Columbia University

Department of Economics
Discussion Paper Series

Profitability of Product Bundling

Yongmin Chen
Michael H. Riordan

Discussion Paper No.: 1011-02

Department of Economics
Columbia University
New York, NY 10027

March 2011

Profitability of Product Bundling [*]

Yongmin Chen[†] and Michael H. Riordan[‡]

March 2011

Abstract. Using copulas to model the stochastic dependence of values, this paper establishes new general conditions on the profitability of product bundling. A multiproduct monopolist generally achieves higher profit from mixed bundling than from separate selling if consumer values for two products are negatively dependent, independent, or have limited positive dependence. With more than two goods, the same conditions are sufficient for an optimal monopoly selling scheme to include a bundle of at least two products. The profitability of monopoly bundling also extends to situations where a multiproduct firm competes with a single-product rival.

Keywords: product bundling, mixed bundling, preference dependence.

[†]University of Colorado, Boulder; yongmin.chen@colorado.edu
[‡]Columbia University; mhr21@columbia.edu

[*]Helpful comments on earlier versions from participants of the 2010 Workshop on Law and Economics at Hitotsubashi University, the 2010 Shanghai Microeconomics Workshop (SHUFE), the 2010 Summer Conference on Industrial Organization (UBC, Vancouver), and seminars at Columbia University, University of Rochester, University of Virginia, and Yonsei University are gratefully acknowledged.

1. INTRODUCTION

When is product bundling more profitable than separate selling? The question has long intrigued economists. Stigler (1963) showed with a simple example that bundling can be profitable even without demand complementarities or scope economies. Adams and Yellen (1976) expanded on this view, showing, mostly with examples, that mixed bundling can be a profitable way to segment markets. Schmalensee (1984) studied the profitability of bundling when consumer values for two goods have a bivariate normal distribution, and found for the symmetric case sufficient conditions on the marginal distribution for pure bundling to dominate separate selling for any degree of correlation short of perfect positive correlation. Working with an arbitrary bivariate distribution having a continuous density function, Long (1984) found bundling to be profitable when consumer values are negatively dependent or independent. McAfee, McMillan, and Whinston (1989) relaxed the assumption of a continuous density function to develop a general sufficient condition for the profitability of bundling, albeit one that apparently is difficult to interpret in terms of dependence relations beyond saying bundling is optimal in a broader range of cases than just independence. Chu, Leslie, and Sorensen (2011) showed with numerical analyses that bundling is profitable in an array of special cases, including some featuring limited positive and negative dependence.

Although considerable attention has been directed at how the correlation of values for products matters for the profitability of bundling, the issue remains generally unclear, not only for positively dependent distributions, but also for negatively dependent distributions that are not absolutely continuous. We revisit the profitability of bundling with a new approach that uses a copula to represent the distribution of consumer values. A copula is a function that couples marginal distributions of random variables to form a joint distribution, making it straightforward to vary dependence while holding marginal distributions constant (Nelsen, 2006). Under general distributions of consumer preferences, and without assuming continuous joint densities, we show that bundling is more profitable than separate selling for a two-product monopolist if consumer values for two products are negatively dependent, independent, or have limited positive dependence.

We then extend the two- product monopoly model in two directions. First, we consider a multiproduct monopolist selling any number of goods. If consumer values for at least two of the goods possess one of the aforementioned dependence properties, then some form of bundling, e.g. selling two of the goods in a bundle as well as on a standalone basis, is more profitable than separate selling. We also consider situations in which a multiproduct firm competes against a single-product rival, with the multiproduct firm producing two distinct products, and the single-product competitor producing a differentiated version of one of them. Under similar dependence conditions as for a multiproduct monopoly, the multiproduct firm optimally chooses bundling in equilibrium, regardless of the dependence relationship between the two differentiated versions of the product that both firms produce.

The rest of the paper is organized as follows. Section 2 sets up a basic model of a monopoly producing two goods, and introduces the copula approach to representing the joint distribution of consumer values for the goods that determines demand. Section 3 establishes a key lemma that provides a general sufficient condition for the profitability of monopoly bundling. The lemma focuses squarely on the properties of the copula, and is employed in subsequent sections in various ways. Section 4 extends Long (1984) by showing product bundling is generally profitable when the distribution of consumer values are negatively dependent or independent even if the joint density function is not continuous. We also illustrate the result and its limits with a parameterized family of copulas. Section 5 introduces a bound on stochastic dependence that depends only on marginal distributions to show that bundling is profitable for any copula exhibiting a limited degree of positive dependence. Section 6 extends these results to a multiproduct monopoly selling any number of goods, and Section 7 to markets where a multiproduct firm competes against a differentiated single-product rival. Section 8 concludes with directions for further research.

2. BASIC MONOPOLY MODEL

Our model of product bundling by a monopolist hews closely to the basic framework of Stigler (1963), Adams and Yellen (1976), Schmalensee (1984), Long (1984) and McAfee, McMillan, and Whinston (1989). There are two goods, X and Y. The size of consumer population is normalized to 1. Each consumer demands at most one unit of each good, and her consumption of one does not affect her demand for the other. A consumer's value for X is u and for Y is v, with marginal distributions $F(u)$ and $G(v)$ on respective supports $[\underline{u}, \bar{u}]$ and $[\underline{v}, \bar{v}]$, with corresponding density functions $f(u) > 0$ and $g(v) > 0$ on $(\underline{u}, \bar{u})$ and $(\underline{v}, \bar{v})$ for $-\infty \leq \underline{u} < \bar{u} \leq \infty$ and $-\infty \leq \underline{v} < \bar{v} \leq \infty$. The value of the outside option is normalized to zero. The constant marginal costs for X and Y are m_X and m_Y, respectively. The value of two goods together is $u + v$, with marginal cost $m_X + m_Y$; thus this framework rules out product complementarity or economies of scale as explanations for bundling. Resale is not possible, and the firm cannot prevent consumers from purchasing both X and Y separately.[1]

[1] McAfee, McMillan, and Whinston (1989) also study the "monitoring case" with no resale, for which the firm can prevent consumers from purchasing both goods separately, and for which they conclude that bundling generally is profitable.

A benchmark for evaluating the profitability of bundling is the profit from simple monopoly pricing when the two goods are sold separately. A consumer can be represented by a point $(x,y) \in I^2$ with values $u(x) = F^{-1}(x)$ and $v(y) = G^{-1}(y)$ for the two goods. If X and Y are sold separately at prices p and q, consumers will purchase X if $u(x) \geq p$, or equivalently $x \geq F(p)$, and will purchase Y if $y \geq G(q)$. Therefore, monopoly prices for X and Y under separate selling respectively satisfy $p^s \in \arg\max_p \{(p - m_X)[1 - F(p)]\}$ and $q^s \in \arg\max_q \{(q - m_Y)[1 - G(q)]\}$. Following McAfee, McMillan, and Whinston (1989), we assume interior solutions under separate selling:[2]

Assumption 1 p^s and q^s satisfy the first-order conditions

$$
\begin{aligned}
1 - F(p^s) - (p^s - m_X) f(p^s) &= 0 \qquad (1)\\
1 - G(q^s) - (q^s - m_Y) g(q^s) &= 0. \qquad (2)
\end{aligned}
$$

with $0 < F(p^s) < 0$ and $0 < G(q^s) < 1$.

For any given (F,G) and (m_x, m_y), p^s and q^s are given, as are $x^s \equiv F(p^s)$ and $y^s \equiv G(q^s)$. For given (m_x, m_y), we shall call any (F,G) for which Assumption 1 holds *admissible*.[3]

Interpreting $(x,y) \in I^2$ as a consumer type, the population of consumers is described by a copula $C(x,y)$. A copula is a bivariate uniform distribution that "couples" arbitrary marginal distributions to form a new joint distribution. Standard uniform margins for x and y imply $C(x,1) = x$ and $C(1,y) = y$. A copula additionally satisfies $C(x,0) = 0 = C(0,y)$. By Sklar's Theorem, it is without loss of generality to represent the joint distribution of consumer values for the two products by a copula and the marginal distributions.[4] Let $x = F(u)$ and $y = G(v)$, and denote the copula associated with the joint distribution of (u,v) by $C(x,y)$. Then the joint distribution of (u,v) is $C(F(u), G(v))$. The partial derivatives, $C_1(x,y) \equiv \partial C(x,y)/\partial x$ and $C_2(x,y) \equiv \partial C(x,y)/\partial y$, exist almost everywhere. Furthermore, $C_1(x,y)$ is the conditional distribution of y given x, and $C_2(x,y)$ is the conditional distribution of x given y (Nelsen, 2006). The main departure of our approach from the previous literature on bundling is to use a copula to describe the population of consumers.[5]

[2] If $\underline{u}$ and $\underline{v}$ are not too much above the marginal costs, then p^s and q^s will be interior values satisfying (1) and (2) below.

