. Then the function $\psi\colon\ \mathrm{Sym}^{2n-2}(\Sigma(S^2)\backslash\{\widetilde{\mathbf{r}}\}) \to \mathbb{R}$ admits a smoothing to a $\mathrm{Sym}^{2n-2}(j)$-convex function $\psi + \chi\colon\ \mathrm{Sym}^{2n-2}(\Sigma(S^2)\backslash\{\widetilde{\mathbf{r}}\}) \to \mathbb{R}$ which is equal to ψ away from a neighborhood of the large diagonal. The symplectic form $\omega_{\phi+\chi}$ is exact and compatible with the complex structure on M.

Finally, on $\mathbb{T}_{\widetilde{\boldsymbol{\alpha}}}$ the map χ_i is identically 0 and $\psi_i = (2n-2)!\widetilde{f}_i|_{\alpha_1^1 \times \alpha_1^2 \times \cdots . \times \alpha_{n-1}^1 \times \alpha_{n-1}^2}$. Therefore $\omega_{\psi+\chi}|_{\mathbb{T}_{\widetilde{\boldsymbol{\alpha}}}} = 0$ and $d^{\mathbb{C}}(\psi_i + \chi_i)|_{\mathbb{T}_{\widetilde{\boldsymbol{\alpha}}}} = (2k)! d^{\mathbb{C}}(\widetilde{f})|_{\alpha_1^1 \times \cdots \times \alpha_{n-1}^2}$ is exact. Ergo $\mathbb{T}_{\widetilde{\boldsymbol{\alpha}}}$ is an exact Lagrangian in the exact symplectic manifold M, and similarly $\mathbb{T}_{\widetilde{\boldsymbol{\beta}}}$ is as well.

The reader may at this point be alarmed that we have failed to check that τ is a symplectic involution. We amend this by replacing $\omega_{\psi+\chi}$ by $\omega = \frac{1}{2}(\omega_{\psi+\chi} + \tau^*\omega_{\psi+\chi})$. Since τ

is a holomorphic involution, the exact form ω is $\mathrm{Sym}^{2n-2}(j)$-compatible and nondegenerate. Moreover, since τ preserves $\mathbb{T}_{\boldsymbol{\alpha}}$ and $\mathbb{T}_{\boldsymbol{\beta}}$, it is still the case that these two submanifolds are exact Lagrangians with respect to ω. This ω is our final choice of symplectic form on M.

5.2 Homotopy Type and Cohomology of Symmetric Products of Punctured Heegaard Surfaces

Recall that to prove the existence of a stable normal trivialization for the triples (M, L_0, L_1) introduced in Section 4.1, it suffices to prove that given a localizable Heegaard diagram pair $\mathcal{D}$ and $\widetilde{\mathcal{D}}$, the standard triple (M_0, L_0, L_1) carries a stable normal trivialization. Here $M_0 = \mathrm{Sym}^{2n-2}(\Sigma(S)\backslash\{\widetilde{\mathbf{w}}\})$, and correspondingly $M_0^{\mathrm{inv}} = \mathrm{Sym}^{n-1}(S^2\backslash\{\mathbf{w}\})$.

Let us now proceed to discuss the homotopy type and cohomology of M_0^{inv}, building up the structure we will use in the proof that this manifold satisfies the complex conditions which imply the existence of a stable normal trivialization.

We claim M_0^{inv} deformation retracts onto each of $\mathbb{T}_{\boldsymbol{\alpha}}$ and $\mathbb{T}_{\boldsymbol{\beta}}$. To check this, we use a lemma whose proof essentially follows an argument of Ong [27].

Lemma 5.2.1. *The rth symmetric product of a wedge of k circles $\vee_{i=1}^{k} S_i^1$ deformation retracts onto the r-skeleton of the k torus $\prod_{i=1}^{k} S_i^1$, where each circle is given a CW structure consisting of the wedge point and a single one-cell, and the torus has the natural product CW structure.*

Proof. Let $\sigma_1, \cdots, \sigma_r$ be the first r elementary symmetric functions of r variables $x_1, \cdots, x_r$, such that

$$\sigma_j(x_1, \cdots, x_r) = \sum_{1 \leq i_1 < \cdots < i_j \leq r} x_{i_1} \cdots x_{i_j}.$$

There is a well-known biholomorphism between $\mathrm{Sym}^r(\mathbb{C})$ and $\mathbb{C}^r$ by evaluating each of the functions σ_j for $1 \leq j \leq r$ on an unordered set of complex numbers $x_1, \cdots, x_r$. This has the effect of mapping to the ordered coefficients of the monic polynomial with roots $x_1, \cdots, x_r$, multiplied by an alternating sign.

$$\begin{aligned}\phi\colon\ \mathrm{Sym}^r(\mathbb{C}) &\to \mathbb{C}^r \\ (x_1\cdots x_r) &\mapsto (\sigma_1(x_1,\cdots,x_r),\cdots,\sigma_r(x_1,\cdots,x_r)).\end{aligned} \tag{5.2.1}$$

Under the map ϕ the submanifold $\mathrm{Sym}^r(\mathbb{C}^*)$ is carried to $\mathbb{C}^{r-1}\times\mathbb{C}^*$. We can use this map to construct the following homotopy equivalence between S^1 and $\mathrm{Sym}^r(S^1)$ for any r.

$$\mathrm{Sym}^r(S^1) \hookrightarrow \mathrm{Sym}^r(\mathbb{C}^*) \xrightarrow{\phi|_{\mathrm{Sym}^r(\mathbb{C}^*)}} \mathbb{C}^{r-1}\times\mathbb{C}^* \longrightarrow \mathbb{C}^* \longrightarrow S^1$$

Here the first inclusion map is a homotopy equivalence and the final two maps are deformation retractions. The total map carries $(e^{i\theta_1}\cdots e^{i\theta_r})\in\mathrm{Sym}^r(S^1)$ to $\sigma_r((e^{i\theta_1},\cdots,e^{i\theta_r}))$, which is exactly the product of the entries, $e^{i(\theta_1+\cdots+\theta_r)}$. Ergo the total map from $\mathrm{Sym}^r(S^1)$ to S^1 which multiplies the entries of an unordered r-tuple of points on the circle is a homotopy equivalence. Indeed, since this homotopy equivalence can be regarded as a retract from $\mathrm{Sym}^r(S^1)$ to its subspace $A_r = \{(e^{i\theta_1}e^{i\theta_2}\cdots e^{i\theta_r}) : \theta_2=\cdots=\theta_r=0\} = S^1\times\{1\}^{r-1}$, there must be a deformation retraction F^r from $\mathrm{Sym}^r(S^1)$ to this subspace.

It will be useful to be slightly more careful concerning our choice of deformation retraction. Suppose $\mathrm{Sym}^{r-1}(S^1)$ is regarded as a subspace $\mathrm{Sym}^{r-1}\times\{1\}$ of $\mathrm{Sym}^r(S^1)$ via the embedding $(e^{i\theta_1}\cdots e^{i\theta_{r-1}})\mapsto((e^{i\theta_1}\cdots e^{i\theta_{r-1}}e^{i0})$. Then both $\mathrm{Sym}^r(S^1)$ and $\mathrm{Sym}^{r-1}(S^1)\times\{1\}$ deformation retract onto the subspace A_r in $\mathrm{Sym}^r(S^1)$, implying that the relative homotopy groups $\pi_i(\mathrm{Sym}^r(S^1),\mathrm{Sym}^{r-1}(S^1)\times\{1\})$ are trivial. Hence since all the spaces involved carry CW structures induced by the CW structure on S^1 and the inclusion $\mathrm{Sym}^{r-1}(S^1)\hookrightarrow\mathrm{Sym}^r(S^1)$ is cellular, there is a deformation retraction $F^{r,0}$ from $\mathrm{Sym}^r(S^1)$ to $\mathrm{Sym}^{r-1}(S^1)\times\{1\}$, which can be taken to run on a time interval $[0,\frac{1}{r-1}]$. By similar logic for $r\geq k\geq 2$, there is a deformation retraction $F^{r,r-k}$ from $\mathrm{Sym}^k(S^1)\times\{1\}^{r-k}$ to $\mathrm{Sym}^{k-1}(S^1)\times\{1\}^{r-k+1}$ whose time input can be taken to be $[\frac{r-k}{r-1},\frac{r-k+1}{r-1}]$. If we take F^r to be the map $\mathrm{Sym}^r(S^1)\times[0,1]\to\mathrm{Sym}^r(S^1)$ which is $F^{r,r-k}$ on $[\frac{r-k}{r-1},\frac{r-k+1}{r-1}]$, then F^r is a deformation retraction from $\mathrm{Sym}^r(S^1)$ to $A_r = \mathrm{Sym}^1(S^1)\times\{1\}^{r-1}$ which preserves each of the subspaces $\mathrm{Sym}^k\times\{1\}^{r-k}$.

We can now deal with the symmetric product of a wedge of circles. To keep track of which points originate in which circle, label the circles $S^1_1, \cdots, S^1_m$ and refer to points $e^{i\theta}_j$ on S^1_j. Take the wedge point to be the basepoint $1 = e^{i0}$ on each circle.

Regard the space $\mathrm{Sym}^r(\bigvee_{i=1}^m S^1_i)$ as the canonical subspace $\mathrm{Sym}^r(\bigvee_{i=1}^m S^1_i) \times \{1\}^{r(m-1)}$ of $\mathrm{Sym}^{rm}(\bigvee_{i=1}^m S^1_i)$. Any point in $\mathrm{Sym}^r(\bigvee_{i=1}^m S^1_i) \times \{1\}^{r(m-1)}$ may be split uniquely into a point in $\mathrm{Sym}^r(S^1_1) \times \mathrm{Sym}^r(S^1_2) \times \cdots \times \mathrm{Sym}^r(S^1_m)$, as there are at most r terms from any circle S^1_i in an mr-tuple in $\mathrm{Sym}^r(\bigvee_{i=1}^m S^1_i) \times \{1\}^{r(m-1)}$. Consider applying $F^r\colon\ \mathrm{Sym}^r(S^1_1) \times [0,1]$ to this submanifold; that is, consider the following map.

$$\begin{aligned} F^r_1\colon\ \mathrm{Sym}^r(S^1_1) \times \mathrm{Sym}^r(S^1_2) \times \cdots \times \mathrm{Sym}^r(S^1_m) \times [0,1] \to& \\ &\mathrm{Sym}^r(S^1_1) \times \mathrm{Sym}^r(S^1_2) \times \cdots \times \mathrm{Sym}^r(S^1_m) \\ ((e^{i\theta_1}_1 \cdots e^{i\theta_r}_1), \cdots, (e^{i\theta_1}_m \cdots e^{i\theta_r}_m)) \times [0,1] \mapsto& \\ &(F^r(e^{i\theta_1}_1 \cdots e^{i\theta_r}_1, t), \cdots, (e^{i\theta_1}_m \cdots e^{i\theta_r}_m)). \end{aligned}$$

Because F^r never increases the number of nonbasepoint terms in $(e^{i\theta_1}_1 \cdots e^{i\theta_r}_1)$, the map F^r_1 preserves the subspace $\mathrm{Sym}^r(\bigvee_{i=1}^m S^1_i) \times \{1\}^{r(m-1)}$ in $\mathrm{Sym}^r(S^1_1) \times \mathrm{Sym}^r(S^1_2) \times \cdots \times \mathrm{Sym}^r(S^1_m)$. Ergo F^r_1 is a deformation retraction from $\mathrm{Sym}^r(\bigvee_{i=1}^m S^1_i)$ to the subspace of r-tuples in $\mathrm{Sym}^r(\bigvee_{i=1}^m S^1_i)$ of which at most one entry is a nonbasepoint point on S^1_1. Running this procedure on each factor of $\mathrm{Sym}^r(S^1_1) \times \mathrm{Sym}^r(S^1_2) \times \cdots \times \mathrm{Sym}^r(S^1_m)$ produces a deformation retraction from $\mathrm{Sym}^r(\bigvee_{i=1}^m S^1_i)$ to the subspace of r-tuples in this space containing only one nontrivial point of each S^1_j for $1 \leq j \leq r$. So we have the following homotopy equivalence.

$$\mathrm{Sym}^r(\bigvee_{i=1}^m S^1_i) \simeq \{(e^{i\theta_1}_1, \cdots, e^{i\theta_m}_m)\} \subset S^1_1 \times \cdots \times S^1_m :\ \text{at most r } \theta_i \text{ are nonzero}\}$$

If $r > m$, this space is all of $S^1_1 \times \cdots \times S^1_m = (S^1)^m$. If $r < m$, this space is exactly $\bigcup_{I=|r|}(S^1)^I \subset (S^1)^m$. $\square$

We will apply this observation to M^{inv}_0. In S^2, let $\nu_i\colon\ [0,1] \to S^2$ be a small closed curve around w_i for $1 \leq i \leq n$, such that ν_i is oriented counterclockwise in the complement of w_0. Then

$$H_1(S^2 \backslash \{\mathbf{z}\}) = \mathbb{Z}\langle \nu_1, \cdots, \nu_{n-1}\rangle.$$

Now $S^2 \backslash \{\mathbf{w}\}$ deformation retracts onto a wedge of $n-1$ circles $\vee_{i=1}^{n-1} \nu_i'$, where ν_i' is a closed curve homotopic to ν_i which passes once through the origin. Ergo $M_0^{\text{inv}} = \text{Sym}^{n-1}(S^2 \backslash \{\mathbf{w}\})$ deformation retracts onto the symmetric product of $\vee_{i=1}^{n-1} \nu_i'$, which in turn deformation retracts onto the product $\prod_{i=1}^{n-1} \nu_i'$. However, this product is homotopy equivalent to $\prod_{i=1}^{n-1} \nu_i$, and indeed to $\prod_{i=1}^{n-1} \alpha_i$ and to $\prod_{i=1}^{n} \beta_i$. We conclude that M_0^{inv} has the homotopy type of an $n-1$-torus and in particular admits a deformation retraction onto each of $\mathbb{T}_{\boldsymbol{\alpha}}$ and $\mathbb{T}_{\boldsymbol{\beta}}$.

