Price Uncertainty and Derivative Securities in a General Equilibrium Model

by
Graciela Chichilnisky, Columbia University
Jayasri Dutta, Cambridge University
Geoffrey Heal, Columbia Business School

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G. Chichilnisky*                      J. Dutta
Columbia University                 Cambridge University

G. M. Heal†                           
Columbia Business School

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Abstract

Consider an exchange economy with multiple competitive equilibria. Agents know the set of equilibria, but not which will be selected. To insure against unfavorable equilibrium outcomes, they trade on markets for commodities contingent on the equilibrium price vector. Such price-contingent contracts allow agents to insure fully against the risk stemming from uncertainty about the equilibrium to be chosen. However they introduce further uncertainty because there may be several possible equilibrium prices for price-index-contingent commodities. The introduction of higher-order derivative products removes this uncertainty, but in turn introduces uncertainty about the prices of these products. We prove that in regular economies this process converges in a finite number of steps to a unique fully-insured Pareto efficient allocation. The introduction of price-contingent commodities or securities and further derivative securities removes all endogenous uncertainty associated with lack of knowledge of equilibrium prices. We thus provide a mechanism for resolving non-uniqueness in economies with multiple equilibria and also give an important resource-allocation role to derivative securities based on price indices.

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1 Introduction

Uncertainty about states of nature drives classical theories of resource-allocation under uncertainty. Examples of uncertain states of nature are the weather, and the occurrence of earthquakes or epidemics. However, the dominant uncertainties facing economic agents are clearly not about such exogenous states: they are about endogenous variables such as interest rates, exchange rates, securities prices, demand levels and employment levels. Certainly most corporate activity in risk markets broadly defined is to hedge against uncertainty about endogenous variables. Kurz [14] has called this endogenous uncertainty, and defined a research agenda and a framework for analysis that has many points in common with this paper. Kurz's work on rational beliefs [15] develops this agenda in a rather different direction from this paper.

Svensson [19] introduced some of these ideas into a temporary general equilibrium model. Chichilnisky and Wu [9] formalize endogenous uncertainty within a general equilibrium model with individual and collective risk. They show that this type of uncertainty is crucially dependent on the information structure and can be generated by financial innovation. Hahn [12] has addressed issues related to the present paper in an incomplete market model: his concern is to exhibit the logical inconsistency of rational expectations in a sequence economy where there are multiple equilibria in the second period. Chichilnisky Hahn and Heal [6] develop similar issues in a more general framework.

In view of the practical importance of endogenous uncertainty, we will not have a satisfactory theory of resource allocation under uncertainty until general equilibrium theory and welfare economics are extended to cover this case. Here our aim is to develop a formal model of institutions through which endogenous uncertainty can be resolved in a general equilibrium framework as close as possible to the usual Arrow-Debreu model. A natural extension of the Arrow-Debreu approach is to define commodities contingent on prices (instead of on exogenous states of nature) and allow agents to trade these. This is precisely the approach that we take: we explore the consequences of trading price-contingent commodities, show that this may indeed

\[\text{1Note that anthropogenic influences on climate, such as carbon dioxide emission and the release of CFCs, suggest that uncertainty about the weather is not truly exogenous.}\]
provide an efficient form of insurance against endogenous risk, and show that trading price-contingent commodities is equivalent to trading index options.

Consider a competitive exchange economy with multiple competitive equilibria. This is an Arrow-Debreu economy except for the structure of information. Agents know the set of market clearing prices, but do not know which will be selected. They thus face a very pristine form of endogenous uncertainty, namely uncertainty about the equilibrium price vector in a competitive model. This uncertainty has a welfare cost to the agents, who are assumed to be risk-averse. The welfare cost is the risk of lower levels of welfare at unfavorable equilibrium prices. To reduce this cost, agents are allowed to trade on markets for commodities that are contingent on the equilibrium price vector, or contingent on an index number derived from this price vector. The introduction of such price-contingent contracts allows agents to insure fully against the risk stemming from uncertainty about the equilibrium to be chosen. However further uncertainty arises if there are now several possible equilibrium prices in the market for price-contingent commodities.

To clarify these issues, consider the market for stock index futures. There may be several possible market clearing levels for this index number. This introduces price uncertainty for agents trading in this market. Such uncertainty can be removed by allowing agents to trade securities that payoff according to the index number (call these level 1 securities). However, the market for these level 1 securities may itself have multiple equilibria. So one source of uncertainty is removed only to be replaced by another. Now, this second round of uncertainty can in turn be removed by introducing higher-order or derivative securities, i.e., securities that pay off according to the value of the level 1 securities: these we call level 2 securities. Again, this introduction of new securities allows agents to remove one source of uncertainty only to introduce another. In general, the introduction of securities both removes one type of uncertainty and also introduces another type.