[3] The marginal costs can be normalized to zero without loss of generality. With this normalization, u and v are interpreted as consumer values net of marginal costs, and p, q, and r are interpreted as markups.

[4] Sklar's Theorm holds also for more than two values, where any joint distribution can be represented by marginal distributions and a multivariate copula (Nelsen, 2006). We will use this to generalize our results to $n \geq 2$ products and to bundling under competition.

[5] An exception is Chu, Leslie, and Sorenson (2011) who use a Gaussian copula to model limited correlation

Under bundling, X and Y are sold individually at prices p and q, respectively, and the XY bundle is sold at price $r \leq p + q$. Pure bundling is a degenerate case in which p and q are high enough to choke off the standalone sales, while mixed bundling admits both bundled and separate sales. Consumers are willing to purchase the bundle if $u(x) + v(y) - r \geq \max\{0, u(x) - p, v(y) - q\}$, X alone if $u(x) - p \geq 0$ and $v(y) \leq r - p$, and Y alone if $v(y) - q \geq 0$ and $u(x) \leq r - q$. Consequently, demands for each good and the bundle at an interior solution are, respectively: [6]

$$\begin{aligned} Q_X(p,r) &\equiv G(r-p) - C(F(p), G(r-p)); &(3)\\ Q_Y(q,r) &\equiv F(r-q) - C(F(r-q), G(q)); &(4)\\ Q_{XY}(p,q,r) &\equiv \textstyle\int_{F(r-q)}^{F(p)} \left[1 - C_1(x, G(r-u(x))\right] dx + [1-F(p)] - Q_X(p,r). &(5) \end{aligned}$$

Therefore, the profit function under mixed bundling is

$$\pi\left(p,q,r\right) = \left(p - m_X\right) Q_X\left(p,r\right) + \left(q - m_Y\right) Q_Y\left(q,r\right) + \left(r - m_X - m_Y\right) Q_{XY}\left(p,q,r\right). \quad (6)$$

The multiproduct monopolist chooses (p, q, r) to maximize profit subject to $r \leq p + q$. Bundling has higher profit than separate selling if $r < p + q$ at the solution. If $r = p + q$, then profit is the same as under separate selling.

3. PRELIMINARY RESULTS

Our approach to finding a sufficient condition for the profitability of bundling is similar to Long (1984) and McAfee, McMillan, and Whinston (1989). Starting at monopoly pricing under separate selling, the analysis asks whether it is profitable to discount the bundle by a small amount.[7] Thus consider the profit function

$$\psi\left(\varepsilon\right) \equiv \pi\left(p^s, q^s, p^s + q^s - \varepsilon\right)$$

for different marginal distribution functions.

[6] Allowance for corner solutions, in which demand for any of the three options is zero, is a straightforword extension of these formulas. Riemann integrability of $C_1(x, G(r - u(x))$ in the formula for Q_{XY} requires continuity almost everywhere, as in the cases of positive or negative dependence examined below.

[7] McAfee, McMillan, and Whinston (1989) also considered raising one of the standalone prices by a small amount, which yields an equivalent condition for profitability. Long (1984) proved his result by interpreting mixed bundling as a two-part tariff and deriving conditions under which it is profitable to raise the fixed fee above zero. This is equivalent to raising the standalone prices *and* the bundle price all by ε, which also yields an equivalent condition.

for $\varepsilon \geq 0$. If $\psi(\varepsilon) > \psi(0)$ for some small positive ε, then some form of bundling must be profitable compared to separate selling. If $\pi(p, q, r)$ is differentiable at $(p^s, q^s, p^s + q^s)$, then $\psi'(0) > 0$ is a sufficient condition.

The following lemma is proved by relating the sign of $\psi'(0)$ to the properties of the copula. Indeed, $\psi'(0)$ has the same sign as

$$\Delta(x, y) \equiv (1 - x)[1 - C_1(x, y)] + (1 - y)[1 - C_2(x, y)] - \bar{C}(x, y). \tag{7}$$

where

$$\bar{C}(x, y) \equiv 1 - x - y + C(x, y).$$

$\bar{C}(x, y)$ is the joint survival function for two standard uniform random variables whose joint distribution is $C(x, y)$, i.e., the probability that a consumer's values for X and Y are above $u(x)$ and $v(y)$ respectively (Nelsen, 2006). The lemma provides a general sufficient indicator for the profitability of product bundling in terms of the dependence of consumer values as summarized by the copula.

Lemma 1 *(a) For any given admissible* (F, G), *bundling has a higher profit than separate selling if* $\Delta(x^s, y^s) > 0$.[8] *(b) If* $\Delta(x, y) > 0$ *for (almost) all* $(x, y) \in$ *int* I^2, *then bundling has a higher profit than separate selling for (almost) all admissible* (F, G).

Proof. (a) Mixed bundling is more profitable than separate selling if $\psi'(0) > 0$. We have

$$\begin{aligned} \psi'(0) &= -(p^s - m_X)\frac{\partial Q_X(p^s, p^s + q^s)}{\partial r} - (q^s - m_Y)\frac{\partial Q_Y(q^s, p^s + q^s)}{\partial r} \\ &\quad - (p^s + q^s - m_X - m_Y)\frac{\partial Q_{XY}(p^s, q^s, p^s + q^s)}{\partial r} - Q_{XY}(p^s, q^s, p^s + q^s). \end{aligned}$$

From (3), (4), and (5), simple differentiation and substitution yield:

$$\begin{aligned} \psi'(0) &= (p^s - m_x) f(p^s)[1 - C_1(F(p^s), G(q^s))] + (q^s - m_y) g(p^s)[1 - C_2(F(p^s), G(q^s))] \\ &\quad - [1 - F(p^s) - G(q^s) + C(F(p^s), G(q^s))]. \end{aligned}$$

Using first order conditions (1) and (2), and substituting $x^s = F(p^s)$ and $y^s = G(q^s)$, we obtain $\psi'(0) = \Delta(x^s, y^s)$.

(b) $\Delta(x, y)$ exists almost everywhere on int I^2. Therefore, if $\Delta(x, y) > 0$ almost everywhere, then bundling necessarily is profitable for almost all admissible (F, G). Furthermore,

[8] $\Delta(x^s, y^s) > 0$ implicitly requires $C_1(x, y)$ and $C_2(x, y)$ to exist at (x^s, y^s), which is almost surely satisfied $C(x, y)$ is differentiable almoste everywhere (Nelsen, 2006) . McAfee, McMillan and Whinston (1989) take a step further and assume that $C_1(x, y)$ (or $C_2(x, y)$) exists for all x (or for all y).

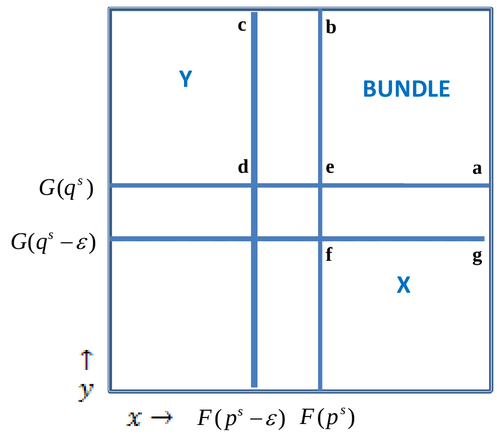

FIG. 1. Monopoly Mixed Bundling with a Small Discount for the Bundle

if $\Delta(x, y) > 0$ everywhere on int I^2, then bundling must be profitable for all admissible (F, G). ■

Lemma 1 provides an elegant, powerful, and useful condition for the profitability of bundling: assuming an interior solution to the monopoly separate-pricing problem, a sufficient condition for profitable bundling is stated only in terms of the copula; the condition dispenses with a joint density function, and is a sufficient condition for the profitability of bundling for all admissible marginal distributions rather than just for a given joint probability distribution; and the condition can be verified numerically for a given copula by evaluating an indicator function on the unit square.