We will require a concrete description of the cohomology rings of M_0^{inv}, $\mathbb{T}_{\boldsymbol{\alpha}}$, and $\mathbb{T}_{\boldsymbol{\beta}}$ for the computations of Chapter 7, so we pause to supply one now. Consider the one cycles

$$\begin{aligned} \overline{\nu_i}\colon\ [0,1] &\to M^{\text{inv}} \\ t &\mapsto (\nu_i(t) x_0 \cdots x_0) \end{aligned}$$

where x_0 is any choice of basepoint. The $[\overline{\nu_i}]$ form a basis for $H_1(M^{\text{inv}})$. Ergo, letting $\widehat{[\overline{\nu_i}]}$ denote the dual of $\overline{\nu_i}$, we have

$$\begin{aligned} H^1(M^{\text{inv}}) &= \mathbb{Z}\langle \widehat{[\overline{\nu_1}]}, \cdots, \widehat{[\overline{\nu_{n-1}}]} \rangle \\ H^k(M^{\text{inv}}) &= \bigwedge\nolimits^k H^1(M^{\text{inv}}) \end{aligned}$$

Similarly, we can write down the homology of the tori $\mathbb{T}_{\boldsymbol{\alpha}}$ and $\mathbb{T}_{\boldsymbol{\beta}}$. Through a slight abuse of notation, let us insist that we have parametrizations $\alpha_i\colon\ [0,1] \to S^2$ running counterclockwise in $S^2 \backslash \{w_0\}$ and β_i running clockwise. The first homology of $\mathbb{T}_{\boldsymbol{\alpha}}$ is generated by one-cycles

$$\begin{aligned} \overline{\alpha_i}\colon\ [0,1] &\to \alpha_1 \times \cdots \times \alpha_{n-1} \\ t &\mapsto (y_1, \cdots, y_{i-1}, \alpha_i(t), y_{i+1}, \cdots, y_{n-1}) \end{aligned}$$

where $y_j = \alpha_j(0)$, and thus the cohomology of this torus is

$$\begin{aligned} H^1(\mathbb{T}_{\boldsymbol{\alpha}}) &= \mathbb{Z}\langle \widehat{[\overline{\alpha_1}]}, \cdots, \widehat{[\overline{\alpha_{n-1}}]} \rangle \\ H^k(\mathbb{T}_{\boldsymbol{\alpha}}) &= \bigwedge\nolimits^k H^1(\mathbb{T}_{\boldsymbol{\alpha}}) \end{aligned}$$

We apply analagous naming conventions to $\mathbb{T}_{\boldsymbol{\beta}}$, obtaining

$$H^1(\mathbb{T}_{\boldsymbol{\beta}}) = \mathbb{Z}\langle\widehat{[\overline{\beta_1}]}, \cdots, \widehat{[\overline{\beta_{n-1}}]}\rangle$$
$$H^k(\mathbb{T}_{\boldsymbol{\beta}}) = \bigwedge^k H^1(\mathbb{T}_{\boldsymbol{\beta}})$$

Moreover, observe that under the map on homology induced by inclusion $\iota\colon X \hookrightarrow M^{\mathrm{inv}} \times [0,1]$, both $[\overline{\alpha_i}]$ and of $[\overline{\beta_i}]$ are sent to $[\overline{\nu_i}]$. Therefore the map ι^* on cohomology $H^k(M^{\mathrm{inv}} \times [0,1]) \to H^k(X)$ is precisely the diagonal map, with

$$\bigwedge_{j=1}^m \widehat{[\overline{\nu_{i_j}}]} \mapsto \bigwedge_{j=1}^m \widehat{[\overline{\alpha_{i_j}}]} + \bigwedge_{j=1}^m \widehat{[\overline{\beta_{i_j}}]}$$

Corollary 5.2.2. *The relative cohomology $H^*(M_0^{\mathrm{inv}} \times [0,1], X)$ is the cohomology of the torus $(S^1)^{n-1}$, and in particular is torsion-free.*

Proof. Consider the long exact sequence

$$\cdots H^{m-1}(X) \longrightarrow H^m(M_0^{\mathrm{inv}} \times [0,1], X) \xrightarrow{q^*} H^m(M_0^{\mathrm{inv}}) \xrightarrow{\iota^*} H^m(X) \cdots$$

Taking into account that the obvious isomorphism between $H^*(\mathbb{T}_{\boldsymbol{\alpha}})$ and $H^*(\mathbb{T}_{\boldsymbol{\beta}})$ respects our labelling of the cohomology classes, ι^* is the diagonal map, and in particular an injection. When $m \geq 2$ we have the following short exact sequences.

$$0 \to H^{m-1}(M_0^{\mathrm{inv}} \times [0,1]) \xrightarrow{\iota^*} H^{m-1}(X) \longrightarrow H^m(M_0^{\mathrm{inv}} \times [0,1], X) \to 0$$
$$\bigwedge_{j=1}^{m-1} \widehat{[\overline{\nu_{i_j}}]} \longmapsto \bigwedge_{j=1}^{m-1} \widehat{[\overline{\alpha_{i_j}}]} + \bigwedge_{j=1}^{m-1} \widehat{[\overline{\beta_{i_j}}]}$$

Ergo for $m \geq 2$, $H^m(M_0^{\mathrm{inv}} \times [0,1], X) \cong H^{m-1}(\mathbb{T}_{\boldsymbol{\alpha}}) \cong H^{m-1}(T_{\boldsymbol{\beta}})$. (We will have occasion to be careful about the generators in Chapter 7.) The same result follows for $H^1(M_0^{\mathrm{inv}} \times [0,1], X)$ trivially.

$\square$

Chapter 6

Important Constructions from K-Theory

Central to our argument that the triples (M, L_0, L_1) introduced in Section 4.1 admit stable normal trivializations will be several useful results from complex K-theory, most trivial but one rather less so. A detailed treatment of the subject, along with proofs of all results up to Proposition 6.0.10, may be found in [16].

Let V be a complex vector bundle over a base space X, which for purposes of this paper we take to be a compact Hausdorff topological space. We let $\text{Vect}_n^{\mathbb{C}}(X)$ denote the set of isomorphism classes of n-dimensional vector bundles over X. This is a monoid under the direct sum of vector bundles. For any n, as before we refer to the trivial complex vector bundle of degree n over X simply as $\mathbb{C}^n$. Then there is an equivalence relation $\approx_S$ on $\text{Vect}_n^{\mathbb{C}}(X)$ defined as follows: given V, W two n-dimensional vector bundles over X, we say that $V \approx_S W$ if and only if there is some m such that $V \oplus \mathbb{C}^m \cong W \oplus \mathbb{C}^m$. The two vector bundles V and W are said to be *stably isomorphic*. This relation respects the direct sums and tensor products of vector bundles, such that the set of equivalence classes of bundles under $\approx_S$ inherits two abelian laws of composition $[V]_S + [W]_S = [V \oplus W]_S$ and $[V]_S \times [W]_S = [V \otimes W]_S$. This set of equivalence classes may be given the structure of a ring by formally adjoining the inverse of each element under the direct sum. More precisely, we set

$$K^0(X) = \{[V]_S - [W]_S : [V]_S, [W]_S \text{ are equivalence classes with respect to } \approx_S\}.$$

Commonly $[V]_S - [\mathbb{C}^0]_S$ will be written simply as $[V]_S$ and its additive inverse $[\mathbb{C}^0]_S - [V]_S$ simply as $-[V]_S$.

Lemma 6.0.3. *$K^0(X)$ is a ring with respect to the operations $[V]_S + [W]_S = [V \oplus W]_S$ and $[V]_S \times [W]_S = [V \otimes W]_S$.*

There is also a reduced form of this ring $\widetilde{K}(X)$ constructed as follows. Let $\sim$ be a second equivalence relation on $\bigcup_{n\in\mathbb{N}} \text{Vect}_n^{\mathbb{C}}$ such that $V \sim W$ if there is some m_1, m_2 such that $V \oplus \mathbb{C}^{m_1} \cong W \oplus \mathbb{C}^{m_2}$. In this case one can show that the set of equivalence classes with respect to $\sim$ contains additive inverses without adjoining any additional elements. Let the equivalence class of a vector bundle V with respect to $\sim$ be $[V]$.

Lemma 6.0.4. *$\widetilde{K}^0(X)$ is a ring with respect to the operations $[V] + [W] = [V \oplus W]$ and $[V] \times [W] = [V \otimes W]$.*

Then $K^0(X) \cong \widetilde{K}^0(X) \oplus \mathbb{Z}$. In both cases a vector bundle in the same equivalence class as $\mathbb{C}^m$ is said to be *stably trivial*. Notice as a most basic case that for $\{x_0\}$ a one point space $K^0(x_0) = \mathbb{Z}$ and $\widetilde{K}^0(x_0) = 0$.

Given a continuous map $f\colon X \to Y$, the corresponding map $f^*\colon \text{Vect}_n^{\mathbb{C}}(Y) \to \text{Vect}_n^{\mathbb{C}}(X)$ which maps a vector bundle V over Y to its pullback $f^*(V)$ descends to maps $f^*\colon K^0(Y) \to K^0(Y)$ and $f^*\colon \widetilde{K}^0(Y) \to \widetilde{K}^0(X)$. Recalling that if maps $f, g\colon X \to Y$ are homotopic then $f^*(V)$ and $g^*(V)$ the two pullbacks of a bundle V over Y are isomorphic, we see that homotopic maps f and g induce the same maps $f^* = g^*$ on $K^0(Y)$ and $\widetilde{K}^0(Y)$. In particular, we have the following lemma.

Lemma 6.0.5. *If $f\colon X_1 \to X_2$ is a homotopy equivalence with homotopy inverse $g\colon X_2 \to X_1$, the induced map $f^*\colon K^0(X_2) \to K^0(X_1)$ is an isomorphism with inverse g^*. The same is true of the reduced theory.*

We will use this lemma to deal with the minor problem that M^{inv} is not actually compact. The three spaces whose K-theory will be of interest to us are $M^{\text{inv}} \times [0,1]$, its subspace $X = L_0^{\text{inv}} \times \{0\} \cup L_1^{\text{inv}} \times \{1\}$, and the quotient space M^{inv}/X. The space X is a disjoint union of two tori, hence compact. We have seen previously that M^{inv} deformation retracts onto a compact subspace homeomorphic to the $n-1$ skeleton of $(S^1)^{2n-1}$. For purposes of dealing with $K^0(M^{\text{inv}})$ let us choose a slightly different deformation retraction. Let $Y = (\cup\alpha_i)\bigcup(\cup\beta_i)$. Then let $F\colon\ (S^2\backslash\{\mathbf{w},\mathbf{z}\})\times[0,1]\to S^2\backslash\{\mathbf{w},\mathbf{z}\}$ be a deformation retraction from the punctured sphere to the union of Y and all components of $S^2\backslash Y$ not containing any point of $\{\mathbf{w},\mathbf{z}\}$. This is a compact subspace of the punctured sphere; let it be Z. Then F induces a deformation retraction $\text{Sym}^{n-1}(F)$ of $M^{\text{inv}} = \text{Sym}^{n-1}(S^2\backslash\{\mathbf{w},\mathbf{z}\})$ onto the compact set $\text{Sym}^{n-1}(Z)$, so we can legitimately refer to the K-theory of M^{inv} by identifying it with $K^0(\text{Sym}^{n-1}(Z))$. Notice further that F is the identity on $Y\times I$, so the induced deformation retraction from M^{inv} preserves $X\subset\text{Sym}^{n-1}(Y)$. Therefore we may produce a deformation retract $\overline{\text{Sym}^{n-1}(F)}$ from M^{inv}/X to $\text{Sym}^{n-1}(Z)/X$, and $\text{Sym}^{n-1}(Z)/X$ is compact Hausdorff. So we identify the K-groups of M^{inv}/X with those of $\text{Sym}^{n-1}(Z)/X$, and from now on make no further reference to this technical subtlety.

The relationship between homotopy classes of maps and pullbacks of vector bundles in fact gives us the following deeper proposition, which follows from the theorem that any n-dimensional complex vector bundle over X is a pullback of the canonical n-dimensional bundle over the Grassmanian $G_n(\mathbb{C}^\infty)$ along some homotopy class of maps $X\to G_n(\mathbb{C}^\infty)$.

Proposition 6.0.6. $\widetilde{K}^0(X)\cong[X,BU]$

Here BU is the classifying space of the infinite unitary group.

We extend our definition of $\widetilde{K}^0(X)$ as follows. Let $\widetilde{K}^{-i}(X)=\widetilde{K}^0(\Sigma^i(X))$, where $\Sigma(X)$ is (for this section only) the reduced suspension of X. Then we have the following.

Proposition 6.0.7. *(The Bott Periodicity Theorem)* $\widetilde{K}^0(X)\cong\widetilde{K}^0(\Sigma^2(X))$.

We use this two-periodicity to define $\widetilde{K}^i(X)$ for positive i. Now notice that $\widetilde{K}^0(X) = K^0(X^+)$, where X^+ denotes the one-point compactification of X (when X is compact, this will mean X together with a point at infinity). In particular, $K^1(X)=\widetilde{K}^0(\Sigma(X))$. Then the groups $K^i(X)$ are also doubly-periodic.

With $K^i(X)$ now defined for all $i \in \mathbb{Z}$, we are ready for the following proposition, which deals with $K^*(X) = \cup_i K^i(X)$ merely as a collection of abelian groups and (momentarily) ignores their ring structure.

Proposition 6.0.8. *The groups $K^i(X)$ form a generalized cohomology theory on compact topological spaces. Likewise, the groups $\widetilde{K}^i(X)$ form a reduced cohomology theory.*

A relevant note is that if $\{x_0\}$ is a one point space, we have

$$K^i(\{x_0\}) = \begin{cases} \mathbb{Z} \text{ if } i \text{ even} \\ 0 \text{ if } i \text{ odd .} \end{cases}$$

One consequence of this structure on the K-theory of a space is that there is a notion of the relative K-theory of a compact space X and a closed subspace Y. We let $K^0(X, Y)$ be the ring of isomorphism classes of vector bundles over X which restrict to a trival bundle over Y. Equivalently, $K^0(X, Y) \cong \widetilde{K}^0(X/Y)$; notice that if X is compact Hausdorff and Y is a closed subspace, X/Y is compact Hausdorff, so this statement makes sense. With respect to this construction, the long exact sequence of a pair (X, x_0) consisting of a space X together with a basepoint x_0 becomes a long exact sequence relating the groups $K^i(X)$ and $\widetilde{K}^i(X)$.

$$\cdots \widetilde{K}^i(X) \to K^i(X) \to K^i(x_0) \to \widetilde{K}^{i-1}(X) \cdots$$

For our purposes, the crucial results from K-theory will be those relating $K^*(X)$ to the rational cohomology $H(X; \mathbb{Q})$ of X. The exact sequence above, and the corresponding exact sequence on the reduced and unreduced rational cohomology of X, will allow us to move between maps on $K^*(X)$ and $\widetilde{K}^*(X)$ with relative ease.

Recall that the Chern classes of a vector bundle V over a space X are a set of natural characteristic classes $c_i(V) \in H^{2i}(X)$. The total Chern class of an n-dimensional vector bundle V is $c(V) = c_0(V) + c_1(V) + \cdots + c_n(X)$. If X has the structure of a complex manifold, as do most spaces of interest to us, we will often use $c(X)$ to refer to the total Chern class of the tangent bundle TX of X. The Chern classes may be used to produce a ring homomorphism $K^0(X) \cup K^1(X) \to H^*(X; \mathbb{Q})$.