We show that in regular economies this process of introducing successive levels of derivative securities will remove all endogenous uncertainty in a finite number of steps and lead to a unique fully-insured and risk-free Pareto efficient allocation. Hence the introduction of price-contingent commodities (or equivalently securities), and further derivative securities, will remove all endogenous uncertainty associated with lack of knowledge of the equilibrium prices to be selected. We thus provide a mechanism for resolving non-uniqueness in economies with multiple equilibria, and also give an important resource-allocation role to derivative securities. We show that the payoff functions of the derivative securities that we introduce, can be replicated as the limit of payoff patterns emerging from trading combinations of options. In fact they are the payoff patterns of what Rubinstein [18] terms exotic options. Henrotte [13] has also considered the use of options as a method of reducing endogenous price uncertainty.

With several possible equilibria and agents' welfares depending on which is selected, there is an incompleteness in the market structure, in the sense of existence of uninsured risks, even though all of the normal Arrow-Debreu markets are present.
The introduction of markets for price-contingent goods (or securities) can resolve this and complete the market.

A fundamental constraint which distinguishes this situation from an Arrow-Debreu market is, however, that an auctioneer cannot clear simultaneously all markets. Goods, and securities or commodities contingent on the prices of those goods, cannot all be traded in the same market\(^2\). Arbitrage between the goods, and the securities whose payoffs are contingent on the prices of those goods, will restrict the set of possible securities prices in a way that ensures that the market remains incomplete (see example 1 below). In general one cannot trade in the same market both goods and also securities whose payoffs depend on the prices of those goods. For example, the Treasury could not simultaneously and in the same market auction Treasury bills and options on the prices of these Treasury bills at that date. In real markets a derivative security whose value depends on the price of an asset is always traded prior to the date at which the trading occurs to determine the price of the asset. However, in an Arrow-Debreu framework, there is only one date at which trading occurs. This makes it difficult to replicate the pattern of market access that one encounters in real markets for assets and associated derivatives. It is therefore necessary to work with a structure of budget constraints within the Arrow-Debreu model that replicates this pattern of market access. This is done by defining a set of assets contingent on the prices of other assets, some assets being logically prior to others. This structure, defined in detail in section 2, ensures that the set of possible equilibria in markets for logically prior assets is known (although the precise equilibrium to be selected is not) before derivatives based on these can be traded.

**Example 1.** Consider an economy with 2 goods, goods price vectors \( p \in \mathbb{R}^2 \), and several possible equilibrium price vectors. Let \( s(p) \) be a security that pays $1 iff the equilibrium price vector is \( p \), and \( q_{s(p)} \) be the price of this security. Suppose that goods and all possible securities are simultaneously traded in the same market, so that overall a price vector for this market is \( \{ p \in \mathbb{R}^2, (q_{s(p)}) \forall p \} \). Now if \( p = (1, 1) \), we must have \( q_{s(p)} = 0 \ \forall p \neq (1, 1) \) and \( q_{s(1,1)} = 1 \). Otherwise there would be infinite arbitrage opportunities.

This example illustrates the fact that the prices of securities are linearly dependent on those of goods when all are traded in the same market, so that no extra insurance possibilities are afforded by the introduction of securities. This issue is pursued at greater length in Chichilnisky Hahn and Heal [6].

### 2 A Framework for Endogenous Uncertainty

Consider an economy with \( i \) agents indexed by \( i \in I = \{1, .., I\} \) and \( J \) goods indexed by \( j \in J = \{1, .., J\} \). \( w_{ij} \) is agent \( i \)'s endowment of good \( j \) and \( w_i \) is agent \( i \)'s

\(^2\)We owe this observation to Olivier Compte.
endowment vector in $R^J$. Preferences are represented by utility functions $U_i : R^J \rightarrow R$, and consumption vectors are $c_i \in R^J$. We make the following assumptions:

A1. $\forall i$, $U_i$ is strictly concave, $C^2$ (twice continuously differentiable), monotonically increasing and has non-zero gradients.

A2. If $\{\pi^k, c^k_i\}$ is a lottery over consumption vectors $c^k_i$ for agent $i$ with probabilities $\pi^k$ then $i$'s utility from this lottery is $\sum_k \pi^k U_i(c^k_i)$.

The exchange economy defined thus far will be denoted $E^1$ and referred to as the underlying economy: in $E^1$ endowment are $w_i$, preferences are $U_i$, and the commodity space is $R^J$. $CE(E^1)$ will denote the set of competitive equilibria of $E^1$, with $p^k$ being the price vector at the $k$-th equilibrium and $c^k_i$ being agent $i$'s consumption at the $k$-th equilibrium.

A3. $E^1$ is regular, so that the cardinality of $CE(E^1)$, $\#^1$, is finite.

Agents know $CE(E^1)$, that is, they know the set of possible equilibria of the underlying economy. They also know that one of the equilibria of $E^1$ will be chosen randomly according to a commonly-known exogenous probability distribution $\pi^1 = \pi^1_k$, $k = 1, \ldots, \#^1$. They are allowed to hedge against the risks associated with the random selection of an equilibrium by trading goods contingent on the equilibrium selected in $E^1$. The market for these contingent goods will typically have in its turn multiple equilibria, and commodities contingent on the prices of these contingent goods will be needed to remove uncertainty thus introduced. In order to define this construction concisely, we introduce the concept of a multi-level economy $E$, in which the underlying economy $E^1$ forms the first level. $Y$ denotes the set of levels in $E$, with $y \in Y$ denoting a typical level. Levels are defined inductively as follows:

1. Level 1 is the underlying exchange economy $E^1$.

2. Level 2, denoted $E^2$, is a set of markets on which agents trade goods contingent on which element of $CE(E^1)$ is chosen according to the exogenous probabilities $\pi^1 = \pi^1_k$, $k = 1, \ldots, \#^1$. The number of states in $E^2$ is $\#^1$, the number of equilibria in $E^1$. Endowments in $E^2$ are consumption vectors at the equilibria of $E^1$, so that $c^1_{i,k}$ is agent $i$'s endowment at the $k$-th state of level 2, i.e., $w^2_{i,k} = c^1_{i,k} \in R^J$. The overall endowment vector of agent $i$ is $w^2_i = (c^1_{i,k})$ $k = 1, \ldots, \#^1 \in R^J\#^1$. Agent $i$'s preferences are $\sum_k \pi^1_k U_i(c^2_{i,k})$. $CE(E^2)$ is the set of competitive equilibria of $E^2$, which has cardinality $\#^2$, with a typical price vector being $p^2_k \in R^J\#^1$, and a typical consumption vector for agent $i$ in state $k$ being $c^2_{i,k}$ with $c^2_i = (c^2_{i,k})$ $k = 1, \ldots, \#^1 \in R^J\#^1$ being the overall consumption vector of agent $i$ across all states.

\textsuperscript{3}For a definition of regularity and proof that the number of equilibria in regular economies is generically finite, see Debreu [11]. In a regular economy, the number of equilibria is locally constant with respect to the initial endowments.
3. Level \( y \), denoted \( E^y \), is a set of markets on which agents trade goods contingent on which element of \( CE(E^{y-1}) \) is chosen according to the exogenous probabilities \( \pi^{y-1} = \pi_k^{y-1}, k = 1, ..., \#^{y-1} \). The number of states in \( E^y \) is \( \#^{y-1} \), the number of equilibria in \( E^{y-1} \). Endowments in \( E^y \) are consumption vectors at the equilibria of \( E^{y-1} \), so that \( c_{i,k}^{y-1} \) is agent \( i \)'s endowment at the \( k \)-th state of level \( y \), i.e., \( w_{i,k}^y = c_{i,k}^{y-1} \). The overall endowment vector of agent \( i \) is \( w_i^y = (c_{i,k}^{y-1}) k = 1, ..., \#^1 \in R^{\#^{y-1}} \). Agent \( i \)'s preferences are \( \sum_k \pi_k^{y-1} U_i(c_{i,k}^y) \). \( CE(E^y) \) is the set of competitive equilibria of \( E^y \), which has cardinality \( \#^y \), with a typical price vector being \( p_k^y \in R^{\#^y} \), and a typical consumption vector for agent \( i \) in state \( k \) being \( c_{i,k}^y \) with \( c_i^y = (c_{i,k}^y), k = 1, ..., \#_1 \in R^{\#^{y-1}} \), being the overall consumption vector of agent \( i \) across all states.

4. At every level \( y \), agents know the set \( CE(E^y) \) of competitive equilibria of that level, with cardinality \( \#^y \). They also know that one of these will be selected according to a commonly-known exogenous probability distribution \( \pi^y = \pi_k^y, k = 1, ..., \#^y \).

This multi-level economy provides a framework for analyzing endogenous uncertainty (see also Chichilnisky Hahn and Heal [6]). A realization \( s \) is the selection of an equilibrium at every level \( y \in Y \). It can be described by a list of \( \#(Y) \) integers drawn from a set \( S \) of \( \prod_{y=1}^{\#^y} \#^y \) possible such lists. \( \#(Y) \) is the cardinality of \( Y \): \( \#^y \) is as before the number of states at level \( y \), i.e., the number of equilibria at level \( y - 1 \). A realization at level \( y \) is the selection of an equilibrium at every level less than \( y \), and is a list of \( y \) integers drawn a set of \( \prod_{y=1}^{\#^y-1} \#^y \) such lists. A realization at level \( y \) induces realizations at all levels less than \( y \). In our model, level 2 realizations will be the equilibria of the underlying exchange economy \( E^1 \), level 3 realizations are pairs of equilibria, one from \( E^1 \) and one from \( E^2 \), the markets for goods contingent on equilibria in \( E^1 \). Level 4 realizations are lists of three equilibria, level 2 realizations plus a level 3 equilibrium, which is an equilibrium in a market for goods contingent on the prices at level 2, i.e. contingent on the prices of goods that are themselves contingent on the prices of goods in the underlying economy. Each level of the economy corresponds to a different class of derivative security, i.e. to a different class of securities whose payoffs depend on the values of other securities. These derivative securities are introduced in an order of logical priority, so that the payoff of each depends on the values of the prior ones.