Lemma 1(a) is analogous to McAfee, McMillan, and Whinston (1989)'s general sufficient condition for profitable bundling (Proposition 1), but uses the copula to describe consumer preferences while also relaxing technical conditions. Some intuition is gained from Fig. 1, which maps McAfee, McMillan, and Whinston (1989)'s Figure III to the consumer type space (I^2). The condition $\Delta(x^s, y^s) > 0$ weighs two effects of a vanishingly small ε discount of the XY bundle relative to separate pricing. The negative first-order effect of an ε discount is to lower the price to those consumers purchasing both products under separate pricing, corresponding to area *abe* in the figure and to the joint survival term $-\bar{C}(x, y)$ in the definition of $\Delta(x, y)$. The positive effect is to cause some consumers purchasing a single product under separate pricing to purchase the bundle instead, corresponding to area *bcde* and *aefg* and to the remaining terms of $\Delta(x, y)$. The lemma states a general condition for the positive effect to outweigh the negative effect as ε goes to zero.

4. NEGATIVE DEPENDENCE

We show that a multiproduct monopolist generally achieves higher profit from bundling than from separate selling under negative dependence or independence. Stigler (1963) and Adams and Yellen (1976) found by various examples that bundling can be more profitable than separate selling when values for products are negatively dependent. While the intuition from these studies seem to suggest that bundling generally is profitable under negative dependence, the precise conditions for such a conclusion are subtle and remain unsettled. Long (1984) showed that bundling is profitable if the distribution of consumer values has a continuous density and is negatively dependent in a particular way, while without assuming a continuous density McAfee, McMillan, and Whinston (1989) did not reach the same strong conclusion.[9]

Long (1984) derived the profitability of bundling under negative dependence by interpreting bundling as a two-part pricing scheme and analyzing demand elasticities. The copula approach provides an alternative statement and extension of Long (1984)'s result. The particular dependence condition identified by Long (1984) is the following:

$$\Pr\{v > q | u > p\} \text{ is nonincreasing in } p \text{ and } \Pr\{u > p | v > q\} \text{ is nonincreasing in } q.$$

In the language of modern statistics, u is *right tail decreasing* in v, and v is *right tail decreasing* in u (Nelsen 2006). Furthermore, if u and v are continuous random variables with a copula $C(x,y)$, then these two properties are equivalent respectively to the two conditions in the following definition (Nelsen, 2006):

Definition 1 *$C(x,y)$ is right tail decreasing (RTD) at $(x,y) \in I^2$ if*

$$C_1(x,y) \leq \frac{y - C(x,y)}{1-x}, \text{ and } C_2(x,y) \leq \frac{x - C(x,y)}{1-y}. \tag{8}$$

The following proposition extends the negative dependence result in Long (1984), replacing the assumption of a continuous density of consumer values with the following weaker condition on the copula at $x = x^s$ and $y = y^s$:

$$C_i(x,y) < 1 \text{ for at least one } i, i = 1, 2, \tag{9}$$

[9] McAfee, McMillan, and Whinston (1989) (pp. 379-380) argued informally that their sufficient condition for profitable bundling (Proposition 1) is satisfied if the monopoly price for good Y conditional on knowing the consumer reservation value for good X is decreasing in the value of good X, but concluded that this "cannot be tied solely to the correlation of reservation values."

The result expresses negative dependence as a property of the copula.

Proposition 1 *For any admissible* (F, G), *bundling is profitable if* $C(x, y)$ *is right tail decreasing and (9) holds at* (x^s, y^s).

Proof. Under right tail decreasing:

$$\begin{aligned}
\Delta(x^s, y^s) &\equiv (1-x^s)\left[1-C_1\left(x^s, y^s\right)\right]+(1-y^s)\left[1-C_2\left(x^s, y^s\right)\right]-\bar{C}\left(x^s, y^s\right) \\
&= 1-C\left(x^s, y^s\right)-(1-x^s)\, C_1\left(x^s, y^s\right)-(1-y^s)\, C_2\left(x^s, y^s\right) \\
&\geq 1-C\left(x^s, y^s\right)-(1-x^s)\, \frac{y^s-C\left(x^s, y^s\right)}{(1-x^s)}-(1-y^s)\, \frac{x^s-C\left(x^s, y^s\right)}{(1-y^s)} \\
&= 1-C\left(x^s, y^s\right)-y^s+C\left(x^s, y^s\right)-x^s+C\left(x^s, y^s\right) \\
&= 1-y^s-x^s+C\left(x^s, y^s\right) \equiv \bar{C}\left(x^s, y^s\right) \geq 0.
\end{aligned}$$

If $\bar{C}\left(x^s, y^s\right) > 0$, then $\Delta(x^s, y^s) > 0$. If $\bar{C}\left(x^s, y^s\right) = 0$, then

$$\Delta(x^s, y^s) = (1-x^s)\left[1-C_1(x^s, y^s)\right]+(1-y^s)\left[1-C_2(x^s, y^s)\right] > 0$$

by condition (9). In either case the result is immediate from Lemma 1(a). ■

Notice that Proposition 1 applies to $C\left(x, y\right) = xy$, i.e., bundling is profitable if consumer values for the two goods are independent.

From Lemma 2 (b) and Proposition 1, we immediately have the following generic condition on the profitability of bundling under negative dependence, which takes into account the fact that $C_i\left(x, y\right)$ might fail to exist on $(x, y) \in I^2$ only on a set of zero measure.

Corollary 1 *If* $C(x, y)$ *is right tail decreasing and (9) holds for (almost) all* $(x, y) \in$ *int* I^2, *then bundling is profitable for (almost) all admissible* (F, G).

If $C(x, y)$ is right tail decreasing at any $(x, y) \in$ int I^2, then condition (9) is satisfied if $\bar{C}\left(x, y\right) > 0$.[10] The property $\bar{C}\left(x, y\right) > 0$ for $(x, y) \in$ int I^2 means that some consumers purchase both goods for any interior solution of the independent pricing problem, and is satisfied if $C\left(x, y\right)$ has positive support as $x \to 1$ and $y \to 1$ from below. While this clearly holds if $C\left(x, y\right)$ has full support on I^2, as implicitly assumed by Long (1984), it is useful to have a more general statement of the negative tail dependence result, because many standard copula families do not have full support (Nelsen 2006).

[10] To see this, suppose to the contrary that $\bar{C}(x, y) > 0$ but $C_1(x, y) = 1$.Then,by negative right tail dependence, $1 = C_1\left(x, y\right) \leq \frac{y-C(x,y)}{1-x}$, or $1-x-y+C\left(x, y\right) = \bar{C}\left(x, y\right) \leq 0$,which is a contradiction.

To illustrate our results under negative dependence and gain additional insights, consider the following example:

Example 1 *Let*

$$C(x,y;\alpha) = \alpha \max\{x+y-1,0\} + (1-\alpha)\,xy, \;\; \textit{where } \alpha \in [0,1].$$

This defines a family of copulas parameterized by α*, combining two familiar copulas corresponding to perfect negative dependence and independence.*[11] *The entire family of copulas,* $C(x,y;\alpha)$*, lacks continuous densities except when* $\alpha = 0$*. For all* $\alpha \in [0,1)$*:*

$$\begin{aligned} C_1(x,y;\alpha) &= \begin{cases} \alpha + (1-\alpha)\,y < 1 & if \quad x+y-1>0 \\ (1-\alpha)\,y < 1 & if \quad x+y-1<0 \end{cases}; \\ C_2(x,y;\alpha) &= \begin{cases} \alpha + (1-\alpha)\,x < 1 & if \quad x+y-1>0 \\ (1-\alpha)\,x < 1 & if \quad x+y-1<0 \end{cases}, \end{aligned}$$

and both $C_1(x,y;\alpha) \leq \frac{y-C(x,y)}{1-x}$ *and* $C_2(x,y;\alpha) \leq \frac{x-C(x,y)}{1-y}$ *are satisfied when* $x^s+y^s-1 \neq 0$ *because*

$$\begin{aligned} \frac{y-C(x,y;\alpha)}{1-x} &= \begin{cases} = \alpha + (1-\alpha)\,y & if \quad x+y-1 \geq 0 \\ = \left[1+\frac{x\alpha}{1-x}\right] y & if \quad x+y-1<0 \end{cases}, \\ \frac{x-C(x,y;\alpha)}{1-y} &= \begin{cases} = \alpha + (1-\alpha)\,x & if \quad x+y-1 \geq 0 \\ = \left[1+\frac{\alpha y}{1-y}\right] x & if \quad x+y-1<0 \end{cases}. \end{aligned}$$