Let $\sigma_j(x_1, \cdots, x_k)$ be the elementary symmetric functions on k elements as in Chapter 5. Then we assert the existence of a new set of polynomials $s(x_1, \cdots, x_k)$ with the following properties.

Lemma 6.0.9. *There exists a unique set of polynomials $s_j(x_1, \cdots, x_k)$ with the property that $s_j(\sigma_1(x_1, \cdots, x_k), \cdots, \sigma_k(x_1, \cdots, x_k)) = x_1^j + \cdots + x_k^j$.*

The polynomials s_j are defined recursively in terms of the elementary symmetric functions and the s_i of lower degree by the following relation.

$$s_j = \sigma_1 s_{j-1} - \sigma_2 s_{j-2} + \cdots + (-1)^{j-2}\sigma_{j-2}s_2 + (-1)^{j-1}\sigma_{j-1}s_1$$

We can now define the Chern character of a vector bundle.

$$\begin{aligned} \mathrm{ch}\colon\ K^0(X) &\to H^{\text{even}}(X; \mathbb{Q}) \\ [V]_S &\mapsto n + \sum_{j>0} s_k(c_1(V), \cdots, c_k(V))/k! \end{aligned}$$

While this is the quickest way to produce this definition, it is not clear that it is necessarily the most intuitive. To clarify: this definition is explicitly chosen so that if L is a line bundle over X and the total Chern class of L is $1 + c_1(L)$, then $\mathrm{ch}(L) = 1 + c_1(L) + \frac{c_1(L)^2}{2!} + \frac{c_1(L)^3}{3!} + \cdots$, the "exponential" of the total Chern class of L. As the map ch was intended to be a ring isomorphism on $K^0(X)$, we next require that for a product of line bundles $L_1 \otimes L_2 \otimes \cdots \otimes L_k$ over X we have $ch(L_1 \otimes \cdots \otimes L_k) = \mathrm{ch}(L_1) \cdots \mathrm{ch}(L_k)$. A computation leads to the formula above.

The Chern character descends to a reduced map $\widetilde{\mathrm{ch}}\colon\ \widetilde{K}^0(X) \to \widetilde{H}^{\text{even}}(X; \mathbb{Q})$. We may also consider the Chern character of vector bundles on $\Sigma(X)$, which yields the following map into the odd cohomology of X.

$$\mathrm{ch}\colon\ K^1(X) = \widetilde{K}^0(\Sigma(X)) \to \widetilde{H}^{\text{even}}(\Sigma(X); \mathbb{Q}) \cong H^{\text{odd}}(X; \mathbb{Q})$$

Similarly, we define $\widetilde{\mathrm{ch}}\colon\ \widetilde{K}^1(X) \to \widetilde{H}^{\text{odd}}(X; \mathbb{Q})$. We then have the following extremely useful result.

Proposition 6.0.10. *The Chern character induces a rational isomorphism*

$$ch\colon\ K^0(X) \cup K^1(X) \otimes \mathbb{Q} \to H^*(X;\mathbb{Q}).$$

In particular, the rank of $K^0(X) \cup K^1(X)$ is equal to the rank of $H^*(X)$. The analogous statement holds for $\widetilde{ch}$ and the two reduced theories.

The proof of this proposition takes Bott periodicity as a base case and is laid out in [16, Thm 4.5].

We will last need a K-theoretic statement about relationship between the torsion of $H^*(X)$ and $K^*(X)$. As it is a rather high-powered result, we'll say a few words about the proof, which relies on the Atiyah-Hirzebruch spectral sequence, a device for computing an arbitrary cohomology theory first described by its eponymous introducers in [2].

Proposition 6.0.11. *Let $h^n(X)$ be any cohomology theory. Then there is a spectral sequence which converges to $h^n(X)$ whose E^2 page is given by*

$$E^2_{i,j} = H^i(X; h^j(\{pt\}).$$

In the particular case of the cohomology theory $K^*(X)$ this spectral sequence collapses rationally, leading to a useful statement concerning the torsion groups of $K^*(X)$.

Proposition 6.0.12. *[2, Section 2.5] Let X be a compact Hausdorff space. If $H^i(X;\mathbb{Z})$ is torsion-free for all i and finitely generated, then $K^0(X) \cup K^1(X)$ is torsion-free of the same rank.*

Proof. Recall that if $\{x_0\}$ is a one-point space then $K^q(x_0)$ is $\mathbb{Z}$ for q even and 0 for q odd. Since unreduced K-theory is a cohomology theory, Proposition 6.0.11 gives first and second quadrant spectral sequence converging to $K^*(X)$ whose E^2 page has entries

$$E^2_{p,q} = H^p(X; K^q(\{x_0\})) = \begin{cases} H^p(X) \text{ if } q \text{ is even} \\ 0 \text{ if } q \text{ is odd} \end{cases}.$$

The rank of $K^i(X)$ is the sum $\sum_{p+q=i} rk(E^\infty_{p,q})$. Moreover, the ranks of the corresponding entries on the E_2 page of the spectral sequence sum to $\sum_{p+q=i} rk(E^2_{p,q}) = \sum_{p\equiv n \bmod 2} H^p(X)$. By Proposition 6.0.10, the rank of $K^i(X)$ and the total rank of the integer cohomology groups of degree the same parity as i are equal, so $\sum_{p+q=i} rk(E^\infty_{p,q}) = \sum_{p+q=i} rk(E^2_{p,q})$. As $rk(E^\infty_{p,q}) \leq rk(E^2_{p,q})$ for all p, q, we see that in fact this is an equality for all pairs (p, q).

Now notice that every entry on the E_2 page of this spectral sequence is free abelian. Suppose $E^\infty_{p,q}$ has a nontrivial torsion subgroup for some fixed p, q such that q is even. Then there must be some first $E^r_{p,q}$ such that $r > 2$ with this property. Because $E^r_{p,q}$ is a quotient of a subgroup of the free abelian group $E^{r-1}_{p,q}$, if it contains a torsion subgroup it must have lower rank than $E^r_{p,q}$, implying that $E^\infty_{p,q}$ has strictly lower rank than $E^2_{p,q}$. We have seen that this never happens. Therefore the E^∞ page of our spectral sequence is torsion-free, implying that $K^*(X)$, which is filtered by the entries on the E^∞ page, is also free abelian.

This argument, which is similar to the original proof of [2, Section 2.5] was suggested by Dan Ramras. □

Our proof that $(M^{\text{inv}}, L_0^{\text{inv}}, L_1^{\text{inv}})$ carries a stable normal trivialization will contain as a crucial step a proof that $H^*(M^{\text{inv}}/(L_0^{\text{inv}} \times \{0\} \cup L_1^{\text{inv}} \times \{1\}))$ is torsion-free, and thus that the K-theory of this space must likewise be free abelian.

Chapter 7

The Existence of Stable Normal Trivializations

Recall that in Chapter 5, we took $\mathcal{D}$, $\widetilde{\mathcal{D}}$ to be a localizable pair of Heegaard diagrams in the sense of 3.3.1, and $\{\mathbf{r}\}$ to be a subset of the basepoints $\{\mathbf{w}, \mathbf{z}\}$ on $\mathcal{D}$ containing $\{\mathbf{w}\}$. We then let $\{\widetilde{\mathbf{r}}\}$ be the lift of $\{\mathbf{r}\}$ on $\widetilde{\mathcal{D}}$. We saw that the triple (M, L_0, L_1), where M is the symmetric product of $\widetilde{\mathcal{D}}$ punctured along $\{\mathbf{r}\}$ and L_0, L_1 are the Lagrangians $\mathbb{T}_{\widetilde{\boldsymbol{\beta}}}$, $\mathbb{T}_{\widetilde{\boldsymbol{\alpha}}}$, satisfies the symplectic geometry conditions of Seidel–Smith localization. We now proceed to check that $(M^{\text{inv}}, L_0^{\text{inv}}, L_1^{\text{inv}})$ fulfills the complex conditions that, by Lemma 4.0.11, imply the existence of a stable normal trivialization. In fact it suffices to check this fact for the case that the set of punctures is $\{\mathbf{r}\}$ is equal to $\{\mathbf{w}\}$. As in Section 5.2, let us refer to the manifold generated in this particular case as M_0. Then the complex relative trivialization we produce then restricts to a relative trivialization on the submanifold M of M_0 generated by considering any larger set $\{\mathbf{r}\}$.

Proposition 7.0.13. *Consider the complex manifold $M_0 = \text{Sym}^{2n-1}(\Sigma(S^2)\backslash\{\widetilde{\mathbf{w}}\})$ together with its totally real submanifolds $L_0 = \mathbb{T}_{\boldsymbol{\beta}}$ and $L_1 = \mathbb{T}_{\boldsymbol{\alpha}}$, and the holomorphic involution τ which preverves L_0 and L_1. The map*

$$(M_0^{\text{inv}} \times [0,1], (L_0 \times \{0\}) \cup (L_1 \times \{1\})) \to (BU, BO)$$

which classifies the pullback $\Upsilon(M_0^{\text{inv}}) = N(M_0^{\text{inv}}) \times [0,1]$ *of the complex normal bundle of* M_0^{inv} *together with the totally real subbundles* $NL_0^{\text{inv}} \times \{0\}$ *over* $L_0^{\text{inv}} \times \{0\}$ *and* $J(NL_1^{\text{inv}}) \times \{1\}$ *over* $L_1^{\text{inv}} \times \{1\}$ *is nulhomotopic.*

As a first step, we must establish the complex triviality of NM_0^{inv} (and thus of $\Upsilon(M_0^{\text{inv}})$) and the real triviality of NL_i^{inv} for $i = 1, 2$.

Lemma 7.0.14. *The complex bundle* NM_0^{inv} *is stably trivial.*

Proof. The inclusion map $\iota_1 \colon (S^2 \backslash \{\mathbf{w}\}) \hookrightarrow S^2$ is nulhomotopic. Therefore the induced inclusion $\mathrm{Sym}^{n-1}(\iota_1) \colon \mathrm{Sym}^{n-1}(S^2 \backslash \{\mathbf{w}\}) \hookrightarrow \mathrm{Sym}^{n-1}(S^2)$ is also nulhomotopic. The normal bundle of $M_0^{\text{inv}} = \mathrm{Sym}^{n-1}(S^2 \backslash \{\mathbf{w}\})$ in $M_0 = \mathrm{Sym}^{2n-2}(\Sigma(S^2) \backslash \{\widetilde{\mathbf{w}}\})$ is exactly the restriction of the normal bundle to $\mathrm{Sym}^{n-1}(S^2)$ in $\mathrm{Sym}^{2n-2}(\Sigma(S^2))$ along the inclusion map $\mathrm{Sym}^{n-1}(\iota_1)$. As the map $\mathrm{Sym}^{n-1}(\iota_1)$ is nulhomotopic, NM_0^{inv} is stably trivial. $\square$

We must also show that the real normal bundles of the invariant Lagrangians are trivial.

Lemma 7.0.15. $N(L_i^{\text{inv}})$ *is trivial for* $i = 0, 1$.

Proof. Recall that $L_1 = \mathbb{T}_{\widetilde{\boldsymbol{\alpha}}} \subset \mathrm{Sym}^{2n-2}(\Sigma(S^2) \backslash \{\widetilde{\mathbf{w}}\})$ is the totally real torus $\alpha_1^1 \times \alpha_1^2 \times \cdots \times \alpha_{n-1}^1 \times \alpha_{n-1}^2$. Thus the tangent bundle $TL_1 = T(\alpha_1^1) \times T(\alpha_1^2) \times \cdots \times T(\alpha_{n-1}^1) \times T(\alpha_{n-1}^2)$ is the trivial real tangent bundle to $(S^1)^{2n-2}$. The invariant set $L_1^{\text{inv}} = i(\mathbb{T}_{\boldsymbol{\alpha}}) = i(\alpha_1 \times \cdots \times \alpha_{n-1})$ is embedded in L_1 via

$$\begin{aligned} i|_{\mathbb{T}_\alpha} \colon \mathbb{T}_\alpha &\hookrightarrow \mathbb{T}_{\widetilde{\alpha}} \\ (x_1, \cdots, x_{n-1}) &\mapsto (x_1^1, x_1^2, \cdots, x_{n-1}^1, x_{n-1}^2). \end{aligned}$$

We conclude that

$$\begin{aligned} T(L_1^{\text{inv}}) &= i_*(T(\mathbb{T}_\alpha)) = \{(v_1^1, v_1^2, \cdots, v_{n-1}, v_{n-1}^2) : (v_1, v_2, \cdots, v_{n-1}) \in T(\mathbb{T}_\alpha)\} \subset TL_1 \\ N(L_1^{\text{inv}}) &= \{(v_1^1, -v_1^2), \cdots, v_{n-1}^1, -v_{n-1}^2) : (v_1, \cdots, v_{n-1}) \in T(\mathbb{T}_\alpha\}) \subset TL_1. \end{aligned}$$

The point is that $T(L_1^{\text{inv}}) \simeq T((S^1)^{n-1})$ is trivial, and there is an isomorphism

$$\begin{aligned} T(L_1^{\text{inv}}) &\to N(L_1^{\text{inv}}) \\ (v_1^1, v_1^2, \cdots, v_{n-1}^1, v_{n-1}^2) &\mapsto (v_1^1, -v_1^2, \cdots, v_{n-1}^1, -v_{n-1}^2). \end{aligned}$$

Triviality of $N(L_0^{\text{inv}})$ is proven analogously. □

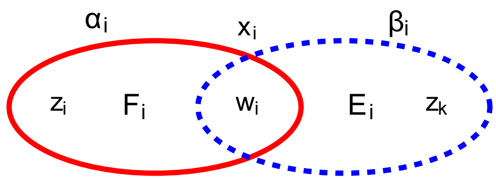
We now turn to the question of relative triviality. Let $X = (L_0 \times \{0\}) \cup (L_1 \times \{1\})$ as in Chapter 5. Choose preferred trivializations of the totally real bundles $NL_0^{\text{inv}} \times \{0\}$ and $J(NL_1^{\text{inv}}) \times \{1\}$ and tensor with $\mathbb{C}$ to extend to a preferred trivialization of the complex bundle $\Upsilon(M_0^{\text{inv}})|_X$. We use this trivialization to pull back $[\Upsilon(M_0^{\text{inv}})] \in \widetilde{K}^0(M_0^{\text{inv}} \times [0,1])$ to a relative bundle $[\Upsilon(M_0^{\text{inv}})]_{\text{rel}} \in \tilde{K}^0((M_0^{\text{inv}} \times [0,1])/X)$. Because the reduced cohomology, and therefore the reduced K-theory, of $(M_0^{\text{inv}} \times [0,1])/X$ have no torsion, to verify triviality of the relative bundle, it suffices to show that the Chern classes of $[\Upsilon(M_0^{\text{inv}})]_{\text{rel}}$ are trivial.