There is another important aspect of the order in which derivative securities are introduced. This is that the extent of price uncertainty must first be clarified before agents can decide how to insure against it. So agents must know the set of equilibria in the underlying economy before they can decide how much insurance is appropriate. Likewise, they must know the extent of the uncertainty in the market for level 2 securities, i.e., the set of possible equilibria in this market, before they can choose the appropriate amounts of insurance. So before positions can be taken in any market, it
is necessary that the extent of the risks in logically prior markets be established, i.e., in our framework, that the set of equilibria in these markets be established, as it is this set which determines the risk faced by agents. It is of course also necessary that the uncertainty in these markets should not be resolved, i.e., a particular price chosen, until after agents have chosen quantities of insurance. These requirements determine the inductive structure of the levels of the economy defined above. In this structure, the set of equilibria for one level, which determines the extent of the uncertainty at that level, has to be established before agents take positions in securities at the next level that insure against this uncertainty. A set of equilibria, one per level, also known as a realization, is chosen (for example by the auctioneer) after positions have been taken by agents in all insurance markets.

It is worth emphasizing that when in \( E^1 \) the auctioneer ask agents for their net trades at various price vectors, agents do not of course know whether equilibrium in this economy is unique or multiple. Nor does the auctioneer. This information is only available after the excess demand function of \( E^1 \) has been investigated. Hence the only response an agent can reasonably give to a price vector quoted by the auctioneer, is the usual utility-maximizing net trade vector. This is true for the first level \( E^1 \) and for all other levels: when responding to prices in level \( y \), agents do not know whether there will be further levels or what these will be.

In this framework, it is natural that the endowments at each level should be the equilibrium consumptions of the previous level. These consumptions are the positions that agents need to insure. Any other specification of endowments for the next level would in effect be allowing agents to change their positions in securities traded in a level after trading at that level has been completed. To be more specific, the contracts traded at level 2 are goods contingent on the equilibrium price vector selected at level 1. Agents are trading, *inter alia*, "goods if the price vector is \( p_1 \)". An agent’s endowments in the original economy do not entitle her or him to any amount of "goods if the price vector is \( p_1 \)". However, the contracts entered into in the economy \( E^1 \) do entitle an agent to goods at each price vector in the amounts of the equilibrium consumption vectors at that equilibrium. Hence the agents' endowments of "goods if the price vector is \( p_1 \)" are naturally the equilibrium consumptions at these prices.

Within the multi-level economy \( E \), a realization \( s_y \) at level \( y \) specifies a price vector for commodities and one for each level of derivative securities up to level \( y \): it also specifies for each agent a consumption level for commodities and consumption levels for, or positions in, all types of contingent commodities up to level \( y \). Equilibrium \( k \) at level \( y \) then determines a realization-dependent consumption vector for each agent \( i \) at level \( y \), which we shall denote by \( c_{i,k}^y \). The overall consumption of agent \( i \) in realization \( s \) is the sum of the consumption vectors of agent \( i \) at every equilibrium selected in this realization. It is thus a consumption vector corresponding to the equilibrium selected in \( E^1 \), plus a consumption vector chosen in \( E^2 \) contingent on the equilibrium of \( E^1 \), plus a consumption vector chosen in \( E^3 \) contingent on the equilibrium in \( E^2 \), etc. Posterior levels of price-contingent contracts entitle agents to
delivery of goods vectors modifying their prior positions, and the overall consumption vector is the sum of all of these. Formally, define \( c_i(s) \) as agent i's overall consumption in realization \( s \). Then

\[
c_i(s) = \sum_{y=1}^{\#(Y)} c_{i,k(y,s)}^y
\]

where \( c_{i,k(y,s)}^y \) is agent i's consumption in equilibrium \( k(y,s) \) of level \( y \) and \( k(y,s) \) is the state chosen at level \( y \) in realization \( s \).

### 3 Preliminary Results

An important preliminary step in our argument is establishing that all equilibria at any level of the multi-level economy \( E \) are fully insured, i.e., they give consumption vectors which are independent of the equilibria selected in all previous levels.

**Definition:** We say that \( E \) achieves full insurance and its equilibria at any level are fully insured if for any level \( y \), and for any equilibrium \( k \) selected at level \( y \), agent i's consumption vector at level \( y \) in equilibrium \( k \) in state \( s_y \), \( c_{i,k}^y \), is independent of the state \( s_y \), i.e., of which equilibria are selected in all levels prior to \( y \). Equivalently, consumption in equilibrium \( k \) at level \( y \) is independent of the realization by which the chosen equilibrium at level \( y \) is reached.

**Lemma 1** Under assumptions (A1) to (A3) all equilibria at all levels of \( E \) are fully insured.

**Proof.** The strategy of the proof is as follows. We show that by strict concavity of utility functions and the fact that the total endowment of the economy is realization-independent, any realization-dependent allocation is dominated by one consisting of its expected values. Hence any Pareto efficient allocation must give realization-independent consumption vectors.