Therefore, with $\alpha \in [0,1)$*,* $C(x,y;\alpha)$ *is right tail decreasing and (9) holds for almost all* $(x,y) \in int\ I^2$*. It follows from Corollary 1 that bundling is profitable for almost all admissible* (F,G)*. Only for* (F,G) *with* $x^s+y^s-1=0$ *does Proposition 1 (or Corollary 1) not determine the profitability of bundling.*

Suppose for instance that $F(t) = G(t) = t$ *on* $[0,1]$ *and* $m_x, m_y \in [0,1)$*. Then, under separate selling, the firm's optimal prices are*

$$p^s(m_x) = \frac{1+m_x}{2} = x^s \in [\frac{1}{2},1); \qquad q^s(m_y) = \frac{1+m_y}{2} = y^s \in [\frac{1}{2},1),$$

and, for all $m_x, m_y \in (0,1)$*, both* (F,G) *are admissible and* $x+y-1 \neq 0$*. Therefore,*

[11] A useful result in the theory of copulas is that a convex linear combination of two copulas is a copula (Nelsen, 2006).

from Proposition 1, bundling is profitable for all joint distributions of consumer values corresponding to $C(x, y; \alpha)$ *with* $\alpha \in [0, 1)$ *and with* $F(t) = G(t) = t$ *for* $m_x, m_y \in (0, 1)$.

When $\alpha = 1$, $C(x, y; 1)$ *corresponds to the "Hotelling case" of perfect negative dependence:*

$$C(x, y; 1) = \max\{x + y - 1, 0\},$$

where $C_1(x, y) = 1 = C_2(x, y)$ *for all* $x + y - 1 > 0$. *Then condition (9) does not hold. Under separate selling, if* $m_x = m_y \equiv m$, *then maximum profit from the two products is* $\pi^S = 2\left(\frac{1-m}{2}\right)^2$. *Under bundling, the optimal pricing strategy is pure bundling with* $r^* = 1$, *which fully extracts consumer surplus.*[12] *So the maximum profit under bundling is* $\pi^B = 1 - 2m$. *Hence*

$$\pi^B - \pi^S = 1 - 2m - 2\left(\frac{1-m}{2}\right)^2 = \frac{1}{2}\left(1 - 2m - m^2\right) \gtreqqless 0 \text{ if } m \lesseqqgtr \sqrt{2} - 1 \approx 0.414\,21.$$

Therefore bundling is profitable in the Hotelling case if and only if $m < \sqrt{2} - 1$.

Several interesting points emerge from Example 1. First, the profitability of bundling under negative dependence applies to a much larger set of distributions of consumer values than identified in the previous literature. Second, the sufficient condition in our result, which is easy to check, is fairly tight. When it is violated, as is in the Hotelling case, it becomes possible that bundling is not profitable. Third, while from the intuition of Stigler (1963), Adams and Yellen (1976), and Long (1984) one might expect more negative dependence to make bundling more profitable, this is not true generally. In Example 1, with $F(t) = G(t) = t$ and $m_x, m_y \in (0, 1)$, bundling always is more profitable than separate selling except for $\alpha = 1$, the case perfect negative dependence, for which bundling is not profitable if $m \geq \sqrt{2} - 1$.

We conclude this section by discussing a somewhat stronger negative dependence property. Value u (v) is *stochastically decreasing* in v (u) if the conditional distribution of v (u) is nondecreasing in u (v). These stochastic monotonicity conditions are equivalent to the conditions of the following definition (Nelsen, 2006):

Definition 2 $C(x, y)$ *is negatively stochastic dependent at* $(x, y) \in I^2$ *if* $C_1(x, y)$ *is nondecreasing in* x *and* $C_2(x, y)$ *is nondecreasing in* y.

Negative stochastic dependence for $(x, y) \in I^2$ implies negative right tail dependence

[12] As in the Hotelling model of product differentiation, consumer preferences can be represented as a uniform distribution of locations on the unit line. The Hotelling case for the bundling model is similar to the negative dependence examples of Stigler (1963) and Adams and Yellen (1976).

(Nelsen, 2006), and, together with condition (9), is thus also generally sufficient for the profitability of bundling.

Strict negative dependence (i.e. $C(x,y)$ strictly convex in x and in y) implies an even stronger conclusion about the profitability of bundling. Armstrong (2010) considers the case of independent firms selling X and Y separately, and shows that, starting from separate monopoly prices, with strict negative stochastic dependence, at least one of the two firms has an incentive to offer a discount to consumers buying the other product. Indeed, in our setting, the firm selling Y has an incentive to offer a small $\varepsilon > 0$ discount to consumers buying X if

$$\begin{aligned} & \Delta(x,y) - (1-x^s)\left[1 - C_1(x^s,y^s)\right] \\ = \;& (1-y^s)\left[1 - C_2(x^s,y^s)\right] - \bar{C}(x^s,y^s) \\ = \;& C(x^s,1) - C(x^s,y^s) - (1-y^s)\,C_2(x^s,y^s) > 0, \end{aligned} \tag{10}$$

which follows from the strict convexity of $C(x,y)$ in y. The graphical interpretation in Fig. 1 helps explain this result: condition (10) states that *bcde* alone exceeds area *abe*. In other words, with strict negative stochastic dependence, the gain from increased sales of only one of the two products alone outweighs the cost of the discount on the bundle.

5. POSITIVE DEPENDENCE

Less is known about positive dependence. Since bundling is strictly more profitable for the independence copula $C(x,y) = xy$ and any admissible marginal distributions, the same must be true for copulas that are "close" to the independence copula. Therefore, as observed by McAfee, McMillan, and Whinston (1989), bundling is optimal in the neighborhood of the independence case. Schmalensee (1984) showed for the symmetric bivariate normal case that bundling is always profitable if demand under separate selling is sufficiently strong (i.e. in our framework if $x^s = y^s$ is sufficiently small) except in the case of perfect positive correlation, for which bundling never has an advantage over separate selling.[13] Beyond Schmalensee (1984)'s bivariate normal results, it remains an open question whether bundling dominates separate selling for any degree of positive dependence short of perfect.

Our main result here is that bundling is profitable if positive stochastic dependence is not

[13] With perfect positive dependence, any feasible mixed bundling scheme is equivalent to a separate selling scheme. This follows from the fact that with perfect positive dependence mixed bundling can have positive standalone sales of only one of the two goods, implying that mixed bundling is equivalent to separately selling one good at p and the other at $r-p$. Consequently, mixed bundling and separate selling have the same outcomes.

too great. The result puts a bound on the degree of positive stochastic dependence that assures profitable bundling. Value u is *stochastically increasing* in v if $\Pr\{u > p|v\}$ is nondecreasing in v; similarly, v is *stochastically increasing* in u if $\Pr\{v > q|u\}$ is nondecreasing in u. These properties are equivalent to the following definition (Nelsen, 2006):

Definition 3 *$C(x,y)$ is positively stochastic dependent at $(x,y) \in I^2$ if $C_1(x,y)$ is nonincreasing in x and $C_2(x,y)$ is nonincreasing in y.*

Given this definition, it is natural to measure the degree of positive stochastic dependence by how negative are $C_{11}(x,y) \equiv \frac{\partial^2 C(x,y)}{\partial x^2}$ and $C_{22}(x,y) \equiv \frac{\partial^2 C(x,y)}{\partial x^2}$, because these second derivatives determine the degree of concavity of $C(x,y)$ in x and in y.[14] Furthermore, if $C(x,y)$ is positively stochastic dependent on the interior of I^2, then these second derivatives exist almost everywhere.