Remark 7.0.16. It may be helpful to draw attention to a minor problem of notation: in earlier sections, $\widetilde{K}$ is a doubly periodic knot, but also in the present section $\widetilde{K}^0(B)$ is the reduced K-theory of the topological space B. We hope this will not occasion confusion.

Fundamentally, the argument for relative triviality rests on the fact that each of the $n-1$ linearly independent periodic domains in $S^2\backslash\{\mathbf{w}\}$ has Maslov index zero. Therefore, we pause here to recall the notation of Section 3.3. Let x_i be the single positive intersection point in $\alpha_i \cap \beta_i$ and y_i the negative intersection point. Let F_i be the closure of the component of $S - \alpha_i - \beta_i$ containing z_i and E_i be the closure of the component of $S - \alpha_i - \beta_i$ containing another basepoint z_k for some k. Then $P_i = E_i - F_i$ is a periodic domain of index zero on $\mathcal{D}$ with boundary $\beta_i - \alpha_i$. Finally, let γ_i be the union of the arc of α_i running from x_i to y_i and the arc of β_i running from x_i to y_i. In particular, this specifies that γ_i has no intersection with any α or β curves other than α_i and β_i, and moreover the component of $S - \gamma_i$ which does not contain w_0 contains only a single basepoint w_i. See Figure 5 for an illustration of the domain P_i.

The structure of the argument is as follows: for each even k such that $1 \leq k \leq 2n-2$, we will use the periodic domains P_i to produce a set of k-chains $\{W_{\mathbf{I}}\}$ in $(M_0^{\text{inv}} \times [0,1], X)$ whose relative homology classes generate the kth homology of $H_k(M_0^{\text{inv}} \times [0,1], X)$. We will then show that the restriction of $\Upsilon(M_0^{\text{inv}})|_{\text{rel}}$ to each $W_{\mathbf{I}}$ is trivial as a relative vector bundle, and that therefore $\langle c_k(\Upsilon(M_0^{\text{inv}})|_{\text{rel}}), [W_{\mathbf{I}}]\rangle = 0$. Since the $[W_{\mathbf{I}}]$ generate $H_k(M_0^{\text{inv}} \times [0,1], X)$, we will

Figure 5: The periodic domain $P_i = E_i - F_i$ has Maslov index zero.



have proven that $c_k(\Upsilon(M_0^{\text{inv}})|_{\text{rel}})$ is identically zero.

More specifically, we will describe the chains $W_{\mathbf{I}}$ as a subset of a product of two-chains Y_i in $(S^2 \backslash \{\mathbf{w}\}) \times [0,1]$ such that the projection to $S^2 \backslash \{\mathbf{w}\}$ is the periodic domain P_i and the Y_i are pairwise disjoint. We will then show that the restriction of the bundle $\Upsilon(M_0^{\text{inv}})|_{\text{rel}}$ to $W_{\mathbf{I}}$ also breaks up as the restriction of a product of relative bundles $\Upsilon(Y_i)$ over the Y_i which are known to be trivial through Maslov index arguments. Let us begin by constructing these manifolds Y_i.

For $1 \leq i \leq n-1$, let Y_i be a subspace of $S^2 \times [0,1]$ with the following properties:

- Y_i is topologically $S^1 \times [0,1]$, and $Y_i \cap (S^2 \times \{t\})$ is S^1 for all t.
- The boundary of Y_i is $\beta_i \times \{0\} - \alpha_i \times \{1\}$.
- The projection of Y_i to the punctured sphere is a copy of the periodic domain P_i.
- $Y_i \cap Y_j = \emptyset$ if $i \neq j$.
- If x_i is the point of positive intersection of α_i and β_i, $\{x_i\} \times [0,1] \subset Y_i$.

The 2-chains Y_i are constructed as follows: for $t \in [0, \frac{1}{2}]$, choose a linear homotopy H_t from β_i to γ_i inside the domain E_i which fixes $\beta_i \cap \gamma_i$. Let the intersection of Y_i with $S^2 \times \{t\}$ be the embedded circle $H_t(\beta_i) \times \{t\}$. Similarly, for $t \in [\frac{1}{2}, 1]$, choose a linear homotopy J_t from γ_i to α_i inside the domain F_i which fixes $\alpha_i \cap \gamma_i$, and let the intersection of Y_i with $S^2 \times \{t\}$ be $J_t(\gamma_i)$.

Observe that this description of Y_i has the following properties: first, Y_i is contained in $(S^2\backslash\{\mathbf{w}\}) \times [0,1]$ as promised. Second, Y_i contains the line segment $\{x_i\} \times [0,1]$. Third, the intersection of Y_i with $(\alpha_i \cup \beta_i) \times (0,1)$ is entirely contained in the cylinder $\gamma_i \times (0,1)$. Finally, the projection of $Y_i \cap (S^2 \times [0, \frac{1}{2}])$ to S^2 lies entirely "inside" β_i - that is, on the component of $S^2\backslash\beta_i$ not containing w_0 - implying that the sets $Y_i \cap (S^2 \times [0, \frac{1}{2}])$ are pairwise disjoint. Similarly, the projection of $Y_i \cap (S^2 \times [\frac{1}{2}, 1])$ to S^2 lies entirely "inside" α_i, and therefore the sets $Y_i \cap (S^2 \times [\frac{1}{2}, 1])$ are pairwise disjoint. Ergo the Y_i are pairwise disjoint.

We are now ready to define the complex line bundles $\Upsilon(Y_i)$. Recall from Section 4.1 that there is a holomorphic embedding

$$\iota\colon\ M_0^{\text{inv}} = \text{Sym}^{n-1}(\Sigma(S^2)\backslash\{\mathbf{w}\}) \to \text{Sym}^{n-1}(S^2\backslash\{\widetilde{\mathbf{w}}\}) = M_0$$
$$(x_1 ... x_{n-1}) \mapsto (x_1^1 x_1^2 ... x_{n-1}^1 x_{n-1}^2).$$

Similarly, we have a holomorphic embedding

$$\iota'\colon\ (S^2\backslash\{\mathbf{w}\}) \to \text{Sym}^2(\Sigma(S^2)\backslash\{\widetilde{\mathbf{w}}\})$$
$$x \mapsto (x^1 x^2)$$

which takes a point x on $S^2\backslash\{\mathbf{w}\}$ to the unordered pair consisting of its two (not necessarily distinct) lifts on $\Sigma(S^2)\backslash\{\widetilde{\mathbf{w}}\}$ under the projection map $\pi\colon\ (\Sigma(S^2)\backslash\{\widetilde{\mathbf{w}}\}) \to (S^2\backslash\{\mathbf{w}\})$. Notice that for each $1 \leq i \leq n-1$, $\iota'(\alpha_i) \subset \alpha_i^1 \times \alpha_i^2$ and $\iota'(\beta_i) \subset \beta_i^1 \times \beta_i^2$. Let $N(S^2\backslash\{\mathbf{w}\})$ be the normal bundle to $S^2\backslash\{\mathbf{w}\}$ in the second symmetric product $\text{Sym}^2(\Sigma(S^2)\backslash\{\widetilde{\mathbf{w}}\})$. This is a complex line bundle over a punctured sphere, hence trivial. For each $1 \leq i \leq n-1$, let $N\alpha_i$ be the real normal bundle to α_i in $\alpha_i^1 \times \alpha_i^2$ and $N\beta_i$ be the real normal bundle to β_i in $\beta_i^1 \times \beta_i^2$. An argument similar to that of Lemma 7.0.15 shows that $N\alpha_i$ and $N\beta_i$ are trivial real line bundles for all i.

Consider the pullback of the normal bundle $N(S^2\backslash\{\mathbf{w}\})$ to $(S^2\backslash\{\mathbf{w}\}) \times [0,1]$. The complex line bundle over Y_i that interests us is the restriction of this pullback to Y_i, which we shall by analogy denote $\Upsilon(Y_i)$. That is, $\Upsilon(Y_i) = (N(S^2\backslash\{\mathbf{w}\}) \times [0,1])|_{Y_i}$. This complex line bundle has totally real subbundles $J(N\alpha_i \times \{1\})$ over $\alpha_i \times \{1\}$ and $N\beta_i \times \{0\}$ over $\beta_i \times \{0\}$. Choose preferred real trivializations of these real line bundles, and extend to a complex trivialization of $\Upsilon(Y_i)|_{(\beta_i\times\{0\})\cup(\alpha_i\times\{1\})}$ by tensoring with $\mathbb{C}$. For convenience, let this subspace $(\beta_i \times \{0\}) \cup (\alpha_i \times \{1\})$ be X_i.

Lemma 7.0.17. *For $1 \leq i \leq n-1$, given any preferred trivialization of $\Upsilon(Y_i)|_{X_i}$ as a complex vector bundle, the relative vector bundle $\Upsilon(Y_i)|_{\mathrm{rel}}$ over (Y_i, X_i) is stably trivial.*

Proof. Let ι_i be the inclusion of Y_i into $(S^2\backslash\{\mathbf{w}\})\times[0,1]$, and p be the projection of $(S^2\backslash\{\mathbf{w}\})\times[0,1]$ to $S^2\backslash\{\mathbf{w}\}$. Then the image of $p\circ\iota_i(Y_i)$ is the periodic domain P_i.

Consider the commutative diagram below. The top horizontal inclusion of $S^2\backslash\{w\}$ into $\mathrm{Sym}^{n-1}(S^2\backslash\{\mathbf{w}\})$ is defined by mapping a point x to $(xx_1...\widehat{x_i}...x_{n-1})$, where again each x_j is the positively oriented point in $\alpha_j\cup\beta_j$. The bottom inclusion of $\mathrm{Sym}^2(\Sigma(S^2)\backslash\{\widetilde{\mathbf{w}}\})$ into $\mathrm{Sym}^{2n-2}(\Sigma(S^2)\backslash\{\widetilde{\mathbf{w}}\})$ sends an unordered pair (xy) to $(xyx_i^1x_i^2...\widehat{x_i^1}\widehat{x_i^2}...x_{n-1}^1x_{n-1}^2)$.

$$\begin{array}{ccccc}
Y_i & \xrightarrow{p\circ\iota_i} & S^2\backslash\{\mathbf{w}\} & \hookrightarrow & \mathrm{Sym}^{n-1}(S^2\backslash\{\mathbf{w}\}) \\
 & \searrow & \downarrow \iota' & & \downarrow \iota \\
 & & \mathrm{Sym}^2(S^2\backslash\{\widetilde{\mathbf{w}}\}) & \hookrightarrow & \mathrm{Sym}^{2n-1}(S^2\backslash\{\widetilde{\mathbf{w}}\})
\end{array}$$

Consider the map $\phi\colon\, Y_i \to \mathrm{Sym}^{n-1}(S^2\backslash\{\mathbf{w}\})$ given by composition along the top row of the diagram. This is a topological annulus representing the periodic domain P_i. Since P_i has Maslov index zero, by the discussion in Chapter 2, the first Chern class of the pullback of the complex tangent bundle $T\mathrm{Sym}^{n-1}(S^2\backslash\{\mathbf{w}\})$ to Y_i relative to the complexification of the pullbacks of the real tangent bundles $J(T(\mathbb{T}_{\boldsymbol{\alpha}}))$ to one component of the boundary of Y_i and $T(\mathbb{T}_{\boldsymbol{\beta}})$ to the other is zero. However, the pullback $\phi^*(T\mathrm{Sym}^{n-1}(S^2\backslash\{\mathbf{w}\}))$ of the tangent bundle of the total symmetric product to Y_i is exactly $(p\circ\iota)_*(T(S^2\backslash\{\mathbf{w}\})\oplus\mathbb{C}^{n-2})$, where the factors of $\mathbb{C}$ are the restriction of the tangent bundle of the punctured sphere to the points x_j such that $j\neq i$. Moreover, $(p\circ\iota)^*(T(S^2\backslash\{\mathbf{w}\}))$ is precisely the restriction of the pullback bundle $p^*(T(S^2\backslash\{\mathbf{w}\})) = T((S^2\backslash\{\mathbf{w}\})\times[0,1])$ over $(S^2\backslash\{\mathbf{w}\})\times[0,1]$ to the subspace Y_i. Therefore $\phi^*(T\mathrm{Sym}^{n-1}(S^2\backslash\{\mathbf{w}\})) = T((S\backslash\{\mathbf{w}\})\times[0,1])|_{Y_i}\oplus\mathbb{C}^{n-2}$.

Similarly, the pullback $\phi^*(J(T(\mathbb{T}_{\boldsymbol{\alpha}})))$ to the boundary component $\alpha_i\times\{1\}\subset Y_i$ is $(p\circ\iota_i)^*(J(T(\alpha_i)\oplus\mathbb{R}^{n-1-1}))$, where the factors of $\mathbb{R}$ are the canonical real subspace of the tangent bundle to the punctured sphere at the points x_j for $j \neq i$. The pullback of this bundle to $\alpha_i \times \{1\} \subset Y_i$ is $J((T(\alpha_i) \times \{1\}) \oplus \mathbb{R}^{n-1-1})$. A similar argument shows that $\phi^*(J(T(\mathbb{T}_{\boldsymbol{\beta}})))$ is $(T(\beta_i) \times \{0\}) \oplus \mathbb{R}^{n-2}$. Therefore we have seen that the complex vector bundle $T((S^2 \backslash \{\mathbf{w}\}) \times [0,1])|_{Y_i} \oplus \mathbb{C}^{n-2}$ relative to the complexification of its totally real subbundles $(T(\beta_i) \times \{0\}) \oplus \mathbb{R}^{n-2}$ over $\beta_i \times \{0\}$ and $J((T(\alpha_i) \times \{1\}) \oplus \mathbb{R}^{n-2})$ over $\alpha_i \times \{1\}$ has relative first Chern class zero. Hence the same is true of the complex line bundle $T((S^2 \backslash \{\mathbf{w}\}) \times [0,1])|_{Y_i}$ relative to the complexification of its totally real subbundles $J(T(\alpha_i) \times \{1\})$ and $T(\beta_i) \times \{0\}$. As this is a line bundle, triviality of the first relative Chern class suffices to show stable triviality of the relative vector bundle.