Let \( p_{ik}^y \) be the equilibrium price vector of the \( k \)-th equilibrium of the \( y \)-th level, \( c_{i,k}^y \) be the associated consumption vector of agent i, and \( c_{i,k}^{y*} \) be the consumption vector of agent i at the \( k \)-th equilibrium of level \( y \) in the \( s \)-th realization at that level. Note that for any realization \( s \), \( \sum_i c_{i,k}^{y*} = \sum_i w_i \) as the total endowment of the economy is the same at all realizations and all levels. Define \( Ec_{i,k}^y = \sum_{s_y} \pi_{s_y} c_{i,k}^{y*} \) as the expected consumption of agent i at level \( y \) in equilibrium \( k \) where the expectation is taken over realizations \( s_y \). By strict concavity of utility functions (A1), we have \( U_i(Ec_{i,k}^y) > \sum_{s_y} \pi_{s_y} U_i(c_{i,k}^{y*}) \) provided that the equilibrium consumption vector is not fully insured. It is feasible for agent i to consume \( Ec_{i,k}^y \) in each realization, since \( \sum_i Ec_{i,k}^y = \sum_{s_y} \pi_{s_y} c_{i,k}^y = \sum_i w_i \). Hence \( Ec_{i,k}^y \) forms a feasible allocation that is Pareto superior to \( Ec_{i,k}^y \), proving that the equilibrium must be fully insured.

Lemma 1 establishes that at any equilibrium in level \( y \), all endogenous uncertainty arising from uncertainty about the selection of equilibria in prior levels is removed.
So if this level had a unique equilibrium, agents would be fully insured against endogenous uncertainty. However, this level will in general have multiple equilibria. In Theorem 1 we show that in fact there is a finite number $N$ such that the equilibrium at level $N$ will be unique and thus full insurance against endogenous uncertainty will in fact be established. Analogous results about fully insured equilibria were established in Malinvaud [16] [17] and in Cass and Shell [4].

We shall use the concepts of utility possibility set and utility possibility frontier for the economy $E^1$. The utility possibility set (UPS) is a subset of $R^4$ consisting of utility vectors $\{U_1(c_1), U_2(c_2), \ldots, U_1(c_I)\}$ corresponding to feasible allocations in $E^1$, i.e., allocations $\{c_1, \ldots, c_I\}$ satisfying $\sum_i c_i \leq \sum_i w_i$. The utility possibility frontier (UPF) is the efficient frontier of the utility possibility set. We assume that the UPS is a compact set in $R^4$. Closedness is automatic: boundedness requires extra conditions. For boundedness it would suffice if consumption sets were bounded below, or in the case that they are unbounded, that preferences satisfy for example the limited arbitrage condition of Chichilnisky [5] or condition (C) of Chichilnisky and Heal [7].

Lemma 2 establishes that if we add an infinite sequence of levels to the economy $E^1$ then in the limit the resulting equilibrium consumption allocations are Pareto efficient, and the associated utility vector is in the UPF. In Theorem 1 we tighten this result to show that in regular economies it in fact holds after the addition of only a finite number of levels. Recall that $c_{i,k}^y \in R^4$ is agent $i$'s consumption at the $k$-th equilibrium of the $y$-th level: $\sum_y \pi_y U_i(c_{i,k}^y)$ is the expected utility of this vector, where the expectation is taken over all equilibria at level $y$. We abbreviate it to $EU_i[y]$ : the vector of expected utility levels for all agents is then $EU[y]$.

**Lemma 2** The vector of agents' expected utilities of consumption across equilibria of the $y$-th level converges to the UPF as the number of levels becomes infinite, i.e., $EU[y] \rightarrow UPF$ as $y \rightarrow \infty$. Furthermore, the sequence of expected utility vectors $EU[y]_{y=1,2,\ldots}$ is Pareto improving, i.e., $EU[y+1] \geq EU[y]$ (where $\geq$ denotes greater than or equal to in all coordinates and greater in some).

**Proof.** $U_i(c_{i,k}^y)$ denotes agent $i$'s utility from the $k$-th equilibrium in the realization $y$: $U_i(c_{i,k}^y)$ is the expected utility from this equilibrium over all realizations. By Lemma 1, $c_{i,k}^y$ is realization-independent. So we let $c_{i,k}^y$ stand for the consumption of agent $i$ at equilibrium $k$ of level $y$, without specifying the realization. Define the following subsets of the UPS:

$$I^v = \left\{ x \in UPS \subset R^4 : \forall i, x_i \geq \min_k U_i(c_{i,k}^y) \right\}$$

$I^v$ is the set of utility vectors that give each agent a utility level at least as great as that which the agent obtains at the equilibrium which is worst for that agent at level $y$. We define $B^v$ similarly, except that the minimum utility level across equilibria is
replaced by the expected utility level across equilibria. $\pi^y_k$ is as usual the probability of equilibrium $k$ being selected at level $y$.

$$B^y = \{ x \in UPS \subset R^I : \forall i, x_i \geq \sum_k \pi^y_k U_i(c^y_{i,k}) \}$$

This is the set of utility vectors in the UPS that give each agent at least the expected utility associated with level $y$. By construction of the levels, $\sum_k \pi^y_k U_i(c^y_{i,k})$ equals the expected utility of the endowment vectors of agent $i$ at level $y$. Clearly we have $B^y \subset I^y$ unless minimum and expected utility levels are equal. In this case all equilibria give the same utility values and we have a unique equilibrium. By Lemma 1 this is fully insured. As each equilibrium is Pareto efficient by normal arguments this gives a utility vector in the UPF and we are done. From now on we assume that there are multiple equilibria at each level. As the equilibria of level $y + 1$ are weakly Pareto superior to the endowments of this level, $I^{y+1} \subseteq B^y$. Hence we have the sequence:

$$I^1 \supset B^1 \supset I^2 \supset B^2 \supset I^3 \supset B^3 \ldots I^y \supset B^y \ldots$$