Positive stochastic dependence for $(x,y) \in I^2$ implies positive right-tail dependence, and both in turn imply positive quadrant dependence (Nelsen, 2006):

Definition 4 *$C(x,y)$ is positively quadrant dependent at $(x,y) \in I^2$ if $C(x,y) \geq xy$.*

Positive quadrant dependence is used in the proof of the following proposition. The result says that bundling is profitable under positive dependence if $C_{ii}(x,y)$ are not too negative on the boundaries of the set of consumer types purchasing both goods under separate pricing.

Proposition 2 *For any given admissible (F,G), define the constant*

$$\delta^s \equiv \frac{2(1-x^s)(1-y^s)}{(1-y^s)^2 + (1-x^s)^2} > 0. \tag{11}$$

If $C(x,y)$ is positively quadrant dependent at (x^s, y^s), and

$$\min\{C_{11}(x,y^s), C_{22}(x^s,y))|x \geq x^s, y \geq y^s\} > -\delta^s, \tag{12}$$

then bundling is more profitable than separate selling.

[14] For many parameterized copula families, C_{ii} decreases in a parameter that indexes the range of positive dependence. This is true, for example, for the FGM, Clayton and Frank copula families discussed later.

Proof. For any given admissible (F, G),

$$\begin{aligned}
\Delta(x^s, y^s) &= (1-y^s)\left[1 - C_2(x^s, y^s)\right] + \int_{x^s}^1 [C_1(x, y^s) - C_1(x^s, y^s)]dx \\
&= (1-y^s)\left[1 - C_2(x^s, y^s)\right] + \int_{x^s}^1 \int_{x^s}^x C_{11}(z, y^s)\, dz dx \\
&= 1 - y^s - x^s + C(x^s, y^s) + \int_{y^s}^1 C_2(x^s, y)\, dy \\
&\quad - (1-y^s) C_2(x^s, y^s) + \int_{x^s}^1 \int_{x^s}^x C_{11}(z, y^s)\, dz dx \\
&= 1 - y^s - x^s + C(x^s, y^s) \\
&\quad + \int_{x^s}^1 \int_{x^s}^x C_{11}(z, y^s)\, dz dx + \int_{y^s}^1 \int_{y^s}^y C_{22}(x^s, z)\, dz dy.
\end{aligned}$$

Furthermore, $C(x^s, y^s) \geq x^s y^s$ implies

$$1 - y^s - x^s + C(x^s, y^s) \geq (1-x^s)(1-y^s); \tag{13}$$

and $\min\{C_{11}, C_{22}\} > -\delta^s$ further implies

$$\begin{aligned}
\Delta(x^s, y^s) &\geq (1-x^s)(1-y^s) + \int_{y^s}^1 \int_{y^s}^y C_{22}(x^s, z)\, dz dy + \int_{x^s}^1 \int_{x^s}^x C_{11}(z, y^s)\, dz dx \\
&> (1-x^s)(1-y^s) - \left[\frac{(1-y^s)^2}{2} + \frac{(1-x^s)^2}{2}\right] \frac{2(1-x^s)(1-y^s)}{(1-y^s)^2 + (1-x^s)^2} = 0.
\end{aligned}$$

■

Note that the bound δ^s is independent of $C(x, y)$ and reaches a maximum of 1 when $x^s = y^s$. Thus, for all admissible marginal distributions, there is some range of positive dependence for which bundling is profitable. This range is larger when market shares under separate pricing are closer together. The result goes substantially beyond McAfee, McMillan, and Whinston (1989)'s observation that bundling is profitable in the neighborhood of independence.

To illustrate the substantial ranges of positive dependence allowed by (12), consider an example where the copula belongs to the Fairlie-Gumbel-Morgenstern (FGM) family:

$$C(x, y; \theta) = xy\left[1 + \theta(1-x)(1-y)\right],$$

where $\theta \in [-1, 1]$. Notice that $C(x, y; \theta)$ is positively (or negatively) dependent if $\theta > 0$ (or

$\theta < 0$), and $C_{ii}(x, y; \theta)$ decreases in θ. For $\theta \leq 0$, bundling is profitable from Proposition 1. Now consider positive dependence. It follows from $C_{11} = -2\theta y (1-y)$ that $C_{11} > -\delta^s$ if and only if

$$1 - x^s > \theta \left[y^s (1-y^s)^2 + y^s (1-x^s)^2 \right],$$

which holds for all $\theta \leq 1$ if

$$1 - x^s - y^s (1-y^s)^2 - y^s (1-x^s)^2 > 0,$$

which in turn holds if $x^s = y^s$ or if $\max\{x^s, y^s\} \leq 5/6$; and similarly for C_{22}. In other words, for all admissible marginal distributions with $F(p^s) = G(q^s)$ or with $F(p^s) \leq 5/6$ and $G(q^s) \leq 5/6$, the degrees of positive dependence for all FGM family copulas fall below the bound given in (12). For any such admissible (F, G), bundling is profitable for any member of the FGM copula family.[15]

The FGM copula family exhibits a limited range of negative and positive dependence, and $\theta = 1$ does not correspond to perfect positive dependence. For distributions with higher degrees of positive dependence short of perfect dependence, it is easy to check Lemma 1 (b)'s condition for the profitability of bundling numerically. Consider, for instance, the Frank copula family

$$C(x, y; \theta) = -\frac{1}{\theta} \ln \left(1 + \frac{\left(e^{-\theta x} - 1\right)\left(e^{-\theta y} - 1\right)}{\left(e^{-\theta} - 1\right)} \right) \text{ for } \theta \in (-\infty, \infty)/0,$$

and the Clayton copula family

$$C(x, y; \theta) = \max \left\{ \left[x^{-\theta} + y^{-\theta} - 1 \right]^{-1/\theta}, 0 \right\} \text{ for } \theta \in [-1, \infty)/0.$$

Both a Frank copula and a Clayton copula exhibit positive stochastic dependence if $\theta > 0$ and $\theta = \infty$ corresponds to perfect dependence. For both copula families, numerical analysis shows that $\Delta(x, y) > 0$ for $(x, y) \in \text{int } I^2$. Therefore, from Lemma 1(b), bundling appears to be profitable for *all* Frank and Clayton copulas for *any* admissible marginal distributions. Referring back to Fig. 1, for positively dependent Frank and Clayton copulas, the gains

[15] In fact, for all $(x, y) \in \text{Int } I^2$, it is easy to show

$$\Delta(x, y) = (1-x)(1-y)(3xy\theta - y\theta - x\theta + 1) > 0.$$

From Lemma 1(b) we conclude that bundling is always profitable for all admissible (F, G) and for all members of the FGM copula family.

from consumers in ($bcde$ + $aefg$) always exceed the loss from consumers in abe.[16] This again shows the power of the copula approach and Lemma 1(b): instead of checking all joint distributions that can be formed by these copulas and all admissible (F, G), which would be virtually impossible to do numerically, all we need is to compute $\Delta(x, y)$ on $(x, y) \in I^2$, which is solely determined by the copula function.

So far, we have not found a counterexample of the profitability of bundling under positive dependence outside the limiting case of perfect positive dependence.

6. ANY NUMBER OF GOODS

The profitability conditions for bundling by a monopolist with two goods can be extended to any number of products. To proceed, we generalize the model in Section 2 to any $n \geq 2$ products: X_1, ..., X_n.[17] Let u_i denote the consumer value for X_i, $F_i(u_i)$ the marginal distribution of values, m_i the marginal cost, and p_i^s the single product monopoly price. As with $n = 2$, p_i^s is assumed to be an interior solution of the profit maximization problem, and satisfies

$$1 - F_i(p_i^s) - (p_i^s - m_i) f_i(p_i^s) = 0.$$

Let $\tilde{C}(x_1, ...x_n)$ denote the multivariate copula describing joint distribution of $x_i = F_i(u_i)$ for $i = 1, ...n$. By Sklar's Theorem, the joint distribution of consumer values for the n goods is therefore $\tilde{C}(F_1(u_1), ..., F_n(u_n))$. Assume that the value of two goods X_i and X_j together is $u_i + u_j$, with constant marginal cost $m_i + m_j$, and the values and marginal costs are similarly obtained for l goods together, $l \leq n$. Again, this framework rules out product complementarity or economies of scale as explanations for bundling.