Now consider the map $\widetilde{\phi} = \iota \circ \phi$ from Y_i to $\mathrm{Sym}^{2n-2}(\Sigma(S^2) \backslash \{\widetilde{\mathbf{w}}\})$ given by any path through the diagram from Y_i to $\mathrm{Sym}^{2n-2}(\Sigma(S^2) \backslash \{\widetilde{\mathbf{w}}\})$. This is a topological annulus representing the periodic domain $\pi^{-1}(P_i)$ in $\widetilde{\mathcal{D}}$, which also has Maslov index zero because of our insistence that either both z basepoints in the image of P_i are branch points or neither is. Therefore the pullback along $\widetilde{\phi}$ of the complex tangent bundle $T(\mathrm{Sym}^{2n-2}(\Sigma(S^2) \backslash \{\widetilde{\mathbf{w}}\}))$ to Y_i relative to complexifications of pullbacks of the totally real subbundles $J(T(\mathbb{T}_{\widetilde{\boldsymbol{\alpha}}}))$ and $T(\mathbb{T}_{\widetilde{\boldsymbol{\beta}}})$ to $\alpha_i \times \{1\}$ and $\beta_i \times \{0\}$ has trivial relative first Chern class. However, once again the pullback of this relative bundle along the inclusion map $\mathrm{Sym}^2(\Sigma(S^2) \backslash \{\widetilde{\mathbf{w}}\}) \hookrightarrow \mathrm{Sym}^{2n-2}(\Sigma(S^2) \backslash \{\widetilde{\mathbf{w}}\})$ decomposes into a much smaller tangent bundle together with a trivial summand. Observe that $\widetilde{\phi}^*(T(\mathrm{Sym}^{2n-2}(\Sigma(S^2) \backslash \{\widetilde{\mathbf{w}}\}))$ is $(\iota' \circ (p \circ \iota_i))^*(T(\mathrm{Sym}^2(\Sigma(S^2) \backslash \{\widetilde{\mathbf{w}}\})) \oplus \mathbb{C}^{2n-4}))$. Pulling back along the inclusion map ι', we see that this bundle breaks up as $T(S^2 \backslash \{\mathbf{w}\}) \oplus N(\Sigma(S^2) \backslash \{\mathbf{w}\}) \oplus \mathbb{C}^{2n-4}$ over $\Sigma(S^2) \backslash \{\mathbf{w}\}$, and that therefore its ultimate pullback to Y_i is $T((\Sigma(S^2) \backslash \{\mathbf{w}\}) \times [0,1]) \oplus \Upsilon(Y_i) \oplus \mathbb{C}^{2n-4}$.

Similarly, the pullback along $\widetilde{\phi}$ of $J(T(\mathbb{T}_{\widetilde{\boldsymbol{\alpha}}}))$ to $\alpha_i \times \{1\}$ is precisely $J(T\alpha_i \times \{1\}) \oplus J(N\alpha_i \times \{1\}) \oplus J(\mathbb{R}^{2n-4})$. Finally, the pullback along $\widetilde{\phi}$ of $T(\mathbb{T}_{\widetilde{\boldsymbol{\beta}}})$ to $\beta_i \times \{0\}$ is $(T\beta_i \times \{0\}) \oplus (N_{\beta_i} \times \{0\}) \oplus \mathbb{R}^{2n-4}$. Dropping the trivial summands, we conclude that the first relative Chern class of $T((S^2 \backslash \{\mathbf{w}\}) \times [0,1]) \oplus \Upsilon(Y_i)$ relative to complexifications of the totally real subbundle $(T\beta_i \times \{0\}) \oplus (N_{\beta_i} \times \{0\})$ over $\beta_i \times \{0\}$ and the totally real subbundle $J(T\alpha_i \times \{1\}) \oplus J(N\alpha_i \times \{1\})$ over $\alpha_i \times \{1\}$ has relative first Chern class zero. This, combined with our previous conclusions concerning relative triviality of $T((S^2\backslash\{\mathbf{w}\}) \times [0,1])$ with respect to its subbundles $T\beta_i \times \{1\}$ over $\beta_i \times \{1\}$ and $J(T\alpha_i \times \{0\})$ over $\alpha_i \times \{0\}$, implies that $\Upsilon(Y_i)$ has relative first Chern class zero with respect to complexifications of its subbundles $N\beta_i \times \{0\}$ and $J(N\alpha_1 \times \{1\})$, as promised.

□

We are now ready to construct generators for $H_k(M_0^{\text{inv}} \times [0,1], X)$, and use them to prove that the relative Chern classes of $[\Upsilon(M_0^{\text{inv}})]$ are identically zero. Recall that for $k > 1$ there are short exact sequences on homology

$$0 \longrightarrow H_{k+1}(M_0^{\text{inv}} \times [0,1], X) \longrightarrow H_k(\mathbb{T}_{\boldsymbol{\beta}}) \oplus H_k(\mathbb{T}_{\boldsymbol{\alpha}}) \xrightarrow{\iota_*} H_k(M_0^{\text{inv}}) \longrightarrow 0.$$

As in Section 5.2, for $k > 1$, the degree $k+1$ homology $H_{k+1}(M_0^{\text{inv}} \times [0,1], X)$ is the kernel of the map (ι_*). Let us take a closer look at generators for this group. We have seen that the kernel of ι_* is $\mathbb{Z}\langle \bigwedge_{j=1}^k [\beta_{i_j}] \oplus -\bigwedge_{j=1}^k [\alpha_{i_j}] : 1 \leq i_1 < \cdots < i_k \leq n-1 \rangle$. Therefore for each $\mathbf{I} = (i_1, ..., i_k)$ such that $1 \leq i_1 < \cdots < i_k \leq n-1$, there is a $(k+1)$-chain $W_{\mathbf{I}}$ in $M_0^{\text{inv}} \times [0,1]$ whose boundary is $\prod_{j=1}^k \beta_{i_j} - \prod_{j=1}^k \alpha_{i_j}$. We will describe this chain in terms of the two-chains Y_i.

Construct the k chain $W_{\mathbf{I}}$ as follows. Let $W_{\mathbf{I}}$ be topologically $(S^1)^k \times [0,1]$, and insist that the intersection of $W_{\mathbf{I}}$ with $M_0^{\text{inv}} \times \{t\}$ is the product $\prod_{j=1}^k Y_{i_j} \cap (S^2\backslash\{\mathbf{w}\}) \times \prod_{i \notin \mathbf{I}} \{x_i\}$. Since the intersection of each Y_{i_j} with $(S^2\backslash\{\mathbf{w}\}) \times \{t\}$ is a circle for each t, this says that the intersection of $W_{\mathbf{I}}$ with $M_0^{\text{inv}} \times \{t\}$ is a k-torus for all t.

Notice that for all t, $W_{\mathbf{I}} \cap (M_0^{\text{inv}} \times \{t\})$ is a subset of the product $\prod_{i=1}^n (Y_i \cap ((S^2\{\mathbf{w}\}) \times [0,1])$. (Recall here that the line segments $\{x_i\} \times [0,1]$ are subsets of Y_i for each i.) Since the Y_i are pairwise disjoint, each $W_{\mathbf{I}} \cap (M_0^{\text{inv}} \times \{t\})$ is a submanifold of $M_0^{\text{inv}} = \text{Sym}^{n-1}(S^2\backslash\{\mathbf{w}\})$ and is disjoint from the fat diagonal. Furthermore, since $W_{\mathbf{I}}$ is homeomorphic to $(S^1)^k \times [0,1]$ and $\partial W_{\mathbf{I}}$ is $\prod_{j=1}^k \beta_{i_j} - \prod_{j=1}^k \alpha_{i_j}$, we see that the collection $\{W_{\mathbf{I}} : 1 \leq i_1 < \cdots < i_k \leq n-1\}$ generates $H^k(M_0^{\text{inv}} \times [0,1], X)$.

Let us consider the restriction of $\Upsilon(M_0^{\text{inv}})$ to $W_{\mathbf{I}}$ for some particular $\mathbf{I}$. We pick preferred trivializations of $N(L_0) \times \{0\}) = N(\mathbb{T}_{\boldsymbol{\beta}}) \times \{0\}$ and $J(N(L_1) \times \{1\}) = J(N(\mathbb{T}_{\boldsymbol{\alpha}}) \times \{1\})$ which are products of preferred trivializations of the real subbundles $N(\beta_i)\times\{0\}$ and $J(N(\alpha_i)\times\{1\})$ in $\Upsilon(Y_i)$.

Because $W_i \cap (M_0^{\mathrm{inv}} \times \{t\})$ lies entirely off the fat diagonal for each t, the restriction of $\Upsilon(M_0^{\mathrm{inv}})$ to each W_i also decomposes as a product bundle. In other words, regard W_i as a subset of the abstract product $\prod_{i=1}^{n-1} Y_i$ (which is *not* a submanifold of $M_0^{\mathrm{inv}} \times [0,1]$). Then $\Upsilon(M_0^{\mathrm{inv}})|_{W_{\mathbf{I}}}$ is the restriction of the product bundle $(\prod \Upsilon(Y_i))|_{W_{\mathbf{I}}}$. Therefore, since each $\Upsilon(Y_i)$ admits a trivialization with respect to a choice of trivialization of the bundle restricted to α_i and β_i, we conclude that $\Upsilon(M_0^{\mathrm{inv}})|_{W_{\mathbf{I}}}$ is trivializable with respect to preferred trivializations of the bundle over $\mathbb{T}_{\boldsymbol{\beta}} \times \{0\}$ and $J(\mathbb{T}_{\boldsymbol{\alpha}} \times \{1\})$.

Remark 7.0.18. We could as easily have shown that $(\mathrm{Sym}^{2k}(\Sigma(S^2)\backslash\{\widetilde{\mathbf{z}}\}), \mathbb{T}_{\widetilde{\boldsymbol{\beta}}}, T_{\widetilde{\boldsymbol{\alpha}}})$ has a stable normal trivialization. In the case of double branched covers this would produce a spectral sequence from $\widehat{HF}(\Sigma(L)) \otimes W^{\otimes n-1}$ to $\widehat{HF}(S^3) \otimes W^{\otimes n-1}$. The author is not presently aware of any interesting applications of this fact.

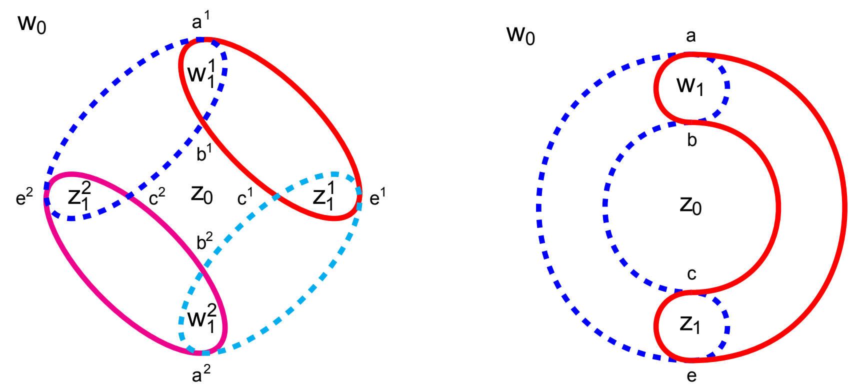
Chapter 8

Sample Computations for Two-Periodic Knots

In this section we produce some examples of the behavior of the spectral sequences of Theorems 1.2.2 and 1.2.3. We present the cases of the unknot and the trefoil as doubly-periodic knots. In general, it is difficult to show that the Seidel-Smith action on $\widetilde{HFL}(\widetilde{\mathcal{D}})$ agrees with the action induced by the map $\tau^{\#}$ on $\widehat{CFL}(\mathcal{D})$ derived from the Heegaard diagram, and therefore difficult to compute the higher differentials in the spectral sequence. However, in these simple examples the higher differentials are entirely constrained by the homological grading, so the spectral sequences we compute here are in fact the same as the Seidel-Smith spectral sequence.

To simplify matters, we will compute all spectral sequences using appropriate chain complexes tensored with $\mathbb{Z}_2((\theta))$, instead of tensoring with $\mathbb{Z}_2[[\theta]]$ and later further tensoring the E^∞ page with θ^{-1}. Then the E^1 page of each spectral sequence is a free module with generators the link Floer or knot Floer homology of our doubly periodic knot, and the E^∞ page is a free module over $\mathbb{Z}((\theta))$ with generators the link or knot Floer homology of the quotient. (The only information we lose by this change is that the E^∞ page is not the Borel Heegaard Floer link or knot homology, but rather the Borel homology tensored with θ^{-1}.)

Figure 6: An equivariant Heegaard diagram for the unknot together with an axis (i.e. a Hopf link), and its quotient Heegaard diagram (another Hopf link).



8.0.1 The case of the Unknot

In the simplest possible case, let $\widetilde{K}$ be an unknot, and K its quotient knot, a second unknot. Consider the diagram $\mathcal{D}$ for $K \cup U$ on the sphere $S^2 = \mathbb{C}^2$ in which K is the unit circle with basepoints w_0, z_0 on U and basepoints w_1, z_1 on K such that w_0 lies at ∞, z_0 lies at 0, w_1 lies at i, and z_1 lies at $-i$. Supply a single α_1 and β_1 as in Figure 6 coherently with a clockwise orientation of K.

Label the intersection points a, b, c, e vertically down the diagram as in the figure. (Since we plan to discuss the differentials d_i in the spectral sequence, we will not also use d to label any intersection points.) There are no differentials that count for $\widetilde{HFL}(\mathcal{D})$ – which is exactly the link Floer homology of the Hopf link with positive linking number – and three differentials that count for $\widetilde{HFK}(\mathcal{D}) = \widehat{HFK}(S^3, K) \otimes W$. See the table of Figure 7 for the Alexander gradings of these entries.

Now lift to a diagram $\widetilde{\mathcal{D}}$ for the same Hopf link, which has basepoints w_0, z_0 on the axis U and $w_1^1, w_1^2, z_1^1, z_1^2$ on the lifted unknot $\widetilde{K}$. These basepoints lie on $S^2 = \mathbb{C} \cup \{\infty\}$ as follows: w_0 and z_0 lie at 0 and ∞ as previously, and $w_1^1, w_1^2, z_1^1, z_1^2$ lie at $i, -i, 1, -1$ respectively. There are two curves α_1^1 and α_1^2 encircling the pairs w_1^1, z_1^1 and w_1^2, z_1^2, and two curves β_1^1 and β_1^2 encircling pairs z_1^2, w_1^1 and z_1^1, w_1^2. The intersection points a, b, c, e lift to eight points

Figure 7: Alexander gradings and differentials for $\widehat{CFL}(\mathcal{D})$ (left) and $\widehat{CFK}(\mathcal{D})$ (right).

$A_2 \backslash A_1$	$-\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	b	a
$-\frac{1}{2}$	c	e

A_1	-1	0
	b	a
	c	e

Figure 8: Alexander gradings and differentials of $\widehat{CFL}(\widetilde{\mathcal{D}})$ (left) and $\widehat{CFK}(\widetilde{\mathcal{D}})$ (right).