The subsequence $\{I^y\}, y = 1, 2, \ldots$ defines a strictly nested sequence of subsets of the UPS, which is itself a compact set bounded above by the UPF. In each dimension $i$ of $R^I$ the greatest lower bound of $I^{y+1}$ exceeds that of $I^y$ (as we have assumed that the equilibrium is not unique), as

$$\forall i, \min_k U_i(c^y_{i,k}) < \sum_k \pi^y_k U_i(c^y_{i,k}) \leq \min_k U_i(c^{y+1}_{i,k})$$

But $\lim_{y \to \infty} \{ \min_k U_i(c^y_{i,k}) - \min_k U_i(c^{y+1}_{i,k}) \} = 0$, because the sets $I^y$ are bounded above. Now $\min_k U_i(c^y_{i,k}) - \min_k U_i(c^{y+1}_{i,k}) = 0$ implies $\min_k U_i(c^y_{i,k}) = \sum_k \pi^y_k U_i(c^y_{i,k})$. This is true only if the equilibrium at level $y$ is unique. Hence in the limit the equilibria at level $y$ converge to a unique equilibrium, which is Pareto efficient. By (2) above expected utility vectors are Pareto improving as $y$ increases. This completes the proof. •

We have assumed in (A3) that the underlying exchange economy $E^1$ is regular, so that it has a finite number of equilibria. We now also require the set of markets at each level to form a regular economy, so that at each level the number of equilibria is finite. We assume this explicitly:

**A4. At every level, the set of equilibria is finite.**

Such a property could be formally derived from more basic assumptions, including smoothness assumptions on demand functions. An outline of how this could be done follows: we offer only an outline of the derivation of this property, as a complete proof would involve an excursion into differential topology which is not the purpose of this paper. In outline the argument is as follows.
The endowments of all agents in \( E_1 \) can be represented by a vector \( W^1 \) in \( R^{IJ} \). Debreu [11] establishes that the set of endowments \( W^1 \) for which \( E_1 \) has a finite set of equilibria, is a "generic set", having a closure that is of Lebesgue measure one. Furthermore, he shows that on a neighborhood of an endowment vector \( W^1 \) in this generic set, the map \( \Theta : W^1 \in R^{IJ} \rightarrow (R^{IJ})^{#^1} \), which sends endowment vectors \( W^1 \) into \( #^1 \) vectors of equilibrium consumptions, can be represented by \( #^1 \) continuously differentiable functions \( \Theta_k : W^1 \in R^{IJ} \rightarrow R^{IJ}, k = 1, \ldots, #^1 \), such that the equilibria are the \( #^1 \) distinct points \( \Theta_1 (W^1), \ldots, \Theta_{#^1} (W^1) \).

The equilibria of \( E_1 \) are the endowments of \( E_2 \) : call a vector of endowments in \( E_2 \), \( W^2 \in (R^{IJ})^{#^1} \). Then for endowments in \( E_1 \) in a neighborhood of a regular endowment vector, \( W^2 = \Theta_1 (W^1), \ldots, \Theta_{#^1} (W^1) \) depends differentiably on the endowment of \( E_1 \). Furthermore, by Debreu's theorem on economies with a finite set of equilibria, for a generic set of \( W^2 \in (R^{IJ})^{#^1} \), the number of equilibria in \( E_2 \) is finite. The consumption vectors of this finite set of equilibria form the endowments of \( E_3 \).

Suppose that \( W^2* \in (R^{IJ})^{#^1} \) is such that \( E_2 \) in fact has an infinite number of equilibria. Then within any neighborhood of \( W^2* \) there exists \( W^2* \) such that the number of equilibria arising from \( W^2* \) is instead finite. Such a \( W^2* \) can be obtained as the endowment vector in \( E_2 \) by a small modification of the endowments in \( E_1 \) from \( W^1 \) to \( W^1* \) : formally, \( W^2* = \Theta_1 (W^1*), \ldots, \Theta_{#^1} (W^1* \) where \( W^1* \) is close to \( W^1 \). This follows from the fact that the Walras correspondence is a manifold (Balasko [2]) and from the strict concavity of preferences.

In general, suppose that at all levels up to level \( k \) the set of equilibria is finite, but that there are infinitely many equilibria at level \( k \). A small change in the endowments at level \( k \) will generate an economy with a finite set of equilibria, and this small change can be produced by a small change in the endowments of the economy \( E_1 \).

By such arguments, one can establish that the finiteness of the set of equilibria at every level, for any finite number of levels, is at least an open property in the underlying exchange economy \( E_1 \).

4 Insurance against Endogenous Uncertainty

We can now state and prove the main theorem of this paper, which establishes that the introduction of a finite number of derivative securities at a finite number of different levels in the economy will suffice to remove all endogenous uncertainty and to provide a unique Pareto efficient and fully-insured allocation of resources in the underlying economy \( E_1 \). Note that this allocation is not one of the competitive equilibria of the underlying economy \( E_1 \).