Our previous results on the profitability of bundling for the two-good monopolist extend readily to the n good case. Consider the profitability of selling a two-good bundle $\{\mathrm{X}_1, \mathrm{X}_2\}$ together with individually-priced goods $\mathrm{X}_1, ..., \mathrm{X}_n$. Suppose the prices for goods $\mathrm{X}_3, ...\mathrm{X}_n$ are set at $p_i = p_i^s$, $i = 3, ..., n$, so the profits from the sale of these $(n-2)$ goods is by hypothesis the same as from separate selling. It then suffices to show that profit from goods X_1 and X_2 will be higher under the proposed bundling than under separate selling. Notice that the joint distribution of consumer values for X_1 and X_2 can be represented

[16] Both copula families also allow independence and the full range of negative dependence, but these cases are covered by Corollary 1.

[17] In reality, a firm sometimes sells multiple groups of products, and goods within each product group could be substitutes such that a consumer may purchase only one of them. For ease of exposition, we do not explicitly model this situation, but we can accommodate this possibility by allowing the interpretation of X_i, if appropriate, as any (symmetric) good from product group i, $i = 1, ..., n$, where goods within group i are substitutes.

by $C(F_1(u_1), F_2(u_2))$, where $C(x,y) \equiv \tilde{C}(x,y,1,...,1)$ is a bivariate copula. Therefore, Lemma 1 applies, and under the conditions of Propositions 1 and 2, profits from X_1 and X_2 are higher under mixed bundling than under separate selling. Hence:

Corollary 2 *For a multiproduct monopolist selling $n \geq 2$ products, if consumer values for at least two goods are negatively dependent, independent, or have limited positive dependence, then some form of bundling will have higher profits than separate selling.*

7. PARTIAL COMPETITION

The profitability of bundling under multiproduct monopoly also extends to markets where a multiproduct firm competes against a single-product firm. We focus on the case where the multiproduct firm, A, offers two products X and Y_A, whereas a single-product firm, B, offers a symmetrically differentiated version of product Y, Y_B. The two firms compete by simultaneously choosing prices, where for firm A the prices can either be those under separate selling or those under mixed bundling. We assume a pure strategy equilibrium exists for this model of price competition, and consider whether in equilibrium the multiproduct firm has higher profits from bundling than from separate selling.

A consumer's value for X is $u(x)$, and for Y_i is $v(y_i)$ with $(x, y_A, y_B) \in I^3$. Therefore, the marginal distribution of consumer values for X is $F(u)$, and the symmetric distribution for each variety of product Y is $G(v)$, with corresponding density functions $f(u) > 0$ and $g(v) > 0$ on their respective supports. The copula $C(x, y_A, y_B)$, with y_A and y_B exchangeable, describes the population of consumers. Adopting stochastic monotonicity dependence concepts, and assuming differentiability, we say that values for X and Y are positively dependent, independent, or negatively dependent when respectively $C_{11}(x, y_A, y_B) \leq 0$, $C_{11}(x, y_A, y_B) = 0$ or $C_{11}(x, y_A, y_B) \geq 0$ for almost all $(x, y_A, y_B) \in I^3$.

Under mixed bundling, let p denote the standalone price of X, q_i the standalone price of Y_i for $i \in \{A, B\}$, and $r \leq p + q_A$ the price of Firm A's bundle. Separate pricing is equivalent to $r = p + q_A$, in which case the demands for X and Y_A are respectively

$$\begin{aligned} Q_X^s(p) &= 1 - F(p), \\ Q_{Y_A}^s(q_A, q_B) &= 1 - C(1, G(q_A), G(q_B)) - \textstyle\int_{G(q_B)}^1 C_3(1, G(q_A + v(y) - q_B), y)\, dy. \end{aligned}$$

The marginal costs for products X and Y are m_X and m_Y, and interior equilibrium prices p^s and q^s in the two product markets satisfy

$$1 - F(p^s) = (p^s - m_X) f(p^s) \tag{14}$$

and

$$
\begin{aligned}
& \frac{1}{2}\left[1-C\left(1, G\left(q^{s}\right), G\left(q^{s}\right)\right)\right] \qquad (15)\\
= \; & \left(q^{s}-m_{Y}\right)\left[C_{2}\left(1, G\left(q^{s}\right), G\left(q^{s}\right)\right) g(q^{s})+\int_{G(q^{s})}^{1} C_{23}\left(1, y, y\right) g(v(y)) dy\right]
\end{aligned}
$$

respectively. See Chen and Riordan (2010) for details on symmetric equilibrium in the Y market.

Return to mixed bundling, consumers will purchase the bundle if

$$
\begin{aligned}
u(x)+v(y_{A})-r & \geq \max\left\{0, v(y_{B})-q_{B}\right\}, \\
u(x)+v(y_{A})-r & \geq u(x)-p+\max\left\{0, v(y_{B})-q_{B}\right\}, \\
u(x)+v(y_{A})-r & \geq v(y_{A})-q_{A},
\end{aligned}
$$

or, equivalently,

$$
\begin{aligned}
y_{A} & \geq G\left(r-u(x)+\max\left\{0, v(y_{B})-q_{B}\right\}\right), \\
y_{A} & \geq G\left(r-p+\max\left\{0, v(y_{B})-q_{B}\right\}\right), \\
x & \geq F\left(r-q_{A}\right).
\end{aligned}
$$

Consumers will purchase X as a standalone product, rather than as part of the bundle, if

$$
\begin{aligned}
x & \geq F(p), \\
y_{A} & < G\left(r-p+\max\left\{0, v(y_{B})-q_{B}\right\}\right);
\end{aligned}
$$

and Y_A as a standalone product if

$$
\begin{aligned}
x & < F\left(r-q_{A}\right), \\
y_{A} & \geq G\left(q_{A}+\max\left\{0, v(y_{B})-q_{B}\right\}\right).
\end{aligned}
$$

To evaluate the demand for the products offered by Firm A, consider a type (x, y_A, y_B) consumer who is willing to purchase good X at price p, i.e. $x \geq F(p)$. This consumer also has the opportunity to acquire good Y_A as part of the bundle by paying an incremental price $r-p$, or to purchase Y_B at price q_B. The consumer's choice in the Y market depends on (y_A, y_B) as illustrated in the unit square of Fig. 2. Consumers in region XY_A purchase the bundle, those in in region XY_B separately purchase X and Y_B, and those in region X only purchase good X. Therefore, consumers making standalone purchases of X are those in the

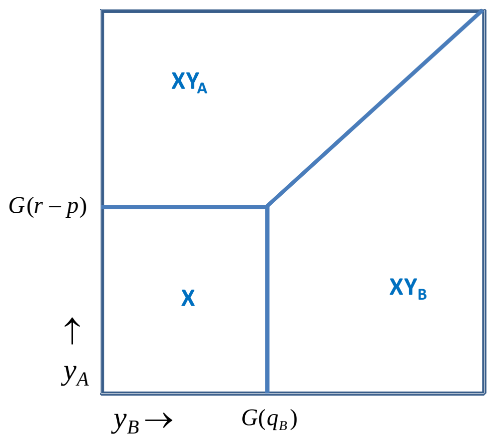

FIG. 2. Demand in Market Y for Consumers Willing to Buy X

union of regions XY_B and X. The following demand function aggregates these consumer for all $x \geq F(p)$:

$$\begin{aligned} & Q_X(p, q_B, r) \\ = \; & C(1, G(r-p), G(q_B)) - C(F(p), G(r-p), G(q_B)) \\ & + \int_{G(q_B)}^{1} [C_3(1, G(r-p+v(y)-q_B), y) - C_3(F(p), G(r-p+v(y)-q_B), y)]\, dy. \end{aligned} \tag{16}$$

Similarly, the standalone demand for Y_A

$$\begin{aligned} & Q_{Y_A}(q_A, q_B, r) \\ = \; & C(F(r-q_A), 1, G(q_B)) - C(F(r-q_A), G(q_A), G(q_B)) \\ & + \int_{G(q_B)}^{1} [C_3(F(r-q_A), 1, y) - C_3(F(r-q_A), G(q_A+v(y)-q_B), y)]\, dy, \end{aligned} \tag{17}$$

and the demand for the bundle is

$$\begin{aligned} & Q_{XY_A}(p, q_A, q_B, r) \\ = \; & 1 - F(p) - Q_X(p, q_B, r) \\ & + \int_{F(r-q_A)}^{F(p)} [C_1(x, 1, G(q_B)) - C_1(x, G(r-u(x)), G(q_B))]\, dx \\ & + \int_{F(r-q_A)}^{F(p)} \int_{G(q_B)}^{1} [C_{13}(x, 1, y) - C_{13}(x, G(r-u(x)+v(y)-q_B), y)]\, dy dx. \end{aligned} \tag{18}$$

The demand for Y_B is analogous to the demand for Y_A.