$A_2 \backslash A_1$	$-\frac{3}{2}$	$-\frac{1}{2}$		$\frac{1}{2}$
$\frac{1}{2}$	b^1b^2	b^1a^2	a^1b^2	a^1a^2
$-\frac{1}{2}$	c^1c^2	c^1e^2	e^1c^2	e^1e^2

A_1	-2	-1		0
	b^1b^2	b^1a^2	a^1b^2	a^1a^2
	c^1c^2	c^1e^2	e^1c^2	e^1e^2

$a^1, a^2, b^1, b^2, c^1, c^2, e^1, e^2$ on the diagram in $\cup(\alpha_1^i \cap \beta_1^j)$. (The numbering of each pair is arbitrarily determined by insisting that a^i lie on α_1^i, and so on.) The complex $\widehat{CFL}(\widetilde{\mathcal{D}})$ has eight generators, whose Alexander gradings are laid out in the left table of Figure 8; there are no differentials that count for the theory $\widetilde{HFL}(\widetilde{\mathcal{D}}) = \widehat{HFL}(S^3, \widetilde{K} \cup U) \otimes V_1$. Allowing differentials that pass over the basepoint z_0, we obtain the complex of the right table Figure 8 which computes $\widehat{HFK}(\mathcal{D}) = \widehat{HFK}(S^3, \widetilde{K}) \otimes V_1 \otimes W$.

First, consider the spectral sequence for link Floer homology derived from the double complex $(\widehat{CFL}(\widetilde{\mathcal{D}} \otimes \mathbb{Z}_2, \partial + (1+\tau^{\#})\theta)$. Since $\partial = 0$ on this complex, the only nontrivial differential occurs on the E^1 page and is exactly $d_1 = 1 + \tau^{\#}$. Therefore the E^2 page of the spectral sequence is $\mathbb{Z}_2((\theta))\langle a^1a^2, b^1b^2, c^1c^2, e^1e^2\rangle$, which is isomorphic to $\widehat{HFL}(S^3, K \cup U) \otimes \mathbb{Z}_2((\theta))$, as expected.

Next, consider the spectral sequence of the double complex $(\widehat{CFK}(\widetilde{\mathcal{D}} \otimes \mathbb{Z}_2, \partial_U + (1 + \tau^{\#})\theta)$. We have the complex of Figure 8; the E^1 page of the spectral sequence is equal to $\mathbb{Z}_2\langle [a^1a^2], [e^1e^2], [a^1b^2 + b^1a^2], [c^1e^2]\rangle \otimes \mathbb{Z}_2((\theta))$. Bearing in mind that $[c^1e^2] = [c^2e^1]$, we see that τ^* is the identity on each of these four elements, and therefore the E^2 page of the spectral sequence is the same as the E^1 page. Since the differential d_2 must raise the Maslov grading by two, the only possible nontrivial differential on the E^2 page is $d_2([c^1e^2]\theta^n)$. Let us compute this differential. On the chain level, we have $(1+\tau^{\#})(c^1e^2) = (c^1e^2 + c^2e^1)$. We observe that $c^1e^2 + c^2e^1 = \partial(a^1b^2)$, so

$$\begin{aligned} d_2([c^1e^2]\theta^n) &= [(1+\tau^{\#})(a^1b^2)]\theta^{n+1} \\ &= [a^1b^2 + a^2b^1]\theta^{n+2} \end{aligned}$$

Therefore the E^3 page of this spectral sequence is exactly $\mathbb{Z}_2((\theta))\langle[a^1a^2],[e^1e^2]\rangle$, which is isomorphic to $\left(\widehat{HFK}(S^3,K)\otimes W\right)\otimes\mathbb{Z}_2((\theta))$ as promised, and unchanged thereafter.

8.0.2 The case of the trefoil

Let us now compute some of the spectral sequence for the trefoil as a doubly-periodic knot with quotient the unknot, using the Heegaard diagrams $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ of Figure 3. Recall that $\mathcal{D}$ is a Heegaard diagram for a link L in S^3 consisting of two unknots with linking number $\lambda = 3$ and $\widetilde{\mathcal{D}}$ is a Heegaard diagram for a link $\widetilde{L}$ in S^3 consisting of the left-handed trefoil and the unknotted axis also with linking number $\lambda = 3$. We label the twelve intersection points of α_1 and β_1 in $\mathcal{D}$ as shown in Figure 9, and lift to twenty-four intersection points in $\widetilde{\mathcal{D}}$ in Figure 10. The Alexander gradings and differentials of $\widehat{CFL}(\mathcal{D})$ and $\widehat{CFK}(\mathcal{D})$ are laid out in the tables of Figure 11.

There are no differentials on $\widehat{CFL}(\mathcal{D})$, so $\widetilde{HFL}(\mathcal{D}) = \widehat{HFL}(S^3,L)$ has the twelve generators and gradings of Figure 9. From the remainder of that diagram, we observe that the group

Figure 9: Intersection points in $\mathcal{D}$.

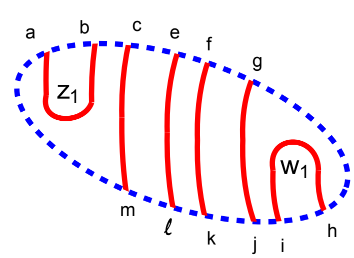


Figure 10: Intersection points of α and β curves in $\widetilde{\mathcal{D}}$.

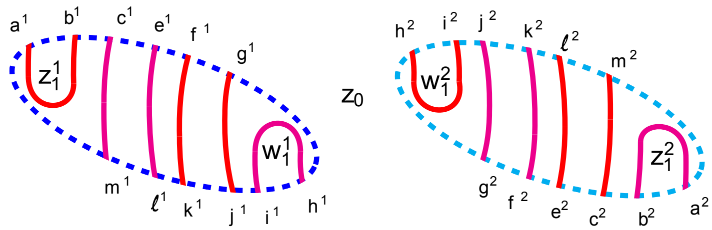


Figure 11: Alexander gradings and differentials of $\widehat{CFL}(\mathcal{D})$ (left) and $\widehat{CFK}(\mathcal{D})$ (right).

A_2 \ A_1	$-\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$
$\frac{3}{2}$	h	i		
$\frac{1}{2}$	g	f j	k	
$-\frac{1}{2}$		e	c ℓ	m
$-\frac{3}{2}$			b	a

A_1	-3	-2	-1	0
	h	i		
	g	f j	k	
		e	c ℓ	m
			b	a

$\widetilde{HFK}(\mathcal{D}) = \widehat{HFK}(S^3, U) \otimes W$ is $\mathbb{Z}_2\langle [a], [m] \rangle$.

Now consider the seventy-two generators of $\widehat{CFL}(\widetilde{\mathcal{D}})$, whose Alexander A_1 and A_2 gradings are laid out in Figure 12. It so happens that $\widetilde{\mathcal{D}}$ is a nice diagram in the sense of Sarkar and Wang [39], although the equivariant diagrams for periodic knots introduced in Section 3.2 are not in general, so we may compute $\widetilde{HFL}(\widetilde{\mathcal{D}})$ with relative ease. The chain complexes in each Alexander A_1 grading are shown in Figures 19, 20, 21, 22, 23, 24 and 25 at the close of this section. These convention for these figures is as follows: the generators shown are those in the A_1 grading of $\widehat{CFL}(\widetilde{\mathcal{D}})$, which are also the generators of the $A_1 - \frac{3}{2}$ grading of $\widehat{CFK}(\widetilde{\mathcal{D}})$. Solid arrows denote differentials that count for the differential ∂ and thus exist in both complexes, whereas dashed arrows denote differentials corresponding to disks with nontrivial intersection with the divisor $V_{z_0} = \{z_0\} \times \mathrm{Sym}^{2n_1}(S^2)$. Therefore dashed differentials only count for the knot Floer complex $(\widehat{CFK}(\widetilde{\mathcal{D}}), \partial_U)$.

The link Floer homology spectral sequence associated to $\widetilde{\mathcal{D}}$ arises from the double complex

Figure 12: Alexander A_1 and A_2 gradings of the generators of $\widehat{CFL}(\widetilde{D})$. These generate the E^0 page of the link Floer homology spectral sequence.

A_2 \ A_1	$-\frac{7}{2}$	$-\frac{5}{2}$	$-\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$
$\frac{3}{2}$	h^1h^2	h^1i^2 i^1h^2	i^1i^2				
$\frac{1}{2}$	g^1g^2	g^1j^2 j^1g^2 g^1f^2 f^1g^2 h^1e^2 e^1h^2	f^1f^2 j^1j^2 e^1i^2 i^1e^2 f^1j^2 j^1f^2 g^1k^2 k^1g^2 h^1c^2 h^2c^1 $h^1\ell^2$ ℓ^1h^2	f^1k^2 k^1f^2 j^1k^2 k^1j^2 $i^1\ell^2$ ℓ^1i^2 c^1i^2 i^1c^2 h^1m^2 m^1h^2	k^1k^2 i^1m^2 m^1i^2		
$-\frac{1}{2}$			e^1e^2 b^1g^2 g^1b^2	e^1c^2 c^1e^2 $e^1\ {}^2$ ${}^1e^2$ b^1f^2 f^1b^2 b^1j^2 j^1b^2 a^1g^2 g^1a^2	c^1c^2 $\ell^1\ell^2$ $c^1\ell^2$ ℓ^1c^2 b^1k^2 k^1b^2 a^1f^2 f^1a^2 e^1m^2 m^1e^2 a^1j^2 j^1a^2	c^1m^2 m^1c^2 ℓ^1m^2 $m^1\ell^2$ a^1k^2 k^1a^2	m^1m^2
$-\frac{3}{2}$					b^1b^2	a^1b^2 b^1a^2	a^1a^2

$(\widehat{CFL}(\widetilde{\mathcal{D}}), \partial + (1+\tau^{\#})\theta)$. Computing homology of $\widehat{CFL}(\widetilde{\mathcal{D}})$ with respect to the differential ∂, we obtain a set of generators for the E^1 page of the spectral sequence, which is $\widetilde{HFL}(\widetilde{\mathcal{D}}) \otimes \mathbb{Z}_2((\theta)) = (\widehat{HFL}(S^3, 3_1 \cup U) \otimes V_1) \otimes \mathbb{Z}_2((\theta))$. These generators and their gradings may be found in Figure 13. Whenever an element of $\widetilde{HFK}(\widetilde{\mathcal{D}})$ is invariant under the involution τ^* but has no representative which is invariant under the chain map $\tau^{\#}$, we have included two representatives of that element to make the τ^* invariance clear. (For example, observe that $[h^1c^2] = [c^1h^2]$ is invariant under τ^*.)

The differential d_1 on the E_1 page of the link Floer spectral sequence for $\widetilde{\mathcal{D}}$ is $(1+\tau^*)\theta$; in particular, computing the homology of d_1 has the effect of killing all elements of $\widetilde{HFL}(\mathcal{D})$ not invariant under τ^*. Ergo we see that the E^2 page of this spectral sequence is generated as a $\mathbb{Z}_2((\theta))$ module by the elements of Figure 14. Notice that the ranks of the E^2 page in each Alexander grading $A_1 = 2k + \frac{1}{2}$ correspond precisely to the ranks of each Alexander grading $A_1 = k + \frac{1}{2}$ of Figure 11, which is the link Floer homology $\widetilde{HFL}(\mathcal{D})$. Therefore the link Floer homology spectral sequence converges on the E^2 page.

Figure 13: The homology $\widetilde{HFL}(\widetilde{\mathcal{D}})$. These elements generate the E^1 page of the link Floer spectral sequence as a $\mathbb{Z}_2((\theta))$ module.

$A_2 \backslash A_1$	$-\frac{7}{2}$	$-\frac{5}{2}$	$-\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$
$\frac{3}{2}$	$[h^1h^2]$	$[h^1i^2]$ $[i^1h^2]$	$[i^1i^2]$				
$\frac{1}{2}$	$[g^1g^2]$	$[g^1j^2 + f^1g^2]$ $[j^1g^2 + g^1f^2]$	$[f^1f^2 + k^1g^2 + g^1k^2 + j^1j^2]$ $[h^1c^2] = [c^1h^2]$	$[i^1c^2]$ $[c^1i^2]$	$[i^1m^2] = [m^1i^2]$		
$-\frac{1}{2}$			$[b^1g^2 + g^1b^2]$	$[a^1g^2]$ $[g^1a^2]$	$[c^1c^2 + e^1m^2 + m^1e^2 + \ell^1\ell^2]$ $[a^1f^2] = [f^1a^2]$	$[c^1m^2 + m^1\ell^2]$ $[m^1c^2 + \ell^1m^2]$	$[m^1m^2]$
$-\frac{3}{2}$					$[b^1b^2]$	$[a^1b^2]$ $[b^1a^2]$	$[a^1a^2]$

We now turn our attention to the knot Floer homology spectral sequence for $\widetilde{\mathcal{D}}$, which arises from the double complex $(\widehat{CFK}(\mathcal{D}) \otimes \mathbb{Z}_2((\theta)), \partial_U + (1+\tau^{\#})\theta))$. The Alexander A_1 gradings of the seventy-two generators in $\widehat{CFK}(\widetilde{\mathcal{D}})$ are laid out in Figure 15. Notice that these gradings are exactly the A_1 gradings of $\widehat{CFL}(\widetilde{\mathcal{D}})$ shifted downward by $\frac{\ell k(3_1,U)}{2} = \frac{3}{2}$. These elements generate the E^0 page of the link Floer spectral sequence as a $\mathbb{Z}_2((\theta))$ module.

As before, the chain complexes in each Alexander grading may be found in Figures Figures 19, 20, 21, 22, 23, 24 and 25. Computing the homology of these complexes, we obtain $\widetilde{\widehat{HFK}}(\widetilde{\mathcal{D}}) = \widehat{HFK}(3_1) \otimes V_1 \otimes W$, whose generators are described in Figure 16. These elements generate the E^1 page of the knot Floer homology spectral sequence of $\widetilde{\mathcal{D}}$ as a $\mathbb{Z}_2((q))$-module. Once again, homology classes which are equivalent under the induced involution τ^* but have no representative which is invariant under the chain map $\tau^{\#}$ have been included with two equivalent descriptions to emphasize their invariance.

The differential $d_1 = (1+\tau^*)\theta$ on the E^1 page of the spectral sequence has the effect of

Figure 14: Generators for the $E^2 = E^\infty$ page of the link Floer spectral sequence for $\widetilde{\mathcal{D}}$ as a $\mathbb{Z}_2((\theta))$-module.

A_2 \ A_1	$-\frac{7}{2}$	$-\frac{5}{2}$	$-\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$
$\frac{3}{2}$	$[h^1h^2]$		$[i^1i^2]$				
$\frac{1}{2}$	$[g^1g^2]$		$[f^1f^2 + k^1g^2 + g^1k^2 + j^1j^2]$ $[h^1c^2] = [c^1h^2]$		$[i^1m^2] = [m^1i^2]$		
$-\frac{1}{2}$			$[b^1g^2 + g^1b^2]$		$[c^1c^2 + e^1m^2 + m^1e^2 + \ell^1\ell^2]$ $[a^1f^2] = [f^1a^2]$		$[m^1m^2]$
$-\frac{3}{2}$					$[b^1b^2]$		$[a^1a^2]$

Figure 15: Alexander A_1 gradings of elements of $\widehat{CFK}(\widetilde{\mathcal{D}})$. These elements generate the E^0 page of the link Floer spectral sequence as a $\mathbb{Z}_2((\theta))$ module.