**Theorem 1** Let the underlying exchange economy \( E_1 \) satisfy (A1) to (A4). Then there is a finite number \( N \) such that an \( N \)-level economy defined as in Section 2 will have a unique equilibrium consumption vector that is fully insured and is Pareto
efficient in $E^1$. The addition of extra levels of derivative security markets up to level $N$ leads to Pareto improvements.

Proof. If an initial endowment vector $W^1 \in R^{IJ}$ is Pareto efficient in $E^1$, then there is a unique no-trade equilibrium associated with this, which is just the initial endowment vector. By Balasko [3] an initial endowment vector that is Pareto efficient is a generic vector in the sense of Debreu’s theorem [11]. By Debreu’s theorem, there is a neighborhood of this initial endowment within which the equilibrium is still unique. Hence there is a neighborhood, denoted $\Psi$, of the set of Pareto efficient allocations in $E^1$ such that if the initial endowment is in $\Psi$ then the equilibrium is unique.

An equilibrium consumption vector for an agent at level $y$ is an element of $R^J \#^{y-1}$, where $\#^{y-1}$ is the number of states at level $y$, i.e. the number of equilibria at level $y - 1$. However, as by Lemma 1 equilibria are fully insured against uncertainty about the equilibria to be selected at previous levels, the equilibrium consumption vector is fully identified by its first $J$ components. It can be regarded as an element of $R^J$ and a consumption vector in $E^1$. Hence we shall denote by $c_{i,k}^y \in R^J$ agent $i$'s consumption at equilibrium $k$ in level $y$.

By Lemma 2, we know that the properties asserted by the theorem hold as $N \to \infty$. In particular, the vector of agents’ expected utility levels across equilibria $EU[y]$ converges to a point in the UPF as $y \to \infty$. As the vector of agents’ expected utility levels converges to a point in the UPF, the associated consumption vectors $(c_{i,k}^y)_{k=1, \ldots, \#^y}$ must converge to a Pareto efficient allocation in $E^1$. Hence for sufficiently large $N$, the equilibrium consumption vector will be an element of $\Psi$ and so the competitive equilibrium at level $N$ will be unique. This proves the theorem.

We have now established that introducing a finite number of levels of securities, and their derivatives, will provide full insurance against the endogenous uncertainty arising from lack of knowledge of the equilibrium price vector. Many levels are needed because the introduction of each level of securities will in general remove the uncertainty associated with not knowing the equilibrium price of the previous level, but will introduce further uncertainty arising from lack of knowledge of the price at this new level. The securities and their derivatives have to be traded as part of a hierarchical multi-level economy because, as discussed in the introduction and in Chichilnisky Hahn and Heal [6], goods, price-contingent goods and their derivatives cannot all be traded in an economy with one single budget constraint. In this case arbitrage would enforce relations between the various prices that would make the price-contingent goods and their derivatives redundant. Theorem 1 establishes that markets for price-contingent goods will lead each agent to a fully-insured consumption vector, i.e., a vector that is independent of the realization selected in the multi-level economy with $N$ levels. The consumption vector of any agent corresponding to a realization is the sum of the agent’s consumption vectors at the equilibria chosen at each level in that realization.

We conjecture that the number $N$ of levels needed to achieve uniqueness and full
insurance depends on the probability distributions over equilibria at each level. The intuition for this conjecture is as follows. In the extreme case that the distribution assigns probability one to a single equilibrium at level one, clearly no further levels are needed. Consider now a case in which the distribution assigns a probability close to one to a single equilibrium \( CE_1 \) at level one. In this case the vector of expected utilities across equilibria of level one, \( EU[1] \), is close to the utility possibility frontier UPF because it is close to the utility vector associated with the single equilibrium \( CE_1 \) at which the probability is concentrated, and this equilibrium is Pareto efficient. Hence the expected utility of endowments at level 2, which equals \( EU[1] \), is also close to the UPF. The expected utility vector across the equilibria of level two, \( EU[2] \), is Pareto superior to \( EU[1] \), and so is closer to the UPF than is \( EU[1] \). Hence if \( EU[1] \) were close enough to the UPF, i.e., the probability distribution at level one were sufficiently concentrated at the single equilibrium \( CE_1 \), the equilibria of level two and so the endowments of level three would be within the neighborhood \( \Psi \) of the set of Pareto efficient allocations within which equilibria are unique. In this case a unique and fully insured equilibrium would be attained with three levels.

In the next section, we turn to an institutional interpretation of Theorem 1. We relate it to securities based on price indices and to trading strategies based on options.

## 5 Institutional Interpretations

There are two results that help give a concrete institutional interpretation to Lemma 2 and Theorem 1. One is that contracts that are contingent on the value of a price vector, can also be designed to be contingent on the value of a price index. Contracts contingent on a price vector can therefore be interpreted as index contracts. The second is that the pattern of payoffs as a function of goods prices exhibited by our price-contingent securities, can be replicated by a limit of trading strategies based on index options. Overall, these two results imply that the securities (price-contingent commodities) traded in our multi-level economy, can be understood as options on price indices and their derivatives.

**Lemma 3** Consider a set of distinct price vectors \( p^1, p^2, \ldots, p^k \in R^l \). Then for an open dense set of \( \Omega \in R^l \), each price vector gives a different value to the index \( p'.\Omega \).