With no economies of scope, the profit of Firm A is

$$\begin{aligned}\pi_A(p, q_A, q_B, r) &= (p - m_X) Q_X(p, q_B, r) + (q_A - m_Y) Q_{Y_A}(q_A, q_B, r) \\ &\quad + (r - m_X - m_Y) Q_{XY_A}(p, q_A, q_B, r).\end{aligned}$$

We now extend the profitability of bundling under multiproduct monopoly to this partial competition model, by establishing that in equilibrium the multiproduct firm will optimally choose bundling if values for X and Y are negatively dependent, independent, or have sufficiently limited positive dependence.

Proposition 3 *Let*

$$\bar{\delta} \equiv (q^s - m_Y) g(q^s) [C_2(1, G(q^s), G(q^s)) - C_2(F(p^s), G(q^s), G(q^s))] > 0. \tag{19}$$

In equilibrium, bundling is more profitable for the multiproduct firm than separate selling if $C_{11} > -4\bar{\delta}$.

Proof. See the appendix. ■

Thus a multiproduct firm facing a single product competitor will optimally choose bundling in equilibrium for a range of dependence conditions, similarly to a multiproduct monopolist. The main contrast with Proposition 1 is that Proposition 3 employs a stronger negative dependence property, and the main contrast with Proposition 2 is that the bound on the degree of positive dependence in Proposition 3 depends on the copula.[18] Nevertheless, the results are the same for negative dependence and independence. A strength of Proposition 3 is that it allows any dependence relation between Y_A and Y_B.

We have confined our analysis of bundling under competition to situations where a multiproduct firm competes with a single-product rival. The profitability of bundling is relatively simple in this case, because only the multiproduct firm can choose to bundle its products. In markets where the competition is between multiproduct firms, the issue of bundling is more complex, since the profitability of bundling by one firm may depend on whether or not the other firm bundles. The issue also is more complex because there are dependence relations both between values for different products and between values for products by different firms. We leave it for future research to address the issue of equilibrium product bundling by competing multiproduct firms under general preference dependence conditions.[19]

[18] With (minor) additional restrictions, it's possible to find a lower bound on C_{11} that is a fixed number. For instance, if $Q^s_{Y_A}$, $g(\cdot)$, and $C_{123}(\cdot)$ are all bound above zero, then $\bar{\delta}$ is bound above zero, and the lower bound on C_{11} in Proposition 3 can be stated as $C_{11} \geq -\bar{\delta}$ for some fixed $\bar{\delta} > 0$.

[19] As McAfee, McMillan, and Whinston (1989) observed, under competition between multiproduct firms,

8. CONCLUSION

This paper shows that a multiproduct firm achieves higher profit from mixed bundling than from separate selling if consumer values for two products are negatively dependent, independent, or have limited positive dependence. When the firm sells more than two products, profit is higher under bundling if values for at least two of the goods possess one of these dependence properties. Furthermore, the profitability of monopoly bundling extends to markets where a multiproduct firm competes against a single-product rival.

There are several directions for future research. For instance, while monopoly bundling often increases the firm's profit, its effects on consumer and social welfare are less clear. It would be desirable to find general conditions with which the consumer and welfare effects of monopoly bundling can be evaluated. Stigler (1963) and our Example 1 show that consumers may be worse off with bundling, but it is unclear how robust is this possibility. It would also be interesting to further study the incentives for and effects of bundling under competition. For instance, according to the existing literature, whereas bundling can be an effective entry barrier, it sometimes may also be entry-accommodating by creating (or increasing) product differentiation. It would be desirable to develop an understanding of when bundling forecloses competition and when it softens competition in a more general framework of preference dependence.[20]

if consumer values for all goods are independently distributed, then a firm will find it optimal to engage in mixed bundling if the other firm does not, so that it cannot be an equilibrium for all firms to choose separate selling.

[20] The foreclosure theory of bundling was first formalized in Whinston (1990). Other contributions on the foreclosure effects of bundling include Carlton and Waldman (2002), Choi and Stefanadis (2001), and Nalebuff (2004). Our result here shows that even without the foreclosure motive, a multiproduct firm can often profit from bundling its products in competing with a single-product rival. With competition between multiproduct firms, firms may also choose to offer (different) bundles in order to create endogenous product differentiation (Carbajo, De Meza, and Seidman, 1990; and Chen, 1997).

APPENDIX

Proof of Proposition 3

We show that, under the conditions specified in the Proposition, at any equilibrium firm A must choose bundling rather than separate selling; i.e., in equilibrium bundling must have higher profit than separate selling for firm A. This would be true if separate selling cannot be part of any equilibrium.

Let (p^s, q^s) denote the prices of X and Y products in a separate-pricing equilibrium. Then

$$\bar{\psi}(\varepsilon) \equiv \pi_A (p^s + \epsilon, q^s, q^s, p^s + q^s)$$

is Firm A's profit from increasing the standalone price of X, while holding constant the standalone prices of the Y product and the price of the bundle at $r = p^s + q^s$. Separate selling cannot be part of any equilibrium if $\bar{\psi}'(0) > 0$. Noticing that $\partial Q_{Y_A}(q_A, q_B, r)/\partial p = 0$, we have

$$\begin{aligned}
\bar{\psi}'(0) &= \frac{\partial \pi_A (p^s, q^s, q^s, p^s + q^s)}{\partial p} \\
&= Q_X (p^s, q^s, p^s + q^s) + (p^s - m_X) \frac{\partial Q_X (p^*, q^*, p^* + q^*)}{\partial p} \\
&\quad + (p^s + q^s - m_X - m_Y) \frac{\partial Q_{XY_A} (p^*, q^*, q^*, p^* + q^*)}{\partial p}
\end{aligned}$$

with

$$Q_X (p^s, q^s, p^s + q^s) \equiv \frac{1}{2}[1 - F(p^s)] + \frac{1}{2}[C(1, G(q^s), G(q^s)) - C(F(p^s), G(q^s), G(q^s))],$$

$$\begin{aligned}
\frac{\partial Q_X (p^s, q^s, p^s + q^s)}{\partial p} \equiv\ & -[C_2(1, G(q^s), G(q^s)) - C_2(F(p^s), G(q^s), G(q^s))]\, g(q^s) \\
& - C_1(F(p^s), G(q^s), G(q^s))\, f(p^s) \\
& - \int_{G(q^s)}^{1} [C_{23}(1, y, y) - C_{23}(F(p^s), y, y)]\, g(v(y))\, dy \\
& - \int_{G(q^s)}^{1} C_{13}(F(p^s), y, y)\, dy f(p^s),
\end{aligned}$$

$$\begin{aligned}\frac{\partial Q_{XY_A}\left(p^s,q^s,p^s+q^s\right)}{\partial p} &\equiv -f\left(p^s\right)-\frac{\partial Q_X\left(p^s,q^s,p^s+q^s\right)}{\partial p} \\ &\quad +\left[C_1\left(F\left(p^s\right),1,G\left(q^s\right)\right)-C_1\left(F\left(p^s\right),G\left(q^s\right),G\left(q^s\right)\right)\right]f\left(p^s\right) \\ &\quad +\int_{G(q^s)}^1\left[C_{13}\left(F\left(p^s\right),1,y\right)-C_{13}\left(F\left(p^s\right),y,y\right)\right]dyf\left(p^s\right)\end{aligned}$$

$$\begin{aligned} &= -f\left(p^s\right)+\left[C_2\left(1,G\left(q^s\right),G\left(q^s\right)\right)-C_2\left(F\left(p^s\right),G\left(q^s\right),G\left(q^s\right)\right)\right]g\left(q^s\right) \\ &\quad +\int_{G(q^s)}^1\left[C_{23}\left(1,y,y\right)-C_{23}\left(F\left(p^s\right),y,y\right)\right]g\left(v\left(y\right)\right)dy+C_1\left(F\left(p\right),1,1\right)f\left(p\right).\end{aligned}$$