A_1	-5	-4	-3	-2	-1	0	1
	h^1h^2	h^1i^2 i^1h^2	i^1i^2				
	g^1g^2	g^1j^2 j^1g^2 g^1f^2 f^1g^2 h^1e^2 e^1h^2	f^1f^2 j^1j^2 e^1i^2 i^1e^2 f^1j^2 j^1f^2 g^1k^2 k^1g^2 h^1c^2 h^2c^1 $h^1\ell^2$ ℓ^1h^2	f^1k^2 k^1f^2 j^1k^2 k^1j^2 $i^1\ ^2$ ℓ^1i^2 c^1i^2 i^1c^2 h^1m^2 m^1h^2	k^1k^2 i^1m^2 m^1i^2		
			e^1e^2 b^1g^2 g^1b^2	e^1c^2 c^1e^2 $e^1\ ^2$ $^1e^2$ b^1f^2 f^1b^2 b^1j^2 j^1b^2 a^1g^2 g^1a^2	c^1c^2 $\ell^1\ell^2$ $c^1\ell^2$ ℓ^1c^2 b^1k^2 k^1b^2 a^1f^2 f^1a^2 e^1m^2 m^1e^2 a^1j^2 j^1a^2	c^1m^2 m^1c^2 ℓ^1m^2 $m^1\ell^2$ a^1k^2 k^1a^2	m^1m^2
					b^1b^2	a^1b^2 b^1a^2	a^1a^2

Figure 16: The homology $\widetilde{\widehat{HFK}}(\widetilde{\mathcal{D}})$, which is $\widehat{HFK}(3_1) \otimes V_1 \otimes W$. These elements generate the E^1 page of the knot Floer homology spectral sequence of $\widetilde{\mathcal{D}}$ as a $\mathbb{Z}_2((q))$-module.

A_1	-5	-4	-3	-2	-1	0	1
				$[h^1m^2 + m^1h^2]$	$[i^1m^2 + m^1e^2]$ $[m^1i^2 + e^1m^2]$		
				$[a^1g^2] = [g^1a^2]$	$[a^1f^2]$ $[f^1a^2]$	$[\ell^1m^2 + m^1c^2]$ $[m^1\ell^2 + c^1m^2]$	$[m^1m^2]$
						$[a^1b^2]$ $[b^1a^2]$	$[a^1a^2]$

Figure 17: Generators for the E^2 page of the knot Floer spectral sequence associated to $\widetilde{\mathcal{D}}$ as a $\mathbb{Z}_2((q))$ module.

A_1	-5	-4	-3	-2	-1	0	1
				$[h^1m^2 + m^1h^2]$			
				$[a^1g^2] = [g^1a^2]$			$[m^1m^2]$
							$[a^1a^2]$

eliminating all elements of $\widetilde{HFK}(\widetilde{\mathcal{D}})$ which are not invariant under the action of τ^*. Computing homology with respect to d_1 yields the set of generators of Figure 17.

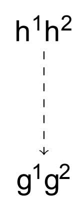
The knot Floer spectral sequence stabilizes on the E^2 page in every Alexander grading except $A_1 = -2$. There is a single nontrivial d_2 differential which behaves similarly to the nontrivial d_2 differential we earlier saw in the case of the unknot as a doubly-periodic knot. In particular: consider $d_2([a^1g^2])$. We compute this differential as follows: first, apply $1 + \tau^{\#}$ to a^1g^2, obtaining $(a^1g^2 + g^1a^2)\theta$. Next, we choose an element whose boundary under ∂_U is $(a^1g^2 + g^1a^2)\theta$; one such is $(h^1m^2)\theta$. Then $d_2([a^1g^2]) = [1 + \tau^{\#}(h^1m^2)]\theta = [h^1m^2 + m^1h^2]\theta^2$. Moreover, we then have $d_2([a^1g^2]\theta^n) = [h^1m^2 + m^1h^2]\theta^{n+2}$ in general. Therefore the Alexander grading -2 vanishes on the E^3 page of the knot Floer spectral sequence, and the $E^3 = E^\infty$ page is isomorphic to $\widetilde{HFK}(\mathcal{D})$ after an appropriate shift and rescaling in Alexander A_1 gradings. Generators for this page of the knot Floer spectral sequence appear in Figure 18.

Notice that for the left-handed trefoil considered as a two-periodic knot, Edmonds' condition is sharp: $g(3_1) = 1 = 2(0) + \frac{3-1}{2} = 2g(U) + \frac{\lambda-1}{2}$. We also see here the realization of

Figure 18: Generators for the $E^3 = E^\infty$ page of the knot Floer spectral sequence associated to $\widetilde{\mathcal{D}}$.

A_1	-5	-4	-3	-2	-1	0	1
							$[m^1m^2]$
							$[a^1a^2]$

Figure 19: The chain complex $\widehat{CFL}(\widetilde{\mathcal{D}})$ in Alexander grading $A_1 = -\frac{7}{2}$, and the chain complex $\widehat{CFK}(\widetilde{\mathcal{D}})$ in Alexander grading $A_1 = -5$. Dashed arrows denote differentials appearing only in the latter.



Corollary 1.2.5: the left-handed trefoil is fibred, and the highest nontrivial Alexander grading in $\widetilde{HFK}(\widetilde{\mathcal{D}})$, namely $A_1 = 1$ has rank two. The two generators in this Alexander grading, $[a_1a_2]$ and $[m_1m_2]$, are preserved over the course of the spectral sequence and become the two generators of $\widetilde{HFK}(\mathcal{D})$ in grading $A_1 = 0$ under the isomorphism between the E^∞ page of the knot Floer spectral sequence and $\widetilde{HFK}(\mathcal{D})$. Then the highest nontrivial Alexander grading of $\widetilde{HFK}(\mathcal{D})$, $A_1 = 0$, also has rank two, corresponding to the unknot being a fibred knot.

Figure 20: The chain complex $\widehat{CFL}(\widetilde{\mathcal{D}})$ in Alexander grading $A_1 = -\frac{5}{2}$, and the chain complex $\widehat{CFK}(\widetilde{\mathcal{D}})$ in Alexander grading $A_1 = -4$. Dashed arrows denote differentials appearing only in the latter.

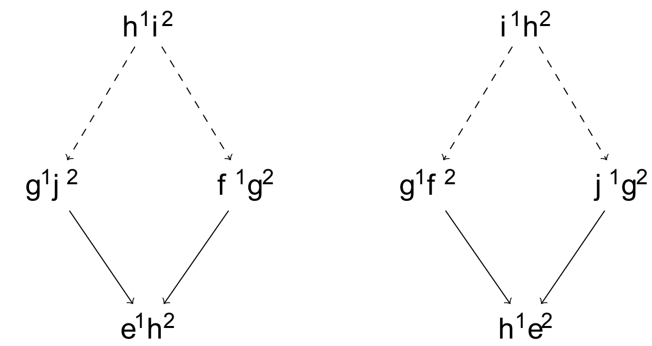


Figure 21: The chain complex $\widehat{CFL}(\widetilde{\mathcal{D}})$ in Alexander grading $A_1 = -\frac{3}{2}$, and the chain complex $\widehat{CFK}(\widetilde{\mathcal{D}})$ in Alexander grading $A_1 = -3$. Dashed arrows denote differentials appearing only in the latter.

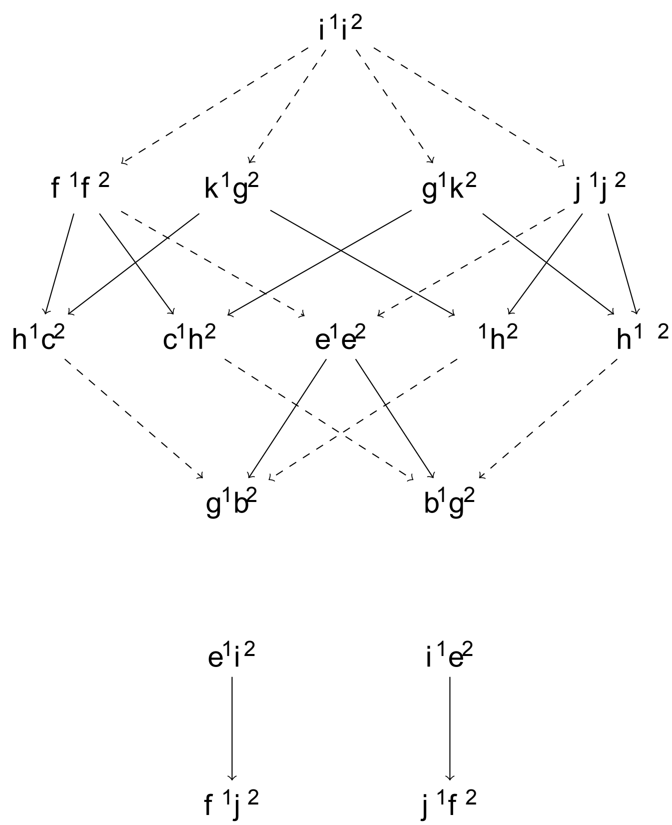


Figure 22: The chain complex $\widehat{CFL}(\widetilde{\mathcal{D}})$ in Alexander grading $A_1 = -\frac{1}{2}$, and the chain complex $\widehat{CFK}(\widetilde{\mathcal{D}})$ in Alexander grading $A_1 = -2$. Dashed arrows denote differentials appearing only in the latter.

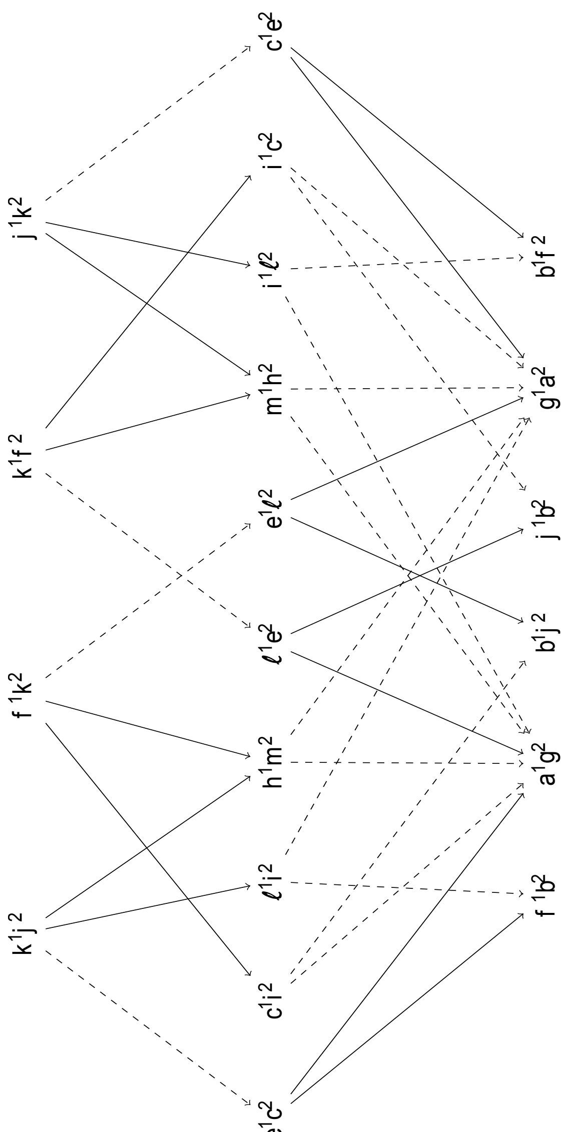


Figure 23: The chain complex $\widehat{CFL}(\widetilde{\mathcal{D}})$ in Alexander grading $A_1 = \frac{1}{2}$, and the chain complex $\widehat{CFK}(\widetilde{\mathcal{D}})$ in Alexander grading $A_1 = -1$. Dashed arrows denote differentials appearing only in the latter.

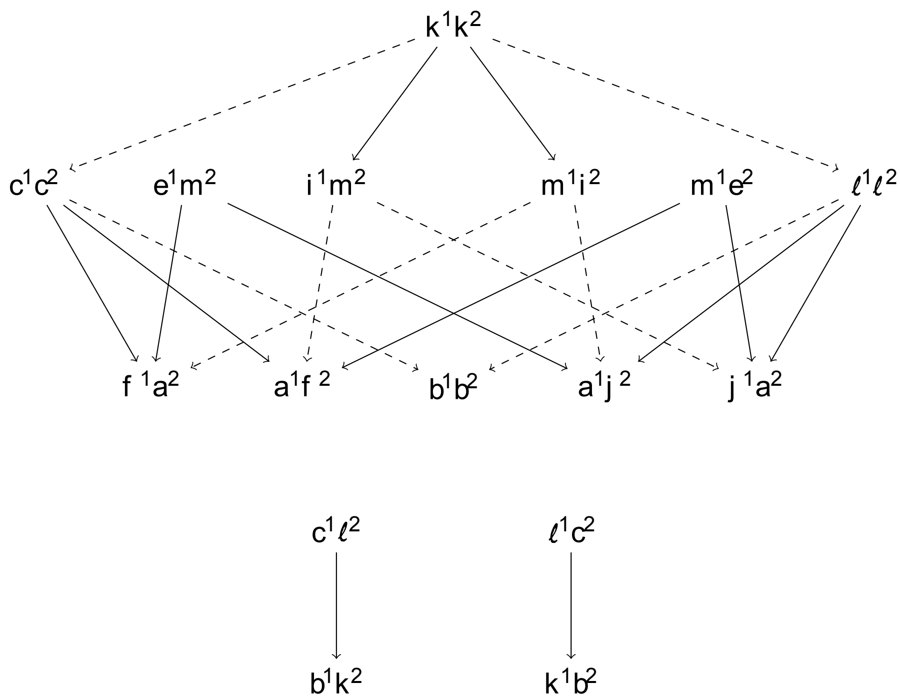


Figure 24: The chain complex $\widehat{CFL}(\widetilde{\mathcal{D}})$ in Alexander grading $A_1 = \frac{3}{2}$, and the chain complex $\widehat{CFK}(\widetilde{\mathcal{D}})$ in Alexander grading $A_1 = 0$. Dashed arrows denote differentials appearing only in the latter.

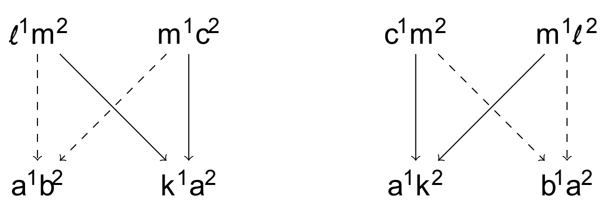


Figure 25: The chain complex $\widehat{CFL}(\widetilde{\mathcal{D}})$ in Alexander grading $A_1 = \frac{5}{2}$, and the chain complex $\widehat{CFK}(\widetilde{\mathcal{D}})$ in Alexander grading $A_1 = 1$. Dashed arrows denote differentials appearing only in the latter.