**Proof.** Assume \( \Omega \neq 0 \). Let \( p^1, \ldots, p^g \) be such that

\[
p^i.\Omega \neq p^j.\Omega, \forall i \neq j \text{ and } i, j \in \{1, \ldots, g\}
\]

(2)

For \( p^{g+1}, \ldots, p^k \) we have

\[
p^{g+1}.\Omega = p^{g+2}.\Omega = \ldots = p^k.\Omega
\]

(3)

Property (2) is an open property, so that there is a neighborhood \( N_\Omega \) of \( \Omega \) within which it holds. Property (3) is a closed property and implies that \( (p^i - p^j).\Omega = 0 \forall i, j \)
in \{g+1, ..., k\}. Now for any \(x \neq 0\), \(\{\Omega : x.\Omega = 0\}\) has measure zero and has no interior in \(R^J\). Hence in the neighborhood \(N_\Omega\) of \(\Omega\) there exists a \(\Omega'\) such that (3) fails and (2) still holds. The set of such \(\Omega'\) is open. This proves the theorem.

This result assures us that distinct equilibrium price vectors map into distinct index numbers using almost any set of weights. Hence we can refer interchangeably to contracts that pay contingent on the value of a price vector or to contracts that pay contingent on the value of a price index.

Finally, we relate the payoff patterns of the contracts traded in the multi-level economy to those that can be realized by trading index options. The contracts traded in level 2 are goods contingent on the equilibrium price vector selected in \(E^1\). By Theorem 2 above, they can be interpreted as goods contingent on the value of a price index. Now, from the work of Arrow [1], we know that a market for contingent goods can be replaced by a market for securities that pay contingent on the same event plus spot markets for the uncontingent goods (see also the discussion in Chichilnisky Hahn and Heal [6]). Hence in level two we can trade securities that pay if and only if a price index attains a particular value, i.e. their payoff as a function of the index number is zero everywhere except at a single point. This payoff structure can be replicated by the limit of a sequence of option trading strategies. Assume that the critical value of the price index is \(V\), i.e. we require a non-zero payoff if and only if the index assumes value \(V\). The basic option trading strategy is as follows:

- Buy \(n\) index call options with exercise price \(\big(V - \frac{1}{n}\big)\).
- Sell \(2n\) index call options with exercise price \(V\).
- Buy \(n\) index call options with exercise price \(\big(V + \frac{1}{n}\big)\).

It is routine to verify that as \(n \to \infty\), the payoff function from this strategy converges pointwise to a function that is zero everywhere except at \(V\), where it assumes value 1. Hence the desired payoff structure can be approximated arbitrarily closely by the above strategy for \(n\) sufficiently large\(^4\). It is also worth noting that the payoff functions that characterize our index-contingent securities, are those associated with what Rubinstein [18] calls "binary options": these are also discussed in Cox and Rubinstein [10]. It follows from this that the contracts traded in level 2, goods contingent on the equilibrium price vector selected in level 1, can be replaced by a combination of spot markets and markets for options based on the price index in level 1. The same reinterpretation can of course be carried out at other levels.

\(^4\)This analysis of course ignores transaction costs. If the cost of each option trade are positive, then they will become infinite in the limit. The relative payoffs at different values of \(V\) are not affected by transaction costs, but their absolute values are affected and may be made negative.
6 Concluding Comments

There is some connection between our results and those on sunspots (Cass and Shell [4]). Both literatures study uncertainty which does not affect the economy’s total endowments. Cass and Shell call this extrinsic uncertainty: we call it endogenous uncertainty. There is therefore a common element in the motivation of the studies. However, there are also big differences in the way the analysis is conducted. Sunspots by assumption do not directly affect any real variables: any effect that they have is via agents’ beliefs and their impact on agents’ behavior. Here, however, the state selected by definition has an effect on real variables because it determines the equilibrium chosen. A clear contrast is with Proposition 3 of [4], which states that with complete markets ("unrestricted market participation") and agreement on probabilities, sunspots do not matter. All of our results occur in the context of complete markets and agreement on probabilities, precisely the case in which sunspot phenomena are not important. In addition, sunspots can matter in economies with unique equilibria ([4], appendix), a case about which we have nothing to say. So the two strands of literature are complementary.

There are several natural directions for further research. One is to study the sensitivity of our results to the assumption that all agents know and agree on the probabilities of the endogenous states. It appears that Lemma 1 (on full insurance) depends strongly on this assumption, and subsequent results in turn depend strongly on Lemma 1. So the treatment of endogenous uncertainty with diverse beliefs appears to be an open issue. Another natural development of our present analysis, is in the direction of a model of asset pricing. Derivative securities play a natural and integral role in our model: it would be of interest to investigate the relationship between their prices at equilibrium, the equilibrium prices of goods in the underlying economy $E^1$, and the probability distributions over equilibria. It would be of particular interest to have formulae for equilibrium derivative securities prices emerge from a model in which these securities are not redundant.\footnote{In the usual arbitrage pricing models, the securities being priced are spanned by others and so from a risk-bearing perspective are redundant.}

References