Therefore,

$$\bar{\psi}'\left(0\right)=Q_X\left(p^s,q^s,p^s+q^s\right)$$

$$+\left(p^s-m_X\right)\left\{\begin{array}{c}-f\left(p^s\right)+\left[C_1\left(F\left(p^s\right),1,G\left(q^s\right)\right)-C_1\left(F\left(p^s\right),G\left(q^s\right),G\left(q^s\right)\right)\right]f\left(p^s\right) \\ +\int_{G(q^s)}^1\left[C_{13}\left(F\left(p\right),1,y_2\right)-C_{13}\left(F\left(p\right),y_2,y_2\right)\right]f\left(p\right)dy_2\end{array}\right\}$$

$$+\left(q^s-m_Y\right)\left\{\begin{array}{c}-f\left(p^s\right)+\left[C_2\left(1,G\left(q^s\right),G\left(q^s\right)\right)-C_2\left(F\left(p^s\right),G\left(q^s\right),G\left(q^s\right)\right)\right]g\left(q^s\right)+ \\ \int_{G(q^s)}^1\left[C_{23}\left(1,y,y\right)-C_{23}\left(F\left(p^s\right),y,y\right)\right]g\left(v\left(y\right)\right)dy+C_1\left(F\left(p\right),1,1\right)f\left(p\right)\end{array}\right\}$$

Substituting $C_1\left(F\left(p^s\right),1,1\right)=1$ and $C_{13}\left(1,y_2,y_2\right)=\frac{1}{2}\frac{dC_1(1,y_2,y_2)}{dy_2}$, and simplifying, we have

$$\begin{aligned}\bar{\psi}'\left(0\right) &= Q_X\left(p^s,q^s,p^s+q^s\right) \\ &\quad +\left(p^s-m_X\right)\left\{\begin{array}{c}\left[C_1\left(F\left(p^s\right),1,G\left(q^s\right)\right)-C_1\left(F\left(p^s\right),G\left(q^s\right),G\left(q^s\right)\right)\right]f\left(p^s\right) \\ -C_1\left(F\left(p\right),1,G\left(q^s\right)\right)f\left(p^s\right)-\int_{G(q^s)}^1\frac{1}{2}dC_1\left(F\left(p\right),y_2,y_2\right)f\left(p\right)\end{array}\right\} \\ &\quad +\left(q^s-m_Y\right)\left\{\begin{array}{c}\left[C_2\left(1,G\left(q^s\right),G\left(q^s\right)\right)-C_2\left(F\left(p^s\right),G\left(q^s\right),G\left(q^s\right)\right)\right]g\left(q^s\right) \\ +\int_{G(q^s)}^1\left[C_{23}\left(1,y,y\right)-C_{23}\left(F\left(p^s\right),y,y\right)\right]g\left(v\left(y\right)\right)dy\end{array}\right\}\end{aligned}$$

$$\begin{aligned} &= Q_X\left(p^s,q^s,p^s+q^s\right)+\left(p^s-m_X\right)\left\{\begin{array}{c}-C_1\left(F\left(p^s\right),G\left(q^s\right),G\left(q^s\right)\right)f\left(p^s\right)- \\ \frac{1}{2}\left[C_1\left(F\left(p^s\right),1,1\right)-C_1\left(F\left(p^s\right),G\left(q^s\right),G\left(q^s\right)\right)\right]f\left(p^s\right)\end{array}\right\} \\ &\quad +\left(q^s-m_Y\right)\left\{\begin{array}{c}\left[C_2\left(1,G\left(q^s\right),G\left(q^s\right)\right)-C_2\left(F\left(p^s\right),G\left(q^s\right),G\left(q^s\right)\right)\right]g\left(q^s\right) \\ +\int_{G(q^s)}^1\left[C_{23}\left(1,y,y\right)-C_{23}\left(F\left(p^s\right),y,y\right)\right]g\left(v\left(y\right)\right)dy\end{array}\right\}.\end{aligned}$$

Substituting for $Q_X(p^s, q^s, p^s + q^s)$, using $(p^s - m_X) f(p^s) = 1 - F(p^s)$, and simplifying

$$\begin{aligned}
\bar{\psi}'(0) &= \frac{1}{2}\int_{F(p^s)}^{1} \left[C_1(x, G(q^s), G(q^s)) - C_1(F(p^s), G(q^s), G(q^s))\right] dx \\
&\quad + (q^s - m_Y)\left\{ \begin{array}{c} \left[C_2(1, G(q^s), G(q^s)) - C_2(F(p^s), G(q^s), G(q^s))\right] g(q^s) \\ + \int_{G(q^s)}^{1} \left[C_{23}(1, y, y) - C_{23}(F(p^s), y, y)\right] g(v(y))\, dy \end{array} \right\} \\
&\geq \frac{1}{2}\int_{F(p^s)}^{1}\int_{F(p^s)}^{x} C_{11}(z, G(q^s), G(q^s))\, dz dx + \bar{\delta},
\end{aligned}$$

since $\int_{G(q^s)}^{1} \left[C_{23}(1, y, y) - C_{23}(F(p^s), y, y)\right] g(v(y))\, dy \geq 0$, where

$$\bar{\delta} = (q^s - m_Y) g(q^s) \left[C_2(1, G(q^s), G(q^s)) - C_2(F(p^s), G(q^s), G(q^s))\right] > 0.$$

Thus, $\bar{\psi}'(0) > 0$ if values for X and Y are negatively dependent ($C_{11} \geq 0$) or independent ($C_{11} = 0$). Now, suppose that values for X and Y are positively dependent but $C_{11} > -4\bar{\delta}$. Then

$$\begin{aligned}
\bar{\psi}'(0) &> -\frac{1}{2}\int_{F(q^s)}^{1}\int_{F(q^s)}^{x} 4\bar{\delta} dz dx + \bar{\delta} \\
&= -\frac{1}{4}(1 - F(q^s))^2 4\bar{\delta} + \bar{\delta} > -\bar{\delta} + \bar{\delta} = 0.
\end{aligned}$$

Q.E.D.

REFERENCES

[1] Adams, William J. and Janet L. Yellen (1976), "Commodity Bundling and the Burden of Monopoly," *Quarterly Journal of Economics*, 90, 475-98.

[2] Armstrong, Mark (2010), " "Bundling Revisited: Substitute Products and Inter-firm Discounts," MPRA Working Paper 26782.

[3] Carbajo, Jose, David De Meza, and Daniel J. Seidman (1990), "A Strategic Motivation for Commodity Bundling," *Journal of Industrial Economics*, 38, 283-298.

[4] Carlton, D. and M. Waldman (2002), "The Strategic Use of Tying to Preserve and Create Market Power in Evolving Industries", *RAND Journal of Economics*, 32(2), pp. 194-220.

[5] Chen, Yongmin (1997), "Equilibrium Product Bundling," *Journal of Business*, 70, 85-103.

[6] Choi, J. and C. Stefanadis (2001), "Tying, Investment, and the Dynamic Leverage Theory", *RAND Journal of Economics,* 32, pp. 52-71.

[7] Chen, Yongmin, and Michael H. Riordan (2010), "Preferences, Prices and Performance in Multiproduct Industries", working paper.

[8] Chu, Sean, Phillip Leslie, and Alan Sorensen (2011), "Bundle-Size Pricing as an Approximation to Mixed Bundling," *American Economic Review*, forthcoming.

[9] Long, John B. Jr. (1984), "Comments on 'Gaussian Demand and Commodity Bundling," *Journal of Business*, 57, S235-S246.

[10] McAfee, R. Preston, John McMillan, and Michael D. Whinston (1989), "Multiproduct Monopoly, Commodity Bundling, and Correlation of Values," *Quarterly Journal of Economics*, 104, 371-83.

[11] Nalebuff, Barry (2004), "Bundling as an Entry Barrier," *Quarterly Journal of Economics*, 119, 159-87.

[12] Nelsen, Roger B. (2006), *An Introduction to Copulas*, 2^{nd} edition, Springer-Verlag.

[13] Schmalensee, Richard (1984), "Gaussian Demand and Commodity Bundling,"*Journal of Business*, 57, S211-S230.

[14] Stigler, George (1963), "A Note on Block Booking," U.S. Supreme *Court Review*, 1963, 152-175.

[15] Whinston, Michael D. (1990), "Tying, Foreclosure, and Exclusion," *American Economic Review*, 80, pp. 837-59.