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Appendix 1: Holomorphic Embeddings

Since the proof that the map

$$\begin{aligned} h : \mathrm{Sym}^{n-1}(S^2) &\to \mathrm{Sym}^{2n-2}(\Sigma(S^2)) \\ (x_1 \cdots x_{n-1}) &\mapsto (x_1^1 x_1^2 \cdots x_{n-1}^1 x_{n-1}^2) \end{aligned}$$

is holomorphic is a computation on charts in the symmetric product, we place it here so as not to disrupt the flow of the preceding chapters.

We claim h is continuous; indeed, holomorphic. Let $\mathbf{x} = (x_1 \cdots x_{n-1})$ be a point in $\mathrm{Sym}^{n-1}(S^2)$. Collect repeated points, so that $\mathbf{x}$ has the form $(y_1, \cdots, y_1, \cdots, y_\ell, \cdots, y_\ell)$ with $n-1$ total entries but ℓ unique entries. Moreover, if $A \subset \{\mathbf{w}, \mathbf{z}\}$ is the set of branch points of the double branched cover, list points which are not in A first, so that $y_1, \cdots, y_s \notin A$ and $y_{s+1}, \cdots, y_\ell \in A$ for some s.

First consider y_i such that $1 \leq i \leq s$, so that y_i is not a branch point of the double cover. Then let y_i^j be a lift of y_i for $j = 1, 2$. There is a neighborhood U_i of y_i which admits homeomorphic lifts to neighborhoods U_i^j of y_i^j such that $(\pi|_{U_i^j})^{-1} : U_i \to U_i^j$ is holomorphic. Moreover, we may pick U_i sufficiently small such that there is a chart $f_i : U_i \to \mathbb{D}$, where $\mathbb{D}$ is the unit disk in the complex plane, and corresponding charts $f_i^j = f_i \circ \pi : U_i^j \to \mathbb{D}$. In total this gives local biholomorphisms between U_i and U_i^j expressed on charts as follows.

$$\begin{array}{ccc} U_i^j & \xhookrightarrow{(\pi|_{U_i^j})^{-1}} & U_i^j \\ \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_i^j} \\ \mathbb{D} & \xrightarrow{\mathrm{Id}} & \mathbb{D} \end{array}$$

Similarly, there is a neighborhood $\tau(\widetilde{U}_i)$ of the second lift $\tau(\widetilde{y}_i)$ of y_i which is homeomorphic to U_i via $(\pi|_{\tau(\widetilde{U}_i)})^{-1}$ and has a chart $\tau(\widetilde{f}_i) : \tau(\widetilde{U}_i) \to \mathbb{D}$.

Now suppose $s + 1 \leq i \leq \ell$, so that y_i is a branch point for the double branched cover map. Then there is a chart $f_i : U_i \to \mathbb{D}$ around y_i and a chart $g_i : \pi^{-1}(U_i) \to \mathbb{D}$ with respect to which

$f_i \circ \pi \circ (g_i)^{-1}$ is $x \mapsto x^2$.

$$\begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow{\pi|_{\pi^{-1}(U_i)}} & U_i \\ \downarrow{\scriptstyle g_i} & & \downarrow{\scriptstyle f_i} \\ \mathbb{D} & \xrightarrow{x \mapsto x^2} & \mathbb{D} \end{array}$$

In particular, if $y \in U_i$ and $f_i(y) = x \in \mathbb{D}$, then if y^1 and y^2 are the two lifts of y in $\pi^{-1}(U_i)$, then $g_i(y^1)$ and $g_i(y^2)$ are $\sqrt{x}$ and $-\sqrt{x}$ in some order.

We will insist that all of our choices of neighborhoods U_i about each y_i on which we choose charts be made such that if $i_1 \neq i_2$, then $U_{i_1} \cap U_{i_2} = \emptyset$, by shrinking if necessary. This in turn implies that $U_1^1, U_1^2), \cdots, U_s^1, U_s^2), \pi^{-1}(U_{s+1}), \cdots, \pi^{-1}(U_\ell)$ are all pairwise disjoint.

We are now ready to discuss the map h. Recall that we began with a point $\mathbf{x} = (x_1 \cdots x_{n-1})$ $= (y_1 \cdots y_1 y_2 \cdots y_2 \cdots y_\ell \cdots y_\ell)$ in $\mathrm{Sym}^{n-1}(S^2)$. Let k_i be the number of times y_i appears in $\mathbf{x}$. Then $\mathbf{x}$ is contained in the open neighborhood $\mathrm{Sym}^{k_1}(U_1) \times \cdots \times \mathrm{Sym}^{k_\ell}(U_\ell)$. Here the product notation arises due to our unwavering insistence that the U_i be pairwise disjoint. Moreover, we see $h(\mathbf{x}) = (x_1^1 x_1^2 \cdots x_{n-1}^1 x_{n-1}^2)$ is contained in the analogous open neighborhood

$$\prod_{i=1}^{s} (\mathrm{Sym}^{k_i}(U_i^1) \times \mathrm{Sym}^{k_i}(U_i^2)) \times \prod_{i=s+1}^{\ell} \mathrm{Sym}^{2k_i}(\pi^{-1}(U_i)).$$

We see that locally the map h is a product of maps of two forms. If $1 \leq i \leq s$, we have maps h_i of the form

$$\begin{aligned} h_i : \mathrm{Sym}^{k_i}(U_i) &\to \mathrm{Sym}^{k_i}(U_i^1) \times \mathrm{Sym}^{k_i}(U_i^2) \\ (z_1 \cdots z_{k_i}) &\mapsto ((z_1^1 \cdots z_{n-1}^1), (z_1^2 \cdots z_{k_i}^2)) \end{aligned}$$

where z_i^1 is the lift of z_i in U_i^1 and similarly for U_i^2, If however $s+1 \leq i \leq \ell$, we have maps h_i of the form

$$\begin{aligned} h_i : \mathrm{Sym}^{k_i}(U_i) &\to \mathrm{Sym}^{2k_i}(\pi^{-1}(U_i)) \\ (z_1 \cdots z_{k_i}) &\mapsto (z_1^1 z_1^2 \cdots z_{k_i}^1 z_{k_i}^2) \end{aligned}$$

Our goal is to show h_i is holomorphic in each of these cases. We begin with the first case, in which x_i is not a branch point of the double branch covering. In this case h_i carries a point $(z_1 \cdots z_{k_i})$ in $\mathrm{Sym}^{k_i}(U_i)$ to the product of its lifts $(z_1^1 \cdots z_{k_i}^1)$ in $\mathrm{Sym}^{k_i}(U_i^1)$ and $(z_1^2 \cdots z_{k_i}^2)$ in $\mathrm{Sym}^{k_i}(U_i^2)$. Ergo $h_i = \mathrm{Sym}^{k_i}((\pi|_{U_i^1})^{-1}) \times \mathrm{Sym}^{k_i}((\pi|_{U_i^2})^{-1})$. We already know how to express this map in terms of the biholomorphisms $\mathrm{Sym}^{k_i}(f_i) : \mathrm{Sym}^{k_i}(U_i) \to \mathrm{Sym}^{k_i}(\mathbb{D}^{k_i})$ and the corresponding maps $\mathrm{Sym}^{k_i}(f_i^1)$ on $\mathrm{Sym}^{k_i}(U_i^1)$ and $\mathrm{Sym}^{k_i}(f_i^2))$ on U_i^2.

$$\begin{array}{ccc}
\mathrm{Sym}^{k_i}(U_i) & \xrightarrow{h_i=\mathrm{Sym}((\pi|_{U_i^1})^{-1})\times \mathrm{Sym}^{k_i}((\pi|_{U_i^2})^{-1})} & \mathrm{Sym}^{k_i}(U_i^1)\times \mathrm{Sym}^{k_i}(U_i^2)) \\
\Big\downarrow{\scriptstyle \mathrm{Sym}^{k_i}(f_i)} & & \Big\downarrow{\scriptstyle \mathrm{Sym}^{k_i}(f_i^1)\times \mathrm{Sym}^{k_i}(f_i^2)} \\
\mathrm{Sym}^{k_i}(\mathbb{D}) & \xrightarrow{\mathrm{Id}\times\mathrm{Id}} & \mathrm{Sym}^{k_i}(\mathbb{D})\times \mathrm{Sym}^{k_i}(\mathbb{D})
\end{array}$$

We see j_i is holomorphic. For the second case we will need to be slightly more subtle, producing actual charts for $\mathrm{Sym}^{k_i}(U_i)$ and $\mathrm{Sym}^{2k_i}(\pi^{-1}(U_i))$. We can assign $\mathrm{Sym}^{k_i}(U_j) \cong \mathrm{Sym}^{k_i}(\mathbb{D})$ a holomorphic chart using the familiar biholomorphism (5.2.1) which maps a point $(r_1 \cdots r_{k_i})$ in $\mathrm{Sym}^{k_i}(\mathbb{D})$ to the k_i elementary symmetric functions of its coordinates in $\mathbb{D}$. Let's see what this produces in our particular case. Let $\phi_{k_i}(\mathbb{D}) \subset \mathbb{C}^{k_i}$ be the image of $\mathrm{Sym}^{k_i}(\mathbb{D})$ under the map (5.2.1) from $\mathrm{Sym}^{k_i}(\mathbb{C}) \to \mathbb{C}^{k_i}$, and similarly for $\phi_{2k_i}(\mathbb{D})$.

$$\begin{array}{ccc}
\mathrm{Sym}^{k_i}(U_i) & \xrightarrow{h_i} & \mathrm{Sym}^{2k_i}(\pi^{-1}(U_i)) \\
\Big\downarrow{\scriptstyle \mathrm{Sym}^{k_i}(f_i)} & & \Big\downarrow{\scriptstyle \mathrm{Sym}^{2k_i}(g_i)} \\
\mathrm{Sym}^{k_i}(\mathbb{D}) & \longrightarrow & \mathrm{Sym}^{2k_i}(\mathbb{D}) \\
\Big\downarrow{\scriptstyle \phi_{k_i}} & & \Big\downarrow{\scriptstyle \phi_{2k_i}} \\
\phi_{k_i}(\mathbb{D}) & \longrightarrow & \phi_{2k_i}(\mathbb{D})
\end{array}$$

Let $(z_1 \cdots z_{k_i})$ be an arbitary point of U_i, so that $h_i(z_1 \cdots z_{k_i}) = (z_1^1 z_1^2 \cdots z_{k_i}^1 z_{k_i}^2)$. Then the middle horizontal map takes $(r_1 \cdots r_{k_i}) = (f_i(z_1) \cdots f_i(z_{k_i}))$ to $(g_i(z_1^1)g_i(z_1^2) \cdots g_i(z_{k_i}^1)g_i(z_{k_i}^2))$, which is exactly $(\sqrt{r_1} - \sqrt{r_1} \cdots \sqrt{r_{k_i}} - \sqrt{r_{k_i}})$. Taking symmetric functions of both sides reveals that the bottom horizontal map is expressed in coordinates as below.

$$\begin{aligned}
\phi_{k_i}(\mathbb{D}) &\to \phi_{2k_i}(\mathbb{D}) \\
(\sigma_1(r_1, \cdots, r_{k_i}), \cdots, \sigma_{k_i}(r_1, \cdots, r_{k_i})) &\mapsto (\sigma_1(\sqrt{r_1}, -\sqrt{r_1}, \cdots, \sqrt{r_{k_i}}, -\sqrt{r_{k_i}}), \cdots, \\
&\qquad \sigma_{2k_i}(\sqrt{r_1}, -\sqrt{r_1}, \cdots, \sqrt{r_{k_i}}, -\sqrt{r_{k_i}}))
\end{aligned}$$

Let's consider the symmetric functions σ_j of a set of complex roots and their opposites, to wit $(a_1, \cdots, a_{2k_i}) = (\sqrt{r_1}, -\sqrt{r_1}, \cdots, \sqrt{r_{k_i}}, -\sqrt{r_{k_i}})$. Recall from Chapter 6 that the functions σ_j are defined to be the sums

$$\sigma_j(a_1, a_2, \cdots, a_{2k_i}) = \sum_{1 \le i_1 < \cdots < i_j \le 2k_i} a_{i_1} \cdots a_{i_j}.$$

If the set of indices $(i_1, \cdots, i_j) \subset (i_1, \cdots, i_{2k})$ contains only one of i_{2t-1} and i_{2t} for any r, then there is a set of indices $(i_1', \cdots, i_j')$ identical to $(i_1, \cdots, i_j)$ except that either i_{2t} is replaced with i_{2t-1} or vice versa. Moreover $a_{i_1} \cdots a_{i_j} = -a'_{i_1'} \cdots a_{i_j'}$, so these two terms cancel each other out in the sum which comprises σ_j. Therefore $\sigma_j(a_1, \cdots, a_{2k_i})$ is a sum of terms $a_{i_1} \cdots a_{i_j}$ for sets $1 \leq i_1 < \cdots < i_j < 1$ which contain either both i_{2t-1} and i_{2t} or neither, for every $1 \leq t \leq \frac{j}{2}$. In particular, $\sigma_j(\sqrt{r_1}, -\sqrt{r_1}, \cdots, \sqrt{r_{k_i}}, -\sqrt{r_{k_i}}) = 0$ for j odd. When j is even we may make the following computation.

$$\begin{aligned}
\sigma_j(a_1, -a_1, \cdots, a_k, -a_k) &= \sum_{1 \leq i_1 < \cdots < i_j \leq k_i} a_{2i_1-1}a_{2i_1} \cdots a_{2i_j-1}a_{2i_j} \\
&= \sum_{1 \leq i_1 < \cdots < i_j \leq k_i} (\sqrt{r_{i_1}})(-\sqrt{r_{i_1}}) \cdots (\sqrt{r_{i_j}})(-\sqrt{r_{i_j}}) \\
&= \sum_{1 \leq i_1 < \cdots < i_j \leq k_i} (-1)^j r_{i_1} \cdots r_{i_j} \\
&= (-1)^j \sigma_j(r_1, \cdots, r_{k_i})
\end{aligned}$$

So the bottom horizontal map in the diagram above is of the form

$$\begin{aligned}
\phi_{k_i}(\mathbb{D}) &\to \phi_{2k_i}(\mathbb{D}) \\
(\sigma_1(r_1, \cdots, r_{k_i}), \cdots, \sigma_{k_i}(r_1, \cdots, r_{k_i})) &\mapsto \\
&(0, -\sigma_1(r_1, \cdots, r_{k_i}), 0, \cdots, (-1)^{k_i}\sigma_{k_i}(r_1, \cdots, r_{k_i}))
\end{aligned}$$

This is holomorphic, implying that h_i is as well. Since all the maps h_i are holomorphisms, and h is locally the product $h_1 \times \cdots \times h_\ell$, h is holomorphic.